EGGs are adhesive!

Roberto Biondo[a], Davide Castelnovo[b], Fabio Gadducci[c]

[a]*Department of Computer Science, University of Pisa, , Pisa 56127, , Italy*
[b]*Department of Computer Science, University of Pisa, , Pisa 56127, , Italy*
[c]*Department of Computer Science, University of Pisa, , Pisa 56127, , Italy*

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1. Introduction

The introduction of *adhesive categories* marked a watershed moment for the algebraic approaches to the rewriting of graph-like structures [25, 15]. Until then, key results of the approaches on e.g. parallelism and confluence had to be proven over and over again for each different formalism at hand, despite the obvious similarity of the procedure. Adhesive categories provides such a disparate set of formalisms with a common abstract framework where many of these general results can be recast and uniformly proved once and for all.

In short, following the double-pushout (DPO) approach to graph transformation [13, 15], a rule is given by two arrows $l : K \to L$ and $r : K \to R$ and its application requires a match $m : L \to G$: the rewriting step from G to H is then given by the diagram aside, whose halves are pushouts.

$$\begin{tikzcd} L \arrow[d, "m"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d] & R \arrow[d] \\ G & C \arrow[l] \arrow[r] & H \end{tikzcd}$$

Thus, L and R are the left- and right-hand side of the rule, respectively, while K witnesses those parts that must be present for the rule to be executed, yet that are not affected by the rule itself. Should a category be adhesive, and the arrows of the rules monomorphisms, the presence of a match ensures that if the pushout complement $C \to G$ exists then it is unique, hence a rewriting step can be deterministically performed. The theory of $\mathcal{M}$-adhesivity [2, 22] extends the core framework, ensuring that if the arrows of the rules are in a suitable class $\mathcal{M}$ of monomorphisms then the benefits of adhesivity can be recovered [16, 17]. If only the left-hand side belongs to $\mathcal{M}$, the theory is still under development, as witnessed e.g. by [3]. However, despite the elegance and effectiveness, proving that a given category satisfies the conditions for being $\mathcal{M}$-adhesive can be a daunting task. For this reason, sufficient criteria have been provided for the core framework, e.g. that every elementary topos is adhesive [26], as well as for the extended one of $\mathcal{M}$-adhesivity [11]. For some structures such as *hypergraphs with equivalence* in [4], the question of their $\mathcal{M}$-adhesivity has not yet been settled.

E-graphs (shortly, EGGs) are an up-and-coming formalism for program optimisation and synthesis via a compact representation and efficient implementation of equality saturation. Albeit a classical data structure [14], EGGs received new impulse after the seminal [33] and developed a thriving community, as witnessed by the official website [31]. The key idea of rewriting-based program optimisation is to perform the manipulation of a syntactical description of a program, replacing some of its components in such a way that the semantics is preserved while the computational costs of its actual execution are improved. Instead of directly removing sub-programs, EGGs just add the new components and link them to the older ones via the equivalence relation, until an optimal program is reached and extracted.

EGGs can be concisely defined as term graphs equipped with a notion of equivalence on nodes that is closed under the operators of a signature [14, Section 4.2]. In the presentation of term graphs via string diagrams [11], EGGs are (hyper)trees whose edges are labelled by operators and with the possibility of sharing subtrees, with an additional equivalence relation $\equiv$ on nodes that is closed under composition. In plain words and using a toy example: if a and b are two constants such that $a \equiv b$, then $f(a) \equiv f(b)$ for any unary operator f.

Building on the criterion developed in [11], this work proves that both hypergraphs with equivalence and EGGs form an $\mathcal{M}$-adhesive category for a suitable choice of $\mathcal{M}$. The advantages from this characterisation are two-fold. On the one side, we put the benefits *per se* of a formal presentation, making precise the properties of the data structure. On the other side, describing the optimisation steps via the DPO approach offers the tools for modelling their parallel and concurrent execution and for proving their confluence and termination.

Synopsis The paper has the following structure. In Section 2 we briefly recall the theory of $\mathcal{M}$-adhesive categories and of kernel pairs. In Section 3 we present the graphical structures of our interest, (labelled) hypergraphs and term graphs, and we provide a functorial characterisation, which allows for proving their adhesivity properties. This is expanded in Section 4 for proving the $\mathcal{M}$-adhesivity of hypergraphs and term graphs with equivalence and in Section 5 of their variants where equivalences are closed with respect to operator application, thus subsuming EGGs. In Section 6 we put the machinery at work, showing how the optimisation steps can be rephrased as the application of term graph rewriting rules. Finally, in Section 7 we draw our conclusions, hint at future endeavours and offer some brief remarks on related works. For the sake of space, the proofs appear in the appendices.

2. Facts about $\mathcal{M}$-adhesive categories and kernel pairs

We open this background section by fixing some notation. Given a category $\mathbf{X}$ we do not distinguish notationally between $\mathbf{X}$ and its class of objects, so "$X \in \mathbf{X}$" means that X is an object of $\mathbf{X}$. We let $\mathsf{Mor}(\mathbf{X})$, $\mathsf{Mono}(\mathbf{X})$ and $\mathsf{Reg}(\mathbf{X})$ denote the class of all arrows, monos and regular monos of $\mathbf{X}$, respectively. Given an object X, we denote by $?_X$ the unique arrow from an initial object into X and by $!_X$ that unique arrow from X into a terminal one. We will also use the notation $e\colon X \twoheadrightarrow Y$ to denote that an arrow $e\colon X \to Y$ is a regular epi.

2.1. $\mathcal{M}$-adhesivity

The key property of $\mathcal{M}$-adhesive categories is the *Van Kampen condition* [7, 23, 25], and for defining it we need some notions. Let $\mathbf{X}$ be a category. A subclass $\mathcal{A}$ of $\mathsf{Mor}(\mathbf{X})$ is said to be

- *stable under pushouts (pullbacks)* if for every pushout (pullback) square as the one aside, if $m \in \mathcal{A}$ ($n \in \mathcal{A}$) then $n \in \mathcal{A}$ ($m \in \mathcal{A}$);
- *closed under composition* if $h, k \in \mathcal{A}$ implies $h \circ k \in \mathcal{A}$ whenever h and k are composable;
- *closed under decomposition*, or *left-cancellable*, if whenever g and $g \circ f$ belong to $\mathcal{A}$, then $f \in \mathcal{A}$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Definition 2.1. Let $\mathcal{A} \subseteq \mathsf{Mor}(\mathbf{X})$ be a class of arrows in a category $\mathbf{X}$ and consider the cube below on the right.

We say that the bottom square is an $\mathcal{A}$-*Van Kampen square* if

1. it is a pushout square;
2. whenever the cube above has pullbacks as back and left faces and the vertical arrows belong to $\mathcal{A}$, then its top face is a pushout if and only if the front and right faces are pullbacks.

$$\begin{tikzcd}[row sep=small, column sep=small] & A' \arrow[rr, "f'"] \arrow[dl, "m'"'] \arrow[dd, "a" near start] & & B' \arrow[dl, "n'"'] \arrow[dd, "b"] \\ C' \arrow[rr, "g'" near start, crossing over] \arrow[dd, "c"'] & & D' & \\ & A \arrow[rr, "f" near start] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\ C \arrow[rr, "g"'] & & D \arrow[from=uu, "d" near start, crossing over] & \end{tikzcd}$$

Pushout squares that enjoy only the "if" half of item (2) above are called $\mathcal{A}$-*stable*. A $\mathsf{Mor}(\mathbf{X})$-Van Kampen square is called *Van Kampen* and a $\mathsf{Mor}(\mathbf{X})$-stable square *stable*.

We can now define $\mathcal{M}$-adhesive categories.

Definition 2.2 ([2, 16, 17, 25, 22])**.** Let $\mathbf{X}$ be a category and $\mathcal{M}$ a subclass of $\mathsf{Mono}(\mathbf{X})$ including all isos, closed under composition, decomposition, and stable under pullbacks and pushouts. The category $\mathbf{X}$ is said to be $\mathcal{M}$-*adhesive* if

1. it has $\mathcal{M}$-*pullbacks*, i.e. pullbacks along arrows of $\mathcal{M}$;
2. it has $\mathcal{M}$-*pushouts*, i.e. pushouts along arrows of $\mathcal{M}$;
3. $\mathcal{M}$-pushouts are $\mathcal{M}$-Van Kampen squares.

A category $\mathbf{X}$ is said to be *strictly* $\mathcal{M}$-*adhesive* if $\mathcal{M}$-pushouts are Van Kampen. We write $m \colon X \rightarrowtail Y$ to denote that an arrow $m \colon X \to Y$ belongs to $\mathcal{M}$.

Remark 2.3. Our notion of $\mathcal{M}$-adhesive category corresponds to what in [15] is called *weak adhesive HLR category*, while a strict $\mathcal{M}$-adhesive categories corresponds to *adhesive HLR* ones. Finally, *adhesivity* and *quasiadhesivity* [25, 18] coincide with strict $\mathsf{Mono}(\mathbf{X})$-adhesivity and strict $\mathsf{Reg}(\mathbf{X})$-adhesivity, respectively.

$\mathcal{M}$-adhesivity is well-behaved with respect to the construction of (co-)slice and functor categories [27], as well with respect to subcategories, as shown by the following properties, taken from [15, Thm. 4.15], [25, Prop. 3.5] and [11, Thm. 2.12].

Proposition 2.4. *Let $\mathbf{X}$ be an (strict) $\mathcal{M}$-adhesive category. Then the following hold*

1. *if $\mathbf{Y}$ is an (strict) $\mathcal{N}$-adhesive category, $L \colon \mathbf{Y} \to \mathbf{A}$ a functor preserving $\mathcal{N}$-pushouts and $R \colon \mathbf{X} \to \mathbf{A}$ one preserving $\mathcal{M}$-pullbacks, then $L{\downarrow}R$ is (strictly) $\mathcal{N}{\downarrow}\mathcal{M}$-adhesive, where*

$$\mathcal{N}{\downarrow}\mathcal{M} := \{(h, k) \in \mathsf{Mor}(L{\downarrow}R) \mid h \in \mathcal{N}, k \in \mathcal{M}\}$$

2. *for every object X the categories $\mathbf{X}/X$ and $X/\mathbf{X}$ are, respectively, (strictly) $\mathcal{M}/X$-adhesive and (strictly) $X/\mathcal{M}$-adhesive, where*

$$\mathcal{M}/X := \{m \in \mathsf{Mor}(\mathbf{X}/X) \mid m \in \mathcal{M}\} \qquad X/\mathcal{M} := \{m \in \mathsf{Mor}(X/\mathbf{X}) \mid m \in \mathcal{M}\}$$

3. *for every small category $\mathbf{Y}$, the category $\mathbf{X}^{\mathbf{Y}}$ of functors $\mathbf{Y} \to \mathbf{X}$ is (strictly) $\mathcal{M}^{\mathbf{Y}}$-adhesive, where $\mathcal{M}^{\mathbf{Y}} := \{\eta \in \mathsf{Mor}(\mathbf{X}^{\mathbf{Y}}) \mid \eta_Y \in \mathcal{M}$ for every $Y \in \mathbf{Y}\}$;*
4. *if $\mathbf{Y}$ is a full subcategory of $\mathbf{X}$ closed under pullbacks and $\mathcal{M}$-pushouts, then $\mathbf{Y}$ is (strictly) $\mathcal{N}$-adhesive for every class of arrows $\mathcal{N}$ of $\mathbf{Y}$ contained in $\mathcal{M}$ that is stable under pullbacks and pushouts, contains all isos, and is closed under composition and decomposition.*

We briefly list some examples of $\mathcal{M}$-adhesive categories.

Example 2.5. **Set** is adhesive, and, more generally, every topos is adhesive [26]. By the closure properties above, every presheaf $[\mathbf{X}, \mathbf{Set}]$ is adhesive, thus the category $\mathbf{Graph} = [E \rightrightarrows V, \mathbf{Set}]$ is adhesive where $E \rightrightarrows V$ is the two objects category with two morphisms $s, t \colon E \to V$. Similarly, various categories of hypergraphs can be shown to be adhesive, such as term graphs and hierarchical graphs [11]. Note that the category **sGraphs** of simple graphs, i.e. graphs without parallel edges, is $\mathsf{Reg}(\mathbf{sGraphs})$-adhesive [5] but not quasiadhesive.

We can state some useful properties of $\mathcal{M}$-adhesive category (see, for instance, [15, Thm. 4.26] or [2, Fact 2.6]). A proof is provided in Section Appendix A.1.

Proposition 2.6. *Let $\mathbf{X}$ be an $\mathcal{M}$-adhesive category. Then the following hold*

1. *every $\mathcal{M}$-pushout square is also a pullback;*
2. *every arrow in $\mathcal{M}$ is a regular mono.*

2.2. Some properties of kernel pairs and regular epimorphisms

In this section we recall the definition and some properties of *kernel pairs.*

Definition 2.7. A *kernel pair* for an arrow $f \colon A \to B$ is an object K_f together with two arrows $\pi^1_f, \pi^2_f \colon K_f \rightrightarrows A$, denoted as (K_f, π^1_f, π^2_f), such that the square aside is a pullback.

$$\begin{tikzcd} K_f \arrow[r, "\pi^2_f"] \arrow[d, "\pi^1_f"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Remark 2.8. If (K_f, π^1_f, π^2_f) is a a kernel pair for $f \colon X \to Y$ and a product of X with itself exists, then the canonical arrow $\langle \pi^1_f, \pi^2_f \rangle \colon K_f \to X \times X$ is a mono.

Remark 2.9. An arrow $m \colon M \to X$ is a mono if and only if it admits $(M, \mathsf{id}_M, \mathsf{id}_M)$ as a kernel pair.

Together with Theorem Appendix A.1, the previous remarks allow us to prove the following result.

Proposition 2.10. *Let $f \colon X \to Y$ be an arrow and $m \colon Y \to Z$ a mono. If (K_f, π^1_f, π^2_f) is a kernel pair for $f \colon X \to Y$, then it is also a kernel pair for $m \circ f$.*

Regular epis are particular well-behaved with respect to their kernel pairs.

Proposition 2.11. *Let $e\colon X \twoheadrightarrow Y$ be a regular epi in a category $\mathbf{X}$ with a kernel pair (K_e, π_e^1, π_e^2). Then, e is the coequalizer of π_e^1 and π_e^2.*

We conclude this section exploring some properties of kernel pairs in an $\mathcal{M}$-adhesive category. The results below are simple, yet they appear to be original, and we give their proofs in Section Appendix A.1.

Lemma 2.12. *Let $f\colon X \to Y$ and $g\colon Z \to W$ be arrows admitting kernel pairs and suppose that the solid part of the four squares below is given. If the leftmost square is commutative, then there is a unique arrow $k_h\colon K_f \to K_g$ making the other three commutative.*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "t"'] & W \end{tikzcd} \qquad \begin{tikzcd} K_f \arrow[r, dotted, "k_h"] \arrow[d, "\pi_f^1"'] & K_g \arrow[d, "\pi_g^1"] \\ X \arrow[r, "h"'] & Z \end{tikzcd} \qquad \begin{tikzcd} K_f \arrow[r, dotted, "k_h"] \arrow[d, "\pi_f^2"'] & K_g \arrow[d, "\pi_g^2"] \\ X \arrow[r, "h"'] & Z \end{tikzcd} \qquad \begin{tikzcd} K_f \arrow[r, dotted, "k_h"] \arrow[d, "{(\pi_f^1,\pi_f^2)}"'] & K_g \arrow[d, "{(\pi_g^1,\pi_g^2)}"] \\ X \times Z \arrow[r, "h\times h"'] & X \times Z \end{tikzcd}$$

Moreover, the following hold

1. *if h is a mono then k_h is a mono;*
2. *if the leftmost square is a pullback then the central two are pullbacks;*
3. *if h is mono and the leftmost square is a pullback then the rightmost is a pullback.*

The previous result allows us to deduce the following lemma in an $\mathcal{M}$-adhesive context.

Proposition 2.13. *Let $\mathbf{X}$ be a strict $\mathcal{M}$-adhesive category with all pullbacks, and suppose that in the cube aside the top face is an $\mathcal{M}$-pushout and all the vertical faces are pullbacks. Then the right square is a pushout.*

$$\begin{tikzcd}[row sep=small, column sep=small] & A' \arrow[dl, tail, "m'"'] \arrow[rr, "f'"] \arrow[dd, "a"' near start] & & B' \arrow[dl, tail, "n'"'] \arrow[dd, "b"] \\ C' \arrow[rr, "g'" near start, crossing over] \arrow[dd, "c"'] & & D' & \\ & A \arrow[dl, tail, "m"] \arrow[rr, "f" near start] & & B \arrow[dl, tail, "n"] \\ C \arrow[rr, "g"'] & & D \arrow[from=uu, "d" near start, crossing over] & \end{tikzcd} \qquad \begin{tikzcd} K_a \arrow[r, "k_{f'}"] \arrow[d, "k_{m'}"'] & K_b \arrow[d, "k_{n'}"] \\ K_c \arrow[r, "k_{g'}"'] & K_d \end{tikzcd}$$

Focusing on **Set**, we can prove a sharper result (see Section Appendix A.1 for the proof).

Lemma 2.14. *Suppose that in* **Set** *the commuting cube in the diagram on the left is given, whose top face is a pushout, the left and bottom faces are pullbacks, and $n\colon B \rightarrowtail D$ is an injection. Then the following hold*

1. *the right face of the cube is a pullback;*
2. *the right square, made by the kernel pairs of the vertical arrows, is a pushout.*

$$\begin{tikzcd}[row sep=small, column sep=small] & A' \arrow[dl, tail, "m'"'] \arrow[rr, "f'"] \arrow[dd, "a"' near start] & & B' \arrow[dl, tail, "n'"'] \arrow[dd, "b"] \\ C' \arrow[rr, "g'" near start, crossing over] \arrow[dd, "c"'] & & D' & \\ & A \arrow[dl, tail, "m"] \arrow[rr, "f" near start] & & B \arrow[dl, tail, "n"] \\ C \arrow[rr, tail, "g"'] & & D \arrow[from=uu, "d" near start, crossing over] & \end{tikzcd} \qquad \begin{tikzcd} K_a \arrow[r, "k_{f'}"] \arrow[d, "k_{m'}"'] & K_b \arrow[d, "k_{n'}"] \\ K_c \arrow[r, "k_{g'}"'] & K_d \end{tikzcd}$$

3. Hypergraphical structures

In this section we briefly recall the notion of *hypergraph*. In order to do so, a pivotal role is played by the *Kleene star monad* $(-)^\star\colon \mathbf{Set} \to \mathbf{Set}$, also known as *list monad*, sending a set to the free monoid on it [30, 32]. We recall some of its proprieties.

Proposition 3.1. *Let X be a set and $n \in \mathbb{N}$. Then the following facts hold*

1. *there are arrows $v_n \colon X^n \to X^\star$ such that $(X^\star, \{v_n\}_{n\in\mathbb{N}})$ is a coproduct;*
2. *for every arrow $f \colon X \to Y$, $f^\star \colon X^\star \to Y^\star$ is the coproduct of the family $\{f^n\}_{n\in\mathbb{N}}$;*
3. *$(-)^\star$ preserves all* connected limits *[8], in particular it preserves pullbacks and equalizers.*

Remark 3.2. Preservation of pullbacks implies that $(-)^\star$ sends monos to monos.

Remark 3.3. Notice that $1^\star$ can be canonically identified with $\mathbb{N}$, thus for every set X the arrow $!_{\mathbf{X}} \colon X \to 1$ induces a *length function* $\mathsf{lg}_X \colon X^\star \to \mathbb{N}$, which sends a word to its length.

3.1. The category of hypergraphs

We open this section with the definition of hypergraphs and we show how to label them with an algebraic signature.

Definition 3.4. An *hypergraph* is a 4-uple $\mathcal{G} := (E_\mathcal{G}, V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G})$ made by two sets $E_\mathcal{G}$ and $V_\mathcal{G}$, called respectively the sets of *hyperedges* and *nodes*, plus a pair of *source* and *target* arrows $s_\mathcal{G}, t_\mathcal{G} \colon E_\mathcal{G} \rightrightarrows V_\mathcal{G}^\star$.

A *hypergraph morphism* $(E_\mathcal{G}, V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G}) \to (E_\mathcal{H}, V_\mathcal{H}, s_\mathcal{H}, t_\mathcal{H})$ is a pair (h, k) of functions $h \colon E_\mathcal{G} \to E_\mathcal{H}$, $k \colon V_\mathcal{G} \to V_\mathcal{H}$ such that the diagrams on the right are commutative.

