

From undirected to directed diffusive networks of dynamical agents

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Outline

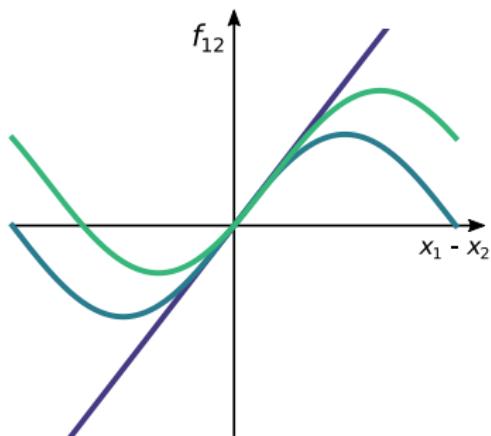
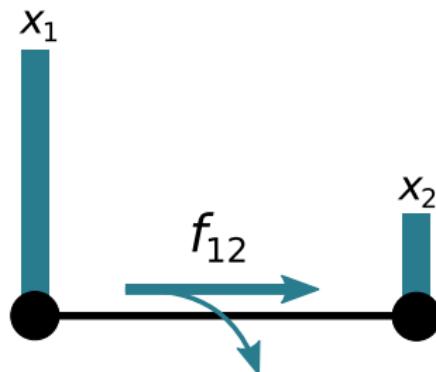
- ▶ Introduction
- ▶ Power Flows and the Kuramoto model
- ▶ Main properties
- ▶ Synchronization and Multistability

Diffusive networks

How is a commodity transmitted over a network?



Pairwise interaction



$a_{ij} \neq a_{ji}$ or $h(x) \neq -h(-x)$

$$f_{ij} = a_{ij} h(x_i - x_j)$$

$$f_{ji} = a_{ji} h(x_j - x_i)$$

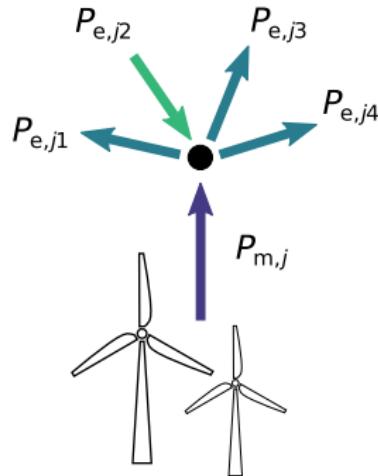
Active power flows:

$$P_{ij} = B_{ij} \sin(\theta_i - \theta_j - \phi)$$

Power flows and the Kuramoto-Sakaguchi model

The power flow equations

- ▶ Voltage: $V_j e^{i\theta_j}$.
- ▶ Power: $P_j + iQ_j$.
- ▶ Admittance: $G_{jk} + iB_{jk}$.
- ▶ Electrical power flow: $P_{e,jk}$.

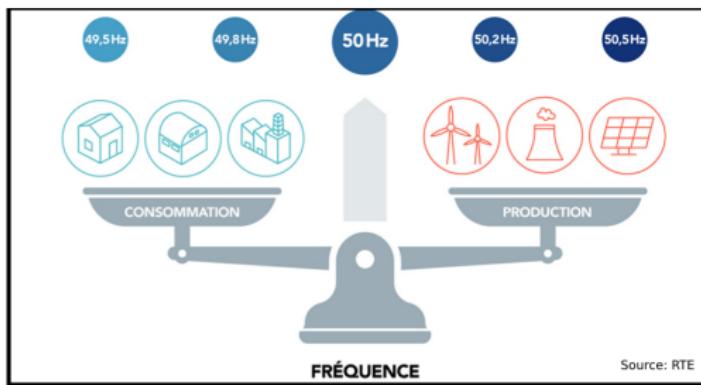
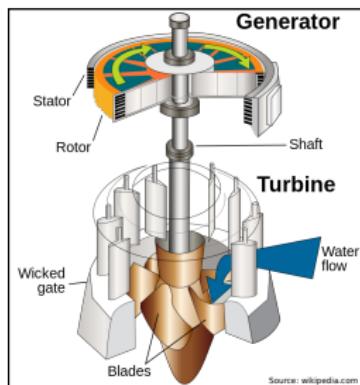


$$P_j = \sum_k V_j V_k [B_{jk} \sin(\theta_j - \theta_k) + G_{jk} \cos(\theta_j - \theta_k)] ,$$

$$Q_j = \sum_k V_j V_k [G_{jk} \sin(\theta_j - \theta_k) - B_{jk} \cos(\theta_j - \theta_k)] .$$

