

On the adhesivity of EGGS

Author: Please provide author information

Abstract

a very nice abstract

2012 ACM Subject Classification Author: Please fill in 1 or more \ccsdesc macro

Keywords and phrases Author: Please fill in \keywords macro

Digital Object Identifier 10.4230/LIPIcs...

1 Introduction

A very nice introduction

2 \mathcal{M} -adhesive categories

This section briefly recalls \mathcal{M} -adhesive categories [1, 10, 11, 16, 14]. Given a category \mathbf{X} we do not distinguish notationally between \mathbf{X} and its class of objects, so “ $X \in \mathbf{X}$ ” means that X is an object of \mathbf{X} . We let $\text{Mor}(\mathbf{X})$, $\text{Mono}(\mathbf{X})$ and $\text{Reg}(\mathbf{X})$ denote the class of all arrows, monos and regular monos of \mathbf{X} , respectively. Given an object X , we denote by $?_X$ the unique arrow from an initial object into X and by $!_X$ that unique arrow from X into a terminal one.

2.1 \mathcal{M} -adhesivity

The key property of \mathcal{M} -adhesive categories is the *Van Kampen condition* [4, 15, 16], and to define it we introduce some terminology. Let \mathbf{X} be a category. A subclass \mathcal{A} of $\text{Mor}(\mathbf{X})$ is

- *stable under pushouts (pullbacks)* if for every pushout (pullback) square as the one aside, if $m \in \mathcal{A}$ ($n \in \mathcal{A}$) then $n \in \mathcal{A}$ ($m \in \mathcal{A}$);
- *closed under composition* if $h, k \in \mathcal{A}$ implies $h \circ k \in \mathcal{A}$ whenever h and k are composable.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ m \downarrow & & \downarrow n \\ C & \xrightarrow{g} & D \end{array}$$

► **Definition 2.1.** Let $\mathcal{A} \subseteq \text{Mor}(\mathbf{X})$ be a class of arrows in a category \mathbf{X} and consider the cube below on the right.

We say that the bottom square is an \mathcal{A} -Van Kampen square if

1. it is a pushout square;
2. whenever the cube above has pullbacks as back and left faces and the vertical arrows belong to \mathcal{A} , then its top face is a pushout if and only if the front and right faces are pullbacks.

$$\begin{array}{ccccc} & & A' & \xrightarrow{f'} & B' \\ m' \swarrow & & \downarrow a & \swarrow g' & \downarrow b \\ C' & \xrightarrow{a} & D' & & \\ c \downarrow & & \downarrow d & \searrow f & \downarrow \\ C & \xrightarrow{m} & A & \xrightarrow{f} & B \\ & \searrow g & \downarrow n & & \end{array}$$

Pushout squares that enjoy only the “if” half of item (2) above are called \mathcal{A} -stable. A $\text{Mor}(\mathbf{X})$ -Van Kampen square is called Van Kampen and a $\text{Mor}(\mathbf{X})$ -stable square stable.

We can now define \mathcal{M} -adhesive categories.

► **Definition 2.2.** Let \mathbf{X} be a category and \mathcal{M} a subclass of $\text{Mono}(\mathbf{X})$ including all isomorphisms, closed under composition, and stable under pullbacks and pushouts. The category \mathbf{X} is said to be \mathcal{M} -adhesive if

1. it has \mathcal{M} -pullbacks, i.e. pullbacks along arrows of \mathcal{M} ;
2. it has \mathcal{M} -pushouts, i.e. pushouts along arrows of \mathcal{M} ;
3. \mathcal{M} -pushouts are \mathcal{M} -Van Kampen squares.



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Leibniz International Proceedings in Informatics

LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

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A category \mathbf{X} is said to be strictly \mathcal{M} -adhesive if \mathcal{M} -pushouts are Van Kampen. We write $m: X \rightrightarrows Y$ to denote that an arrow $m: X \rightarrow Y$ belongs to \mathcal{M} .

► **Remark 2.3.** Adhesivity and quasiadhesivity [16, 12] coincide with strict $\mathbf{Mono}(\mathbf{X})$ -adhesivity and strict $\mathbf{Reg}(\mathbf{X})$ -adhesivity, respectively.

\mathcal{M} -adhesivity is well-behaved with respect to the construction of slice and functor categories [17], as shown by the following theorems [9, 16].

► **Theorem 2.4.** Let \mathbf{X} be an (strict) \mathcal{M} -adhesive category, then the following hold true:
 1. if \mathbf{Y} is an (strict) \mathcal{N} -adhesive category $L: \mathbf{Y} \rightarrow \mathbf{A}$ a functor preserving \mathcal{N} -pushouts and $R: \mathbf{X} \rightarrow \mathbf{A}$ one preserving pullbacks, then $L \downarrow R$ is (strictly) $\mathcal{N} \downarrow \mathcal{M}$ -adhesive, where

$$\mathcal{N} \downarrow \mathcal{M} := \{(h, k) \in \mathbf{Mor}(L \downarrow R) \mid h \in \mathcal{N}, k \in \mathcal{M}\}$$

2. for every object X the categories \mathbf{X}/X and X/X are, respectively, (strictly) \mathcal{M}/X -adhesive and (strictly) X/\mathcal{M} -adhesive, where

$$\mathcal{M}/X := \{m \in \mathbf{Mor}(\mathbf{X}/X) \mid m \in \mathcal{M}\} \quad X/\mathcal{M} := \{m \in \mathbf{Mor}(X/\mathbf{X}) \mid m \in \mathcal{M}\}$$

3. for every small category \mathbf{Y} , the category $\mathbf{X}^{\mathbf{Y}}$ of functors $\mathbf{Y} \rightarrow \mathbf{X}$ is (strictly) $\mathcal{M}^{\mathbf{Y}}$ -adhesive, where $\mathcal{M}^{\mathbf{Y}} := \{\eta \in \mathbf{Mor}(\mathbf{X}^{\mathbf{Y}}) \mid \eta_Y \in \mathcal{M} \text{ for every } Y \in \mathbf{Y}\}$;

4. if \mathbf{Y} is a full subcategory of \mathbf{X} closed in it under pullbacks and \mathcal{M} -pushouts, then \mathbf{Y} is (strictly) \mathcal{N} -adhesive for every class of arrows \mathcal{N} of \mathbf{Y} contained in \mathcal{M} and stable under pullbacks and pushout, containing all isomorphisms and closed under composition and decomposition.

We will briefly list some examples of \mathcal{M} -adhesive categories.

► **Example 2.5.** **Set** is adhesive, and, more generally, every topos is adhesive [?]. By the closure properties above, every presheaf $[\mathbf{X}, \mathbf{Set}]$ is adhesive, thus the category **Graph** = $[E \rightrightarrows V, \mathbf{Set}]$ is adhesive where $E \rightrightarrows V$ is the two objects category with two morphisms $s, t: E \rightarrow V$. Similarly, various categories of hypergraphs can be shown to be adhesive, such as term graphs and hierarchical graphs [7]. Note that the category **sGraphs** of simple graphs, i.e. graphs without parallel edges, is $\mathbf{Reg}(\mathbf{sGraphs})$ -adhesive [2] but not quasiadhesive.

We can state some useful properties of \mathcal{M} -adhesive category.

► **Proposition 2.6.** If \mathbf{X} is \mathcal{M} -adhesive then the following are true:

1. every \mathcal{M} -pushout square is also a pullback;
2. every arrow in \mathcal{M} is a regular mono.

[Proof in Appendix A.1]

2.2 Kernel Pairs and Regular Epimorphisms

In this section we recall the definition and some properties of *kernel pairs*.

► **Definition 2.7.** A kernel pair for an arrow $f: A \rightarrow B$ is an object K_f together with two arrows $\pi_f^1, \pi_f^2: K_f \rightrightarrows A$, denoted as (K_f, π_f^1, π_f^2) , such that the square aside is a pullback.

$$\begin{array}{ccc} K_f & \xrightarrow{\pi_f^2} & A \\ \pi_f^1 \downarrow & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

► **Remark 2.8.** If (K_f, π_f^1, π_f^2) is a kernel pair for $f: X \rightarrow Y$ and a product of X with itself exists, then the canonical arrow $\langle \pi_f^1, \pi_f^2 \rangle: K_f \rightarrow X \times X$ is a mono.

► **Remark 2.9.** An arrow $m: M \rightarrow X$ is a mono if and only if it admits $(M, \text{id}_M, \text{id}_M)$ as a kernel pair.

Together with Lemma A.1, the previous remarks allow us to prove the following result.

► **Proposition 2.10.** *Let $f: X \rightarrow Y$ be an arrow and $m: Y \rightarrow Z$ a mono. If (K_f, π_f^1, π_f^2) is a kernel pair for $f: X \rightarrow Y$, then it is also a kernel pair for $m \circ f$.*

We are now going to explore some further properties of kernel pairs

► **Lemma 2.11.** *Let $f: X \rightarrow Y$ and $g: Z \rightarrow W$ be two arrows admitting kernel pairs and suppose that the solid part of the three squares below is given. Then there exists a unique arrow $k_h: K_f \rightarrow K_g$ completing them.*

$$\begin{array}{ccccc}
 X & \xrightarrow{h} & Z & & K_f & \xrightarrow{k_h} & K_g & & K_f & \xrightarrow{k_h} & K_g \\
 f \downarrow & & \downarrow g & & \pi_f^1 \downarrow & & \downarrow \pi_g^1 & & \pi_f^2 \downarrow & & \downarrow \pi_g^2 \\
 Y & \xrightarrow{t} & W & & X & \xrightarrow{h} & Z & & X & \xrightarrow{h} & Z
 \end{array}$$

Moreover, if the leftmost is a pullback, then also the other two are so.

[Proof in Appendix A.1]

The previous result allows us to deduce the following lemma in an \mathcal{M} -adhesive context.

► **Lemma 2.12.** *Let \mathbf{X} be a strict \mathcal{M} -adhesive category with all pullbacks, and suppose that in the cube aside the top face is an \mathcal{M} -pushout. Then the right square is a pushout.*

$$\begin{array}{ccccc}
 & & A' & \xrightarrow{f'} & B' \\
 & m' \swarrow & \downarrow a & \searrow n' & \\
 C' & \xrightarrow{g'} & D' & & K_a & \xrightarrow{k_{f'}} & K_b \\
 & \downarrow c & \downarrow d & & \downarrow k_{m'} & & \downarrow k_{n'} \\
 & & A & \xrightarrow{f} & B & & \\
 & m \swarrow & \downarrow g & \searrow n & & & \\
 C & \xrightarrow{g} & D & & K_c & \xrightarrow{k_{g'}} & K_d
 \end{array}$$

[Proof in Appendix A.1]

As a final step, we explore the link between regular epis and kernel pairs.

► **Proposition 2.13.** *Let $e: X \rightarrow Y$ be a regular epi in a category \mathbf{X} with a kernel pair (K_e, π_e^1, π_e^2) . Then, e is the coequalizer of π_e^1 and π_e^2 .*

[Proof in Appendix A.1]

► **Corollary 2.14.** *Let \mathbf{X} be a category with pullbacks and $\phi: F \rightarrow G$ a natural transformation between functors $F, G: \mathbf{D} \rightarrow \mathbf{X}$. If ϕ_d is a regular epi for every d in \mathbf{D} , then ϕ is a regular epi.*

[Proof in Appendix A.1]

From the previous result we deduce that the class of regular epis is closed under colimits.

► **Lemma 2.15.** *Let $F, G: \mathbf{D} \rightarrow \mathbf{X}$ be two diagrams, and suppose that \mathbf{X} has all colimits of shape \mathbf{D} . Let $(X, \{x_d\}_{d \in \mathbf{D}})$ and $(Y, \{y_d\}_{d \in \mathbf{D}})$ be the colimits of F and G , respectively. If $\phi: F \rightarrow G$ is a natural transformation whose components are regular epi, then the arrow induced by ϕ from X to Y is a regular epi too.*

[Proof in Appendix A.1]

3 Hypergraphical structures

In this section we briefly recall the notion \mathbf{X} -hypergraph. It is necessary to have a monad $(-)^*$ playing a role analogous to the usual Kleene star $(-)^*: \mathbf{Set} \rightarrow \mathbf{Set}$, also known as *list monad*, sending a set to the free monoid on it [18, 19]. We recall some of its properties.

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101 ► **Proposition 3.1.** *The following facts are true:*

- 102 1. *for every set X and $n \in \mathbb{N}$ there are arrows $v_n: X^n \rightarrow X^*$ such that $(X^*, \{v_n\}_{n \in \mathbb{N}})$ is a*
 103 *coproduct;*
- 104 2. *for every arrow $f: X \rightarrow Y$, f^* is the coproduct of the family $\{f^n\}_{n \in \mathbb{N}}$;*
- 105 3. *$(-)^*$ preserves all connected limits [5], in particular it preserves pullbacks and equalizers.*

106 ► **Remark 3.2.** Preservation of pullbacks implies that $(-)^*$ sends monos to monos.

107 ► **Remark 3.3.** Notice that 1^* can be canonically identified with \mathbb{N} , thus for every set X the
 108 arrow $!_X: X \rightarrow 1$ induces a *length function* $!_X^*: X^* \rightarrow \mathbb{N}$, which sends a word to its length.

109 3.1 The category of hypergraphs

110 We open this section with the definition of hypergraphs and we will see how to label them
 111 with an algebraic signature.

112 ► **Definition 3.4.** *An hypergraph is a 4-uple $\mathcal{G} := (E_{\mathcal{G}}, V_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}})$ made by two sets $E_{\mathcal{G}}$ and*
 113 *$V_{\mathcal{G}}$, called respectively the set of hyperedges and nodes, plus a pair of source and target*
 114 *arrows $s_{\mathcal{G}}, t_{\mathcal{G}}: E_{\mathcal{G}} \rightrightarrows V_{\mathcal{G}}^*$. A hypergraph morphism $(E_{\mathcal{G}}, V_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}}) \rightarrow (E_{\mathcal{H}}, V_{\mathcal{H}}, s_{\mathcal{H}}, t_{\mathcal{H}})$ is a*
 115 *pair (h, k) of functions $h: E_{\mathcal{G}} \rightarrow E_{\mathcal{H}}$, $k: V_{\mathcal{G}} \rightarrow V_{\mathcal{H}}$ such that the following diagrams commute*

$$\begin{array}{ccc}
 E_{\mathcal{G}} & \xrightarrow{s_{\mathcal{G}}} & V_{\mathcal{G}}^* \\
 h \downarrow & & \downarrow k^* \\
 E_{\mathcal{G}} & \xrightarrow{s_{\mathcal{H}}} & V_{\mathcal{H}}^*
 \end{array}
 \qquad
 \begin{array}{ccc}
 E_{\mathcal{G}} & \xrightarrow{t_{\mathcal{G}}} & V_{\mathcal{G}}^* \\
 h \downarrow & & \downarrow k^* \\
 E_{\mathcal{G}} & \xrightarrow{t_{\mathcal{H}}} & V_{\mathcal{H}}^*
 \end{array}$$

117 We define **Hyp** to be the resulting category.

118 Let prod^* be the functor sending X to $X^* \times X^*$: we can get **Hyp** as a comma category.

119 ► **Proposition 3.5.** **Hyp** is isomorphic to $\text{id}_{\text{Set}} \downarrow \text{prod}^*$

120 ► **Corollary 3.6.** **Hyp** is an adhesive category.