We define **Hyp** to be the resulting category.

$$\begin{tikzcd} E_\mathcal{G} \arrow[r, "s_\mathcal{G}"'] \arrow[d, "h"'] & V_\mathcal{G}^\star \arrow[d, "k^\star"] \\ E_\mathcal{G} \arrow[r, "s_\mathcal{H}"] & V_\mathcal{H}^\star \end{tikzcd}$$

$$\begin{tikzcd} E_\mathcal{G} \arrow[r, "t_\mathcal{G}"'] \arrow[d, "h"'] & V_\mathcal{G}^\star \arrow[d, "k^\star"] \\ E_\mathcal{G} \arrow[r, "t_\mathcal{H}"] & V_\mathcal{H}^\star \end{tikzcd}$$

Let $\mathsf{prod}^\star$ be the functor $\mathbf{Set} \to \mathbf{Set}$ sending X to the product $X^\star \times X^\star$. We can use it to present **Hyp** as a comma category.

Proposition 3.5. **Hyp** *is isomorphic to* $\mathsf{id}_{\mathbf{Set}} {\downarrow} \mathsf{prod}^\star$

Note that by hypothesis $(-)^\star$ preserves pullbacks, while prod is continuous by definition, hence by Proposition 3.5 and Corollary Appendix B.5 we can deduce the following result.

Corollary 3.6. *A morphism (h, k) is a mono in* **Hyp** *if and only if both its components are injective functions.*

Applying Theorem 2.4 we also get the next corollary (cfr. [15, Fact 4.17]).

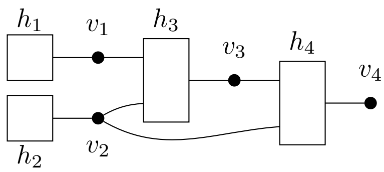
Corollary 3.7. **Hyp** *is an adhesive category.*

Propositions Appendix B.6 and 3.5 allow us to deduce immediately the following.

Proposition 3.8. *The forgetful functor $U_{\mathbf{Hyp}}$ which sends a hypergraph $\mathcal{G}$ to its set of nodes has a left adjoint $\Delta_{\mathbf{Hyp}}$.*

Example 3.9. Since the initial object of **Set** is the empty set, $\Delta_{\mathbf{Hyp}}(X)$ is the hypergraph which has X as set of nodes, $\emptyset$ as set of hyperedges, and $?_X$ as source and target function.

Example 3.10. We represent hypergraphs visually: dots denote nodes and boxes hyperedges. Should we be interested in their identity, we put a name near the corresponding dot or box. Sources and targets are represented by lines between dots and squares: the lines from the sources of a hyperedge comes from the left of the box, while the lines to the targets exit from the right of the box. Let us look at the picture below. It represent a hypergraph $\mathcal{G}$ with sets $V_{\mathcal{G}} = \{v_1, v_2, v_3, v_4\}$ and $E_{\mathcal{G}} = \{h_1, h_2, h_3, h_4\}$ of nodes and edges, respectively, such that h_1, h_2 have no source and h_3, h_4 a pair of nodes



Remark 3.11. It is worth to point out, as first noted in [6], that **Hyp** is equivalent to a category of presheaves. Indeed, consider the category $\mathbf{H}$ in which the set of objects is given by $(\mathbb{N} \times \mathbb{N}) \cup \{\bullet\}$ and arrows are given by the identities $\mathsf{id}_{k,l}$, $\mathsf{id}_\bullet$ and exactly $k+l$ arrows $f_i \colon (k,l) \to \bullet$, where i ranges from 0 to $k+l-1$. The functors $\mathbf{H} \to \mathbf{Set}$ corresponds exactly to hypergraphs: nodes correspond to the image of $\bullet$ while the set of hyperedges with source of length k and target of length l corresponds to the image of (k,l) (see [11]).

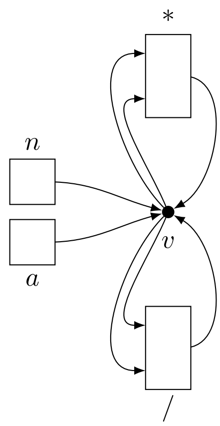
In particular, Theorem 3.11 entails the following result.

Proposition 3.12. **Hyp** *has all limits and colimits.*

3.1.1. Labelling hypergraph with an algebraic signature

Our interest for hypergraphs stems from their use as a graphical representation of algebraic terms. We thus need a way to label hyperedges with symbols taken from a signature.

Definition 3.13. An *algebraic signature* Σ is a pair $(O_\Sigma, \mathsf{ar}_\Sigma)$ given by a *set of operations* O_Σ and an *arity function* $\mathsf{ar}_\Sigma \colon O_\Sigma \to \mathbb{N}$. We define the *hypergraph* $\mathcal{G}_\Sigma$ *associated with* Σ taking O_Σ as set of hyperedges, 1 as set of nodes, so that $1^\star$ is $\mathbb{N}$, ar_Σ as the source function and γ_1, which always picks the element 1, as target function. The category $\mathbf{Hyp}_\Sigma$ of *algebraically labelled hypergraphs* is the slice category $\mathbf{Hyp}/\mathcal{G}^\Sigma$.



Example 3.14. Let $\Sigma = (O_\Sigma, \mathsf{ar}_\Sigma)$ be the signature with $O_\Sigma = \mathbb{N} \uplus A \uplus \{*, /\}$, where n stands for any natural number and a for any element in A, both sets of constants, and $\mathsf{ar}_\Sigma(*) = \mathsf{ar}_\Sigma(/) = 2$. Then the hypergraph $\mathcal{G}^\Sigma$ is depicted as the picture aside.

Corollary Appendix B.5 and Theorem 2.4 give us immediately an adhesivity result for $\mathbf{Hyp}_\Sigma$ and a characterisation of monos in it.

Proposition 3.15. *Let Σ be an algebraic signature. Then the following hold*

1. *an arrow (h,k) in $\mathbf{Hyp}_\Sigma$ is a mono if and only if h and k are injective;*
2. $\mathbf{Hyp}_\Sigma$ *is an adhesive category.*

Remark 3.16. Let $\mathcal{H} = (E, V, s, t)$ be a hypergraph, by definition we know that $U_{\mathbf{Hyp}}(\mathcal{G}^{\Sigma})$ is the terminal object 1, so an arrow $\mathcal{H} \to \mathcal{G}^{\Sigma}$, is determined by a function $h \colon E_{\mathcal{H}} \to O_{\Sigma}$ making the two squares on the right commutative (cfr. Remark 3.3).

$$\begin{tikzcd} E_{\mathcal{H}} \arrow[r, "s_{\mathcal{H}}"'] \arrow[d, "h"'] & V_{\mathcal{H}}^{\star} \arrow[d, "\mathsf{lg}_{V_{\mathcal{H}}}"] \\ O_{\Sigma} \arrow[r, "\mathsf{ar}_{\Sigma}"] & \mathbb{N} \end{tikzcd}$$

$$\begin{tikzcd} E_{\mathcal{H}} \arrow[r, "t_{\mathcal{H}}"'] \arrow[d, "h"'] & V_{\mathcal{H}}^{\star} \arrow[d, "\mathsf{lg}_{V_{\mathcal{H}}}"] \\ O_{\Sigma} \arrow[r, "\gamma_1"] & \mathbb{N} \end{tikzcd}$$

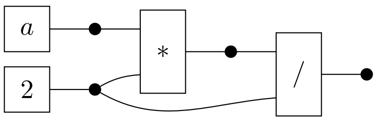
Now, consider a coprojection $v_n \colon V_{\mathcal{H}}^n \to V_{\mathcal{H}}^{\star}$. By the second diagram above entails that $t_{\mathcal{H}}$ factors via the inclusion $v_1 \colon V_{\mathcal{H}} \to V_{\mathcal{H}}^{\star}$ of words of length 1, i.e. all hyperedges must have a single target vertex, that is $t_{\mathcal{H}} = v_1 \circ \tau_{\mathcal{H}}$ for some $\tau_{\mathcal{H}} \colon E_{\mathcal{H}} \to V_{\mathcal{H}}$.

$\mathbf{Hyp}_{\Sigma}$ has a forgetful functor $U_{\Sigma} \colon \mathbf{Hyp}_{\Sigma} \to \mathbf{X}$ which sends $(h, k) \colon \mathcal{H} \to \mathcal{G}^{\Sigma}$ to $U_{\mathbf{Hyp}}(\mathcal{H})$. Now, since $U_{\mathbf{Hyp}}(\mathcal{G}^{\Sigma}) = 1$ for every set X we can define $\Delta_{\Sigma}(X) \colon \Delta_{\mathbf{Hyp}}(X) \to \mathcal{G}^{\Sigma}$ as $(?_{O_{\Sigma}}, !_X)$. It is straightforward to see that in this way we get a left adjoint to U_{Σ}.

Proposition 3.17. *U_{Σ} has a left adjoint Δ_{Σ}.*

We extend our graphical notation of hypergraphs to labeled ones putting the label of an hyperedge h inside its corresponding square.

Example 3.18. Consider again Σ the signature of Example 3.14, then the hypergraph $\mathcal{G}$ of Example 3.10 can be labeled by a morphism $(l, !_{V_{\mathcal{G}}}) : \mathcal{G} \to \mathcal{G}^{\Sigma}$ that is characterised by the image of the edges. If $l(h_1) = a$, $l(h_2) = 2$, $l(h_3) = *$, and $l(h_4) = /$, we represent it visually by putting the labels of the edges in $\mathcal{G}^{\Sigma}$ inside the boxes representing the edges of $\mathcal{G}$.



3.2. Term Graphs

Term graphs have been adopted as a convenient way to represent terms over a signature with an explicit sharing of sub-terms, and as such have been a convenient tool for an efficient implementation of term rewriting [29]. We elaborate here on the presentation given in [11].

Definition 3.19. Given an algebraic signature Σ, we say that a labelled hypergraph $(l, !_{V_{\mathcal{G}}}) \colon \mathcal{G} \to \mathcal{G}^{\Sigma}$ is a *term graph* if $t_{\mathcal{G}}$ is a mono. We define $\mathbf{TG}_{\Sigma}$ to be the full subcategory of $\mathbf{Hyp}_{\Sigma}$ given by term graphs and denote by I_{Σ} the inclusion. Restricting $U_{\Sigma} \colon \mathbf{Hyp}_{\Sigma} \to \mathbf{Set}$ we get a forgetful functor $U_{\mathbf{TG}_{\Sigma}} \colon \mathbf{TG}_{\Sigma} \to \mathbf{Set}$.

Remark 3.20. By Remark 3.16, we know that if $\mathcal{G}$ is a term graph then $t_{\mathcal{G}} = v_1 \circ \tau_{\mathcal{G}}$, where v_1 is the coprojection of $V_{\mathcal{G}}$ into $V_{\mathcal{G}}^{\star}$. Notice that since $t_{\mathcal{G}}$ is a mono then $\tau_{\mathcal{G}}$ is a mono.

Example 3.21. The labelled hypergraph of Example 3.18 is a term graph.

We now examine some properties of $\mathbf{TG}_{\Sigma}$, in order to study its adhesivity properties. We begin noticing that, for every set X, $\Delta_{\Sigma}(X)$ has no hyperedges and so is a term graph. this yields at once the following.

Proposition 3.22. *The forgetful functor $U_{\mathbf{TG}_{\Sigma}}$ has a left adjoint $\Delta_{\mathbf{TG}_{\Sigma}}$.*

We can list some other categorical properties of $\mathbf{TG}_\Sigma$ (see [11, Sec. 5]).

Proposition 3.23. *Let Σ be an algebraic signature. Then the following hold*

1. *if $(i,j)\colon \mathcal{H} \to \mathcal{G}$ is a mono from $(l, !_{V_\mathcal{G}})\colon \mathcal{G} \to \mathcal{G}^\Sigma$ to $(l', !_{V_\mathcal{H}})\colon \mathcal{H} \to \mathcal{G}^\Sigma$ in $\mathbf{Hyp}_\Sigma$ and the latter is in $\mathbf{TG}_\Sigma$, then also the former is in $\mathbf{TG}_\Sigma$*
2. *$\mathbf{TG}_\Sigma$ has equalizers, binary products and pullbacks and they are created by I_Σ.*

Remark 3.24. $\mathbf{TG}_\Sigma$ in general does not have terminal objects. Since $U_{\mathbf{TG}_\Sigma}$ preserves limits, if a terminal object exists it must have the singleton as set of nodes, therefore the set of hyperedges must be empty or a singleton. Hence, for a counterexample, it suffices to take a signature with two operations a and b, both of arity 0. $\mathbf{TG}_\Sigma$ is not an adhesive category, either. In particular, as noted in e.g. [11], it does not have pushouts along all monos.

Definition 3.25. Let $(l, !_{V_\mathcal{G}})\colon \mathcal{G} \to \mathcal{G}^\Sigma$ be a term graph. A *input node* is an element of $V_\mathcal{G}$ not in the image of $\tau_\mathcal{G}$. A morphism (f,g) between $(l, !_{V_\mathcal{G}})\colon \mathcal{G} \to \mathcal{G}^\Sigma$ and $(l, !_{V_\mathcal{H}})\colon \mathcal{H} \to \mathcal{G}^\Sigma$ in $\mathbf{TG}_\Sigma$, is said to *preserve input nodes* if g sends input nodes to input nodes.

Preservation of input nodes characterizes regular monos in $\mathbf{TG}_\Sigma$.

Proposition 3.26. *Let (i,j) be a mono between two term graphs $(l, !_{V_\mathcal{G}})\colon \mathcal{G} \to \mathcal{G}^\Sigma$ and $(l', !_{V_\mathcal{H}})\colon \mathcal{H} \to \mathcal{G}^\Sigma$. Then it is a regular mono if and only if it preserves the input nodes.*

This characterization, in turn, provides us with the following result [11, 10].

Lemma 3.27. *Consider three term graphs $(l_0, !_{V_\mathcal{G}})\colon \mathcal{G} \to \mathcal{G}^\Sigma$, $(l_1, !_{V_\mathcal{H}})\colon \mathcal{H} \to \mathcal{G}^\Sigma$ and $(l_2, !_{V_\mathcal{K}})\colon \mathcal{K} \to \mathcal{G}^\Sigma$. Given $(f_1, g_1)\colon (l_0, !_{V_\mathcal{G}}) \to (l_1, !_{V_\mathcal{H}})$, $(f_2, g_2)\colon (l_0, !_{V_\mathcal{G}}) \to (l_2, !_{V_\mathcal{K}})$, if (f_1, g_1) is a regular mono, then its pushout $(p, !_{V_\mathcal{P}})\colon \mathcal{P} \to \mathcal{G}^\Sigma$ in $\mathbf{Hyp}_\Sigma$ along (f_2, g_2) is a term graph.*

Theorem 2.4 and proposition 3.15, Proposition 3.26 and Lemma 3.27 allow us to recover the following result, previously proved by direct computation in [12, Thm. 4.2] (see also [11, Cor. 5.15] for the details of the argument presented here).

Corollary 3.28. *The category $\mathbf{TG}_\Sigma$ is quasiadhesive.*

4. Adding equivalences to hypergraphical structures

The use of hypergraphs equipped with an equivalence relation over their nodes has been argued as a convenient way to express concurrency in the DPO approach to rewriting [4]. This section presents the framework by means of adhesive categories, including also its version for term graphs, as a stepping stone towards an analogous characterisation for e-graphs.

4.1. Hypergraphs with equivalence

Let us start with the case of general hypergraphs. These were introduced in [4], even if no general consideration about their structure as a category was proved, and adhesivity, which is the main focus here, was yet to be presented to the world.

Definition 4.1. A *hypergraph with equivalence* $\mathcal{G} = (E_\mathcal{G}, V_\mathcal{G}, Q_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G}, q_\mathcal{G})$ is a 6-tuple such that $\mathcal{G} = (E_\mathcal{G}, V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G})$ is a hypergraph, $Q_\mathcal{G}$ is a set and $q_\mathcal{G} : V_\mathcal{G} \twoheadrightarrow Q_\mathcal{G}$ is a surjection called *quotient map.* A morphism $h\colon \mathcal{G} \to \mathcal{H}$ is a triple (h_E, h_V, h_Q) such that the following diagrams commute

$$\begin{tikzcd} E_\mathcal{G} \arrow[r, "s_\mathcal{G}"] \arrow[d, "h_E"'] & V_\mathcal{G}^\star \arrow[d, "h_V^\star"] \\ E_\mathcal{H} \arrow[r, "s_\mathcal{H}"'] & V_\mathcal{H}^\star \end{tikzcd} \qquad \begin{tikzcd} E_\mathcal{G} \arrow[r, "t_\mathcal{G}"] \arrow[d, "h_E"'] & V_\mathcal{G}^\star \arrow[d, "h_V^\star"] \\ E_\mathcal{H} \arrow[r, "t_\mathcal{H}"'] & V_\mathcal{H}^\star \end{tikzcd} \qquad \begin{tikzcd} V_\mathcal{G} \arrow[r, two heads, "q_\mathcal{G}"] \arrow[d, "h_V"'] & Q_\mathcal{G} \arrow[d, "h_Q"] \\ V_\mathcal{H} \arrow[r, two heads, "q_\mathcal{H}"'] & Q_\mathcal{H} \end{tikzcd}$$

The category of hypergraphs with equivalence and their morphisms is denoted **EqHyp**.

Remark 4.2. Notice that in **Set** the classes of surjections, epis and regular epis coincide.

Remark 4.3. Morphisms of hypergraphs with equivalences are uniquely determined by the first two components. That is, if $h_1 = (h_E, h_V, f)$ and $h_2 = (h_E, h_V, g)$ are two morphisms $\mathcal{G} \rightrightarrows \mathcal{H}$, then we have $f \circ q_\mathcal{G} = q_\mathcal{H} \circ h_V = g \circ q_\mathcal{G}$. Since $q_\mathcal{G}$ is epi, we obtain $f = g$.

Forgetting the quotient part yields a functor $T\colon \mathbf{EqHyp} \to \mathbf{Hyp}$ sending a hypergraph with equivalence $(E_\mathcal{G}, V_\mathcal{G}, C_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G}, q_\mathcal{G})$ to $(E_\mathcal{G}, V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G})$. We now explore some of its properties to deduce information on the structure of **EqHyp**. A proof is in Section Appendix A.2.

Proposition 4.4. *Consider the forgetful functor $T\colon$ **EqHyp** $\to$ **Hyp**. Then the following hold*

1. *T is faithful;*
2. *T has a left adjoint;*
3. *T has a right adjoint.*

Corollary 4.5. *The functor T preserves limits and colimits.*

From the previous results we get the following characterization of monos in **EqHyp**.

Corollary 4.6. *An arrow $(h_E, h_V, h_Q) : \mathcal{G} \to \mathcal{H}$ in* **EqHyp** *is a mono if and only if (h_E, h_V) is a mono in* **Hyp**.

Now, we can consider the forgetful functor $U_{\mathsf{eq}}\colon \mathbf{EqHyp} \to \mathbf{Set}$ obtained by composing T and $U_{\mathbf{Hyp}}$. By Theorem 3.8 and the second point of Theorem 4.4 we get the following.

Corollary 4.7. *U_{eq} has a left adjoint $\Delta_{\mathsf{eq}}\colon$ **Set** $\to$ **EqHyp**.*

Notice that there is another functor $K : \mathbf{EqHyp} \to \mathbf{Set}$ sending (E, V, Q, s, t, q) to Q, and a morphism (h_E, h_V, h_Q) to h_Q. We exploit it to compute limits and colimits in **EqHyp**. A full proof of the following lemma can be found in Section Appendix A.2.

Lemma 4.8. *Consider a diagram* $F\colon \mathbf{D} \to \mathbf{EqHyp}$ *and let* $(E_D, V_D, Q_D, s_D, t_D, q_D)$ *be the image of an object* D*. Then the following hold*

$$\begin{tikzcd} V \arrow[r, "\pi_V^D"] \arrow[d, two heads, "q"'] & V_D \arrow[dd, two heads, "q_D"] \\ Q \arrow[d, tail, dotted, "m"'] & \\ L \arrow[r, "l_d"'] & Q_D \end{tikzcd}$$

1. *F has a colimit, which is preserved by K;*
2. *consider a cone $(L, \{l_D\}_{D \in \mathbf{D}})$ limiting for $K \circ F$ and let $((E, V), \{(\pi_E^D, \pi_V^D)\}_{D \in \mathbf{D}})$ be one for $T \circ F$, then F has a limit (E, V, Q, s, t, q) and there is a mono $m\colon Q \rightarrowtail L$ such that the diagram on the right commutes for every $D \in \mathbf{D}$.*

Corollary 4.9. *An arrow* $(h_E, h_V, h_Q) : \mathcal{G} \to \mathcal{H}$ *in* **EqHyp** *is a regular mono if and only if all its components are injective functions.*

A proof is in Section Appendix A.2. We have now all the ingredients to study the adhesivity properties of **EqHyp**. As a first step we need to introduce a class of monos.