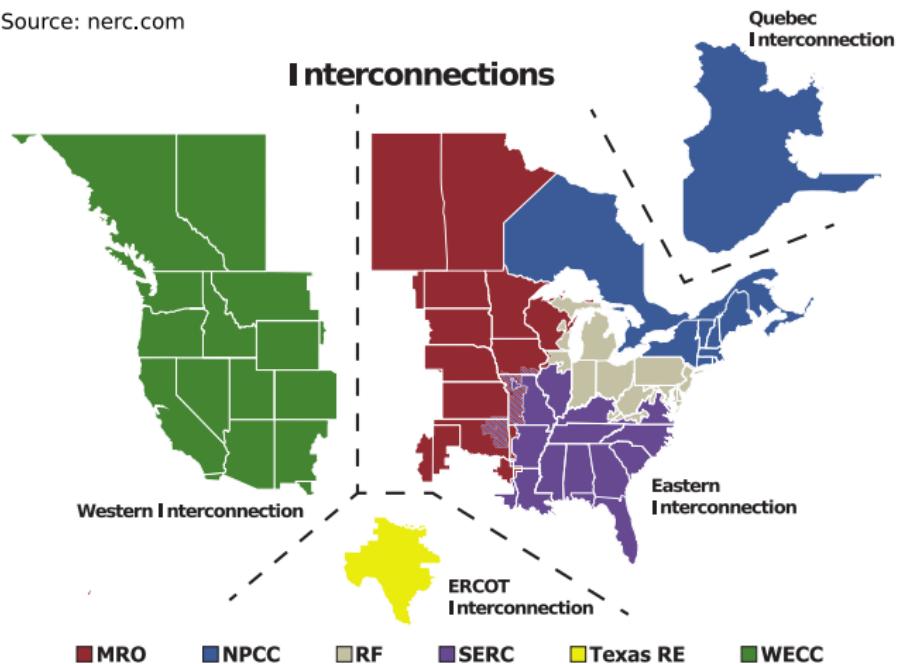
The swing equations

$$\begin{aligned} m_j \ddot{\theta}_j + d_j \dot{\theta}_j &= P_{m,j} - P_{e,j} \\ &= P_j - \sum_k B_{jk} \sin(\theta_j - \theta_k) + G_{jk} \cos(\theta_j - \theta_k) \end{aligned}$$