121 **Proof.** By hypothesis $(-)^*$ preserves pullbacks, while prod is continuous by definition, thus
 122 the thesis follows from Theorem 2.4 and Proposition 3.5. ◀

123 Another useful corollary of Proposition 3.5 is the following one.

124 ► **Corollary 3.7.** *A morphism (h, k) is a mono in **Hyp** if and only if both its components*
 125 *are injective functions.*

126 Propositions 3.5 and B.6 allow us to deduce immediately the following.

127 ► **Proposition 3.8.** *The forgetful functor U_{Hyp} which sends an hypergraph \mathcal{G} to its object of*
 128 *nodes has a left adjoint Δ_{Hyp} .*

129 ► **Example 3.9.** Since the initial object of **Set** is the empty set, $\Delta_{\text{Set}}(X)$ is the hypergraph
 130 which has X as set of nodes, \emptyset as set of hyperedges, and $?_X$ as source and target function.

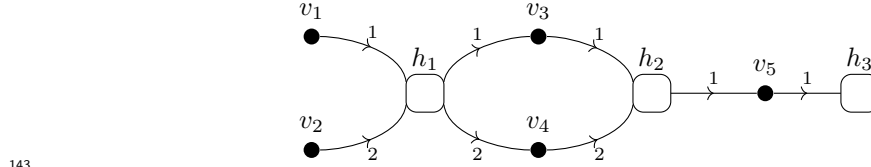
forse possiamo
 anche cancel-
 lare tutti gli
 esempi sotto

131 We can represent hypergraphs graphically. We will use dots to denote nodes and squares to
 132 denote hyperedges, the name of a node or of an hyperedge will be put near the corresponding
 133 dot or square. Sources and targets are represented by lines between dots and squares: the
 134 lines from the sources of an hyperedge will have an arrowhead in the middle pointing towards
 135 the hyperedge, while the lines to the targets will have arrowheads pointing to the target
 136 nodes. We will decorate the arrow corresponding to the i^{th} letter of a target or a source with
 137 a label i .
 138

139 ► **Example 3.10.** Take $V_{\mathcal{G}}$ to be $\{v_1, v_2, v_3, v_4, v_5\}$ and $E_{\mathcal{G}}$ to be $\{h_1, h_2, h_3\}$. Sources
 140 and targets are given by:

$$\begin{array}{llll}
 s_{\mathcal{G}}(h_1): 2 \rightarrow V_{\mathcal{G}} & 0 \mapsto v_1 & s_{\mathcal{G}}(h_2): 2 \rightarrow V_{\mathcal{G}} & 0 \mapsto v_3 \\
 & 1 \mapsto v_2 & & 1 \mapsto v_4 \\
 s_{\mathcal{G}}(h_3): 1 \rightarrow V_{\mathcal{G}} & 0 \mapsto v_5 & & \\
 t_{\mathcal{G}}(h_1): 2 \rightarrow V_{\mathcal{G}} & 0 \mapsto v_3 & t_{\mathcal{G}}(h_2): 2 \rightarrow V_{\mathcal{G}} & 0 \mapsto v_5 \\
 & 1 \mapsto v_4 & & t_{\mathcal{G}}(h_3): 0 \rightarrow V_{\mathcal{G}} \quad t_{\mathcal{G}}(h_3) = ?_{V_{\mathcal{G}}}
 \end{array}$$

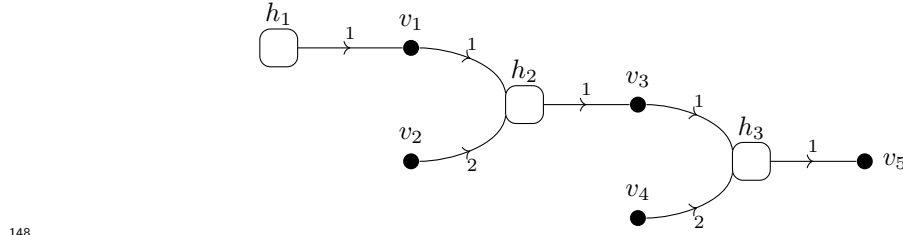
142 We can draw the resulting \mathcal{G} as follows:



144 ► **Example 3.11.** Let $V_{\mathcal{G}}$ be as in the previous example and $E_{\mathcal{G}} = \{h_1, h_2, h_3\}$. Then we
 145 define

$$\begin{array}{llll}
 s_{\mathcal{G}}(h_1): 0 \rightarrow V_{\mathcal{G}} & s_{\mathcal{G}}(h_1) = ?_{V_{\mathcal{G}}} & s_{\mathcal{G}}(h_2): 2 \rightarrow V_{\mathcal{G}} & 0 \mapsto v_1 \\
 & & & 1 \mapsto v_2 \\
 s_{\mathcal{G}}(h_3): 2 \rightarrow V_{\mathcal{G}} & 0 \mapsto v_1 & & 1 \mapsto v_4 \\
 t_{\mathcal{G}}(h_1): 1 \rightarrow V_{\mathcal{G}} & 0 \mapsto v_1 & t_{\mathcal{G}}(h_2): 1 \rightarrow V_{\mathcal{G}} & 0 \mapsto v_3 \\
 & & & t_{\mathcal{G}}(h_3): 1 \rightarrow V_{\mathcal{G}} \quad 1 \mapsto v_5
 \end{array}$$

147 Now we can depict \mathcal{G} as



149 3.1.1 Hyp as a category of functors

150 Following [3], we can present **Hyp** as a category of functor over a suitable category.

151 ► **Definition 3.12.** Let **H** be the category isuch that

- 152 ■ the set of objects is $(\mathbb{N} \times \mathbb{N}) \cup \{\bullet\}$;
- 153 ■ arrows are given by the identities $\text{id}_{k,l}$ and id_{\bullet} and exactly $k + l$ arrows $f_i: (k, l) \rightarrow \bullet$,
 154 where i ranges from 0 to $k + l - 1$;
- 155 ■ composition is defined by putting $f_i = f_i \circ \text{id}_{k,l}$ and $f_i = \text{id}_{\bullet} \circ f_i$ for every $f_i: (k, l) \rightarrow \bullet$.

156 The idea is that for every functor $F: \mathbf{H} \rightarrow \mathbf{Set}$ we can define

$$157 \quad E_F := \sum_{k,l \in \mathbb{N}} F(k, l)$$

158 Now, for every k, l, i and j in \mathbb{N} with $i < k$ and $j < l$ we define $s_{k,l}^F: F(k, l) \rightarrow F(\bullet)^k$ and
 159 $t_{k,l}^F: F(k, l) \rightarrow F(\bullet)^l$ as the unique arrows fitting in the diagrams below, where the vertical
 160 arrows are the projections

$$\begin{array}{ccc}
 F(k, l) & \xrightarrow{s_{k,l}^F} & F(\bullet)^k \\
 \searrow F(f_i) & & \downarrow \pi_{k,i}^F \\
 & & F(\bullet)
 \end{array}
 \quad
 \begin{array}{ccc}
 F(k, l) & \xrightarrow{t_{k,l}^F} & F(\bullet)^l \\
 \searrow F(f_{k+j}) & & \downarrow \pi_{l,j}^F \\
 & & F(\bullet)
 \end{array}$$

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In turn, these arrows allow us to consider $s_F, t_F: E_F \rightrightarrows F(\bullet)^*$ as the unique arrows fitting in the diagrams below, where the vertical arrows are coprojections

$$\begin{array}{ccc}
 F(k, l) & \xrightarrow{s_{k,l}^F} & F(\bullet)^k \\
 a_{k,l}^F \downarrow & & \downarrow b_k^F \\
 E_F & \xrightarrow{s_F} & F(\bullet)^*
 \end{array}
 \quad
 \begin{array}{ccc}
 F(k, l) & \xrightarrow{t_{k,l}^F} & F(\bullet)^l \\
 a_{k,l}^F \downarrow & & \downarrow b_l^F \\
 E_F & \xrightarrow{t_F} & F(\bullet)^*
 \end{array}$$

Let \mathcal{G}_F be the resulting hypergraph. One can now show that sending F to \mathcal{G}_F can be extended to an equivalence $\mathcal{G}_-: \mathbf{Set}^{\mathbf{H}} \rightarrow \mathbf{Hyp}$ (see [6, 7] for details).

► **Proposition 3.13.** *Hyp is equivalent to the category $\mathbf{Set}^{\mathbf{H}}$.*

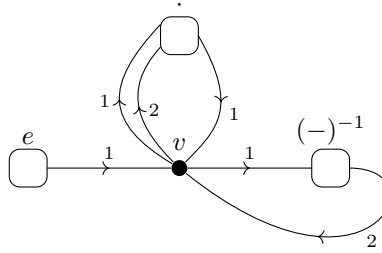
3.1.2 Labelling hypergraph with an algebraic signature

Our interest for hypergraphs stems from their use as a graphical representation of algebraic terms. We thus need a way to label hyperedges with symbols taken from a signature.

► **Definition 3.14.** *An algebraic signature Σ is a pair $(O_\Sigma, \text{ar}_\Sigma)$ given by a set of operations O_Σ and an arity function $\text{ar}_\Sigma: O_\Sigma \rightarrow \mathbb{N}$.*

We define the hypergraph \mathcal{G}_Σ associated with Σ taking O_Σ as set of hyperedges, 1 as set of nodes, so that 1^ is \mathbb{N} , ar_σ as the source function and δ_1 as target function, where δ_1 picks the element 1. The category \mathbf{Hyp}_Σ of algebraically labelled hypergraphs is the slice category $\mathbf{Hyp}/\mathcal{G}_\Sigma$.*

► **Example 3.15.** Let $\Sigma = (O_\Sigma, \text{ar}_\Sigma)$ be an algebraic signature in \mathbf{Set} . This simply amount to a set of *operations* with an associated natural number, called *arity*. For instance let Σ_G be the signature of groups, then \mathcal{G}^{Σ_G} can be depicted as



180

Corollary B.5 and Theorem 2.4 give us immediately an adhesivity result for \mathbf{Hyp}_Σ and a characterisation of monos in it.

► **Proposition 3.16.** *For every algebraic signature Σ , the following are true:*

1. *a morphism (h, k) between two object of \mathbf{Hyp}_Σ is a mono if and only if h and k are injective functions;*
2. *\mathbf{Hyp}_Σ is an adhesive category.*

► **Remark 3.17.** Let $\mathcal{H} = (E, V, s, t)$ be an hypergraph, by definition we know that $U_{\mathbf{Hyp}}(\mathcal{G}^\Sigma)$ is the terminal object 1, so an arrow $\mathcal{H} \rightarrow \mathcal{G}^\Sigma$, is determined by a morphism $h: E_{\mathcal{H}} \rightarrow O_\Sigma$

188

189 making the two squares below commute (cfr. Remark 3.3).

$$\begin{array}{ccc}
 E_{\mathcal{H}} & \xrightarrow{h} & O_{\Sigma} \\
 s_{\mathcal{H}} \downarrow & & \downarrow \text{ar}_{\Sigma} \\
 V_{\mathcal{H}}^* & \xrightarrow{\text{lg}_{V_{\mathcal{H}}}} & \mathbb{N}
 \end{array}
 \quad
 \begin{array}{ccc}
 E_{\mathcal{H}} & \xrightarrow{h} & O_{\Sigma} \\
 t_{\mathcal{H}} \downarrow & & \downarrow \delta_1 \\
 V_{\mathcal{H}}^* & \xrightarrow{\text{lg}_{V_{\mathcal{H}}}} & \mathbb{N}
 \end{array}$$

191 Let $v_n: V_{\mathcal{H}}^n \rightarrow V_{\mathcal{H}}^*$ be a coprojection. The second diagram in particular above entails that
 192 $t_{\mathcal{H}}$ factors through the inclusion $v_1: V_{\mathcal{H}} \rightarrow V^*$ of words of length 1, i.e. that $t_{\mathcal{H}} = v_1 \circ \tau_{\mathcal{H}}$
 193 for some $\tau_{\mathcal{H}}: E_{\mathcal{H}} \rightarrow \tau_{\mathcal{H}}$.

194 **Hyp $_{\Sigma}$** , has a forgetful functor $U_{\Sigma}: \mathbf{Hyp}_{\Sigma} \rightarrow \mathbf{X}$ which sends $(h, k): \mathcal{H} \rightarrow \mathcal{G}^{\Sigma}$ to $U_{\mathbf{X}}(\mathcal{H})$.
 195 Now, $U_{\mathbf{X}}(\mathcal{G}^{\Sigma}) = 1$ thus, for every object X , there is only one arrow $X \rightarrow U_{\mathbf{X}}(\mathcal{G}^{\Sigma})$. Define
 196 $\Delta_{\Sigma}(X): \Delta_{\mathbf{X}}(X) \rightarrow \mathcal{G}^{\Sigma}$ as the transpose of this arrow. Explicitly, $\Delta_{\mathbf{X}}(X) = (0, X, ?_{X^*}, ?_{X^*})$
 197 and $\Delta_{\Sigma}(X)$ is simply $(?_{O_{\Sigma}}, !_X)$.

198 ► **Proposition 3.18.** U_{Σ} has a left adjoint Δ_{Σ} .

