UNIVERSITÀ DI PISA

DIPARTIMENTO DI INFORMATICA

Corso di Laurea Triennale in Informatica

On Adhesivity of EGGs

Relatori:

Fabio Gadducci

Davide Castelnovo

Candidato:

Roberto Biondo

ANNO ACCADEMICO 2023/2024

al pettirosso,
agli specchi delle case dei vecchi.

Contents

Introduction

The use of graph-like structures as specification tools, as well as formal models, in rewriting-based formalisms and dependency analysis has a long tradition in computer science. These structures offer an intuitive, flexible, and expressive framework for describing the behaviour and the evolution of complex systems.

For instance, graphs can represent a set of entities as nodes and the relationships between them as edges, providing a clear depiction of the state of a system at any given moment. Rewriting steps then model computational processes, simulating changes in the system. These transformations might involve the addition, removal, or reconfiguration of nodes and edges, enabling the representation of diverse phenomena such as modifications in the architecture of a software system, evolution steps of a biological network, or updates within a dependency graph.

The development of the visual formalism of *equality graphs* (e-graphs) can be traced back to the 1980s, originating in the context of automatic theorem provers (ATPs) [Nel80]. E-graphs provide a compact and efficient way to represent equivalence classes of expressions while maintaining the crucial property of closure under congruence. This ensures that equivalences are preserved across all contexts in which an expression might appear, making e-graphs a robust tools for reasoning about equivalence of algebraic expressions.

Dependency and concurrency harden the process of program optimization, because application order of rewrites can significantly improve the quality of the overall program. Such problem is known as *phase-ordering problem*, and a recent approach, known as equality saturation, was developed in [TSTL11]. This technique provides an alternative to heuristic methods, consisting of keeping track of a set of candidate equivalent optimized programs, computed by repeatedly inferring equivalences between program fragments. E-Graphs provide a compact representation of this set, making operation feasible in concrete applications.

The utility of e-graphs thus extends beyond theoretical applications, finding practical relevance in fields such as system optimization and component upgrading. Examples of uses can be found in [Che21]. However, the more a formalism is used, the greater is the need for formal frameworks to guide its development and application. In particular, addressing issues of parallelism and concurrency is critical for a mathematically sound understanding of these structures.

Among the rewriting paradigms for graph-like structures, one of the leading proposal is the Double Pushout (DPO) approach [EG06, EEPT06], since it provides, paired with the framework of $\mathcal{M}$-adhesive categories [LS05, ACR19, BHK22], a robust mathematical foundation for the definition and the application of rules, that us the identification of a subgraph to be removed and the insertion of a new subgraph. Integrating the theory of e-graphs manipulation into the broader subject of DPO rewriting would put on firmer ground the status of e-graphs. Moreover, it would allow to apply to this new context the well known results about concurrency and parallelism already known provided by the abstract theory of DPO rewriting systems [BCE+99, ERMK99].

Such an integration is the main aim of the thesis. On the technical level, to reach our aim we need to prove some adhesivity property of e-graphs. In order to do so, some consideration on graphs with equivalences are in order. A graph with equivalence is nothing but a graph endowed with an equivalence relation among its nodes. The main result on this kind of graphs is a weak adhesivity property that holds for regular monomorphisms. This adhesivity result, in turn, extends naturally, via the results of [CGM22], to e-graphs, which in fact can be interpreted as a specialised class of graphs with equivalences that satisfy additional structural constraints. An operational presentation of such structures can be found in [WNW+21]. Specifically, e-graphs enforce a condition that edges must respect equivalence relations: if two edges have equivalent target nodes, then their source nodes must also be equivalent.

Synopsis. Chapter 1 introduces the basic concepts of category theory that will be used throughout this thesis. The main goal of this chapter is to build the categorical toolbox necessary for the remainder of the work. Category theory provides a general and uniform formalism, which has the side effect of making many definitions seem quite abstract. To address this, we will provide concrete examples to help clarify and illustrate the concepts. Chapter 2 focuses on graphs. The structure we will use to model graphs are presheaves, which not only provides a straightforward formalisation, putting in evidence the categorical properties of graphs that we will need. We will then endow graphs with an equivalence relation among vertices, and proof that the resulting category satisfies an adhesivity property. Finally, in the last part we will examine e-graphs, showing that they inherit the above adhesivity property from graphs with equivalences.

Chapter 1

Background

In this chapter the building blocks for this work, almost entirely based on categories, will be defined. The aim of what follows is not only to introduce concepts that will be used later, but also to understand how category theory is general enough to give the abstraction of known notions (mainly from set theory) to reuse them in different contexts. This is not a complete tutorial on categories, but instead a sufficient compendium of definitions to make clear what will be done in the next chapters.

1.1 Basic Notions

This section is all about basic definitions and examples, to get familiar with the formalism of categories.

1.1.1 Categories

Definition 1.1.1 (Category)**.** A *category* $\mathscr{C}$ comprises:

1. A collection of *objects* $\mathcal{O}b(\mathscr{C})$;
2. A collection of *arrows* (or *morphisms*) $\mathcal{H}om(\mathscr{C})$, often called *homset*.

Two operators, *dom* and *cod*, that map every morphism $f \in \mathcal{H}om(\mathscr{C})$ to two objects, respectively, its *domain* and its *codomain*. In case $dom\ f = A$ and $cod\ f = B$, we will write $f : A \to B$. The collection of morphisms from an object A to an object B is denoted as $\mathscr{C}(A, B)$. An operator $\circ$ of *composition* maps every couple of morphisms f, g with $cod\ f = dom\ g$ (in this case f and g are said to be composable) to a morphism $g \circ f : dom\ f \to cod\ g$. The composition operator is associative, i.e., for each composable arrows f, g and h, it holds that

$$h \circ (g \circ f) = (h \circ g) \circ f$$

For each object A, an *identity* morphism $id_A : A \to A$ such that, for each $f : A \to B$

$$id_B \circ f = f = f \circ id_A$$

The most important thing here is not the structure of the objects, but instead how this structure is preserved by the morphisms.

Example 1.1.2. A trivial example of category is the one with no objects, and hence no morphisms. Such category is denoted by **0** and is called *empty category.*

Example 1.1.3. The category with just one object and just one arrow, the identity arrow on that object, is denoted **1**. In particular, the only object of this category is $\bullet$, and the only arrow is $id_\bullet$.

Given an arrow $f : A \to B$ in a category $\mathscr{C}$, we say that f *factors through* $g : C \to B$ if there exists an arrow $h : A \to C$ such that $f = h \circ g$.

Definition 1.1.4. [Dual Category] Given a category $\mathscr{C}$, there exist a category $\mathscr{C}^{op}$ such that:

- $Ob(\mathscr{C}^{op}) = Ob(\mathscr{C})$;
- if $f : A \to B$ is a morphism in $\mathscr{C}$, then $f : B \to A$ is a morphism in $\mathscr{C}^{op}$.

Hence, given $f : A \to B$ and $g : B \to C$ arrows in $\mathscr{C}$, as $g \circ f : A \to C$ is an arrow in $\mathscr{C}$, then $f \circ g : C \to A$ is an arrow in $\mathscr{C}^{op}$. Such category is called *dual category* or *opposite category.*

Duality is a concept that we will encounter most of the time. Given a property P valid for a category $\mathscr{C}$, we will refer to the same property in the opposite category $\mathscr{C}^{op}$ as the *dual* of P, without explicitly constructing $\mathscr{C}^{op}$. There exist some properties that coincide exactly with their dual, and such properties are said to be *self dual* properties.

To represent morphisms of a category $\mathscr{C}$ it is possible to use *diagrams*, as the one below, in which the vertices are objects of $\mathscr{C}$, and the edges are morphisms of $\mathscr{C}$.

$$
\begin{tikzcd}
X \arrow[r, "f'"] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\
W \arrow[r, "f"'] & Y
\end{tikzcd}
$$

The diagram is said to commute whenever $f \circ g' = g \circ f'$. Unique morphisms are represented with dashed arrows. A more rigorous definition of what a diagram is will be given later (Definition 1.2.3).

Example 1.1.5. It is easy to see that taking sets as objects and (total) functions as arrows, we obtain a category. In fact, given two functions $f : A \to B$ and $g : B \to C$, it is possible to compose them obtaining an arrow $g \circ f : A \to C$, and the composition is associative. For each set A there exists an identity function $id_A : A \to A$ such that $id_A(a) = a$ for each $a \in A$. This category is denoted as **Set**.

Remark 1.1.6. It is important to note that the Definition 1.1.1 above does not specify what kind of collections $\mathcal{O}b(\mathscr{C})$ and $\mathcal{H}om(\mathscr{C})$ are. Taking **Set** as example, the collection $\mathcal{O}b(\mathbf{Set})$ cannot be a set itself, due to Russel's paradox. It would be more appropriate referring to a category $\mathscr{C}$ which $\mathcal{Ob}(\mathscr{C})$ and $\mathcal{Hom}(\mathscr{C})$ are both sets as a *small category*, but it is assumed in this work, except where it is made explicit, for a category to be small. Another clarification must to be done, still considering **Set**. Given two sets A and B, it is possible to construct the set B^A of all functions from A to B. This is isomorphic to $\mathbf{Set}(A, B)$, for each pair of sets A and B. A category $\mathscr{C}$ where, for each pair of objects A and B, $\mathscr{C}(A, B)$ is a set is said to be *locally small.*

Definition 1.1.7 (Subcategory)**.** A category $\mathscr{D}$ is a *subcategory* of a category $\mathscr{C}$ if:

1. each object of $\mathscr{D}$ is an object of $\mathscr{C}$;

2. each morphism between two objects of $\mathscr{D}$ is a morphism of $\mathscr{C}$; and

3. composites and identities of $\mathscr{D}$ are the same of $\mathscr{C}$.

If the inclusion at 2 is an equality (i.e. $\mathscr{D}(A, B) = \mathscr{C}(A, B)$ for each couple of objects A, B of $\mathscr{D}$), then $\mathscr{D}$ is said to be a *full subcategory* of $\mathscr{C}$. Another way to express that composites are the same (point 3) is to say that if $f, g \in \mathcal{Hom}(\mathscr{D})$ are composable, then $g \circ f \in \mathcal{Hom}(\mathscr{D})$, i.e., $\mathcal{Hom}(\mathscr{D})$ is *closed under composition.*

1.1.2 Mono, Epi and Iso

Between the morphisms of a category, it is possible to distinguish some that have certain properties, as functions between sets can be surjective, injective or bijective.

Definition 1.1.8 (Monomorphism)**.** An arrow $f : B \to C$ in a category $\mathscr{C}$ is a *monomorphism* if, for any pair of arrows of $\mathscr{C}$ $g : A \to B$, $h : A \to B$, the equality $f \circ g = f \circ h$ implies $g = h$. The class of monomorphisms of $\mathscr{C}$ is denoted $\mathcal{Mono}(\mathscr{C})$.

