On the adhesivity of EGGS

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Abstract

a very nice abstract

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1 Introduction

A very nice introduction

- general on adhesive categories
- general on the use of eggs

In the original definition, e-graphs are defined as term graphs with an additional notion of equivalence on nodes. Adopting the more modern presentation via string diagrams, they are tree with sharing of subtrees, and with an an equivalence $\equiv$ on nodes that is closed under xxx. In plain words, if a and b are two constants such that $a \equiv b$, then $f(a) \equiv f(b)$ for any unary operator f.

[9, Section 4.2] [21]

Synopsis The paper has the following structure. In Section 2 we briefly recall the theory of $\mathcal{M}$*-adhesive categories* and of *kernel pairs*. In Section 3 we present the graphical structures of our interest, namely (labelled) hypergraphs and term graphs, and we provide a functorial characterisation, which allows for proving their adhesivity properties. This is further expended in Section 4 for describing hypergraphs and term graphs with equivalence and in Section 5 for capturing their variants where the equivalences are closed with respect to operator application, thus subsuming EGGs. All the proofs have been moved to the appendix. For the sake of completeness, and in order to fix the notation, we prove all the results recalled in the background section, besides those that are original to our work.

2 $\mathcal{M}$-adhesive categories

This section briefly recalls $\mathcal{M}$*-adhesive categories* [1, 11, 12, 17, 15]. Given a category $\mathbf{X}$ we do not distinguish notationally between $\mathbf{X}$ and its class of objects, so "$X \in \mathbf{X}$" means that X is an object of $\mathbf{X}$. We let $\mathsf{Mor}(\mathbf{X})$, $\mathsf{Mono}(\mathbf{X})$ and $\mathsf{Reg}(\mathbf{X})$ denote the class of all arrows, monos and regular monos of $\mathbf{X}$, respectively. Given an object X, we denote by $?_X$ the unique arrow from an initial object into X and by $!_X$ that unique arrow from X into a terminal one.

2.1 $\mathcal{M}$-adhesivity

The key property of $\mathcal{M}$-adhesive categories is the *Van Kampen condition* [4, 16, 17], and for defining it we need some notions. Let $\mathbf{X}$ be a category. A subclass $\mathcal{A}$ of $\mathsf{Mor}(\mathbf{X})$ is said to be

- *stable under pushouts (pullbacks)* if for every pushout (pullback) square as the one aside, if $m \in \mathcal{A}$ ($n \in \mathcal{A}$) then $n \in \mathcal{A}$ ($m \in \mathcal{A}$);
- *closed under composition* if $h, k \in \mathcal{A}$ implies $h \circ k \in \mathcal{A}$ whenever h and k are composable.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

▶ **Definition 2.1.** *Let $\mathcal{A} \subseteq \mathsf{Mor}(\mathbf{X})$ be a class of arrows in a category $\mathbf{X}$ and consider the cube below on the right.*

*We say that the bottom square is an $\mathcal{A}$-*Van Kampen square *if*

1. *it is a pushout square;*
2. *whenever the cube above has pullbacks as back and left faces and the vertical arrows belong to $\mathcal{A}$, then its top face is a pushout if and only if the front and right faces are pullbacks.*

$$
\begin{tikzcd}
& A' \arrow[rr, "f'"] \arrow[dl, "m'"'] \arrow[dd, "a"' near start] & & B' \arrow[dl, "n'"'] \arrow[dd, "b"] \\
C' \arrow[rr, "g'" near start, crossing over] \arrow[dd, "c"'] & & D' & \\
& A \arrow[rr, "f" near end] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\
C \arrow[rr, "g"'] & & D \arrow[from=uu, crossing over, "d" near start] &
\end{tikzcd}
$$

*Pushout squares that enjoy only the "if" half of item (2) above are called $\mathcal{A}$-*stable*. A* $\mathsf{Mor}(\mathbf{X})$*-Van Kampen square is called* Van Kampen *and a* $\mathsf{Mor}(\mathbf{X})$*-stable square* stable*.*

We can now define $\mathcal{M}$-adhesive categories.

▶ **Definition 2.2.** *Let $\mathbf{X}$ be a category and $\mathcal{M}$ a subclass of $\mathsf{Mono}(\mathbf{X})$ including all isomorphisms, closed under composition, and stable under pullbacks and pushouts. The category $\mathbf{X}$ is said to be $\mathcal{M}$-*adhesive *if*

1. *it has $\mathcal{M}$-*pullbacks*, i.e. pullbacks along arrows of $\mathcal{M}$;*
2. *it has $\mathcal{M}$-*pushouts*, i.e. pushouts along arrows of $\mathcal{M}$;*
3. *$\mathcal{M}$-pushouts are $\mathcal{M}$-Van Kampen squares.*

A category $\mathbf{X}$ is said to be strictly $\mathcal{M}$-adhesive *if $\mathcal{M}$-pushouts are Van Kampen. We write $m \colon X \rightarrowtail Y$ to denote that an arrow $m \colon X \to Y$ belongs to $\mathcal{M}$.*

▶ Remark 2.3. *Adhesivity* and *quasiadhesivity* [17, 13] coincide with strict $\mathsf{Mono}(\mathbf{X})$-adhesivity and strict $\mathsf{Reg}(\mathbf{X})$-adhesivity, respectively.

$\mathcal{M}$-adhesivity is well-behaved with respect to the construction of slice and functor categories [18], as shown by the following theorems [10, 17].

▶ **Proposition 2.4.** *Let $\mathbf{X}$ be an (strict) $\mathcal{M}$-adhesive category. Then it holds*

1. *if $\mathbf{Y}$ is an (strict) $\mathcal{N}$-adhesive category $L \colon \mathbf{Y} \to \mathbf{A}$ a functor preserving $\mathcal{N}$-pushouts and $R \colon \mathbf{X} \to A$ one preserving pullbacks, then $L{\downarrow}R$ is (strictly) $\mathcal{N}{\downarrow}\mathcal{M}$-adhesive, where*
$$\mathcal{N}{\downarrow}\mathcal{M} := \{(h, k) \in \mathsf{Mor}(L{\downarrow}R) \mid h \in \mathcal{N}, k \in \mathcal{M}\}$$
2. *for every object X the categories $\mathbf{X}/X$ and X/X are, respectively, (strictly) $\mathcal{M}/X$-adhesive and (strictly) $X/\mathcal{M}$-adhesive, where*
$$\mathcal{M}/X := \{m \in \mathsf{Mor}(\mathbf{X}/X) \mid m \in \mathcal{M}\} \quad X/\mathcal{M} := \{m \in \mathsf{Mor}(X/\mathbf{X}) \mid m \in \mathcal{M}\}$$
3. *for every small category $\mathbf{Y}$, the category $\mathbf{X}^{\mathbf{Y}}$ of functors $\mathbf{Y} \to \mathbf{X}$ is (strictly) $\mathcal{M}^{\mathbf{Y}}$-adhesive, where $\mathcal{M}^{\mathbf{Y}} := \{\eta \in \mathsf{Mor}(\mathbf{X}^{\mathbf{Y}}) \mid \eta_Y \in \mathcal{M}$ for every $Y \in \mathbf{Y}\}$;*
4. *if $\mathbf{Y}$ is a full subcategory of $\mathbf{X}$ closed in it under pullbacks and $\mathcal{M}$-pushouts, then $\mathbf{Y}$ is (strictly) $\mathcal{N}$-adhesive for every class of arrows $\mathcal{N}$ of $\mathbf{Y}$ contained in $\mathcal{M}$ that is stable under pullbacks and pushouts, contains all the isomorphisms, and is closed under composition and decomposition.*

We will briefly list some examples of $\mathcal{M}$-adhesive categories.

▶ **Example 2.5.** **Set** is adhesive, and, more generally, every topos is adhesive [**?**]. By the closure properties above, every presheaf $[\mathbf{X}, \mathbf{Set}]$ is adhesive, thus the category $\mathbf{Graph} = [E \rightrightarrows V, \mathbf{Set}]$ is adhesive where $E \rightrightarrows V$ is the two objects category with two morphisms $s, t \colon E \to V$. Similarly, various categories of hypergraphs can be shown to be adhesive, such as term graphs and hierarchical graphs [7]. Note that the category **sGraphs** of simple graphs, i.e. graphs without parallel edges, is $\mathsf{Reg}(\mathbf{sGraphs})$-adhesive [2] but not quasiadhesive.

We can state some useful properties of $\mathcal{M}$-adhesive category.

▶ **Proposition 2.6.** *If $\mathbf{X}$ is $\mathcal{M}$-adhesive then it holds*

1. *every $\mathcal{M}$-pushout square is also a pullback;*
2. *every arrow in $\mathcal{M}$ is a regular mono.*

[Proof in Appendix A.1]

2.2 Kernel Pairs and Regular Epimorphisms

In this section we recall the definition and some properties of *kernel pairs.*

▶ **Definition 2.7.** *A* kernel pair *for an arrow $f\colon A \to B$ is an object K_f together with two arrows $\pi_f^1, \pi_f^2\colon K_f \rightrightarrows A$, denoted as (K_f, π_f^1, π_f^2), such that the square aside is a pullback.*

$$\begin{tikzcd} K_f \arrow[r, "\pi_f^2"'] \arrow[d, "\pi_f^1"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

▶ Remark 2.8. If (K_f, π_f^1, π_f^2) is a a kernel pair for $f\colon X \to Y$ and a product of X with itself exists, then the canonical arrow $\langle \pi_f^1, \pi_f^2 \rangle\colon K_f \to X \times X$ is a mono.

▶ Remark 2.9. An arrow $m\colon M \to X$ is a mono if and only if it admits $(M, \mathsf{id}_M, \mathsf{id}_M)$ as a kernel pair.

Together with Lemma A.1, the previous remarks allow us to prove the following result.

▶ **Proposition 2.10.** *Let $f\colon X \to Y$ be an arrow and $m\colon Y \to Z$ a mono. If (K_f, π_f^1, π_f^2) is a kernel pair for $f\colon X \to Y$, then it is also a kernel pair for $m \circ f$.*

We are now going to explore some further properties of kernel pairs.

▶ **Lemma 2.11.** *Let $f\colon X \to Y$ and $g\colon Z \to W$ be two arrows admitting kernel pairs and suppose that the solid part of the three squares below is given. Then there exists a a unique arrow $k_h : K_f \to K_g$ completing them. Moreover, if the leftmost is a pullback, then also the other two are so.*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "t"'] & W \end{tikzcd} \qquad \begin{tikzcd} K_f \arrow[r, dotted, "k_h"] \arrow[d, "\pi_f^1"'] & K_g \arrow[d, "\pi_g^1"] \\ X \arrow[r, "h"'] & Z \end{tikzcd} \qquad \begin{tikzcd} K_f \arrow[r, dotted, "k_h"] \arrow[d, "\pi_f^2"'] & K_g \arrow[d, "\pi_g^2"] \\ X \arrow[r, "h"'] & Z \end{tikzcd}$$

[Proof in Appendix A.1]

The previous result allows us to deduce the following lemma in an $\mathcal{M}$-adhesive context.

▶ **Lemma 2.12.** *Let $\mathbf{X}$ be a strict $\mathcal{M}$-adhesive category with all pullbacks, and suppose that in the cube aside the top face is an $\mathcal{M}$-pushout. Then the right square is a pushout.*

$$\begin{tikzcd} & A' \arrow[rr, "f'"] \arrow[dl, "m'"'] \arrow[dd, "a" near end] & & B' \arrow[dl, "n'"'] \arrow[dd, "b"] \\ C' \arrow[rr, crossing over, "g'" near start] \arrow[dd, "c"'] & & D' & \\ & A \arrow[rr, "f" near start] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\ C \arrow[rr, "g"'] & & D \arrow[from=uu, crossing over, "d" near start] & \end{tikzcd} \qquad \begin{tikzcd} K_a \arrow[r, "k_{f'}"] \arrow[d, "k_{m'}"'] & K_b \arrow[d, "k_{n'}"] \\ K_c \arrow[r, "k_{g'}"'] & K_d \end{tikzcd}$$

[Proof in Appendix A.1]

As a final step, we explore the link between regular epis and kernel pairs.

▶ **Proposition 2.13.** *Let $e\colon X \to Y$ be a regular epi in a category $\mathbf{X}$ with a kernel pair (K_e, π_e^1, π_e^2). Then, e is the coequalizer of π_e^1 and π_e^2.*

[Proof in Appendix A.1]

▶ **Corollary 2.14.** *Let $\mathbf{X}$ be a category with pullbacks and $\phi\colon F \dot{\to} G$ a natural transformation between functors $F, G : \mathbf{D} \rightrightarrows \mathbf{X}$. If ϕ_d is a regular epi for every d in $\mathbf{D}$, then ϕ is a regular epi.*

[Proof in Appendix A.1]

From the previous result we deduce that the class of regular epis is closed under colimits.

▶ **Lemma 2.15.** *Let $F, G\colon \mathbf{D} \rightrightarrows \mathbf{X}$ be two diagrams, and suppose that $\mathbf{X}$ has all colimits of shape $\mathbf{D}$. Let $(X, \{x_d\}_{d\in\mathbf{D}})$ and $(Y, \{y_d\}_{d\in D})$ be the colimits of F and G, respectively. If $\phi\colon F \to G$ is a natural transformation whose components are regular epis, then the arrow induced by ϕ from X to Y is a regular epi.*

[Proof in Appendix A.1]

3 Hypergraphical structures

In this section we briefly recall the notion $\mathbf{X}$-*hypergraph*. It is necessary to have a monad $(-)^\star\colon \mathbf{Set} \to \mathbf{Set}$, also known as *list monad*, sending a set to the free monoid on it [19, 20] and playing a role analogous to the usual *Kleene star*. We recall some of its proprieties.

▶ **Proposition 3.1.** *The following facts hold*
1. *for every set X and $n \in \mathbb{N}$ there are arrows $v_n\colon X^n \to X^\star$ such that $(X^\star, \{v_n\}_{n\in\mathbb{N}})$ is a coproduct;*
2. *for every arrow $f\colon X \to Y$, $f^\star$ is the coproduct of the family $\{f^n\}_{n\in\mathbb{N}}$;*
3. *$(-)^\star$ preserves all* connected limits *[5], in particular it preserves pullbacks and equalizers.*

▶ Remark 3.2. Preservation of pullbacks implies that $(-)^\star$ sends monos to monos.

▶ Remark 3.3. Notice that $1^\star$ can be canonically identified with $\mathbb{N}$, thus for every set X the arrow $!_\mathbf{X}\colon X \to 1$ induces a *length function* $!^\star_X\colon X^\star \to \mathbb{N}$, which sends a word to its length.

3.1 The category of hypergraphs

We open this section with the definition of hypergraphs and we will see how to label them with an algebraic signature.