Definition 4.10. We define Pb as the class of regular monos $(h_E, h_V, h_Q)\colon \mathcal{G} \to \mathcal{H}$ in **EqHyp** such that the square on the right is a pullback

$$\begin{tikzcd} V_{\mathcal{G}} \arrow[r, two heads, "q_{\mathcal{G}}"'] \arrow[d, tail, "h_V"'] & Q_{\mathcal{G}} \arrow[d, tail, "h_Q"] \\ V_{\mathcal{H}} \arrow[r, two heads, "q_{\mathcal{H}}"] & Q_{\mathcal{H}} \end{tikzcd}$$

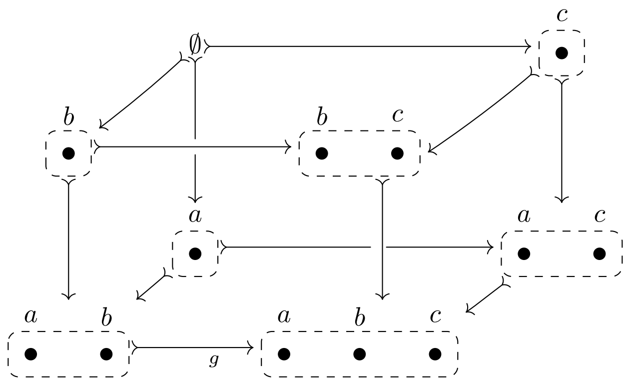
Now, we show that Pb is a suitable class for adhesivity. A proof is in Section Appendix A.2.

Lemma 4.11. *The class* Pb *contains all isomorphisms, it is closed under composition, decomposition and it is stable under pullbacks and pushouts.*

Finally, we show the key lemma for Pb-adhesivity of **EqHyp**. A proof is in Section Appendix A.2.

Lemma 4.12. *In* **EqHyp***,* Pb*-pushouts are stable.*

Example 4.13. It is noteworthy to show that pushouts along regular monos are not stable. Consider e.g. the cube aside: the vertices are just graphs without edges, and in the graphs themselves the equivalence classes are denoted by encircling nodes with a dotted line. All the arrows are regular monos. It is immediate to see that the bottom face is a pushout and the side faces are pullback. Unfortunately, the top face fails to be a pushout



Having proved stability of Pb-pushouts, we now turn to prove that they are Van Kampen with respect to regular monos. A proof is found in Section Appendix A.2.

Lemma 4.14. *In* $\mathbf{EqHyp}$*, pushouts along arrows in* Pb *are* $\mathsf{Reg}(\mathbf{EqHyp})$*-Van Kampen.*

Corollary 4.15. $\mathbf{EqHyp}$ *is* Pb*-adhesive.*

4.2. Term graphs with equivalence

We are now going to generalize the work done in the previous section equipping term graphs with equivalence relation. First of all we need a notion of labelling for $\mathbf{EqHyp}$.

Definition 4.16. Let Σ be an algebraic signature and $\mathcal{G}^\Sigma$ the hypergraph associated to it. A *labelled hypergraph with equivalence* is a pair $(\mathcal{H}, l)$ where $\mathcal{H}$ is an object of $\mathbf{EqHyp}$ and l a morphism $T(\mathcal{H}) \to \mathcal{G}^\Sigma$ of $\mathbf{Hyp}$. A *morphism of labelled hypergraphs with equivalence* between $(\mathcal{H}, l)$ and $(\mathcal{H}', l')$ is an arrow $h\colon \mathcal{H} \to \mathcal{H}'$ such that $l = l' \circ T(h)$.

We denote the resulting category by $\mathbf{EqHyp}_\Sigma$.

Let $(\mathcal{H}, l)$ be an object of $\mathbf{EqHyp}_\Sigma$: since T has a right adjoint R by Theorem 4.4, $l\colon T(\mathcal{H}) \to \mathcal{G}^\Sigma$ corresponds to the arrow $(l, !_{Q_\mathcal{H}})\colon \mathcal{H} \to R(\mathcal{G}^\Sigma)$. It is immediate to see that this correspondence extends to an equivalence with the slice over $R(\mathcal{G}^\Sigma)$.

Proposition 4.17. $\mathbf{EqHyp}_\Sigma$ *is equivalent to* $\mathbf{EqHyp}/R(\mathcal{G}^\Sigma)$.

Let $V_\Sigma\colon \mathbf{EqHyp}_\Sigma \to \mathbf{EqHyp}$ be the functor forgetting the labelling and Pb_Σ the class of morphisms in $\mathbf{EqHyp}_\Sigma$ whose image in $\mathbf{EqHyp}$ is in Pb. From Theorem 4.8 and corollary Appendix B.11, the second point of Theorems 2.4 and 4.15, we can deduce the following.

Proposition 4.18. *Consider the forgetful functor* $V_\Sigma\colon \mathbf{EqHyp}_\Sigma \to \mathbf{EqHyp}$*. Then the following hold*

1. $\mathbf{EqHyp}_\Sigma$ *has all colimits and all connected limits, which are created by* V_Σ*;*
2. $\mathbf{EqHyp}_\Sigma$ *is* Pb_Σ*-adhesive.*

Using Theorems 4.6 and 4.9 we immediately get the following result.

Corollary 4.19. *Let* $h = (h_E, h_V, h_Q)$ *be an arrow in* $\mathbf{EqHyp}_\Sigma$*. Then the following hold*

1. *h is mono if and only if h_E and h_V are injective;*
2. *h is a regular mono if and only if h_E, h_V and h_Q are injective.*

We can now easily define term graphs with equivalence.

Definition 4.20. Let Σ be an algebraic signature. An object $(\mathcal{H}, l)$ of $\mathbf{EqHyp}_\Sigma$ is a *term graph with equivalence* if $l\colon T(\mathcal{H}) \to \mathcal{G}^\Sigma$ is a term graph. We denote by $\mathbf{EqTG}_\Sigma$ the full subcategory of $\mathbf{EqHyp}_\Sigma$ so defined and by J_Σ the corresponding inclusion functor.

Remark 4.21. Let $T_\Sigma : \mathbf{EqHyp}_\Sigma \to \mathbf{Hyp}_\Sigma$ the forgetful functor lifting $T : \mathbf{EqHyp} \to \mathbf{Hyp}$. Notice that, by definition, $(\mathcal{H}, l)$ is in $\mathbf{EqTG}_\Sigma$ amounts to say that $(T(\mathcal{H}), l)$ is in $\mathbf{TG}_\Sigma$. Thus there exists a functor S_Σ as in the diagram on the right.

$$\begin{tikzcd} \mathbf{EqTG}_\Sigma \arrow[r, "J_\Sigma"'] \arrow[d, "S_\Sigma"'] & \mathbf{EqHyp}_\Sigma \arrow[d, "T_\Sigma"] \\ \mathbf{TG}_\Sigma \arrow[r, "I_\Sigma"] & \mathbf{Hyp}_\Sigma \end{tikzcd}$$

Remark 4.22. Notice that, by Theorem 4.5 and Theorem 4.18, the functor T_Σ preserves all connected limits and all colimits.

The previous remark allows us to prove an analog of Theorem 3.23. We refer the reader to Section Appendix A.2 for details.

Proposition 4.23. $\mathbf{EqTG}_\Sigma$ *has equalizers, binary products and pullbacks and they are created by* J_Σ.

Let now $\mathcal{T}$ be the class of morphism in $\mathbf{EqTG}_\Sigma$ whose image through J_Σ is in Pb_Σ and whose image through S_Σ is a regular mono in $\mathbf{TG}_\Sigma$. By Proposition 3.26 and theorem 4.19, we have that a morphism $(h_E, h_V, h_Q) \colon (\mathcal{G}, l) \to (\mathcal{H}, l')$ is in $\mathcal{T}$ if and only if all of its components are injections and the square on the right is a pullback.

$$\begin{tikzcd} V_\mathcal{G} \arrow[r, tail, "h_V"] \arrow[d, two heads, "q_\mathcal{H}"'] & V_\mathcal{H} \arrow[d, two heads, "q_\mathcal{H}"] \\ Q_\mathcal{G} \arrow[r, tail, "h_Q"'] & Q_\mathcal{H} \end{tikzcd}$$

In particular, by Theorem 3.28 and Theorem 4.12, $\mathcal{T}$ contains all isomorphisms, is closed under composition and decomposition and stable under pushout and pullbacks.

The following proposition is now an easy corollary of Lemma 3.27, Theorem 4.18, and Theorem 4.21. We provide the details in Section Appendix A.2.

Proposition 4.24. $\mathbf{EqTG}_\Sigma$ *has all* $\mathcal{T}$*-pushouts, which are created by* J_Σ.

Corollary 4.25. $\mathbf{EqTG}_\Sigma$ *is* $\mathcal{T}$*-adhesive.*

5. EGGs

The previous section proved some adhesivity property for the categories $\mathbf{EqHyp}$ and $\mathbf{EqTG}_\Sigma$. We extend these results to encompass equivalence classes which are closed under the target arrow, i.e. under operator composition for term graphs, thus precisely capturing EGGs.

5.1. E-hypergraphs

We start introducing the notion of *e-hypergraphs*, hypergraphs equipped with an equivalence relation that is closed under the target arrow: in other words, whenever the relation identifies the source of two hyperedges, it identifies their targets too.

Definition 5.1. Let $\mathcal{G} = (E, V, Q, s, t, q)$ be a hypergraph with equivalence and (S, π_1, π_2) a kernel pair for $q^\star \circ s$. We will say that $\mathcal{G}$ is an *e-hypergraph* if $q^\star \circ t \circ \pi_1 = q^\star \circ t \circ \pi_2$.

We will denote by e-$\mathbf{EqHyp}$ the full subcategory of $\mathbf{EqHyp}$ whose objects are e-hypergraphs, and by $I \colon \mathsf{e}$-$\mathbf{EqHyp} \to \mathbf{EqHyp}$ the associated inclusion functor.

Example 5.2. Consider the hypergraph $\mathcal{G}$ of Example 3.10 and consider as quotient the identity $\mathsf{id}_{V_\mathcal{G}} \colon V_\mathcal{G} \colon \to \mathcal{G}$. Then the kernel pair of $\mathsf{id}_{V^\star_\mathcal{G}} \circ s$ concide with the kernel pair of s, which is empty. Thus $\mathcal{G}$ is, trivially, an e-hypergraph

A first result that we need is that limits in e-**EqHyp** are computed as in **EqHyp**. Full proofs are in Section Appendix A.3.

Lemma 5.3. e-**EqHyp** *has all limits and I creates them.*

Corollary 5.4. *If an arrow $h : \mathcal{G} \to \mathcal{H}$ in* e-**EqHyp** *is a regular mono in* e-**EqHyp** *then $I(h)$ is a regular mono in* **Hyp**.

We can now turn to pushouts. We refer again the reader to Section Appendix A.3 for the details.

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, "h"] \arrow[d, tail, "m"'] & \mathcal{G}_2 \arrow[d, "n"] \\ \mathcal{G}_3 \arrow[r, "z"'] & \mathcal{P} \end{tikzcd}$$

Lemma 5.5. *Suppose that the square on the right is a pushout in* **EqHyp** *and that m is a mono in* Pb. *If $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are e-hypergraphs, then $\mathcal{P}$ is is an e-hypergraph.*

Let now ePb be the class of arrows in e-**EqHyp** that are sent to Pb by I. By definition and Theorem 4.11 we have that such class is closed under composition and decomposition, it contains all isomorphisms and it is stable under pullbacks. Using Theorems 4.11 and 5.5 we can further deduce that it is stable under pushouts. Then Theorems 2.4 and 4.15 yield the following.

Corollary 5.6. e-**EqHyp** *is* ePb*-adhesive.*

5.2. E-term graphs, a.k.a. EGGs

We now turn our attention to the labelled context.

Definition 5.7. We say that an object $(\mathcal{H}, l)$ of $\mathbf{EqHyp}_\Sigma$ is a *labelled e-hypergraph* if $\mathcal{H}$ is an e-hypergraph. We define the category e-$\mathbf{EqHyp}_\Sigma$ as the full subcategory of $\mathbf{EqHyp}_\Sigma$ given by labelled e-hypergraphs. We denote by Z_Σ the corresponding inclusion functor .

To prove some adhesivity property of e-$\mathbf{EqHyp}_\Sigma$, we begin with the following elementary, yet useful observation.

Remark 5.8. Given a signature $\Sigma = (O_\Sigma, \mathsf{ar}_\Sigma)$, the hypergraph $R(\mathcal{G}^\Sigma) = (O_\Sigma, 1, 1, \mathsf{ar}_\Sigma, \gamma_1, \mathsf{id}_1)$ is an object of e-**EqHyp**. Indeed, under the identification of $1^\star$ with $\mathbb{N}$, the kernel (S, π_1, π_2) of $\mathsf{id}_\mathbb{N} \circ \mathsf{ar}_\Sigma$, is given by $S := \{(o_1, o_2) \in O_\Sigma \times O_\Sigma \mid \mathsf{ar}_\Sigma(o_1) = \mathsf{ar}_\Sigma(o_2)\}$ equipped with the two projections. On the other hand, both $\mathsf{id}_\mathbb{N} \circ \gamma_1 \circ \pi_1$ and $\mathsf{id}_\mathbb{N} \circ \gamma_1 \circ \pi_2$ are the function $O_\Sigma \to \mathbb{N}$ constant in 1.

Theorem 5.8 now implies at once the following result.

Proposition 5.9. *Let Σ be an algebraic signature. Then the following hold*

1. e-$\mathbf{EqHyp}_\Sigma$ *is equivalent to* e-$\mathbf{EqHyp}/R(\mathcal{G}^\Sigma)$*;*
2. *there exists a functor* $W_\Sigma \colon \mathsf{e}\text{-}\mathbf{EqHyp}_\Sigma \to \mathsf{e}\text{-}\mathbf{EqHyp}$ *forgetting the labeling which creates all colimits, pullbacks and equalizers.*

Let ePb_Σ be the class of morphisms in e-$\mathbf{EqHyp}_\Sigma$ whose image in e-**EqHyp** lies in ePb. Notice that ePb_Σ is also the class of arrows whose image through Z_Σ is in Pb_Σ. By Theorem 2.4 we get the following result.

Corollary 5.10. e-$\mathbf{EqHyp}_\Sigma$ *is* ePb_Σ*-adhesive.*

We turn now to term graphs with equivalence.

Definition 5.11. Given a signature Σ, we say that an object $(\mathcal{H}, l)$ of $\mathbf{EqTG}_\Sigma$ is an *e-term graph* if $\mathcal{H}$ is an e-hypergraph. We define the category $\mathbf{EGG}$ as the full subcategory of $\mathbf{EqTG}_\Sigma$ given by e-term graphs and denote by K_Σ the corresponding inclusion.

Remark 5.12. By definition we also have an inclusion functor $Y_\Sigma \colon \mathbf{EGG} \to \text{e-}\mathbf{EqHyp}_\Sigma$.

Remark 5.13. Now that we have put all the structures of this work in place, it is worthwhile to give a visual map of all the categories that we have discussed/introduced and of the relationships between them. We will use the curved arrows to denote full and faithful inclusions.

$$\begin{tikzcd}
\mathbf{EGG} \arrow[r, hook, "Y_\Sigma"] \arrow[d, hook, "K_\Sigma"'] & \text{e-}\mathbf{EqHyp}_\Sigma \arrow[r, "W_\Sigma"] \arrow[d, hook, "Z_\Sigma"] & \text{e-}\mathbf{EqHyp} \arrow[d, hook, "I"] \\
\mathbf{EqTG}_\Sigma \arrow[r, hook, "J_\Sigma"'] \arrow[d, "S_\Sigma"'] & \mathbf{EqHyp}_\Sigma \arrow[r, "V_\Sigma"'] \arrow[d, "T_\Sigma"'] & \mathbf{EqHyp} \arrow[d, "T"'] \\
\mathbf{TG}_\Sigma \arrow[r, hook, "I_\Sigma"'] & \mathbf{Hyp}_\Sigma \arrow[r, "U_\Sigma"'] & \mathbf{Hyp}
\end{tikzcd}$$

We can now prove our last three results. The reader can find the proof in Section Appendix A.3.

Proposition 5.14. $\mathbf{EGG}$ *has equalizers, binary products and pullbacks and they are created by* K_Σ.

Let now $\mathcal{T}_\Sigma$ be the class of morphisms of $\mathbf{EGG}$ which are sent by K_Σ to the class $\mathcal{T}$. Then Lemma Appendix B.3 and theorems 4.24 and 5.5 allow us to deduce the following result.

Proposition 5.15. $\mathbf{EGG}$ *has* $\mathcal{T}_\Sigma$*-pushouts, which are created by* K_Σ.

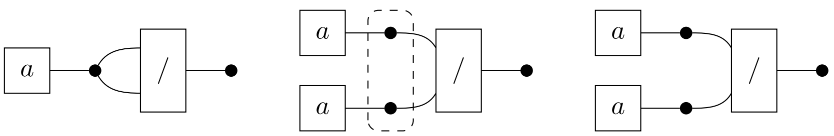
Reasoning as in the previous sections, Theorems 5.14 and 5.15 now give us the following.

Corollary 5.16. $\mathbf{EGG}$ *is* $\mathcal{T}_\Sigma$*-adhesive.*

6. Pros and cons of adhesive rewriting

The previous sections have shown how hypergraphs and term graphs with equivalence can be described as suitable $\mathcal{M}$-adhesive categories. The same fact holds for their sub-categories where the equivalence is closed with respect to operator composition, and this allows to model EGGs, as originally presented in [33].

Sharing. Terms are trees, thus different term graphs may represent the same term, up-to the sharing of sub-terms. Consider e.g. a constant a and a binary operator $/$: the term a/a admits a few different representations as a term graph with equivalence, as shown below

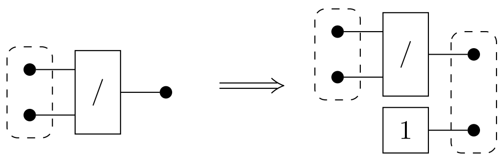


The left-most and the right-most images represent just ordinary term graphs, the middle one is a term graph with equivalence. As such, they are all objects of $\mathbf{EqTG}_\Sigma$. However, note that the right-most image is not an object of $\mathbf{EGG}$: constants have no input, and nodes that are targets of edges with the same label and whose source is the empty word must be equivalent and possibly coincide, as in the middle and in the left-most image, respectively.

In general, for (acyclic) term graphs, a maximally and a minimally shared representation exist, and for our toy example are the left-most and the right-most diagram above, respectively: it is a standard result for term graph rewriting, see e.g. [1]. These two representations can be interpreted as the final and the initial object of a suitable comma category. respectively. The same properties carry on for term graphs with equivalence and for EGGs. However, while in the former case the two representations are the same of those for ordinary term graphs, in the latter case the minimally shared representation is now the middle diagram.

Rewriting. The theory of $\mathcal{M}$-adhesivity ensures that if the rules are spans of arrows in $\mathcal{M}$, then we can lift the standard properties of the DPO approach that hold in the category of graphs. However, as we recalled in the introduction, instead of removing sub-terms, the EGG approach chooses to just add new terms and link them to the older ones via the equivalence relation [14]. So, the corresponding DPO rules are spans $L \leftarrow L \rightarrow R$, where the first component is the identity, thus in Pb_Σ, while the second component may not belong to Pb_Σ.

Consider e.g. an EGG rule such $x/x \rightarrow 1$, from the introductory example in [33]. Variables in a term graph are represented as nodes that do not occur among the targets of an edge, thus the rule can be modelled as the DPO rule below, concisely given by the arrow $L \rightarrow R$



In this case L is the minimally shared EGG corresponding to the term x/x, thus the rule could be applied to the two EGGs depicted above with a match that is injective on equivalence classes. In general, if the term graph underlying the EGG to whom the DPO rule is applied is acyclic, the same property holds for the EGG obtained as the result of the rule application. The right-hand side is a regular mono, though, hence it is not an arrow in Pb_Σ. The asymmetry between left-hand and right-hand sides falls in the current research about left-linear rules for adhesive categories, as pursued e.g. in [3].