Remark 1.1.9. For a morphism, from an algebraic point of view, being mono means being *left cancelable.* Let A be an object in a category $\mathscr{C}$. Given two monomorphism $m : X \to A$ and $n : Y \to A$, then if $h : X \to Y$ is a morphism such that $m = n \circ h$, then is the unique one: suppose k is another morphism such that $m = n \circ k$. We can conclude $h = k$ observing that $n \circ h = n \circ k$ implies $h = k$ when n is mono, which is by hypothesis.

Definition 1.1.10 (Subobject)**.** Starting from this consideration, we can define a preorder on monomorphisms, placing $m \leq n$ if $m = n \circ h$ for some h. Such preorder induces an equivalence relation $\equiv$ on monomorphisms with codomain A, where $m \equiv n$ whenever $m \leq n$ and $n \leq m$, and the corresponding equivalence class is called *subobject* of A.

Definition 1.1.11 (Epimorphism)**.** An arrow $f : A \to B$ in a category $\mathscr{C}$ is an *epimorphism* if, for any pair of arrows of $\mathscr{C}$ $g : B \to C$, $h : B \to C$, the equality $g \circ f = h \circ f$ implies $g = h$.

Definition 1.1.12 (Isomorphism)**.** An arrow $f : A \to B$ is an *isomorphism* if there is an arrow $f^{-1} : B \to A$, called the *inverse* of f, such that $f^{-1} \circ f = id_A$ and $f \circ f^{-1} = id_B$. Two objects are said to be *isomorphic* if there is an isomorphism between them.

Example 1.1.13. In **Set**, monomorphisms are injective functions, epimorphisms are surjective functions and isomorphisms are bijections.

Remark 1.1.14. Mono and epi are dual concepts. This fact is easily shown by considering how a monomorphism m in a category $\mathscr{C}$ behaves in the dual category $\mathscr{C}^{op}$. In $\mathscr{C}$ we have that $m \circ f = m \circ g$ implies $f = g$. In $\mathscr{C}^{op}$, the we can state that $f \circ m = g \circ m$ implies $f = g$, obtaining the definition of epi.

Proposition 1.1.15. *The following statements hold for every pair of composable arrows f and g for any category $\mathscr{C}$:*

1. *if both f and g are mono, then $g \circ f$ is mono;*
2. *if $g \circ f$ is mono, then f is mono;*
3. *if both f and g are epi, then $g \circ f$ is epi;*
4. *if $g \circ f$ is epi, then g is epi.*

Observation 1.1.16. Consider two monomorphisms $m : X \to A$ and $n : Y \to A$, and suppose $m \equiv n$ (in the sense of Definition 1.1.10). This corresponds to having a isomorphism between X and Y. in fact, since $m \leq n$, there exists s unique $h : Y \to X$ such that $m = n \circ h$, and, since $n \leq m$, there exists a unique $k : X \to Y$ such that $n = m \circ k$. But then

$$\begin{aligned} m &= n \circ h & \qquad n &= m \circ k \\ &= m \circ h \circ k & &= n \circ h \circ k \end{aligned}$$

Since m is mono, we obtain $id_X = h \circ k$, and, since n is mono, $id_Y = k \circ h$, thus, an isomorphism.

The next proposition will be useful later.

Proposition 1.1.17. *In* **Set***, for every commutative square as the one below, if $e : X \to Y$ is epi and $m : M \to Z$ is mono, then there exists a unique morphism $h : Y \to M$ making the whole diagram below commutative.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "e"'] & M \arrow[d, "m"] \\ Y \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Z \end{tikzcd}$$

Proof. Let $y \in Y$. Since e is epi (i.e., surjective), there exists $x \in X$ such that $e(x) = y$. Then, we put $h(y) = f(x)$. h is well defined, in fact, let $x' \in X$ be such that $e(x') = y$. In this case, we have $h(x) = f(x')$, and so

$$\begin{aligned} (m \circ f)(x') &= (g \circ e)(x') \\ &= g(y) \\ &= (g \circ e)(x) \\ &= (m \circ f)(x) \end{aligned}$$

since m is mono by hypothesis, $f(x) = f(x')$. $\square$

Corollary 1.1.18. *In* **Set***, let $f : A \to B$ be a morphism, and let $f = m \circ e$, $f = m' \circ e'$ be two factorizations of f, with $e : A \to X$ and $e' : A \to Y$ epis and $m : X \to B$, $m' : Y \to B$ monos. Then, there exists an isomorphism $\phi : X \to Y$.*

Proof. Representing the hypothesis in diagrams, we have the following situation

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "e'"'] & X \arrow[d, "m"] \\ Y \arrow[r, "m'"'] \arrow[ur, dashed, "h"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "e'"] \arrow[d, "e"'] & Y \arrow[d, "m'"] \\ X \arrow[r, "m"'] \arrow[ur, dashed, "k"] & B \end{tikzcd}$$

where the dashed arrows are yielded by Proposition 1.1.17. Hence, we have that

$$\begin{aligned} id_X \circ e &= h \circ e' \\ &= h \circ k \circ e \end{aligned} \qquad \begin{aligned} id_Y \circ e' &= k \circ e \\ &= k \circ h \circ e' \end{aligned}$$

Hence, ìsince e and e' are epimorphisms, $h \circ k = id_X$, $k \circ h = id_Y$, obtaining the isomorphism. □

1.2 Functors, Natural Transformations, Adjoints

1.2.1 Functors and Natural Transformations

A functor is a structure preserving map between categories.

Definition 1.2.1 (Functor)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *functor* $F : \mathscr{C} \to \mathscr{D}$ is a map taking each object of $A \in \mathcal{O}b(\mathscr{C})$ to an object $F(A) \in \mathcal{O}b(\mathscr{D})$ and each arrow $f : A \to B$ of $\mathscr{C}$ to a arrow $F(f) : F(A) \to F(B)$ of $\mathscr{D}$, such that, for all objects $A \in \mathcal{O}b(\mathscr{C})$ and composable arrows f and g of $\mathscr{C}$:

- $F(id_A) = id_{F(A)}$;
- $F(g \circ f) = F(g) \circ F(f)$.

In this case, $\mathscr{C}$ is called *domain* and $\mathscr{D}$ is called *codomain* of the functor F.

Example 1.2.2. A first example of functor is the *identity functor*. Given a category $\mathscr{C}$, the identity functor $Id_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ is the functor that maps each object on itself and each arrow onto itself.

Once defined what a functor is, we can give a more rigorous definition of diagram. Although this may seem extremely technical, it will be useful, especially in the definition of limits (Definition 1.3.3).

Definition 1.2.3 (Diagram)**.** A *diagram in a category $\mathscr{C}$ of shape $\mathscr{I}$* is a functor $D : \mathscr{I} \to \mathscr{C}$. The category $\mathscr{I}$ can be considered as the category indexing the objects and the morphisms of $\mathscr{C}$ shaped in $\mathscr{I}$.

Example 1.2.4. A diagram of shape $\Lambda = (L \xleftarrow{l} X \xrightarrow{r} R)$ is said to be a *span*, and is denoted by $(l, X, r) : L \rightharpoonup R$. A span can be viewed as the generalization of relations between sets. In fact, in **Set**, a relation $R \subseteq A \times B$ is a span, with the projections $\pi_A : R \to A$ and $\pi_B : R \to B$ as arrows.

The dual notion of span is a *cospan*, namely, a diagram of shape $\Lambda^{op} = (L \xrightarrow{l} X \xleftarrow{r} R)$, and is denoted by $(l, X, r) : L \rightharpoondown R$.

Functor are often used to generalize some structural behaviour that constructions in categories have. An important example of this fact is the universal property. The definition is not straightforward, but it gives the abstraction of a property that will be useful in further definitions

Definition 1.2.5 (Universal property)**.** Let $F : \mathscr{C} \to \mathscr{D}$ be a functor, and let $B \in \mathcal{O}b(\mathscr{D})$. A pair (u, A), with $A \in \mathcal{O}b(\mathscr{C})$ and $u : B \to F(A)$ is said to be an *universal map for B with respect to F* if for each $A' \in \mathcal{O}b(\mathscr{C})$ and each $f : B \to F(A')$ there exists a unique morphism $h \in \mathscr{C}(A, A')$ such that the following triangle commute:

$$\begin{tikzcd}
B \arrow[r, "u"] \arrow[dr, "f"'] & F(A) \arrow[d, dashed, "F(h)"] & A \arrow[d, dashed, "h"] \\
& F(A') & A'
\end{tikzcd}$$

– i.e. there exists a unique h such that $F(h) \circ u = f$. In this case, (u, A) is said to have the *universal property.*

Dually, if $G : \mathscr{C} \to \mathscr{D}$ is a functor and $B \in \mathcal{O}b(\mathscr{D})$, then a pair (A, u) is a *co-universal map for B with respect to G* if $u : G(A) \to B$ and for each $A' \in \mathcal{O}b(\mathscr{C})$ and each $f : G(A') \to B$ there exists a unique morphism $h \in \mathscr{C}(A', A)$ such that the following diagram commutes:

$$\begin{tikzcd}
A' \arrow[d, dashed, "h"'] & G(A') \arrow[d, dashed, "G(h)"'] \arrow[dr, "f"] & \\
A & G(A) \arrow[r, "u"'] & B
\end{tikzcd}$$

Some interesting properties of certain functors depend strictly on how they behave on the homsets of the domain and the codomain categories. The following definitions are about this particular type of functors.

Definition 1.2.6 (Full functor, faithful functor, fully faithful functor)**.** Let $F : \mathscr{C} \to \mathscr{D}$ be a functor, and consider the inducted function

$$F_{A,B} : \mathscr{C}(A, B) \to \mathscr{D}(F(A), F(B))$$

If, for each A, B objects of $\mathscr{C}$, $F_{A,B}$ is surjective, then F is said to be *full*, if it is injective, F is said to be *faithful*, if it is both injective and surjective, F is said to be *fully faithful.*

Observation 1.2.7. Properties such as fullness and faithfulness are so called *self-dual*, because the dual notion coincide with the same notion. This fact can be advantageous because if for example the faithfulness implies the preservation of some property, then the dual property is implied at the same way.

Example 1.2.8. Let $\mathscr{C}$ be a category and $\mathscr{D}$ a subcategory. The inclusion functor $I : \mathscr{D} \to \mathscr{C}$, mapping each object and each arrow onto itself. I is a faithful functor, because, given any pair of objects A and B of $\mathscr{D}$, $I_{A,B}$ is injective. If $\mathscr{D}$ is a full subcategory, then I is fully faithful.

Having such classification among functors turns out to be useful in many contexts. For example, consider $F(m) : F(B) \to F(C)$ be a monomorphism in a category $\mathscr{D}$, where $F : \mathscr{C} \to \mathscr{D}$ is a faithful functor. Then, if $f, g : A \to B$ are two morphisms in $\mathscr{C}$ such that $m \circ f = m \circ g$, then $F(m \circ f) = F(m) \circ F(f) = F(m) \circ F(g) = F(m \circ g)$. Since $F(m)$ is mono, then $F(f) = F(g)$, and, since $F_{A,B}$ is injective, $f = g$. Together with the fact that faithfulness is a self-dual concept, we have a proof for what follows [HS79].