▶ **Definition 3.4.** *An* hypergraph *is a 4-uple $\mathcal{G} := (E_\mathcal{G}, V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G})$ made by two sets $E_\mathcal{G}$ and $V_\mathcal{G}$, called respectively the set of* hyperedges *and* nodes*, plus a pair of* source *and* target *arrows $s_\mathcal{G}, t_\mathcal{G}\colon E_\mathcal{G} \rightrightarrows V^\star_\mathcal{G}$. A* hypergraph morphism *$(E_\mathcal{G}, V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G}) \to (E_\mathcal{H}, V_\mathcal{H}, s_\mathcal{H}, t_\mathcal{H})$ is a pair (h, k) of functions $h\colon E_\mathcal{G} \to E_\mathcal{H}$, $k\colon V_\mathcal{G} \to V_\mathcal{H}$ such that the following diagrams commute*

$$\begin{tikzcd} E_\mathcal{G} \arrow[r, "s_\mathcal{G}"] \arrow[d, "h"'] & V^\star_\mathcal{G} \arrow[d, "k^\star"] \\ E_\mathcal{G} \arrow[r, "s_\mathcal{H}"'] & V^\star_\mathcal{H} \end{tikzcd} \qquad \begin{tikzcd} E_\mathcal{G} \arrow[r, "t_\mathcal{G}"] \arrow[d, "h"'] & V^\star_\mathcal{G} \arrow[d, "k^\star"] \\ E_\mathcal{G} \arrow[r, "t_\mathcal{H}"'] & V^\star_\mathcal{H} \end{tikzcd}$$

We define $\mathbf{Hyp}$ to be the resulting category.

Let $\mathsf{prod}^\star$ be the functor sending X to $X^\star \times X^\star$: we can get $\mathbf{Hyp}$ as a comma category.

▶ **Proposition 3.5.** *$\mathbf{Hyp}$ is isomorphic to $\mathsf{id}_{\mathbf{Set}}{\downarrow}\mathsf{prod}^\star$*

▶ **Corollary 3.6.** *$\mathbf{Hyp}$ is an adhesive category.*

Proof. By hypothesis $(-)^\star$ preserves pullbacks, while prod is continuous by definition, thus the thesis follows from Proposition 2.4 and Proposition 3.5. ◀

Another useful corollary of Proposition 3.5 is the following one.

▶ **Corollary 3.7.** *A morphism (h, k) is a mono in* **Hyp** *if and only if both its components are injective functions.*

Propositions 3.5 and B.6 allow us to deduce immediately the following.

▶ **Proposition 3.8.** *The forgetful functor $U_{\mathbf{Hyp}}$ which sends an hypergraph $\mathcal{G}$ to its object of nodes has a left adjoint $\Delta_{\mathbf{Hyp}}$.*

▶ **Example 3.9.** Since the initial object of **Set** is the empty set, $\Delta_{\mathbf{Set}}(X)$ is the hypergraph which has X as set of nodes, $\emptyset$ as set of hyperedges, and $?_X$ as source and target function.

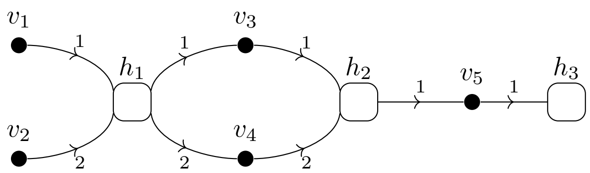
forse possiamo anche cancellare tutti gli esempi sotto

We can represent hypergraphs graphically. We will use dots to denote nodes and squares to denote hyperedges, the name of a node or of an hyperedge will be put near the corresponding dot or square. Sources and targets are represented by lines between dots and squares: the lines from the sources of an hyperedge will have an arrowhead in the middle pointing towards the hyperedge, while the lines to the targets will have arrowheads pointing to the target nodes. We will decorate the arrow corresponding to the i^{th} letter of a target or a source with a label i.

▶ **Example 3.10.** Take $V_\mathcal{G}$ to be be $\{v_1, v_2, v_3, v_4, v_5\}$ and $E_\mathcal{G}$ to be $\{h_1, h_2, h_3\}$. Sources and targets are given by:

$$
\begin{array}{llllll}
s_\mathcal{G}(h_1)\colon 2 \to V_\mathcal{G} & \begin{array}{l} 0 \mapsto v_1 \\ 1 \mapsto v_2 \end{array} & s_\mathcal{G}(h_2)\colon 2 \to V_\mathcal{G} & \begin{array}{l} 0 \mapsto v_3 \\ 1 \mapsto v_4 \end{array} & s_\mathcal{G}(h_3)\colon 1 \to V_\mathcal{G} & 0 \mapsto v_5 \\
t_\mathcal{G}(h_1)\colon 2 \to V_\mathcal{G} & \begin{array}{l} 0 \mapsto v_3 \\ 1 \mapsto v_4 \end{array} & t_\mathcal{G}(h_2)\colon 2 \to V_\mathcal{G} & 0 \mapsto v_5 & t_\mathcal{G}(h_3)\colon 0 \to V_\mathcal{G} & t_\mathcal{G}(h_3) = ?_{V_\mathcal{G}}
\end{array}
$$

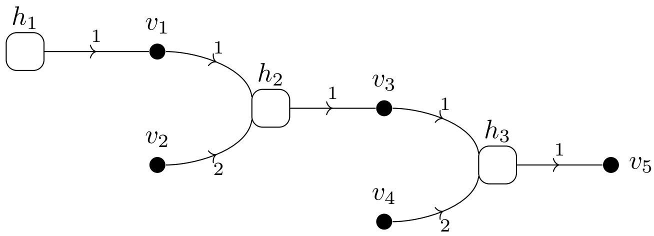
We can draw the resulting $\mathcal{G}$ as follows:



▶ **Example 3.11.** Let $V_\mathcal{G}$ be as in the previous example and $E_\mathcal{G} = \{h_1, h_2, h_3\}$. Then we define

$$
\begin{array}{llllll}
s_\mathcal{G}(h_1)\colon 0 \to V_\mathcal{G} & s_\mathcal{G}(h_1) = ?_{V_\mathcal{G}} & s_\mathcal{G}(h_2)\colon 2 \to V_\mathcal{G} & \begin{array}{l} 0 \mapsto v_1 \\ 1 \mapsto v_2 \end{array} & s_\mathcal{G}(h_3)\colon 2 \to V_\mathcal{G} & \begin{array}{l} 0 \mapsto v_1 \\ 1 \mapsto v_4 \end{array} \\
t_\mathcal{G}(h_1)\colon 1 \to V_\mathcal{G} & 0 \mapsto v_1 & t_\mathcal{G}(h_2)\colon 1 \to V_\mathcal{G} & 0 \mapsto v_3 & t_\mathcal{G}(h_3)\colon 1 \to V_\mathcal{G} & 1 \mapsto v_5
\end{array}
$$

Now we can depict $\mathcal{G}$ as



3.1.1 Hyp as a category of functors

Following [3], we can present **Hyp** as a category of functor over a suitable category.

▶ **Definition 3.12.** *Let* $\mathbf{H}$ *be the category such that*
- *the set of objects is* $(\mathbb{N} \times \mathbb{N}) \cup \{\bullet\}$*;*
- *arrows are given by the identities* $\mathsf{id}_{k,l}$ *and* $\mathsf{id}_\bullet$ *and exactly* $k+l$ *arrows* $f_i \colon (k,l) \to \bullet$*, where* i *ranges from* 0 *to* $k+l-1$*;*
- *composition is defined by putting* $f_i = f_i \circ \mathsf{id}_{k,l}$ *and* $f_i = \mathsf{id}_\bullet \circ f_i$ *for every* $f_i \colon (k,l) \to \bullet$*.*

The idea is that for every functor $F \colon \mathbf{H} \to \mathbf{Set}$ we can define

$$E_F := \sum_{k,l \in \mathbb{N}} F(k,l)$$

Now, for every k, l, i and j in $\mathbb{N}$ with $i < k$ and $j < l$ we define $s^F_{k,l} \colon F(k,l) \to F(\bullet)^k$ and $t^F_{k,l} \colon F(k,l) \to F(\bullet)^l$ as the unique arrows fitting in the diagrams below, where the vertical arrows are the projections

$$\begin{tikzcd} F(k,l) \arrow[r, dotted, "s^F_{k,l}"] \arrow[dr, "F(f_i)"'] & F(\bullet)^k \arrow[d, "\pi^F_{k,i}"] \\ & F(\bullet) \end{tikzcd} \qquad \begin{tikzcd} F(k,l) \arrow[r, dotted, "t^F_{k,l}"] \arrow[dr, "F(f_{k+j})"'] & F(\bullet)^l \arrow[d, "\pi^F_{l,j}"] \\ & F(\bullet) \end{tikzcd}$$

In turn, these arrows allow us to consider $s_F, t_F \colon E_F \rightrightarrows F(\bullet)^\star$ as the unique arrows fitting in the diagrams below, where the vertical arrows are coprojections

$$\begin{tikzcd} F(k,l) \arrow[r, "s^F_{k,l}"] \arrow[d, "a^F_{k,l}"'] & F(\bullet)^k \arrow[d, "b^F_k"] \\ E_F \arrow[r, dotted, "s_F"'] & F(\bullet)^\star \end{tikzcd} \qquad \begin{tikzcd} F(k,l) \arrow[r, "t^F_{k,l}"] \arrow[d, "a^F_{k,l}"'] & F(\bullet)^l \arrow[d, "b^F_l"] \\ E_F \arrow[r, dotted, "t_F"'] & F(\bullet)^\star \end{tikzcd}$$

Let $\mathcal{G}_F$ be the resulting hypergraph. One can now show that sending F to $\mathcal{G}_F$ can be extended to an equivalence $\mathcal{G}_- \colon \mathbf{Set}^{\mathbf{H}} \to \mathbf{Hyp}$ (see [6, 7] for details).

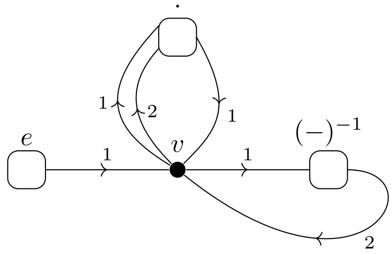
▶ **Proposition 3.13.** $\mathbf{Hyp}$ *is equivalent to the category* $\mathbf{Set}^{\mathbf{H}}$*.*

3.1.2 Labelling hypergraph with an algebraic signature

Our interest for hypergraphs stems from their use as a graphical representation of algebraic terms. We thus need a way to label hyperedges with symbols taken from a signature.

▶ **Definition 3.14.** *An* algebraic signature Σ *is a pair* $(O_\Sigma, \mathsf{ar}_\Sigma)$ *given by a* set of operations O_Σ *and an* arity function $\mathsf{ar}_\Sigma \colon O_\Sigma \to \mathbb{N}$*. We define the* hypergraph $\mathcal{G}_\Sigma$ associated with Σ *taking* O_Σ *as set of hyperedges,* 1 *as set of nodes, so that* $1^\star$ *is* $\mathbb{N}$*,* ar_σ *as the source function and* δ_1 *as target function, where* δ_1 *picks the element* 1*. The category* $\mathbf{Hyp}_\Sigma$ *of* algebraically labelled hypergraphs *is the slice category* $\mathbf{Hyp}/\mathcal{G}^\Sigma$*.*

▶ **Example 3.15.** Let $\Sigma = (O_\Sigma, \mathsf{ar}_\Sigma)$ be an algebraic signature in **Set**. This simply amount to a set of *operations* with an associated natural number, called *arity*. For instance let Σ_G be the signature of groups, then $\mathcal{G}^{\Sigma_G}$ can be depicted as



Corollary B.5 and Proposition 2.4 give us immediately an adhesivity result for $\mathbf{Hyp}_\Sigma$ and a characterisation of monos in it.

▶ **Proposition 3.16.** *Let Σ be an algebraic signature. Then it holds*

1. *a morphism (h,k) between two object of $\mathbf{Hyp}_\Sigma$ is a mono if and only if h and k are injective functions;*
2. *$\mathbf{Hyp}_\Sigma$ is an adhesive category.*

▶ Remark 3.17. Let $\mathcal{H} = (E, V, s, t)$ be an hypergraph, by definition we know that $U_{\mathbf{Hyp}}(\mathcal{G}^\Sigma)$ is the terminal object 1, so an arrow $\mathcal{H} \to \mathcal{G}^\Sigma$, is determined by a morphism $h\colon E_\mathcal{H} \to O_\Sigma$ making the two squares below commute (cfr. Remark 3.3).

$$\begin{tikzcd} E_\mathcal{H} \arrow[r, "h"] \arrow[d, "s_\mathcal{H}"'] & O_\Sigma \arrow[d, "\mathsf{ar}_\Sigma"] \\ V^\star_\mathcal{H} \arrow[r, "\mathsf{lg}_{V_\mathcal{H}}"'] & \mathbb{N} \end{tikzcd} \qquad \begin{tikzcd} E_\mathcal{H} \arrow[r, "h"] \arrow[d, "t_\mathcal{H}"'] & O_\Sigma \arrow[d, "\delta_1"] \\ V^\star_\mathcal{H} \arrow[r, "\mathsf{lg}_{V_\mathcal{H}}"'] & \mathbb{N} \end{tikzcd}$$

Let $v_n\colon V^n_\mathcal{H} \to V^\star_\mathcal{H}$ be a coprojection. The second diagram above entails that $t_\mathcal{H}$ factors via the inclusion $v_1\colon V_\mathcal{H} \to V^\star$ of words of length 1, i.e. $t_\mathcal{H} = v_1 \circ \tau_\mathcal{H}$ for some $\tau_\mathcal{H}\colon E_\mathcal{H} \to \tau_\mathcal{H}$.

$\mathbf{Hyp}_\Sigma$, has a forgetful functor $U_\Sigma\colon \mathbf{Hyp}_\Sigma \to \mathbf{X}$ which sends $(h,k)\colon \mathcal{H} \to \mathcal{G}^\Sigma$ to $U_\mathbf{X}(\mathcal{H})$. Now, $U_\mathbf{X}(\mathcal{G}^\Sigma) = 1$ thus, for every object X, there is only one arrow $X \to U_\mathbf{X}(\mathcal{G}^\Sigma)$. Define $\Delta_\Sigma(X)\colon \Delta_\mathbf{X}(X) \to \mathcal{G}^\Sigma$ as the transpose of this arrow. Explicitly, $\Delta_\mathbf{X}(X) = (0, X, ?_{X^\star}, ?_{X^\star})$ and $\Delta_\Sigma(X)$ is simply $(?_{O_\Sigma}, !_X)$.

