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Chapter 1

Categories of Graphs

This chapter is about graphs, and how it is possible to formalize them using categories in order to point out their properties from an abstract point of view. Starting from the set-theoretical definition of graphs, we will give an abstraction via functor categories, in which a graph is nothing but a functor between a category to another.

1.1 Graphs

A (directed graph) $\mathcal{G}$ is a mathematical structure consisting of a set of edge, a set of nodes and two functions, one assigning a *source* node and one assigning a *target* node to an edge. Formally, $\mathcal{G}$ is a quadruple $(V_{\mathcal{G}}, E_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}})$, where $V_{\mathcal{G}}$ is the set of nodes, $E_{\mathcal{G}}$ is the set of edges, and $s_{\mathcal{G}}, t_{\mathcal{G}} : E_{\mathcal{G}} \to V_{\mathcal{G}}$ are the source and the target functions.

A *graph homomorphism* $h : \mathcal{G} \to \mathcal{H}$ is then a pair of functions $h = (h_V : V_{\mathcal{G}} \to V_{\mathcal{H}}, h_E : E_{\mathcal{G}} \to E_{\mathcal{H}})$ such that

$$h_V \circ s_{\mathcal{G}} = s_{\mathcal{H}} \circ h_E$$

and

$$h_V \circ t_{\mathcal{G}} = t_{\mathcal{H}} \circ h_E$$

that is, a structure preserving map.

We can then generalize such notion to something more abstract, considering a graph to be nothing more than a presheaf from the category $(E \rightrightarrows V)$ to the category of sets. Having two of such presheaves, a natural transformation from one to another encapsulates the behavior of a graph morphism due to naturality. We can now define the category of graphs.

Definition 1.1.1 (Category of Graphs)**.** We denote as **Graph** the category

$$[E \overset{s}{\underset{t}{\rightrightarrows}} V, \mathbf{Set}]$$

Since **Graph** is a category of presheaves, **??** guarantees the existence of limits and colimits, and gives us an easy way to compute them.

Corollary 1.1.2. **Graph** *has all limits and colimits.*

Example 1.1.3. The initial object in **Graph** is the empty graph, i.e., the graph with an empty set of vertices and an empty set of edges. The initial object instead is the graph with exactly one node and a single edge from that node to itself.

Example 1.1.4. Given two graphs $G = (V_G, E_G, s_G, t_G)$ and $H = (V_H, E_H, s_H, t_H)$, the graph $G \times H = (V_G \times V_H, E_G \times E_H, (s_G, s_H), (t_G, t_H))$, where $(s_G, s_H), (t_G, t_H) : V_G \times V_H \to E_G \times E_H$ are the pairwise sources and targets, is the categorical product in **Graph**, together with the two projections $\pi_G : G \times H \to G$, $\pi_H : G \times H \to H$ defined in the obvious way.

Example 1.1.5. The equalizer of two morphisms $h, k : G \to H$ in **Graph** is defined as in **Set**, that is, a graph Q together with a graph morphism q that is the restriction of G to all the vertices and all the arcs that are mapped on the same vertices and edges both from h and k. Formally, (Q, q) is an equalizer for $h, k : G \to H$, $h = (h_V, h_E), k = (k_V, k_E)$ where $V(Q) = \{n \in V(G) \mid h_V(n) = k_V(n)\}$, $E(Q) = \{e \in E(G) \mid h_E(e) = k_E(e)\}$, $s_Q = s_G\, |_{V(Q)}$, $t_E = t_G\, |_{V(Q)}$.

Remark 1.1.6. TODO: Si può generalizzare a tutte le categorie regolari per evitare di perdere le proprietà che usiamo (da eq.rel. a quot.).

1.2 Graphs with equivalences - notes and results

1.2.1 Some results on kernel pair and regular epis

Questi risultati vanno completati e distribuiti lungo la tesi: quello sui kernel pair nella parte sui pullback, quello sugli epi regolari nella parte sui coequalizzatori, quelli che riguardano l'adesività nella relativa sezione

Lemma 1.2.1. *Suppose that the following diagram is given and its right half is a pullback. Then the whole rectangle is a pullback if and only if its left half is a pullback.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "t"'] & Y \arrow[r, "g"] \arrow[d, "k"] & Z \arrow[d, "h"] \\
A \arrow[r, "a"'] & B \arrow[r, "b"'] & C
\end{tikzcd}$$

Proof. □ Esercizio per te

Corollary 1.2.2. *Let $\mathscr{C}$ be a category and suppose that the solid part of the following cube is given*

$$\begin{tikzcd}
 & Y' \arrow[rr, "g'"] \arrow[dl, dotted, "q'"'] \arrow[dd, "y" near end] & & Z' \arrow[dl, "r'"'] \arrow[dd, "z"] \\
B' \arrow[rr, "k'" near start] \arrow[dd, "b"'] & & C' \arrow[dd, "c" near start] & \\
 & Y \arrow[rr, "g" near end] \arrow[dl, "q"] & & Z \arrow[dl, "r"] \\
B \arrow[rr, "k"'] & & C &
\end{tikzcd}$$

If the front face is a pullback then there is a unique $q' \colon Y' \to B'$ filling the diagram. If, moreover, the other two vertical faces are also pullbacks, then the following square is a pullback too.

$$\begin{tikzcd}
Y' \arrow[r, "q'"] \arrow[d, "y"'] & B' \arrow[d, "b"] \\
Y \arrow[r, "q"'] & B
\end{tikzcd}$$

Proof. Let us compute:

$$\begin{aligned}
c \circ r' \circ g' &= r \circ z \circ g' \\
&= r \circ g \circ y \\
&= k \circ q \circ y
\end{aligned}$$

Since the front face is a pullback, this guarantees the existence of q'. The second half of the thesis follows applying lemma 1.2.1 to the following rectangle.

$$\begin{tikzcd}
Y' \arrow[r, "q'"'] \arrow[d, "y"'] \arrow[rr, bend left, "r' \circ g'"] & B' \arrow[r, "k'"'] \arrow[d, "b"'] & C' \arrow[d, "c"] \\
Y \arrow[r, "q"] \arrow[rr, bend right, "r \circ g"'] & B \arrow[r, "k"] & C
\end{tikzcd}$$

□

Definition 1.2.3. A *kernel pair* $(K_f, p_{f,1}, p_{f,2})$ for an arrow $f\colon X \to Y$ is an object K_f with two arrows $p_{f,1}, p_{f,2}\colon K_f \to X$ making the following square a pullback.

$$\begin{tikzcd} K_f \arrow[r, "p_{f,1}"] \arrow[d, "p_{f,2}"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Remark 1.2.4. If a category $\mathscr{C}$ has pullbacks then every arrow has a kernel pair.