Application conditions. As argued above, the DPO rules that are suitable for the EGG approach have identities as left-hand side, thus they belong to any choice of $\mathcal{M}$. However, this would allow for the repeated application of the same rule, hence the possibility to keep on performing the same rewriting step. This is forbidden by using rules *with negative application conditions*, given as the usual span $L \leftarrow K \rightarrow R$ plus an additional arrow $n : L \rightarrow N$, such that a match $m : L \rightarrow G$ is admissible if it cannot be factorised through n [21]. For EGGs and rules $L \leftarrow L \rightarrow R$, it suffices to choose n as the right-hand side itself. The theory of $\mathcal{M}$-adhesivity carries on for rules with negative application conditions, see [16, 17].

7. Conclusions and further and related works

The aim of our paper was to extend the theory of $\mathcal{M}$-adhesive categories in order to include EGGs, a formalism for program optimisation. To do so, we revisited the notions of hyper-graphs and term graphs with equivalence, proving that they are $\mathcal{M}$-adhesive categories, and we extended these results in order to prove the same property for EGGs as term graphs with equivalence satisfying a suitable closure constraint. Summing up, we proved that EGGs are objects of an $\mathcal{M}$-adhesive category **EGG** and that optimisation steps are obtained via DPO rules (possibly with negative application conditions) whose left-hand side is in $\mathcal{M}$, and that allows for exploiting the properties of the $\mathcal{M}$-adhesive framework.

Future works. Our result on **EGG** opens a few threads of research. The first is to check how the $\mathcal{M}$-adhesivity of EGGs can be pushed to model their rewriting via the double-pushout (DPO) approach. We have seen that the rules adopted in the literature of EGGs appears to be spans whose left-hand side is an identity and right-hand side is a regular mono (possibly with negative application conditions), and as such they fit the mould of rewriting on left-linear rewriting in $\mathcal{M}$-adhesive categories. However, it still needs to be shown how parallelism and causality, the key features for DPO rewriting on $\mathcal{M}$-adhesive categories, can be exploited in the context of implementing the EGGs updates. Moreover, extensions of the EGGs formalism could be suggested by the adhesive machinery we developed. In fact, most of the results presented here for hypergraphs can be generalised to hierarchical hypergraphs, that is, hypergraphs with a hierarchy (a partial order) among edges [20, 11]. The additional structure is useful for modelling properties such as encapsulation and sandboxing, and it seems worthwhile to check the expressiveness and applications of hierarchical EGGS.

Related works. Despite the interest they have ben raising as an efficient data structure, we are not aware of any attempt to provide an algebraic characterisation of EGGs, the only exception being [19]. We leave for future work an in-depth comparison among the two proposals, which appear to be related yet quite different. In fact, for both proposals the key intuition is to use string diagrams to represents term graphs. We adopted a more down-to-earth approach by equipping concretely the nodes of a term graph with an equivalence relation, thus directly corresponding to the original presentation [14], at the same time extending and generalising it [4] in the contemporary jargon of adhesive categories. The route chosen in [19] is to consider term graphs as arrows of a symmetric monoidal category, and equipping them with an enrichment over semi-lattices, the resulting formalism being reminiscent of hierarchical hypergraphs. Thus, the solution in [19] is more general than ours since it can be lifted to term graphs defined over categories other than **Set**. At the same time, we both consider the DPO approach for rewriting, even if only our solution guarantees that the resulting category is actually $\mathcal{M}$-adhesive, thus allowing to exploit all the features of the framework as they hold for DPO graph transformation.

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Appendix A. Omitted proofs

This section contains the proofs which are omitted from the main body of the paper. We begin recalling a well-known fact about composition and decomposition of pullbacks [25, Lem. 1.1].

Lemma Appendix A.1. *Let* $\mathbf{X}$ *be a category, and consider the diagram aside, in which the right square is a pullback. Then the whole rectangle is a pullback if and only if the left square is one.*

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d, "a"'] & Y \arrow[r, "g"'] \arrow[d, "b"] & Z \arrow[d, "c"] \\ A \arrow[r, "h"] & B \arrow[r, "k"] & C \end{tikzcd}$$

Appendix A.1. Proofs for Section 2

Proposition Appendix A.2. *Let* $\mathbf{X}$ *be an* $\mathcal{M}$*-adhesive category. Then the following hold*

1. *every* $\mathcal{M}$*-pushout square is also a pullback;*
2. *every arrow in* $\mathcal{M}$ *is a regular mono.*

Proof. 1. Consider the following cube in which the bottom face is an $\mathcal{M}$-pushout

$$\begin{tikzcd} & A \arrow[rr, "g"] \arrow[dl, "\mathsf{id}_A"'] \arrow[dd, "\mathsf{id}_A"', near start] & & B \arrow[dl, "\mathsf{id}_B"'] \arrow[dd, "\mathsf{id}_B"] \\ A \arrow[rr, "g", near start] \arrow[dd, "m"', tail] & & B \arrow[dd, "n", near start, tail] & \\ & A \arrow[rr, "g", near start] \arrow[dl, "m", tail] & & B \arrow[dl, "n", tail] \\ C \arrow[rr, "f"'] & & D & \end{tikzcd}$$

By construction the top face of the cube is a pushout and the back one a pullback. The left face is a pullback because m is mono, thus the Van Kampen property yields that the front and the right faces are pullbacks too and the claim follows.

2. Let $m\colon X \rightarrowtail Y$ be an arrow in $\mathcal{M}$, we can then take its pushout along itself, which, by the previous point, is also a pullback

$$\begin{tikzcd} X \arrow[r, tail, "m"] \arrow[d, tail, "m"'] & Y \arrow[d, tail, "h"] \\ Y \arrow[r, tail, "k"] & Z \end{tikzcd}$$

It is now immediate to see that m is the equalizer of h and k. □

Lemma Appendix A.3. *Let* $f\colon X \to Y$ *and* $g\colon Z \to W$ *be arrows admitting kernel pairs and suppose that the solid part of the four squares below is given. If the leftmost square is commutative, then there is a unique arrow* $k_h\colon K_f \to K_g$

making the other three commutative.

$$
\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\
Y \arrow[r, "t"'] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
K_f \arrow[r, dotted, "k_h"] \arrow[d, "\pi_f^1"'] & K_g \arrow[d, "\pi_g^1"] \\
X \arrow[r, "h"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
K_f \arrow[r, dotted, "k_h"] \arrow[d, "\pi_f^2"'] & K_g \arrow[d, "\pi_g^2"] \\
X \arrow[r, "h"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
K_f \arrow[r, dotted, "k_h"] \arrow[d, "(\pi_f^1, \pi_f^2)"'] & K_g \arrow[d, "(\pi_g^1, \pi_g^2)"] \\
X \times Z \arrow[r, "h \times h"'] & X \times Z
\end{tikzcd}
$$

Moreover, the following hold

1. *if h is a mono then k_h is a mono;*
2. *if the leftmost square is a pullback then the central two are pullbacks;*
3. *if h is mono and the leftmost square is a pullback then the rightmost is a pullback.*

Proof. We begin by computing

$$g \circ h \circ \pi_f^1 = t \circ f \circ \pi_f^1 = t \circ f \circ \pi_f^2 = g \circ h \circ \pi_f^2$$

so that existence and uniqueness of the wanted k_h follow at once from the universal property of K_g as the pullback of g along itself.

1. Let $a, b \colon T \rightrightarrows K_f$ be two arrows such that $k_h \circ a = k_h \circ b$, then we have

$$
\begin{aligned}
h \circ \pi_f^1 \circ a = \pi_g^1 \circ k_h \circ a = \pi_g^1 \circ k_h \circ b = h \circ \pi_f^1 \circ b \\
h \circ \pi_f^2 \circ a = \pi_g^2 \circ k_h \circ a = \pi_g^2 \circ k_h \circ b = h \circ \pi_f^2 \circ b
\end{aligned}
$$

   Since h is mono this entails that

$$\pi_f^1 \circ a = \pi_f^1 \circ b \qquad \pi_f^2 \circ a = \pi_f^2 \circ b$$

   and thus a must coincide with b.
2. To prove the second half of the claim, we can notice that, by Theorem Appendix A.1, two rectangles below are pullbacks

$$
\begin{tikzcd}
K_f \arrow[r, "\pi_f^2"] \arrow[d, "\pi_f^1"'] & X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y \arrow[r, "t"'] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
K_f \arrow[r, "\pi_f^1"] \arrow[d, "\pi_f^2"'] & X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y \arrow[r, "t"'] & W
\end{tikzcd}
$$

   But then the outer rectangle in the following diagrams are pullbacks too

$$
\begin{tikzcd}
K_f \arrow[r, "k_h"'] \arrow[rr, bend left, "h \circ \pi_f^2"] \arrow[d, "\pi_f^1"'] & K_g \arrow[r, "\pi_g^2"'] \arrow[d, "\pi_g^1"'] & Z \arrow[d, "g"'] \\
X \arrow[r, "h"] \arrow[rr, bend right, "t \circ f"'] & Y \arrow[r, "g"] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
K_f \arrow[r, "k_h"'] \arrow[rr, bend left, "h \circ \pi_f^2"] \arrow[d, "\pi_f^1"'] & K_g \arrow[r, "\pi_g^2"'] \arrow[d, "\pi_g^1"'] & Z \arrow[d, "g"'] \\
X \arrow[r, "h"] \arrow[rr, bend right, "t \circ f"'] & Y \arrow[r, "g"] & W
\end{tikzcd}
$$

   Thus the left halves of the rectangle above are pullbacks again by Theorem Appendix A.1.
3. For the last square, let $(t_1, t_2) \colon T \to X \times X$ and $s \colon T \to K_g$ be two arrows

such that

$$(\pi_g^1, \pi_g^2) \circ s = (h \times h) \circ (t_1, t_2)$$

Thus, by point 2, there exist two arrows $x, y \colon T \rightrightarrows K_f$ fitting in the diagrams below

$$\begin{tikzcd}
T \arrow[r, dotted, "x"'] \arrow[rr, bend left, "s"] \arrow[dr, bend right, "t_1"'] & K_f \arrow[r, "k_h"'] \arrow[d, "\pi_f^1"'] & K_g \arrow[d, "\pi_g^1"] \\
& X \arrow[r, tail, "h"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, dotted, "y"'] \arrow[rr, bend left, "s"] \arrow[dr, bend right, "t_2"'] & K_f \arrow[r, "k_h"'] \arrow[d, "\pi_f^2"'] & K_g \arrow[d, "\pi_g^2"] \\
& X \arrow[r, tail, "h"'] & Y
\end{tikzcd}$$

By the first point k_h is mono, thus we can deduce that $x = y$. By construction we have

$$(\pi_f^1, \pi_f^2) \circ x = (t_1, t_2) \qquad k_h \circ x = s$$

Uniqueness now follows once again from the fact that k_h is mono. $\square$

Proposition Appendix A.4. *Let* $\mathbf{X}$ *be a strict* $\mathcal{M}$*-adhesive category with all pullbacks, and suppose that in the cube aside the top face is an* $\mathcal{M}$*-pushout and all the vertical faces are pullbacks. Then the right square is a pushout.*

$$\begin{tikzcd}
& A' \arrow[rr, "f'"] \arrow[dl, tail, "m'"'] \arrow[dd, "a"' near start] & & B' \arrow[dl, tail, "n'"'] \arrow[dd, "b"] \\
C' \arrow[rr, crossing over, "g'" near start] \arrow[dd, "c"'] & & D' & \\
& A \arrow[rr, "f" near end] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\
C \arrow[rr, "g"'] & & D \arrow[from=uu, crossing over, "d" near start] &
\end{tikzcd}
\qquad
\begin{tikzcd}
K_a \arrow[r, "k_{f'}"] \arrow[d, "k_{m'}"'] & K_b \arrow[d, "k_{n'}"] \\
K_c \arrow[r, "k_{g'}"'] & K_d
\end{tikzcd}$$

$$\begin{tikzcd}
& K_a \arrow[rr, "k_{f'}"] \arrow[dl, tail, "k_{m'}"'] \arrow[dd, "\pi_a^1"' near start] & & K_b \arrow[dl, tail, "k_{n'}"'] \arrow[dd, "\pi_b^1"] \\
K_c \arrow[rr, crossing over, "k_{g'}" near start] \arrow[dd, "\pi_c^1"'] & & K_d & \\
& A' \arrow[rr, "f'" near end] \arrow[dl, tail, "m'"] & & B' \arrow[dl, tail, "n'"] \\
C' \arrow[rr, "g'"'] & & D' \arrow[from=uu, crossing over, "\pi_d^1" near start] &
\end{tikzcd}$$

Proof. By Theorem 2.6 we know that the top face of the original cube is a pullback. Thus Theorem 2.12 entails that in the following cube the vertical faces are pullbacks. The claim now follows from strict $\mathcal{M}$-adhesivity. $\square$

To prove Theorem 2.14 we need some results about pushouts and coproducts in **Set**.

Lemma Appendix A.5. *Suppose that the square aside is a pushout, with* $m \colon A \rightarrowtail B$ *an injection. Let* $\iota_m \colon E \rightarrowtail B$ *be the inclusion of the complement of the image of* m*. Then* $(D, \{n, h \circ \iota_m\})$ *is a coproduct. In particular,* $h \circ \iota_m$ *is mono.*

$$\begin{tikzcd}
A \arrow[r, tail, "m"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
C \arrow[r, tail, "n"'] & D
\end{tikzcd}$$

Proof. Let $f \colon E \to Z$ and $k \colon C \to Z$ be two arrows with the same codomain. By definition of complement and since m is mono we know that $(B, \{m, \iota_m\})$ is a coproduct. Define $\phi \colon B \to Z$ as the unique arrow such that

$$\phi \circ m = k \circ g \qquad \phi \circ \iota_m = f$$

By the universal property of pushouts there is a unique arrow $\psi \colon D \to Z$ such that $\psi \circ n = k$ and $\psi \circ h = \phi$, thus

$$\psi \circ h \circ \iota_m = \phi \circ \iota_m = f$$

To conclude we have to show that ψ is the unique arrow $D \to Z$ such that $\psi \circ n = k$ and $\psi \circ h \circ \iota_m = f$. Let $\psi' \colon D \to Z$ be another arrow such that

$\psi' \circ n = k$ and $\psi' \circ h \circ \iota_m = f$. Then we have

$$\psi' \circ h \circ m = \psi' \circ n \circ g = k \circ g = \phi \circ m = \psi \circ h \circ m$$

Since m is mono then $\psi' \circ h = \psi \circ h$, but $(D, \{n, h\})$ is a pushout and so $\psi' = \psi$. $\square$

The category of **Set** enjoys two other remarkable properties; it is *distributive* and *extensive* [9]. Distributivity amount to the following property: for every family $\{X_i\}_i$ and an object Y, the unique morphisms ϕ and ψ fitting in the diagrams below, where j_i, k_i and h_i are coprojections, are isomorphisms

$$\begin{tikzcd} & Y \times X_i \arrow[dl, tail, "j_i"'] \arrow[dr, tail, "\mathsf{id}_Y \times h_i"] & \\ \sum_{i\in I}(Y \times X_i) \arrow[rr, dotted, "\phi"'] & & Y \times (\sum_{i\in I} X_i) \end{tikzcd} \qquad \begin{tikzcd} & X_i \times Y \arrow[dl, tail, "k_i"'] \arrow[dr, tail, "h_i \times \mathsf{id}_Y"] & \\ \sum_{i\in I}(X_i \times Y) \arrow[rr, dotted, "\psi"'] & & (\sum_{i\in I} X_i) \times Y \end{tikzcd}$$

Extensivity means that given a family of objects $\{X_i\}_{i\in I}$ and a family of commuting squares as the one on the right, where j_i is a coprojection, then all the squares are pullbacks if and only if $(Z, \{k_i\}_{i\in I})$ is a coproduct.

$$\begin{tikzcd} Z_i \arrow[r, "k_i"'] \arrow[d, "f_i"'] & Z \arrow[d, "f"] \\ X_i \arrow[r, tail, "j_i"] & X \end{tikzcd}$$

Remark Appendix A.6. Notice that extensivity entails that coproducts are *disjoint*, i.e. the pullback between two coprojections is given by the initial object (with initial maps as coprojections). To see this, let $\{X, \{j_i\}_{i\in I}\}$ be the coproduct of the family $\{X_i\}_{i\in I}$ and k an element of I. Then $(X_k, \{t_i\}_{i\in I})$ such that $t_i = \mathsf{id}_{X_k}$ if $i = k$, and $?_{X_k}$ otherwise, is a coproduct and so the squares on the right are pullbacks.

$$\begin{tikzcd} \emptyset \arrow[r, tail, "?_{X_k}"'] \arrow[d, tail, "?_{X_i}"'] & X_k \arrow[d, tail, "j_k"] \\ X_i \arrow[r, tail, "j_i"] & X \end{tikzcd} \qquad \begin{tikzcd} X_k \arrow[r, "\mathsf{id}_X"'] \arrow[d, "\mathsf{id}_X"'] & X_k \arrow[d, tail, "j_k"] \\ X_k \arrow[r, tail, "j_k"] & X \end{tikzcd}$$

In particular, the previous remark yields at once the following fact.

Proposition Appendix A.7. *Suppose that the square on the right is a pullback. Let $\iota\colon E \rightarrowtail Y$ be the inclusion of the complement of the image of p_1. If there exist two arrows $t_1\colon T \to E$ and $t_2\colon T \to X$ such that $g \circ \iota \circ t_1 = f \circ t_2$ then T is empty.*

$$\begin{tikzcd} P \arrow[r, "p_1"'] \arrow[d, "p_2"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

Proof. By the universal property of pullbacks we get an arrow $t\colon T \to P$ such that

$$p_1 \circ t = \iota \circ t_1 \qquad p_2 \circ t = t_2$$

$$\begin{tikzcd} T \arrow[r, dotted, "s"] \arrow[rr, bend left, "t_1"] \arrow[dr, "\pi"'] & \emptyset \arrow[r, tail, "?_E"'] \arrow[d, tail, "?_I"'] & E \arrow[d, tail, "\iota"] \\ & I \arrow[r, tail, "i"'] & Y \end{tikzcd}$$

Let $i\colon I \rightarrowtail Y$ be the inclusion of the image of p_1, so that we have also an epi $q\colon P \twoheadrightarrow I$. By definition of complement $(Y, \{i, \iota\})$ is a coproduct, thus by Theorem Appendix A.6 the inner square in the diagram on the left is a pullback and we get a dotted arrow $s\colon T \to \emptyset$ filling the diagram. But a set with an arrow towards the empty set must be empty and we conclude. $\square$

Extensivity and distributivity, moreover, yield the following fact.

Lemma Appendix A.8. *Let $f, g\colon A \rightrightarrows B+C$ be two arrows with a coproduct as codomain. Then $(A, \{j_i\}_{i=0}^{3})$ is a coproduct, where $j_i\colon A_i \rightarrowtail A$ is defined by the following pullback squares*

$$
\begin{tikzcd}
A_0 \arrow[r, tail, "j_0"] \arrow[d, "p_0"'] & A \arrow[d, "{(f,g)}"] \\
B \times B \arrow[r, tail, "i_B \times i_B"'] & (B+C) \times (B+C)
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, tail, "j_1"] \arrow[d, "p_1"'] & A \arrow[d, "{(f,g)}"] \\
B \times C \arrow[r, tail, "i_B \times i_C"'] & (B+C) \times (B+C)
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_2 \arrow[r, tail, "j_2"] \arrow[d, "p_2"'] & A \arrow[d, "{(f,g)}"] \\
C \times B \arrow[r, tail, "i_C \times i_B"'] & (B+C) \times (B+C)
\end{tikzcd}
\qquad
\begin{tikzcd}
A_3 \arrow[r, tail, "j_3"] \arrow[d, "p_3"'] & A \arrow[d, "{(f,g)}"] \\
C \times C \arrow[r, tail, "i_C \times i_C"'] & (B+C) \times (B+C)
\end{tikzcd}
$$

We are now ready to exploit the previous properties to prove Theorem 2.14.

Lemma Appendix A.9. *Suppose that in* **Set** *the commuting cube in the diagram on the left is given, whose top face is a pushout, the left and bottom faces are pullbacks, and $n\colon B \rightarrowtail D$ is an injection. Then the following hold*

1. *the right face of the cube is a pullback;*
2. *the right square, made by the kernel pairs of the vertical arrows, is a pushout.*

$$
\begin{tikzcd}
& A' \arrow[rr, "f'"] \arrow[dl, tail, "m'"'] \arrow[dd, "a" near start] & & B' \arrow[dl, tail, "n'"'] \arrow[dd, "b"] \\
C' \arrow[rr, "g'" near start, crossing over] \arrow[dd, "c"'] & & D' & \\
& A \arrow[rr, "f" near start] \arrow[dl, tail, "m"'] & & B \arrow[dl, tail, "n"] \\
C \arrow[rr, "g"'] & & D \arrow[from=uu, "d" near start, crossing over] &
\end{tikzcd}
\qquad
\begin{tikzcd}
K_a \arrow[r, "k_{f'}"] \arrow[d, "k_{m'}"'] & K_b \arrow[d, "k_{n'}"] \\
K_c \arrow[r, "k_{g'}"'] & K_d
\end{tikzcd}
$$

Proof. We can start noticing that m, being the pullback of n is mono too. This, in turn, entails that m', being the pullback of m, is mono, and so n' is injective too because in **Set**, as in any adhesive category, monomorphisms are stable under pushouts.