Proposition 1.2.9. *Let $F : \mathscr{C} \to \mathscr{D}$ be a faithful functor. Then F reflects monomorphisms and epimorphisms.*

Given two functors that share domain and codomain categories, it is possible to define a transformation between them, taking each object of the domain of the functors to an arrow in the codomain of the functors that represent the action of "changing the functor acting on that object".

Definition 1.2.10 (Natural transformation)**.** Let $F, G : \mathscr{C} \to \mathscr{D}$ be two functors. A *natural transformation* η between them, denoted $\eta : F \dot{\to} G$, is a function $\eta : \mathcal{O}b(\mathscr{C}) \to \mathcal{H}om(\mathscr{D})$ taking each $A \in \mathcal{O}b(\mathscr{C})$ to a morphism $\eta_A : F(A) \to G(A)$ in $\mathscr{D}$, such that, for each morphism $f : A \to B$ of $\mathscr{C}$, the following diagram commutes:

$$\begin{tikzcd}
F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\eta_B"'] & G(B)
\end{tikzcd}$$

– i.e., such that $G(f) \circ \eta_A = \eta_B \circ F(f)$.

We say that $\eta : F \dot{\to} G$ is a *natural isomorphism* if, for each $A \in \mathcal{O}b(\mathscr{C})$, η_A is an isomorphism in $\mathscr{D}$. In this case, F and G are said to be *naturally isomorphic*, and is denoted $F \cong G$.

Observation 1.2.11. It is easy to see that, given two natural transformations $\eta : F \dot{\to} G$, $\theta : G \dot{\to} H$, it is possible to compose them obtaining a new natural transformation $\xi = \theta \circ \eta : F \dot{\to} H$. This follows by the fact that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[r, "\theta_A"] \arrow[d, "G(f)"] & H(A) \arrow[d, "H(f)"] \\ F(B) \arrow[r, "\eta_B"'] & G(B) \arrow[r, "\theta_B"'] & H(B) \end{tikzcd}$$

commutes because the two inner squares do. Sticking another diagram on the right of the one above, it is possible to show associativity of composition of natural transformations.

Functor Categories

The Observation 1.2.11 shows that natural transformations recreate on the functors the same structure that morphisms in a category have on objects. This leads us to define a particular kind of category, in which objects are functors between two categories, and arrow are natural transformations.

Definition 1.2.12 (Functor Category)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be categories. The category whose objects are functors between $\mathscr{C}$ and $\mathscr{D}$ and whose arrows are natural transformations between them is said to be a *functor category*, and it is denoted by $[\mathscr{C}, \mathscr{D}]$.

Lemma 1.2.13. *Let $\mathscr{C}, \mathscr{D}, \mathscr{I}$ be categories. Then, it holds that*

$$[\mathscr{I}, [\mathscr{C}, \mathscr{D}]] \cong [\mathscr{I} \times \mathscr{C}, \mathscr{D}]$$

A functor with a small category as domain (Remark 1.1.6) and **Set** as codomain is said to be a *presheaf* on that category. Given a category $\mathscr{C}$, it is possible to construct the functor category of the presheaves on $\mathscr{C}$, i.e. $[\mathscr{C}, \mathbf{Set}]$.

Remark 1.2.14. What we are calling here a presheaf is not totally accurate, because technically a presheaf on a small category $\mathscr{C}$ is a functor $F : \mathscr{C}^{op} \to \mathbf{Set}$. This technicality would bring more complexity, and it is beyond the scope of this work, so we will continue adopting the definition given above.

1.2.2 Adjoints

Definition 1.2.15 (Right Adjoint)**.** Let $R : \mathscr{C} \to \mathscr{D}$ be a functor. R is said *right adjoint* if, for each object A of $\mathscr{D}$, there exists an object $L(A)$ and an arrow $\eta_A : A \to R(L(A))$ in $\mathscr{C}$ such that, for each arrow $f : A \to R(B)$ of $\mathscr{D}$, there is a unique arrow $g : L(A) \to B$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & R(L(A)) \arrow[d, dashed, "R(g)"] \\
& R(B)
\end{tikzcd}$$

–i.e., $R(g) \circ \eta_A = f$.

Proposition 1.2.16. *In Definition 1.2.15, the map that takes an object A to an object $L(A)$ can be extended to a functor $L : \mathscr{D} \to \mathscr{C}$. Moreover, there exists a natural transformation $id_{\mathscr{D}} \to R \circ L$.*

Proof. Let R be the right adjoint as in Definition 1.2.15. Given $f \colon X \to Y$, we can define $L(f)$ as the unique arrow $L(X) \to L(Y)$ whose image through R fits in the diagram below.

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & R(L(X)) \arrow[d, dashed, "R(L(f))"] \\
Y \arrow[r, "\eta_Y"'] & R(L(Y))
\end{tikzcd}$$

To see that in this way we get a functor it is now enough to notice the commutativity of the following diagrams.

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "id_X"'] & R(L(X)) \arrow[d, "R(id_{L(X)})"] \\
Y \arrow[r, "\eta_Y"'] & R(L(Y))
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[r, "g"] \arrow[d, "\eta_Y"'] & Z \arrow[d, "\eta_Z"] \\
R(L(X)) \arrow[r, "R(L(f))"'] & R(L(Y)) \arrow[r, "R(L(g))"'] & R(L(Z))
\end{tikzcd}$$

Finally, by construction the family given by all the $\eta_A \colon A \to R(L(A))$ is natural and we can conclude. $\square$

Remark 1.2.17. The family above mentioned is called *unit* of the adjunction.

Definition 1.2.18 (Left Adjoint)**.** Let $L : \mathscr{D} \to \mathscr{C}$ be a functor. L is a *left adjoint* if, for each object B of $\mathscr{C}$, there exists an object $R(B)$ and an arrow $\epsilon_B : L(R(B)) \to B$ in $\mathscr{D}$ such that, for each arrow $g : L(A) \to B$ of $\mathscr{C}$, there exists a unique arrow $f : A \to R(B)$ such that the following diagram commutes.

$$\begin{tikzcd}
L(R(B)) \arrow[r, "\epsilon_B"] & B \\
L(A) \arrow[u, dashed, "L(f)"] \arrow[ur, "g"'] &
\end{tikzcd}$$

– i.e., $\epsilon_B \circ L(f) = g$.

As we have shown before, it is possible to extend the mapping $A \to R(B)$ to a functor R, whose functoriality follows placing $\epsilon_X \circ L(R(f)) = f \circ \epsilon_Y$ for each $f : X \to Y$. The family $\epsilon_B : L(R(B)) \to B$ is natural and it is called *counit* of the adjunction.

The connection between left and right adjoints is expressed in the following proposition.

Proposition 1.2.19. *Let L be the functor of Proposition 1.2.16. Then, L is a left adjoint.*

Proof. Given an object B in $\mathscr{C}$, we can consider the solid part of the diagram below. Since R is a right adjoint, we get a unique arrow whose image through R make the triangle commutative.

$$\begin{tikzcd}
R(B) \arrow[r, "\eta_{R(B)}"] \arrow[dr, "id_{R(B)}"'] & R(L(R(B))) \arrow[d, dashed, "R(\epsilon_B)"] \\
& R(B)
\end{tikzcd}$$

Let now A be an object of $\mathscr{D}$ and $g\colon L(A) \to B$ an arrow in $\mathscr{C}$. We can consider the composite $R(g) \circ \eta_A\colon A \to R(B)$. Then we have

$$\begin{aligned}
R(\epsilon_B) \circ R(L(R(g))) \circ R(L(\eta_A)) \circ \eta_A &= R(\epsilon_B) \circ R(L(R(g))) \circ \eta_{R(L(A))} \circ \eta_A \\
&= R(\epsilon_B) \circ \eta_{R(B)} \circ R(g) \circ \eta_A \\
&= R(g) \circ \eta_A
\end{aligned}$$

Since R is a right adjoint and η its unit, it follows that $\epsilon_B \circ L(R(g) \circ \eta_A)$ coincides with g as wanted. $\square$

1.3 Limits and Universal Constructions

1.3.1 Limits and Colimits

Definition 1.3.1 (Cones)**.** Let $D : \mathscr{I} \to \mathscr{C}$ be a diagram in $\mathscr{C}$ of shape $\mathscr{I}$. A *cone* for D is an object X of $\mathscr{C}$, together with arrows $f_i : X \to D(i)$ indexed by $\mathscr{I}$ (i.e. one for each object i of $\mathscr{I}$), such that, for each morphism $\alpha : i \to j$ of $\mathscr{I}$, the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\ D(i) \arrow[rr, "D(\alpha)"'] & & D(j) \end{tikzcd}$$

– i.e., $D(\alpha) \circ f_i = f_j$. We denote such cone as $(X, (f_i)_{i\in\mathscr{I}})$.

Observation 1.3.2. Given a diagram D, the category of the cones for D, denoted $\mathbf{Cone}(D)$, is defined to have cones for D as objects and cone morphisms as arrows, where a cone morphism $\phi : C \to C'$ from $C = (X, (f_i)_{i\in\mathscr{I}})$ to $C' = (X', (f'_i)_{i\in\mathscr{I}})$ is a morphism $\phi : X \to X'$ such that the following diagram commutes for each i:

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "f_i"'] & & X' \arrow[dl, "f'_i"] \\ & D(i) & \end{tikzcd}$$

Definition 1.3.3 (Limits)**.** Let $D : \mathscr{I} \to \mathscr{C}$ be a diagram in $\mathscr{C}$ of shape $\mathscr{I}$. A cone $(X, (f_i)_{i\in\mathscr{I}})$ is a *limit* provided that, for any other cone $(X', (f'_i)_{i\in\mathscr{I}})$ for D, then there exists a unique morphism $k : X' \to X$ such that the following diagram commutes for each object i of $\mathscr{I}$:

$$\begin{tikzcd} X' \arrow[rr, dashed, "k"] \arrow[dr, "f'_i"'] & & X \arrow[dl, "f_i"] \\ & D(i) & \end{tikzcd}$$

– i.e., $f_i \circ k = f'_i$ for each object i of $\mathscr{I}$. Sometimes we will refer to the unique arrow k as the *mediating arrow*.

Observation 1.3.4. Given a diagram D, a limit for D is exactly the terminal object of the category $\mathbf{Cone}(D)$, defined in Observation 1.3.2.

The dual notions of cones and limits are that of cocones and colimits.

Definition 1.3.5. (Cocones, Colimits) A *cocone* for a diagram $D : \mathscr{I} \to \mathscr{C}$ is an object Y of $\mathscr{C}$ together with arrows $f_i : D(i) \to Y$ such that, for each $g : D(i) \to D(j)$ of $\mathscr{C}$, $f_j \circ g = f_i$. A cocone is denoted $((f_i)_{i\in\mathscr{I}}, Y)$. A *colimit* for D is a cocone $((f_i)_{i\in\mathscr{I}}, Y)$ with the universal property – i.e., if $((f'_i)_{i\in\mathscr{I}}, Y')$ is another cone for D, then there exists a unique arrow $h : Y \to Y'$ such that, for each i, $h \circ f_i = f'_i$.