Proposition 1.2.5. *An arrow $m\colon M \to X$ is mono if and only if (M, id_M, id_M) is a kernel pair for it.*

esercizio

Proof. □

Corollary 1.2.6. *Let $(K_f, p_{f,1}, p_{f,2})$ be a kernel pair for $f\colon X \to Y$. Then for every mono $m\colon Y \to Z$, $(K_f, p_{f,1}, p_{f,2})$ is a kernel pair also for $m \circ f$.*

Proof. It is enough to see that, by Lemma 1.2.1 and proposition 1.2.5 the outer boundary of the following square is a pullback.

$$\begin{tikzcd} K_f \arrow[r, "p_{f,1}"] \arrow[d, "p_{f,2}"'] & X \arrow[r, "id_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] \arrow[d, "id_X"'] & Y \arrow[r, "id_Y"'] \arrow[d, "id_Y"'] & Y \arrow[d, "m"] \\ X \arrow[r, "f"'] & Y \arrow[r, "m"'] & Z \end{tikzcd}$$

□

Lemma 1.2.7. *Suppose that the following square is given and that $f\colon X \to Y$ and $g\colon Z \to W$ have kernel pairs.*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "t"'] & W \end{tikzcd}$$

Then there exists a unique arrow $k_h \colon K_f \to K_g$ making the squares below commutes.

$$\begin{tikzcd} K_f \arrow[r, dotted, "k_h"] \arrow[d, "p_{f,1}"'] & K_g \arrow[d, "p_{g,1}"] \\ X \arrow[r, "h"'] & Z \end{tikzcd} \qquad \begin{tikzcd} K_f \arrow[r, dotted, "k_h"] \arrow[d, "p_{f,2}"'] & K_g \arrow[d, "p_{g,2}"] \\ X \arrow[r, "h"'] & Z \end{tikzcd}$$

Moreover, if the beginning square is a pullback, then also the preceding ones are so.

Proof. Computing we have

$$\begin{aligned} g \circ h \circ p_{f,1} &= t \circ f \circ p_{f,1} \\ &= t \circ f \circ p_{f,2} \\ &= g \circ h \circ p_{f,2} \end{aligned}$$

So that the wanted k_h exists, and it is unique, by the universal property of K_g as the pullback of g along itself.

To prove the second half of the thesis, let us consider the two rectangles below, which, by Lemma 1.2.1 are pullbacks.

$$\begin{tikzcd} K_f \arrow[r, "p_{f,1}"] \arrow[d, "p_{f,2}"'] & X \arrow[r, "h"] \arrow[d, "f"] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \arrow[r, "t"'] & W \end{tikzcd} \qquad \begin{tikzcd} K_f \arrow[r, "p_{f,2}"] \arrow[d, "p_{f,1}"'] & X \arrow[r, "h"] \arrow[d, "f"] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \arrow[r, "t"'] & W \end{tikzcd}$$

But then the following ones are pullbacks too.

$$\begin{tikzcd} K_f \arrow[r, "k_h"'] \arrow[rr, bend left, "h \circ p_{f,2}"] \arrow[d, "p_{f,1}"'] & K_g \arrow[r, "p_{g,2}"'] \arrow[d, "p_{g,1}"'] & Z \arrow[d, "g"] \\ X \arrow[r, "h"] \arrow[rr, bend right, "t \circ f"'] & Z \arrow[r, "g"] & W \end{tikzcd} \qquad \begin{tikzcd} K_f \arrow[r, "k_h"'] \arrow[rr, bend left, "h \circ p_{f,1}"] \arrow[d, "p_{f,2}"'] & K_g \arrow[r, "p_{g,1}"'] \arrow[d, "p_{g,2}"'] & Z \arrow[d, "g"] \\ X \arrow[r, "h"] \arrow[rr, bend right, "t \circ f"'] & Z \arrow[r, "g"] & W \end{tikzcd}$$

The thesis now follows again by Lemma 1.2.1. □

Lemma 1.2.8. *Let $\mathscr{C}$ be an $\mathcal{M}$-adhesive category with all pullbacks and suppose that the cube below is given, in which every face is a pullback and the bottom one is an $\mathcal{M}$-pushout.*

$$
\begin{tikzcd}
& A' \arrow[rr, "f'"] \arrow[dl, "m'"'] \arrow[dd, "a"' near end] & & B' \arrow[dl, "n'"'] \arrow[dd, "b"] \\
C' \arrow[rr, "g'" near end, crossing over] \arrow[dd, "c"'] & & D' & \\
& A \arrow[rr, "f" near end] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\
C \arrow[rr, "g"'] & & D \arrow[from=uu, "d" near start, crossing over] &
\end{tikzcd}
$$

Then the square below is a pushout.

$$
\begin{tikzcd}
K_a \arrow[r, "k_{f'}"] \arrow[d, "k_{m'}"'] & K_b \arrow[d, "k_{n'}"] \\
K_c \arrow[r, "k_{g'}"'] & K_d
\end{tikzcd}
$$

Proof. By Lemma 1.2.7 in the following cube the vertical faces are all pullbacks.

$$
\begin{tikzcd}
& K_a \arrow[rr, "k_{f'}"] \arrow[dl, "k_{m'}"'] \arrow[dd, "p_{a,1}"' near end] & & K_b \arrow[dl, "k_{n'}"'] \arrow[dd, "p_{b,1}"] \\
K_c \arrow[rr, "k_{g'}" near end, crossing over] \arrow[dd, "p_{c,1}"'] & & K_d & \\
& A' \arrow[rr, "f'" near end] \arrow[dl, "m'"] & & B' \arrow[dl, "n'"] \\
C' \arrow[rr, "g'"'] & & D' \arrow[from=uu, "p_{d,1}" near start, crossing over] &
\end{tikzcd}
$$

f' is in $\mathcal{M}$ as it is the pullback of $\mathcal{M}$, thus the bottom face of the cube is a Van Kampen pushout and the thesis follows. $\square$

Proposition 1.2.9. *Let $e\colon X \to Y$ be a regular epi in a category $\mathscr{C}$ with a kernel pair $p_1, p_2\colon P \rightrightarrows X$, then e is the coequalizer of p_1 and p_2.*

Proof. By hypothesis there exists a pair $f, g\colon Z \rightrightarrows X$ of which e is the coequalizer, since $e \circ f = e \circ g$ we have a diagram

$$
\begin{tikzcd}
Z \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] \arrow[dr, dotted, "k"] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "e"] \\
& X \arrow[r, "e"'] & Y
\end{tikzcd}
$$

and thus there exists the dotted $k\colon Z \to P$. Let $h\colon Z \to V$ be an arrow such that $h \circ p_1 = h \circ p_2$, then

$$\begin{aligned} h \circ f &= h \circ p_1 \circ k \\ &= h \circ p_2 \circ k \\ &= h \circ g \end{aligned}$$

and thus there exists a unique $l\colon Y \to V$ such that $l \circ e = h$. $\square$

Corollary 1.2.10. *Let $\mathscr{C}$ be a category with pullbacks and $\phi\colon D \to D'$ be a natural transformation between two functor $D, D'\colon \mathscr{I} \to \mathscr{C}$. If ϕ_i is a regular epi for every i, then ϕ is a regular epi.*

Proof. Let K_i be the kernel pair of ϕ_i, with projections $p_{1,i}, p_{2,i}\colon K_i \rightrightarrows D(i)$. Given an arrow $f\colon i \to j$ in $\mathscr{I}$, we have

$$\begin{aligned} \phi_j \circ D(f) \circ p_{1,i} &= D'(f) \circ \phi_i \circ p_{1,i} \\ &= D'(f) \circ \phi_i \circ p_{2,i} \\ &= \phi_j \circ D(f) \circ p_{2,i} \end{aligned}$$