Let $\iota'\colon E' \rightarrowtail C'$ and $\iota\colon E \to C$ be the inclusions of the complement of the images of of m' and m respectively . By Theorem Appendix A.5, $(D', \{n', g' \circ \iota\})$ is a coproduct.

$$
\begin{tikzcd}
E' \arrow[r, tail, "\iota'"] \arrow[d, dotted, "w"'] & C' \arrow[d, "c"] \\
E \arrow[r, tail, "\iota"] & C
\end{tikzcd}
$$

We can also notice that $c \circ \iota'$ factors through E, as shown by the square on the right.To see this, let e be in E' and suppose that $c(\iota'(e)) = m(x)$ for some $x \in A$. Then we can apply the universal property of pullbacks to build the dotted arrow v in the diagram aside, where v_0 and v_1 are the arrows picking x and $\iota'(e)$, respectively. But then e must belong to the image of m', which is a contradiction.

$$
\begin{tikzcd}
1 \arrow[r, dotted, "v"'] \arrow[rr, bend left, "v_1"] \arrow[dr, "v_0"'] & A' \arrow[r, tail, "m'"'] \arrow[d, "a"'] & C' \arrow[d, "c"] \\
& A \arrow[r, tail, "m"] & C
\end{tikzcd}
$$

1. Consider two arrows $t_1\colon T \to D'$ and $t_2\colon T \to B$ such that $d \circ t_1 = n \circ t_2$. By extensivity

   of **Set**, we already know that $(T, \{l_i\}_{i=0}^{1})$ is a coproduct, where $l_i\colon T_i \rightarrowtail T$ are defined by the diagram aside, whose two halves are pullbacks.

$$
\begin{tikzcd}
T_0 \arrow[r, tail, "l_0"'] \arrow[d, "h_0"'] & T \arrow[d, "t_1"] & T_1 \arrow[l, tail, "l_1"'] \arrow[d, "h_1"] \\
E' \arrow[r, tail, "g' \circ \iota'"] & D' & B' \arrow[l, tail, "n'"']
\end{tikzcd}
$$

If we compute we get

$$g \circ c \circ \iota' \circ h_0 = d \circ g' \circ \iota' \circ h_0 = d \circ t_1 \circ l_0 = n \circ t_2 \circ l_0$$

By hypothesis the bottom face of the given cube is a pullback, therefore there exists an arrow $u \colon T_0 \to A$ such that

$$f \circ u = n \circ t_2 \circ l_0 \qquad m \circ u = c \circ \iota' \circ h_0$$

By Theorem Appendix A.7 we conclude that T_0 is empty. Therefore l_1 is an isomorphism and we have $n' \circ h_1 \circ l_1^{-1} = t_1$. On the other hand

$$n \circ b \circ h_1 \circ l_1^{-1} = d \circ n' \circ h_1 \circ l_1^{-1} = d \circ t_1 = n \circ t_2$$

And we can conclude that $b \circ h_1 \circ l_1^{-1} = t_2$ because n is a mono. The claim now follows from the fact that also n' is mono.

2. By Theorems Appendix A.5 and Appendix A.8, the previous point and the third point of Theorem 2.12, we can decompose K_d as the coproduct of K_b K_1, K_2 and K_3 with coprojections given by, respectively, $k_{n'}$ and j_1, j_2 and j_3, where these objects and arrows fit in the following four pullbacks

$$\begin{tikzcd}
K_b \arrow[r, tail, "k_{n'}"] \arrow[d, "(\pi_b^1, \pi_b^2)"'] & K_d \arrow[d, "(\pi_d^1, \pi_d^2)"] \\
B' \times B' \arrow[r, tail, "n' \times n'"'] & D' \times D'
\end{tikzcd}
\qquad
\begin{tikzcd}
K_1 \arrow[r, tail, "j_1"] \arrow[d, "(p_1, q_1)"'] & K_d \arrow[d, "(\pi_d^1, \pi_d^2)"] \\
B' \times E' \arrow[r, tail, "n' \times (g' \circ \iota')"'] & D' \times D'
\end{tikzcd}$$

$$\begin{tikzcd}
K_2 \arrow[r, tail, "j_2"] \arrow[d, "(p_2, q_2)"'] & K_d \arrow[d, "(\pi_d^1, \pi_d^2)"] \\
E' \times B' \arrow[r, tail, "(g' \circ \iota') \times n'"'] & D' \times D'
\end{tikzcd}
\qquad
\begin{tikzcd}
K_3 \arrow[r, tail, "j_3"] \arrow[d, "(p_3, q_3)"'] & K_d \arrow[d, "(\pi_d^1, \pi_d^2)"] \\
E' \times E' \arrow[r, tail, "(g' \circ \iota') \times (g' \circ \iota')"'] & D' \times D'
\end{tikzcd}$$

Let us now examine K_1 and K_2. We have

$$n \circ b \circ p_1 = d \circ n' \circ p_1 = d \circ \pi_d^1 \circ j_1 = d \circ \pi_d^2 \circ j_1 = d \circ g' \circ \iota' \circ q_1 = g \circ c \circ \iota' \circ q_1$$
$$n \circ b \circ q_2 = d \circ n' \circ p_2 = d \circ \pi_D^2 \circ j_2 = d \circ \pi_d^1 \circ j_2 = d \circ g' \circ \iota' \circ p_2 = g \circ c \circ \iota' \circ p_2$$

Since the bottom face of the original cube is a pullback (by hypothesis), we conclude that there exist arrows $k_1 \colon K_1 \to A$ and $k_2 \colon K_2 \to A$ such that

$$m \circ k_1 = c\iota' \circ q_1 \quad f \circ k_1 = b \circ p_1 \quad m \circ k_2 = c\iota' \circ q_2 \quad f \circ k_2 = b \circ q_2$$

By Theorem Appendix A.7 we conclude that K_1 and K_2 are both empty. In particular, this implies that $(K_d, \{k_{n'}, j_3\})$ is a coproduct.
We focus now on K_3. By computing we get

$$\begin{aligned} g \circ \iota \circ w \circ p_3 &= g \circ c \circ \iota' \circ p_3 = d \circ g' \circ \iota' \circ p_3 = d \circ \pi_d^1 \circ j_3 \\ &= d \circ \pi_d^2 \circ j_3 = d \circ g' \circ \iota' \circ q_3 = g \circ c \circ \iota' \circ q_3 = g \circ \iota \circ w \circ q_3 \end{aligned}$$

$$\begin{tikzcd}
K_3 \arrow[r, "p_3"] \arrow[d, "q_3"'] \arrow[dr, dotted, "\phi_3"] & E' \arrow[dr, "\iota'"] & \\
E' \arrow[dr, "\iota'"'] & K_c \arrow[r, "\pi_c^1"] \arrow[d, "\pi_c^2"'] & C' \arrow[d, "c"] \\
& C' \arrow[r, "c"'] & C
\end{tikzcd}$$

By Theorem Appendix A.5 $g \circ \iota$ is a monomorphism, thus $w \circ p_3 = w \circ q_3$ and therefore

$$c \circ \iota' \circ p_3 = \iota \circ w \circ p_3 = \iota \circ w \circ q_3 = c \circ \iota' \circ q_3$$

Thus we get the dotted $\phi_3 \colon K_3 \to K_c$ in the diagram aside.

Moreover, $k_{g'} \circ \phi_3 = j_3$ as shown by the following computation.

$$
\begin{aligned}
&(\pi_d^1, \pi_d^2) \circ k_{g'} \circ \phi_3 = (g' \times g') \circ (\pi_c^1, \pi_c^2) \circ \phi_3 = (g' \times g') \circ (\iota' \circ p_3, \iota' \circ q_3) \\
=&((g' \circ \iota') \times (g' \circ \iota')) \circ (p_3, q_3) = (\pi_d^1, \pi_d^2) \circ j_3
\end{aligned}
$$

By definition of complement, we also know that $(C', \{\iota', m'\})$ is a coproduct. We can then apply again Theorem Appendix A.8, decomposing K_c in four parts, given by the pullbacks below

$$
\begin{tikzcd}
H_0 \arrow[r, tail, "x_0"] \arrow[d, "{(y_0,z_0)}"'] & K_c \arrow[d, "{(\pi_c^1,\pi_c^2)}"] \\
A' \times A' \arrow[r, tail, "m' \times m'"'] & C' \times C'
\end{tikzcd}
\qquad
\begin{tikzcd}
H_1 \arrow[r, tail, "x_1"] \arrow[d, "{(y_1,z_1)}"'] & K_c \arrow[d, "{(\pi_c^1,\pi_c^2)}"] \\
A' \times E' \arrow[r, tail, "m' \times \iota'"'] & C' \times C'
\end{tikzcd}
$$

$$
\begin{tikzcd}
H_2 \arrow[r, tail, "x_2"] \arrow[d, "{(y_2,z_2)}"'] & K_c \arrow[d, "{(\pi_c^1,\pi_c^2)}"] \\
E' \times A' \arrow[r, tail, "\iota' \times m'"'] & C' \times C'
\end{tikzcd}
\qquad
\begin{tikzcd}
H_3 \arrow[r, tail, "x_3"] \arrow[d, "{(y_3,z_3)}"'] & K_c \arrow[d, "{(\pi_c^1,\pi_c^2)}"] \\
E' \times E' \arrow[r, tail, "\iota' \times \iota'"'] & C' \times C'
\end{tikzcd}
$$

Let us examine H_0. We start noticing that

$$
\begin{aligned}
(\pi_d^1, \pi_d^2) \circ k_{g'} \circ x_0 =&(g' \times g') \circ (\pi_c^1, \pi_c^2) \circ x_0 \\
=&(g' \times g') \circ (m' \times m') \circ (y_0, z_0) \\
=&((g' \circ m') \times (g' \circ m')) \circ (y_0, z_0) \\
=&((n' \circ f') \times (n' \circ f')) \circ (y_0, z_0) \\
=&(n' \times n') \circ (f' \times f') \circ (y_0, z_0)
\end{aligned}
$$

$$
\begin{tikzcd}
H_0 \arrow[rr, "{(y_0,z_0)}"] \arrow[d, "x_0"'] \arrow[dr, dotted, "h_0"] & & A' \times A' \arrow[d, "f' \times f'"] \\
K_c \arrow[dr, "k_{g'}"'] & K_b \arrow[r, "{(\pi_b^1,\pi_b^2)}"'] \arrow[d, "k_{n'}"] & B' \times B' \arrow[d, "n' \times n'"] \\
& K_d \arrow[r, "{(\pi_d^1,\pi_d^2)}"] & D' \times D'
\end{tikzcd}
$$

Thus we get the dotted arrow $h_0 \colon H_0 \to K_b$ in the diagram above. Moreover, we have

$$
m \circ a \circ y_0 = c \circ m' \circ y_0 = c \circ \pi_c^1 \circ x_0 = c \circ \pi_c^2 \circ x_0 = c \circ m' \circ z_0 = m \circ a \circ z_0
$$

Since m is a mono $a \circ y_0 = a \circ z_0$ and there exists an arrow $h'_0 \colon H_0 \to K_a$ such that $\pi_a^1 \circ h'_0 = y_0$ and $\pi_a^2 \circ h'_0 = z_0$. We now can conclude that $k_{f'} \circ h'_0 = h_0$, noticing that

$$
(\pi_b^1, \pi_b^2) \circ h_0 = (f' \times f') \circ (y_0, z_0) = (f' \times f') \circ (\pi_a^1, \pi_a^2) \circ h'_0 = (\pi_b^1, \pi_b^2) \circ k_{f'} \circ h'_0
$$

We can prove another property of h'_0. By computing we have

$$
(\pi_c^1, \pi_c^2) \circ k_{m'} \circ h'_0 = (m' \times m') \circ (\pi_a^1, \pi_a^2) \circ h'_0 = (m' \times m') \circ (y_0, z_0) = (\pi_c^1, \pi_c^2) \circ x_0
$$

so that $k_{m'} \circ h'_0$ must coincide with x_0.
As a next step, let us focus on H_1 and H_2. Two computations yield

$$
\begin{aligned}
(\pi_d^1, \pi_d^2) \circ k_{g'} \circ x_1 =& (g' \times g') \circ (\pi_c^1, \pi_c^2) \circ x_1 \\
=&(g' \times g') \circ (m' \times \iota') \circ (y_1, z_1) \\
=&((g' \circ m') \times (g' \circ \iota')) \circ (y_1, z_1) \\
=&((n' \circ f') \times (g' \circ \iota')) \circ (y_1, z_1) \\
=&(n' \times (g' \circ \iota')) \circ (f' \circ y_1, z_1)
\end{aligned}
\qquad
\begin{aligned}
(\pi_d^1, \pi_d^2) \circ k_{g'} \circ x_2 =& (g' \times g') \circ (\pi_c^1, \pi_c^2) \circ x_2 \\
=&(g' \times g') \circ (\iota' \times m') \circ (y_2, z_2) \\
=&((g' \circ \iota') \times (g' \circ m')) \circ (y_2, z_2) \\
=&((g' \circ \iota') \times (n' \circ f')) \circ (y_2, z_2) \\
=&((g' \circ \iota') \times n') \circ (y_2, f' \circ z_2)
\end{aligned}
$$

Hence, we have arrows $h_1 \colon H_1 \to K_1$ and $h_2 \colon H_2 \to K_2$, showing that H_1

and H_2 are empty. For H_3, let us consider the two diagrams below

$$\begin{tikzcd}
H_3 \arrow[r, "x_3"] \arrow[d, dotted, "\alpha_3"] \arrow[dd, bend right, "{(y_3,z_3)}"'] & K_c \arrow[d, "k_{g'}"'] \\
K_3 \arrow[r, tail, "j_3"] \arrow[d, "{(p_3,q_3)}"] & K_d \arrow[d, "{(\pi^1_d,\pi^2_d)}"] \\
E' \times E' \arrow[r, tail, "(g'\circ\iota')\times(g'\circ\iota')"'] & D' \times D'
\end{tikzcd}
\qquad
\begin{tikzcd}
K_3 \arrow[dr, bend left, "\phi_3"] \arrow[d, dotted, "\beta_3"] \arrow[dd, bend right, "{(p_3,q_3)}"'] & \\
H_3 \arrow[r, tail, "x_3"] \arrow[d, "{(y_3,z_3)}"] & K_c \arrow[d, "{(\pi^1_c,\pi^2_c)}"] \\
E' \times E' \arrow[r, tail, "\iota'\times\iota'"'] & C' \times C'
\end{tikzcd}$$

Their solid part commute. Indeed, we have

$$\begin{aligned}(\pi^1_d, \pi^2_d) \circ k_{g'} \circ x_3 = &(g' \times g') \circ (\pi^1_c, \pi^2_c) \circ x_3 \\ =&(g' \times g') \circ (\iota' \times \iota') \circ (y_3 \times z_3) \\ =&(g' \circ \iota') \times (g' \circ \iota') \circ (y_3, z_3)\end{aligned}
\qquad
\begin{aligned}(\pi^1_c, \pi^2_c) \circ \phi_3 =&(\iota' \circ p_3, \iota' \circ q_3) \\ =&(\iota' \times \iota') \circ (p_3, q_3)\end{aligned}$$

Thus we get the dotted arrows α_3 and β_3 which are one the inverse of the other. Indeed, on the one hand we have

$$j_3 \circ \alpha_3 \circ \beta_3 = k_{g'} \circ x_3 \circ \beta_3 = k_{g'} \circ \phi_3 = j_3 \qquad (p_3, q_3) \circ \alpha_3 \circ \beta_3 = (y_3, z_3) \circ \beta_3 = (p_3, q_3)$$

On the other hand, notice that

$$(\pi^1_c, \pi^2_c) \circ \phi_3 \circ \alpha_3 = (\iota' \times \iota') \circ (p_3, q_3) \circ \alpha_3 = (\iota' \times \iota') \circ (y_3, z_3) = (\pi^1_c, \pi^2_c) \circ x_3$$

Therefore $\phi_3 \circ \alpha_3 = x_3$ and thus $x_3 \circ \beta_3 \circ \alpha_3 = x_3$. Moreover

$$(y_3, z_3) \circ \beta_3 \circ \alpha_3 = (p_3, q_3) \circ \alpha_3 = (y_3, z_3)$$

Summing up, we have just proved that $(K_c, \{x_0, \phi_3\})$ is a coproduct.
Let now $\gamma\colon K_b \to Z$ and $\delta\colon K_c \to Z$ be two arrows such that $\gamma \circ k_{f'} = \delta \circ k_{m'}$. We want to construct an arrow $\theta\colon K_d \to Z$ such that $\theta \circ k_{g'} = \delta$ and $\theta \circ k_{n'} = \gamma$.
We have already proved that $(K_d, \{k_n, j_3\})$ is a coproduct, thus there is a unique arrow $\theta\colon K_d \to Z$ such that $\theta \circ k_{n'} = \gamma$ and $\theta \circ j_3 = \delta \circ \phi_3$. Now, on the one hand we have

$$\theta \circ k_{g'} \circ \phi_3 = \theta \circ j_3 = \delta \circ \phi_3$$

On the other hand, we can conclude that $\theta \circ k_{g'} = \delta$ as wanted, since

$$\theta \circ k_{g'} \circ x_0 = \theta \circ k_{n'} \circ h_0 = \gamma \circ h_0 = \gamma \circ k_{f'} \circ h'_0 = \delta \circ k_{m'} \circ h'_0 = \delta \circ x_0$$

We are left with uniqueness. If θ' is another arrow $K_d \to Z$ such that $\theta' \circ k_{n'} = \gamma$ and $\theta' \circ k_{g'} = \delta$. If we compute we get

$$\theta' \circ j_3 = \theta' \circ k_{g'} \circ \phi_3 = \delta \circ \phi_3 = \theta \circ j_3$$

We already know that $\theta' \circ k_{n'} = \theta \circ k_{n'}$, so the previous identity entails that $\theta = \theta'$. $\square$

Appendix A.2. Proofs for Section 4

Proposition Appendix A.10. *Consider the forgetful functor $T\colon \mathbf{EqHyp} \to \mathbf{Hyp}$. Then the following hold*

1. *T is faithful;*
2. *T has a left adjoint;*
3. *T has a right adjoint.*

Proof. 1. This follows at once from Theorem 4.3.

2. Let $\mathcal{H}$ be a hypergraph, and define $L(\mathcal{H}) := (E_{\mathcal{H}}, V_{\mathcal{H}}, V_{\mathcal{H}}, s_{\mathcal{H}}, t_{\mathcal{H}}, \mathsf{id}_{V_{\mathcal{H}}})$. By construction we have $T(L(\mathcal{H})) = \mathcal{H}$, thus we can define $\eta_{\mathcal{H}}\colon \mathcal{H} \to T(L(\mathcal{H}))$ as the identity $\mathsf{id}_{\mathcal{H}}$.
   To see that in this way we get a unit, take an arrow $(h,k)\colon \mathcal{H} \to T(\mathcal{G})$ for some $\mathcal{G}$ in $\mathbf{EqHyp}$. Then $(h,k,q_{\mathcal{G}} \circ k)$ is an arrow $L(\mathcal{H}) \to \mathcal{G}$ and the unique one such that $T(h,k,q_{\mathcal{G}} \circ k) \circ \eta_{\mathcal{H}} = (h,k)$
3. For every hypergraph $\mathcal{H}$ define $R(\mathcal{H})$ as $(E_{\mathcal{H}}, V_{\mathcal{H}}, 1, s_{\mathcal{H}}, t_{\mathcal{H}}, !_{V_{\mathcal{H}}})$, so that $T(R(\mathcal{H}))$ is again $\mathcal{H}$. Now, let $\epsilon_{\mathcal{H}}\colon T(R(\mathcal{H})) \to \mathcal{H}$ be the identity and take an arrow $(h,k)\colon T(\mathcal{G}) \to \mathcal{H}$ for some $\mathcal{G}$ in $\mathbf{EqHyp}$. Notice that $!_{Q_{\mathcal{G}}} \circ q_{\mathcal{G}} = !_{V_{\mathcal{H}}} \circ k$ so that $(h,k,!_{Q_{\mathcal{G}}})$ is an arrow $\mathcal{G} \to R(\mathcal{H})$ of $\mathbf{EqHyp}$ such that $\epsilon_{\mathcal{H}} \circ T(h,k,!_{Q_{\mathcal{G}}}) = (h,k)$.
   Uniqueness of such an arrow follows at once from the first point and the fact that 1 is terminal. □

Remark Appendix A.11. Before proceeding further, let us recall the following result about the classes of regular epis (i.e. surjections) and of monos in $\mathbf{Set}$. In particular, we need the fact that they form a a *factorization system* [24] on $\mathbf{Set}$. This amounts to ask that

1. every arrow $f\colon X \to Y$ can be factored as $m \circ e$, where $e\colon X \twoheadrightarrow Q$ is a regular epi and $m\colon Q \rightarrowtail Y$ is a mono;
2. for every commuting square as the one on the right, with $e\colon X \twoheadrightarrow E$ a surjection and $m\colon N \rightarrowtail Y$ a mono, there exists a unique $k\colon E \to N$ making it commutative.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "e"'] & N \arrow[d, tail, "m"] \\ E \arrow[r, "g"'] \arrow[ur, dotted, "k"] & Y \end{tikzcd}$$

From these properties, one can deduce that if $f = m \circ e$ and $f = m' \circ e'$ are two factorizations of f then there is a bijection ϕ such that $m = m' \circ \phi$ and $\phi \circ e = e'$.