Remark 1.3.6. It makes sense to refer to a (co)limit as *the* (co)limit. Suppose $(P, p_i)_{i\in\mathscr{I}}$ and $(Q, q_i)_{i\in\mathscr{I}}$ be limits for a diagram $D : \mathscr{I} \to \mathscr{C}$. Then, there exists a unique morphism $h : Q \to P$ such that $p_i \circ k = q_i$ for each i. At the same way, there exists a unique morphisms $k : P \to Q$ such that $q_i \circ k = p_i$ for each i. From the existence of the identity, must be $k \circ h = id_Q$ and $h \circ k = id_P$, that is, P and Q are isomorphic.

Proposition 1.3.7. *Let $D, D' : \mathscr{I} \to \mathscr{C}$ be two functors, and let $((c_i)_{i\in\mathscr{I}}, C)$ and $((c'_i)_{i\in\mathscr{I}}, C')$ be, respectively, the colimit of D and D'. Then, a natural transformation $\phi : D \dot{\to} D'$ induces a unique arrow $c : C \to C'$.*

Proof. Consider the following situation.

$$\begin{tikzcd}
 & C & \\
D(i) \arrow[ur, "c_i"] \arrow[rr, "D(\alpha)"] \arrow[d, "\phi_i"'] & & D(j) \arrow[ul, "c_j"'] \arrow[d, "\phi_j"] \\
D'(i) \arrow[rr, "D'(\alpha)"'] \arrow[dr, "c'_i"'] & & D'(j) \arrow[dl, "c'_j"] \\
 & C' &
\end{tikzcd}$$

To prove the statement, we note that $((c'_i \circ \phi_i)_{i\in\mathscr{I}}, C')$ is a cocone for D. Computing, we have, for each i, j and $\alpha : i \to j$

$$\begin{aligned}
c'_j \circ \phi_j \circ D(\alpha) &= c'_j \circ D'(\alpha) \circ \phi_i && \textit{Naturality of } \phi \\
&= c'_i \circ \phi_i && ((c'_i)_{i\in\mathscr{I}}, C') \textit{ is a colimit for } D'
\end{aligned}$$

Since $((c'_i)_{i\in\mathscr{I}}, C)$ is a colimit by hypothesis, then must exists a unique arrow $c : C \to C'$ such that $c \circ c_i = \phi_i \circ c'_i$, having the thesis. $\square$

Notion such limits and colimits are generalization of more particular cases that will be now introduced, that we will often call *universal constructions*.

Definition 1.3.8 (Initial Object, Terminal Object)**.** Consider the empty diagram (i.e., a diagram $D : \mathbf{0} \to \mathscr{C}$ where $\mathbf{0}$ is the empty category Example 1.1.2). Then, the limit of D is called *terminal object* and the colimit of D is called *initial object*, denoted, respectively, $\mathbb{1}_{\mathscr{C}}$ and $\mathbb{0}_{\mathscr{C}}$. (Subscripts are omitted when they are clear from the context).

Example 1.3.9. In **Set**, the initial object is the empty set $\varnothing$, because, for each set S, there exists the empty function from $\varnothing$ to S. The terminal object of **Set** is the singleton $\{\bullet\}$, because there is exactly one function from a set S to $\{\bullet\}$, namely, the function which sends each $s \in S$ to $\bullet$.

We now illustrate a result on functor categories (Definition 1.2.12) that will be useful later.

In particular, every presheaf has an initial and a terminal object, because **Set** does (Example 1.3.9).

Definition 1.3.10 (Product, Coproduct)**.** Let D be the following diagram:

$$A \qquad\qquad\qquad\qquad B$$

Then, a cone for D is an object X and two arrows $f : X \to A$, $g : X \to B$ (i.e., a span, defined in Example 1.2.4):

$$A \xleftarrow{\quad f \quad} X \xrightarrow{\quad g \quad} B$$

If it exists, a limit for D is called *product* of A and B, usually denoted as $(A \times B, \pi_A, \pi_B)$, while whose arrows are called *projections*. The colimit of D is called *coproduct* of A and B, usually denoted as $(\iota_A, \iota_B, A + B)$.

Example 1.3.11. Set has both products and coproduts. Given two sets A and B, the categorical product is the set-theoretic cartesian product $A \times B$, together with the two projections π_A and π_B, while the coproduct is the disjoint sum $A \amalg B = \{(x, 0) \mid x \in A\} \cup \{(y, 1) \mid y \in B\}$, together with the two canonical injections ι_A and ιB, where $\iota_A(a) = (a, 0)$ and $\iota_B(b) = (b, 1)$.

The notions of product and coproduct can be easily generalized, extending the definition to the product (and coproduct) of a family of objects, together with appropriate arrows (e.g., the projection arrows for each object in the product). We will denote the product of a collection of objects indexed by a (finite) category $\mathscr{I}$ as $\left(\prod_{i\in\mathcal{O}b(\mathscr{I})} X_i, (\pi_i)_{i\in\mathcal{O}b(\mathscr{I})}\right)$, and the coproduct as $\left((\iota_i)_{i\in\mathcal{O}b(\mathscr{I})}, \coprod_{i\in\mathcal{O}b(\mathscr{I})} X_i\right)$.

Definition 1.3.12 (Equalizer, Coequalizer)**.** Let D be the diagram below.

$$A \overset{f}{\underset{g}{\rightrightarrows}} B$$

The limit of D is called *equalizer*, and its colimit is called *coequalizer*.

Proposition 1.3.13. *Let $e : E \to A$ be the arrow that equalizes $f, g : A \to B$ in a category $\mathscr{C}$. Then, e is a monomorphism.*

Proof. Suppose X be an object and $x, y : X \to E$ be two morphisms in $\mathscr{C}$ such that $e \circ x = e \circ y$, and let $z = e \circ x$. Then, since e is an equalizer, $f \circ e = g \circ e$, and $f \circ z = g \circ z$. But, for the universal property of limits, there must be exactly one $u : Z \to E$ such that $z = e \circ u$. It follow that $x = u$ and $y = u$, hence $x = y$. $\square$

The dual of the proposition above states that a coequalizer is an epimorphism.

Of all monomorphisms, an interesting subclass of them is the one that contains only the equalizers.

Definition 1.3.14 (Regular Monomorphism, Regular Epimorphism)**.** A monomorphism that is an equalizer for a pair of arrows is said *regular monomorphism.* The class of all regular monomorphisms of a category $\mathscr{C}$ is denoted $\mathcal{R}eg(\mathscr{C})$. An epimorphism that is a coequalizer for a pair of arrows is said *regular epimorphism.*

Definition 1.3.15 (Pullback, Pushout)**.** Let D be the cospan $(f, C, g) : A \rightharpoondown B$. A cone for D is an object P and three arrows $\phi : P \to A$, $\psi : P \to B$, and $h : P \to C$, but the latter is uniquely determined by the other ones ($f \circ \phi = h = g \circ \psi$). Thus, the following diagram is a cone:

$$\begin{tikzcd}
P \arrow[r, "\psi"] \arrow[d, "\phi"'] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

Then, the limit of D is called *pullback* of f and g. Given a span $S = (l, X, r) : L \rightharpoonup R$, shown in the diagram below,

$$\begin{tikzcd}
L & X \arrow[l, "l"'] \arrow[r, "r"] & R
\end{tikzcd}$$

a cocone for S is any commutative square of the form

$$\begin{tikzcd}
 & C & \\
L \arrow[ur, "f"] & X \arrow[l, "l"'] \arrow[r, "r"] & R \arrow[ul, "g"']
\end{tikzcd}$$

(the morphism $X \to C$ is uniquely determined by the relation $f \circ l = g \circ r$). The colimit for S is called *pushout* of l and r.

Example 1.3.16. In **Set**, given two functions $f : A \to C$ and $g : B \to C$, a pullback of f and g exists and is exactly the set $P = \{(x, y) \in A \times B \mid f(x) = g(y)\}$, with $\pi_f : P \to B$ and $\pi_g : P \to C$ defined, respectively, by $\pi_f((x, y)) = y$ and $\pi_g((x, y)) = x$. In this way, we have then, $\forall (x, y) \in P$:

$$\begin{aligned}
(f \circ \pi_g)((x, y)) &= f(\pi_g((x, y))) \\
&= f(x) && \textit{Definition of } \pi_g \\
&= g(y) && (x, y) \in P \\
&= g(\pi_f((x, y))) && \textit{Definition of } \pi_f \\
&= (g \circ \pi_f)((x, y))
\end{aligned}$$

thus, $f \circ \pi_g = g \circ \pi_f$.

Another important example to our aims is a concrete definition of what is a pushout in the category of sets, and why morally we can regard a pushout as *the way to identify part of an object with a part of another* [BW95].

Example 1.3.17. In **Set**, given two functions $f : A \to B$ and $g : A \to C$, the pushout of them is the set $X = (B \amalg C)/_{\sim}$, where $\sim$ is the least equivalence relation such that $f(a) \sim g(a)$ for each $a \in A$, with $\iota_g : B \to X$ and $\iota_f : C \to X$ as arrows, sending each element of the domain in the corresponding equivalence class in X. In particular, for each $a \in A$:

$$\begin{aligned}
(\iota_g \circ f)(a) &= \iota_g(f(a)) && \\
&= [(f(a), 0)] && \textit{Definition of } \iota_g \\
&= [(g(a), 1)] && f(a) \sim g(a) \\
&= \iota_f(g(a)) && \textit{Definition of } \iota_f \\
&= (\iota_f \circ g)(a) &&
\end{aligned}$$

When both f and g are monos (that is, injections), then we can construct the pushout in the same way we have done above, with $(f(a), 0) \sim (g(a), 1)$ when such a exists and $(b, 0) \sim (c, 1)$ on each b and c with no preimage in A, with ι_f and ι_g injective. An easy way to see this fact is considering the following situation: let $f : A \to A \cup B$ and $g : A \to A \cup C$, with A disjoint from B and C, $f(a) = a$ and $g(a) = a$. Then the pushout is the object $A \cup B \cup C$, with the inclusions as arrows, that are also injective. A more general case is what happens considering functions $f : A \to B$ and $g : A \to C$ injective. Differently from the previous example, in this case is not possible to take just the union of codomains as the pushout, but rather the disjoint union of them and then identify the elements $f(a)$ with $g(a)$, as we have done above. In the category of sets and functions, we have the certainty that the pullback arrows are injective. In fact, taking the equivalence relation $\sim$, we have that $f(a) \sim f(a')$ if and only if $a = a'$ by hypothesis, and then $x \sim x'$ if and only if $x = x'$, then the pushout morphisms sends each element in an equivalence class composed only by the element itself, thus are injective.

Given a subclass of morphisms of a category, an important property is *stability* under certain type of constructions. In our case, we are interested in stability under pullbacks and under pushouts.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\
C \arrow[r, "g"'] & D
\end{tikzcd} \qquad (*)$$

Definition 1.3.18 (Stability under pullbacks, pushouts)**.** Given a category $\mathscr{C}$, a subclass $\mathcal{A} \subseteq \mathcal{H}om(\mathscr{C})$ is said to be *stable under pullbacks* if, for every pullback square as the one in $(*)$, if $n \in \mathcal{A}$, then $m \in \mathcal{A}$. $\mathcal{A}$ is said to be *stable under pushouts* if, for every pushout square as the one in $(*)$, if $m \in \mathcal{A}$, then $n \in \mathcal{A}$.