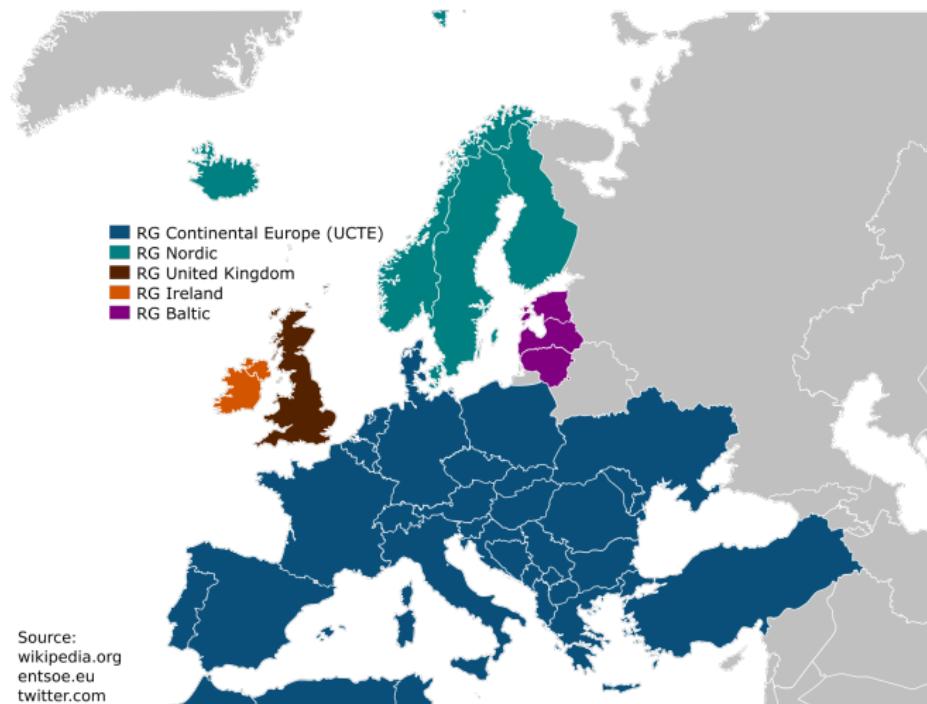


Synchronous power grid

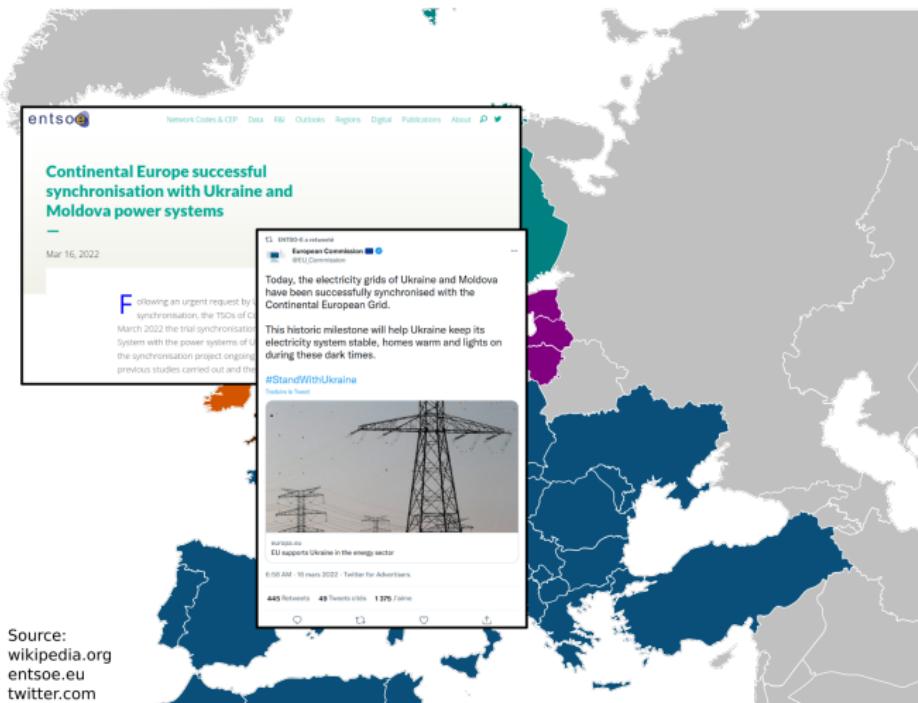
Source: nerc.com



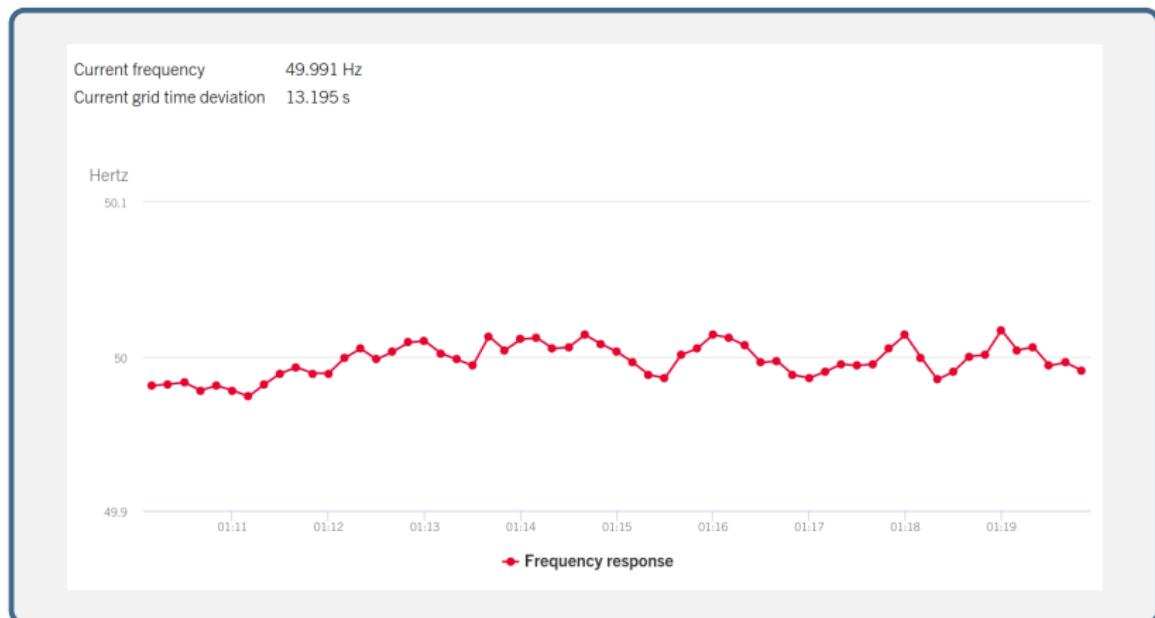
Synchronous power grid



Synchronous power grid



Grid frequency



Source: www.swissgrid.ch (Apr. 26, 2022)

The Kuramoto-Sakaguchi model

$$m_j \ddot{\theta}_j + d_j \dot{\theta}_j = P_j - \sum_k B_{jk} \sin(\theta_j - \theta_k) + G_{jk} \cos(\theta_j - \theta_k)$$

$$\dot{\theta}_j = P_j - \sum_k a_{jk} \sin(\theta_j - \theta_k - \phi)$$

$$\dot{\theta}_j = P_j - \sum_k a_{jk} \sin(\theta_j - \theta_k).$$

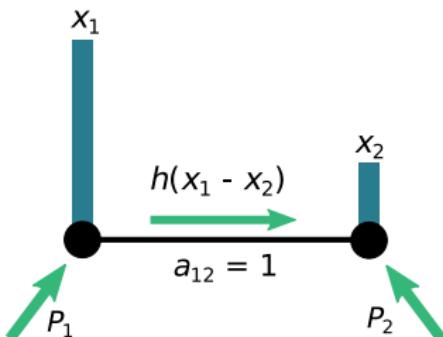
Assumptions:

$$m_j = 0, \quad d_j = 1, \quad a_{jk} = \sqrt{B_{jk}^2 + G_{jk}^2}, \quad \phi = \arctan(G_{jk}/B_{jk}).$$