Thus the outer boundary of the diagram below commutes, yielding the dotted arrows $K(f)$.

$$\begin{tikzcd}
K_i \arrow[r, "p_{1,i}"] \arrow[d, "p_{2,i}"'] \arrow[dr, dotted, "K(f)"] & D(i) \arrow[dr, "D(f)"] & \\
D(i) \arrow[dr, "D(f)"'] & K_j \arrow[r, "p_{1,j}"] \arrow[d, "p_{2,j}"'] & D(j) \arrow[d, "\phi_j"] \\
& D(j) \arrow[r, "\phi_j"'] & D'(j)
\end{tikzcd}$$

In this way we get a functor $E\colon \mathscr{I} \to \mathscr{C}$ with two natural transformations $p_1, p_2\colon E \rightrightarrows D$. By Proposition 1.2.9 every component ϕ_i of ϕ is the coequalizer of $p_{1,i}, p_{2,i}\colon E \rightrightarrows D$ and so ϕ is the coequalizer of p_1 and p_2. $\square$

esercizio per te

Lemma 1.2.11. *Let $D, D'\colon \mathscr{I} \rightrightarrows \mathscr{C}$ be two diagrams diagram with colimiting cocone $(C, \{c_i\}_{i\in\mathscr{I}})$ and $(Q, \{q_i\}_{i\in\mathscr{I}})$. If $\mathscr{C}$ has all colimits for diagrams of shape $\mathscr{I}$ and $\phi\colon D \to D'$ is a natural transformation in which all components are regular epis, then the canonical arrow $c\colon C \to Q$ is a regular epi to.*

COSE DA FARE: nel capitolo 1 metti una proposizione in cui mostri che da ogni trasf naturale $D \to D'$ puoi ricavare una freccia tra i colimiti, così in questa e nelle altre proposizioni puoi citarla.

richiama prop precedente

Proof. By Corollary 1.2.10 we know that $\phi\colon D \to D'$ is a regular epi, so that there is a functor $E\colon \mathscr{I} \to \mathscr{C}$ and $\alpha, \beta\colon E \rightrightarrows D$ such that ϕ is a coequalizer for α and β. Let $(P, \{p_i\}_{i\in\mathscr{I}})$ be the colimit of E, by ??? we have arrows $a, b\colon P \rightrightarrows C$ fitting in the diagram belows:

richiama prop sull
freccia tra i colimi

$$\begin{tikzcd}
E(i) \arrow[r, "p_i"] \arrow[d, "\alpha_i"'] & P \arrow[d, dotted, "a"] \\
D(i) \arrow[r, "c_i"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
E(i) \arrow[r, "p_i"] \arrow[d, "\alpha_i"'] & P \arrow[d, dotted, "b"] \\
D(i) \arrow[r, "c_i"'] & C
\end{tikzcd}$$

We want to show that c coequalizes a and b. Let thus $t\colon C \to T$ be an arrow such that $t \circ a = t \circ b$. Then for every $i \in I$ we have

$$\begin{aligned} t \circ c_i \circ \alpha_i &= t \circ a \circ p_i \\ &= t \circ b \circ p_i \\ &= t \circ c_i \circ \beta_i \end{aligned}$$

Verificalo!

Thus there is $t_i\colon D'(i) \to T$ such that $t \circ c_i = t_i \circ \phi_i$. It is now easy to see that $(T, \{t_i\}_{i\in\mathscr{I}})$ is a cocone on D'. Thus we have an arrow $k\colon Q \to T$ such that $k \circ q_i = t_i$. But then we have

$$\begin{aligned} k \circ c \circ c_i &= k \circ q_i \circ \phi_i \\ &= t_i \circ \phi \\ &= t \circ c_i \end{aligned}$$

Showing that $k \circ c = t$.

For uniqueness, let k' be another arrow $Q \to T$ such that $k' \circ c = t$, then we have

$$\begin{aligned} k' \circ q_i \circ \phi_i &= k' \circ c \circ c_i \\ &= t \circ c_i \\ &= t_i \circ \phi_i \end{aligned}$$

mettere nel capitolo 1 una proposizione che mostra che epi regolari sono epi

Since ϕ_i is a regular epi, by ??? this entails $k' \circ q_i = t_i$. By construction $k \circ q_i = t_i$ and so $k = k'$ since $(Q, \{q_i\}_{i\in\mathscr{I}})$ is a colimiting cocone. $\square$

Definition 1.2.12. A *graph with equivalence* is a 6-uple (A, B, C, s, t, q) where A, B and C are set, $s, t\colon A \rightrightarrows B$ are functions and $q\colon B \to C$ is another surjective function.

A morphism $(A, B, C, s, t, q) \to (A', B', C', s', t', q')$ is a triple (h_1, h_2, h_3) of functions $h_1\colon A \to A'$, $h_2\colon B \to B'$ and $h_3\colon C \to C'$ making the following diagrams commute.

$$
\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "h_1"'] & B \arrow[d, "h_2"] \\
A' \arrow[r, "s'"'] & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "h_1"'] & B \arrow[d, "h_2"] \\
A' \arrow[r, "t'"'] & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "q"] \arrow[d, "h_2"'] & C \arrow[d, "h_3"] \\
B' \arrow[r, "q'"'] & C'
\end{tikzcd}
$$

In this way, defining the composition componentwise, we get a category **EqGrph**.

Remark 1.2.13. There is a faithful functor $U\colon$ **EqGrph** $\to$ **Graph**, forgetting the quotient part.

Questo è per te da dimostrare (forse meglio come proposizione che come remark).

Remark 1.2.14. There is another functor $V\colon$ **EqGrph** $\to$ **Set** sending (A, B, C, s, t, q) to C and a morphism to its last component.

Proposition 1.2.15. **EqGrph** *is complete, cocomplete and* U *preserves limits and colimits.*

Remark 1.2.16. In **Set** we have the following property: for every square as the one below, if $e\colon X \to Y$ is epi and $m\colon M \to Z$ is mono, then there exists a unique dotted arrow $Y \to M$ making the diagram below commutative.

Dimostrala

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "e"'] & M \arrow[d, "m"] \\
Y \arrow[r, "g"'] \arrow[ur, dotted, "h"] & Z
\end{tikzcd}
$$

Proof. NOTATION: $D(i)$ is $(A_i, B_i, Q_i, s_i, t_i, q_i)$. Let (A, B, s, t) be the limit of $U \circ D$, with projections $(p_{1,i}, p_{2,i})\colon (A, B, s, t) \to (A_i, B_i, s_i, t_i)$. Let $(L, \{l_i\}_{i\in\mathscr{I}})$ be a limiting cone for $V \circ D$.

Per generalizzare ad altre categorie: serve poter fattorizzare con un epi regolare.

Now, notice that $(B, \{q_i \circ p_{2,i}\}_{i\in\mathscr{I}})$ is a cone over $V \circ D$, so that we have an arrow $l\colon B \to L$. This arrow is not epi in general, let Q be its image and $q\colon B \to Q$ be the resulting epi and $m\colon Q \to L$ the corresponding mono. By definition the external square in the diagram below commutes, so for every $i \in \mathscr{I}$, Remark 1.2.16 yields the dotted arrow $p_{3,i}$.