We will also need the following well-known fact about regular epis.

Lemma Appendix A.12. *Let $F, G\colon \mathbf{D} \rightrightarrows \mathbf{X}$ be two diagrams and suppose that $\mathbf{X}$ has all colimits of shape $\mathbf{D}$. Let $(X, \{x_d\}_{d\in\mathbf{D}})$ and $(Y, \{y_d\}_{d\in D})$ be the colimits of F and G, respectively. If $\phi\colon F \to G$ is a natural transformation whose components are regular epis, then the arrow induced by ϕ from X to Y*

is a regular epi.

Lemma Appendix A.13. *Consider a diagram $F\colon \mathbf{D} \to \mathbf{EqHyp}$ and let $(E_D, V_D, Q_D, s_D, t_D, q_D)$ be the image of an object D. Then the following hold*

1. *F has a colimit, which is preserved by K;*
2. *consider a cone $(L, \{l_D\}_{D\in\mathbf{D}})$ limiting for $K \circ F$ and let $((E, V), \{(\pi_E^D, \pi_V^D)\}_{D\in\mathbf{D}})$ be one for $T \circ F$, then F has a limit (E, V, Q, s, t, q) and there is a mono $m\colon Q \rightarrowtail L$ such that the diagram on the right commutes for every $D \in \mathbf{D}$.*

$$\begin{tikzcd}
V \arrow[r, "\pi_V^D"] \arrow[d, two heads, "q"'] & V_D \arrow[dd, two heads, "q_D"] \\
Q \arrow[d, tail, dotted, "m"'] & \\
L \arrow[r, "l_d"'] & Q_D
\end{tikzcd}$$

Proof. 1. We know by Theorem 3.12 that **Hyp** is cocomplete. Thus, let (E, V, s, t) together with $\{(\kappa_E^D, \kappa_V^D)\}_{D\in\mathbf{D}}$ be a colimit for $T \circ F$. By Lemma Appendix B.3 we also know that $(V, \{\kappa_V^D\}_{D\in\mathbf{D}})$ is a colimit for $U_{\mathsf{eq}} \circ F$. Moreover, since **Set** is cocomplete too we can also take a colimit $(C, \{c_d\}_{d\in\mathbf{D}})$ for $K \circ F$. Now, let α be an arrow $D \to D'$ in $\mathbf{D}$, and suppose that $F(\alpha)$ is (h_1, h_2, h_3). By definition of morphisms in **EqHyp**, the square on the right commutes.

$$\begin{tikzcd}
V_d \arrow[r, "h_2"] \arrow[d, two heads, "q_D"'] & V_{D'} \arrow[d, two heads, "q_{D'}"] \\
Q_D \arrow[r, "h_3"] & Q_{D'}
\end{tikzcd}$$

Thus the family $\{q_D\}_{D\in\mathbf{D}}$ form a natural transformation $U_{\mathsf{eq}} \circ F \to K \circ F$. By Theorem Appendix A.12, the induced arrow $q : V \to C$ between the colimits is a surjection. We can then consider the object (E, V, C, s, t, q) of **EqHyp**, together with the family $\{(\kappa_E^D, \kappa_V^D, c_D)\}_{D\in\mathbf{D}}$, which by construction is a cocone on F. Let $((E_\mathcal{G}, V_\mathcal{G}, Q_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G}, q_\mathcal{G}), \{(h_E^D, h_V^D, h_Q^D)\}_{D\in\mathbf{D}})$ be a cocone, then $((E_\mathcal{G}, V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G}), \{(h_E^D, h_V^D)\}_{D\in\mathbf{D}})$ and $(Q_\mathcal{G}, \{h_Q^D\}_{D\in\mathbf{D}})$ are cocone on, respectively, $T \circ F$ and $K \circ D$, giving arrows (h_E, h_V) in **Hyp** and h_Q in **Set** such that

$$(h_E, h_V) \circ (\kappa_E^D, \kappa_V^D) = (h_E^D, h_V^D) \qquad h_Q \circ c_D = h_Q^D$$

Now, to show that (h_E, h_V, h_Q) is an arrow of **EqHyp** we can compute

$$h_Q \circ q \circ \kappa_V^D = h_Q \circ c_D \circ q_D = h_Q^D \circ q_D = q_\mathcal{G} \circ h_V^D = q_\mathcal{G} \circ h_V \circ \kappa_V^D$$

Uniqueness of such arrow follows form the fact that T is faithful and Theorem 4.3.

2. **Hyp** is complete, again by Theorem 3.12, we can then consider a limiting cone $((E, V, s, t), \{(\pi_E^D, \pi_V^D)\}_{D\in\mathbf{D}})$ over of $T \circ F$. Now, $(V, \{q_D \circ \pi_V^D\}_{D\in\mathbf{D}})$, is a cone for $K \circ F$: indeed, if $\alpha\colon D \to D'$ is an arrow of $\mathbf{D}$, and $F(\alpha) = (h_1, h_2, h_3)$, then we have

$$h_3 \circ q_D \circ \pi_V^D = q_{D'} \circ h_2 \circ \pi_V^D = q_{D'} \circ \pi_V^{D'}$$

Thus there is an arrow $l\colon V \to L$ such that $l_D \circ l = q_D \circ \pi_V^D$. By Theorem Appendix A.11, we know that there exist $m\colon Q \rightarrowtail L$ and $q\colon X \twoheadrightarrow Q$ such that $m \circ q = l$. Since the identity is mono, Theorem Appendix A.11 yield a unique arrow π_Q^D fitting in the rectangle aside.

$$\begin{tikzcd}
V \arrow[r, "\pi_V^D"] \arrow[d, two heads, "q"'] & V_D \arrow[r, two heads, "q_D"] & Q_D \arrow[d, "\mathsf{id}_{Q_D}"] \\
Q \arrow[urr, dotted, "\pi_Q^D"] \arrow[r, tail, "m"'] & L \arrow[r, "l_D"'] & Q_D
\end{tikzcd}$$

Let α be an arrow $D \to D'$ in $\mathbf{D}$. Then we have

$$T(F\alpha\circ(\pi_E^D, \pi_V^D, \pi_Q^D)) = T(F(\alpha))\circ(\pi_E^D, \pi_V^D) = (\pi_E^{D'}, \pi_V^{D'}) = T(\pi_E^{D'}, \pi_V^{D'}, \pi_Q^{D'})$$

Thus, by faitfhulness of T, $((E, V, C, s, t, q), \{(\pi_E^D, \pi_V^D, \pi_Q^D)\}_{D\in\mathbf{D}})$ is a cone over F. To see that it is terminal, let $((E_\mathcal{G}, V_\mathcal{G}, Q_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G}, q_\mathcal{G}), \{(h_E^D, h_V^D, h_Q^D)\}_{D\in\mathbf{D}})$ be another cone. In particular there is an arrow $(h_E, h_V)\colon (E_\mathcal{G}, V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G}) \to (E, V, s, t)$ in **Hyp** such that

$$\pi_E^D \circ h_E = h_E^D \qquad \pi_V^D \circ h_V = h_V^D$$

Moreover, applying K we get that $(Q_\mathcal{G}, \{h_Q^D\}_{D\in\mathbf{D}})$ is a cone over $K \circ F$, so that there is an arrow $h\colon Q_\mathcal{G} \to L$ such that $l_D \circ h = h_Q^D$. Thus

$$l_D \circ h \circ q_\mathcal{G} = h_Q^D \circ q_\mathcal{G} = q_D \circ h_V^D = q_D \circ \pi^D \circ h_V = l_D \circ l \circ h_V$$

$$\begin{tikzcd} V_\mathcal{G} \arrow[r, "h_V"] \arrow[d, two heads, "q_\mathcal{G}"'] & V \arrow[r, two heads, "q"] & Q \arrow[d, tail, "m"] \\ Q_\mathcal{G} \arrow[urr, dotted, "k_Q"] \arrow[rr, "h"'] & & L \end{tikzcd}$$

Hence the outer boundary of the square on the left commutes, yielding the dotted diagonal arrow $k_Q\colon Q_\mathcal{G} \to Q$. We have therefore built an arrow (h_E, h_V, k_Q) of **EqHyp** such that

$$(\pi_E^D, \pi_V^D, \pi_Q^D) \circ (h_E, h_V, k_Q) = (h_E^D, h_V^D, h_Q^D)$$

As in the point above, uniqueness is guaranteed by faithfulness of T and Theorem 4.3. $\square$

Corollary Appendix A.14. *An arrow $(h_E, h_V, h_Q) : \mathcal{G} \to \mathcal{H}$ in* **EqHyp** *is a regular mono if and only if all its components are injective functions.*

Proof. ($\Rightarrow$) If (h_E, h_V, h_Q) is mono, from Theorem 4.6 we have that h_E and h_V are monos.

Let now $(f_E, f_V, f_Q), (g_E, g_V, g_Q) : \mathcal{H} \rightrightarrows \mathcal{K}$ be arrows equalized by (h_E, h_V, h_Q). Let $e\colon E_Q \rightarrowtail Q_\mathcal{H}$ be the equalizer in **Set** of $f_Q, g_Q\colon Q_\mathcal{H} \rightrightarrows Q_\mathcal{K}$. From the third point of Theorem 3.1 and Corollary Appendix B.4 we know that $h_V\colon V_\mathcal{G} \rightarrowtail V_\mathcal{H}$ is the equalizer of $f_V, g_V\colon V_\mathcal{H} \rightrightarrows V_\mathcal{K}$, thus by the second point of Theorem 4.8 we know that there is a mono $m\colon Q_\mathcal{G} \rightarrowtail E_Q$ fitting in the diagram aside. Now, notice that $e \circ m \circ q_\mathcal{G} = q_\mathcal{H} \circ h_V = h_Q \circ q_\mathcal{G}$. Since $q_\mathcal{G}$ is epi we conclude that $e \circ m = h_Q$, from which the claim follows.

$$\begin{tikzcd} V_\mathcal{G} \arrow[r, tail, "h_V"] \arrow[d, two heads, "q_\mathcal{G}"'] & V_\mathcal{H} \arrow[dd, two heads, "q_\mathcal{H}"] \\ Q_\mathcal{G} \arrow[d, tail, dotted, "m"'] & \\ E_Q \arrow[r, tail, "e"'] & Q_\mathcal{H} \end{tikzcd}$$

($\Leftarrow$) Suppose that h_E, h_V, and h_Q are monos. We can build the cokernel pair of the three arrows, taking the pushout of h_E, h_V and h_Q along itself, obtaining the three diagrams below, which by Theorem 2.6 are also pullbacks

$$\begin{tikzcd} E_\mathcal{G} \arrow[r, tail, "h_E"] \arrow[d, tail, "h_E"'] & E_\mathcal{H} \arrow[d, tail, "g_E"] \\ E_\mathcal{H} \arrow[r, tail, "f_E"'] & E_\mathcal{K} \end{tikzcd} \qquad \begin{tikzcd} V_\mathcal{G} \arrow[r, tail, "h_V"] \arrow[d, tail, "h_V"'] & V_\mathcal{H} \arrow[d, tail, "g_V"] \\ V_H \arrow[r, tail, "f_V"'] & V_\mathcal{K} \end{tikzcd} \qquad \begin{tikzcd} Q_\mathcal{G} \arrow[r, tail, "h_Q"] \arrow[d, tail, "h_Q"'] & Q_\mathcal{H} \arrow[d, tail, "g_Q"] \\ Q_\mathcal{H} \arrow[r, tail, "f_Q"'] & Q_\mathcal{K} \end{tikzcd}$$

Now, we know by Proposition 3.5 and Lemma Appendix B.3 that there exists arrows $s, t\colon E_\mathcal{K} \rightrightarrows V_\mathcal{K}$ such that the resulting hypergraph is the cokernel pair of (h_E, h_V) in **Hyp**. Moreover, by Theorem Appendix A.12 we know that the unique arrow $q\colon V_\mathcal{K} \to Q_\mathcal{K}$ induced by $q_Q \circ q_\mathcal{H}$ and $f_Q \circ q_\mathcal{H}$ is a regular epi. Let thus $\mathcal{K}$ be the resulting hypergraph with equivalence.

By construction we have two arrows $(f_E, f_V, f_Q), (g_E, g_V, g_Q) : \mathcal{H} \rightrightarrows \mathcal{K}$ such that

$$(f_E, f_V, f_Q) \circ (h_E, h_V, h_Q) = (g_E, g_V, g_Q) \circ (h_E, h_V, h_Q)$$

Now, in **Set** every mono is regular [28], so by the dual of Theorem 2.11 we deduce that h_E, h_V and h_Q are the equalizers of the arrows built above. By Proposition 3.5 and corollary Appendix B.4 we also know that (h_E, h_V) is the equalizer in **Hyp** of (f_E, f_V) and (g_E, g_V). Hence, if $(z_E, z_V, z_Q)\colon \mathcal{Z} \to \mathcal{H}$ is an arrow such that

$$(f_E, f_V, f_Q) \circ (z_E, z_V, z_Q) = (g_E, g_V, g_Q) \circ (z_E, z_V, z_Q)$$

then we get a unique morphism $(a_E, a_V)\colon (E_{\mathcal{Z}}, V_{\mathcal{Z}}) \to (E_{\mathcal{G}}, V_{\mathcal{G}})$ such that $(h_E, h_V) \circ (a_E, a_V) = (z_E, z_V)$. Moreover, since $g_Q \circ z_Q = f_Q \circ z_Q$ there is a unique $a_Q\colon Q_{\mathcal{Z}} \to Q_{\mathcal{G}}$ such that $h_Q \circ a_Q = z_Q$. Thus

$$h_Q \circ a_Q \circ q_{\mathcal{Z}} = z_Q \circ q_{\mathcal{Z}} = q_{\mathcal{H}} \circ z_V = q_{\mathcal{H}} \circ h_V \circ a_V = h_Q \circ q_{\mathcal{G}} \circ a_V$$

Since h_Q is mono we conclude that $a_Q \circ q_{\mathcal{Z}} = q_{\mathcal{G}} \circ a_V$ and we conclude. $\square$

Lemma Appendix A.15. *The class* Pb *contains all isomorphisms, it is closed under composition, decomposition and it is stable under pullbacks and pushouts.*

Proof. Closure under composition and decomposition follows from Theorem Appendix A.1, while stability follows from the first point of Theorem 2.14 because pushouts along injections are also pullbacks by Theorem 2.6. We are left with stability under pullbacks. Let $h\colon \mathcal{G}_1 \to \mathcal{G}_3$ be an arrow in Pb and $k\colon \mathcal{G}_2 \to \mathcal{G}_3$ be another morphism of **EqHyp**, with $\mathcal{G}_i = (E_i, V_i, Q_i, s_i, t_i, q_i)$.

Consider the diagrams below, in which the square on the left is a pullback in **EqHyp**, with $\mathcal{P} = (E_{\mathcal{P}, V_{\mathcal{P}}}, Q_{\mathcal{P}}, s_{\mathcal{P}}, t_{\mathcal{P}}, q_{\mathcal{P}})$, and in which the right half of the right rectangle is a pullback too, so that the existence of $m_{\mathcal{P}}$ is guaranteed by Theorem 4.8.

Again by Theorem 4.8 the two halves of the central rectangle are pullbacks and so by Theorem Appendix A.1 the whole rectangle is a pullback too. This, in turn entails that the whole right rectangle is a pullback and so, also its left half is a pullback (again by Theorem Appendix A.1)

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "p"] \arrow[d, "r"'] & \mathcal{G}_1 \arrow[d, "h"] \\
\mathcal{G}_2 \arrow[r, "k"'] & \mathcal{G}_3
\end{tikzcd}
\qquad
\begin{tikzcd}
V_{\mathcal{P}} \arrow[r, "p_V"'] \arrow[d, tail, "r_V"'] \arrow[rr, bend left, "p_Q \circ q_{\mathcal{P}}"] & V_1 \arrow[r, two heads, "q_1"'] \arrow[d, tail, "h_V"'] & Q_1 \arrow[d, tail, "h_Q"] \\
V_2 \arrow[r, "k_V"] \arrow[rr, bend right, "k_Q \circ q_2"'] & V_3 \arrow[r, two heads, "q_3"] & Q_3
\end{tikzcd}
\qquad
\begin{tikzcd}
V_{\mathcal{P}} \arrow[r, two heads, "q_{\mathcal{P}}"'] \arrow[d, tail, "r_V"'] & Q_{\mathcal{P}} \arrow[r, tail, "m_{\mathcal{P}}"'] \arrow[dr, tail, "r_Q"'] \arrow[rr, bend left, "p_Q"] & T \arrow[r, "t_1"'] \arrow[d, tail, "t_2"'] & Q_1 \arrow[d, tail, "h_Q"] \\
V_2 \arrow[rr, two heads, "q_2"] & & Q_2 \arrow[r, "k_Q"] & Q_3
\end{tikzcd}$$

Let now $z_1\colon Z \to Q_{\mathcal{P}}$ and $z_2\colon Z \to V_2$ be two arrows such that $q_2 \circ z_2 = r_Q \circ z_1$. Then there exists a unique arrow $z\colon Z \to V_{\mathcal{P}}$ such that

$$m_{\mathcal{P}} \circ q_{\mathcal{P}} \circ z = m_{\mathcal{P}} \circ z_1 \qquad r_V \circ z = z_2$$

Since $m_{\mathcal{P}}$ is mono we conclude that $q_{\mathcal{P}} \circ z = z_1$ as wanted.

