BOPM

$$S \begin{cases} \xrightarrow{q} S \cdot u \\ \xrightarrow{1-q} S \cdot d \end{cases} \qquad \text{where } 0 < q < 1 \ , \ d < u$$

$$C \begin{cases} \xrightarrow{q} C_u = \max(0, S_u - X) \\ \xrightarrow{1-q} C_d = \max(0, S_d - X) \end{cases}$$

Set up a protfolio of h shares of stock and B dollars in riskless bonds

① Call

$\Rightarrow$ Cost : $hS + B \quad = C$

$$\underbrace{\text{payoff} : \begin{cases} hS_u + RB \\ hS_d + RB \end{cases}}_{\text{Portfolio}} \quad \underbrace{\begin{matrix} = C_u \\ = C_d \end{matrix}}_{\text{European Call}} \quad \Rightarrow \quad h = \frac{C_u - C_d}{S_u - S_d} \ , \ B = \frac{uC_d - dC_u}{(u-d)R}$$

② Put

$$h = \frac{P_u - P_d}{S_u - S_d} \ , \quad B = \frac{uP_d - dP_u}{(u-d)R}$$

American

① Call

$$c = \max(hS + B, S - X)$$

② put

$$p = \max(hS + B, X - S)$$

Risk : $qS_u + (1-q)S_d \rightarrow$ expected value of the stock

$$\because \quad h = \frac{C_u - C_d}{S_u - S_d}, \quad B = \frac{uC_d - dC_u}{(u-d)R}$$

$$\Rightarrow \quad hS + B = \frac{C_u - C_d}{u-d} + \frac{uC_d - dC_u}{(u-d)R}$$

$$= \frac{1}{R}\left(\frac{1}{u-d}\left(RC_u - RC_d + uC_d - dC_u\right)\right)$$

$$= \left[\left(\frac{R-d}{u-d}\right)C_u + \left(\frac{u-R}{u-d}\right)C_d\right] \cdot \frac{1}{R}$$

$$\downarrow$$

$$\Rightarrow \quad p \triangleq \frac{R-d}{u-d}$$

Risk-Neutral Probability : $pS_u + (1-p)S_d = RS$