Diffusive network - summary

$$\dot{x}_i = P_i - \sum_j a_{ij} h(x_i - x_j)$$

Potentially: $a_{ij} \neq a_{ji}$,
 $h(x) \neq -h(-x)$.



- ▶ P_i : Natural frequency, commodity injection,...
- ▶ a_{ij} : Element of the adjacency matrix;
- ▶ h : Coupling function, flow function,...
- ▶ x_i : Agent's state.

Main properties

- ▶ **Flow conservation**
- ▶ Vectorial form
- ▶ Spectrum of the Jacobian

Example: 6-cycle

Cycle of 6 identical Kuramoto-Sakaguchi oscillators.

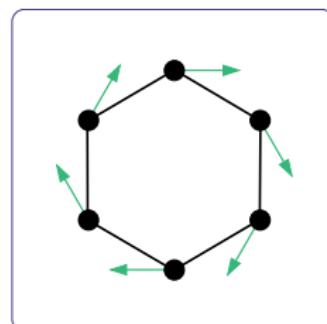
$$\begin{aligned}\dot{x}_i &= - \sum_j a_{ij} \sin(x_i - x_j - \phi) \\ &= -\sin(x_i - x_{i-1} - \phi) - \sin(x_i - x_{i+1} - \phi)\end{aligned}$$

$$x_0 = (0, 0, 0, 0, 0, 0)$$

$$\implies \dot{x}_i = 2 \sin \phi$$

$$x_1 = (0, \pi/3, 2\pi/3, \pi, -2\pi/3, -\pi/3)$$

$$\implies \dot{x}_i = \sin \phi$$



Flow conservation

$$\dot{x}_i = P_i - \sum_j a_{ij} h(x_i - x_j)$$

Undirected coupling: $a_{ij} h(x) = -a_{ji} h(-x)$

[e.g., Kuramoto: $h(x) = \sin(x)$].



Directed coupling: $a_{ij} h(x) \neq -a_{ji} h(-x)$

[Kuramoto-Sakaguchi: $h(x) = \sin(x - \phi)$]



Average velocity

$$\dot{x}_i = P_i - \sum_j a_{ij} h(x_i - x_j)$$

Undirected coupling: $a_{ij} h(x) = -a_{ji} h(-x)$

$$\frac{1}{n} \sum_i \dot{x}_i = \frac{1}{n} \sum_i P_i = \bar{P}$$

Directed coupling: State-dependent \implies time-dependent.

Main properties

- ▶ Flow conservation
 - *No conservation of flows and average velocity.*
- ▶ **Vectorial form**
- ▶ Spectrum of the Jacobian

Vectorial form (lossless case)

Linear diffusion:

$$\dot{x}_i = P_i - \sum_j a_{ij}(x_i - x_j), \quad \rightsquigarrow \quad \dot{x} = P - Lx \\ = P - BB^\top x.$$

Nonlinear lossless diffusion [$h(x) = -h(-x)$]:

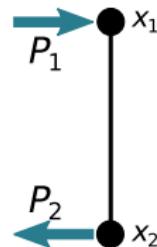
$$\dot{x}_i = P_i - \sum_j a_{ij}h(x_i - x_j), \quad \rightsquigarrow \quad \dot{x} = P - Bh(B^\top x).$$

$$h(B^\top x) = \begin{pmatrix} \vdots \\ h(x_i - x_j) \\ \vdots \end{pmatrix} \quad \begin{aligned} \dot{x}_i &= \cdots - h(x_i - x_j) \cdots \\ \dot{x}_j &= \cdots + h(x_i - x_j) \cdots \end{aligned}$$

Example: 2 agents

$$\dot{x}_1 = P_1 - \sin(x_1 - x_2 - \phi)$$

$$\dot{x}_2 = P_2 - \sin(x_2 - x_1 - \phi)$$



Reminder: $\dot{x} = P - Bh(B^\top x)$.