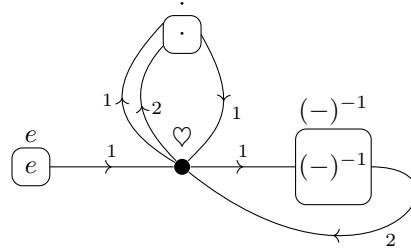
199 **Proof.** Let $(h, !_{V_{\mathcal{H}}}): \mathcal{H} \rightarrow \mathcal{G}^{\Sigma}$ be an object of **Hyp $_{\Sigma}$** , and suppose that there exists $f: X \rightarrow$
 200 $U_{\Sigma}(\mathcal{H})$. Since, $U_{\Sigma}(\mathcal{H}) = U_{\mathbf{X}}(\mathcal{H})$ and the identity is the unit of $\Delta_{\mathbf{Hyp}} \dashv U_{\mathbf{Hyp}}$, we get a
 201 morphism $(?_{E_{\mathcal{H}}}, f): \Delta_{\mathbf{X}}(X) \rightarrow \mathcal{H}$ of **Hyp**. But then we have

$$202 \quad (h, !_{V_{\mathcal{H}}}) \circ (?_{E_{\mathcal{H}}}, f) = (h \circ ?_{E_{\mathcal{H}}}, !_{V_{\mathcal{H}}} \circ f) = (?_{O_{\Sigma}}, !_X) = \Delta_{\mathbf{Hyp}}(X)$$

203 and the thesis follow. ◀

204 We will extend our graphical notation of hypergraphs to labeled ones putting the label of
 205 an hyperedge h inside its corresponding square.

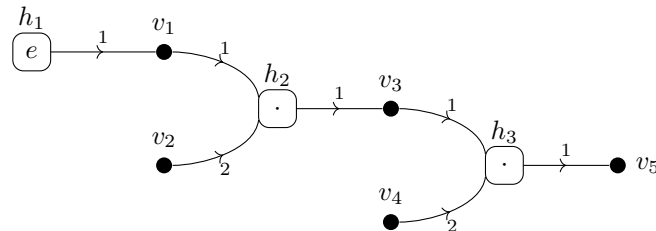
206 ► **Example 3.19.** The simplest example is given by the identity $\text{id}_{\mathcal{G}^{\Sigma}}: \mathcal{G}^{\Sigma} \rightarrow \mathcal{G}^{\Sigma}$. If Σ is the
 207 signature of groups Σ_G we get



209 ► **Example 3.20.** Take again Σ_G the signature of groups, then the hypergraph \mathcal{G} of Ex-
 210 ample 3.11 can be labeled defining

$$211 \quad e = f(h_1) \quad \cdot = f(h_2) \quad \cdot = f(h_3)$$

212 In this case we get the following picture



215

anche questo
forse val la
pena toglierlo

3.2 Term Graphs

Let us start using labelled hypergraphs to define term graphs.

► **Definition 3.21.** Given an algebraic signature Σ , we say that a labelled hypergraph $(l, !_{V_G}): \mathcal{G} \rightarrow \mathcal{G}^\Sigma$ is a term graph if t_G is mono. We define \mathbf{TG}_Σ to be the full subcategory of \mathbf{Hyp}_Σ and denote by I_Σ the inclusion. Restricting $U_\Sigma: \mathbf{Hyp}_\Sigma \rightarrow \mathbf{Set}$ we get a forgetful functor $U_{\mathbf{TG}_\Sigma}: \mathbf{TG}_\Sigma \rightarrow \mathbf{Set}$.

► **Remark 3.22.** By Remark 3.17, we know that if \mathcal{G} is a term graph then $t_G = v_1 \circ \tau_G$, where v_1 is the coprojection of V_G into V_G^* . Notice that since t_G is mono then τ_G is mono too.

We are now going back to examine the properties of \mathbf{TG}_Σ , with the purpose of studying its adhesivity properties.

► **Proposition 3.23.** The forgetful functor $U_{\mathbf{TG}_\Sigma}$ has a left adjoint $\Delta_{\mathbf{TG}_\Sigma}$.

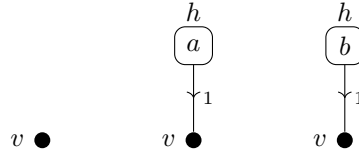
Proof. This follows noticing that $\Delta_\Sigma(X)$ is a term graph for every object X . ◀

We can list some categorical properties of \mathbf{TG}_Σ

► **Proposition 3.24.** For every algebraic signature Σ , the following are true:

1. if $(i, j): \mathcal{H} \rightarrow \mathcal{G}$ is a mono between $(l, !_{V_G}): \mathcal{G} \rightarrow \mathcal{G}^\Sigma$ and $(l', !_{V_H}): \mathcal{H} \rightarrow \mathcal{G}^\Sigma$ in \mathbf{Hyp}_Σ and the latter is a term graph, then also the former is in \mathbf{TG}_Σ
2. \mathbf{TG}_Σ has equalizers, binary products and pullbacks and they are created by I_Σ .

► **Remark 3.25.** \mathbf{TG}_Σ in general does not have terminal objects. Consider an algebraic signature in \mathbf{Set} . Since $U_{\mathbf{TG}_\Sigma}$ preserves limits, if a terminal object exists it must have the singleton as set of nodes, therefore the set of hyperedges must be empty or a singleton $\{h\}$. Now take as signature the one given by two operations a and b , both of arity 0; we have three term graphs with only one node v : $\Delta_{\mathbf{TG}_\Sigma}(\{v\})$, $(l_a, !_{V_G}): \mathcal{G}_a \rightarrow \mathcal{G}^\Sigma$ and $(l_b, !_{V_G}): \mathcal{G}_b \rightarrow \mathcal{G}^\Sigma$.



There are no morphisms in \mathbf{TG}_Σ between the last two and from the last two to the first one, therefore none of them can be terminal.

► **Remark 3.26.** \mathbf{TG}_Σ is not an adhesive category. In particular it does not have pushouts along all monos. For instance, if we take the three term graphs of the previous remark, then have two arrows $(?_{\{h\}}, \text{id}_{\{v\}}): \Delta_{\mathbf{TG}_\Sigma}(\{v\}) \rightarrow (l_a, !_{V_{\mathcal{G}_a}})$ and $(?_{\{h\}}, \text{id}_{\{v\}}): \Delta_{\mathbf{TG}_\Sigma}(\{v\}) \rightarrow (l_b, !_{V_{\mathcal{G}_b}})$ which cannot be completed to a square. Indeed if $(q, !_{V_H}): \mathcal{H} \rightarrow \mathcal{G}^\Sigma$ is another term graph with $(g_E, g_V): (l_a, !_{V_G}) \rightarrow (q, !_{V_H})$ and $(k_E, k_V): (l_b, !_{V_G}) \rightarrow (q, !_{V_H})$ such that

$$(g_E, g_V) \circ (?_{\{h\}}, \text{id}_{\{v\}}) = (k_E, k_V) \circ (?_{\{h\}}, \text{id}_{\{v\}})$$

then $g_V = k_V$ and

$$t_{\mathcal{H}}(g_E(h)) = g_V^*(t_{\mathcal{G}}(h)) = g_V^*(\delta_v) = k_V^*(\delta_v) = k_V^*(t_{\mathcal{G}}(h)) = t_{\mathcal{H}}(k_E(h))$$

so that we also have $g_E = k_E$, but then

$$a = l_a(h) = q(g_E(h)) = q(k_E(h)) = l_b(h) = b$$

► **Definition 3.27.** Let $(l, !_{V_G}): \mathcal{G} \rightarrow \mathcal{G}^\Sigma$ be a term graph. A input node is an element of V_G not in the image of $\tau_{\mathcal{H}}$. A morphism (f, g) between $(l, !_{V_G}): \mathcal{G} \rightarrow \mathcal{G}^\Sigma$ and $(l', !_{V_{\mathcal{H}}}): \mathcal{H} \rightarrow \mathcal{G}^\Sigma$ in \mathbf{TG}_Σ , is said to preserve input nodes if g sends input nodes to input nodes.

► **Remark 3.28.** Suppose that $(f, g): ((l, !_{V_G})) \rightarrow (l', !_{V_{\mathcal{H}}})$ preserves input nodes. Then if $\tau_{\mathcal{H}}(h) = g(v)$ for some $v \in V_G$ then h belongs to the image of f . Indeed, by hypothesis v must be in the image of τ_G and so there exists k such that $\tau_G(k) = v$. But then $\tau_{\mathcal{H}}(f(k)) = g(v)$ and we can conclude that $f(k) = h$.

Preservation of inputs, characterizes regular monos in \mathbf{TG}_Σ .

► **Proposition 3.29.** The following are equivalent for a mono (i, j) between two term graphs $(l, !_{V_G}): \mathcal{G} \rightarrow \mathcal{G}^\Sigma$ and $(l', !_{V_{\mathcal{H}}}): \mathcal{H} \rightarrow \mathcal{G}^\Sigma$:

1. (i, j) is a regular mono;
2. (i, j) preserves input nodes.

This characterization, in turn, provides us with the following result [7, 6].

► **Lemma 3.30.** Consider three term graphs $(l_0, !_{V_G}): \mathcal{G} \rightarrow \mathcal{G}^\Sigma$, $(l_1, !_{V_{\mathcal{H}}}): \mathcal{H} \rightarrow \mathcal{G}^\Sigma$ and $(l_2, !_{V_{\mathcal{K}}}): \mathcal{K} \rightarrow \mathcal{G}^\Sigma$. Given $(f_1, g_1): (l_0, !_{V_G}) \rightarrow (l_1, !_{V_{\mathcal{H}}})$, $(f_2, g_2): (l_0, !_{V_G}) \rightarrow (l_2, !_{V_{\mathcal{K}}})$, if (f_1, g_1) is a regular mono, then its pushout along (f_2, g_2) , then their pushout $(p, !_{V_{\mathcal{P}}}): \mathcal{P} \rightarrow \mathcal{G}^\Sigma$ in \mathbf{Hyp}_Σ is a term graph too.

Theorem 2.4, Proposition 3.29 and Lemma 3.30 allow us to recover the following result, previously proved by direct computation in [8, Thm. 4.2].

► **Corollary 3.31.** The category \mathbf{TG}_Σ is quasiadhesive.

4 Hypergraphs and term graphs with equivalences

► **Definition 4.1.** A hypergraph with equivalence $\mathcal{G} = (E_{\mathcal{G}}, V_{\mathcal{G}}, C_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}}, q_{\mathcal{G}})$ is a 6-tuple such that $\mathcal{G} = (E_{\mathcal{G}}, V_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}})$ is a hypergraph, $C_{\mathcal{G}}$ is an object and $q_{\mathcal{G}}: V_{\mathcal{G}} \rightarrow C_{\mathcal{G}}$ is a regular epi called quotient map. A morphism $h: \mathcal{G} \rightarrow \mathcal{H}$ is a triple (h_E, h_V, h_C) such that the following diagrams commute

$$\begin{array}{ccccc} E_{\mathcal{G}} & \xrightarrow{s_{\mathcal{G}}} & V_{\mathcal{G}}^* & & E_{\mathcal{G}} & \xrightarrow{t_{\mathcal{G}}} & V_{\mathcal{G}}^* & & V_{\mathcal{G}} & \xrightarrow{q_{\mathcal{G}}} & C_{\mathcal{G}} \\ h_E \downarrow & & \downarrow h_V^* & & h_E \downarrow & & \downarrow h_V^* & & h_V \downarrow & & \downarrow h_C \\ E_{\mathcal{H}} & \xrightarrow{s_{\mathcal{H}}} & V_{\mathcal{H}}^* & & E_{\mathcal{H}} & \xrightarrow{t_{\mathcal{H}}} & V_{\mathcal{H}}^* & & V_{\mathcal{H}} & \xrightarrow{q_{\mathcal{H}}} & C_{\mathcal{H}} \end{array}$$

The category of hypergraphs with equivalences and their morphisms is denoted \mathbf{EqHyp} .

► **Remark 4.2.** Morphisms of hypergraphs with equivalences are uniquely determined by the first two components. That is, if $h_1 = (h_E, h_V, f)$ and $h_2 = (h_E, h_V, g)$ are two morphisms $\mathcal{G} \rightarrow \mathcal{H}$, then we have

$$\begin{array}{ccccc} V_{\mathcal{G}} & \xrightarrow{h_V} & V_{\mathcal{H}} & \xleftarrow{h_V} & V_{\mathcal{G}} \\ q_{\mathcal{G}} \downarrow & & \downarrow q_{\mathcal{H}} & & \downarrow q_{\mathcal{G}} \\ C_{\mathcal{G}} & \xrightarrow{f} & C_{\mathcal{H}} & \xleftarrow{g} & C_{\mathcal{G}} \end{array}$$

Hence $f \circ q_{\mathcal{G}} = q_{\mathcal{H}} \circ h_V = g \circ q_{\mathcal{G}}$, and since $q_{\mathcal{G}}$ is epi, we obtain $f = g$.

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EqHyp has a forgetful functor $U_{\mathbf{EqHyp}} : \mathbf{EqHyp} \rightarrow \mathbf{Set}$, which sends each $\mathcal{G} = (E_{\mathcal{G}}, V_{\mathcal{G}}, C_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}}, q_{\mathcal{G}})$ into $V_{\mathcal{G}}$, and each $h = (h_E, h_V, h_C)$ onto h_V .

► **Proposition 4.3.** $U_{\mathbf{EqHyp}}$ has a left adjoint $\Delta_{\mathbf{EqHyp}} : \mathbf{Set} \rightarrow \mathbf{EqHyp}$.

Proof. For each set X , define $\Delta_{\mathbf{EqHyp}}(X) := (\emptyset, X, \{\bullet\}, ?_X, ?_X, !_X)$. Consider now $h : \Delta_{\mathbf{EqHyp}}(X) \rightarrow \mathcal{H}$.

$$\begin{array}{ccc} \Delta_{\mathbf{EqHyp}}(X) & & \\ \Delta_{\mathbf{EqHyp}}(f) \downarrow & \searrow h & \\ \Delta_{\mathbf{EqHyp}}(U_{\mathbf{EqHyp}}(\mathcal{H})) & \xrightarrow{\epsilon_{\mathcal{H}}} & \mathcal{H} \end{array}$$

Where $\Delta_{\mathbf{EqHyp}}(U_{\mathbf{EqHyp}}(\mathcal{H})) = (\emptyset, V_{\mathcal{H}}, \{\bullet\}, ?_{V_{\mathcal{H}}}, ?_{V_{\mathcal{H}}}, !_V)$ and $\epsilon_{\mathcal{H}} = (?_{E_{\mathcal{H}}}, \text{id}_{V_{\mathcal{H}}}, g)$. Note that, since $\Delta_{\mathbf{EqHyp}}(X)$ has the empty set as object of edges, $h_E = ?_{E_{\mathcal{H}}}$, then, the unique arrow that fits in the diagram is $\Delta_{\mathbf{EqHyp}}(f) = (?_{E_{\mathcal{H}}}, h_V, \text{id}_{\{\bullet\}})$. ◀

We now define another functor $T : \mathbf{EqHyp} \rightarrow \mathbf{Hyp}$, which “forgets” the quotient part, mapping each hypergraph with equivalence $\mathcal{G} = (E_{\mathcal{G}}, V_{\mathcal{G}}, C_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}}, q_{\mathcal{G}})$ onto $T(\mathcal{G}) = (E_{\mathcal{G}}, V_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}})$. Then, we have the following result.