▶ **Proposition 3.18.** *U_Σ has a left adjoint Δ_Σ.*

Proof. Let $(h, !_{V_\mathcal{H}})\colon \mathcal{H} \to \mathcal{G}^\Sigma$ be an object of $\mathbf{Hyp}_\Sigma$, and suppose that there exists $f\colon X \to U_\Sigma(\mathcal{H})$. Since, $U_\Sigma(\mathcal{H}) = U_\mathbf{X}(\mathcal{H})$ and the identity is the unit of $\Delta_\mathbf{Hyp} \dashv U_\mathbf{Hyp}$, we get a morphism $(?_{E_\mathcal{H}}, f)\colon \Delta_\mathbf{X}(X) \to \mathcal{H}$ of $\mathbf{Hyp}$. And then the thesis follows since we have

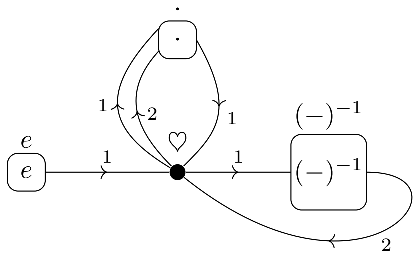
$$(h, !_{V_\mathcal{H}}) \circ (?_{E_\mathcal{H}}, f) = (h \circ ?_{E_\mathcal{H}}, !_{V_\mathcal{H}} \circ f) = (?_{0_\Sigma}, !_X) = \Delta_\mathbf{Hyp}(X)$$

◀

We will extend our graphical notation of hypergraphs to labeled ones putting the label of an hyperedge h inside its corresponding square.

anche questo forse val la pena toglierlo

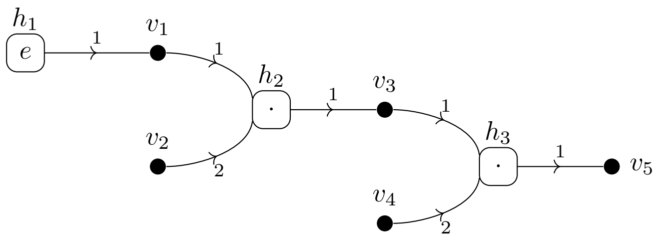
▶ **Example 3.19.** The simplest example is given by the identity $\mathsf{id}_{\mathcal{G}^\Sigma}\colon \mathcal{G}^\Sigma \to \mathcal{G}^\Sigma$. If Σ is the signature of groups $\Sigma_G$we get



▶ **Example 3.20.** Take again Σ_G the signature of groups, then the hypergraph $\mathcal{G}$ of Example 3.11 can be labeled defining

$$e = f(h_1) \quad \cdot = f(h_2) \quad \cdot = f(h_3)$$

In this case we get the following picture



3.2 Term Graphs

a very nice introduction

Let us start using labelled hypergraphs to define term graphs.

▶ **Definition 3.21.** *Given an algebraic signature Σ, we say that a labelled hypergraph $(l, !_{V_\mathcal{G}}) \colon \mathcal{G} \to \mathcal{G}^\Sigma$ is a* term graph *if $t_\mathcal{G}$ is mono. We define $\mathbf{TG}_\Sigma$ to be the full subcategory of $\mathbf{Hyp}_\Sigma$ and denote by I_Σ the inclusion. Restricting $U_\Sigma \colon \mathbf{Hyp}_\Sigma \to \mathbf{Set}$ we get a forgetful functor $U_{\mathbf{TG}_\Sigma} \colon \mathbf{TG}_\Sigma \to \mathbf{Set}$.*

▶ Remark 3.22. By Remark 3.17, we know that if $\mathcal{G}$ is a term graph then $t_\mathcal{G} = v_1 \circ \tau_\mathcal{G}$, where v_1 is the coprojection of $V_\mathcal{G}$ into $V^\star_\mathcal{G}$. Notice that since $t_\mathcal{G}$ is mono then $\tau_\mathcal{G}$ is mono too.

We now examine some properties of $\mathbf{TG}_\Sigma$, in order to study its adhesivity properties.

▶ **Proposition 3.23.** *The forgetful functor $U_{\mathbf{TG}_\Sigma}$ has a left adjoint $\Delta_{\mathbf{TG}_\Sigma}$.*

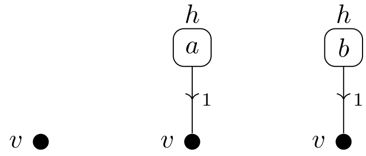
Proof. This follows noticing that $\Delta_\Sigma(X)$ is a term graph for every object X. ◀

We can list some categorical properties of $\mathbf{TG}_\Sigma$

▶ **Proposition 3.24.** *Let Σ be an algebraic signature. Then it holds*

1. *if $(i, j) \colon \mathcal{H} \to \mathcal{G}$ is a mono between $(l, !_{V_\mathcal{G}}) \colon \mathcal{G} \to \mathcal{G}^\Sigma$ and $(l', !_{V_\mathcal{H}}) \colon \mathcal{H} \to \mathcal{G}^\Sigma$ in $\mathbf{Hyp}_\Sigma$ and the latter is a term graph, then also the former is in $\mathbf{TG}_\Sigma$*
2. *$\mathbf{TG}_\Sigma$ has equalizers, binary products and pullbacks and they are created by I_Σ.*

▶ Remark 3.25. $\mathbf{TG}_\Sigma$ in general does not have terminal objects. Consider an algebraic signature in $\mathbf{Set}$. Since $U_{\mathbf{TG}_\Sigma}$ preserves limits, if a terminal object exists it must have the singleton as set of nodes, therefore the set of hyperedges must be empty or a singleton $\{h\}$. Now take as signature the one given by two operations a and b, both of arity 0; we have three term graphs with only one node v: $\Delta_{\mathbf{TG}_\Sigma\Sigma}(\{v\})$, $(l_a, !_{V_\mathcal{G}}) \colon \mathcal{G}_a \to \mathcal{G}^\Sigma$ and $(l_b, !_{V_\mathcal{G}}) \colon \mathcal{G}_b \to \mathcal{G}^\Sigma$.



There are no morphisms in $\mathbf{TG}_\Sigma$ between the last two and from the last two to the first one, therefore none of them can be terminal.

▶ Remark 3.26. $\mathbf{TG}_\Sigma$ is not an adhesive category. In particular it does not have pushouts along all monos. For instance, if we take the three term graphs of the previous remark, then have two arrows $(?_{\{h\}}, \mathsf{id}_{\{v\}})\colon \Delta_{\mathbf{TG}_\Sigma}(\{v\}) \to (l_a, !_{V_{\mathcal{G}_a}})$ and $(?_{\{h\}}, \mathsf{id}_{\{v\}})\colon \Delta_{\mathbf{TG}_\Sigma}(\{v\}) \to (l_b, !_{V_{\mathcal{G}_a}})$ which cannot be completed to a square. Indeed if $(q, !_{V_\mathcal{H}})\colon \mathcal{H} \to \mathcal{G}^\Sigma$ is another term graph with $(g_E, g_V)\colon (l_a, !_{V_\mathcal{G}}) \to (q, !_{V_\mathcal{H}})$ and $(k_E, k_V)\colon (l_a, !_{V_\mathcal{G}}) \to (q, !_{V_\mathcal{H}})$ such that

$$(g_E, g_V) \circ (?_{\{h\}}, \mathsf{id}_{\{v\}}) = (k_E, k_V) \circ (?_{\{h\}}, \mathsf{id}_{\{v\}})$$

then $g_V = k_V$ and

$$t_\mathcal{H}(g_E(h)) = g_V^\star(t_\mathcal{G}(h)) = g_V^\star(\delta_v) = k_V^\star(\delta_V) = k_V^\star(t_\mathcal{G}(h)) = t_\mathcal{H}(k_E(h))$$

so that we also have $g_E = k_E$, but then

$$a = l_a(h) = q(g_E(h)) = q(k_E(h)) = l_b(h) = b$$

▶ **Definition 3.27.** *Let $(l, !_{V_\mathcal{G}})\colon \mathcal{G} \to \mathcal{G}^\Sigma$ be a term graph. A* input node *is an element of $V_\mathcal{G}$ not in the image of $\tau_\mathcal{H}$. A morphism (f, g) between Let $(l, !_{V_\mathcal{G}})\colon \mathcal{G} \to \mathcal{G}^\Sigma$ and $(l, !_{V_\mathcal{H}})\colon \mathcal{H} \to \mathcal{G}^\Sigma$ in $\mathbf{TG}_\Sigma$, is said to* preserve input nodes *if g sends input nodes to input nodes.*

▶ Remark 3.28. Suppose that $(f, g)\colon ((l, !_{V_\mathcal{G}})) \to (l', !_{V_\mathcal{H}})$ preserves input nodes. Then if $\tau_\mathcal{H}(h) = g(v)$ for some $v \in V_\mathcal{G}$ then h belongs to the image of f. Indeed, by hypothesis v must be in the image of $\tau_\mathcal{G}$ and so there exists k such that $\tau_\mathcal{G}(k) = v$. But then $\tau_\mathcal{H}(f(k)) = g(v)$ and we can conclude that $f(k) = h$.

Se questo remark sotto non serve nel pezzo sulle equivalenze possiamo toglierlo

Preservation of inputs characterizes regular monos in $\mathbf{TG}_\Sigma$.

▶ **Proposition 3.29.** *Let (i, j) be a mon between two term graphs $(l, !_{V_\mathcal{G}})\colon \mathcal{G} \to \mathcal{G}^\Sigma$ and $(l', !_{V_\mathcal{H}})\colon \mathcal{H} \to \mathcal{G}^\Sigma$. Then it is a regular mono if and only if it preserves the input nodes.*

This characterization, in turn, provides us with the following result [7, 6].

▶ **Lemma 3.30.** *Consider three term graphs $(l_0, !_{V_\mathcal{G}})\colon \mathcal{G} \to \mathcal{G}^\Sigma$, $(l_1, !_{V_\mathcal{H}})\colon \mathcal{H} \to \mathcal{G}^\Sigma$ and $(l_2, !_{V_\mathcal{K}})\colon \mathcal{K} \to \mathcal{G}^\Sigma$. Given $(f_1, g_1)\colon (l_0, !_{V_\mathcal{G}}) \to (l_1, !_{V_\mathcal{H}})$, $(f_2, g_2)\colon (l_0, !_{V_\mathcal{G}}) \to (l_2, !_{V_\mathcal{K}})$, if (f_1, g_1) is a regular mono, then its pushout along (f_2, g_2), then their pushout $(p, !_{V_\mathcal{P}})\colon \mathcal{P} \to \mathcal{G}^\Sigma$ in $\mathbf{Hyp}_\Sigma$ is a term graph too.*

Proposition 2.4, Proposition 3.29 and Lemma 3.30 allow us to recover the following result, previously proved by direct computation in [8, Thm. 4.2].

▶ **Corollary 3.31.** *The category $\mathbf{TG}_\Sigma$ is quasiadhesive.*

4 Hypergraphs and term graphs with equivalences

▶ **Definition 4.1.** *A* hypergraph with equivalence *$\mathcal{G} = (E_\mathcal{G}, V_\mathcal{G}, C_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G}, q_\mathcal{G})$ is a 6-tuple such that $\mathcal{G} = (E_\mathcal{G}, V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G})$ is a hypergraph, $C_\mathcal{G}$ is a set and $q_\mathcal{G} : V_\mathcal{G} \to C_\mathcal{G}$ is a regular epi called* quotient map. *A morphism $h : \mathcal{G} \to \mathcal{H}$ is a triple (h_E, h_V, h_C) such that the following diagrams commute*

$$\begin{tikzcd} E_\mathcal{G} \arrow[r, "s_\mathcal{G}"] \arrow[d, "h_E"'] & V_\mathcal{G}^\star \arrow[d, "h_V^\star"] \\ E_\mathcal{H} \arrow[r, "s_\mathcal{H}"'] & V_\mathcal{H}^\star \end{tikzcd} \qquad \begin{tikzcd} E_\mathcal{G} \arrow[r, "t_\mathcal{G}"] \arrow[d, "h_E"'] & V_{\mathcal{G}^\star} \arrow[d, "h_V^\star"] \\ E_\mathcal{H} \arrow[r, "t_\mathcal{H}"'] & V_{\mathcal{H}^\star} \end{tikzcd} \qquad \begin{tikzcd} V_\mathcal{G} \arrow[r, "q_\mathcal{G}"] \arrow[d, "h_V"'] & C_\mathcal{G} \arrow[d, "h_C"] \\ V_\mathcal{H} \arrow[r, "q_\mathcal{H}"'] & C_\mathcal{H} \end{tikzcd}$$

The category of hypergraphs with equivalences and their morphisms is denoted **EqHyp**.

▶ Remark 4.2. Morphisms of hypergraphs with equivalences are uniquely determined by the first two components. That is, if $h_1 = (h_E, h_V, f)$ and $h_2 = (h_E, h_V, g)$ are two morphisms $\mathcal{G} \to \mathcal{H}$, then we have

$$\begin{tikzcd} V_\mathcal{G} \arrow[r, "h_V"] \arrow[d, "q_\mathcal{G}"'] & V_\mathcal{H} \arrow[d, "q_\mathcal{H}"] & V_\mathcal{G} \arrow[l, "h_V"'] \arrow[d, "q_\mathcal{G}"] \\ C_\mathcal{G} \arrow[r, "f"'] & C_\mathcal{H} & C_\mathcal{G} \arrow[l, "g"] \end{tikzcd}$$

Hence $f \circ q_\mathcal{G} = q_\mathcal{H} \circ h_V = g \circ q_\mathcal{G}$, and since $q_\mathcal{G}$ is epi, we obtain $f = g$.

EqHyp has a forgetful functor $U_{\mathbf{EqHyp}} : \mathbf{EqHyp} \to \mathbf{Set}$, which sends each $\mathcal{G} = (E_\mathcal{G}, V_\mathcal{G}, C_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G}, q_\mathcal{G})$ into $V_\mathcal{G}$, and each $h = (h_E, h_V, h_C)$ onto h_V.

▶ **Proposition 4.3.** *$U_{\mathbf{EqHyp}}$ has a left adjoint $\Delta_{\mathbf{EqHyp}} : \mathbf{Set} \to \mathbf{EqHyp}$.*

Proof. For each set X, define $\Delta_{\mathbf{EqHyp}}(X) := (\emptyset, X, \{\bullet\}, ?_X, ?_X, !_X)$. Consider now $h : \Delta_{\mathbf{EqHyp}}(X) \to \mathcal{H}$.

$$\begin{tikzcd} \Delta_{\mathbf{EqHyp}}(X) \arrow[d, dotted, "\Delta_{\mathbf{EqHyp}}(f)"'] \arrow[dr, "h"] & \\ \Delta_{\mathbf{EqHyp}}(U_{\mathbf{EqHyp}}(\mathcal{H})) \arrow[r, "\epsilon_\mathcal{H}"'] & \mathcal{H} \end{tikzcd}$$

Where $\Delta_{\mathbf{EqHyp}}(U_{\mathbf{EqHyp}}(\mathcal{H})) = (\emptyset, V_\mathcal{H}, \{\bullet\}, ?_{V_\mathcal{H}}, ?_{V_\mathcal{H}}, !_{V_\mathcal{H}})$ and $\epsilon_\mathcal{H} = (?_{E_\mathcal{H}}, \mathsf{id}_{V_\mathcal{H}}, g)$. Note that, since $\Delta_{\mathbf{EqHyp}}(X)$ has the empty set as object of edges, $h_E = ?_{E_\mathcal{H}}$, then, the unique arrow that fits in the diagram is $\Delta_{\mathbf{EqHyp}}(f) = (?_{E_\mathcal{H}}, h_V, \mathsf{id}_{\{\bullet\}})$. ◀

We now define another functor $T : \mathbf{EqHyp} \to \mathbf{Hyp}$, which "forgets" the quotient part, mapping each hypergraph with equivalence $\mathcal{G} = (E_\mathcal{G}, V_\mathcal{G}, C_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G}, q_\mathcal{G})$ onto $T(\mathcal{G}) = (E_\mathcal{G}, V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G})$. Then, we have the following result.