Let us try: $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$, $B = \begin{pmatrix} +1 \\ -1 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

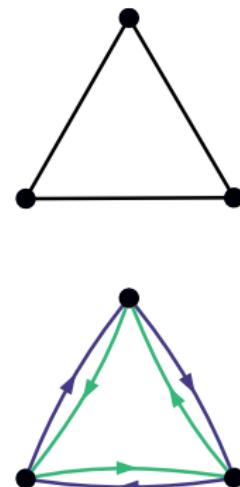
$$\dot{x} = P - B \sin(B^\top x - \phi 1_m) = \begin{pmatrix} P_1 - \sin(x_1 - x_2 - \phi) \\ P_2 - \sin(x_2 - x_1 + \phi) \end{pmatrix}$$

Directed incidence matrices

$$B = \begin{pmatrix} +1 & 0 & +1 \\ -1 & +1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

$$B_d = (B, -B) = \begin{pmatrix} +1 & 0 & +1 & -1 & 0 & -1 \\ -1 & +1 & 0 & +1 & -1 & 0 \\ 0 & -1 & -1 & 0 & +1 & +1 \end{pmatrix}$$

$$B_o = [B_d]_+ = \begin{pmatrix} +1 & 0 & +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & +1 \end{pmatrix}$$

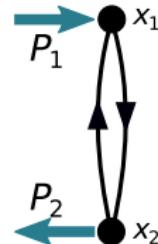


$$L = BB^\top = B_o B_d^\top$$

Example: 2 agents, cont.'d

$$\dot{x}_1 = P_1 - \sin(x_1 - x_2 - \phi)$$

$$\dot{x}_2 = P_2 - \sin(x_2 - x_1 - \phi)$$



Let us try: $B_d = \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}$, $B_o = \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix}$

$$\dot{x} = P - B_o \left[\sin(B_d^\top x - \phi 1_{2m}) \right] = \begin{pmatrix} P_1 - \sin(x_1 - x_2 - \phi) \\ P_2 - \sin(x_2 - x_1 - \phi) \end{pmatrix}$$

$$\boxed{\dot{x} = P - B_o h(B_d^\top x)}$$

Main properties

- ▶ Flow conservation
 - *No flow conservation anymore.*
- ▶ Average velocity
 - *Average velocity is state-/time-dependent.*
- ▶ Vectorial form
 - *There exists a natural vectorial form.*
- ▶ **Spectrum of the Jacobian**

The Jacobian

- (i) $h(x) = -h(-x)$ $\implies \frac{\partial}{\partial x_j} h(x_i - x_j) = \frac{\partial}{\partial x_i} h(x_j - x_i).$
- (ii) $a_{ij} = a_{ji}.$

$$\text{(i) \& (ii)} \implies [J(x)]_{ij} = [J(x)]_{ji}.$$

1. $\lambda_1 = 0;$
2. Eigenvalues are real;
3. Eigenvectors are orthonormal.

For directed coupling:

1. Preserved;
2. No guarantee;
3. Not true in general.

Main properties

- ▶ Flow conservation
 - *No flow conservation anymore.*
- ▶ Average velocity
 - *Average velocity is state-/time-dependent.*
- ▶ Vectorial form
 - *There exists a natural vectorial form.*
- ▶ Spectrum Jacobian
 - *Linearization is not symmetric.*

Synchronization of oscillator systems

Dynamics on the n -torus

Kuramoto: 2π -periodic coupling.

$$\dot{x}_i = P_i - \sum_j a_{ij} \sin(x_i - x_j)$$

From Euclidean space to the torus:

$$x_i \in \mathbb{R}$$

$$\rightarrow$$

$$\theta_i \in \mathbb{S}^1 = [-\pi, \pi)$$

$$x \in \mathbb{R}^n$$

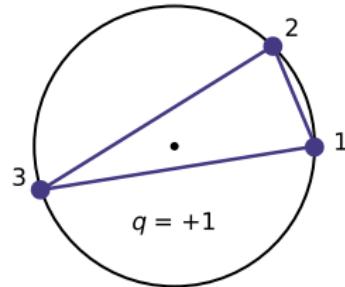
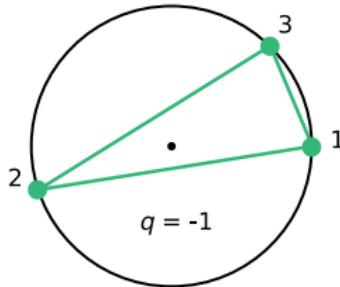
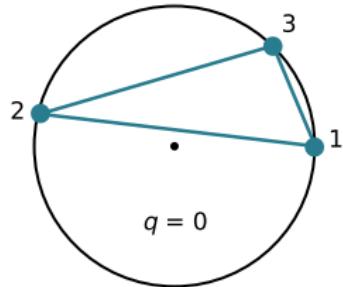
$$\rightarrow$$

$$\theta \in (\mathbb{S}^1)^n = \mathbb{T}^n$$

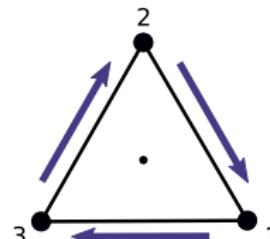
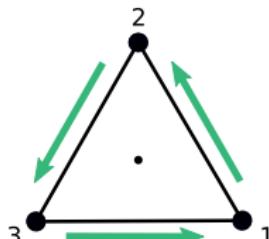
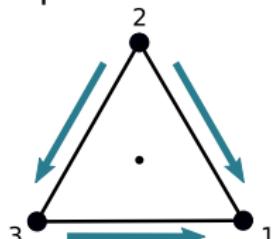


Winding number and loop flows

Given a cycle $\sigma = (i_1, \dots, i_\ell)$: $q = (2\pi)^{-1} \sum_{k=1}^{\ell} d_{cc}(\theta_{i_k}, \theta_{i_{k-1}}) \in \mathbb{Z}$.