► **Proposition 4.4.** T has a left adjoint $L : \mathbf{Hyp} \rightarrow \mathbf{EqHyp}$.

Proof. Let \mathcal{G} be a hypergraph, and define $L(\mathcal{G}) := (E_{\mathcal{G}}, V_{\mathcal{G}}, \{\bullet\}, s_{\mathcal{G}}, t_{\mathcal{G}}, !_V)$. Let now $h : L(\mathcal{G}) \rightarrow \mathcal{H}$ be a morphism in \mathbf{EqHyp} , and consider the following situation.

$$\begin{array}{ccc} L(\mathcal{G}) & & \\ L(f) \downarrow & \searrow h & \\ L(T(\mathcal{H})) & \xrightarrow{\epsilon_{\mathcal{H}}} & \mathcal{H} \end{array}$$

for $L(T(\mathcal{H})) = (E_{\mathcal{H}}, V_{\mathcal{H}}, \{\bullet\}, s_{\mathcal{H}}, t_{\mathcal{H}}, !_V)$. Then, $\epsilon_{\mathcal{H}} = (\text{id}_{E_{\mathcal{H}}}, \text{id}_{V_{\mathcal{H}}}, h_C)$ (by Remark 4.2, the last component is uniquely determined by the first two), and $L(f)$ must be $(h_E, h_V, \text{id}_{\{\bullet\}})$. ◀

► **Remark 4.5.** T is faithful. Indeed, consider two morphisms $h = (h_E, h_V, h_C)$ and $k = (k_E, k_V, k_C)$, and suppose $T(h) = T(k)$, that is, $(h_E, h_V) = (k_E, k_V)$. By Remark 4.2, we can conclude also $h_C = k_C$, and hence the faithfulness of T .

Let now $K : \mathbf{EqHyp} \rightarrow \mathbf{Set}$ be the functor which sends each hypergraph with equivalence $\mathcal{G} = (E, V, C, s, t, q)$ onto $K(\mathcal{G}) = C$, and each morphism (h_E, h_V, h_C) to h_C .

► **Proposition 4.6.** \mathbf{EqHyp} is complete and cocomplete, and T preserves limits and colimits.

Proof. Let $D : \mathbf{I} \rightarrow \mathbf{EqHyp}$ be a diagram, and, for each $i \in \mathbf{I}$, $D(i) = (E_i, V_i, C_i, s_i, t_i, q_i)$. Suppose now (E, V, s, t) , together with morphisms (π_i^E, π_i^V) , be the limit of $T \circ D$. Then, V , together with $(q_i \circ \pi_i^V)_{i \in \mathbf{I}}$, is a cone for $K \circ D$. Indeed, let $\alpha : i \rightarrow j$ be an arrow of \mathbf{I} , $D(\alpha) = (h_E, h_V, h_C)$. By definition of T , $(T \circ D)(\alpha) = (h_E, h_V)$, hence we have

$$\begin{array}{ccc} & V & \\ \pi_V^i \swarrow & & \searrow \pi_V^j \\ V_i & \xrightarrow{h_V} & V_j \\ q_i \downarrow & & \downarrow q_j \\ C_i & \xrightarrow{h_C} & C_j \end{array}$$

Suppose now that L , with morphisms $(l_i)_{i \in \mathbf{I}}$ be the limit of $K \circ D$. Hence, we have an arrow $l : V \rightarrow L$, which is not epi in general. Let then $l = m \circ q$ be the epi-mono factorization of it. Consider the following situation, where the outer rectangle commutes by definition, and the dotted arrow is yielded by (cite left lifting prop).

$$\begin{array}{ccccc}
 V & \xrightarrow{\pi_V^i} & V_i & \xrightarrow{q_i} & C_i \\
 q \downarrow & & \searrow \pi_C^i & \nearrow & \downarrow \text{id}_{C_i} \\
 C & \xrightarrow{m} & L & \xrightarrow{l_i} & C_i
 \end{array}$$

Thus, (E, V, C, s, t, q) , together with $(\pi_E^i, \pi_V^i, \pi_C^i)$ is a cone over D . remain to show that this cone is terminal

Suppose now (E', V', s', t') , together with $(\kappa_E^i, \kappa_V^i)_{i \in \mathbf{I}}$, be the colimit of $T \circ D$, and C' , with $(c_i)_{i \in \mathbf{I}}$ be the colimit of $K \circ D$. Then, we have the following situation.

$$\begin{array}{ccccc}
 & & V' & & \\
 & \nearrow \kappa_i & & \nwarrow \kappa_j & \\
 V_i & \xrightarrow{h_V} & V_j & & \\
 q_i \downarrow & & & & \downarrow q_j \\
 C_i & \xrightarrow{h_C} & C_j & & \\
 c_i \searrow & & \swarrow c_j & & \\
 & & C' & &
 \end{array}$$

Then, C' with morphisms $(c_i \circ q_i)_{i \in \mathbf{I}}$ is a conone for $U \circ D$. Then, there exists a unique morphism $q' : V' \rightarrow C'$ such that $q' \circ \kappa_V^i = c_i \circ q_i$. Such morphism is epi (cite Lemma 1.3.45 of the thesis), and thus (E', V', C', s', t', q') , together with $(\kappa_E^i, \kappa_V^i, c_i)_{i \in \mathbf{I}}$ is the colimit of D .

► **Corollary 4.7.** *An arrow $h = (h_E, h_V, h_C) : \mathcal{G} \rightarrow \mathcal{H}$ is mono if and only if $T(h)$ is mono.*

Proof. The “if” part is given by the faithfulness of T . The “only if” part is given by Remark 4.2.

► **Corollary 4.8.** *If $h = (h_E, h_V, h_C) : \mathcal{G} \rightarrow \mathcal{H}$ is a regular mono in **EqHyp**, then h_E, h_V and h_C are all monos.*

Proof. If h is mono, from Corollary 4.7 we have that h_E and h_V are monos. Suppose now $f, g : \mathcal{H} \rightrightarrows \mathcal{K}$ be the arrows equalized by h . Then, we have:

$$\begin{aligned}
 f_C \circ h_C \circ q_G &= f_C \circ q_{\mathcal{H}} \circ h_V \\
 &= q_{\mathcal{K}} \circ g_V \circ h_V \\
 &= q_{\mathcal{K}} \circ f_V \circ h_V \\
 &= g_C \circ h_C \circ q_G
 \end{aligned}$$

Since $q_{\mathcal{G}}$ is epi, we have $f_C \circ h_C = g_C \circ h_C$, hence h_C is an equalizer for f_C and g_C , and thus a mono.

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► **Proposition 4.9.** *Let $h = (h_E, h_V, h_C) : \mathcal{G} \rightarrow \mathcal{H}$ be a regular mono in **EqHyp**. Then, h_E and h_V are monos and (K, π_1, π_2) is the kernel pair of $q_{\mathcal{H}} \circ h_V$ if and only if (K, π_1, π_2) is the kernel pair of $q_{\mathcal{G}}$.*

Proof. By Corollary 4.8, we have that h_E, h_V and h_C are all monos. Hence, by Proposition 2.10, (K, π_1, π_2) is the kernel pair of $q_{\mathcal{G}}$ if and only if it is the kernel pair also of $h_C \circ q_{\mathcal{G}}$, since h_C is mono by hypothesis. The thesis follows from $h_C \circ q_{\mathcal{G}} = q_{\mathcal{H}} \circ h_V$, and from the hypothesis of h_E mono. ◀

► **Remark 4.10.** It is possible to restate the last proposition, by ??, as

h_E and h_V are mono and, for every $v, v' \in V_H$, $q_H(h_V(v)) = q_H(h_V(v'))$ if and only if $q_G(v) = q_G(v')$

That is, a regular mono in **EqHyp** is a morphism that both reflects and preserves equivalences.

Let us turn to another functor **EqHyp** \rightarrow **Hyp**.

► **Definition 4.11.** *The quotient functor $Q : \mathbf{EqHyp} \rightarrow \mathbf{Hyp}$ is defined as the one sending (E, V, C, s, t, q) to $(E, C, q^* \circ s, q^* \circ t)$ and an arrow (h_E, h_V, h_C) to (h_E, h_C) .*

► **Remark 4.12.** The action of the functor on a morphism of hypergraphs with equivalences gives a morphism of hypergraphs, in fact $q_{\mathcal{H}}^* \circ s_{\mathcal{H}} \circ h_E = q_{\mathcal{H}}^* \circ h_V^* \circ s_{\mathcal{G}} = h_C^* \circ q_{\mathcal{G}}^* \circ s_{\mathcal{G}}$. The same is valid for $t_{\mathcal{H}}$ and $t_{\mathcal{G}}$.

► **Lemma 4.13.** *Q is a left adjoint.*

Proof. Let $R((A, B, s, t))$ be $(A, B, B, s, t, \text{id}_B)$, so that $Q(R((A, B, s, t))) = (A, B, s, t)$. Now, suppose that $h = (h_E, h_V) : Q((E, V, C, s', t', q)) \rightarrow (A, B, s, t)$ is an arrow in **Hyp**, and consider the triple $(h_E, h_V, h_V \circ q)$. Since h is a morphism of **Hyp**, we have $h_V^* \circ q^* \circ s' = s \circ h_E$ and $h_V^* \circ q^* \circ t' = t \circ h_E$. Then we have the following squares

$$\begin{array}{ccc} \begin{array}{ccc} E & \xrightarrow{h_E} & A \\ s' \downarrow & & \downarrow s \\ V^* & \xrightarrow{h_V^* \circ q^*} & B^* \end{array} & \begin{array}{ccc} E & \xrightarrow{h_E} & A \\ t' \downarrow & & \downarrow t \\ V^* & \xrightarrow{h_V^* \circ q^*} & B^* \end{array} & \begin{array}{ccc} V & \xrightarrow{h_V \circ q} & B \\ q \downarrow & & \downarrow \text{id}_B \\ C & \xrightarrow{h_V} & B \end{array} \end{array}$$

We have therefore found a morphism $(E, V, C, s', t', q) \rightarrow R((A, B, s, t))$ whose image through Q fits in the diagram below

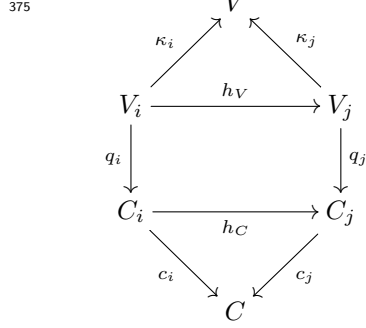
$$\begin{array}{ccc} (A, B, s, t) & \xrightarrow{(\text{id}_A, \text{id}_B)} & (A, B, s, t) \\ \uparrow Q(h_E, h_V \circ q, h_V) & \nearrow (h_E, h_V) & \\ (E, C, q^* \circ s', q^* \circ t') & & \end{array}$$

Such arrow is unique. Suppose $f = (f_E, f_V, f_C)$ to be another arrow with such property. Then, it must be $(\text{id}_A, \text{id}_B) \circ Q(f) = (f_E, f_C) = (h_E, h_C)$. Finally, $f_C = f_V \circ q = h_V \circ q$. ◀

► **Proposition 4.14.** *Q creates colimits.*

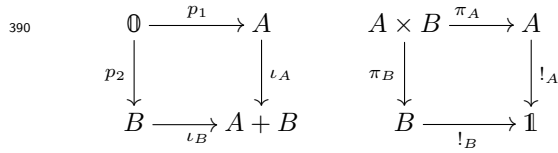
Proof. Since Q is a left adjoint, it preserves colimits. Let $D : \mathbf{I} \rightarrow \mathbf{EqHyp}$ be a diagram, and let \mathcal{C} , together with $(c_i)_{i \in \mathbf{I}}$ be the colimit of $Q \circ D$, where $\mathcal{C} = (A, C, q \circ s, q \circ t)$, and

373 $D(i)$ is $(A_i, B_i, C_i, s_i, t_i, q_i)$. Let $((\kappa_i)_{i \in \mathbf{I}}, V)$ be the colimit of $U_{\mathbf{EqHyp}} \circ D$. Consider the
 374 following situation



376 Now, since $((c_C^i \circ q_i)_{i \in \mathbf{I}}, C)$ is a cocone for $U_{\mathbf{EqHyp}} \circ D$, there exists a unique $q : V \rightarrow C$,
 377 which is epi by Lemma 2.15. Consider now the functor $W : \mathbf{EqHyp} \rightarrow \mathbf{Set}$ mapping each
 378 (X, Y, Z, x, y, z) onto X , and each morphism on its first component. By Proposition 4.6
 379 and ??, we have that $((c_E^i)_{i \in \mathbf{I}}, E)$ is the colimit of $W \circ D$. Notice that $((\kappa_i \circ s_i)_{i \in \mathbf{I}}, B)$ and
 380 $((\kappa_i \circ t_i)_{i \in \mathbf{I}}, B)$ are cocones for $W \circ D$, so let s and t be, respectively, the mediating arrow
 381 for the first one and the mediating arrow for the second one. It remains now to show that
 382 (E, V, C, s, t, q) , together with $(c_E^i, \kappa_i, c_C^i)_{i \in \mathbf{I}}$, is a colimit for D , but this follows by the proof
 383 of ??.