▶ **Proposition 4.4.** *T has a left adjoint $L : \mathbf{Hyp} \to \mathbf{EqHyp}$.*

Proof. Let $\mathcal{G}$ be a hypergraph, and define $L(\mathcal{G}) := (E_\mathcal{G}, V_\mathcal{G}, \{\bullet\}, s_\mathcal{G}, t_\mathcal{G}, !_{V_\mathcal{G}})$. Let now $h : L(\mathcal{G}) \to \mathcal{H}$ be a morphism in **EqHyp**, and consider the following situation

$$\begin{tikzcd} L(\mathcal{G}) \arrow[d, dotted, "L(f)"'] \arrow[dr, "h"] & \\ L(T(\mathcal{H})) \arrow[r, "\epsilon_\mathcal{H}"'] & \mathcal{H} \end{tikzcd}$$

for $L(T(\mathcal{H})) = (E_\mathcal{H}, V_\mathcal{H}, \{\bullet\}, s_\mathcal{H}, t_\mathcal{H}, !_{V_\mathcal{H}})$. Then, $\epsilon_\mathcal{H} = (\mathsf{id}_{E_\mathcal{H}}, \mathsf{id}_{V_\mathcal{H}}, h_C)$ (by Remark 4.2, the last component is uniquely determined by the first two), and $L(f)$ must be $(h_E, h_V, \mathsf{id}_{\{\bullet\}})$. ◀

▶ Remark 4.5. T is faithful. Indeed, consider two morphisms $h = (h_E, h_V, h_C)$ and $k = (k_E, k_V, k_C)$, and suppose $T(h) = T(k)$, that is, $(h_E, h_V) = (k_E, k_V)$. By Remark 4.2, we can conclude also $h_C = k_C$, and hence the faithfulness of T.

Let now $K : \mathbf{EqHyp} \to \mathbf{Set}$ be the functor which sends each hypergraph with equivalence $\mathcal{G} = (E, V, C, s, t, q)$ onto $K(\mathcal{G}) = C$, and each morphism (h_E, h_V, h_C) to h_C.

▶ **Proposition 4.6.** $\mathbf{EqHyp}$ *is complete and cocomplete, and T preserves limits and colimits.*

Proof. Let $D : \mathbf{I} \to \mathbf{EqHyp}$ be a diagram, and, for each $i \in \mathbf{I}$, $D(i) = (E_i, V_i, C_i, s_i, t_i, q_i)$. Suppose now (E, V, s, t), together with morphisms (π_i^E, π_i^V), be the limit of $T \circ D$. Then, V, together with $(q_i \circ \pi_V^i)_{i \in \mathbf{I}}$, is a cone for $K \circ D$. Indeed, let $\alpha : i \to j$ be an arrow of $\mathbf{I}$, $D(\alpha) = (h_E, h_V, h_C)$. By definition of T, $(T \circ D)(\alpha) = (h_E, h_V)$, hence we have

$$\begin{tikzcd}
 & V \arrow[dl, "\pi_V^i"'] \arrow[dr, "\pi_V^j"] & \\
V_i \arrow[rr, "h_V"] \arrow[d, "q_i"'] & & V_j \arrow[d, "q_j"] \\
C_i \arrow[rr, "h_C"'] & & C_j
\end{tikzcd}$$

Suppose now that L, with morphisms $(l_i)_{i \in \mathbf{I}}$ be the limit of $K \circ D$. Hence, we have an arrow $l : V \to L$, which is not epi in general. Let then $l = m \circ q$ be the epi-mono factorization of it. Consider the following situation, where the outer rectangle commutes by definition, and the dotted arrow is yielded by (cite left lifting prop).

$$\begin{tikzcd}
V \arrow[r, "\pi_V^i"] \arrow[d, "q"'] & V_i \arrow[r, "q_i"] & C_i \arrow[d, "\mathsf{id}_{C_i}"] \\
C \arrow[r, "m"'] \arrow[urr, dotted, "\pi_C^i"] & L \arrow[r, "l_i"'] & C_i
\end{tikzcd}$$

Thus, (E, V, C, s, t, q), together with $(\pi_E^i, \pi_V^i, \pi_C^i)$ is a cone over D. remain to show that this cone is terminal

Suppose now (E', V', s', t'), together with $(\kappa_E^i, \kappa_V^i)_{i \in \mathbf{I}}$, be the colimit of $T \circ D$, and C', with $(c_i)_{i \in \mathbf{I}}$ be the colimit of $K \circ D$. Then, we have the folliwing situation.

$$\begin{tikzcd}
 & V' & \\
V_i \arrow[ur, "\kappa_i"] \arrow[rr, "h_V"] \arrow[d, "q_i"'] & & V_j \arrow[ul, "\kappa_j"'] \arrow[d, "q_j"] \\
C_i \arrow[rr, "h_C"'] \arrow[dr, "c_i"'] & & C_j \arrow[dl, "c_j"] \\
 & C' &
\end{tikzcd}$$

Then, C' with morphisms $(c_i \circ q_i)_{i\in\mathbf{I}}$ is a conone for $U \circ D$. Then, there exists a unique morphism $q' : V' \to C'$ such that $q' \circ \kappa_V^i = c_i \circ q_i$. Such morphism is epi (cite Lemma 1.3.45 of the thesis), and thus (E', V', C', s', t', q'), together with $(\kappa_E^i, \kappa_V^i, c_i)_{i\in\mathbf{I}}$ is the colimit of D. ◀

▶ **Corollary 4.7.** *Let $h = (h_E, h_V, h_C) : \mathcal{G} \to \mathcal{H}$ be an arrow in* **EqHyp***. Then it is a mono if and only if $T(h)$ is a mono.*

Proof. The "if" part is given by the faithfulness of T. The "only if" part is given by Remark 4.2. ◀

▶ **Corollary 4.8.** *If $h = (h_E, h_V, h_C) : \mathcal{G} \to \mathcal{H}$ is a regular mono in* **EqHyp***, then h_E, h_V and h_C are all monos.*

Proof. If h is mono, from Corollary 4.7 we have that h_E and h_V are monos. Suppose now $f, g : \mathcal{H} \rightrightarrows \mathcal{K}$ be the arrows equalized by h. Then, we have:

$$\begin{aligned} f_C \circ h_C \circ q_{\mathcal{G}} &= f_C \circ q_{\mathcal{H}} \circ h_V \\ &= q_{\mathcal{K}} \circ g_V \circ h_V \\ &= q_{\mathcal{K}} \circ f_V \circ h_V \\ &= g_C \circ h_C \circ q_{\mathcal{G}} \end{aligned}$$

Since $q_{\mathcal{G}}$ is epi, we have $f_C \circ h_C = g_C \circ h_C$, hence h_C is an equalizer for f_C and g_C, and thus a mono. ◀

▶ **Proposition 4.9.** *Let $h = (h_E, h_V, h_C) : \mathcal{G} \to \mathcal{H}$ be a regular mono in* **EqHyp***. Then, h_E and h_V are monos and (K, π_1, π_2) is the kernel pair of $q_{\mathcal{H}} \circ h_V$ if and only if (K, π_1, π_2) is the kernel pair of $q_{\mathcal{G}}$.*

Proof. By Corollary 4.8, we have that h_E, h_V and h_C are all monos. Hence, by Proposition 2.10, (K, π_1, π_2) is the kernel pair of $q_{\mathcal{G}}$ if and only if it is the kernel pair also of $h_C \circ q_{\mathcal{G}}$, since h_C is mono by hypothesis. The thesis follows from $h_C \circ q_{\mathcal{G}} = q_H \circ h_V$, and from the hypothesis of h_E mono. ◀

▶ Remark 4.10. It is possible to restate the last proposition, by **??**, as

> *h_E and h_V are mono and, for every $v, v' \in V_H$, $q_H(h_V(v)) = q_H(h_V(v'))$ if and only if $q_G(v) = q_G(v')$*

That is, a regular mono in **EqHyp** is a morphism that both reflects and preserves equivalences.

Let us turn to another functor **EqHyp** $\to$ **Hyp**.

▶ **Definition 4.11.** *The* quotient functor Q : **EqHyp** $\to$ **Hyp** *is defined as the one sending (E, V, C, s, t, q) to $(E, C, q^\star \circ s, q^\star \circ t)$ and an arrow (h_E, h_V, h_C) to (h_E, h_C).*

▶ Remark 4.12. The action of the functor on a morphism of hypergraphs with equivalences gives a morphism of hypergraphs, in fact $q_{\mathcal{H}}^\star \circ s_{\mathcal{H}} \circ h_E = q_{\mathcal{H}}^\star \circ h_V^\star \circ s_{\mathcal{G}} = h_C^\star \circ q_{\mathcal{G}}^\star \circ s_{\mathcal{G}}$. The same is valid for $t_{\mathcal{H}}$ and $t_{\mathcal{G}}$.

▶ **Lemma 4.13.** *Q is a left adjoint.*

Proof. Let $R((A, B, s, t))$ be $(A, B, B, s, t, \mathsf{id}_B)$, so that $Q(R((A, B, s, t))) = (A, B, s, t)$. Now, suppose that $h = (h_E, h_V) : Q((E, V, C, s', t', q)) \to (A, B, s, t)$ is an arrow in **Hyp**, and consider the triple $(h_E, h_V, h_V \circ q)$. Since h is a morphism of **Hyp**, we have $h_V^\star \circ q^\star \circ s' = s \circ h_E$ and $h_V^\star \circ q^\star \circ t' = t \circ h_E$. Then we have the following squares

$$\begin{tikzcd} E \arrow[r, "h_E"] \arrow[d, "s'"'] & A \arrow[d, "s"] \\ V^\star \arrow[r, "h_V^\star \circ q^\star"'] & B^\star \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "h_E"] \arrow[d, "t'"'] & A \arrow[d, "t"] \\ V^\star \arrow[r, "h_V^\star \circ q^\star"'] & B^\star \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "h_V \circ q"] \arrow[d, "q"'] & B \arrow[d, "\mathsf{id}_B"] \\ C \arrow[r, "h_V"'] & B \end{tikzcd}$$

We have therefore found a morphism $(E, V, C, s', t', q) \to R((A, B, s, t))$ whose image through Q fits in the diagram below

$$\begin{tikzcd} (A, B, s, t) \arrow[r, "(\mathsf{id}_A, \mathsf{id}_B)"] & (A, B, s, t) \\ (E, C, q^\star \circ s', q^\star \circ t') \arrow[u, "Q(h_E, h_V \circ q, h_V)"] \arrow[ur, "(h_E, h_V)"'] & \end{tikzcd}$$

Such arrow is unique. Suppose $f = (f_E, f_V, f_C)$ to be another arrow with such property. Then, it must be $(\mathsf{id}_A, \mathsf{id}_B) \circ Q(f) = (f_E, f_C) = (h_E, h_C)$. Finally, $f_C = f_V \circ q = h_V \circ q$. ◀

▶ **Proposition 4.14.** *Q creates colimits.*

Proof. Since Q is a left adjoint, it preserves colimits. Let $D : \mathbf{I} \to \mathbf{EqHyp}$ be a diagram, and let $\mathcal{C}$, together with $(c_i)_{i \in \mathbf{I}}$ be the colimit of $Q \circ D$, where $\mathcal{C} = (A, C, q \circ s, q \circ t)$, and $D(i)$ is $(A_i, B_i, C_i, s_i, t_i, q_i)$. Let $((\kappa_i)_{i \in \mathbf{I}}, V)$ be the colimit of $U_{\mathbf{EqHyp}} \circ D$. Consider the following situation

$$\begin{tikzcd} & V & \\ V_i \arrow[ur, "\kappa_i"] \arrow[rr, "h_V"] \arrow[d, "q_i"'] & & V_j \arrow[ul, "\kappa_j"'] \arrow[d, "q_j"] \\ C_i \arrow[rr, "h_C"'] \arrow[dr, "c_i"'] & & C_j \arrow[dl, "c_j"] \\ & C & \end{tikzcd}$$

Now, since $((c_C^i \circ q_i)_{i \in \mathbf{I}}, C)$ is a cocone for $U_{\mathbf{EqHyp}} \circ D$, there exists a unique $q : V \to C$, which is epi by Lemma 2.15. Consider now the functor $W : \mathbf{EqHyp} \to \mathbf{Set}$ mapping each (X, Y, Z, x, y, z) onto X, and each morphism on its first component. By Proposition 4.6 and **??**, we have that $((c_E^i)_{i \in \mathbf{I}}, E)$ is the colimit of $W \circ D$. Notice that $((\kappa_i \circ s_i)_{i \in \mathbf{I}}, B)$ and $((\kappa_i \circ t_i)_{i \in \mathbf{I}}, B)$ are cocones for $W \circ D$, so let s and t be, respectively, the mediating arrow for the first one and the mediating arrow for the second one. It remains now to show that (E, V, C, s, t, q), together with $(c_E^i, \kappa_i, c_C^i)_{i \in \mathbf{I}}$, is a colimit for D, but this follows by the proof of **??**. ◀

▶ **Example 4.15.** Q does not preserve limits. Indeed, let $\mathcal{G}_1 = (E_1, A, A, s_1, t_1, id_A)$, $\mathcal{G}_2 = (E_2, B, B, s_2, t_2, id_B)$ and $\mathcal{G}_3 = (E_3, A + B, \mathbb{1}, s_3, t_3, !_{A+B})$, and let $h = (h_E, \iota_A, !_A) : \mathcal{G}_1 \to \mathcal{G}_3$,

$k = (k_E, \iota_B, !_B) : \mathcal{G}_2 \to \mathcal{G}_3$, where $(\iota_A, \iota_B, A+B)$ is the coproduct of A and B, $\mathbb{1}$ is the intial object (in **Set**, the singleton set as shown in **??**), and $!_X$ the unique arrow $X \to \mathbb{1}$. The following two diagrams show the pullback of h and k and the pullback of $Q(h)$ and $Q(k)$, on the second component (the vertices of the graphs)

$$
\begin{tikzcd}
\mathbb{0} \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "\iota_A"] \\
B \arrow[r, "\iota_B"'] & A+B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times B \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \arrow[d, "!_A"] \\
B \arrow[r, "!_B"'] & \mathbb{1}
\end{tikzcd}
$$

But the arrow $\mathbb{0} \to A \times B$ is not an epi in general (this is easy to see taking **Set** as example), hence such pullback is not preserved by Q.

