$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$$

let $f = \log S_t$, $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial S_t} = \frac{1}{S_t}$, $\frac{\partial^2 f}{\partial S_t^2} = -\frac{1}{S_t^2}$

$$d\log S_t = \left(\frac{1}{S_t}\cdot \mu S_t - \frac{1}{S_t^2}\cdot\frac{1}{2}(\sigma S_t)^2\right)dt + \frac{1}{S_t}(\sigma S_t)\,dW_t$$

$$= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma\,dW_t \quad , \quad dW_t \sim N(0, dt)$$

$$d\log S_t \sim N\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)dt\ ,\ \sigma^2 dt\right)$$

$$\Delta \log S_T = \log S_T - \log S_0 \quad , \quad dt = T$$

$$\log S_T \sim N\left(\log S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T\ ,\ \sigma^2 T\right)$$

$$d\log S_u = \left(\mu - \tfrac{1}{2}\sigma^2\right)du + \sigma\,dW_u$$

$$\int_t^T d\log S_u = \int_t^T \left(\mu - \tfrac{1}{2}\sigma^2\right)du + \int_t^T \sigma\,dW_u$$

$$\log S_T - \log S_t = \left(\mu - \tfrac{1}{2}\sigma^2\right)\cdot(T-t) + \sigma(W_T - W_t)$$

$$\log S_T = \log S_t + \left(\mu - \tfrac{1}{2}\sigma^2\right)\cdot(T-t) + \sigma(W_T - W_t)$$

$$S_T = S_t \cdot e^{\left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\Delta W_{T-t}}$$

$$\Delta W_{T-t} = W_T - W_t \sim N(0, (T-t)\sigma^2) \sim N(0, T-t)$$

Hull White Term Structure Simulations

$$dr = (\theta(t) - ar)\,dt + \sigma\,dW$$

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + a f(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)$$