Portfolios : define f as the current date t value of the foward contract and $\tau \equiv T - t$ as its time until maturity.

⟨date t⟩

A : A long position in one foward contract written on an asset having current value of S and having a forward price of K.

B : One share of the underlying asset plus borrowing an amount $e^{-r\tau} K$

⟨date T⟩

$A : \tilde{S}_T - K$
$B : \tilde{S}_T - K$
same cash flow at date T

$$f = e^{-r\tau} \hat{E}[\tilde{S}_T - K] = e^{-r\tau} \hat{E}[\tilde{S}_T] - e^{-r\tau} K$$

In risk-neutral , $\hat{E}[\tilde{S}_T] = S e^{r\tau} \Rightarrow f = S - e^{-r\tau} K$

Risk-adjusted : $f = e^{-\theta\tau} \cdot \hat{E}[\tilde{S}_T] - K e^{-r\tau}$, $\theta = r + \beta(E[\tilde{r}_m - r]$

Black-Scholes Formula : $c = e^{-r\tau} \hat{E}[\max(0, \tilde{S}_T - X)]$

$$\ln(\tilde{S}_T) \sim N\left(\ln(S) + \left(r - \frac{\sigma^2}{2}\right)\tau,\ \sigma^2\tau\right)$$

$$\Rightarrow c = e^{-r\tau} \int_X^\infty (S_T - X)\, g(S_T)\, dS_T$$

$$\Rightarrow c = S \cdot N(d_1) - X e^{-r\tau} N(d_2)$$

$$d_1 = \frac{\ln(S/X) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

$$p = c + X e^{-r\tau} - S$$

when $\sigma \to 0 \Rightarrow d_1 \to \infty$, $d_2 \to \infty$

$$\text{if } S > Xe^{-r\tau}, \quad \begin{cases} \text{call}: & S - Xe^{-r\tau} \\ \text{put}: & 0 \end{cases}$$

$$S < Xe^{-r\tau}, \quad \begin{cases} \text{call}: & 0 \\ \text{put}: & Xe^{-r\tau} - S \end{cases}$$