▶ **Lemma 4.16.** *In* **EqHyp**, *pushouts along regular monos are stable.*

Proof. Let $\mathcal{G}_i = (A_i, B_i, C_i, s_i, t_i, q_i)$, $\mathcal{G}'_i = (A'_i, B'_i, C'_i, s'_i, t'_i, q'_i)$, for $i \in \{1,2,3,4\}$, be hypergraphs with equivalence, and, in the diagram below, suppose all the vertical faces are pullbacks, the bottom face is a pushout and h is regular mono.

$$
\begin{tikzcd}[row sep=small, column sep=small]
& \mathcal{G}'_1 \arrow[rr, "h'"] \arrow[dl, "k'"'] \arrow[dd, "a" near end] & & \mathcal{G}'_2 \arrow[dl, "t'"] \arrow[dd, "b"] \\
\mathcal{G}'_3 \arrow[rr, "p'" near start, crossing over] \arrow[dd, "c"'] & & \mathcal{G}'_4 & \\
& \mathcal{G}_1 \arrow[rr, "h" near end] \arrow[dl, "k"] & & \mathcal{G}_2 \arrow[dl, "t"] \\
\mathcal{G}_3 \arrow[rr, "p"'] & & \mathcal{G}_4 \arrow[from=uu, crossing over, "d" near start] &
\end{tikzcd}
$$

By Proposition 4.6 and Corollary 4.7, the following cubes in **Set** have pushouts as bottom faces and pullbacks as vertical faces, hence their top faces are pushouts.

$$
\begin{tikzcd}[row sep=small, column sep=small]
& A'_1 \arrow[rr, "h'_E"] \arrow[dl, "k'_E"'] \arrow[dd, "a_E" near end] & & A'_2 \arrow[dl, "t'_E"] \arrow[dd, "b_E"] \\
A'_3 \arrow[rr, "p'_E" near start, crossing over] \arrow[dd, "c_E"'] & & A'_4 & \\
& A_1 \arrow[rr, "h_E" near end] \arrow[dl, "k_E"] & & A_2 \arrow[dl, "t_E"] \\
A_3 \arrow[rr, "p_E"'] & & A_4 \arrow[from=uu, crossing over, "d_E" near start] &
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small, column sep=small]
& B'_1 \arrow[rr, "h'_V"] \arrow[dl, "k'_V"'] \arrow[dd, "a_V" near end] & & B'_2 \arrow[dl, "t'_V"] \arrow[dd, "b_V"] \\
B'_3 \arrow[rr, "p'_V" near start, crossing over] \arrow[dd, "c_V"'] & & B'_4 & \\
& B_1 \arrow[rr, "h_V" near end] \arrow[dl, "k_V"] & & B_2 \arrow[dl, "t_V"] \\
B_3 \arrow[rr, "p_V"'] & & B_4 \arrow[from=uu, crossing over, "d_V" near start] &
\end{tikzcd}
$$

Consider now the following pullbacks.

$$
\begin{tikzcd}
Y \arrow[r, "y_1"] \arrow[d, "y_2"'] & C'_4 \arrow[d, "d_C"] \\
C_3 \arrow[r, "p_C"'] & C_4
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "u_1"] \arrow[d, "u_2"'] & C'_4 \arrow[d, "d_C"] \\
C_2 \arrow[r, "t_C"'] & C_4
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "x_1"] \arrow[d, "x_2"'] & U \arrow[d, "u_2"] \\
C_1 \arrow[r, "h_C"'] & C_2
\end{tikzcd}
$$

inserire in sezione 1 la proposizione

Thus, [???] yields the following situation, in which the bottom face is a pushout, and the vertical faces are pullbacks, hence the top face is a pushout too.

$$
\begin{tikzcd}[row sep=small, column sep=small]
& T \arrow[rr, "x_1"] \arrow[dl, dotted, "w"'] \arrow[dd, "x_2" near end] & & U \arrow[dl, "u_1"] \arrow[dd, "u_2"] \\
Y \arrow[rr, "y_1" near start, crossing over] \arrow[dd, "y_2"'] & & C'_4 & \\
& C_1 \arrow[rr, "h_C" near end] \arrow[dl, "k_C"] & & C_2 \arrow[dl, "t_C"] \\
C_3 \arrow[rr, "p_C"'] & & C_4 \arrow[from=uu, crossing over, "d_C" near start] &
\end{tikzcd}
$$

By the proof of Proposition 4.6, we have that $m_2 \circ q'_3 : B'_3 \to Y$ and $m_3 \circ q'_2 : B'_2 \to U$ are two epi-mono factorizations, with m_2 and m_3 monos. At the same way, let the following square to be a pullback.

$$\begin{tikzcd}
S \arrow[r, "s_1"] \arrow[d, "s_2"'] & C'_2 \arrow[d, "m_3"] \\
T \arrow[r, "x_1"'] & U
\end{tikzcd}$$

Hence, in the following diagram, the outer rectangle is a pullback.

$$\begin{tikzcd}
S \arrow[r, "s_1"] \arrow[d, "s_2"'] & C'_2 \arrow[d, "m_3"] \\
T \arrow[r, "x_1"'] \arrow[d, "x_2"'] & U \arrow[d, "u_2"] \\
C_1 \arrow[r, "h_C"'] & C_2
\end{tikzcd}$$

By the same argument as before, there exists a mono m_1 such that $m_1 \circ q'_1 : B'_1 \to S$.

We have to show that the top face of the cube ate the beginning of the proof is a pushout. Suppose then that $z : \mathcal{G}'_2 \to \mathcal{H}$ and $w : \mathcal{G}'_3 \to \mathcal{H}$, with $\mathcal{H} = (E, V, C, s, t, q)$, are two morphisms such that $z \circ h' = w \circ h'$, and let $v_V : B'_4 \to V$ the arrow induced by z_V and w_V. We want to construct the dotted arrow v_C which fits in the diagram below.

$$\begin{tikzcd}
& B'_1 \arrow[rr, "h'_V"] \arrow[dl, "k'_V"'] \arrow[dd, "q'_1"] & & B'_2 \arrow[dl, "t'_V"'] \arrow[dr, "z_V"] \arrow[dd, "q'_2"] & \\
B'_3 \arrow[rr, "p'_V"] \arrow[dd, "q'_3"'] & & B'_4 \arrow[rr, "v_V"] \arrow[dd, "q'_4"] & & V \arrow[dd, "q"] \\
& C'_1 \arrow[rr, "h'_C"] \arrow[dl, "k'_C"'] & & C'_2 \arrow[dl, "t'_C"'] \arrow[dr, "z_C"] & \\
C'_3 \arrow[rr, "p'_C"'] & & C'_4 \arrow[rr, dotted, "v_C"'] & & C
\end{tikzcd}$$

By Lemma 2.12, we know that the top face of the cube below is a pushout.

$$\begin{tikzcd}
& K_{s_2 \circ m_1 \circ q'_1} \arrow[rr, "k_{h'_2}"] \arrow[dl, "k_{k'_2}"'] \arrow[dd, "\pi^1_{s_2 \circ m_1 \circ q'_1}"] & & K_{m_2 \circ q'_2} \arrow[dl, "k_{t'_2}"'] \arrow[dd, "\pi^1_{m_2 \circ q'_2}"] \\
K_{m_3 \circ q'_3} \arrow[rr, "k_{p'_2}"] \arrow[dd, "\pi^1_{m_3 \circ q'_3}"'] & & K_{q'_4} \arrow[dd, "\pi^1_{q'_4}"] & \\
& B'_1 \arrow[rr, "h'_2"] \arrow[dl, "k'_2"'] & & B'_2 \arrow[dl, "t'_2"] \\
B'_3 \arrow[rr, "p'_2"'] & & C_4 &
\end{tikzcd}$$

And, since m_3 and m_2 are monos,

$$q'_3 \circ \pi^1_{m_3 \circ q'_3} = q'_3 \circ \pi^2_{m_3 \circ q'_3} \qquad q'_2 \circ \pi^1_{m_2 \circ q'_2} = q'_2 \circ \pi^2_{m_2 \circ q'_2}$$

Computing, we obtain

$$\begin{aligned} q\circ v_V\circ\pi^1_{q_4'}\circ k_{p_V'} &= q\circ v_V\circ p_V'\circ\pi^1_{m_3\circ q_3'} \\ &= q\circ w_V\circ\pi^1_{m_3\circ q_3'} \\ &= w_C\circ q_3'\circ\pi^1_{m_3\circ q_3'} \\ &= w_C\circ q_3'\circ\pi^2_{m_3\circ q_3} \\ &= q\circ w_V\circ\pi^2_{m_3\circ q_3'} \\ &= q\circ v_V\circ p_V'\circ\pi^2_{m_3\circ q_3'} \\ &= q\circ v_V\circ\pi^2_{q_4'}\circ k_{p_V'} \end{aligned} \qquad \begin{aligned} q\circ v_V\circ\pi^1_{q_4'}\circ k_{t_V'} &= q\circ v_V\circ t_V'\circ\pi^1_{m_2\circ q_2'} \\ &= q\circ z_V\circ\pi^1_{m_2\circ q_2'} \\ &= z_C\circ q_2'\circ\pi^1_{m_2\circ q_2'} \\ &= z_C\circ q_2'\circ\pi^2_{m_2\circ q_2} \\ &= q\circ z_V\circ\pi^2_{m_2\circ q_2'} \\ &= q\circ v_V\circ t_V'\circ\pi^2_{m_2\circ q_2'} \\ &= q\circ v_V\circ\pi^2_{q_4'}\circ k_{t_V'} \end{aligned}$$

Since the previous cube has a pushout as top face, by universal property, we have

$$q\circ v_V\circ\pi^1_{q_4'} = q\circ v_V\circ\pi^2_{q_4'}$$

hence, v_C is the mediating arrow.

$$v_C\circ q_4'\circ\pi^1_{q_4'} = v_C\circ q_4'\circ\pi^2_{q_4'}$$

◀

▶ **Lemma 4.17.** *In* **EqHyp***, pushouts along regular monos are* Reg(**EqHyp**)*-Van Kampen.*

Proof. In lieu of Lemma 4.16, it is enough to proof that, given a cube as the one below, with pullbacks as back faces, pushouts as bottom and top faces and such that h is a regular mono, the front faces are pullbacks too, where $\mathcal{G}_i = (A_i, B_i, C_i, s_i, t_i, q_i)$, $\mathcal{G}' = (A_i', B_i', C_i', s_i', t_i', q_i')$, for $i = 1, 2, 3, 4$.

$$\begin{tikzcd}
& \mathcal{G}_1' \arrow[rr, "h'"] \arrow[dl, "k'"'] \arrow[dd, "a"] & & \mathcal{G}_2' \arrow[dl, "t'"'] \arrow[dd, "b"] \\
\mathcal{G}_3' \arrow[rr, "p'", crossing over] \arrow[dd, "c"'] & & \mathcal{G}_4' & \\
& \mathcal{G}_1 \arrow[rr, "h"] \arrow[dl, "k"] & & \mathcal{G}_2 \arrow[dl, "t"] \\
\mathcal{G}_3 \arrow[rr, "p"'] & & \mathcal{G}_4 \arrow[from=uu, "d", crossing over] &
\end{tikzcd}$$

By Proposition 4.6 and **??**, the following two cubes have $\mathcal{M}$-pushouts as bottom faces and pullbacks as back faces, thus their front faces are pullbacks too.

$$\begin{tikzcd}
& A_1' \arrow[rr, "h_E'"] \arrow[dl, "k_E'"'] \arrow[dd, "a_E"] & & A_2' \arrow[dl, "t_E'"'] \arrow[dd, "b_E"] \\
A_3' \arrow[rr, "p_E'", crossing over] \arrow[dd, "c_E"'] & & A_4' & \\
& A_1 \arrow[rr, "h_E"] \arrow[dl, "k_E"] & & A_2 \arrow[dl, "t_E"] \\
A_3 \arrow[rr, "p_E"'] & & A_4 \arrow[from=uu, "d_E", crossing over] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& B_1' \arrow[rr, "h_V'"] \arrow[dl, "k_V'"'] \arrow[dd, "a_V"] & & B_2' \arrow[dl, "t_V'"'] \arrow[dd, "b_V"] \\
B_3' \arrow[rr, "p_V'", crossing over] \arrow[dd, "c_V"'] & & B_4' & \\
& B_1 \arrow[rr, "h_V"] \arrow[dl, "k_V"] & & B_2 \arrow[dl, "t_V"] \\
B_3 \arrow[rr, "p_V"'] & & B_4 \arrow[from=uu, "d_V", crossing over] &
\end{tikzcd}$$

On the other hand we can consider the diagrams below, in which the inner squares are pullbacks. Since the outer diagrams commute, by definition of morphism of **EqHyp**, then we have the existence of $m_2\colon C'_2 \to U$, $m_3\colon C'_3 \to Y$, $a_3\colon B'_3 \to Y$ and $a_2\colon B'_2 \to Y$.

$$
\begin{tikzcd}
C'_3 \arrow[dr, dotted, "m_3"] \arrow[drr, bend left, "p'_3"] \arrow[ddr, bend right, "d_3"'] & & \\
& Y \arrow[r, "y_1"'] \arrow[d, "y_2"'] & C'_4 \arrow[d, "d_3"] \\
& C_3 \arrow[r, "p_3"'] & C_4
\end{tikzcd}
\qquad
\begin{tikzcd}
C'_2 \arrow[dr, dotted, "m_2"] \arrow[drr, bend left, "t'_3"] \arrow[ddr, bend right, "d_2"'] & & \\
& U \arrow[r, "u_1"'] \arrow[d, "u_2"'] & C'_4 \arrow[d, "d_3"] \\
& C_2 \arrow[r, "t_3"'] & C_4
\end{tikzcd}
$$

$$
\begin{tikzcd}
B'_3 \arrow[r, "p'_2"] \arrow[d, "q'_3"'] \arrow[dr, dotted, "a_3"] & B'_4 \arrow[dr, "q'_4"] & \\
C'_3 \arrow[dr, "d_3"'] & Y \arrow[r, "y_1"'] \arrow[d, "y_2"'] & C'_4 \arrow[d, "d_3"] \\
& C_3 \arrow[r, "p_3"'] & C_4
\end{tikzcd}
\qquad
\begin{tikzcd}
B'_2 \arrow[r, "t'_2"] \arrow[d, "q'_2"'] \arrow[dr, dotted, "a_2"] & B'_4 \arrow[dr, "q'_4"] & \\
C'_2 \arrow[dr, "d_2"'] & U \arrow[r, "u_1"'] \arrow[d, "u_2"'] & C'_4 \arrow[d, "d_3"] \\
& C_2 \arrow[r, "t_3"'] & C_4
\end{tikzcd}
$$

Now, notice that m_3 and m_2 are monos because d_3 and d_2 are regular monos. By the proof of Proposition 4.6, to conclude it is enough to show that

$$m_3 \circ q'_3 = a_3 \qquad m_2 \circ q'_2 = a_2$$

Indeed, if the previous equations hold, then C'_3 and C'_2 are epi-mono factorizations of a_3 and a_2 and the thesis follows from **??** and the proof of Proposition 4.6.