Dimostralo

$$
\begin{tikzcd}
B \arrow[r, "p_{2,i}"] \arrow[d, "q"'] & B_i \arrow[r, "q_i"] & Q_i \arrow[d, "id_{Q_i}"] \\
Q \arrow[r, "m"'] \arrow[urr, dotted, "p_{3,i}"] & L \arrow[r, "l_i"'] & Q_i
\end{tikzcd}
$$

We have to show that in this way we get a cone over the diagram D. Let $f\colon i \to j$ be an arrow of $\mathscr{I}$, then we have:

$$\begin{aligned} U(D(f) \circ (p_{1,i}, p_{2,i}, p_{3,i})) &= U(D(f)) \circ (p_{1,i}, p_{2,i}) \\ &= (p_{1,j}, p_{2,j}) \\ &= U(D(f) \circ (p_{1,j}, p_{2,j}, p_{3,j})) \end{aligned}$$

And faithfulness of U yields the thesis.

To see that this cone is terminal, let (E, F, G, a, b, c) be another graph with the vertex of a cone with sides $(t_{1,i}, t_{2,i}, t_{3,i})$. By construction, we have an arrow $(t_1, t_2)\colon (E, F, a, b) \to (A, B, s, t)$ such that

$$\begin{tikzcd} & E \arrow[dl, dotted, "t_1"'] \arrow[dr, "t_{1,i}"] & \\ A \arrow[rr, "p_{1,i}"'] & & A_i \end{tikzcd} \qquad \begin{tikzcd} & F \arrow[dl, dotted, "t_2"'] \arrow[dr, "t_{2,i}"] & \\ B \arrow[rr, "p_{2,i}"'] & & B_i \end{tikzcd}$$

Verificalo

Moreover $(G, \{t_{3,i}\}_{i \in \mathscr{I}})$ is a cone over $V \circ D$, thus there exists an arrow $t\colon G \to L$ such that $l_i \circ t = t_{3,i}$. Now, precomposing with c we get

$$\begin{aligned} l_i \circ t \circ c &= t_{3,i} \circ c \\ &= q_i \circ t_{2,i} \\ &= q_i \circ p_{2,i} \circ t_2 \\ &= l_i \circ l \circ t_2 \end{aligned}$$

Therefore the solid part of the diagram below commutes and Remark 1.2.16 yields the dotted arrow $t_3\colon G \to Q$.

$$\begin{tikzcd} F \arrow[r, "t_2"] \arrow[d, "c"'] & B \arrow[r, "q"] & Q \arrow[d, "m"] \\ G \arrow[urr, dotted, "t_3"] \arrow[rr, "t"'] & & L \end{tikzcd}$$

esercizio per te: scrivere la dimostrazione di queste ultime due righe

Faithfulness of U now guarantees that (t_1, t_2, t_3) is the unique arrow such that $(p_{1,i}, p_{2,i}, p_{3,i}) \circ (t_1, t_2, t_3) = (t_{1,i}, t_{2,i}, t_{3,i})$. $\square$

Corollary 1.2.17. *An arrow $(h_1, h_2, h_3)\colon (A, B, C, s, t, q) \to (E, F, G, a, b, c)$ in* **EqGrph** *is mono if and only if h_1 and h_2 are mono in* **Set**.

esercizio per te: fare i colimiti. Hint: la dimostrazione è diversa ma più semplice. Se (A, B, s, t) è il colimite di $U \circ D$ puoi considerare il colimite Q dei vari Q_i. Per la proprietà del colimite hai una freccia $B \to Q$, mostra che è epi (segue in una riga

Proof. $\square$

Corollary 1.2.18. *Let $(h_1, h_2, h_3)\colon (A, B, C, s, t, q) \to (E, F, G, a, b, c)$ be a morphism of* **EqGrph**, *then the following are equivalent:*

1. *(h_1, h_2, h_3) is a regular mono;*
2. *h_1, h_2, h_3 are all monos;*
3. *h_1 and h_2 are mono and for every $f, f' \in F$, $c(h_2(f)) = c(h_2(f'))$ if and only if $q(f) = q(f')$.*

Proof. $1 \Rightarrow 2$.

Esercizio

$2 \Rightarrow 3$.

Esercizio

$3 \Rightarrow 1$. □

Esercizio

Let us turn to another functor **EqGrph** $\to$ **Graph**.

Definition 1.2.19. The *quotient functor* Q : **EqGrph** $\to$ **Graph** sends (A, B, C, s, t, q) to $(A, C, q \circ s, q \circ t)$ and an arrow $(h_1, h_2, h_3)\colon (A, B, C, s, t, q) \to (E, F, G, a, b, c)$ to (h_1, h_3).

Verifica che questa cosa funziona (h_1, h_3) è un morfismo di grafi?

Lemma 1.2.20. *Q is a left adjoint.*

Proof. Let us start proving that Q is a left adjoint. Let $R(A, B, s, t)$ be (A, B, B, s, t, id_B) , so that $Q(R(A, B, s, t)) = (A, B, s, t)$. Now, suppose that an arrow $(h_1, h_2)\colon Q(E, F, G, a, b, c) \to (A, B, s, t)$ is given. Consider the triple $(h_1, h_2, h_2 \circ c)$. Notice that, since (h_1, h_2) is an arrow in **Graph**:

$$h_2 \circ c \circ a = s \circ h_1 \qquad h_2 \circ c \circ b = t \circ h_1$$

Then we have three squares:

$$\begin{tikzcd} E \arrow[r, "a"] \arrow[d, "h_1"'] & F \arrow[d, "h_2 \circ c"] \\ A \arrow[r, "s"'] & B \end{tikzcd} \quad \begin{tikzcd} E \arrow[r, "b"] \arrow[d, "h_1"'] & F \arrow[d, "h_2 \circ c"'] \\ A \arrow[r, "t"'] & B \end{tikzcd} \quad \begin{tikzcd} B \arrow[r, "c"] \arrow[d, "h_2 \circ c"'] & C \arrow[d, "h_2"] \\ B \arrow[r, "id_B"'] & B \end{tikzcd}$$

We have therefore found a morphism $(E, F, G, a, b, c) \to R(A, B, s, t)$ whose image through Q fits in the diagram below.

$$\begin{tikzcd} (A, B, s, t) \arrow[r, "(id_A, id_B)"] & (A, B, s, t) \\ (E, G, q \circ s, q \circ t) \arrow[u, "Q(h_1, h_2 \circ q, h_2)"] \arrow[ur, "(h_1, h_2)"'] & \end{tikzcd}$$

□

Esercizio: prova unicità (è facile)

1.2.2 Rimanenze da integrare

Proposition 1.2.21. *Q creates colimits.*

Proof. □

esercizio per te (d
verificare solo la r
lessione: ma dato
diagramma in gra
fatto di quozienti,
come costruisci un
insieme di vertici
il colimite?)

Q non preserva i limiti

Example 1.2.22.

1.2.3 Adhesivity of EqGrph

Lemma 1.2.23. *In* **EqGrph** *pushouts along regular monos are stable.*

Proof. Suppose that the cube below is given, in which all the vertical faces are pullbacks and the bottom face is a pushout, with $(h_1, h_2, h_3)\colon (A_1, B_1, C_1, s_1, t_1, q_1) \to (A_2, B_2, C_2, s_2, t_2, q_2)$ a regular mono.