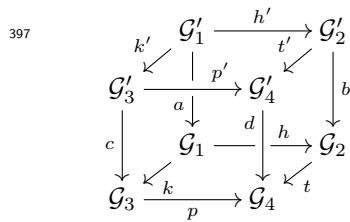
384 ► **Example 4.15.** Q does not preserve limits. Indeed, let $\mathcal{G}_1 = (E_1, A, A, s_1, t_1, id_A)$, $\mathcal{G}_2 =$
 385 $(E_2, B, B, s_2, t_2, id_B)$ and $\mathcal{G}_3 = (E_3, A + B, \mathbb{1}, s_3, t_3, !_{A+B})$, and let $h = (h_E, \iota_A, !_A) : \mathcal{G}_1 \rightarrow \mathcal{G}_3$,
 386 $k = (k_E, \iota_B, !_B) : \mathcal{G}_2 \rightarrow \mathcal{G}_3$, where $(\iota_A, \iota_B, A + B)$ is the coproduct of A and B , $\mathbb{1}$ is the initial
 387 object (in \mathbf{Set} , the singleton set as shown in ??), and $!_X$ the unique arrow $X \rightarrow \mathbb{1}$. The
 388 following two diagrams show the pullback of h and k and the pullback of $Q(h)$ and $Q(k)$, on
 389 the second component (the vertices of the graphs)



391 But the arrow $0 \rightarrow A \times B$ is not epi in general (this is easy to see taking \mathbf{Set} as example),
 392 hence such pullback is not preserved by Q .

393 ► **Lemma 4.16.** In \mathbf{EqHyp} , pushouts along regular monos are stable.

394 **Proof.** Let $\mathcal{G}_i = (A_i, B_i, C_i, s_i, t_i, q_i)$, $\mathcal{G}'_i = (A'_i, B'_i, C'_i, s'_i, t'_i, q'_i)$, for $i \in \{1, 2, 3, 4\}$, be hy-
 395 pergraphs with equivalence, and, in the diagram below, suppose all the vertical faces are
 396 pullbacks, the bottom face is a pushout and h is regular mono.



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By Proposition 4.6 and Corollary 4.7, the following cubes in **Set** have pushouts as bottom faces and pullbacks as vertical faces, hence their top faces are pushouts.

$$\begin{array}{ccc}
 & A'_1 & \xrightarrow{h'_E} A'_2 \\
 k'_E \swarrow & \downarrow p'_E & \swarrow t'_E \\
 A'_3 & \xrightarrow{a_E} A'_4 & \\
 c_E \downarrow & \downarrow d_E & \downarrow b_E \\
 A_3 & \xrightarrow{p_E} A_4 & \\
 & \downarrow h_E & \\
 & A_2 &
 \end{array}
 \quad
 \begin{array}{ccc}
 & B'_1 & \xrightarrow{h'_V} B'_2 \\
 k'_V \swarrow & \downarrow p'_V & \swarrow t'_V \\
 B'_3 & \xrightarrow{a_V} B'_4 & \\
 c_V \downarrow & \downarrow d_V & \downarrow b_V \\
 B_3 & \xrightarrow{p_V} B_4 & \\
 & \downarrow h_V & \\
 & B_2 &
 \end{array}$$

Consider now the following pullbacks.

$$\begin{array}{ccc}
 Y & \xrightarrow{y_1} & C'_4 \\
 y_2 \downarrow & & \downarrow d_C \\
 C_3 & \xrightarrow{p_C} & C_4
 \end{array}
 \quad
 \begin{array}{ccc}
 U & \xrightarrow{u_1} & C'_4 \\
 u_2 \downarrow & & \downarrow d_C \\
 C_2 & \xrightarrow{t_C} & C_4
 \end{array}
 \quad
 \begin{array}{ccc}
 T & \xrightarrow{x_1} & U \\
 x_2 \downarrow & & \downarrow u_2 \\
 C_1 & \xrightarrow{h_C} & C_2
 \end{array}$$

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proposizione

Thus, [???] yields the following situation, in which the bottom face is a pushout, and the vertical faces are pullbacks, hence the top face is a pushout too.

$$\begin{array}{ccccc}
 & & T & \xrightarrow{x_1} & U \\
 & & \downarrow y_1 & \swarrow u_1 & \downarrow u_2 \\
 Y & \xrightarrow{y_1} & C'_4 & & \\
 y_2 \downarrow & \swarrow x_2 & \downarrow d_C & \downarrow h_C & \\
 C_3 & \xrightarrow{p_C} & C_4 & & C_2
 \end{array}$$

By the proof of Proposition 4.6, we have that $m_2 \circ q'_3 : B'_3 \rightarrow Y$ and $m_3 \circ q'_2 : B'_2 \rightarrow U$ are two epi-mono factorizations, with m_2 and m_3 monos. At the same way, let the following square to be a pullback.

$$\begin{array}{ccc}
 S & \xrightarrow{s_1} & C'_2 \\
 s_2 \downarrow & & \downarrow m_3 \\
 T & \xrightarrow{x_1} & U
 \end{array}$$

Hence, in the following diagram, the outer rectangle is a pullback.

$$\begin{array}{ccc}
 S & \xrightarrow{s_1} & C'_2 \\
 s_2 \downarrow & & \downarrow m_3 \\
 T & \xrightarrow{x_1} & U \\
 x_2 \downarrow & & \downarrow u_2 \\
 C_1 & \xrightarrow{h_C} & C_2
 \end{array}$$

By the same argument as before, there exists a mono m_1 such that $m_1 \circ q'_1 : B'_1 \rightarrow S$.

We have to show that the top face of the cube at the beginning of the proof is a pushout.

Suppose then that $z : \mathcal{G}'_2 \rightarrow \mathcal{H}$ and $w : \mathcal{G}'_3 \rightarrow \mathcal{H}$, with $\mathcal{H} = (E, V, C, s, t, q)$, are two morphisms

such that $z \circ h' = w \circ h'$, and let $v_V : B'_4 \rightarrow V$ the arrow induced by z_V and w_V . We want to construct the dotted arrow v_C which fits in the diagram below.

$$\begin{array}{ccccc}
 & B'_1 & \xrightarrow{h'_V} & B'_2 & \\
 & \swarrow k'_V & \downarrow p'_V & \swarrow k'_V & \downarrow z_V \\
 B'_3 & \xrightarrow{q'_1} & B'_4 & \xrightarrow{q'_2} & V \\
 \downarrow q'_3 & \downarrow k'_C & \downarrow q'_4 & \downarrow h'_C & \downarrow z_C \\
 C'_3 & \xrightarrow{p'_C} & C'_4 & \xrightarrow{t'_C} & C \\
 & & & \swarrow v_C & \\
 & & & & q
 \end{array}$$

By Lemma 2.12, we know that the top face of the cube below is a pushout.

$$\begin{array}{ccccc}
 & K_{s_2 \circ m_1 \circ q'_1} & \xrightarrow{k_{h'_2}} & K_{m_2 \circ q'_2} & \\
 & \swarrow k_{k'_2} & \downarrow k_{p'_2} & \swarrow k_{t'_2} & \downarrow \pi^1_{m_2 \circ q'_2} \\
 K_{m_3 \circ q'_3} & \xrightarrow{\pi^1_{s_2 \circ m_1 \circ q'_1}} & K_{q'_4} & \xrightarrow{\pi^1_{q'_4}} & B'_2 \\
 \downarrow \pi^1_{m_3 \circ q'_3} & \downarrow \pi^1_{q'_4} & \downarrow h'_2 & \downarrow t'_2 & \\
 B'_3 & \xrightarrow{p'_2} & C_4 & \xrightarrow{t'_2} & \\
 & & & \swarrow k'_2 & \\
 & & & & B'_1
 \end{array}$$

And, since m_3 and m_2 are monos,

$$q'_3 \circ \pi^1_{m_3 \circ q'_3} = q'_3 \circ \pi^2_{m_3 \circ q'_3} \quad q'_2 \circ \pi^1_{m_2 \circ q'_2} = q'_2 \circ \pi^2_{m_2 \circ q'_2}$$

Computing, we obtain

$$\begin{aligned}
 q \circ v_V \circ \pi^1_{q'_4} \circ k_{p'_V} &= q \circ v_V \circ p'_V \circ \pi^1_{m_3 \circ q'_3} & q \circ v_V \circ \pi^1_{q'_4} \circ k_{t'_V} &= q \circ v_V \circ t'_V \circ \pi^1_{m_2 \circ q'_2} \\
 &= q \circ w_V \circ \pi^1_{m_3 \circ q'_3} & &= q \circ z_V \circ \pi^1_{m_2 \circ q'_2} \\
 &= w_C \circ q'_3 \circ \pi^1_{m_3 \circ q'_3} & &= z_C \circ q'_2 \circ \pi^1_{m_2 \circ q'_2} \\
 &= w_C \circ q'_3 \circ \pi^2_{m_3 \circ q'_3} & &= z_C \circ q'_2 \circ \pi^2_{m_2 \circ q'_2} \\
 &= q \circ w_V \circ \pi^2_{m_3 \circ q'_3} & &= q \circ z_V \circ \pi^2_{m_2 \circ q'_2} \\
 &= q \circ v_V \circ p'_V \circ \pi^2_{m_3 \circ q'_3} & &= q \circ v_V \circ t'_V \circ \pi^2_{m_2 \circ q'_2} \\
 &= q \circ v_V \circ \pi^2_{q'_4} \circ k_{p'_V} & &= q \circ v_V \circ \pi^2_{q'_4} \circ k_{t'_V}
 \end{aligned}$$

Since the previous cube has a pushout as top face, by universal property, we have

$$q \circ v_V \circ \pi^1_{q'_4} = q \circ v_V \circ \pi^2_{q'_4}$$

hence, v_C is the mediating arrow.

$$v_C \circ q'_4 \circ \pi^1_{q'_4} = v_C \circ q'_4 \circ \pi^2_{q'_4}$$

428

► **Lemma 4.17.** *In EqHyp, pushouts along regular monos are Reg(EqHyp)-Van Kampen.*

429

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Proof. In lieu of Lemma 4.16, it is enough to proof that, given a cube as the one below, with pullbacks as back faces, pushouts as bottom and top faces and such that h is a regular mono, the front faces are pullbacks too, where $\mathcal{G}_i = (A_i, B_i, C_i, s_i, t_i, q_i)$, $\mathcal{G}' = (A'_i, B'_i, C'_i, s'_i, t'_i, q'_i)$, for $i = 1, 2, 3, 4$.

$$\begin{array}{ccccc}
 & & \mathcal{G}'_1 & \xrightarrow{h'} & \mathcal{G}'_2 \\
 & \swarrow k' & \downarrow p' & \swarrow t' & \downarrow b \\
 \mathcal{G}'_3 & \xrightarrow{a} & \mathcal{G}'_4 & & \\
 \downarrow c & & \downarrow d & \searrow h & \\
 \mathcal{G}_1 & \xrightarrow{\quad} & \mathcal{G}_2 & & \\
 \downarrow k & & \downarrow t & & \\
 \mathcal{G}_3 & \xrightarrow{p} & \mathcal{G}_4 & &
 \end{array}$$

By Proposition 4.6 and ??, the following two cubes have \mathcal{M} -pushouts as bottom faces and pullbacks as back faces, thus their front faces are pullbacks too.

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 & & A'_1 & \xrightarrow{h'_E} & A'_2 \\
 & \swarrow k'_E & \downarrow p'_E & \swarrow t'_E & \downarrow b_E \\
 A'_3 & \xrightarrow{a_E} & A'_4 & & \\
 \downarrow c_E & & \downarrow d_E & \searrow h_E & \\
 A_1 & \xrightarrow{\quad} & A_2 & & \\
 \downarrow k_E & & \downarrow t_E & & \\
 A_3 & \xrightarrow{p_E} & A_4 & &
 \end{array}
 &
 &
 \begin{array}{ccccc}
 & & B'_1 & \xrightarrow{h'_V} & B'_2 \\
 & \swarrow k'_V & \downarrow p'_V & \swarrow t'_V & \downarrow b_V \\
 B'_3 & \xrightarrow{a_V} & B'_4 & & \\
 \downarrow c_V & & \downarrow d_V & \searrow h_V & \\
 B_1 & \xrightarrow{\quad} & B_2 & & \\
 \downarrow k_V & & \downarrow t_V & & \\
 B_3 & \xrightarrow{p_V} & B_4 & &
 \end{array}
 \end{array}$$

On the other hand we can consider the diagrams below, in which the inner squares are pullbacks. Since the outer diagrams commute, by definition of morphism of **EqHyp**, then we have the existence of $m_2: C'_2 \rightarrow U$, $m_3: C'_3 \rightarrow Y$, $a_3: B'_3 \rightarrow Y$ and $a_2: B'_2 \rightarrow Y$.

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 C'_3 & & & & \\
 \downarrow d_3 & \searrow m_3 & \searrow p'_3 & & \\
 & Y & \xrightarrow{y_1} & C'_4 & \\
 & \downarrow y_2 & & \downarrow d_3 & \\
 & C_3 & \xrightarrow{p_3} & C_4 &
 \end{array}
 &
 &
 \begin{array}{ccccc}
 C'_2 & & & & \\
 \downarrow d_2 & \searrow m_2 & \searrow t'_3 & & \\
 & U & \xrightarrow{u_1} & C'_4 & \\
 & \downarrow u_2 & & \downarrow d_3 & \\
 & C_2 & \xrightarrow{t_3} & C_4 &
 \end{array}
 \\
 \\
 \begin{array}{ccccc}
 B'_3 & \xrightarrow{p'_2} & B'_4 & & \\
 \downarrow q'_3 & \searrow a_3 & \searrow q'_4 & & \\
 C'_3 & & Y & \xrightarrow{y_1} & C'_4 \\
 \downarrow d_3 & \searrow y_2 & & \downarrow d_3 & \\
 & C_3 & \xrightarrow{p_3} & C_4 &
 \end{array}
 &
 &
 \begin{array}{ccccc}
 B'_2 & \xrightarrow{t'_2} & B'_4 & & \\
 \downarrow q'_2 & \searrow a_2 & \searrow q'_4 & & \\
 C'_2 & & U & \xrightarrow{u_1} & C'_4 \\
 \downarrow d_2 & \searrow u_2 & & \downarrow d_3 & \\
 & C_2 & \xrightarrow{t_3} & C_4 &
 \end{array}
 \end{array}$$

Now, notice that m_3 and m_2 are monos because d_3 and d_2 are regular monos. By the proof of Proposition 4.6, to conclude it is enough to show that

$$m_3 \circ q'_3 = a_3 \quad m_2 \circ q'_2 = a_2$$

Indeed, if the previous equations hold, then C'_3 and C'_2 are epi-mono factorizations of a_3 and a_2 and the thesis follows from ?? and the proof of Proposition 4.6.