Proposition 1.3.19. *Let $f : A \to C$, $g : B \to C$ be arrows in any category $\mathscr{C}$, and consider the following pullback square:*

$$\begin{tikzcd} P \arrow[r, "\pi_f"] \arrow[d, "\pi_g"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

If g is mono, then so is π_g.

The proposition above can be dualised stating that pushouts preserves epimorphisms.

The following lemma is a classical result.

Lemma 1.3.20 (Pullback Lemma)**.** *Suppose that the following diagram is given and its right half is a pullback. Then the whole rectangle is a pullback if and only if its left half is a pullback.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "t"'] & Y \arrow[r, "g"] \arrow[d, "k"] & Z \arrow[d, "h"] \\ A \arrow[r, "a"'] & B \arrow[r, "b"'] & C \end{tikzcd}$$

Proof. For the "only if" part, suppose the left square to be a pullback. To verify the outer rectangle is a pullback, consider the following situation:

$$\begin{tikzcd} P \arrow[drr, bend left, "p_1"] \arrow[ddr, bend right, "p_2"'] & & \\ & & Z \arrow[d, "h"] \\ & A \arrow[r, "a"'] \quad B \arrow[r, "b"'] & C \end{tikzcd}$$

But the right square is a pullback implies that there exists a unique u such that

$$\begin{tikzcd} P \arrow[drr, bend left, "p_1"] \arrow[dr, dashed, "u"] \arrow[ddl, bend right, "p_2"'] & & \\ & Y \arrow[r, "g"] \arrow[d, "k"] & Z \arrow[d, "h"] \\ A \arrow[r, "a"'] & B \arrow[r, "b"'] & C \end{tikzcd}$$

And, since the left square is a pullback, there exists a unique v such that

$$
\begin{tikzcd}
P \arrow[dr, dashed, "v"] \arrow[drr, bend left, "u"] \arrow[drrr, bend left, "p_1"] \arrow[ddr, bend right, "p_2"'] & & & \\
& X \arrow[r, "f"] \arrow[d, "t"'] & Y \arrow[r, "g"] \arrow[d, "k"] & Z \arrow[d, "h"] \\
& A \arrow[r, "a"'] & B \arrow[r, "b"'] & C
\end{tikzcd}
$$

Hence, the whole rectangle is a pullback.

For the "if" part, consider the following situation.

$$
\begin{tikzcd}
P \arrow[drr, bend left, "p_1"] \arrow[ddr, bend right, "p_2"'] & & \\
& & Y \arrow[d, "k"] \\
& B \arrow[r, "b"'] & C
\end{tikzcd}
$$

We have now to show that the unique arrow $v : P \to X$ (of the outer rectangle) is such that $f \circ v = p_1$, but this follows from the fact that the right square is a pullback, having the following situation.

$$
\begin{tikzcd}
P \arrow[drr, dashed, "u"] \arrow[drrr, bend left, "g \circ p_1"] \arrow[ddr, bend right, "p_2"'] & & & \\
& & Y \arrow[r, "g"] \arrow[d, "k"] & Z \arrow[d, "h"] \\
& A \arrow[r, "a"'] & B \arrow[r, "b"'] & C
\end{tikzcd}
$$

□

Corollary 1.3.21. *Let $\mathscr{C}$ be a category and suppose that the solid part of the following cube is given*

$$
\begin{tikzcd}
& Y' \arrow[rr, "g'"] \arrow[dl, dashed, "q'"'] \arrow[dd, "y"'] & & Z' \arrow[dl, "r'"] \arrow[dd, "z"] \\
B' \arrow[rr, "k'"] \arrow[dd, "b"'] & & C' \arrow[dd, "c"] & \\
& Y \arrow[rr, "g"] \arrow[dl, "q"'] & & Z \arrow[dl, "r"] \\
B \arrow[rr, "k"'] & & C &
\end{tikzcd}
$$

If the front face is a pullback then there is a unique $q' \colon Y' \to B'$ filling the diagram. If, moreover, the other two vertical faces are also pullbacks, then the following square *is a pullback too.*

$$
\begin{tikzcd}
Y' \arrow[r, "q'"] \arrow[d, "y"'] & B' \arrow[d, "b"] \\
Y \arrow[r, "q"'] & B
\end{tikzcd}
$$

Proof. Let us compute:

$$
\begin{aligned}
c \circ r' \circ g' &= r \circ z \circ g' \\
&= r \circ g \circ y \\
&= k \circ q \circ y
\end{aligned}
$$

Since the front face is a pullback, this guarantees the existence of q'. The second half of the thesis follows applying Lemma 1.3.20 to the following rectangle.

$$
\begin{tikzcd}
Y' \arrow[r, "q'"'] \arrow[d, "y"'] \arrow[rr, bend left, "r' \circ g'"] & B' \arrow[r, "k'"'] \arrow[d, "b"] & C' \arrow[d, "c"] \\
Y \arrow[r, "q"] \arrow[rr, bend right, "r \circ g"'] & B \arrow[r, "k"] & C
\end{tikzcd}
$$

□

The connection between constructions as products and equalizers and limits is made clear by the following theorem. The idea behind the proof is the fact that, given a diagram $D : \mathscr{I} \to \mathscr{C}$, if each subset of objects $X = \{D(i) \mid i \in \mathcal{O}b(\mathscr{I})\} \subseteq \mathcal{O}b(\mathscr{C})$ has a product $(\prod_{i \in I} D(i), (\pi_i)_{i \in \mathcal{O}b(\mathscr{I})})$ and each pair of arrows $f, g \in \mathscr{C}(D(i), D(j))$ has an equalizer, then one can construct the cone taking the equalizer of the arrows that has as domain the product of the objects of the diagram, and as codomain the product of the codomains of the arrows of the diagram. This construction has the universal property because equalizers and products do.

Theorem 1.3.22 (Limit theorem)**.** *Let $\mathscr{C}$ be a category. Then $\mathscr{C}$ has all finite limits if and only if $\mathscr{C}$ has all finite products and all finite equalizers.*

Proof. Let $D : \mathscr{I} \to \mathscr{C}$ a diagram, with $\mathscr{I}$ finite.

The *if* statement follows from definitions of products and equalizers (Definition 1.3.15, Definition 1.3.12)

To satisfy the *only if* statement, we want an object L together with morphisms $p_i : L \to D(j)$ such that:

1. $\{p_i : L \to D(i)\}$ is a cone – i.e., for each morphism of $\mathscr{I}$ $\alpha : i \to j$, $D(\alpha) \circ p_i = p_j$; and

2. for each E and $q_i : E \to D(j)$ in $\mathscr{C}$, with $D(\alpha) \circ q_i = q_j$ for each $\alpha : i \to j$ of $\mathscr{I}$, there exists a unique $f : E \to L$ such that $q_i = p_i \circ f$ for each $i \in \mathcal{O}b(\mathscr{I})$.

Consider the two products (which exist by hypothesis) $\prod_{j \in \mathcal{O}b(\mathscr{I})} D(j)$, the product of the objects of the diagram, and $\prod_{\alpha \in \mathcal{H}om(\mathscr{I})} D(cod\ \alpha)$, the product of the codomains of the morphisms in D, where π_x is the x-th projection of the product. Let now:

$$\gamma, \varepsilon : \prod_{j \in \mathcal{O}b(\mathscr{I})} D(j) \longrightarrow \prod_{\alpha \in \mathcal{H}om(\mathscr{I})} D(cod\ \alpha)$$

be defined by $\gamma_\alpha = \pi_{D(cod\ \alpha)}$ (the projection on the codomain of α) and $\varepsilon_\alpha = D(\alpha) \circ \pi_{D(dom\ \alpha)}$. Let now $e : L \to \prod_{j \in \mathcal{O}b(\mathscr{I})} D(j)$ the equalizer of γ and ε (which exists by hypothesis), and, for each $j \in \mathcal{O}b(\mathscr{I})$, $p_j : L \to D(j)$, defined by $p_j = \pi_{D(j)} \circ e$.

What we want now is to show that $(L, (p_i))_{i \in \mathscr{I}}$ is the limit of D, namely, to prove that the conditions given at the beginning are valid.

For condition 1, we have to show that, for each $\alpha : i \to j$ of $\mathscr{I}$, we have $D(\alpha) \circ p_i = p_j$:

$$\begin{aligned}
D(\alpha) \circ p_i &= D(\alpha) \circ \pi_{D(i)} \circ e && \textit{Definition of } p_j \\
&= \varepsilon_\alpha \circ e && \textit{Definition of } \varepsilon \\
&= \gamma_\alpha \circ e && e \textit{ is an equalizer of } \pi, \varepsilon \\
&= \pi_{D(j)} \circ e && \textit{Definition of } \pi \\
&= p_j && \textit{Definition of } p_j
\end{aligned}$$

For condition 2, suppose that $(E, (q_i)_{i \in \mathcal{O}b(\mathscr{I})})$ has the properties stated. By definition of product, there exists a (unique) arrow $q : E \to \prod_{j \in \mathcal{O}b(\mathscr{I})} D(j)$. For each arrow $\alpha : i \to j$, we have:

$$\begin{aligned}
\gamma_\alpha \circ q &= \pi_{D(j)} \circ q && \textit{Definition of } \pi \\
&= q_j && \textit{Definition of } q_j \\
&= D(\alpha) \circ q_i && \textit{Assumption on } q_j \\
&= D(\alpha) \circ \pi_{D(j)} \circ q && \textit{Definition of } q_i \\
&= \varepsilon_\alpha \circ q && \textit{Definition of } \varepsilon
\end{aligned}$$

Since e equalizes π and ε, there exists a unique $f : E \to L$ in $\mathscr{C}$ such that $q = e \circ f$. Then, for each $j \in \mathcal{O}b(\mathscr{I})$, we have $\pi_{D(j)} \circ q = \pi_{D(j)} \circ e \circ f$, hence, $q_i = p_i \circ f$. $\square$

Remark 1.3.23. The theorem above (and its relative proof) can be stated in its dual form leading to a theorem on existence of colimits, and a relative criterion to calculate them (taking the dual of the proof).

Example 1.3.24. Limit theorem gives us an easy way to calculate limits. An example of this fact is how limits are computed in **Set**. Given a diagram $D : \mathscr{I} \to$ **Set**, where $\mathscr{I}$ is a small category and $I = Ob(\mathscr{I})$, its limit is the set L defined as follows:

$$L = \{(d_i)_{i \in I} \in \prod_{i \in I} D(i) \mid \forall \phi \in \mathscr{I}(i, i'), D(\phi)(d_i) = d_{i'}\}$$

with projections as arrows.