Questo diagramma va sistemato per farlo stare nella pagina (val la pena magari dire "sia $\mathcal{G}_1$ il grafo....")

$$\begin{tikzcd}
& (A_1', B_1', C_1', s_1', t_1', q_1') \arrow[rr, "(h_1', h_2', h_3')"] \arrow[dl, "(k_1', k_2', k_3')"'] \arrow[dd, "(a_1, a_2, a_3)"] & & (A_2', B_2', C \arrow[dl, "(t_1', t_2', t_3')"] \\
(A_3', B_3', C_3', s_3', t_3', q_3') \arrow[rr, "(p_1', p_2', p_3')"] \arrow[dd, "(c_1, c_2, c_3)"'] & & (A_4', B_4', C_4', s_4', t_4', q_4') \arrow[dd, "(d_1, d_2, d_3)"] & \\
& (A_1, B_1, C_1, s_1, t_1, q_1) \arrow[rr, "(h_1, h_2, h_3)"] \arrow[dl, "(k_1, k_2, k_3)"] & & (A_2, B_2, C \arrow[dl, "(t_1, t_2, t_3)"] \\
(A_3, B_3, C_3, s_3, t_3, q_3) \arrow[rr, "(p_1, p_2, p_3)"'] & & (A_4, B_4, C_4, s_4, t_4, q_4) &
\end{tikzcd}$$

By Proposition 1.2.15 and corollary 1.3.7 the following two cubes have $\mathcal{M}$-pushouts as bottom faces and pullbacks as vertical faces, thus their top faces are $\mathcal{M}$-pushouts.

$$\begin{tikzcd}
& A_1' \arrow[rr, "h_1'"] \arrow[dl, "k_1'"'] \arrow[dd, "a_1"'] & & A_2' \arrow[dl, "t_1'"'] \arrow[dd, "b_1"] \\
A_3' \arrow[rr, "p_1'"] \arrow[dd, "c_1"'] & & A_4' \arrow[dd, "d_1"] & \\
& A_1 \arrow[rr, "h_1"] \arrow[dl, "k_1"] & & A_2 \arrow[dl, "t_1"] \\
A_3 \arrow[rr, "p_1"'] & & A_4 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& B_1' \arrow[rr, "h_2'"] \arrow[dl, "k_2'"'] \arrow[dd, "a_2"'] & & B_2' \arrow[dl, "t_2'"'] \arrow[dd, "b_2"] \\
B_3' \arrow[rr, "p_2'"] \arrow[dd, "c_2"'] & & B_4' \arrow[dd, "d_2"] & \\
& B_1 \arrow[rr, "h_2"] \arrow[dl, "k_2"] & & B_2 \arrow[dl, "t_2"] \\
B_3 \arrow[rr, "p_2"'] & & B_4 &
\end{tikzcd}$$

Now, using Corollary 1.2.2, we can consider a third cube, which, by Proposition 1.2.21, has a bottom face an $\mathcal{M}$-pushout and pullbacks as vertical faces, so that its top face is an $\mathcal{M}$-pushout too.

$$\begin{tikzcd}
 & T \arrow[rr, "x_1"] \arrow[dl, "w"'] \arrow[dd, "x_2"] & & U \arrow[dl, "u_1"'] \arrow[dd, "u_2"] \\
Y \arrow[rr, "y_1"] \arrow[dd, "y_2"'] & & C'_4 \arrow[dd, "d_3"] & \\
 & C_1 \arrow[rr, "h_3"] \arrow[dl, "k_3"] & & C_2 \arrow[dl, "t_3"] \\
C_3 \arrow[rr, "p_3"'] & & C_4 &
\end{tikzcd}$$

Moreover, by the proof of Proposition 1.2.15 we know that there are monos $m_2\colon C'_2 \to U$ and $m_3\colon C'_3 \to Y$ fitting in the diagrams

$$\begin{tikzcd}
B'_3 \arrow[r, "p'_2"] \arrow[d, "q'_3"] \arrow[dd, bend right, "c_2"'] & B'_4 \arrow[dr, bend left, "q'_4"] & \\
C'_3 \arrow[r, "m_3"] & Y \arrow[r, "y_1"] \arrow[d, "y_2"'] & C'_4 \arrow[d, "d_3"] \\
B_3 \arrow[r, "q_2"'] & C_3 \arrow[r, "p_3"'] & C_4
\end{tikzcd}
\qquad
\begin{tikzcd}
B'_2 \arrow[r, "t'_2"] \arrow[d, "q'_2"] \arrow[dd, bend right, "b_2"'] & B'_4 \arrow[dr, bend left, "q'_4"] & \\
C'_2 \arrow[r, "m_2"] & U \arrow[r, "u_1"] \arrow[d, "u_2"'] & C'_4 \arrow[d, "d_3"] \\
B_2 \arrow[r, "q_2"'] & C_2 \arrow[r, "t_3"'] & C_4
\end{tikzcd}$$

For C'_1, the we can make a similar argument, let S be the pullback of m_2 along x_1, using Lemma 1.2.1 and, again, the proof of Proposition 1.2.15 we know that q'_1 arise as the factorization of the arrow $B'_1 \to S$ induced by $q'_2 \circ h'_2$ and a_2 so that we have a diagram.

$$\begin{tikzcd}
B'_1 \arrow[r, "h'_2"] \arrow[d, "q'_2"] \arrow[dd, bend right, "a_2"'] & B'_2 \arrow[dr, bend left, "q'_2"] & \\
C'_1 \arrow[r, "m_1"] & S \arrow[r, "s_1"] \arrow[d, "s_2"'] & C'_2 \arrow[d, "m_2"] \\
B_2 \arrow[dr, bend right, "q_2"'] & T \arrow[r, "x_1"] \arrow[d, "x_2"'] & U \arrow[d, "u_2"'] \\
 & C_1 \arrow[r, "h_3"'] & C_2
\end{tikzcd}$$

Moreover, notice that

Esercizio (basta comporre prima e dopo con gli opportuni epi e mono)

$$s_1 \circ m_1 = h'_3 \quad w \circ s_2 \circ m_1 = m_3 \circ k'_3 \quad t'_3 = u_1 \circ m_2 \quad p_3 = y_1 \circ m_3$$

Let now $(z_1, z_2, z_3)\colon (A'_2, B'_2, C'_2) \to (E, F, G, e, f, g)$ and $(w_1, w_2, w_3)\colon (A'_3, B'_3, C'_3) \to (E, F, G, e, f, g)$ be two morphisms such that

$$(z_1, z_2, z_3) \circ (h'_1, h'_2, h'_3) = (w_1, w_2, w_3) \circ (k'_1, k'_2, k'_3)$$