Uniqueness follows at once since r_V, being the pullback of h_V is a mono. $\square$

Lemma Appendix A.16. *In* **EqHyp**, Pb*-pushouts are stable.*

Proof. Let $\mathcal{G}_i = (A_i, B_i, Q_i, s_i, t_i, q_i)$, $\mathcal{G}'_i = (A'_i, B'_i, Q'_i, s'_i, t'_i, q'_i)$, for $i \in \{1, 2, 3, 4\}$, be hypergraphs with equivalence, and suppose that in the first diagram below, all the vertical faces are pullbacks, the bottom face is a pushout h is a regular mono and $k_Q \colon Q_1 \rightarrowtail Q_3$ is mono. By Theorem 4.8 the same is true for the other two cubes, by Theorem 4.9, h_E and h_V are monos and so the top faces of these cubes are pushouts

$$\begin{tikzcd}
& \mathcal{G}'_1 \arrow[rr, "h'"] \arrow[dl, "k'"'] \arrow[dd, "a"'] & & \mathcal{G}'_2 \arrow[dl, "z'"'] \arrow[dd, "b"] \\
\mathcal{G}'_3 \arrow[rr, "p'"] \arrow[dd, "c"'] & & \mathcal{G}'_4 \arrow[dd, "d"] & \\
& \mathcal{G}_1 \arrow[rr, "h"] \arrow[dl, "k"] & & \mathcal{G}_2 \arrow[dl, "z"] \\
\mathcal{G}_3 \arrow[rr, "p"'] & & \mathcal{G}_4 &
\end{tikzcd}
\quad
\begin{tikzcd}
& A'_1 \arrow[rr, tail, "h'_E"] \arrow[dl, "k'_E"'] \arrow[dd, "a_E"'] & & A'_2 \arrow[dl, "z'_E"'] \arrow[dd, "b_E"'] \\
A'_3 \arrow[rr, tail, "p'_E"] \arrow[dd, "c_E"'] & & A'_4 \arrow[dd, "d_E"'] & \\
& A_1 \arrow[rr, tail, "h_E"] \arrow[dl, "k_E"] & & A_2 \arrow[dl, "z_E"] \\
A_3 \arrow[rr, tail, "p_E"'] & & A_4 &
\end{tikzcd}
\quad
\begin{tikzcd}
& B'_1 \arrow[rr, tail, "h'_V"] \arrow[dl, "k'_V"'] \arrow[dd, "a_V"'] & & B'_2 \arrow[dl, "z'_V"'] \arrow[dd, "b_V"] \\
B'_3 \arrow[rr, tail, "p'_V"] \arrow[dd, "c_V"'] & & B'_4 \arrow[dd, "d_V"'] & \\
& B_1 \arrow[rr, tail, "h_V"] \arrow[dl, "k_V"] & & B_2 \arrow[dl, "z_V"] \\
B_3 \arrow[rr, tail, "p_V"'] & & B_4 &
\end{tikzcd}$$

$$\begin{tikzcd}
& T \arrow[rr, tail, "x_1"] \arrow[dl, dotted, tail, "w"'] \arrow[dd, "x_2"'] & & U \arrow[dl, "u_1"'] \arrow[dd, "u_2"] \\
Y \arrow[rr, tail, "y_1"] \arrow[dd, "y_2"'] & & Q'_4 \arrow[dd, "d_Q"'] & \\
& Q_1 \arrow[rr, tail, "h_Q"] \arrow[dl, tail, "k_Q"] & & Q_2 \arrow[dl, "z_Q"] \\
Q_3 \arrow[rr, tail, "p_C"'] & & Q_4 &
\end{tikzcd}$$

Now, if $d = (d_E, d_V, d_Q)$, we can pull back the third component to get the solid part of the cube aside. Notice moreover that the commutativity of the solid diagram yields the existence of the dotted $w \colon T \to Y$. By the usual composition and decomposition properties of pullbacks (cfr. Theorem Appendix A.1) the left face is a pullback. By the first point of Theorem 4.8 the bottom face is a pushout and h_Q is a mono by Theorem 4.9, so the top face is a pushout too.

By the second point of Theorem 4.8 we know that there are monos $m_2 \colon Q'_2 \rightarrowtail U$ and $m_3 \colon Q'_3 \rightarrowtail Y$ fitting in the diagrams below

$$\begin{tikzcd}
B'_3 \arrow[r, tail, "p'_V"] \arrow[d, two heads, "q'_3"] \arrow[dd, bend right, "c_V"'] & B'_4 \arrow[dr, bend left, "q'_4"] & \\
Q'_3 \arrow[r, dotted, tail, "m_3"] & Y \arrow[r, tail, "y_1"] \arrow[d, "y_2"'] & Q'_4 \arrow[d, "d_C"'] \\
B_3 \arrow[r, two heads, "q_3"'] & Q_3 \arrow[r, tail, "p_Q"'] & Q_4
\end{tikzcd}
\qquad
\begin{tikzcd}
B'_2 \arrow[r, "z_V"] \arrow[d, two heads, "q'_2"] \arrow[dd, bend right, "b_V"'] & B'_4 \arrow[dr, bend left, "q'_4"] & \\
Q'_2 \arrow[r, dotted, tail, "m_2"] & U \arrow[r, "u_1"] \arrow[d, "u_2"'] & Q'_4 \arrow[d, "d_C"'] \\
B_2 \arrow[r, two heads, "q_3"'] & Q_2 \arrow[r, "z_Q"'] & Q_4
\end{tikzcd}$$

For Q'_1, we can make a similar argument. Taking (S, s_1, s_2) as the pullback of m_2 along x_1, we can use again the composition properties of pullbacks and the second point of Theorem 4.8 to guarantee the existence of a monomorphism $m_1 \colon Q'_1 \to S$ that makes the diagram aside commutative.

$$\begin{tikzcd}
B'_1 \arrow[r, two heads, "q'_1"'] \arrow[d, tail, "h'_V"'] \arrow[rr, bend left, "a_V"] & Q'_1 \arrow[d, tail, dotted, "m_1"] & B_1 \arrow[dr, "q_1"] & \\
B'_2 \arrow[dr, "q'_2"'] & S \arrow[r, tail, "s_2"] \arrow[d, "s_1"'] & T \arrow[r, "x_2"] \arrow[d, tail, "x_1"'] & Q_1 \arrow[d, tail, "h_Q"] \\
& Q'_2 \arrow[r, tail, "m_2"'] & U \arrow[r, "u_2"'] & Q_2
\end{tikzcd}$$

We have to show that the top face of the cube with which we have begun is a pushout. Let $\mathcal{H}$ be (E, V, Q, s, t, q) and suppose that there exist $o \colon \mathcal{G}'_2 \to \mathcal{H}$ and $w \colon \mathcal{G}'_3 \to \mathcal{H}$ such that $o \circ h' = w \circ k'$. Since T preserves colimits by Theorem 4.4, we know that there exists a morphism $(v_E, v_V) \colon T(\mathcal{G}'_4) \to T(\mathcal{H})$ such that

$$v_E \circ p'_E = w_E \quad v_V \circ p'_V = w_V \quad v_E \circ z'_E = o_e \quad v_V \circ z'_V = o_V$$

We want to extend such a morphism to one of **EqHyp** between $\mathcal{G}'_4 \to \mathcal{H}$. If we are able to do so we can conclude because uniqueness is guaranteed by Theorem 4.3.

Now, consider the cube aside, h is in Pb so by Theorem Appendix A.1 we know that its back face is a pullback. We can then apply the second point of Theorem 2.14 to deduce that the square on the right is a pushout.

$$\begin{tikzcd}
& B'_1 \arrow[rr, tail, "h'_V"] \arrow[dl, "k'_V"'] \arrow[dd, two heads, "s_2 \circ m_1 \circ q'_1"' near end] & & B'_2 \arrow[dl, "z'_V"'] \arrow[dd, "m_2 \circ q'_2"] \\
B'_3 \arrow[rr, tail, "p'_V" near start] \arrow[dd, "m_3 \circ q'_3"'] & & B'_4 \arrow[dd, two heads, "q'_4" near start] & \\
& T \arrow[rr, tail, "x_1" near start] \arrow[dl, "w"'] & & U \arrow[dl, "u_1"] \\
Y \arrow[rr, tail, "y_1"'] & & Q'_4 &
\end{tikzcd}$$

By construction we have that

$$\begin{tikzcd}
K_{s_2 \circ m_1 \circ q'_1} \arrow[r, tail, "k_{h'_V}"] \arrow[d, "k_{k'_V}"'] & K_{m_2 \circ q'_2} \arrow[d, "k_{z'_V}"] \\
K_{m_3 \circ q'_3} \arrow[r, tail, "k_{p'_V}"] & K_{q'_4}
\end{tikzcd}$$

$$m_3 \circ q'_3 \circ \pi^1_{m_3 \circ q'_3} = m_3 \circ q'_3 \circ \pi^2_{m_3 \circ q'_3}$$
$$m_2 \circ q'_2 \circ \pi^1_{m_2 \circ q'_2} = m_2 \circ q'_2 \circ \pi^2_{m_2 \circ q'_2}$$

but since m_3 and m_2 are monos this implies

$$q'_3 \circ \pi^1_{m_3 \circ q'_3} = q'_3 \circ \pi^2_{m_3 \circ q'_3} \qquad q'_2 \circ \pi^1_{m_2 \circ q'_2} = q'_2 \circ \pi^2_{m_2 \circ q'_2}$$

By computing, we obtain

$$\begin{aligned}
&q \circ v_V \circ \pi^1_{q'_4} \circ k_{p'_V} = q \circ v_V \circ p'_V \circ \pi^1_{m_3 \circ q'_3} = q \circ w_V \circ \pi^1_{m_3 \circ q'_3} = w_Q \circ q'_3 \circ \pi^1_{m_3 \circ q'_3} \\
=&w_Q \circ q'_3 \circ \pi^2_{m_3 \circ q_3} = q \circ w_V \circ \pi^2_{m_3 \circ q'_3} = q \circ v_V \circ p'_V \circ \pi^2_{m_3 \circ q'_3} = q \circ v_V \circ \pi^2_{q'_4} \circ k_{p'_V} \\
&q \circ v_V \circ \pi^1_{q'_4} \circ k_{z'_V} = q \circ v_V \circ z'_V \circ \pi^1_{m_2 \circ q'_2} = q \circ o_V \circ \pi^1_{m_2 \circ q'_2} = o_Q \circ q'_2 \circ \pi^1_{m_2 \circ q'_2} \\
=&o_Q \circ q'_2 \circ \pi^2_{m_2 \circ q_2} = q \circ o_V \circ \pi^2_{m_2 \circ q'_2} = q \circ v_V \circ t'_V \circ \pi^2_{m_2 \circ q'_2} = q \circ v_V \circ \pi^2_{q'_4} \circ k_{z'_V}
\end{aligned}$$

As already noticed the square above is a pushout, hence we can conclude that

$$q \circ v_V \circ \pi^1_{q'_4} = q \circ v_V \circ \pi^2_{q'_4}$$

Now, q'_4 is a regular epi and so it is the coequalizer of its kernel pair, hence there exists $v_Q \colon Q'_4 \to Q$ such that $v_Q \circ q'_4 = q \circ v_V$ and we can conclude. $\square$

Lemma Appendix A.17. *In* **EqHyp***, pushouts along arrows in* Pb *are* $\mathsf{Reg}(\mathbf{EqHyp})$*-Van Kampen.*

Proof. Consider a cube as the one below on the left, with regular monos as vertical arrows, pullbacks as back faces, pushouts as bottom and top faces and such that h is in Pb. Given Theorem 4.12, if we show that the front faces are pullbacks too we can conclude.

To fix notation, let $\mathcal{G}_i = (A_i, B_i, Q_i, s_i, t_i, q_i)$, $\mathcal{G}' = (A'_i, B'_i, Q'_i, s'_i, t'_i, q'_i)$, for $i = 1, 2, 3, 4$. By Theorem 4.8 and Theorem 4.9 we know that the central and right cube below have pushouts as bottom faces and pullbacks as back faces, thus their front faces are pullbacks

$$
\begin{tikzcd}
& \mathcal{G}'_1 \arrow[rr, "h'"] \arrow[dl, "k'"'] \arrow[dd, "a" near end] & & \mathcal{G}'_2 \arrow[dl, "z'"] \arrow[dd, "b"] \\
\mathcal{G}'_3 \arrow[rr, "p'" near start, crossing over] \arrow[dd, "c"'] & & \mathcal{G}'_4 & \\
& \mathcal{G}_1 \arrow[rr, "h" near end] \arrow[dl, "k"] & & \mathcal{G}_2 \arrow[dl, "z"] \\
\mathcal{G}_3 \arrow[rr, "p"'] & & \mathcal{G}_4 \arrow[from=uu, "d" near start, crossing over] &
\end{tikzcd}
\quad
\begin{tikzcd}
& A'_1 \arrow[rr, tail, "h'_E"] \arrow[dl, "k'_E"'] \arrow[dd, tail, "a_E" near end] & & A'_2 \arrow[dl, "z'_E"] \arrow[dd, tail, "b_E"] \\
A'_3 \arrow[rr, tail, "p'_E" near start, crossing over] \arrow[dd, tail, "c_E"'] & & A'_4 & \\
& A_1 \arrow[rr, tail, "h_E" near end] \arrow[dl, "k_E"] & & A_2 \arrow[dl, "z_E"] \\
A_3 \arrow[rr, tail, "p_E"'] & & A_4 \arrow[from=uu, tail, "d_E" near start, crossing over] &
\end{tikzcd}
\quad
\begin{tikzcd}
& B'_1 \arrow[rr, tail, "h'_V"] \arrow[dl, "k'_V"'] \arrow[dd, tail, "a_V" near end] & & B'_2 \arrow[dl, "z'_V"] \arrow[dd, tail, "b_V"] \\
B'_3 \arrow[rr, tail, "p'_V" near start, crossing over] \arrow[dd, tail, "c_V"'] & & B'_4 & \\
& B_1 \arrow[rr, tail, "h_V" near end] \arrow[dl, "k_V"] & & B_2 \arrow[dl, "z_V"] \\
B_3 \arrow[rr, tail, "p_V"'] & & B_4 \arrow[from=uu, tail, "d_V" near start, crossing over] &
\end{tikzcd}
$$

Let us now consider the diagrams below, in which the inner squares are pullbacks. Since the outer diagrams commute, by definition of morphism of **EqHyp**, then we have the existence of $m_2 \colon Q'_2 \to U$, $m_3 \colon Q'_3 \to Y$, $a_3 \colon B'_3 \to Y$ and $a_2 \colon B'_2 \to Y$.

Now, notice that m_3 and m_2 are monos because c_Q and b_2 are injections. By the proof of Theorem 4.8, to conclude it is enough to show that

$$m_3 \circ q'_3 = a_3 \qquad m_2 \circ q'_2 = a_2$$

Indeed, if the previous equations hold, then Q'_3 and Q'_2 are images for a_3 and a_2 and the claim follows from Theorem 4.8.

$$
\begin{tikzcd}
Q'_3 \arrow[r, dotted, "m_3"'] \arrow[rr, tail, bend left, "p'_Q"] \arrow[dr, bend right, "c_Q"'] & Y \arrow[r, tail, "y_1"'] \arrow[d, "y_2"'] & Q'_4 \arrow[d, "d_Q"] \\
& Q_3 \arrow[r, tail, "p_3"'] & Q_4
\end{tikzcd}
\qquad
\begin{tikzcd}
Q'_2 \arrow[r, dotted, "m_2"'] \arrow[rr, bend left, "z'_Q"] \arrow[dr, bend right, "b_Q"'] & U \arrow[r, "u_1"'] \arrow[d, "u_2"'] & Q'_4 \arrow[d, "d_Q"] \\
& Q_2 \arrow[r, "z_Q"'] & Q_4
\end{tikzcd}
$$

$$
\begin{tikzcd}
B'_3 \arrow[r, tail, "p'_V"] \arrow[d, tail, "q'_3"'] \arrow[drr, dotted, "a_3"] & B'_4 \arrow[drr, "q'_4"] & & \\
Q'_3 \arrow[drr, tail, "c_Q"'] & & Y \arrow[r, tail, "y_1"'] \arrow[d, "y_2"'] & Q'_4 \arrow[d, "d_Q"] \\
& & Q_3 \arrow[r, tail, "p_Q"'] & Q_4
\end{tikzcd}
\qquad
\begin{tikzcd}
B'_2 \arrow[r, "z'_2"] \arrow[d, tail, "q'_2"'] \arrow[drr, dotted, "a_2"] & B'_4 \arrow[drr, "q'_4"] & & \\
Q'_2 \arrow[drr, tail, "c_Q"'] & & U \arrow[r, "u_1"'] \arrow[d, "u_2"'] & Q'_4 \arrow[d, "d_Q"] \\
& & Q_2 \arrow[r, "z_Q"'] & Q_4
\end{tikzcd}
$$

By computing we have

$$
\begin{aligned}
y_1 \circ a_3 &= q'_4 \circ p'_2 = p'_3 \circ q'_3 = y_1 \circ m_3 \circ q'_3 & \qquad y_2 \circ a_3 &= d_3 \circ q'_3 = y_2 \circ m_3 \circ q'_3 \\
u_1 \circ a_2 &= q'_4 \circ t'_2 = t'_3 \circ q'_3 = u_1 \circ m_2 \circ q'_2 & \qquad u_2 \circ a_2 &= d_2 \circ q'_2 = u_2 \circ m_2 \circ q'_2
\end{aligned}
$$

And we have done. □

Proposition Appendix A.18. $\mathbf{EqTG}_\Sigma$ *has equalizers, binary products and pullbacks and they are created by* J_Σ*.*

Proof. Let $F \colon \mathbf{D} \to \mathbf{EqTG}_\Sigma$ be the diagram of an equalizer, a binary product or of a pullback. By Theorem 4.18 we can consider a limiting cone $((\mathcal{L}, l), \{\pi_d\}_{d \in \mathbf{D}})$. By Theorem 4.22 we know that T_Σ preserves limits, thus by Theorem 4.21 $(l, \{T_\Sigma(\pi_d)\}_{d \in \mathbf{D}})$ is limiting for $I_\Sigma \circ S_\Sigma$. Then by Theorem 3.23 l is a term graph. We conclude, again by Theorem 4.21 that $(\mathcal{L}, l)$ is in $\mathbf{EqTG}_\Sigma$ and the claim follows. $\square$

Proposition Appendix A.19. $\mathbf{EqTG}_\Sigma$ *has all* $\mathcal{T}$*-pushouts, which are created by* J_Σ*.*

Proof. Suppose that the square on the left below is a pushout in $\mathbf{EqHyp}_\Sigma$, with h in $\mathcal{T}$. Then, by Theorem 4.22 the square on the right is a pushout and by the definition of $\mathcal{T}$ and Theorem 4.21, $T_\Sigma(h)$ is a regular mono in $\mathbf{TG}_\Sigma$

$$\begin{tikzcd}
J_\Sigma(\mathcal{G}_0, l_0) \arrow[r, "k"] \arrow[d, "h"'] & J_\Sigma(\mathcal{G}_1, l_1) \arrow[d, "p"] \\
J_\Sigma(\mathcal{G}_1, l_1) \arrow[r, "q"'] & J_\Sigma(\mathcal{H}, l)
\end{tikzcd}
\qquad
\begin{tikzcd}
T_\Sigma(J_\Sigma(\mathcal{G}_0, l_0)) \arrow[r, "T_\Sigma(k)"] \arrow[d, "T_\Sigma(h)"'] & T_\Sigma(J_\Sigma(\mathcal{G}_1, l_1)) \arrow[d, "T_\Sigma(p)"] \\
T_\Sigma(J_\Sigma(\mathcal{G}_1, l_1)) \arrow[r, "T_\Sigma(q)"'] & T_\Sigma(J_\Sigma(\mathcal{H}, l))
\end{tikzcd}$$

We conclude using Lemma 3.27 and theorem 4.21. $\square$

Appendix A.3. Proofs for Section 5

Lemma Appendix A.20. e-$\mathbf{EqHyp}$ *has all limits and* I *creates them.*

Proof. Let $F : \mathbf{D} \to \mathsf{e}\text{-}\mathbf{EqHyp}$ be a diagram, with $F(d) = (A_d, B_d, Q_d, s_d, t_d, q_d)$. Let (U_d, u_1^d, u_2^d) be a kernel pair for $q_d \circ s_d$. Now let (A, B, Q, s, t, q), together with projections $(\pi_E^d, \pi_V^d, \pi_Q^d)_{d \in \mathbf{D}}$, be the limit of $I \circ F$. Suppose that (U, u_1, u_2) is the kernel pair of $q^\star \circ s$ and let $(L, (l_i)_{i \in \mathbf{I}})$ be the limit of $K \circ I \circ F$. If we show that it lies in e-$\mathbf{EqHyp}$ we are done.