448 No if we compute we have:

$$\begin{aligned}
 y_1 \circ a_3 &= q'_4 \circ p'_2 & u_1 \circ a_2 &= q'_4 \circ t'_2 \\
 &= p'_3 \circ q'_3 & &= t'_3 \circ q'_3 \\
 449 \quad &= y_1 \circ m_3 \circ q'_3 & &= u_1 \circ m_2 \circ q'_2 \\
 450 \quad y_2 \circ a_3 &= d_3 \circ q'_3 & u_2 \circ a_2 &= d_2 \circ q'_2 \\
 451 \quad &= y_2 \circ m_3 \circ q'_3 & &= u_2 \circ m_2 \circ q'_2
 \end{aligned}$$

452 And we have done. ◀

453 4.1 Labeled Hypergraphs with Equivalences

454 As we have done in Section 3.1.2, we can define the category of hypergraphs with equivalence
455 labeled over an algebraic signature.

456 ► **Definition 4.18.** Let $\Sigma = (O_\Sigma, \text{ar}_\Sigma)$ be an algebraic signature, and let \mathcal{G}^Σ the hypergraph
457 associated to Σ . Then, the hypergraph with equivalence associated to Σ is $L(\mathcal{G}^\Sigma)$, and the
458 category of hypergraphs with equivalence labeled over Σ is the slice category $\mathbf{EqHyp}_\Sigma :=$
459 $\mathbf{EqHyp}/L(\mathcal{G}^\Sigma)$.

460 By ??, we can deduce the following.

461 ► **Proposition 4.19.** \mathbf{EqHyp}_Σ is $\text{Reg}(\mathbf{EqHyp}_\Sigma)$ -adhesive.

462 We can lift the adjunction given by T and L to \mathbf{EqHyp}_Σ and \mathbf{Hyp}_Σ .

463 4.2 title

464 5 EGGs

introduction

466 ► **Definition 5.1.** Let $\mathcal{G} = (E, V, C, s, t, q)$ be a hypergraph with equivalence, and let (S, π_1, π_2)
467 be the kernel pair of $q^* \circ s$. Then, \mathcal{G} is an e-hypergraph whenever $q^* \circ t \circ \pi_1 = q^* \circ t \circ \pi_2$. **EGG**
468 is the full subcategory of \mathbf{EqHyp} where objects are e-hypergraphs, and $I : \mathbf{EGG} \rightarrow \mathbf{EqHyp}$
469 is the inclusion functor.

470 ► **Lemma 5.2.** \mathbf{EGG} has all limits, and I preserves them.

471 **Proof.** Let $D : \mathbf{I} \rightarrow \mathbf{EGG}$ be a diagram, with $D(i) = (A_i, B_i, C_i, s_i, t_i, q_i)$, let (U_i, u_1^i, u_2^i) be
472 the kernel pair of $q_i \circ s_i$. Let now be (A, B, C, s, t, q) , together with projections $(\pi_E^i, \pi_V^i, \pi_C^i)_{i \in \mathbf{I}}$
473 the limit of $I \circ D$, let (U, u_1, u_2) be the kernel pair of $q \circ s$ and let $(L, (l_i)_{i \in \mathbf{I}})$ be the limit of
474 $K \circ I \circ D$. By construction (proof of Proposition 4.6), there exists a mono $m : C \rightarrow L$ such
475 that $\pi_C^i = l_i \circ m$. Notice that

$$\begin{aligned}
 476 \quad q_i^* \circ s_i \circ \pi_E^i \circ u_1 &= q_i^* \circ (\pi_V^i)^* \circ s \circ u_1 \\
 477 \quad &= (\pi_C^i)^* \circ q^* \circ s \circ u_1 \\
 478 \quad &= (\pi_C^i)^* \circ q^* \circ s \circ u_2 \\
 479 \quad &= q_i^* \circ s_i \circ \pi_E^i \circ u_2 \\
 480
 \end{aligned}$$

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481 Then, for each i , there exists an arrow $a_i : U \rightarrow U_i$ making the following diagram to commute

$$\begin{array}{ccccc}
 482 & U & \xrightarrow{u_1} & A & \\
 & \downarrow u_2 & \searrow a_i & \downarrow \pi_E^i & \\
 & A & & U_i & \xrightarrow{u_1^i} A_i \\
 & \searrow \pi_E^i & & \downarrow u_2^i & \downarrow q_i^* \circ s_i \\
 & & & A_i & \xrightarrow{q_i^* \circ s_i} C_i^*
 \end{array}$$

483 We have then

$$\begin{aligned}
 484 \quad l_i^* \circ m^* \circ q^* \circ t \circ u_1 &= q_i^* \circ (\pi_V^i)^* \circ t \circ u_1 \\
 485 &= q_i^* \circ t_i \circ \pi_E^i \circ u_1 \\
 486 &= q_i^* \circ t_i \circ u_1^i \circ a_i \\
 487 &= q_i^* \circ t_i \circ u_2^i \circ a_i \\
 488 &= q_i^* \circ t_i \circ \pi_E^i \circ u_2 \\
 489 &= q_i^* \circ (\pi_V^i)^* \circ t \circ u_2 \\
 490 &= l_i^* \circ m^* \circ q^* \circ t \circ u_2 \\
 491
 \end{aligned}$$

492 By universal property of limits, we have that $m^* \circ q^* \circ t \circ u_1 = m^* \circ q^* \circ t \circ u_2$, and, since m
 493 is mono, $q^* \circ t \circ u_1 = q^* \circ t \circ u_2$, hence the thesis. \blacktriangleleft

494 **► Corollary 5.3.** *I creates limits.*

495 **► Corollary 5.4.** *$h : \mathcal{G} \rightarrow \mathcal{H}$ is a regular mono in **EGG** if and only if it is a regular mono*
 496 *in **EqHyp**.*

497 **► Lemma 5.5.** *Consider the following pushout square in **EqHyp**.*

$$\begin{array}{ccc}
 498 & \mathcal{G}_1 & \xrightarrow{h} \mathcal{G}_2 \\
 & \downarrow m & \downarrow n \\
 & \mathcal{G}_3 & \xrightarrow{k} \mathcal{P}
 \end{array}$$

499 *with m regular mono. If \mathcal{G}_1 , \mathcal{G}_2 and \mathcal{G}_3 are e-hypergraphs, then \mathcal{P} is an e-hypergraph too,*
 500 *and n is regular mono.*

501 **Proof.** Let $\mathcal{P} = (A, B, C, s, t, q)$, (K_i, π_i^1, π_i^2) the kernel pair of $q_i^* \circ s_i$, and let (U, u_1, u_2) the
 502 kernel pair of $q^* \circ s$. Consider then the following situation.

$$\begin{array}{ccccc}
 503 & & A_1 & \xrightarrow{h_E} & A_2 \\
 & m_E \swarrow & \downarrow k_E & \swarrow k_E & \downarrow q_2^* \circ s_2 \\
 & A_3 & \xrightarrow{q_1^* \circ s_1} & A & \\
 & \downarrow q_3^* \circ s_3 & \downarrow q^* \circ s & \downarrow h_C^* & \downarrow q_2^* \circ s_2 \\
 & C_3^* & \xrightarrow{m_C^*} & C^* & \xrightarrow{n_C^*} C_2^* \\
 & & \downarrow k_C^* & & \\
 & & C^* & &
 \end{array}$$

504 Since m is regular mono, m_E is mono (inserire citazione). Then, by construction, the top
 505 face is a pushout, and since **Set** is adhesive, by Lemma 2.12, the square below is a pushout.

$$\begin{array}{ccc}
 K_1 & \xrightarrow{f_k} & K_2 \\
 f_m \downarrow & & \downarrow f_n \\
 K_3 & \xrightarrow{f_k} & U
 \end{array}$$

507 Computing, we have

$$\begin{aligned}
 q^* \circ t \circ u_1 \circ f_n &= q^* \circ t \circ n_E \circ \pi_2^1 & q^* \circ t \circ u_1 \circ f_k &= q^* \circ t \circ k_E \circ \pi_3^1 \\
 &= n_C^* \circ q_2^* \circ s_2 \circ \pi_2^1 & &= k_C^* \circ q_3^* \circ s_3 \circ \pi_3^1 \\
 &= n_C^* \circ q_2^* \circ s_2 \circ \pi_2^2 & &= k_C^* \circ q_3^* \circ s_3 \circ \pi_3^2 \\
 508 \quad &= q^* \circ t \circ u_2 \circ f_n & &= q^* \circ t \circ u_2 \circ f_k
 \end{aligned}$$

509 By universal property of pushouts, we deduce $q^* \circ t \circ u_1 = q^* \circ t \circ u_2$, and the thesis follows. ◀

510 By direct application of Theorem 2.4, we can conclude what follows.

511 ▶ **Corollary 5.6.** *EGG is $\text{Reg}(\mathbf{EGG})$ -adhesive.*

512 6 Conclusions and further works

513 The aim of our paper was to extend the theory of adhesive categories in order to include
 514 EGGS, an up-and-coming formalism for program optimisation and synthesis via a compact
 515 representation and efficient implementation of equality saturation. To do so, we revisited and
 516 generalised the notions of hyper-graph and term graph with equivalence, and we extended it
 517 in order to capture EGGS as term graphs satisfying a suitable closure property.

518 Our result opens two threads of research. The first is to use the quasi-adhesivity of EGGs
 519 to model their rewriting via the double-pushout (DPO) approach. This seems now easy,
 520 since the rules adopted in the literature of EGGs appears to be span of regular monos, and
 521 such rules perfectly fit the mold of rewriting on quasi-adhesive categories. For example, the
 522 equivalence $x \div x = 1$, from the introductory example in [?], can be modelled as the rule

523 *todrawDPOrule*

524 It still needs to be investigated what concurrency and termination, the key properties for
 525 DPO rewriting on adhesive categories, mean in the context of EGGs. More interestingly,
 526 another venue for development is using the adhesive machinery to extend the EGGs formalism.
 527 In fact, most of the results presented here for hyper-graphs can be generalised to hierarchical
 528 hyper-graphs, that is, hypergraphs with a hierarchy (a partial order) among edges that is
 529 useful for adding structural information, such as encapsulation and sandboxing [?].

530 Finally, we need to draw a comparison with an alternative categorical presentation for
 531 EGGs advanced in [13]. The proposal is quite different from our own. Simplifying, the key
 532 is to equip categories of trees with a lattice on hom-sets. Instead of going through such an
 533 enrichment, we chose to obtain closure via suitable requirements on morphisms, even if it
 534 seems that the former generalises the latter, at the expenses of a more complex machinery.

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576 **A** Omitted proofs

577 This section contains the proofs which are omitted from the main body of the paper. We
578 begin stating a well-known fact about composition and decomposition of pullbacks [17].

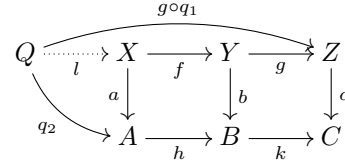
► **Lemma A.1.** *Let \mathbf{X} be a category, and consider the diagram*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ a \downarrow & & \downarrow b & & \downarrow c \\ A & \xrightarrow{h} & B & \xrightarrow{k} & C \end{array}$$
aside, in which the right square is a pullback. Then the whole rectangle is a pullback if and only if the left square is one.

580 **Proof.** (\Rightarrow) Let $q_1: Q \rightarrow Y$ and $q_2: Q \rightarrow A$ be two arrows such that $b \circ q_1 = h \circ q_2$, if we
 581 compute we get

$$582 \quad c \circ g \circ q_1 = k \circ b \circ q_1 = k \circ h \circ q_2$$

Thus by the pullback property of the whole rectangle we
 583 get the dotted l in the diagram on the side. All we have
 to prove is that $f \circ l = q_1$. By construction we know that
 $g \circ f \circ l = g \circ q_1$, while we also have

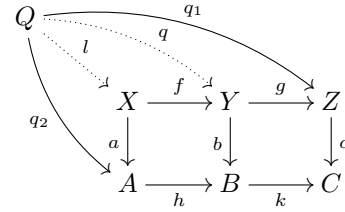


$$584 \quad b \circ f \circ l = h \circ a \circ l = h \circ q_2 = b \circ q_1$$

585 and we can conclude since the right square in the original diagram is a pullback.

586 For uniqueness: if $l': Q \rightarrow X$ is such that $f \circ l' = q_1$ and $a \circ l' = q_2$ then $g \circ f \circ l' = g \circ q_1$
 587 and we can conclude applying the pullback property of the outer rectangle.

(\Leftarrow) Take two arrows $q_1: Q \rightarrow Z$ and $q_2: Q \rightarrow A$ such that
 $c \circ q_1 = k \circ h \circ q_2$. We can apply the pullback property of the
 588 right square to get the dotted $q: Q \rightarrow Y$ in the following
 Now, by construction we have $b \circ q = h \circ q_2$ and thus, since
 the left square is a pullback, we get also a unique $l: Q \rightarrow X$
 such that $f \circ l = q$ and $a \circ l = q_2$ but then we clearly have



$$589 \quad g \circ f \circ l = g \circ q = q_1$$

590 We are left with uniqueness. Let $l': Q \rightarrow X$ be another arrow such that $q_1 = g \circ f \circ l'$
 591 and $q_2 = a \circ l'$, then we must also have

$$592 \quad b \circ f \circ l' = h \circ a \circ l' = h \circ q_2 = b \circ q$$

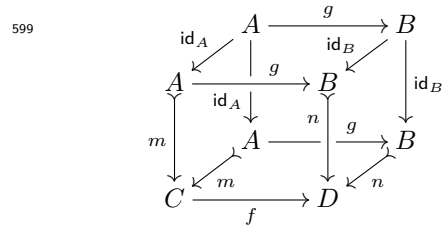
593 which implies $f \circ l' = q$, from which $l = l'$ follows. ◀

594 A.1 Proofs for Section 2

595 ► **Proposition 2.6.** *If \mathbf{X} is \mathcal{M} -adhesive then the following are true:*

- 596 1. *every \mathcal{M} -pushout square is also a pullback;*
- 597 2. *every arrow in \mathcal{M} is a regular mono.*

598 **Proof.** 1. Consider the following cube in which the bottom face is an \mathcal{M} -pushout.



600 By construction the top face of the cube is a pushout and the back one a pullback. The
 601 left face is a pullback because m is mono, thus the Van Kampen property yields that the
 602 front and the right faces are pullbacks too and the thesis follows.

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2. Let $m: X \rightarrowtail Y$ be an arrow in \mathcal{M} , we can then take its pushout along itself, which, by the previous point, is also a pullback.