No if we compute we have:

$$
\begin{aligned}
y_1 \circ a_3 &= q'_4 \circ p'_2 & \qquad u_1 \circ a_2 &= q'_4 \circ t'_2 \\
&= p'_3 \circ q'_3 & &= t'_3 \circ q'_3 \\
&= y_1 \circ m_3 \circ q'_3 & &= u_1 \circ m_2 \circ q'_2
\end{aligned}
$$

$$
\begin{aligned}
y_2 \circ a_3 &= d_3 \circ q'_3 & \qquad u_2 \circ a_2 &= d_2 \circ q'_2 \\
&= y_2 \circ m_3 \circ q'_3 & &= u_2 \circ m_2 \circ q'_2
\end{aligned}
$$

And we have done. ◀

4.1 Labeled Hypergraphs with Equivalences

As we have done in Section 3.1.2, we can define the category of hypergraphs with equivalence labeled over an algebraic signature.

▶ **Definition 4.18.** *Let* $\Sigma = (O_\Sigma, \mathsf{ar}_\Sigma)$ *be an algebraic signature and* $\mathcal{G}^\Sigma$ *the hypergraph associated to* Σ*. Then the* hypergraph with equivalence *associated to* Σ *is* $L(\mathcal{G}^\Sigma)$ *and the category of hypergraphs with equivalence labeled over* Σ *is the slice category* $\mathbf{EqHyp}_\Sigma = \mathbf{EqHyp}/L(\mathcal{G}^\Sigma)$.

By **??**, we can deduce the following.

▶ **Proposition 4.19.** $\mathbf{EqHyp}_\Sigma$ *is* $\mathsf{Reg}(\mathbf{EqHyp}_\Sigma)$*-adhesive.*

We can lift the adjunction given by T and L to $EqHyp_\Sigma$ and $\mathbf{Hyp}_\Sigma$. Non capisco se è uno svarione mio o ha senso

By Corollary 4.7, we can deduce what follows.

▶ **Proposition 4.20.** *A morphism h between two objects of* $\mathbf{EqHyp}_\Sigma$ *is mono if and only if* $T(h)$ *is mono.*

4.1.1 Term Graphs with Equivalences

▶ **Definition 4.21.** *Let Σ be an algebraic signature. A labeled hypergraph with equivalence $l : \mathcal{G} \to L(\mathcal{G}^\Sigma)$ is a* term graph with equivalence *if $t_\mathcal{G}$ is mono. We define category of term graphs with equivalence over Σ, denoted $\mathbf{EqTG}_\Sigma$, as the full subcategory of $\mathbf{EqHyp}_\Sigma$, and the corresponding inclusion functor $I_{\mathbf{EqTG}_\Sigma}$.*

▶ **Proposition 4.22.** *If $l : \mathcal{G} \to \mathcal{G}^\Sigma$ is a term graph, then $L(l)$ is a term graph with equivalence.*

5 EGGs

introduction

▶ **Definition 5.1.** *Let $\mathcal{G} = (E, V, C, s, t, q)$ be a hypergraph with equivalence and (S, π_1, π_2) a kernel pair of $q^\star \circ s$. Then, $\mathcal{G}$ is an* e-hypergraph *whenever $q^\star \circ t \circ \pi_1 = q^\star \circ t \circ \pi_2$. $\mathbf{EGG}$ is the full subcategory of $\mathbf{EqHyp}$ whose objects are e-hypergraphs, and $I : \mathbf{EGG} \to \mathbf{EqHyp}$ is the inclusion functor.*

▶ **Lemma 5.2.** *$\mathbf{EGG}$ has all limits and I preserves them.*

Proof. Let $D : \mathbf{I} \to \mathbf{EGG}$ be a diagram, with $D(i) = (A_i, B_i, C_i, s_i, t_i, q_i)$, let (U_i, u_1^i, u_2^i) be the kernel pair of $q_i \circ s_i$. Let now be (A, B, C, s, t, q), togehter with projections $(\pi_E^i, \pi_V^i, \pi_C^i)_{i \in \mathbf{I}}$ the limit of $I \circ D$, let (U, u_1, u_2) be the kernel pair of $q \circ s$ and let $(L, (l_i)_{i \in \mathbf{I}})$ be the limit of $K \circ I \circ D$. By construction (proof of Proposition 4.6), there exists a mono $m : C \to L$ such that $\pi_C^i = l_i \circ m$. Notice that

$$
\begin{aligned}
q_i^\star \circ s_i \circ \pi_E^i \circ u_1 &= q_i^\star \circ (\pi_V^i)^\star \circ s \circ u_1 \\
&= (\pi_C^i)^\star \circ q^\star \circ s \circ u_1 \\
&= (\pi_C^i)^\star \circ q^\star \circ s \circ u_2 \\
&= q_i^\star \circ s_i \circ \pi_E^i \circ u_2
\end{aligned}
$$

Then, for each i, there exists an arrow $a_i : U \to U_i$ making the following diagram to commute

$$
\begin{tikzcd}
U \arrow[r, "u_1"] \arrow[d, "u_2"'] \arrow[dr, dotted, "a_i"'] & A \arrow[dr, "\pi_E^i"] & \\
A \arrow[dr, "\pi_E^i"'] & U_i \arrow[r, "u_1^i"] \arrow[d, "u_2^i"] & A_i \arrow[d, "q_i^\star \circ s_i"] \\
& A_i \arrow[r, "q_i^\star \circ s_i"'] & C_i^\star
\end{tikzcd}
$$

We have then

$$
\begin{aligned}
l_i^\star \circ m^\star \circ q^\star \circ t \circ u_1 &= q_i^\star \circ (\pi_V^i)^\star \circ t \circ u_1 \\
&= q_i^\star \circ t_i \circ \pi_E^i \circ u_1 \\
&= q_i^\star \circ t_i \circ u_1^i \circ a_i \\
&= q_i^\star \circ t_i \circ u_2^i \circ a_i \\
&= q_i^\star \circ t_i \circ \pi_E^i \circ u_2 \\
&= q_i^\star \circ (\pi_V^i)^\star \circ t \circ u_2 \\
&= l_i^\star \circ m^\star \circ q^\star \circ t \circ u_2
\end{aligned}
$$

By universal property of limits, we have that $m^\star \circ q^\star \circ t \circ u_1 = m^\star \circ q^\star \circ t \circ u_2$, and, since m is mono, $q^\star \circ t \circ u_1 = q^\star \circ t \circ u_2$, hence the thesis. ◀

▶ **Corollary 5.3.** *I creates limits.*

▶ **Corollary 5.4.** *Let $h : \mathcal{G} \to \mathcal{H}$ be an arrow in* **EGG***. Then it is a regular mono if and only if $I(h)$ is a regular mono.*

▶ **Lemma 5.5.** *Consider the following pushout square in* **EqHyp***.*

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[r, "h"] \arrow[d, "m"'] & \mathcal{G}_2 \arrow[d, "n"] \\
\mathcal{G}_3 \arrow[r, "k"'] & \mathcal{P}
\end{tikzcd}$$

with m regular mono. If $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are e-hypergraphs, then $\mathcal{P}$ is an e-hypergraph too, and n is regular mono.

Proof. Let $\mathcal{P} = (A, B, C, s, t, q)$, (K_i, π_i^1, π_i^2) the kernel pair of $q_i^\star \circ s_i$, and let (U, u_1, u_2) the kernel pair of $q^\star \circ s$. Consider then the following situation.

$$\begin{tikzcd}
& A_1 \arrow[rr, "h_E"] \arrow[dl, "m_E"'] \arrow[dd, "q_1^\star \circ s_1"' near end] & & A_2 \arrow[dl, "k_E"'] \arrow[dd, "q_2^\star \circ s_2"] \\
A_3 \arrow[rr, "k_E"] \arrow[dd, "q_3^\star \circ s_3"'] & & A \arrow[dd, "q^\star \circ s" near start] & \\
& C_1^\star \arrow[rr, "h_C^\star"] \arrow[dl] & & C_2^\star \arrow[dl, "n_C^\star"] \\
C_3^\star \arrow[rr, "m_C^\star", "k_C^\star"'] & & C^\star &
\end{tikzcd}$$

Since m is regular mono, m_E is mono (inserire citazione). Then, by construction, the top face is a pushout, and since **Set** is adhesive, by Lemma 2.12 , the square below is a pushout.

$$\begin{tikzcd}
K_1 \arrow[r, "f_k"] \arrow[d, "f_m"'] & K_2 \arrow[d, "f_n"] \\
K_3 \arrow[r, "f_k"'] & U
\end{tikzcd}$$

Computing, we have

$$\begin{aligned}
q^\star \circ t \circ u_1 \circ f_n &= q^\star \circ t \circ n_E \circ \pi_2^1 \\
&= n_C^\star \circ q_2^\star \circ s_2 \circ \pi_2^1 \\
&= n_C^\star \circ q_2^\star \circ s_2 \circ \pi_2^2 \\
&= q^\star \circ t \circ u_2 \circ f_n
\end{aligned}
\qquad
\begin{aligned}
q^\star \circ t \circ u_1 \circ f_k &= q^\star \circ t \circ k_E \circ \pi_3^1 \\
&= k_C^\star \circ q_3^\star \circ s_3 \circ \pi_3^1 \\
&= k_C^\star \circ q_3^\star \circ s_3 \circ \pi_3^2 \\
&= q^\star \circ t \circ u_2 \circ f_k
\end{aligned}$$

By universal property of pushouts, we deduce $q^\star \circ t \circ u_1 = q^\star \circ t \circ u_2$, and the thesis follows. ◀

By direct application of Proposition 2.4, we can conclude what follows.

▶ **Corollary 5.6.** **EGG** *is* Reg(**EGG**)*-adhesive.*

6 Conclusions and further works

The aim of our paper was to extend the theory of adhesive categories in order to include EGGS, an up-and-coming formalism for program optimisation and synthesis via a compact representation and efficient implementation of equality saturation. To do so, we revisited and generalised the notions of hyper-graph and term graph with equivalence, and we extended it in order to capture EGGS as term graphs satisfying a suitable closure property.

Our result opens two threads of research. The first is to use the quasi-adhesivity of EGGs to model their rewriting via the double-pushout (DPO) approach. This seems now easy, since the rules adopted in the literature of EGGs appears to be span of regular monos, and such rules perfectly fit the mold of rewriting on $\mathcal{M}$-adhesive categories. For example, the equivalence $x \div x = 1$, from the introductory example in [?], can be modelled as the rule

$$todrawDPOrule$$

It still needs to be investigated what parallelism and termination, the key properties for DPO rewriting on adhesive categories, mean in the context of EGGs. More interestingly, another venue for development is using the adhesive machinery to extend the EGGs formalism. In fact, most of the results presented here for hyper-graphs can be generalised to hierarchical hyper-graphs, that is, hypergraphs with a hierarchy (a partial order) among edges that is useful for adding structural information, such as encapsulation and sandboxing [?].

Finally, we need to draw a comparison with an alternative categorical presentation for EGGs advanced in [14]. The proposal is quite different from our own. Simplifying, the key is to equip categories of trees with a lattice on hom-sets. It seems that such proposal generalises our own, even if at the expenses of a more complex machinery.

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A Omitted proofs

This section contains the proofs which are omitted from the main body oof the paper. We begin stating a well-known fact about composition and decomposition of pullbacks [18].

▶ **Lemma A.1.** *Let* $\mathbf{X}$ *be a category, and consider the diagram aside, in which the right square is a pullback. Then the whole rectangle is a pullback if and only if the left square is one.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[d, "c"] \\ A \arrow[r, "h"'] & B \arrow[r, "k"'] & C \end{tikzcd}$$

Proof. ($\Rightarrow$) Let $q_1 : Q \to Y$ and $q_2 : Q \to A$ be two arrows such that $b \circ q_1 = h \circ q_2$, if we compute we get

$$c \circ g \circ q_1 = k \circ b \circ q_1 = k \circ h \circ q_2$$

Thus by the pullback property of the whole rectangle we get the dotted l in the diagram on the side. All we have to prove is that $f \circ l = q_1$. By construction we know that $g \circ f \circ l = g \circ q_1$, while we also have

$$\begin{tikzcd} Q \arrow[r, dotted, "l"'] \arrow[rrr, bend left, "g \circ q_1"] \arrow[dr, bend right, "q_2"'] & X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[d, "c"] \\ & A \arrow[r, "h"'] & B \arrow[r, "k"'] & C \end{tikzcd}$$

$$b \circ f \circ l = h \circ a \circ l = h \circ q_2 = b \circ q_1$$

and we can conclude since the right square in the original diagram is a pullback.

For uniqueness: if $l' : Q \to X$ is such that $f \circ l' = q_1$ and $a \circ l' = q_2$ then $g \circ f \circ l' = g \circ q_1$ and we can conclude applying the pullback property of the outer rectangle.

($\Leftarrow$) Take two arrows $q_1 : Q \to Z$ and $q_2 : Q \to A$ such that $c \circ q_1 = k \circ h \circ q_2$ We can apply the pullback property of the right square to get the dotted $q : Q \to Y$ in the following Now, by construction we have $b \circ q = h \circ q_2$ and thus, since the left square is a pullback, we get also a unique $l : Q \to X$ such that $f \circ l = q$ and $a \circ l = q_2$ but then we clearly have

$$\begin{tikzcd} Q \arrow[dr, dotted, "l"] \arrow[drr, dotted, "q"] \arrow[drrr, bend left, "q_1"] \arrow[ddr, bend right, "q_2"'] & & & \\ & X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"'] & Z \arrow[d, "c"] \\ & A \arrow[r, "h"'] & B \arrow[r, "k"'] & C \end{tikzcd}$$

$$g \circ f \circ l = g \circ q = q_1$$

We are left with uniqueness. Let $l' \colon Q \to X$ be another arrow such that $q_1 = g \circ f \circ l'$ and $q_2 = a \circ l'$, then we must also have

$$b \circ f \circ l' = h \circ a \circ l' = h \circ q_2 = b \circ q$$

which implies $f \circ l' = q$, from which $l = l'$ follows. ◀

A.1 Proofs for Section 2

▶ **Proposition 2.6.** *If $\mathbf{X}$ is $\mathcal{M}$-adhesive then it holds*

1. *every $\mathcal{M}$-pushout square is also a pullback;*
2. *every arrow in $\mathcal{M}$ is a regular mono.*

Proof. 1. Consider the following cube in which the bottom face is an $\mathcal{M}$-pushout.

$$\begin{tikzcd}
& A \arrow[rr, "g"] \arrow[dl, "\mathsf{id}_A"'] \arrow[dd, "\mathsf{id}_A"' near start] & & B \arrow[dl, "\mathsf{id}_B"'] \arrow[dd, "\mathsf{id}_B"] \\
A \arrow[rr, "g" near start] \arrow[dd, "m"'] & & B \arrow[dd, "n"' near start] & \\
& A \arrow[rr, "g" near end] \arrow[dl, tail, "m"] & & B \arrow[dl, tail, "n"] \\
C \arrow[rr, "f"'] & & D &
\end{tikzcd}$$

By construction the top face of the cube is a pushout and the back one a pullback. The left face is a pullback because m is mono, thus the Van Kampen property yields that the front and the right faces are pullbacks too and the thesis follows.