By Theorem 4.8 there exists a mono $m \colon Q \rightarrowtail L$ such that $\pi_Q^d = l_d \circ m$. Notice that

$$\begin{aligned}
q_d^\star \circ s_d \circ \pi_E^d \circ u_1 &= q_d^\star \circ (\pi_V^d)^\star \circ s \circ u_1 = (\pi_Q^d)^\star \circ q^\star \circ s \circ u_1 \\
=& (\pi_Q^d)^\star \circ q^\star \circ s \circ u_2 = q_d^\star \circ (\pi_V^d)^\star \circ s \circ u_2 = q_d^\star \circ s_d \circ \pi_E^d \circ u_2
\end{aligned}$$

Thus for each d in $\mathbf{D}$, there exists an arrow $a_d : U \to U_i$ making the diagram on the right commutative. Then we have

$$\begin{tikzcd}
U \arrow[r, "u_1"] \arrow[d, "u_2"'] \arrow[dr, dotted, "a_d"] & A \arrow[dr, "\pi_E^d"] & \\
A \arrow[dr, "\pi_E^d"'] & U_d \arrow[r, "u_1^d"] \arrow[d, "u_2^d"'] & A_d \arrow[d, "q_d^\star \circ s_d"] \\
& A_i \arrow[r, "q_d^\star \circ s_d"'] & Q_d^\star
\end{tikzcd}$$

$$\begin{aligned}
& l_d^\star \circ m^\star \circ q^\star \circ t \circ u_1 = q_d^\star \circ (\pi_V^d)^\star \circ t \circ u_1 \\
=& q_d^\star \circ t_d \circ \pi_E^d \circ u_1 = q_d^\star \circ t_d \circ u_1^d \circ a_d = q_d^\star \circ t_d \circ u_2^d \circ a_d \\
=& q_d^\star \circ t_d \circ \pi_E^d \circ u_2 = q_d^\star \circ (\pi_V^d)^\star \circ t \circ u_2 = l_d^\star \circ m^\star \circ q^\star \circ t \circ u_2
\end{aligned}$$

By universal property of limits, we have that $m^\star \circ q^\star \circ t \circ u_1 = m^\star \circ q^\star \circ t \circ u_2$. Since m is mono we deduce that $q^\star \circ t \circ u_1 = q^\star \circ t \circ u_2$, hence the claim. $\square$

Remark Appendix A.21. Let $\delta_1 \colon 1 \to \mathbb{N}$ be the arrow which picks $1 \in \mathbb{N}$. Consider the inclusion $v_1 \colon X \to X^\star$ defined by the first point of Theorem 3.1. By extensivity we know that the square on the right is a pullback.

$$\begin{tikzcd}
X \arrow[r, "!_X"'] \arrow[d, "!_X"'] & 1 \arrow[d, "\delta_1"] \\
1 \arrow[r, "\mathsf{lg}_X"] & \mathbb{N}
\end{tikzcd}$$

$$\begin{tikzcd}[row sep=small, column sep=small]
& X \arrow[rr, "!_X"] \arrow[dl, "f"'] \arrow[dd, "!_X" near end] & & 1 \arrow[dl, "\mathsf{id}_1"] \arrow[dd, "\delta_1"] \\
Y \arrow[rr, "!_Y" near start, crossing over] \arrow[dd, "!_Y"'] & & 1 & \\
& X^\star \arrow[rr, tail, "\mathsf{lg}_X" near end] \arrow[dl, tail, "f^\star"'] & & \mathbb{N} \arrow[dl, "\mathsf{id}_N"] \\
Y^\star \arrow[rr, tail, "\mathsf{lg}_Y"'] & & \mathbb{N} \arrow[from=uu, crossing over, "\delta_1"' near start] &
\end{tikzcd}$$

Let now $f\colon X \to Y$ be an arrow. Then we can build the cube aside which, by what we have just observed above, has pullbacks as back, front and right faces. Thus, by Theorem Appendix A.1 its left face is a pullback too.

Lemma Appendix A.22. *Suppose that the square on the right is a pushout in* **EqHyp** *and that m is a mono in* Pb. *If $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are e-hypergraphs, then $\mathcal{P}$ is is an e-hypergraph.*

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[r, "h"] \arrow[d, "m"'] & \mathcal{G}_2 \arrow[d, "n"] \\
\mathcal{G}_3 \arrow[r, "z"'] & \mathcal{P}
\end{tikzcd}$$

Proof. Let $\mathcal{P}$ be (A, B, C, s, t, q) and consider the kernel pair (K_i, π_i^1, π_i^2) the kernel pair of $q_i^\star \circ s_i$, for $i \in \{1, 2, 3\}$. Let also (U, u_1, u_2) be the kernel pair of $q^\star \circ s$.

$$\begin{tikzcd}[row sep=small, column sep=small]
& A_1 \arrow[rr, "h_E"] \arrow[dl, tail, "m_E"'] \arrow[dd, "q_1^\star \circ s_1" near end] & & A_2 \arrow[dl, tail, dotted, "n_E"'] \arrow[dd, "q_2^\star \circ s_2"] \\
A_3 \arrow[rr, "z_E" near start, crossing over] \arrow[dd, "q_3^\star \circ s_3"'] & & A & \\
& Q_1^\star \arrow[rr, "h_Q^\star" near end] \arrow[dl, tail, "m_Q^\star"'] & & Q_2^\star \arrow[dl, tail, "n_Q^\star"] \\
Q_3^\star \arrow[rr, "z_Q^\star"'] & & Q^\star \arrow[from=uu, crossing over, "q^\star \circ s"' near start] &
\end{tikzcd}$$

Consider now the cube on the right. By hypothesis and Theorem Appendix A.21 its left face is a pullback, moreover its bottom face, while not a pushout it is still a pullback by the first point of Theorem 2.6 and the third one of Theorem 3.1.

$$\begin{tikzcd}
K_1 \arrow[r, "k_{h_E}"] \arrow[d, tail, "k_{m_E}"'] & K_2 \arrow[d, tail, "k_{n_E}"] \\
K_3 \arrow[r, "k_{z_E}"] & U
\end{tikzcd}$$

By hypothesis and the first point of Theorem 4.8 the top face is a pushout. Thus we can apply Theorem 2.14 to deduce that the square on the left is a pushout.

By computing we obtain

$$q^\star \circ t \circ u_1 \circ f_n = q^\star \circ t \circ n_E \circ \pi_2^1 = n_C^\star \circ q_2^\star \circ s_2 \circ \pi_2^1 = n_C^\star \circ q_2^\star \circ s_2 \circ \pi_2^2 = q^\star \circ t \circ u_2 \circ f_n$$
$$q^\star \circ t \circ u_1 \circ f_k = q^\star \circ t \circ k_E \circ \pi_3^1 = k_C^\star \circ q_3^\star \circ s_3 \circ \pi_3^1 = k_C^\star \circ q_3^\star \circ s_3 \circ \pi_3^2 = q^\star \circ t \circ u_2 \circ f_k$$

We can therefore deduce that $q^\star \circ t \circ u_1 = q^\star \circ t \circ u_2$, and the claim follows. $\square$

Proposition Appendix A.23. **EGG** *has equalizers, binary products and pullbacks and they are created by K_Σ.*

Proof. Let $F\colon \mathbf{D} \to \mathbf{EGG}$ be a diagram of one of the shapes mentioned in the statement. Let $((\mathcal{L}, l), \{\pi_d\}_{d \in \mathbf{D}})$ be a limiting cone for $k_\Sigma \circ F$. By Corollary Appendix B.4, Theorem 4.23, and Section Appendix A.3 we know that $\mathcal{L}$ is an e-termgraph and we can conclude. $\square$

Appendix B. Some properties of comma categories

In this section we briefly recall the definition of the comma category [27] associated to two functors and some of its properties.

Definition Appendix B.1. Let $L\colon \mathbf{A} \to \mathbf{X}$ and $R\colon \mathbf{B} \to \mathbf{X}$ be two functors with the same codomain, the *comma category* $L{\downarrow}R$ is the category in which

- objects are triples (A, B, f) with $A \in \mathbf{A}$, $B \in \mathbf{B}$, and $f \colon L(A) \to R(B)$;
- a morphism $(A, B, f) \to (A', B', g)$ is a pair (h, k) with $h \colon A \to A'$ in $\mathbf{A}$ and $k \colon B \to B'$ in $\mathbf{B}$ such that the diagram aside commutes

$$\begin{tikzcd} L(A) \arrow[r, "L(h)"] \arrow[d, "f"'] & L(A') \arrow[d, "g"] \\ R(B) \arrow[r, "R(k)"'] & R(B') \end{tikzcd}$$

We have two forgetful functors $U_L \colon L{\downarrow}R \to \mathbf{A}$ and $U_R \colon L{\downarrow}R \to \mathbf{B}$ given respectively by

$$\begin{tikzcd} (A,B,f) \arrow[r, mapsto] \arrow[d, "{(h,k)}"'] & A \arrow[d, "h"] \\ (A',B',g) \arrow[r, mapsto] & A' \end{tikzcd} \qquad \begin{tikzcd} (A,B,f) \arrow[r, mapsto] \arrow[d, "{(h,k)}"'] & B \arrow[d, "k"] \\ (A',B',g) \arrow[r, mapsto] & B' \end{tikzcd}$$

Given $L \colon \mathbf{A} \to \mathbf{X}$ and $R \colon \mathbf{B} \to \mathbf{X}$, we can also consider their duals $L^{op} \colon \mathbf{A}^{op} \to \mathbf{X}^{op}$ and $R^{op} \colon \mathbf{B}^{op} \to \mathbf{X}^{op}$. An arrow $f \colon L(A) \to R(B)$ in $\mathbf{X}$ is the same thing as an arrow $f \colon R^{op}(B) \to L^{op}(A)$ in $\mathbf{X}^{op}$. Thus $L{\downarrow}R$ and $R^{op}{\downarrow}L^{op}$ have the same objects. Moreover, the left square below commutes in $\mathbf{X}$ if and only if the right one commutes in $\mathbf{X}^{op}$.

$$\begin{tikzcd} L(A) \arrow[r, "L(h)"] \arrow[d, "f"'] & L(A') \arrow[d, "g"] \\ R(B) \arrow[r, "R(k)"'] & R(B') \end{tikzcd} \qquad \begin{tikzcd} R(B') \arrow[r, "R(k)"] \arrow[d, "g"'] & R(B) \arrow[d, "f"] \\ L(A') \arrow[r, "L(h)"'] & L(A) \end{tikzcd}$$

Summing up we have just proved the following fact.

Proposition Appendix B.2. *$(L{\downarrow}R)^{op}$ is equal to $R^{op}{\downarrow}L^{op}$, and $U_L^{op} = U_{L^{op}}$ and $U_R^{op} = U_{R^{op}}$.*

Lemma Appendix B.3. *Let $L \colon \mathbf{A} \to \mathbf{X}$ and $R \colon \mathbf{B} \to \mathbf{X}$ be functors and $F \colon \mathbf{D} \to L{\downarrow}R$ be a diagram such that L preserves colimits along $U_L \circ F$. Then the family $\{U_L, U_R\}$ jointly creates colimits of F (see [10, Sec. 5.1.3] or [11, Sec. 2.3]).*

Proof. Suppose that $U_L \circ F$ and $U_R \circ F$ have respectively colimiting cocones $(A, \{a_D\}_{D \in \mathbf{D}})$ and $(B, \{b_D\}_{D \in \mathbf{D}})$. By hypothesis $(L(A), \{L(a_D)\}_{D \in \mathbf{D}})$ is colimiting for $L \circ U_L \circ F$. Now, let $F(D)$ be (A_D, B_D, f_D), then we have arrows $R(b_D) \circ f_D \colon L(A_D) \to R(B)$ that forms a cocone on $L \circ U_L \circ F$: if $d \colon D \to D'$ is an arrow in $\mathbf{D}$ then $F(d)$ is an arrow in $L{\downarrow}R$ and so

$$\begin{aligned} & R(b_{D'}) \circ f_{D'} \circ L(U_L(F(d))) = R(b_{D'}) \circ R(U_R(F(d))) \circ f_D \\ = & R(b_{D'} \circ U_R(F(d))) \circ f_D = R(b_D) \circ f_D \end{aligned}$$

Thus there exists $f \colon L(A) \to R(B)$ fitting in the diagram on the right. Notice that f is the unique arrow in $\mathbf{X}$ wich makes (a_D, b_D) an arrow $(A_D, B_D, f_D) \to (A, B, f)$ of $L{\downarrow}R$. If we show that $\big((A, B, f), \{(a_D, b_D)\}_{D \in \mathbf{D}}\big)$ is colimiting for F we are done.

$$\begin{tikzcd} L(A_D) \arrow[r, "L(a_D)"] \arrow[d, "f_D"'] & L(A) \arrow[d, "f"] \\ R(B_D) \arrow[r, "R(b_D)"'] & R(B) \end{tikzcd}$$

First of all, let us show that it is a cocone. Given $d\colon D \to D'$ in $\mathbf{D}$ we have

$$\begin{aligned}(a_{D'}, b_{D'}) \circ F(d) &= (a_{D'}, b_{D'}) \circ (U_L(F(d)), U_R(F(d))) \\ &= (a_{D'} \circ U_L(F(d)), b_{D'} \circ U_R(F(d))) = (a_D, b_D)\end{aligned}$$

For the colimiting property, let $\big((X, Y, g), \{(x_D, y_D)\}_{D\in\mathbf{D}}\big)$ be another cocone on F. In particular, $\big(X, \{x_D\}_{D\in\mathbf{D}}\big)$ and $\big(Y, \{y_D\}_{D\in\mathbf{D}}\big)$ are cocones on $U_L \circ F$ and $U_R \circ F$ respectively, so we have uniquely determined arrows $x\colon A \to X$ and $y\colon B \to Y$ such that

$$x \circ a_D = x_D \qquad y \circ b_D = y_D$$

Let us show that (x, y) is an arrow of $L{\downarrow}R$. Given $D \in \mathbf{D}$ we have

$$\begin{aligned}R(y) \circ f \circ L(a_D) &= R(y) \circ R(b_D) \circ f_D = R(y \circ b_D) \circ f_D \\ =R(y_D) \circ f_D &= g \circ L(x_D) = g \circ L(x \circ a_D) = g \circ L(x) \circ L(a_D)\end{aligned}$$

from which it follows that $g \circ L(x) = R(y) \circ f$ as wanted. □

Proposition Appendix B.2 and Lemma Appendix B.3 now yields the following.

Corollary Appendix B.4. *The family $\{U_L, U_R\}$ jointly creates limits along every diagram $F\colon \mathbf{D} \to L{\downarrow}R$ such that R preserves the limit of $U_R \circ F$.*

We can use Corollary Appendix B.4 to characterize monos in comma categories.

Corollary Appendix B.5. *If R preserves pullbacks then an arrow (h, k) in $L{\downarrow}R$ is mono if and only if both h and k are monos.*

Proof. ($\Rightarrow$) If $(h, k)\colon (A, B, f) \to (A', B', g)$ is a mono then the first rectangle below is a pullback in $L{\downarrow}R$. By Corollary Appendix B.4 then also the other two squares are pullbacks, respectively in $\mathbf{A}$ and $\mathbf{B}$, proving that both h and k are monos

$$\begin{tikzcd}(A,B,f) \arrow[r, "\mathsf{id}_{(A,B,f)}"] \arrow[d, "\mathsf{id}_{(A,B,f)}"'] & (A,B,f) \arrow[d, "{(h,k)}"] \\ (A,B,f) \arrow[r, "{(h,k)}"'] & (A',B',g)\end{tikzcd} \qquad \begin{tikzcd}A \arrow[r, "\mathsf{id}_A"] \arrow[d, "\mathsf{id}_A"'] & A \arrow[d, "h"] \\ A \arrow[r, "h"'] & A'\end{tikzcd} \qquad \begin{tikzcd}B \arrow[r, "\mathsf{id}_B"] \arrow[d, "\mathsf{id}_B"'] & B \arrow[d, "k"] \\ B \arrow[r, "k"'] & B'\end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\mathsf{id}_A"] \arrow[d, "\mathsf{id}_A"'] & A \arrow[d, "h"] \\ A \arrow[r, "h"'] & A' \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\mathsf{id}_B"] \arrow[d, "\mathsf{id}_B"'] & B \arrow[d, "k"] \\ B \arrow[r, "k"'] & B' \end{tikzcd}$$

($\Leftarrow$) Since h and k are monos then we have the two pullback squares on the left. Thus by Corollary Appendix B.4 the pullback of (h, k) along itself has isomorphisms as projections and so (h, k) is mono. $\square$

We end this section pointing out another useful fact, showing that in some cases we can guarantee the existence of a left adjoint to U_R.

Proposition Appendix B.6. *If* $\mathbf{A}$ *has initial objects and* L *preserves them then the forgetful functor* $U_R \colon L{\downarrow}R \to \mathbf{B}$ *has a left adjoint* Δ.

Proof. For an object $B \in \mathbf{B}$ we can define $\Delta(B)$ as $(0, B, ?_{R(B)})$, for 0 is an initial object in $\mathbf{A}$ and $?_{R(B)}$ is the unique arrow $L(0) \to R(B)$. Let $\mathsf{id}_B \colon B \to U_R(\Delta(B))$ be the identity,
and suppose that a $k \colon B \to U_R(A, B', f)$ in $\mathbf{B}$ is given. By initiality of 0, there is only one arrow $?_A \colon 0 \to A$ in $\mathbf{A}$ and, since L preserves initial objects, the square aside commutes. Thus $(?_A, k)$ is the unique morphism $\Delta(B) \to (A, B', f)$ such that $U_R(?_A, k) = k$.

$$\begin{tikzcd} L(0) \arrow[r, "L(?_A)"'] \arrow[d, "?_{R(B)}"'] & L(A) \arrow[d, "f"] \\ R(B) \arrow[r, "R(k)"] & R(B') \end{tikzcd}$$

$\square$

Dualizing we get immediately the following.

Corollary Appendix B.7. *If* $\mathbf{B}$ *has terminal objects preserved by* R *then* $U_L \colon L{\downarrow}R \to \mathbf{A}$ *has a right adjoint.*

Appendix B.1. Slice categories

This section is devoted to recall some basic facts about the so called *slice categories*.

Definition Appendix B.8. Let X be an object of a category $\mathbf{X}$. We define the following two categories.

- The *slice category over* X is the category $\mathbf{X}/X$ which has as objects arrows $f \colon Y \to X$ and in which an arrow $h \colon f \to g$ is $h \colon Y \to Y'$ in $\mathbf{X}$ such that the triangle on the right commutes.

$$\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "f"'] & & Y' \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

- Dually, the *slice category under* X is the category $X/\mathbf{X}$ in which objects are arrows $f \colon X \to Y$ with domain X and a morphism $h \colon f \to g$ is an arrow of $\mathbf{X}$ fitting in a triangle as the one aside.

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y \arrow[rr, "h"'] & & Y' \end{tikzcd}$$

Remark Appendix B.9. For every $X \in X$ we have forgetful functors

$$\begin{array}{ccc} \mathsf{dom}_X \colon \mathbf{X}/X \to \mathbf{X} & \qquad & \mathsf{cod}_X \colon X/\mathbf{X} \to \mathbf{X} \\ \begin{tikzcd} f \arrow[r, mapsto] \arrow[d, "h"'] & \mathsf{dom}(f) \arrow[d, "h"] \\ g \arrow[r, mapsto] & \mathsf{dom}(g) \end{tikzcd} & & \begin{tikzcd} f \arrow[r, mapsto] \arrow[d, "h"'] & \mathsf{cod}(f) \arrow[d, "h"] \\ g \arrow[r, mapsto] & \mathsf{cod}(g) \end{tikzcd} \end{array}$$

We can realize the slice over and under an object $X \in \mathbf{X}$ as comma categories.

Proposition Appendix B.10. *For every object* X *in a category* $\mathbf{X}$, *if* $\delta_X \colon \mathbf{1} \to \mathbf{X}$ *is the constant functor of value* X *from the category with only one object* $*$, *then* $\mathbf{X}/X$ *and* $X/\mathbf{X}$ *are isomorphic to, respectively,* $\mathsf{id}_X{\downarrow}\delta_X$ *and* $\delta_X{\downarrow}\mathsf{id}_X$.

Proof. Define functors $F_1 \colon \mathsf{id}_X \!\downarrow\! \delta_X \to \mathbf{X}/X$ and $G_1 \colon \mathbf{X}/X \to \mathsf{id}_X \!\downarrow\! \delta_X$ as follows

$$\begin{array}{rcl} (Y,*,f) & \longmapsto & f \\ {\scriptstyle (h,\mathsf{id}_*)}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ (Y',*,g) & \longmapsto & g \end{array} \qquad \begin{array}{rcl} f & \longmapsto & (\mathsf{dom}(f),*,f) \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle (h,\mathsf{id}_*)} \\ g & \longmapsto & (\mathsf{dom}(g),*,g) \end{array}$$

Similarly, we have $F_2 \colon \delta_X \!\downarrow\! \mathsf{id}_X \to X/\mathbf{X}$ and $G_2 \colon X/\mathbf{X} \to \delta_X \!\downarrow\! \mathsf{id}_X$

$$\begin{array}{rcl} (*,Y,f) & \longmapsto & f \\ {\scriptstyle (\mathsf{id}_*,h)}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ (*,Y',g) & \longmapsto & g \end{array} \qquad \begin{array}{rcl} f & \longmapsto & (*,\mathsf{cod}(f),f) \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle (\mathsf{id}_*,h)} \\ g & \longmapsto & (*,\mathsf{cod}(g),g) \end{array}$$

It is now obvious to see that F_1, G_1 and F_2, G_2 are pairs of inverses. □

A straightforward application of Corollary Appendix B.4 and lemma Appendix B.3 now yields the following.

Corollary Appendix B.11. *For every object $\mathbf{X}$, $\mathbf{X}/X$ has all colimits and connected limits that $\mathbf{X}$ has. Moreover such limits and colimits are created by* dom_X.

In particular, if $\mathbf{X}$ has pullbacks, equalizers or pushouts, then for every object X, the slice $\mathbf{X}/X$ has such limits and colimits.