Sistema il diagramma mettendo gli opportuni buchi

Let $z\colon B_4' \to F$ be the arrow induced by z_2 and w_2, we want to construct the dotted arrow $v\colon C_4' \to G$ in the diagram below

$$
\begin{tikzcd}
 & B_1' \arrow[rr, "h_2'"] \arrow[dl, "k_2'"'] \arrow[dd, "q_1'"] & & B_2' \arrow[dl, "t_2'"'] \arrow[dr, bend left, "z_2"] \arrow[ddd, "q_2'"] & \\
B_3' \arrow[rr, "p_2'"] \arrow[ddd, "q_3'"'] & & B_4' \arrow[rr, "z"] & & F \arrow[ddddd, "g"] \\
 & C_1' \arrow[d, "m_1"] \arrow[drr, bend left, "h_3'"] \arrow[ddl, bend right, "k_3'"'] & & & \\
 & S \arrow[rr, "s_1"] \arrow[dd, "s_2"] & & C_2' \arrow[dd, "m_2"] \arrow[dddl, "t_3'"'] \arrow[dddr, bend left, "z_3"] & \\
C_3' \arrow[dd, "m_3"'] \arrow[ddrr, bend right, "p_3'"] \arrow[ddrrrr, "w_3"] & & & & \\
 & T \arrow[rr, "x_1"] \arrow[dl, "w"'] \arrow[dd, "x_2"] & & U \arrow[dl, "u_1"'] \arrow[dd, "u_2"] & \\
Y \arrow[rr, "y_1"] \arrow[dd, "y_2"'] & & C_4' \arrow[rr, dotted, "v"] \arrow[dd, "d_3"] & & G \\
 & C_1 \arrow[rr, "h_3"] \arrow[dl, "k_3"'] & & C_2 \arrow[dl, "t_3"] & \\
C_3 \arrow[rr, "p_3"'] & & C_4 & &
\end{tikzcd}
$$

Now by Proposition 1.2.9 d_3 is the coequalizer of its kernel pair. On the other hand, by Lemma 1.2.8 we know that the top face of the cube below is a pushout.

$$
\begin{tikzcd}
 & K_{s_2\circ m_1\circ q_1'} \arrow[rr, "k_{h_2'}"] \arrow[dl, "k_{k_2'}"'] \arrow[dd, "p_{s_2\circ m_1\circ q_1',1}"] & & K_{m_2\circ q_2'} \arrow[dl, "k_{t_2'}"'] \arrow[dd, "p_{m_2\circ q_2',1}"] \\
K_{m_3\circ q_3'} \arrow[rr, "k_{p_2'}"] \arrow[dd, "p_{m_3\circ q_3',1}"'] & & K_{q_4'} \arrow[dd, "p_{q_4',1}"] & \\
 & B_1' \arrow[rr, "h_2'"] \arrow[dl, "k_2'"'] & & B_2' \arrow[dl, "t_2'"] \\
B_3' \arrow[rr, "p_2'"'] & & C_4 &
\end{tikzcd}
$$

Moreover, since m_3 and m_2 are monos, or by Corollary 1.2.6 we also know that

$$
q_3' \circ p_{m_3\circ q_3',1} = q_3' \circ p_{m_3\circ q_3',2} \qquad q_2' \circ p_{m_2\circ q_2',1} = q_2' \circ p_{m_2\circ q_2',2}
$$

Now, we have

$$
\begin{aligned}
g \circ z \circ p_{q'_4,1} \circ k_{p'_2} &= g \circ z \circ p'_2 \circ p_{m_3 \circ q'_3,1} \\
&= g \circ w_2 \circ \circ p_{m_3 \circ q'_3,1} \\
&= w_3 \circ q'_3 \circ p_{m_3 \circ q'_3,1} \\
&= w_3 \circ q'_3 \circ p_{m_3 \circ q'_3,2} \\
&= g \circ w_2 \circ \circ p_{m_3 \circ q'_3,2} \\
&= g \circ z \circ p'_2 \circ p_{m_3 \circ q'_3,2} \\
&= g \circ z \circ p_{q'_4,2} \circ k_{p'_2}
\end{aligned}
\qquad
\begin{aligned}
g \circ z \circ p_{q'_4,1} \circ k_{t'_2} &= g \circ z \circ t'_2 \circ p_{m_2 \circ q'_2,1} \\
&= g \circ z_2 \circ \circ p_{m_2 \circ q'_2,1} \\
&= z_3 \circ q'_2 \circ p_{m_2 \circ q'_2,1} \\
&= z_3 \circ q'_2 \circ p_{m_2 \circ q'_2,2} \\
&= g \circ z_2 \circ \circ p_{m_2 \circ q'_2,2} \\
&= g \circ z \circ t'_2 \circ p_{m_2 \circ q'_2,2} \\
&= g \circ z \circ p_{q'_4,2} \circ k_{t'_2}
\end{aligned}
$$

The thesis now follows

Scrivi perché

Esercizio: prova unicità

□

Lemma 1.2.24. *In* **EqGrph** *pushouts along regular monos are Van Kampen.*

Proof. contenuto... □

From Proposition 1.2.15 and Lemmas 1.2.23 and 1.2.24 we deduce at once the following.

Corollary 1.2.25. **EqGrph** *is quasiadhesive.*

1.3 Graphs with Equivalences

A graph with equivalence is a 6-tuple $\mathbb{G} = (E, V, C, s, t, q)$, where E and V are, respectively, the edges and the vertices sets, and C is the set of the equivalence classes among vertices, $s, t : E \to V$ are the source and target functions and $q : V \to C$ is the *quotient* function, that is, the map from a vertex to its equivalence class. For this definition to make sense, q needs to be surjective. A morphisms h from a graph with equivalence $\mathbb{G} = (E, V, C, s, t, q)$ to another $\mathbb{H} = (E', V', C', s', t', q')$ is a triple $h = (h_E, h_V, h_C)$ of functions $h_V : V \to V'$, $h_E : E \to E'$ and $h_C : C \to C'$ such that:

- $h_E \circ s = s' \circ h_V$;
- $h_E \circ t = t' \circ h_V$;
- $h_C \circ q = q' \circ h_C$.

Remark 1.3.1. A graph with equivalence can be viewed as a graph endowed with an equivalence relation over its set of vertices, $(\mathcal{G}, \sim_{\mathcal{G}})$. An homomorphism between two graphs with equivalences $h : \mathbb{G} = (\mathcal{G}, \sim_{\mathcal{G}}) \to \mathbb{H} = (\mathcal{H}, \sim_{\mathcal{H}})$ is a graph homomorphism $h = (h_V, h_E) : \mathcal{G} \to \mathcal{H}$ such that if $v_1 \sim_{\mathcal{G}} v_2$ then $h_V(v_1) \sim_{\mathcal{H}} h_V(v_2)$. In **Set**, it is possible to formalize an equivalence relation $\sim$ over X as a surjective function sending each element x on its equivalence class $[x]_{\sim}$, and this justify our formalization via surjective functions (i.e., epimorphisms).

As we have done in Section 1.1, we can think to a graph with equivalence as a presheaf, this time from a category $E \rightrightarrows V \to C$, where the image of C along the presheaf is the set of the equivalence classes, requiring that the morphism $V \to C$ is an epimorphism (that is, a surjective function).

Definition 1.3.2 (Category of Graphs with Equivalences)**.** The category **EqGrph** is the subcategory of

$$[E \overset{s}{\underset{t}{\rightrightarrows}} V \xrightarrow{q} C, \mathbf{Set}]$$

such that, for each $\mathbb{G} \in \mathcal{O}b(\mathbf{EqGrph})$, $\mathbb{G}(q)$ is an epimorphism.

Observation 1.3.3. Morphisms of graphs with equivalence are uniquely determined by the first two component. That is, if $h_1 = (h_E, h_V, \phi)$ and $h_2 = (h_E, h_V, \psi)$, then $\phi = \psi$.