2. Let $m \colon X \rightarrowtail Y$ be an arrow in $\mathcal{M}$, we can then take its pushout along itself, which, by the previous point, is also a pullback.

$$\begin{tikzcd}
X \arrow[r, tail, "m"] \arrow[d, tail, "m"'] & Y \arrow[d, tail, "h"] \\
Y \arrow[r, tail, "k"] & Z
\end{tikzcd}$$

It is now immediate to see that m is the equalizer of h and k. ◀

▶ **Lemma 2.11.** *Let $f \colon X \to Y$ and $g \colon Z \to W$ be two arrows admitting kernel pairs and suppose that the solid part of the three squares below is given. Then there exists a a unique arrow $k_h : K_f \to K_g$ completing them. Moreover, if the leftmost is a pullback, then also the other two are so.*

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\
Y \arrow[r, "t"'] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
K_f \arrow[r, dotted, "k_h"] \arrow[d, "\pi_f^1"'] & K_g \arrow[d, "\pi_g^1"] \\
X \arrow[r, "h"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
K_f \arrow[r, dotted, "k_h"] \arrow[d, "\pi_f^2"'] & K_g \arrow[d, "\pi_g^2"] \\
X \arrow[r, "h"'] & Z
\end{tikzcd}$$

Proof. Computing, we have

$$g \circ h \circ \pi_f^1 = t \circ f \circ \pi_f^1 = t \circ f \circ \pi_f^2 = g \circ h \circ \pi_f^2$$

Therefore the existence and uniqueness of the wanted k_h follows at once from the the universal property of K_g as the pullback of g along itself.

To prove the second half of the thesis, we can notice that, by Lemma A.1, two rectangles below are pullbacks.

$$\begin{tikzcd}
K_f \arrow[r, "\pi_f^2"] \arrow[d, "\pi_f^1"'] & X \arrow[r, "h"] \arrow[d, "f"] & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y \arrow[r, "t"'] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
K_f \arrow[r, "\pi_f^1"] \arrow[d, "\pi_f^2"'] & X \arrow[r, "h"] \arrow[d, "f"] & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y \arrow[r, "t"'] & W
\end{tikzcd}$$

But then the following ones are pullbacks too.

$$\begin{tikzcd}
K_f \arrow[r, "k_h"'] \arrow[rr, bend left, "h\circ\pi_f^2"] \arrow[d, "\pi_f^1"'] & K_g \arrow[r, "\pi_g^2"'] \arrow[d, "\pi_g^1"'] & Z \arrow[d, "g"'] \\
X \arrow[r, "h"] \arrow[rr, bend right, "t\circ f"'] & Y \arrow[r, "g"] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
K_f \arrow[r, "k_h"'] \arrow[rr, bend left, "h\circ\pi_f^2"] \arrow[d, "\pi_f^1"'] & K_g \arrow[r, "\pi_g^2"'] \arrow[d, "\pi_g^1"'] & Z \arrow[d, "g"'] \\
X \arrow[r, "h"] \arrow[rr, bend right, "t\circ f"'] & Y \arrow[r, "g"] & W
\end{tikzcd}$$

The thesis now follows again by Lemma A.1. ◀

▶ **Lemma 2.12.** *Let $\mathbf{X}$ be a strict $\mathcal{M}$-adhesive category with all pullbacks, and suppose that in the cube aside the top face is an $\mathcal{M}$-pushout. Then the right square is a pushout.*

$$\begin{tikzcd}
& A' \arrow[rr, "f'"] \arrow[dl, "m'"'] \arrow[dd, "a", near start] & & B' \arrow[dl, "n'"'] \arrow[dd, "b"] \\
C' \arrow[rr, "g'", near start, crossing over] \arrow[dd, "c"'] & & D' & \\
& A \arrow[rr, "f", near start] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\
C \arrow[rr, "g"'] & & D \arrow[from=uu, "d", near start, crossing over] &
\end{tikzcd}
\qquad
\begin{tikzcd}
K_a \arrow[r, "k_{f'}"] \arrow[d, "k_{m'}"'] & K_b \arrow[d, "k_{n'}"] \\
K_c \arrow[r, "k_{g'}"'] & K_d
\end{tikzcd}$$

$$\begin{tikzcd}
& K_a \arrow[rr, "k_{f'}"] \arrow[dl, "k_{m'}"'] \arrow[dd, "\pi_a^1", near start] & & K_n \arrow[dl, "k_{n'}"'] \arrow[dd, "\pi_b^1"] \\
K_c \arrow[rr, "k_{g'}", near start, crossing over] \arrow[dd, "\pi_c^1"'] & & K_d & \\
& A' \arrow[rr, "f'", near start] \arrow[dl, "m'"] & & B' \arrow[dl, "n'"] \\
C' \arrow[rr, "g'"'] & & D' \arrow[from=uu, "\pi_d^1", near start, crossing over] &
\end{tikzcd}$$

Proof. By Proposition 2.6 we know that the the top face of the original cube is a pullback. Thus Lemma 2.11 entails that in the following cube the vertical faces are pullbacks. The thesis now follows from strict $\mathcal{M}$-adhesivity. ◀

▶ **Proposition 2.13.** *Let $e \colon X \to Y$ be a regular epi in a category $\mathbf{X}$ with a kernel pair (K_e, π_e^1, π_e^2). Then, e is the coequalizer of π_e^1 and π_e^2.*

$$\begin{tikzcd}
Z \arrow[r, dotted, "k"'] \arrow[rr, bend left, "f"] \arrow[dr, bend right, "g"'] & K_e \arrow[r, "\pi_e^1"'] \arrow[d, "\pi_e^2"'] & X \arrow[d, "e"] \\
& X \arrow[r, "e"'] & Y
\end{tikzcd}$$

Proof. By hypothesis, there exists a pair $f, g \colon Z \rightrightarrows X$ of which e is the coequalizer. Since $e \circ f = e \circ g$ we get the dotted arrow fitting in the digram aside.
Let now $h : Z \to V$ be an arrow such that $h \circ \pi_1 = h \circ \pi_2$, then

$$h \circ f = h \circ \pi_1 \circ k = h \circ \pi_2 \circ k = h \circ g$$

and thus there exists a unique $l : Y \to V$ such that $l \circ e = h$. ◀

▶ **Corollary 2.14.** *Let $\mathbf{X}$ be a category with pullbacks and $\phi \colon F \dot{\to} G$ a natural transformation between functors $F, G : \mathbf{D} \rightrightarrows \mathbf{X}$. If ϕ_d is a regular epi for every d in $\mathbf{D}$, then ϕ is a regular epi.*

Proof. Let (K_i, π^1_d, π^2_d) be the kernel pair of ϕ_d for each object d in $\mathbf{D}$. Given an arrow $\alpha : d \to d'$ of $\mathbf{D}$, we have

$$\phi_{d'} \circ F(\alpha) \circ \pi^1_d = G(\alpha) \circ \phi_d \circ \pi^1_d = G(\alpha) \circ \phi_d \circ \pi^2_d = \phi_{d'} \circ F(\alpha) \circ \pi^2_d$$

Thus, the solid part of the diagram below commutes, yielding the dotted arrow $K(\alpha)$.

$$\begin{tikzcd}
K_d \arrow[r, "\pi^1_d"] \arrow[d, "\pi^2_d"'] \arrow[drr, dotted, "K(\alpha)"] & F(d) \arrow[drr, "F(\alpha)"] & & \\
F(d) \arrow[drr, "F(\alpha)"'] & & K_{d'} \arrow[r, "\pi^1_{d'}"] \arrow[d, "\pi^2_{d'}"'] & F(d') \arrow[d, "\phi_{d'}"] \\
& & F(d') \arrow[r, "\phi_{d'}"'] & G(d')
\end{tikzcd}$$

In this way, we get a functor $K\colon \mathbf{D} \to \mathbf{X}$ mapping d to K_d and each arrow α onto $K(\alpha)$. Indeed, notice that $K(\mathsf{id}_d)\colon K_d \to K_d$ is the arrow such that

$$\pi^1_d \circ K(\mathsf{id}_d) = F(\mathsf{id}_d) \circ \pi^1_d = \mathsf{id}_{F(d)} \circ \pi^1_d = \pi^1_d$$
$$\pi^2_d \circ K(\mathsf{id}_d) = F(\mathsf{id}_d) \circ \pi^2_d = \mathsf{id}_{F(d)} \circ \pi^2_d = \pi^2_d$$

Thus, by the universal property of pullbacks, $K(\mathsf{id}_d) = \mathsf{id}_{K_d}$.

Let now $\alpha\colon a \to b$ and $\beta\colon b \to c$ be two arrows in $\mathbf{D}$, computing, we have

$$\pi^1_c \circ K(\beta \circ \alpha) = F(\beta) \circ F(\alpha) \circ \pi^1_a = F(\beta) \circ \pi^1_b \circ K(\alpha) = \pi^1_c \circ K(\beta) \circ K(\alpha)$$
$$\pi^2_c \circ K(\beta \circ \alpha) = F(\beta) \circ F(\alpha) \circ \pi^2_a = F(\beta) \circ \pi^2_b \circ K(\alpha) = \pi^2_c \circ K(\beta) \circ K(\alpha)$$

Allowing us to conclude that $K(\beta \circ \alpha) = K(\beta) \circ K(\alpha)$, proving the functoriality of K.

Hence, by construction we have two natural transformations $\pi^1, \pi^2\colon K \rightrightarrows F$. By Proposition 2.13, every component ϕ_d is the coequalizer of $\pi^1_d, \pi^2_d\colon K(d) \rightrightarrows F$, and so ϕ is the coequalizer of π^1 and π^2. ◀

▶ **Lemma 2.15.** *Let $F, G\colon \mathbf{D} \rightrightarrows \mathbf{X}$ be two diagrams, and suppose that $\mathbf{X}$ has all colimits of shape $\mathbf{D}$. Let $(X, \{x_d\}_{d\in\mathbf{D}})$ and $(Y, \{y_d\}_{d\in D})$ be the colimits of F and G, respectively. If $\phi\colon F \to G$ is a natural transformation whose components are regular epis, then the arrow induced by ϕ from X to Y is a regular epi.*

Proof. By Corollary 2.14, we know that $\phi\colon F \to G$ is a regular epi, so that there is a functor $E\colon \mathbf{D} \to \mathbf{X}$ and $\eta, \theta\colon E \rightrightarrows F$ such that ϕ is the coequalizer of η and θ. Let now $(P, \{p_d\}_{d\in\mathbf{D}})$ be the colimit of E and consider the unique arrows $a, b\colon P \rightrightarrows X$ fitting in the squares below

$$\begin{tikzcd}
E(d) \arrow[r, "p_d"] \arrow[d, "\eta_d"'] & P \arrow[d, dotted, "a"] \\
F(d) \arrow[r, "x_d"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
E(d) \arrow[r, "p_d"] \arrow[d, "\theta_d"'] & P \arrow[d, dotted, "b"] \\
F(d) \arrow[r, "x_d"'] & X
\end{tikzcd}$$

We want to show that ϕ coequalizes a and n. Let thus $h\colon X \to Z$ be an arrow such that $h \circ a = h \circ b$. Then, for every d, we have

$$h \circ x_d \circ \eta_d = h \circ a \circ p_d = h \circ b \circ p_d = h \circ x_d \circ \theta_d$$

Thus, there is $h_d : G(d) \to Z$ such that $h \circ x_d = h_d \circ \phi_d$. It is now easy to see that $(Z, \{h_d\}_{d\in\mathbf{D}})$ is a cocone on G. Suppose $\alpha : d \to d'$ is an arrow of $\mathbf{D}$, then

$$h_d \circ \phi_d = h \circ y_d = h \circ y_{d'} \circ F(\alpha) = h_{d'} \circ \phi_{d'} \circ F(\alpha) = h_{d'} \circ G(\alpha) \circ \phi_d$$

By the hypothesis ϕ_d is regular epi for each d and so we can conclude that $h_d = h_{d'} \circ G(\alpha)$.

Therefore, we have an arrow $k\colon Y \to Z$ such that $k \circ y_d = h_d$. But then

$$k \circ \phi \circ x_d = k \circ y_d \circ \phi_d = h_d \circ \phi_d = h \circ x_d$$

Showing that $k \circ \phi = h$.

For the uniqueness, let $k' : Y \to Z$ be another arrow such that $k' \circ \phi = h$. Then we have

$$k' \circ y_d \circ \phi_d = k' \circ \phi \circ x_d = h \circ x_d = h_d \circ \phi_d$$

Since ϕ_d is a regular epi, we have $k' \circ y_d = h_d$ allowing us to conclude. ◀

B Some properties of comma categories

In this section we will briefly recall the definition of the comma category [18] associated to two functors and some of its properties.

▶ **Definition B.1.** *Let $L\colon \mathbf{A}\to\mathbf{X}$ and $R\colon \mathbf{B}\to\mathbf{X}$ be two functors with the same codomain, the* comma category *$L{\downarrow}R$ is the category in which*

- *objects are triples (A,B,f) with $A\in\mathbf{A}$, $B\in\mathbf{B}$, and $f\colon L(A)\to R(B)$;*
- *a morphism $(A,B,f)\to(A',B',g)$ is a pair (h,k) with $h\colon A\to A'$ in $\mathbf{A}$ and $k\colon B\to B'$ in $\mathbf{B}$ such that the following diagram commutes*

$$\begin{tikzcd} L(A) \arrow[r, "L(h)"] \arrow[d, "f"'] & L(A') \arrow[d, "g"] \\ R(B) \arrow[r, "R(k)"'] & R(B') \end{tikzcd}$$

We have two forgetful functors $U_L\colon L{\downarrow}R\to\mathbf{A}$ and $U_R\colon L{\downarrow}R\to\mathbf{B}$ given, respectively by

$$\begin{tikzcd} (A,B,f) \arrow[r, mapsto] \arrow[d, "{(h,k)}"'] & A \arrow[d, "h"] \\ (A',B',g) \arrow[r, mapsto] & A' \end{tikzcd} \qquad \begin{tikzcd} (A,B,f) \arrow[r, mapsto] \arrow[d, "{(h,k)}"'] & B \arrow[d, "k"] \\ (A',B',g) \arrow[r, mapsto] & B' \end{tikzcd}$$

Given $L\colon\mathbf{A}\to\mathbf{X}$ and $R\colon\mathbf{B}\to\mathbf{X}$, we can also consider their duals $L^{op}\colon\mathbf{A}^{op}\to\mathbf{X}^{op}$ and $R^{op}\colon\mathbf{B}^{op}\to\mathbf{X}^{op}$. An arrow $f\colon L(A)\to R(B)$ in $\mathbf{X}$ is the same ting as an arrow $f\colon R^{op}(B)\to L^{op}(A)$ in $\mathbf{X}^{op}$, thus $(L{\downarrow}R)$ and $R^{op}{\downarrow}L^{op}$ have the same objects. Moreover, the commutativity in $\mathbf{X}$ of the square

$$\begin{tikzcd} L(A) \arrow[r, "L(h)"] \arrow[d, "f"'] & L(A') \arrow[d, "g"] \\ R(B) \arrow[r, "R(k)"'] & R(B') \end{tikzcd}$$

is tantamount to the commutativity in $\mathbf{X}^{op}$ of the square

$$\begin{tikzcd} R(B') \arrow[r, "R(k)"] \arrow[d, "g"'] & R(B) \arrow[d, "f"] \\ L(A') \arrow[r, "L(h)"'] & L(A) \end{tikzcd}$$

Summing up we have just proved the following fact.