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ m \downarrow & & \downarrow h \\ Y & \xrightarrow{k} & Z \end{array}$$

It is now immediate to see that m is the equalizer of h and k . ◀

► **Lemma 2.11.** *Let $f: X \rightarrow Y$ and $g: Z \rightarrow W$ be two arrows admitting kernel pairs and suppose that the solid part of the three squares below is given. Then there exists a unique arrow $k_h: K_f \rightarrow K_g$ completing them.*

$$\begin{array}{ccc} X \xrightarrow{h} Z & K_f \xrightarrow{k_h} K_g & K_f \xrightarrow{k_h} K_g \\ f \downarrow & \pi_f^1 \downarrow & \pi_f^2 \downarrow \\ Y \xrightarrow{t} W & X \xrightarrow{h} Z & X \xrightarrow{h} Z \end{array}$$

Moreover, if the leftmost is a pullback, then also the other two are so.

Proof. Computing, we have

$$g \circ h \circ \pi_f^1 = t \circ f \circ \pi_f^1 = t \circ f \circ \pi_f^2 = g \circ h \circ \pi_f^2$$

Therefore the existence and uniqueness of the wanted k_h follows at once from the universal property of K_g as the pullback of g along itself.

To prove the second half of the thesis, we can notice that, by Lemma A.1, two rectangles below are pullbacks.

$$\begin{array}{ccc} K_f \xrightarrow{\pi_f^2} X \xrightarrow{h} Z & K_f \xrightarrow{\pi_f^1} X \xrightarrow{h} Z \\ \pi_f^1 \downarrow & \pi_f^2 \downarrow & \downarrow g \\ X \xrightarrow{f} Y \xrightarrow{t} W & X \xrightarrow{f} Y \xrightarrow{t} W \end{array}$$

But then the following ones are pullbacks too.

$$\begin{array}{ccc} K_f \xrightarrow{h \circ \pi_f^2} K_g \xrightarrow{\pi_g^2} Z & K_f \xrightarrow{h \circ \pi_f^2} K_g \xrightarrow{\pi_g^2} Z \\ \pi_f^1 \downarrow & \pi_f^1 \downarrow & \downarrow g \\ X \xrightarrow{h} Y \xrightarrow{g} W & X \xrightarrow{h} Y \xrightarrow{g} W \end{array}$$

The thesis now follows again by Lemma A.1. ◀

► **Lemma 2.12.** *Let \mathbf{X} be a strict \mathcal{M} -adhesive category with all pullbacks, and suppose that in the cube aside the top face is an \mathcal{M} -pushout. Then the right square is a pushout.*

$$\begin{array}{ccccc} & A' & \xrightarrow{f'} & B' & \\ m' \swarrow & \downarrow g' & \searrow n' & \downarrow b & \\ C' & \xrightarrow{a} & D' & \xrightarrow{f} & B \\ c \downarrow & \downarrow a & \downarrow d & \downarrow f & \\ C & \xrightarrow{m} & D & \xrightarrow{n} & B \end{array} \quad \begin{array}{ccc} K_a & \xrightarrow{k_{f'}} & K_b \\ k_{m'} \downarrow & & \downarrow k_{n'} \\ K_c & \xrightarrow{k_{g'}} & K_d \end{array}$$

$$\begin{array}{ccccc}
& & K_a & \xrightarrow{k_{f'}} & K_n \\
& \swarrow k_{m'} & \downarrow k_{g'} & \searrow k_{n'} & \\
K_c & \xrightarrow{\pi_a^1} & K_d & \xrightarrow{\pi_b^1} & \\
\downarrow \pi_c^1 & & \downarrow \pi_d^1 & & \downarrow \pi_b^1 \\
& A' & \xrightarrow{f'} & B' & \\
\swarrow m' & \downarrow g' & \searrow n' & & \\
C' & \xrightarrow{g'} & D' & &
\end{array}$$

Proof. By Proposition 2.6 we know that the top face of the original cube is a pullback. Thus Lemma 2.11 entails that in the following cube the vertical faces are pullbacks. The thesis now follows from strict \mathcal{M} -adhesivity. \blacktriangleleft

► **Proposition 2.13.** Let $e: X \rightarrow Y$ be a regular epi in a category \mathbf{X} with a kernel pair (K_e, π_e^1, π_e^2) . Then, e is the coequalizer of π_e^1 and π_e^2 .

Proof. By hypothesis, there exists a pair $f, g: Z \rightrightarrows X$ of which e is the coequalizer. Since $e \circ f = e \circ g$ we get the dotted arrow fitting in the digram aside.

Let now $h: Z \rightarrow Y$ be an arrow such that $h \circ \pi_1 = h \circ \pi_2$, then

$$\begin{array}{ccccc}
& & f & & \\
Z & \xrightarrow{\quad} & K_e & \xrightarrow{\pi_e^1} & X \\
& \searrow k & \downarrow \pi_e^2 & & \downarrow e \\
& & X & \xrightarrow{e} & Y \\
& \swarrow g & & &
\end{array}$$

$$h \circ f = h \circ \pi_1 \circ k = h \circ \pi_2 \circ k = h \circ g$$

and thus there exists a unique $l: Y \rightarrow V$ such that $l \circ e = h$. \blacktriangleleft

► **Corollary 2.14.** Let \mathbf{X} be a category with pullbacks and $\phi: F \rightarrow G$ a natural transformation between functors $F, G: \mathbf{D} \rightarrow \mathbf{X}$. If ϕ_d is a regular epi for every d in \mathbf{D} , then ϕ is a regular epi.

Proof. Let (K_d, π_d^1, π_d^2) be the kernel pair of ϕ_d for each object d in \mathbf{D} . Given an arrow $\alpha: d \rightarrow d'$ of \mathbf{D} , we have

$$\phi_{d'} \circ F(\alpha) \circ \pi_d^1 = G(\alpha) \circ \phi_d \circ \pi_d^1 = G(\alpha) \circ \phi_d \circ \pi_d^2 = \phi_{d'} \circ F(\alpha) \circ \pi_d^2$$

Thus, the solid part of the diagram below commutes, yielding the dotted arrow $K(\alpha)$.

$$\begin{array}{ccccc}
K_d & \xrightarrow{\pi_d^1} & F(d) & \xrightarrow{F(\alpha)} & F(d') \\
\downarrow \pi_d^2 & \searrow K(\alpha) & \downarrow \pi_{d'}^1 & \searrow \pi_{d'}^2 & \\
F(d) & & K_{d'} & \xrightarrow{\pi_{d'}^1} & F(d') \\
& \searrow F(\alpha) & \downarrow \phi_{d'} & & \\
& & F(d') & \xrightarrow{\phi_{d'}} & G(d')
\end{array}$$

In this way, we get a functor $K: \mathbf{D} \rightarrow \mathbf{X}$ mapping d to K_d and each arrow α onto $K(\alpha)$. Indeed, notice that $K(\text{id}_d): K_d \rightarrow K_d$ is the arrow such that

$$\begin{aligned}
\pi_d^1 \circ K(\text{id}_d) &= F(\text{id}_d) \circ \pi_d^1 = \text{id}_{F(d)} \circ \pi_d^1 = \pi_d^1 \\
\pi_d^2 \circ K(\text{id}_d) &= F(\text{id}_d) \circ \pi_d^2 = \text{id}_{F(d)} \circ \pi_d^2 = \pi_d^2
\end{aligned}$$

Thus, by the universal property of pullbacks, $K(\text{id}_d) = \text{id}_{K_d}$.

Let now $\alpha: a \rightarrow b$ and $\beta: b \rightarrow c$ be two arrows in \mathbf{D} , computing, we have

$$\begin{aligned}
\pi_c^1 \circ K(\beta \circ \alpha) &= F(\beta) \circ F(\alpha) \circ \pi_a^1 = F(\beta) \circ \pi_b^1 \circ K(\alpha) = \pi_c^1 \circ K(\beta) \circ K(\alpha) \\
\pi_c^2 \circ K(\beta \circ \alpha) &= F(\beta) \circ F(\alpha) \circ \pi_a^2 = F(\beta) \circ \pi_b^2 \circ K(\alpha) = \pi_c^2 \circ K(\beta) \circ K(\alpha)
\end{aligned}$$

Allowing us to conclude that $K(\beta \circ \alpha) = K(\beta) \circ K(\alpha)$, proving the functoriality of K .

Hence, by construction we have two natural transformations $\pi^1, \pi^2: K \rightrightarrows F$. By Proposition 2.13, every component ϕ_d is the coequalizer of $\pi_d^1, \pi_d^2: K(d) \rightrightarrows F$, and so ϕ is the coequalizer of π^1 and π^2 . \blacktriangleleft

► **Lemma 2.15.** Let $F, G: \mathbf{D} \rightarrow \mathbf{X}$ be two diagrams, and suppose that \mathbf{X} has all colimits of shape \mathbf{D} . Let $(X, \{x_d\}_{d \in \mathbf{D}})$ and $(Y, \{y_d\}_{d \in \mathbf{D}})$ be the colimits of F and G , respectively. If $\phi: F \rightarrow G$ is a natural transformation whose components are regular epi, then the arrow induced by ϕ from X to Y is a regular epi too.

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Proof. By Corollary 2.14, we know that $\phi: F \rightarrow G$ is a regular epi, so that there is a functor $E: \mathbf{D} \rightarrow \mathbf{X}$ and $\eta, \theta: E \rightrightarrows F$ such that ϕ is the coequalizer of η and θ . Let now $(P, \{p_d\}_{d \in \mathbf{D}})$ be the colimit of E and consider the unique arrows $a, b: P \rightrightarrows X$ fitting in the squares below

$$\begin{array}{ccc} E(d) & \xrightarrow{p_d} & P \\ \eta_d \downarrow & & \downarrow a \\ F(d) & \xrightarrow{x_d} & X \end{array} \quad \begin{array}{ccc} E(d) & \xrightarrow{p_d} & P \\ \theta_d \downarrow & & \downarrow b \\ F(d) & \xrightarrow{x_d} & X \end{array}$$

We want to show that ϕ coequalizes a and b . Let thus $h: X \rightarrow Z$ be an arrow such that $h \circ a = h \circ b$. Then, for every d , we have

$$h \circ x_d \circ \eta_d = h \circ a \circ p_d = h \circ b \circ p_d = h \circ x_d \circ \theta_d$$

Thus, there is $h_d: G(d) \rightarrow Z$ such that $h \circ x_d = h_d \circ \phi_d$. It is now easy to see that $(Z, \{h_d\}_{d \in \mathbf{D}})$ is a cocone on G . Suppose $\alpha: d \rightarrow d'$ is an arrow of \mathbf{D} , then

$$h_d \circ \phi_d = h \circ y_d = h \circ y_{d'} \circ F(\alpha) = h_{d'} \circ \phi_{d'} \circ F(\alpha) = h_{d'} \circ G(\alpha) \circ \phi_d$$

By the hypothesis ϕ_d is regular epi for each d and so we can conclude that $h_d = h_{d'} \circ G(\alpha)$. Therefore, we have an arrow $k: Y \rightarrow Z$ such that $k \circ y_d = h_d$. But then

$$k \circ \phi \circ x_d = k \circ y_d \circ \phi_d = h_d \circ \phi_d = h \circ x_d$$

Showing that $k \circ \phi = h$.

For the uniqueness, let $k': Y \rightarrow Z$ be another arrow such that $k' \circ \phi = h$. Then we have

$$k' \circ y_d \circ \phi_d = k' \circ \phi \circ x_d = h \circ x_d = h_d \circ \phi_d$$

Since ϕ_d is a regular epi, we have $k' \circ y_d = h_d$ allowing us to conclude. \blacktriangleleft

B Some properties of comma categories

In this section we will briefly recall the definition of the comma category [17] associated to two functors and some of its properties.

Definition B.1. Let $L: \mathbf{A} \rightarrow \mathbf{X}$ and $R: \mathbf{B} \rightarrow \mathbf{X}$ be two functors with the same codomain, the comma category $L \downarrow R$ is the category in which

- objects are triples (A, B, f) with $A \in \mathbf{A}$, $B \in \mathbf{B}$, and $f: L(A) \rightarrow R(B)$;
- a morphism $(A, B, f) \rightarrow (A', B', g)$ is a pair (h, k) with $h: A \rightarrow A'$ in \mathbf{A} and $k: B \rightarrow B'$ in \mathbf{B} such that the following diagram commutes

$$\begin{array}{ccc} L(A) & \xrightarrow{L(h)} & L(A') \\ f \downarrow & & \downarrow g \\ R(B) & \xrightarrow{R(k)} & R(B') \end{array}$$

We have two forgetful functors $U_L: L \downarrow R \rightarrow \mathbf{A}$ and $U_R: L \downarrow R \rightarrow \mathbf{B}$ given, respectively by

$$\begin{array}{ccc} (A, B, f) & \mapsto & A \\ (h, k) \downarrow & & \downarrow h \\ (A', B', g) & \mapsto & A' \end{array} \quad \begin{array}{ccc} (A, B, f) & \mapsto & B \\ (h, k) \downarrow & & \downarrow k \\ (A', B', g) & \mapsto & B' \end{array}$$

Given $L: \mathbf{A} \rightarrow \mathbf{X}$ and $R: \mathbf{B} \rightarrow \mathbf{X}$, we can also consider their duals $L^{op}: \mathbf{A}^{op} \rightarrow \mathbf{X}^{op}$ and $R^{op}: \mathbf{B}^{op} \rightarrow \mathbf{X}^{op}$. An arrow $f: L(A) \rightarrow R(B)$ in \mathbf{X} is the same thing as an arrow

674 $f: R^{op}(B) \rightarrow L^{op}(A)$ in \mathbf{X}^{op} , thus $(L \downarrow R)$ and $R^{op} \downarrow L^{op}$ have the same objects. Moreover,
 675 the commutativity in \mathbf{X} of the square

$$676 \quad \begin{array}{ccc} L(A) & \xrightarrow{L(h)} & L(A') \\ f \downarrow & & \downarrow g \\ R(B) & \xrightarrow{R(k)} & R(B') \end{array}$$

677 is tantamount to the commutativity in \mathbf{X}^{op} of the square

$$678 \quad \begin{array}{ccc} R(B') & \xrightarrow{R(k)} & R(B) \\ g \downarrow & & \downarrow f \\ L(A') & \xrightarrow{L(h)} & L(A) \end{array}$$

679 Summing up we have just proved the following fact.

680 ► **Proposition B.2.** $(L \downarrow R)^{op}$ is equal to $R^{op} \downarrow L^{op}$, moreover $U_L^{op} = U_{L^{op}}$ and $U_R^{op} = U_{R^{op}}$.