Example 1.3.25. As we have done in Example 1.3.24, we illustrate how to construct colimits in the category of sets. Given a small category $\mathscr{I}$, $I = Ob(\mathscr{I})$, and a diagram $D : \mathscr{I} \to \mathbf{Set}$, consider the equivalence relation $\sim$ defined on $\coprod_{i\in I} D(i)$ such that $d_i \sim d_{i'}$ if $d_i \in D(i), d_{i'} \in D(i')$ and there exists some $\phi \in \mathscr{I}(i, i')$ such that $D(\phi)(d_i) = d_{i'}$. Then, a colimit for D is the set

$$C = \big(\coprod_{i\in I} D(i)\big)/\sim$$

with the inclusions as arrows.

Remark 1.3.26. Since a diagram is nothing more than a functor from a "shape" category to another, it makes sense to talk about limits of functors in general, even when they are not intended to be diagrams.

Lemma 1.3.27. *Let $\mathscr{C}, \mathscr{D}$ be a categories, and let $[\mathscr{C}, \mathscr{D}]$ be the category of functors from $\mathscr{C}$ to $\mathscr{D}$. Let $D : \mathscr{I} \to [\mathscr{C}, \mathscr{D}]$ be a diagram of shape $\mathscr{I}$ on the functor category early mentioned. Then,*

1. *The limit of D exists, and it is the functor $L : \mathscr{C} \to \mathscr{D}$ such that, for each object A of $\mathscr{C}$, $L(A)$ is the limit in $\mathscr{D}$ of the values of the functors $D(i)(A)$ for each i.*

2. *The colimit of D exists, and it is the functor $C : \mathscr{C} \to \mathscr{D}$ such that, for each object A of $\mathscr{C}$, $C(A)$ is the colimit in $\mathscr{D}$ of the values of the functors $D(i)(A)$, for each i.*

In the next sections, we will work on a special kind of diagrams with certain properties. In particular, we are interested in how a functor behaves with respect to the constructions defined so far.

Definition 1.3.28. Let $D : \mathscr{I} \to \mathscr{C}$ be a diagram, and $F : \mathscr{C} \to \mathscr{D}$ a functor. We say that F:

1. *preserves limits* of D if, given a limit $(L, l_i)_{i\in\mathscr{I}}$ for D, then $(F(L), F(l_i))_{i\in\mathscr{I}}$ is a limit for $F \circ D$.

2. *reflects limits* of D if a cone $(L, l_i)_{i\in\mathscr{I}}$ is a limit for D whenever $(F(L), F(l_i))_{i\in\mathscr{I}}$ is a limit for $F \circ D$.

3. *lifts limits (uniquely)* of D if, given a limit $(L, l_i)_{i\in\mathscr{I}}$ for $F \circ D$, there exists a (unique) limit $(L', l'_i)_{i\in\mathscr{I}}$ for D such that $(F(L'), F(l'_i))_{i\in\mathscr{I}} = (L, l_i)_{i\in\mathscr{I}}$.

4. *creates limits* of D if D has a limit and F preserves and reflects limits along it.

The dual notions are obtained in the obvious way, namely, substituting the words "limits" and "cones" with "colimits" and "cocones", respectively

Observation 1.3.29. It holds that if a functor creates limits, then lifts uniquely limits [AHS09].

Proposition 1.3.30. *A fully faithful functor reflects all limits and colimits.*

Proposition 1.3.31. *Let $\mathscr{D}$ be a full subcategory of a category $\mathscr{C}$, $D : \mathscr{I} \to \mathscr{D}$ be a diagram and $I : \mathscr{D} \to \mathscr{C}$ be the inclusion functor. Then, if $(L, (l_i)_{i \in \mathscr{I}})$ is the limit of $I \circ D$, and L is an object of $\mathscr{D}$ for each diagram D, then the inclusion functor creates limits.*

The next theorem is about a particular property that adjoint functors have.

Theorem 1.3.32. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor, and $G : \mathscr{D} \to \mathscr{C}$ its right adjoint. Then, G preserves limits.*

Remark 1.3.33. The dual of the theorem above states that, if G is a functor and F is a left adjoint, then F preserves colimits.

1.3.2 Kernel Pairs and Regular Epimorphisms

Definition 1.3.34 (Kernel Pair)**.** A *kernel pair* for an arrow $f : A \to B$ is an object K_f together with two arrows $\pi_f^1, \pi_f^2 : K_f \to A$, denoted as (K_f, π_f^1, π_f^2), such that the following square is a pullback.

$$\begin{tikzcd}
K_f \arrow[r, "\pi_f^1"] \arrow[d, "\pi_f^2"'] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Remark 1.3.35. If a category $\mathscr{C}$ has pullbacks then every arrow has a kernel pair.

Remark 1.3.36. Since a kernel pair is nothing more that a pullback, that is, a limit, by Remark 1.3.6, it make sense to refer to it as *the* kernel pair for a morphism f.

Example 1.3.37. In **Set**, a kernel pair for a function $f : A \to B$ is the set

$$K_f = \{(x, y) \in A \times A \mid f(x) = f(y)\}$$

together with the canonical projection on the first and the second component of the pairs.

Proposition 1.3.38. *Let (K, p_1, p_2) be the kernel pair of $f : X \to Y$, and let $(X \times X, \pi_1, \pi_2)$ be the product of X with itself. Then, the mediating arrow $\langle p_1, p_2 \rangle : K \to X \times X$ is mono.*

Proof. Suppose $\langle p_1, p_2 \rangle \circ f = \langle p_1, p_2 \rangle \circ g$ for $f, g : Z \to K$. Then, we have

$$\begin{aligned} \langle p_1, p_2 \rangle \circ f &= \langle p_1, p_2 \rangle \circ g & \qquad \langle p_1, p_2 \rangle \circ f &= \langle p_1, p_2 \rangle \circ g \\ \pi_1 \circ \langle p_1, p_2 \rangle \circ f &= \pi_1 \circ \langle p_1, p_2 \rangle \circ g & \qquad \pi_2 \circ \langle p_1, p_2 \rangle \circ f &= \pi_2 \circ \langle p_1, p_2 \rangle \circ g \\ p_1 \circ f &= p_1 \circ g & \qquad p_2 \circ f &= p_2 \circ g \end{aligned}$$

Thus, from the universal property of the pullback, $f = g$. $\square$

Proposition 1.3.39. *An arrow $m\colon M \to X$ is mono if and only if (M, id_M, id_M) is a kernel pair for it.*

Proof. To prove the "if" part of the statement, let $f, g : A \to M$ be such that $m \circ f = m \circ g$, and consider the following situation.

$$\begin{tikzcd}
A \arrow[drr, bend left, "f"] \arrow[dr, dashed, "u"] \arrow[ddr, bend right, "g"'] & & \\
& M \arrow[r, "id_M"] \arrow[d, "id_M"] & M \arrow[d, "m"] \\
& M \arrow[r, "m"'] & X
\end{tikzcd}$$

For the universal property of pullbacks, we have that

$$f = id_M \circ u = g$$

Hence, m is mono.

Conversely, if m is mono, then, we have that

$$\begin{aligned} m \circ f = m \circ g &\Rightarrow f = g \\ &\Rightarrow f \circ id_M = g \circ id_M \end{aligned}$$

Hence, f is the unique arrow that makes the commutative square illustrated above a pushout. $\square$

Remark 1.3.40. From characterization of monos via pullbacks in Proposition 1.3.39 and Lemma 1.3.27, we have that a mono in a category of presheaves is a natural transformation of which each component is mono.

Corollary 1.3.41. *(K_f, π^1_f, π^2_f) is a kernel pair for $f\colon X \to Y$ if and only if, for each mono $m\colon Y \to Z$, (K_f, π^1_f, π^2_f) is a kernel pair also for $m \circ f$.*

Proof. It is enough to see that, by Lemma 1.3.20 and Proposition 1.3.39 the outer boundary of the following square is a pullback.

$$\begin{tikzcd}
K_f \arrow[r, "\pi^2_f"] \arrow[d, "\pi^2_f"'] & X \arrow[r, "id_X"] \arrow[d, "f"] & X \arrow[d, "f"] \\
X \arrow[r, "f"] \arrow[d, "id_X"'] & Y \arrow[r, "id_Y"] \arrow[d, "id_Y"] & Y \arrow[d, "m"] \\
X \arrow[r, "f"'] & Y \arrow[r, "m"'] & Z
\end{tikzcd}$$

The leftward part of the statement follows by definition of monomorphism an Lemma 1.3.20. $\square$

Lemma 1.3.42. *Suppose the following situation, and that $f : X \to Y$ and $g : Z \to W$ have kernel pairs.*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "t"'] & W \end{tikzcd}$$

Then, there exists a unique arrow $k_h : K_f \to K_g$ making the squares below commute.

$$\begin{tikzcd} K_f \arrow[r, dashed, "k_h"] \arrow[d, "\pi_f^1"'] & K_g \arrow[d, "\pi_g^1"] \\ X \arrow[r, "h"'] & Z \end{tikzcd} \qquad \begin{tikzcd} K_f \arrow[r, dashed, "k_h"] \arrow[d, "\pi_f^2"'] & K_g \arrow[d, "\pi_g^2"] \\ X \arrow[r, "h"'] & Z \end{tikzcd}$$

Moreover, if the beginning square is a pullback, then also the preceding ones are so.

Proof. Computing, we have

$$\begin{aligned} g \circ h \circ \pi_f^1 &= t \circ f \circ \pi_f^1 \\ &= t \circ f \circ \pi_f^2 \\ &= g \circ h \circ \pi_f^2 \end{aligned}$$

By the universal property of K_g as the pullback of g along itself, such k_h exists and it is unique.

To prove the second half of the thesis, let us consider the two rectangles below, which, by Lemma 1.3.20 are pullbacks.

$$\begin{tikzcd} K_f \arrow[r, "\pi_f^1"] \arrow[d, "\pi_f^2"'] & X \arrow[r, "h"] \arrow[d, "f"] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \arrow[r, "t"'] & W \end{tikzcd} \qquad \begin{tikzcd} K_f \arrow[r, "\pi_f^2"] \arrow[d, "\pi_f^1"'] & X \arrow[r, "h"] \arrow[d, "f"] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \arrow[r, "t"'] & W \end{tikzcd}$$

But then the following ones are pullbacks too.

$$\begin{tikzcd} K_f \arrow[r, "k_h"'] \arrow[rr, bend left, "h \circ \pi_f^2"] \arrow[d, "\pi_f^1"'] & K_g \arrow[r, "\pi_g^2"'] \arrow[d, "\pi_g^1"] & Z \arrow[d, "g"] \\ X \arrow[r, "h"] \arrow[rr, bend right, "t \circ f"'] & Y \arrow[r, "g"] & W \end{tikzcd} \qquad \begin{tikzcd} K_f \arrow[r, "k_h"'] \arrow[rr, bend left, "h \circ \pi_f^1"] \arrow[d, "\pi_f^2"'] & K_g \arrow[r, "\pi_g^1"'] \arrow[d, "\pi_g^2"] & Z \arrow[d, "g"] \\ X \arrow[r, "h"] \arrow[rr, bend right, "t \circ f"'] & Y \arrow[r, "g"] & W \end{tikzcd}$$

The thesis follows again by Lemma 1.3.20. □

Proposition 1.3.43. *Let $e : X \to Y$ be a regular epimorphism in a category $\mathscr{C}$ with a kernel pair (K, π_1, π_2). Then, e is the coequalizer of π_1 and π_2.*