Loop flows:



Winding vectors and partition

For a cycle σ .

The **winding number**:

$$q_\sigma: \mathbb{T}^n \rightarrow \mathbb{Z}$$

$$\theta \mapsto q_\sigma(\theta)$$

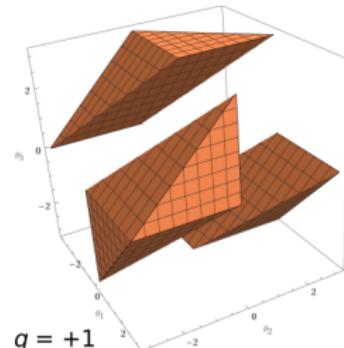
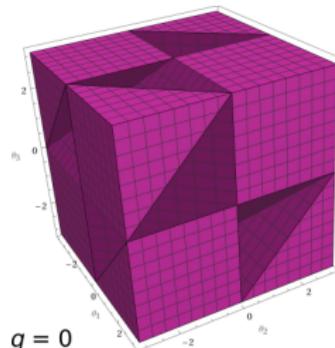
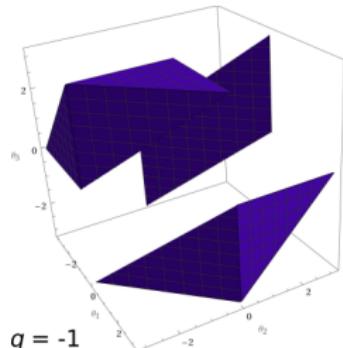
For a cycle basis $\Sigma = (\sigma_1, \dots, \sigma_c)$.

The **winding vector**:

$$q_\Sigma: \mathbb{T}^n \rightarrow \mathbb{Z}^c$$

$$\theta \mapsto [q_{\sigma_1}(\theta), \dots, q_{\sigma_c}(\theta)]$$

Winding cells: $\Omega_u = \{\theta \mid q_\Sigma(\theta) = u\}$.



Synchronization in Kuramoto-Sakaguchi

$$\dot{x} = P - B_o h(B_d^\top x) = P - B_o \left[h_c(B_d^\top x) + h_d(B_d^\top x) \right]$$

Theorem

Consider a connected network of coupled oscillators. For sufficiently low dissipation to coupling ratio,

$$d_{\max} \cdot L_d < \lambda_2 \cdot L_c ,$$

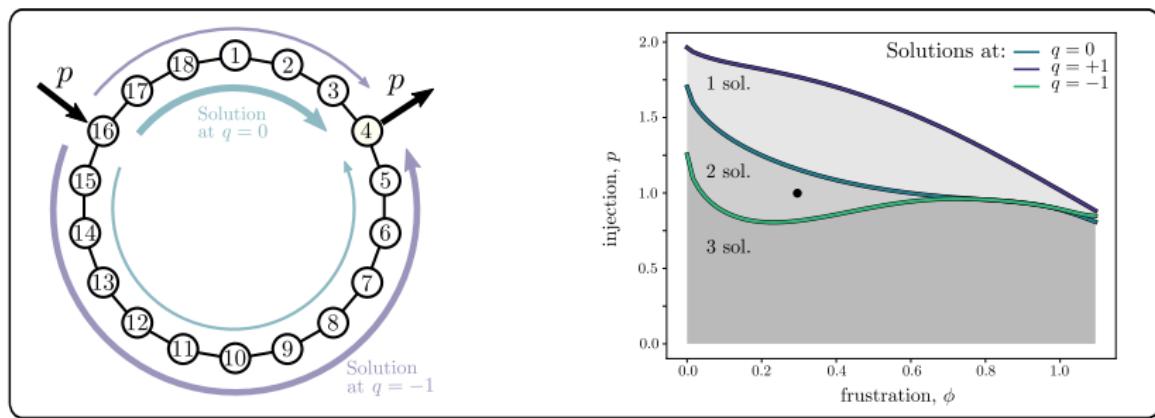
there is at most one stable synchronous state in each winding cell.

RD, S. Jafarpour, and F. Bullo, *Multistability and Paradoxes in Lossy Oscillator Networks*, arXiv preprint: [2202.02439](https://arxiv.org/abs/2202.02439) (2022).