Proof. Let $h_1, h_2 : \mathbb{G} \to \mathbb{H}$, where $\mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G)$ and $\mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$. Then, we have the following situation

$$\begin{tikzcd}
V_G \arrow[r, "h_V"] \arrow[d, "q_G"'] & V_H \arrow[d, "q_H"] & V_G \arrow[l, "h_V"'] \arrow[d, "q_G"] \\
C_G \arrow[r, "\phi"'] & C_H & C_G \arrow[l, "\psi"]
\end{tikzcd}$$

Then, we have:

$$\begin{aligned} \psi \circ q_G &= q_H \circ h_V \\ &= \phi \circ q_G \end{aligned}$$

From the fact that q_G is epi, we can conclude $\phi = \psi$. □

A graph with equivalence is then a graph with an extra structure, the quotient map. Hence, it is possible to get the underlying graph by forgetting it. Such action is described by the *forgetful functor* U : **EqGrph** $\to$ **Graph**, that maps each graph with equivalence $\mathbb{G} = (E, V, C, s, t, q)$ onto $U(\mathbb{G}) = (E, V, s, t)$, and each morphisms $h = (h_E, h_V, h_C)$ onto $U(h) = (h_E, h_V)$. U is effectively a functor, since, on identities, $U((id_E, id_V, id_C)) = (id_E, id_V)$, and on compositions $U(h \circ k) = U((h_E \circ k_E, h_V 1circk_V, h_C \circ k_C)) = (h_E \circ k_E, h_V \circ k_V) = (h_E, h_V) \circ (k_E \circ k_V) = U(h) \circ U(k)$.

Proposition 1.3.4. *The forgetful functor U :* **EqGrph** $\to$ **Graph** *is faithful.*

Proof. Let $\mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G)$ and $\mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$ be two graphs with equivalences, and let $h, k : \mathbb{G} \to \mathbb{H}$. If $U(h) = U(k)$ (i.e., the first two component of h and k are the same), from Observation 1.3.3, we can conclude that $h = k$. Then, the restriction $U_{\mathbb{G},\mathbb{H}}$: **EqGrph**$(\mathbb{G}, \mathbb{H}) \to$ **Graph**$(U(\mathbb{G}), U(\mathbb{H}))$, therefore U is faithful. $\square$

Another functor that will be useful later is V : **EqGrph** $\to$ **Set**, that sends $(E_G, V_G, C_G, s_G, t_G, q_G)$ to C_G, and $h = (h_E, h_V, h_C)$ to h_C.

Proposition 1.3.5. EqGrph *has all limits, colimits and U preserves limits and colimits.*

Proof. Let $D : \mathscr{I} \to$ **EqGrph** be a diagram. In the following, we will denote the graph with equivalence $D(i)$ as $(E_i, V_i, C_i, s_i, t_i, q_i)$. Let now be the graph (A, B, s, t) the limit of $U \circ D$, with projections $(\pi_E^i, \pi_V^i) : (A, B, s, t) \to (E_i, V_i, s_i, t_i)$. Notice now that $(B, (q_i \circ \pi_V^i)_{i \in \mathscr{I}})$ is a cone for $V \circ D$. To see this, let $\alpha : i \to j$ be an arrow of $\mathscr{I}$, $D(\alpha) = (h_E, h_V, h_C)$, $U \circ D(\alpha) = (h_E, h_V)$. From the definition of cone, we have that $U \circ D(\alpha) \circ (\pi_E^i, \pi_V^i) = (\pi_E^j, \pi_V^j)$, hence $h_V \circ \pi_V^i = \pi_V^j$. Consider now the following diagram in **Set**

$$\begin{tikzcd}
& B \arrow[dl, "\pi_V^i"'] \arrow[dr, "\pi_V^j"] & \\
V_i \arrow[rr, "h_V"] \arrow[d, "q_i"'] & & V_j \arrow[d, "q_j"] \\
C_i \arrow[rr, "h_C"'] & & C_j
\end{tikzcd}$$

So we have $q_j \circ h_V \circ \pi^i_V = q_j \circ \pi^j_V$, by definition of graph with equivalence, $h_C \circ q_i \circ \pi^i_V = q_j$, and, by definition of V, $V \circ D(\alpha) \circ q_i \circ \pi^i_V = q_j \circ \pi^j_V$. Suppose now $(L, (l_i)_{i \in \mathscr{I}})$ be a limit for $V \circ D$, so that we have an arrow $l : B \to L$. This arrow is not epi in general, so let Q be its image, $q : Q \to B$ be the resulting epi and $m : Q \to L$ the corresponding mono, as the diagram below shows. By definition, the external rectangle commutes, so, for each i object of $\mathscr{I}$, REMARK yields the dotted arrow π^i_C.

epi-mono factorization in Set (or Regular Cats in general)

$$\begin{tikzcd}
B \arrow[r, "\pi^i_V"] \arrow[d, "q"'] & B_i \arrow[r, "q_i"] & Q_i \arrow[d, "id_{Q_i}"] \\
Q \arrow[r, "m"'] \arrow[urr, dashed, "\pi^i_C"] & L \arrow[r, "l_i"'] & Q_i
\end{tikzcd}$$

We have to show that in this way we get a cone over the diagram D. Let $\alpha : i \to j$ be an arrow of $\mathscr{I}$, then we have:

$$\begin{aligned}
U(D(\alpha) \circ (\pi^i_E, \pi^i_V, \pi^i_C)) &= U(D(\alpha)) \circ (\pi^i_E, \pi^i_V) \\
&= (\pi^j_E, \pi^j_V) \\
&= U(D(\alpha) \circ (\pi^j_E, \pi^j_V, \pi^j_C))
\end{aligned}$$

And faithfulness of U yields the thesis.

To see that this cone is terminal, let $((E, F, G, a, b, c), \tau_i = (\tau^i_E, \tau^i_V, \tau^i_C)_{i \in \mathscr{I}})$ be another cone. By construction, we have an arrow $(\tau_E, \tau_V) : (E, F, a, b) \to (A, B, s, t)$ such that

$$\begin{tikzcd}
& E \arrow[dl, dashed, "\tau_E"'] \arrow[dr, "\tau^i_E"] & \\
A \arrow[rr, "\pi^i_E"'] & & A_i
\end{tikzcd}
\qquad
\begin{tikzcd}
& F \arrow[dl, dashed, "\tau_V"'] \arrow[dr, "\tau^i_V"] & \\
B \arrow[rr, "\pi^i_V"'] & & B_i
\end{tikzcd}$$

For the same reason as before, $(G, (\tau^i_C)_{i \in \mathscr{I}})$ is a cone over $V \circ D$, thus there exists an arrow $\tau : G \to L$ such that $l_i \circ \tau = \tau^i_C$. At this point, we get

$$\begin{aligned}
l_i \circ \tau \circ c &= \tau^i_C \circ c \\
&= q_i \circ \tau^i_V && \textit{τ_i is a morphism in } \mathbf{EqGrph} \\
&= q_i \circ \pi^i_V \circ \tau_V && \textit{Diagram above} \\
&= l_i \circ l \circ \tau_V && (B, (q_i \circ \pi^i_V)_{i \in \mathscr{I}}) \textit{ cone}
\end{aligned}$$

Therefore, the outer part of the rectangle below commutes, and by REMARK there exists a unique $\tau_C : G \to Q$

epi-reg fact in SET

$$\begin{tikzcd}
F \arrow[r, "\tau_V"] \arrow[d, "c"'] & B \arrow[r, "q"] & Q \arrow[d, "m"] \\
G \arrow[urr, dashed, "\tau_C"] \arrow[rr, "\tau"'] & & L
\end{tikzcd}$$

Faithfulness of U guarantees that (τ_E, τ_V, τ_C) is the unique arrow such that $(\pi^i_E, \pi^i_V, \pi^i_C) \circ (\tau_E, \tau_V, \tau_C) = (\tau^i_E, \tau^i_V, \tau^i_C)$. □

Dimostrazione dello statement di sopra, e commiti

Corollary 1.3.6. *An arrow* $h = (h_E, h_V, h_C) : \mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G) \to \mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$ *in* **EqGrph** *is mono if and only if* h_E *and* h_V *are mono in* **Set**.