681 ► **Lemma B.3.** Let $L: \mathbf{A} \rightarrow \mathbf{X}$ and $R: \mathbf{B} \rightarrow \mathbf{X}$ be functors and $F: \mathbf{D} \rightarrow L \downarrow R$ be a diagram
 682 such that L preserves colimits along $U_L \circ F$. Then the family $\{U_L, U_R\}$ jointly creates colimits
 683 of F (see [6, 7]).

684 **Proof.** Suppose that $U_L \circ F$ and $U_R \circ F$ have colimiting cocones $(A, \{a_D\}_{D \in \mathbf{D}})$ and $(B, \{b_D\}_{D \in \mathbf{D}})$
 685 respectively. By hypothesis $(L(A), \{L(a_D)\}_{D \in \mathbf{D}})$ is colimiting for $L \circ U_L \circ F$. Now, if we
 686 define

$$687 \quad F(D) := (A_D, B_D, f_D)$$

688 then we have arrows $R(a_i) \circ f_D: L(A_D) \rightarrow R(B)$ that forms a cocone on $L \circ U_L \circ F$: if
 689 $d: D \rightarrow D'$ is an arrow in \mathbf{D} then $F(d)$ is an arrow in $L \downarrow R$ and so

$$\begin{aligned} 690 \quad R(b_{D'}) \circ f_{D'} \circ L(U_L(F(d))) &= R(b_{D'}) \circ R(U_R(F(d))) \circ f_D \\ 691 &= R(b_{D'} \circ U_R(F(d))) \circ f_D \\ 692 &= R(b_D) \circ f_D \end{aligned}$$

694 Thus there exists $f: L(A) \rightarrow R(B)$ such that

$$695 \quad \begin{array}{ccc} L(A_D) & \xrightarrow{L(a_D)} & L(A) \\ f_D \downarrow & & \downarrow f \\ R(B_D) & \xrightarrow{R(b_D)} & R(B) \end{array}$$

696 Notice that f is the unique arrow in \mathbf{X} which makes (a_D, b_D) an arrow $(A_D, B_D, f_D) \rightarrow$
 697 (A, B, f) of $L \downarrow R$. If we show that $((A, B, f), \{(a_D, b_D)\}_{D \in \mathbf{D}})$ is colimiting for F we are
 698 done.

699 First of all, let us show that it is a cocone. Given $d: D \rightarrow D'$ in \mathbf{D} we have:

$$\begin{aligned} 700 \quad (a_{D'}, b_{D'}) \circ F(d) &= (a_{D'}, b_{D'}) \circ (U_L(F(d)), U_R(F(d))) \\ 701 &= (a_{D'} \circ U_L(F(d)), b_{D'} \circ U_R(F(d))) \\ 702 &= (a_D, b_D) \end{aligned}$$

703

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For the colimiting property, let $((X, Y, g), \{(x_D, y_D)\}_{D \in \mathbf{D}})$ be another cocone on F . In particular $(X, \{x_D\}_{D \in \mathbf{D}})$ and $(Y, \{y_D\}_{D \in \mathbf{D}})$ are cocones on $U_L \circ F$ and $U_R \circ F$ respectively, so we have uniquely determined arrows $x: A \rightarrow X$ and $y: B \rightarrow Y$ such that

$$x \circ a_D = x_D \quad y \circ b_D = y_D$$

Let us show that (x, y) is an arrow of $L \downarrow R$. Given $D \in \mathbf{D}$ we have

$$\begin{aligned} R(y) \circ f \circ L(a_D) &= R(y) \circ R(b_D) \circ f_D \\ &= R(y \circ b_D) \circ f_D \\ &= R(y_D) \circ f_D \\ &= g \circ L(x_D) \\ &= g \circ L(x \circ a_D) \\ &= g \circ L(x) \circ L(a_D) \end{aligned}$$

from which it follows that the following diagram commutes.

$$\begin{array}{ccc} L(A) & \xrightarrow{L(x)} & X \\ f \downarrow & & \downarrow g \\ R(B) & \xrightarrow{R(y)} & Y \end{array}$$

This shows that $((A, B, f), \{(a_D, b_D)\}_{D \in \mathbf{D}})$ is colimiting for F and the thesis follows. \blacktriangleleft

Proposition B.2 and Lemma B.3 now yields the following.

► **Corollary B.4.** *The family $\{U_L, U_R\}$ jointly creates limits along every diagram $F: \mathbf{D} \rightarrow L \downarrow R$ such that R preserves the limit of $U_R \circ I$.*

We can use Corollary B.4 to characterize monos in comma categories.

► **Corollary B.5.** *If R preserves pullbacks then an arrow (h, k) in $L \downarrow R$ is mono if and only if both h and k are monos.*

Proof. (\Rightarrow) If $(h, k): (A, B, f) \rightarrow (A', B', g)$ is a mono then the following square is a pullback in $L \downarrow R$

$$\begin{array}{ccc} (A, B, f) & \xrightarrow{\text{id}_{(A, B, f)}} & (A, B, f) \\ \text{id}_{(A, B, f)} \downarrow & & \downarrow (h, k) \\ (A, B, f) & \xrightarrow{(h, k)} & (A', B', g) \end{array}$$

Using Corollary B.4 we deduce that the following two squares are pullbacks in \mathbf{A} and \mathbf{B} .

$$\begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ \text{id}_A \downarrow & & \downarrow h \\ A & \xrightarrow{h} & A' \end{array} \quad \begin{array}{ccc} B & \xrightarrow{\text{id}_B} & B \\ \text{id}_B \downarrow & & \downarrow k \\ B & \xrightarrow{k} & B' \end{array}$$

From which it follows that h and k are monos.

(\Leftarrow) Since h and k are monos then we have two pullback squares

$$\begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ \text{id}_A \downarrow & & \downarrow h \\ A & \xrightarrow{h} & A' \end{array} \quad \begin{array}{ccc} B & \xrightarrow{\text{id}_B} & B \\ \text{id}_B \downarrow & & \downarrow k \\ B & \xrightarrow{k} & B' \end{array}$$

By Corollary B.4 this implies that

$$\begin{array}{ccc} (A, B, f) & \xrightarrow{\text{id}_{(A, B, f)}} & (A, B, f) \\ \text{id}_{(A, B, f)} \downarrow & & \downarrow (h, k) \\ (A, B, f) & \xrightarrow{(h, k)} & (A', B', g) \end{array}$$

is a pullback in $L \downarrow R$ and we are done. \blacktriangleleft

We end this section pointing out another useful fact, showing that in some cases we can guarantee the existence of a left adjoint to U_R .

► **Proposition B.6.** *If \mathbf{A} has initial objects and L preserves them then the forgetful functor $U_R: L \downarrow R \rightarrow \mathbf{B}$ has a left adjoint Δ .*

Proof. For an object $B \in \mathbf{B}$ we can define $\Delta(B)$ as $(0, B, ?_B)$, where 0 is an initial object in \mathbf{A} and $?_{R(B)}$ is the unique arrow $L(0) \rightarrow R(B)$. Consider $\text{id}_B: B \rightarrow U_R(\Delta(B))$ be the identity, and suppose that a $k: B \rightarrow U_R(A, B', f)$ in \mathbf{B} is given. By initiality of 0 , there is only one arrow $?_A: 0 \rightarrow A$ in \mathbf{A} and, since L preserves initial objects, the following square commutes.

$$\begin{array}{ccc} L(0) & \xrightarrow{L(?_A)} & L(A) \\ ?_{R(B)} \downarrow & & \downarrow f \\ R(B) & \xrightarrow{R(k)} & R(B') \end{array}$$

Thus (h, k) is the unique morphism $\Delta(B) \rightarrow (A, B', f)$ such that $U_R(h, k) = k$. \blacktriangleleft

Dualizing we get immediately the following.

► **Corollary B.7.** *If \mathbf{B} has terminal objects preserved by R then $U_L: L \downarrow R \rightarrow \mathbf{A}$ has a right adjoint.*

B.1 Slice categories

This section is devoted to recall some basic facts about the so called *slice categories*.

► **Definition B.8.** *Let X be an object of a category \mathbf{X} , we will define the following two categories.*

■ *The slice category over X is the category \mathbf{X}/X which has as objects arrows $f: Y \rightarrow X$ and in which an arrow $h: f \rightarrow g$ is $h: Y \rightarrow Y'$ in \mathbf{X} such that the following triangle commutes.*

$$\begin{array}{ccc} Y & \xrightarrow{h} & Y' \\ & \searrow f & \swarrow g \\ & X & \end{array}$$

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758 ■ Dually, the slice category under X is the category X/\mathbf{X} in which objects are arrows
 759 $f: X \rightarrow Y$ with domain X and a morphism $h: f \rightarrow g$ is an arrow of \mathbf{X} fitting in a
 760 triangle as the one below.

$$761 \quad \begin{array}{ccc} & X & \\ f \swarrow & & \searrow g \\ Y & \xrightarrow{h} & Y' \end{array}$$

762 ► Remark B.9. For every $X \in \mathbf{X}$ we have forgetful functors

$$763 \quad \begin{array}{ccc} \text{dom}_X: \mathbf{X}/X \rightarrow \mathbf{X} & \text{cod}_X: X/\mathbf{X} \rightarrow \mathbf{X} \\ f \mapsto \text{dom}(f) & f \mapsto \text{cod}(f) \\ h \downarrow \quad \downarrow h & h \downarrow \quad \downarrow h \\ g \mapsto \text{dom}(g) & g \mapsto \text{cod}(g) \end{array}$$

764 We can realize the slice over and under an object $X \in \mathbf{X}$ as comma categories.

765 ► **Proposition B.10.** For every object X in a category \mathbf{X} , if $\delta_X: \mathbf{1} \rightarrow \mathbf{X}$ is the constant
 766 functor of value X from the category with only one object $*$, then \mathbf{X}/X and X/\mathbf{X} are
 767 isomorphic to, respectively, $\text{id}_X \downarrow \delta_X$ and $\delta_X \downarrow \text{id}_X$.

768 **Proof.** Define functors $F_1: \text{id}_X \downarrow \delta_X \rightarrow \mathbf{X}/X$ and $G_1: \mathbf{X}/X \rightarrow \text{id}_X \downarrow \delta_X$ as follows

$$769 \quad \begin{array}{ccc} (Y, *, f) \mapsto f & f \mapsto (Y, *, f) \\ (h, \text{id}_*) \downarrow \quad \downarrow h & h \downarrow \quad \downarrow (h, \text{id}_*) \\ (Y', *, g) \mapsto g & g \mapsto (Y', *, g) \end{array}$$

770 Similarly, we have $F_2: \delta_X \downarrow \text{id}_X \rightarrow X/\mathbf{X}$ and $G_2: X/\mathbf{X} \rightarrow \delta_X \downarrow \text{id}_X$

$$771 \quad \begin{array}{ccc} (*, Y, f) \mapsto f & f \mapsto (*, Y, f) \\ (\text{id}_*, h) \downarrow \quad \downarrow h & h \downarrow \quad \downarrow (\text{id}_*, h) \\ (*, Y', g) \mapsto g & g \mapsto (*, Y', g) \end{array}$$

772 It is now obvious to see that F_1, G_1 and F_2, G_2 are pairs of inverses. ◀

773 A straightforward application of Corollary B.4 now yields the following.

774 ► **Corollary B.11.** If \mathbf{X} has pullbacks, then for every object X , the slice \mathbf{X}/X has pullbacks
 775 too.

776 In a category \mathbf{X} with pullbacks, each $f: X \rightarrow Y$ induces a functor $f^*: \mathbf{X}/Y \rightarrow \mathbf{X}/X$,
 777 which sends each morphism $a: A \rightarrow Y$ of \mathbf{X} onto its pullback along f , p_a , and each morphism
 778 $h: a \rightarrow b$ onto the unique arrow from the pullback of a along f to the pullback of b along f .
 779 Then, we have the following result.

780 ► **Proposition B.12.** Let \mathbf{X} be a category with pullbacks, $R: \mathbf{Y} \rightarrow \mathbf{X}$ be a functor and
 781 $L: \mathbf{X} \rightarrow \mathbf{Y}$ be its left adjoint, and η the unit of the adjunction. Then, each object X of \mathbf{X}
 782 induces an adjoint pair of functors $L_X: \mathbf{X}/X \rightarrow \mathbf{Y}/L(X)$, $R_X: \mathbf{Y}/L(X) \rightarrow \mathbf{X}/X$, where
 783 L_X is the obvious functor, and R_X is the composite $(\eta_X)^* \circ R$.

Proof. Let $f : L(l) \rightarrow b$ be a morphism of $\mathbf{Y}/L(X)$, where $l : A \rightarrow X$ in \mathbf{X} and $b : B \rightarrow L(X)$ in \mathbf{Y} , as shown below.

$$\begin{array}{ccc} L(A) & \xrightarrow{f} & B \\ & \searrow L(l) \quad \swarrow b & \\ & L(X) & \end{array}$$

Then, we have the following situation in \mathbf{X} .

$$\begin{array}{ccccc} & & g & & \\ & \curvearrowright & & \curvearrowleft & \\ A & \xrightarrow{\eta_A} & R(L(A)) & \xrightarrow{R(f)} & R(B) \\ & \searrow l & \searrow R(L(l)) & \swarrow R(b) & \\ & X & \xrightarrow{\eta_X} & R(L(X)) & \end{array}$$

Where g is the adjunct of f . Consider now the pullback P of $R(p)$ along η_X , as shown below.

$$\begin{array}{ccc} A & \xrightarrow{g} & R(B) \\ \downarrow l & \swarrow v & \downarrow R(b) \\ P & \xrightarrow{q} & R(L(X)) \\ \downarrow \eta_X^*(R(b)) & & \downarrow \eta_X \\ X & \xrightarrow{\eta_X} & R(L(X)) \end{array}$$

By universal property of pullbacks, the two diagrams express the same morphism g . Hence, we can rewrite the first diagram as follows.

$$\begin{array}{ccccc} & & f & & \\ & \curvearrowright & & \curvearrowleft & \\ L(A) & \xrightarrow{L(g)} & (L(R(B))) & \xrightarrow{\epsilon_B} & B \\ \downarrow L(l) & & \downarrow L(R(b)) & & \downarrow b \\ L(X) & \xrightarrow{L(\eta_X)} & L(R(L(X))) & \xrightarrow{\epsilon_{L(X)}} & L(X) \\ & & \text{id}_{L(X)} & & \end{array}$$

where ϵ is the counit of the adjunction. This describes the action of functors and thus the adjunction, obtained considering the isomorphism on hom-sets of the categories.