Proof. By hypothesis, there exists a pair $f, g : Z \to X$ of which e is the coequalizer. Since $e \circ f = e \circ g$, we have

$$\begin{tikzcd}
Z \arrow[dr, dashed, "k"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& K \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \arrow[d, "e"] \\
& X \arrow[r, "e"'] & Y
\end{tikzcd}$$

thus there exists the unique $k : Z \to K$. Let now $h : Z \to V$ be an arrow such that $h \circ \pi_1 = h \circ \pi_2$, then

$$\begin{aligned}
h \circ f &= h \circ \pi_1 \circ k \\
&= h \circ \pi_2 \circ k \\
&= h \circ g
\end{aligned}$$

and thus there exists a unique $l : Y \to V$ such that $l \circ e = h$. □

Corollary 1.3.44. *Let $\mathscr{C}$ be a category with pullbacks and $\phi : D \dot{\to} D'$ be a natural transformation between two functors $D, D' : \mathscr{I} \to \mathscr{C}$. If ϕ_i is a regular epi for every i, then ϕ is a regular epi.*

Proof. Let (K_i, π_i^1, π_i^2) be the kernel pair of ϕ_i for each i. Given an arrow $\alpha : i \to j$ of $\mathscr{I}$, we have

$$\begin{aligned}
\phi_j \circ D(\alpha) \circ \pi_i^1 &= D'(\alpha) \circ \phi_i \circ \pi_i^1 \\
&= D'(\alpha) \circ \phi_i \circ \pi_i^2 \\
&= \phi_j \circ D(\alpha) \circ \pi_i^2
\end{aligned}$$

Thus, the outer boundary of the diagram below commutes, yielding the arrow $K(\alpha)$

$$\begin{tikzcd}
K_i \arrow[r, "\pi_i^1"] \arrow[d, "\pi_i^2"'] \arrow[dr, dashed, "K(\alpha)"] & D(i) \arrow[dr, "D(\alpha)"] & \\
D(i) \arrow[dr, "D(\alpha)"'] & K_j \arrow[r, "\pi_j^1"] \arrow[d, "\pi_j^2"'] & D(j) \arrow[d, "\phi_j"] \\
& D(j) \arrow[r, "\phi_j"'] & D'(j)
\end{tikzcd}$$

In this way, we get a functor $K : \mathscr{I} \to \mathscr{C}$, which maps each i onto K_i and each arrow α onto $K(\alpha)$. We have in fact $K(id_i) : K_i \to K_i$ is the arrow such that

$$\begin{aligned}
D(id_i) \circ \pi_i^1 &= \pi_i^1 \circ K(id_i) & \qquad D(id_i) \circ \pi_i^2 &= \pi_i^2 \circ K(id_i) \\
\pi_i^1 &= \pi_i^1 \circ K(id_i) & \pi_i^2 &= \pi_i^2 \circ K(id_i)
\end{aligned}$$

Thus, for the universal property of pullbacks, $K(id_i) = id_{K_i}$.

Suppose now $\alpha : i \to j$ and $\beta : j \to k$. Computing, we have

$$\begin{aligned} \pi^1_k \circ K(\beta \circ \alpha) &= D(\beta \circ \alpha) \circ \pi^1_i \\ &= D(\beta) \circ D(\alpha) \circ \pi^1_i \\ &= D(\beta) \circ \pi^1_j \circ K(\alpha) \\ &= \pi^1_k \circ K(\beta) \circ K(\alpha) \end{aligned} \qquad \begin{aligned} \pi^2_k \circ K(\beta \circ \alpha) &= D(\beta \circ \alpha) \circ \pi^2_i \\ &= D(\beta) \circ D(\alpha) \circ \pi^2_i \\ &= D(\beta) \circ \pi^2_j \circ K(\alpha) \\ &= \pi^2_k \circ K(\beta) \circ K(\alpha) \end{aligned}$$

Again, for universal property of pullbacks, necessarily we have $K(\beta \circ \alpha) = K(\beta) \circ K(\alpha)$, proving functoriality of K.

Hence, we have two natural transformations $\pi^1, \pi^2 : E \dot{\to} D$. By Proposition 1.3.43, every component ϕ_i is the coequalizer of $\pi^1_i, \pi^2_i : E \to D$, and so ϕ is the coequalizer of π^1 and π^2. □

Lemma 1.3.45. *Let $D, D' : \mathscr{I} \to \mathscr{C}$ be two diagrams, and let $((c_i)_{i\in\mathscr{I}}, C)$ and $((c'_i)_{i\in\mathscr{I}}, C')$ be, respectively, the colimit of D and D'. If $\mathscr{C}$ has all colimits, for diagrams of shape $\mathscr{I}$ and $\phi : D \dot{\to} D'$ is a natural transformation in which all components are regular epimorphisms, then, the arrow induced by ϕ from C to C' (Proposition 1.3.7) is a regular epimorphism too.*

Proof. By Corollary 1.3.44, we know that $\phi : D \dot{\to} D'$ is a regular epimorphism, so that there is a functor $E : \mathscr{I} \to \mathscr{C}$ and $\eta, \theta : E \dot{\to} D$ such that ϕ is the coequalizer of η and θ. Let now $((p_i)_{i\in\mathscr{I}}, P)$ be the colimit of E, by Proposition 1.3.7, we have $a, b : P \to C$ fitting in the diagram below.

$$\begin{tikzcd} E(i) \arrow[r, "p_i"] \arrow[d, "\eta_i"'] & P \arrow[d, dashed, "a"] \\ D(i) \arrow[r, "c_i"'] & C \end{tikzcd} \qquad \begin{tikzcd} E(i) \arrow[r, "p_i"] \arrow[d, "\theta_i"'] & P \arrow[d, dashed, "b"] \\ D(i) \arrow[r, "c_i"'] & C \end{tikzcd}$$

We want to show that c coequalizes η and θ. Let thus $t : C \to T$ be an arrow such that $t \circ a = t \circ b$. Then, for every i, we have

$$\begin{aligned} t \circ c_i \circ \eta_i &= t \circ a \circ p_i \\ &= t \circ b \circ p_i \\ &= t \circ c_i \circ \theta_i \end{aligned}$$

Thus, there is $t_i : D(i) \to T$ such that $t \circ c_i = t_i \circ \phi_i$. It is now easy to see that $((t_i)_{i\in\mathscr{I}}, T)$ is a cocone of D': suppose $\alpha : i \to j$ be an arrow of $\mathscr{I}$, obtaining

$$\begin{aligned} t_i \circ \phi_i &= t \circ c_i \\ &= t \circ c_j \circ D(\alpha) \\ &= t_j \circ \phi_j \circ D(\alpha) \\ &= t_j \circ D'(\alpha) \circ \phi_i \end{aligned}$$

By the hypothesis that ϕ_i is regular epi for each i, therefore epi (by the dual of Proposition 1.3.13), we can conclude $t_i = t_j \circ D'(\alpha)$.

Hence, we have an arrow $k : C' \to T$ such that $k \circ c'_i = t_i$. But then

$$\begin{aligned} c \circ c \circ c_i &= k \circ c'_i \circ \phi_i \\ &= t_i \circ \phi \\ &= t \circ c_i \end{aligned}$$

Showing that $k \circ c = t$.

For the uniqueness, let $k' : C' \to T$ be another arrow such that $k' \circ c = t$. Then we have

$$\begin{aligned} k' \circ c'_i \circ \phi_i &= k' \circ c \circ c_i \\ &= t \circ c_i \\ &= t_i \circ \phi_i \end{aligned}$$

Since ϕ_i is a regular epimorphism, we have $k' \circ c'_i = t_i$, and, because $k \circ c'_i = t_i$ by construction, we can conclude that $k' = k$ since $((c'_i)_{i \in \mathscr{I}}, C')$ is a colimit. $\square$

1.4 Adhesivity

The next section is about adhesivity. An adhesive category is intuitively a category in which pushouts of (some) monomorphisms exist and they behave more or less as they do among sets.

Definition 1.4.1. (Van Kampen property) Let $\mathcal{A}$ be a subclass of $\mathcal{H}om(\mathscr{C})$, and consider the diagram below:

$$\begin{tikzcd}
 & A' \arrow[rr, "f'"] \arrow[dl, "m'"'] \arrow[dd, "a"] & & B' \arrow[dl, "n'"'] \arrow[dd, "b"] \\
C' \arrow[rr, "g'"] \arrow[dd, "c"'] & & D' \arrow[dd, "d"] & \\
 & A \arrow[rr, "f"] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\
C \arrow[rr, "g"'] & & D &
\end{tikzcd}$$

we say that the inner square is an $\mathcal{A}$-*Van Kampen* square if:

- it is a pushout;
- $a, b, c, d \in \mathcal{A}$;
- whenever the back and the left squares of the cube are pullbacks, then the top square is a pullback if and only if the front and the right squares are pullbacks.

when in the last requirement holds only the *if* part, we say that the category is $\mathcal{A}$-stable. Moreover, $\mathcal{H}om(\mathscr{C})$-Van Kampen squares are said to be *Van Kampen*, and $\mathcal{H}om(\mathscr{C})$-stable squares are called *stable*.

We are now ready to give the notion of $\mathcal{M}$-adhesivity.

Definition 1.4.2 ($\mathcal{M}$-adhesivity)**.** Let $\mathscr{C}$ be a category and $\mathcal{M} \subseteq \mathcal{M}ono(\mathscr{C})$ containing all isomorphisms, closed under composition and stable under pullbacks and pushouts (Definition 1.3.18). Then $\mathscr{C}$ is $\mathcal{M}$-*adhesive* if

1. every cospan $C \overset{g}{\to} D \overset{m}{\leftarrow} B$ with $m \in \mathcal{M}$ can be completed to a pullback (such pullbacks are called $\mathcal{M}$-pullbacks);

2. every span $C \overset{m}{\leftarrow} A \overset{f}{\to} B$ with $m \in \mathcal{M}$ can be completed to a pushout (such pushouts are called $\mathcal{M}$-pushouts);

3. pushouts along $\mathcal{M}$-arrows are $\mathcal{M}$-Van Kampen squares.

$\mathscr{C}$ is said to be *strictly* $\mathcal{M}$-*adhesive* if $\mathcal{M}$-pushouts are Van Kampen sqaures. We also say that $\mathscr{C}$ is *adhesive* when it is strictly $\mathcal{M}ono(\mathscr{C})$-adhesive, and *quasiadhesive* when it is strictly $\mathcal{R}eg(\mathscr{C})$-adhesive.

Observation 1.4.3. Set is adhesive.

Here it follows an interesting property of adhesive categories [Lac11].