S. Jafarpour, E. Y. Huang, K. D. Smith, and F. Bullo, *Flow and Elastic Networks on the n-Torus: Geometry, Analysis, and Computation*, SIAM Review **64** (2022). doi: [10.1137/18M1242056](https://doi.org/10.1137/18M1242056)

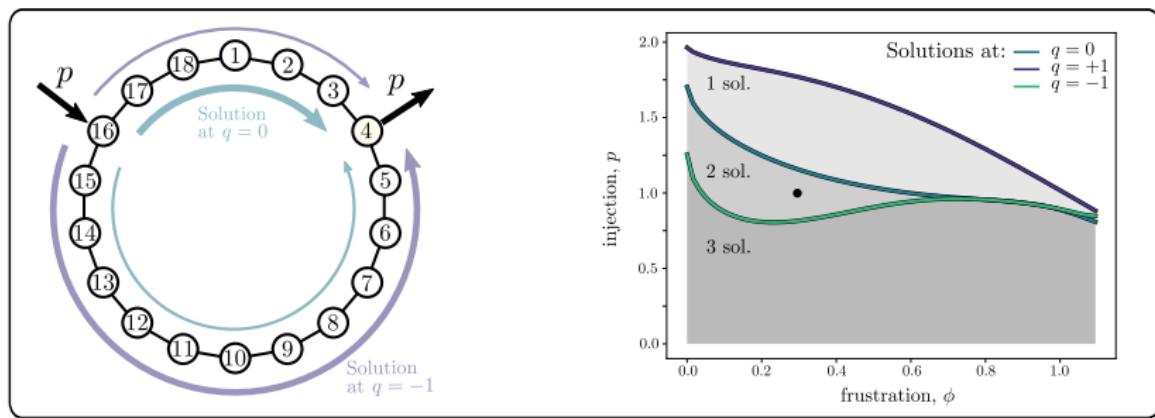
Paradox 1: "Loop flows increase capacity."

$$\dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j - \phi)$$



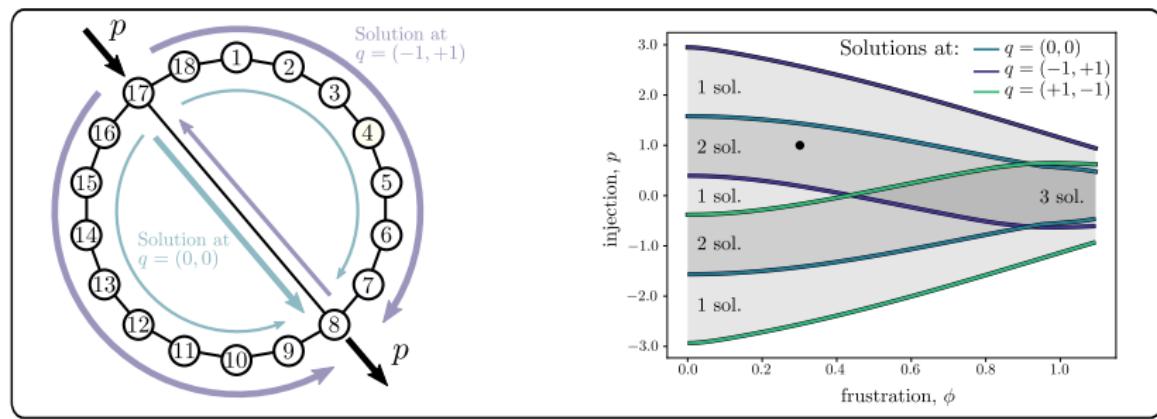
Paradox 2: "Frustration increases capacity."

$$\dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j - \phi)$$



Paradox 3: "Frustration promotes multistability."

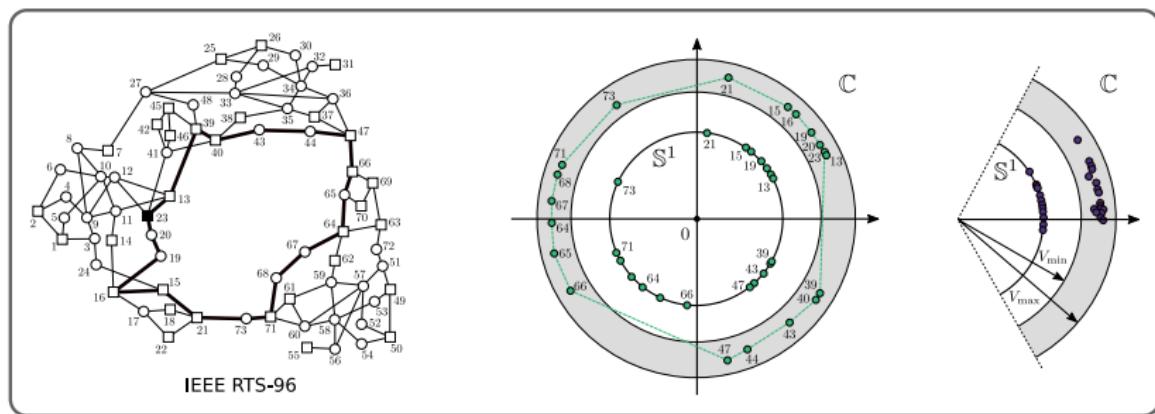
$$\dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j - \phi)$$