Proof. The "if" part is given by the fact that U is faithful, and hence reflects monomorphisms. Since a morphism ain a category of presheaves is mono if and only if it is injective on each component we have that, if $U(h)$ is mono, that is, h_E and h_V are injective in **Set**, then h is mono. For the "only if" part, suppose $f = (f_E, f_V, f_C)$, $g = (g_E, g_V, g_C)$, $f, g : \mathbb{H} \to \mathbb{K}$ be such that $h \circ f = h \circ g$. Then, we have

commentato il link, dimostrare questa cosa

$$\begin{aligned}
h \circ f &= (h_E \circ f_E, h_V \circ f_V, h_C \circ f_C) \\
&= (h_E \circ f_E, h_V \circ f_V, h_V \circ f_V \circ \mathbb{K}(q)) \\
&= (h_E \circ g_E, h_V \circ g_V, h_V \circ g_V \circ \mathbb{K}(q))
\end{aligned}$$

Since $\mathbb{K}(q)$ is epi, we have, on the third component, that $h_V \circ f_V \circ \mathbb{K}(q) = h_V \circ g_V \circ \mathbb{K}(q))$ implies $f_C = g_C$, and hence $f = g$ □

Corollary 1.3.7. *Let* $h = (h_E, h_V, h_C) : \mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G) \to \mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$ *be a morphism of* **EqGrph**, *then the following are equivalent:*

1. *h is a regular mono;*
2. *h_E, h_V, h_C are all monos;*
3. *h_E and h_V are mono and (K, π_1, π_2) is the kernel pair of $q_H \circ h_V$ if and only if (K, π_1, π_2) is the kernel pair of q_G.*

Proof. 1 $\Rightarrow$ 2. If h is mono, from Corollary 1.3.6 we have that h_E and h_V are monos. To derive h_C mono, suppose $f, g : \mathbb{H} \to \mathbb{K}$ to be the arrows equalized by h. Then we have

$$\begin{aligned} f_C \circ h_C \circ \mathbb{G}(q) &= f_C \circ \mathbb{H}(q) \circ h_V \\ &= \mathbb{K}(q) \circ f_V \circ h_V \\ &= \mathbb{K}(q) \circ g_V \circ h_V \\ &= g_C \circ h_C \circ \mathbb{G}(q) \end{aligned}$$

since $\mathbb{G}(q)$ is epi, we have that $f_C \circ h_C = g_C \circ h_C$, hence h_C is an equalizer for f_C and g_C, thus a monomorphism.
2 $\Rightarrow$ 3. We note that, by Corollary 1.2.6, (K, π_1, π_2) is the kernel pair of q_G if and only if it is the kernel pair also of $h_C \circ q_G$, since h_C is mono by hypothesis. The thesis follows from $h_C \circ q_G = q_H \circ h_V$, and from the hypothesis of h_E mono.

Esercizio

3 $\Rightarrow$ 1. idea: force the comm. of the diagram on the last two components to obtain the two arrows that are equalized, and show that the condition in 3 is sufficient to conclude reg. mono ☐

Remark 1.3.8. It is possible to restate the third point of the Corollary 1.3.7, by **??**, as

> *h_E and h_V are mono and, for every $v, v' \in V_H$, $q_H(h_V(v)) = q_H(h_V(v'))$ if and only if $q_G(v) = q_G(v')$*

That is, a regular monomorphism in **EqGrph** is a morphism that reflects equivalences besides preserving them.

Let us turn to another functor **EqGrph** $\to$ **Graph**.

Definition 1.3.9. The *quotient functor* Q : **EqGrph** $\to$ **Graph** sends $(E_G, V_G, C_G, s_G, t_G, q_G)$ to $(E_G, C_G, q_G \circ s_G, q_G \circ t_G)$ and an arrow $(h_E, h_V, h_C) \colon (E_G, V_G, C_G, s_G, t_G, q_G) \to (E_H, V_H, C_H, s_H, t_H, q_H)$ to (h_E, h_C).

Remark 1.3.10. The action of the functor on a morphism of graphs with equivalences gives a morphism of graphs, in fact $q_H \circ s_H \circ h_E = q_H \circ h_V \circ s_G = h_C \circ q_G \circ s_G$. The same is valid for t_H and t_G.

Lemma 1.3.11. *Q is a left adjoint.*

Proof. Let $R((A,B,s,t))$ be (A,B,B,s,t,id_B), so that $Q(R((A,B,s,t))) = (A,B,s,t)$. Now, suppose that $h = (h_E, h_V) : Q((E,V,C,s',t',q)) \to (A,B,s,t)$ is an arrow in **Graph**, and consider the triple $(h_E, h_V, h_V \circ q)$. Since h is a morphism of **Graph**,

$$h_V \circ q \circ s' = s \circ h_E \qquad h_V \circ q \circ t' = t \circ h_E$$

Then we have the following squares:

$$\begin{tikzcd}
E \arrow[r, "h_E"] \arrow[d, "s_G"'] & A \arrow[d, "s"] \\
V \arrow[r, "h_V \circ q"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "h_E"] \arrow[d, "t_G"'] & A \arrow[d, "t"] \\
V \arrow[r, "h_V \circ q"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "h_V \circ q"] \arrow[d, "q"'] & B \arrow[d, "id_B"] \\
C \arrow[r, "h_V"'] & B
\end{tikzcd}$$

We have therefore found a morphism $(E,V,C,s',t',q) \to R((A,B,s,t))$ whose image through Q fits in the diagram below.

$$\begin{tikzcd}
(A,B,s,t) \arrow[r, "{id_A, id_B}"] & (A,B,s,t) \\
(E,C,q \circ s', q \circ t') \arrow[u, "{Q((h_E, h_V \circ q, h_V))}"] \arrow[ur, "{(h_E, h_V)}"'] &
\end{tikzcd}$$

Such arrow is unique. Suppose $f = (f_E, f_V, f_C)$ to be another arrow wit such property. Then, it must be $(id_A, id_B) \circ Q(f) = (f_E, f_C) = (h_E, h_C)$. Finally, $f_C = f_V \circ q = h_V \circ q$. □

Proposition 1.3.12. *Q creates colimits.*

Proof. Preserve from **??**. Remain to see Reflect. □

Bibliography