▶ **Proposition B.2.** *$(L{\downarrow}R)^{op}$ is equal to $R^{op}{\downarrow}L^{op}$, moreover $U_L^{op}=U_{L^{op}}$ and $U_R^{op}=U_{R^{op}}$.*

▶ **Lemma B.3.** *Let $L\colon\mathbf{A}\to\mathbf{X}$ and $R\colon\mathbf{B}\to\mathbf{X}$ be functors and $F\colon\mathbf{D}\to L{\downarrow}R$ be a diagram such that L preserves colimits along $U_L\circ F$. Then the family $\{U_L,U_R\}$ jointly creates colimits of F (see [6, 7]).*

Proof. Suppose that $U_L\circ F$ and $U_R\circ F$ have colimiting cocones $(A,\{a_D\}_{D\in\mathbf{D}})$ and $(B,\{b_D\}_{D\in\mathbf{D}})$ respectively. By hypothesis $(L(A),\{L(a_D)\}_{D\in\mathbf{D}})$ is colimiting for $L\circ U_L\circ F$. Now, if we define

$$F(D) := (A_D, B_D, f_D)$$

then we have arrows $R(a_i) \circ f_D \colon L(A_D) \to R(B)$ that forms a cocone on $L \circ U_L \circ F$: if $d \colon D \to D'$ is an arrow in $\mathbf{D}$ then $F(d)$ is an arrow in $L{\downarrow}R$ and so

$$\begin{aligned} R(b_{D'}) \circ f_{D'} \circ L(U_L(F(d))) &= R(b_{D'}) \circ R(U_R(F(d))) \circ f_D \\ &= R(b_{D'} \circ U_R(F(d))) \circ f_D \\ &= R(b_D) \circ f_D \end{aligned}$$

Thus there exists $f \colon L(A) \to R(B)$ such that

$$\begin{tikzcd} L(A_D) \arrow[r, "L(a_D)"] \arrow[d, "f_D"'] & L(A) \arrow[d, "f"] \\ R(B_D) \arrow[r, "R(b_D)"'] & R(B) \end{tikzcd}$$

Notice that f is the unique arrow in $\mathbf{X}$ wich makes (a_D, b_D) an arrow $(A_D, B_D, f_D) \to (A, B, f)$ of $L{\downarrow}R$. If we show that $\big((A, B, f), \{(a_D, b_D)\}_{D \in \mathbf{D}}\big)$ is colimiting for F we are done.

First of all, let us show that it is a cocone. Given $d \colon D \to D'$ in $\mathbf{D}$ we have:

$$\begin{aligned} (a_{D'}, b_{D'}) \circ F(d) &= (a_{D'}, b_{D'}) \circ (U_L(F(d)), U_R(F(d))) \\ &= (a_{D'} \circ U_L(F(d)), b_{D'} \circ U_R(F(d))) \\ &= (a_D, b_D) \end{aligned}$$

For the colimiting property, let $\big((X, Y, g), \{(x_D, y_D)\}_{D \in \mathbf{D}}\big)$ be another cocone on F. In particular $\big(X, \{x_D\}_{D \in \mathbf{D}}\big)$ and $\big(Y, \{y_D\}_{D \in \mathbf{D}}\big)$ are cocones on $U_L \circ F$ and $U_R \circ F$ respectively, so we have uniquely determined arrows $x \colon A \to X$ and $y \colon B \to Y$ such that

$$x \circ a_D = x_D \qquad y \circ b_D = y_D$$

Let us show that (x, y) is an arrow of $L{\downarrow}R$. Given $D \in \mathbf{D}$ we have

$$\begin{aligned} R(y) \circ f \circ L(a_D) &= R(y) \circ R(b_D) \circ f_D \\ &= R(y \circ b_D) \circ f_D \\ &= R(y_D) \circ f_D \\ &= g \circ L(x_D) \\ &= g \circ L(x \circ a_D) \\ &= g \circ L(x) \circ L(a_D) \end{aligned}$$

from which it follows that the following diagram commutes.

$$\begin{tikzcd} L(A) \arrow[r, "L(x)"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ R(B) \arrow[r, "R(y)"'] & Y \end{tikzcd}$$

This shows that $\big((A, B, f), \{(a_D, b_D)\}_{D \in \mathbf{D}}\big)$ is colimiting for F and the thesis follows. ◀

Proposition B.2 and Lemma B.3 now yields the following.

▶ **Corollary B.4.** *The family $\{U_L, U_R\}$ jointly creates limits along every diagram $F\colon \mathbf{D} \to L{\downarrow}R$ such that R preserves the limit of $U_R \circ I$.*

We can use Corollary B.4 to characterize monos in comma categories.

▶ **Corollary B.5.** *If R preserves pullbacks then an arrow (h,k) in $L{\downarrow}R$ is mono if and only if both h and k are monos.*

Proof. ($\Rightarrow$) If $(h,k)\colon (A,B,f) \to (A',B',g)$ is a mono then the following square is a pullback in $L{\downarrow}R$

$$\begin{tikzcd}
(A,B,f) \arrow[r, "\mathsf{id}_{(A,B,f)}"] \arrow[d, "\mathsf{id}_{(A,B,f)}"'] & (A,B,f) \arrow[d, "(h,k)"] \\
(A,B,f) \arrow[r, "(h,k)"'] & (A',B',g)
\end{tikzcd}$$

Using Corollary B.4 we deduce that the following two squares are pullbacks in $\mathbf{A}$ and $\mathbf{B}$.

$$\begin{tikzcd}
A \arrow[r, "\mathsf{id}_A"] \arrow[d, "\mathsf{id}_A"'] & A \arrow[d, "h"] \\
A \arrow[r, "h"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\mathsf{id}_B"] \arrow[d, "\mathsf{id}_B"'] & B \arrow[d, "k"] \\
B \arrow[r, "k"'] & B'
\end{tikzcd}$$

From which it follows that h and k are monos.

($\Leftarrow$) Since h and k are monos then we have two pullback squares

$$\begin{tikzcd}
A \arrow[r, "\mathsf{id}_A"] \arrow[d, "\mathsf{id}_A"'] & A \arrow[d, "h"] \\
A \arrow[r, "h"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\mathsf{id}_B"] \arrow[d, "\mathsf{id}_B"'] & B \arrow[d, "k"] \\
B \arrow[r, "k"'] & B'
\end{tikzcd}$$

By Corollary B.4 this implies that

$$\begin{tikzcd}
(A,B,f) \arrow[r, "\mathsf{id}_{(A,B,f)}"] \arrow[d, "\mathsf{id}_{(A,B,f)}"'] & (A,B,f) \arrow[d, "(h,k)"] \\
(A,B,f) \arrow[r, "(h,k)"'] & (A',B',g)
\end{tikzcd}$$

is a pullback in $L{\downarrow}R$ and we are done. ◀

We end this section pointing out another useful fact, showing that in some cases we can guarantee the existence of a left adjoint to U_R.

▶ **Proposition B.6.** *If $\mathbf{A}$ has initial objects and L preserves them then the forgetful functor $U_R\colon L{\downarrow}R \to \mathbf{B}$ has a left adjoint Δ.*

Proof. For an object $B \in \mathbf{B}$ we can define $\Delta(B)$ as $(0, B, ?_B)$, where 0 is an initial object in $\mathbf{A}$ and $?_{R(B)}$ is the unique arrow $L(0) \to R(B)$. Consider $\mathsf{id}_B\colon B \to U_R(\Delta(B))$ be the identity, and suppose that a $k\colon B \to U_R(A, B', f)$ in $\mathbf{B}$ is given. By initiality of 0, there is only one arrow $?_A \colon 0 \to A$ in $\mathbf{A}$ and, since L preserves initial objects, the following square commutes.

$$\begin{tikzcd} L(0) \arrow[r, "L(?_A)"] \arrow[d, "?_{R(B)}"'] & L(A) \arrow[d, "f"] \\ R(B) \arrow[r, "R(k)"'] & R(B') \end{tikzcd}$$

Thus (h, k) is the unique morphism $\Delta(B) \to (A, B', f)$ such that $U_R(h, k) = k$. ◀

Dualizing we get immediately the following.

▶ **Corollary B.7.** *If $\mathbf{B}$ has terminal objects preserved by R then $U_L \colon L{\downarrow}R \to \mathbf{A}$ has a right adjoint.*

B.1 Slice categories

This section is devoted to recall some basic facts about the so called *slice categories.*

▶ **Definition B.8.** *Let X be an object of a category $\mathbf{X}$, we will define the following two categories.*

- *The* slice category over X *is the category $\mathbf{X}/X$ which has as objects arrows $f \colon Y \to X$ and in which an arrow $h \colon f \to g$ is $h \colon Y \to Y'$ in $\mathbf{X}$ such that the following triangle commutes.*

$$\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "f"'] & & Y' \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

- *Dually, the* slice category under X *is the category $X/\mathbf{X}$ in which objects are arrows $f \colon X \to Y$ with domain X and a morphism $h \colon f \to g$ is an arrow of $\mathbf{X}$ fitting in a triangle as the one below.*

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y \arrow[rr, "h"'] & & Y' \end{tikzcd}$$

▶ Remark B.9. For every $X \in X$ we have forgetful functors

$$\begin{array}{ccc} \mathsf{dom}_X \colon \mathbf{X}/X \to \mathbf{X} & \qquad & \mathsf{cod}_X \colon X/\mathbf{X} \to \mathbf{X} \\ f \longmapsto \mathsf{dom}(f) & & f \longmapsto \mathsf{cod}(f) \\ h\big\downarrow \qquad\quad \big\downarrow h & & h\big\downarrow \qquad\quad \big\downarrow h \\ g \longmapsto \mathsf{dom}(g) & & g \longmapsto \mathsf{cod}(g) \end{array}$$

We can realize the slice over and under an object $X \in \mathbf{X}$ as comma categories.

▶ **Proposition B.10.** *For every object X in a category $\mathbf{X}$, if $\delta_X \colon \mathbf{1} \to \mathbf{X}$ is the constant functor of value X from the category with only one object $*$, then $\mathbf{X}/X$ and $X/\mathbf{X}$ are isomorphic to, respectively, $\mathsf{id}_X{\downarrow}\delta_X$ and $\delta_X{\downarrow}\mathsf{id}_X$.*

Proof. Define functors $F_1 \colon \mathsf{id}_X{\downarrow}\delta_X \to \mathbf{X}/X$ and $G_1 \colon \mathbf{X}/X \to \mathsf{id}_X{\downarrow}\delta_X$ as follows

$$\begin{array}{rcl} (Y,*,f) & \longmapsto & f \\ {\scriptstyle (h,\mathsf{id}_*)}\downarrow & & \downarrow{\scriptstyle h} \\ (Y',*,g) & \longmapsto & g \end{array} \qquad \begin{array}{rcl} f & \longmapsto & (\mathsf{dom}(f),*,f) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle (h,\mathsf{id}_*)} \\ g & \longmapsto & (\mathsf{dom}(g),*,g) \end{array}$$

Similarly, we have $F_2 \colon \delta_X{\downarrow}\mathsf{id}_X \to X/\mathbf{X}$ and $G_2 \colon X/\mathbf{X} \to \delta_X{\downarrow}\mathsf{id}_X$

$$\begin{array}{rcl} (*,Y,f) & \longmapsto & f \\ {\scriptstyle (\mathsf{id}_*,h)}\downarrow & & \downarrow{\scriptstyle h} \\ (*,Y',g) & \longmapsto & g \end{array} \qquad \begin{array}{rcl} f & \longmapsto & (*,\mathsf{cod}(f),f) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle (\mathsf{id}_*,h)} \\ g & \longmapsto & (*,\mathsf{cod}(g),g) \end{array}$$

It is now obvious to see that F_1, G_1 and F_2, G_2 are pairs of inverses. ◀

A straightforward application of Corollary B.4 now yields the following.

▶ **Corollary B.11.** *If $\mathbf{X}$ has pullbacks, then for every object X, the slice $\mathbf{X}/X$ has pullbacks too.*

In a category $\mathbf{X}$ with pullbacks, each $f : X \to Y$ induces a functor $f^* : \mathbf{X}/Y \to \mathbf{X}/X$, which sends each morphism $a : A \to Y$ of $\mathbf{X}$ onto its pullback along f, p_a, and each morphism $h : a \to b$ onto the unique arrow from the pullback of a along f to the pullback of b along f.

Then, we have the following result.

▶ **Proposition B.12.** *Let $\mathbf{X}$ be a category with pullbacks, $R : \mathbf{Y} \to \mathbf{X}$ be a functor and $L : \mathbf{X} \to \mathbf{Y}$ be its left adjoint, and η the unit of the adjunction. Then, each object X of $\mathbf{X}$ induces an adjoint pair of functors $L_X : \mathbf{X}/X \to \mathbf{Y}/L(X)$, $R_X : \mathbf{Y}/L(X) \to \mathbf{X}/X$, where L_X is the obvious functor, and R_X is the composite $(\eta_X)^* \circ R$.*

Proof. Let $f : L(l) \to b$ be a morphism of $\mathbf{Y}/L(X)$, where $l : A \to X$ in $\mathbf{X}$ and $b : B \to L(X)$ in $\mathbf{Y}$, as shown below.

$$\begin{tikzcd} L(A) \arrow[rr, "f"] \arrow[dr, "L(l)"'] & & B \arrow[dl, "b"] \\ & L(X) & \end{tikzcd}$$

Then, we have the following situation in $\mathbf{X}$.

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "l"'] \arrow[rr, bend left, "g"] & R(L(A)) \arrow[r, "R(f)"] \arrow[dr, "R(L(l))"'] & R(B) \arrow[d, "R(b)"] \\ X \arrow[rr, "\eta_X"'] & & R(L(X)) \end{tikzcd}$$

Where g is the adjunct of f. Consider now the pullback P of $R(p)$ along η_X, as shown below.

$$\begin{tikzcd} A \arrow[dr, dotted, "v"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "l"'] & & \\ & P \arrow[r, "q"] \arrow[d, "\eta_X^*(R(b))"'] & R(B) \arrow[d, "R(b)"] \\ & X \arrow[r, "\eta_X"'] & R(L(X)) \end{tikzcd}$$

By universal property of pullbacks, the two diagrams express the same morphism g. Hence, we can rewrite the first diagram as follows.

$$\begin{tikzcd}
L(A) \arrow[r, "L(g)"] \arrow[d, "L(l)"'] \arrow[rr, bend left, "f"] & (L(R(B))) \arrow[r, "\epsilon_B"] \arrow[d, "L(R(b))"'] & B \arrow[d, "b"] \\
L(X) \arrow[r, "L(\eta_X)"'] \arrow[rr, bend right, "\mathsf{id}_{L(X)}"'] & L(R(L(X))) \arrow[r, "\epsilon_{L(X)}"'] & L(X)
\end{tikzcd}$$

where ϵ is the counit of the adjunction. This descriveb the action of functors and thus the adjunction, obtained considering the isomorphism on hom-sets of the categories.

◀