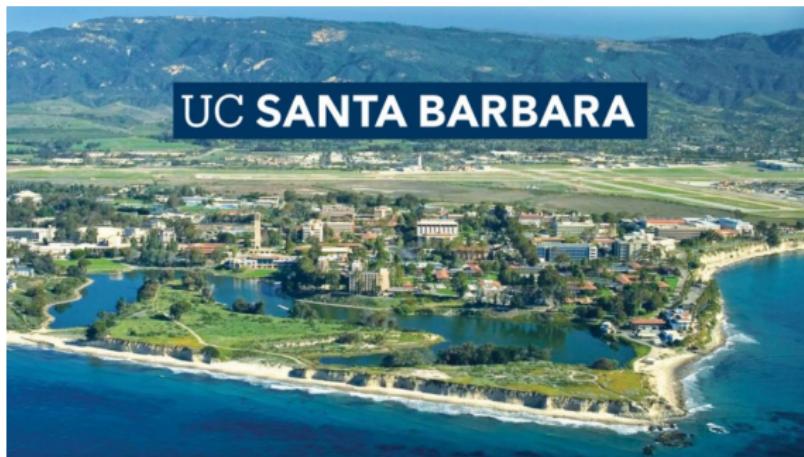
Wrap-up

- Summary:**
- ▶ Properties preserved and not;
 - ▶ Natural framework for directed systems;
 - ▶ Synchronization of networked oscillators.
- Follow-ups:**
- ▶ Relax the assumptions for *at most uniqueness*;
 - ▶ Refine the power flow models.

Loop flows in power grids?



Thank you!



UC SANTA BARBARA

Balbuena et al.

Consider a weighted directed graph and let:

- ▶ A : adjacency matrix;
- ▶ D_o, D_i : out- and in-degree matrices;
- ▶ $L = D_o - A$: Laplacian matrix;
- ▶ B : incidence matrix.
- ▶ $B_o = [B]_+, B_i = [B]_-$: out- and in-incidence matrices;
- ▶ W : edge weight matrix.

Then,

$$D_o = B_o W B_o^\top, \quad D_i = B_i W B_i^\top, \quad A = B_o W B_i^\top, \quad L = B_o W B^\top.$$

Derivation of the power flow equations

Define:

- ▶ Current: $I_j \in \mathbb{C}$;
- ▶ Voltage: $V_j = |V_j|e^{i\theta_j}$;
- ▶ Power: $S_j = P_j + iQ_j$;
- ▶ Impedance: $Z_{jk} = R_{jk} + iX_{jk}$;
- ▶ Admittance: $Y_{jk} = Z_{jk}^{-1} = G_{jk} + iB_{jk}$.

Compute the power:

$$\begin{aligned}P_j + iQ_j &= V_j I_j^* = V_j \left[\sum_k Z_{jk}^{-1} (V_k - V_j) \right]^* = \sum_k Y_{jk}^* |V_j| |V_k| e^{i(\theta_j - \theta_k)} \\&= \sum_k (V_j V_k [B_{jk} \sin(\theta_j - \theta_k) + G_{jk} \cos(\theta_j - \theta_k)]) \\&\quad + i V_j V_k [G_{jk} \sin(\theta_j - \theta_k) - B_{jk} \cos(\theta_j - \theta_k)] .\end{aligned}$$

Idea of the proof

1. Define the iteration map leaving the winding cells invariant:

$$S_\epsilon: \mathbb{R}^m \rightarrow \mathbb{R}^m$$

$$\Delta \mapsto \Delta - \epsilon B^\top L^\dagger [B_o h(\Delta) - P] .$$

2. Decompose the coupling function:

$$h_f(x) = [h(x) - h(-x)] / 2, \quad h_I(x) = [h(x) + h(-x)] / 2$$

3. Construct the state dependent graphs with Laplacian matrices

$$L_f(x) = B \cdot \text{diag}(\dots, h'_f(x_i - x_j), \dots) \cdot B^\top,$$

$$L_I(x) = B \cdot \text{diag}(\dots, |h'_I(x_i - x_j)|, \dots) \cdot B^\top.$$

4. Compare $\lambda_2(L_f)$ and $\text{diag}(L_I)$.

Synchronization in the oriented Kuramoto model

- ▶ "Kuramoto model": $\dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$.
 - ▶ "Oriented": $a_{ij} \neq 0 \implies a_{ji} = 0$.
 - ▶ "Synchronization": $\dot{\theta}_i = \dot{\theta}_j$
-

- ▶ Acyclic;
- ▶ Homogeneous cycle;
- ▶ Combination of the above.