Proposition 1.4.4. *In any adhesive category, the pushout of a monomorphism along any morphism is a monomorphism, and the resulting square is also a pullback. In* $\mathcal{M}$*-adhesive categories,* $\mathcal{M}$*-pushouts are pullbacks.*

Proof. Let the following square be a pushout, with m mono.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}$$

Consider now the following cube.

$$\begin{tikzcd}
& A \arrow[rr, "f"] \arrow[dl, "id_A"'] \arrow[dd, "id_A"' near start] & & B \arrow[dl, "id_B"'] \arrow[dd, "id_B"] \\
A \arrow[rr, "f" near end] \arrow[dd, "m"'] & & B \arrow[dd, "n"' near start] & \\
& A \arrow[rr, "f" near end] \arrow[dl, "m"'] & & B \arrow[dl, "n"] \\
C \arrow[rr, "g"'] & & D &
\end{tikzcd}$$

We have then that the top face is a pushout, and by Proposition 1.3.39, the left face is a pullback. Since the bottom face is a pushout by hypothesis, we have that the front face and the right one are pullbacks (by adhesivity). Hence, if $\mathscr{C}$

is adhesive, we can conclude that the starting square (which is the front and the bottom face of the cube) is a pushout, and by Proposition 1.3.39, since the left face is a pullback, n is mono. If $\mathscr{C}$ is $\mathcal{M}$-adhesive, with $m \in \mathcal{M}$ (that is, the square is a $\mathcal{M}$-pushout), $n \in \mathcal{M}$, and the vertical arrows then in $\mathcal{M}$. Hence, the front face is a pullback. $\square$

Verifying $\mathcal{M}$-adhesivity using the definition above may turn out to be very complex, so we can make use of the following result [CGM22].

Theorem 1.4.5. *Let $\mathscr{C}$ be a category, $\mathcal{M} \subseteq \mathcal{M}ono(\mathscr{C})$ containing all isomorphisms, closed under composition and stable under pullbacks and pushouts. Let now $F : \mathscr{C} \to \mathscr{D}$ be a functor with $\mathscr{D}$ $\mathcal{N}$-adhesive for some $\mathcal{N} \subseteq \mathcal{M}ono(\mathscr{D})$. If F is such that $F(\mathcal{M}) \subseteq \mathcal{N}$ and creates pullbacks and $\mathcal{M}$-pushout, then $\mathscr{C}$ is $\mathcal{M}$-adhesive.*

The idea behind this theorem is to simplify calculations to show that a certain category is adhesive for some subclass of monomorphisms, considering a functor from the category of which we want to prove adhesivity to a category we know it is adhesive, requiring that such functor has some properties.

Proof. In order to prove $\mathcal{M}$-adhesivity of $\mathscr{C}$, we have to verify the condition in Definition 1.4.2.

- Let $C \xrightarrow{g} D \xleftarrow{m} B$ with $m \in \mathcal{M}$ be a cospan in $\mathscr{C}$. Applying F, we obtain $F(C) \xrightarrow{F(g)} F(D) \xleftarrow{F(m)} B$, with $F(m) \in \mathcal{N}$ by hypothesis. Then, there exists a pullback $(P_F, p_{F(B)}, p_{F(D)})$ in $\mathscr{D}$, which is an $\mathcal{N}$-pullback (Definition 1.3.15). Since F creates pullbacks, hence lifts them (Observation 1.3.29), there exist a pullback (P, p_B, p_D) in $\mathscr{C}$.

- Let $C \xleftarrow{m} A \xrightarrow{f} B$ with $m \in \mathcal{M}$ be a cospan in $\mathscr{C}$. Analogously to the previous point, applying the functor F we obtain $F(C) \xleftarrow{F(m)} F(A) \xrightarrow{F(f)} F(B)$ with $F(m) \in \mathcal{N}$, and there exists a $\mathcal{N}$-pushout $(q_{F(C)}, q_{F(B)}, F(Q))$ in $\mathscr{D}$. Since F reflects pushouts, (q_C, q_B, Q) is a $\mathcal{M}$-pushout in $\mathscr{C}$.

- the Van Kampen property of $\mathcal{M}$-pullbacks follows from the closure under pullbacks and pushouts of $\mathcal{M}$ and from the fact that F reflects pullbacks.

$\square$

Corollary 1.4.6. *Let $\mathscr{A}$ be a $\mathcal{M}$-adhesive category for some $\mathcal{M} \subseteq \mathcal{H}om(\mathscr{A})$. Then, if $\mathscr{A}$ is a category whith all the pullbacks, the functor category $[\mathscr{C}, \mathscr{A}]$ is $\mathcal{M}^{\mathscr{C}}$-adhesive, where*

$$\mathcal{M}^{\mathscr{C}} = \{\eta \in \mathcal{H}om([\mathscr{C}, \mathscr{A}]) \mid \eta_C \in \mathcal{M} \textit{ for each object } C \textit{ of } \mathscr{C} \}$$

Since **Set** is adhesive, we can conclude what follows.

Corollary 1.4.7. *Every category of presheaves is adhesive.*

Lemma 1.4.8. *Let $\mathscr{C}$ be a strict $\mathcal{M}$-adhesive category with all pullbacks, and suppose that in the cube below the top face is an $\mathcal{M}$-pushout.*

$$\begin{tikzcd}
& A' \arrow[rr, "f'"] \arrow[dl, "m'"'] \arrow[dd, "a"] & & B' \arrow[dl, "n'"] \arrow[dd, "b"] \\
C' \arrow[rr, "g'"] \arrow[dd, "c"'] & & D' \arrow[dd, "d"] & \\
& A \arrow[rr, "f"] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\
C \arrow[rr, "g"'] & & D &
\end{tikzcd}$$

Then, the square below is a pushout.

$$\begin{tikzcd}
K_a \arrow[r, "k_{f'}"] \arrow[d, "k_{m'}"'] & K_b \arrow[d, "k_{n'}"] \\
K_c \arrow[r, "k_{g'}"'] & K_d
\end{tikzcd}$$

Proof. Since f', being the pullback of f, is in $\mathcal{M}$ then by Proposition 1.4.4 the bottom face of the cube is a pullback. Thus Lemma 1.3.42 entails that in the following cube the vertical faces are pullbacks.

$$\begin{tikzcd}
& K_a \arrow[rr, "k_{f'}"] \arrow[dl, "k_{m'}"'] \arrow[dd, "\pi_a^1"] & & K_b \arrow[dl, "k_{n'}"] \arrow[dd, "\pi_b^1"] \\
K_c \arrow[rr, "k_{g'}"] \arrow[dd, "\pi_c^1"'] & & K_d \arrow[dd, "\pi_d^1"] & \\
& A' \arrow[rr, "f'"] \arrow[dl, "m'"] & & B' \arrow[dl, "n'"] \\
C' \arrow[rr, "g'"'] & & D' &
\end{tikzcd}$$

Now the thesis follows from strong $\mathcal{M}$-adhesivity. □

Chapter 2

Categories of Graphs

This chapter is about graphs, and how it is possible to formalize them using categories in order to point out their properties from an abstract point of view. Starting from the set-theoretical definition of graphs, we will give an abstraction via functor categories, in which a graph is nothing but a functor between categories.

2.1 Graphs with Equivalences

A (directed graph) $\mathcal{G}$ is a structure consisting of a set of edge, a set of nodes and two functions, one assigning a *source* node and one assigning a *target* node to an edge. Formally, $\mathcal{G}$ is a quadruple $(V_{\mathcal{G}}, E_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}})$, where $V_{\mathcal{G}}$ is the set of nodes, $E_{\mathcal{G}}$ is the set of edges, and $s_{\mathcal{G}}, t_{\mathcal{G}} : E_{\mathcal{G}} \to V_{\mathcal{G}}$ are the source and the target functions.

A *graph homomorphism* $h : \mathcal{G} \to \mathcal{H}$ is then a pair of functions $h = (h_V : V_{\mathcal{G}} \to V_{\mathcal{H}}, h_E : E_{\mathcal{G}} \to E_{\mathcal{H}})$ such that

$$h_V \circ s_{\mathcal{G}} = s_{\mathcal{H}} \circ h_E$$

and

$$h_V \circ t_{\mathcal{G}} = t_{\mathcal{H}} \circ h_E$$

that is, a structure preserving map.

We can then generalize such notion to something more abstract, considering a graph to be nothing more than a presheaf from the category $(E \rightrightarrows V)$ to the category of sets. Having two of such presheaves, a natural transformation from one to another encapsulates the behavior of a graph morphism due to naturality. We can now define the category of graphs.

Definition 2.1.1 (Category of Graphs)**.** We denote as **Graph** the category

$$[E \underset{t}{\overset{s}{\rightrightarrows}} V, \mathbf{Set}]$$

Since **Graph** is a category of presheaves, Lemma 1.3.27 guarantees the existence of limits and colimits, and gives us an easy way to compute them.

Corollary 2.1.2. **Graph** *has all limits and colimits.*

A graph with equivalence is a 6-tuple $\mathbb{G} = (E, V, C, s, t, q)$, where E and V are, respectively, the edges and the vertices sets, and C is the set of the equivalence classes among vertices, $s, t : E \to V$ are the source and target functions and $q : V \to C$ is the *quotient* function, that is, the map from a vertex to its equivalence class. For this definition to make sense, q needs to be surjective. A morphisms h from a graph with equivalence $\mathbb{G} = (E, V, C, s, t, q)$ to another $\mathbb{H} = (E', V', C', s', t', q')$ is a triple $h = (h_E, h_V, h_C)$ of functions $h_V : V \to V'$, $h_E : E \to E'$ and $h_C : C \to C'$ such that

$$h_E \circ s = s' \circ h_V \qquad h_E \circ t = t' \circ h'_V \qquad h_C \circ q = q' \circ h_V$$

Remark 2.1.3. A graph with equivalence can be viewed as a graph endowed with an equivalence relation over its set of vertices, $(\mathcal{G}, \sim_{\mathcal{G}})$. An homomorphism between two graphs with equivalences $h : \mathbb{G} = (\mathcal{G}, \sim_{\mathcal{G}}) \to \mathbb{H} = (\mathcal{H}, \sim_{\mathcal{H}})$ is a graph homomorphism $h = (h_V, h_E) : \mathcal{G} \to \mathcal{H}$ such that if $v_1 \sim_{\mathcal{G}} v_2$ then $h_V(v_1) \sim_{\mathcal{H}} h_V(v_2)$. In **Set**, it is possible to formalize an equivalence relation $\sim$ over X as a surjective function sending each element x on its equivalence class $[x]_\sim$, and this justify our formalization via surjective functions (i.e., epimorphisms).

As we have done for graphs, we can think to a graph with equivalence as a presheaf, this time from a category $E \rightrightarrows V \to C$, where the image of C along the presheaf is the set of the equivalence classes, requiring that the morphism $V \to C$ is an epimorphism (that is, a surjective function).

Definition 2.1.4 (Category of Graphs with Equivalences)**.** The category **EqGrph** is the subcategory of

$$[E \overset{s}{\underset{t}{\rightrightarrows}} V \xrightarrow{q} C, \mathbf{Set}]$$

such that, for each $\mathbb{G} \in \mathcal{O}b(\mathbf{EqGrph})$, $\mathbb{G}(q)$ is an epimorphism.

Observation 2.1.5. Morphisms of graphs with equivalences are uniquely determined by the first two components. That is, if $h_1 = (h_E, h_V, \phi)$ and $h_2 = (h_E, h_V, \psi)$, then $\phi = \psi$. Indeed, consider two arrows $h_1, h_2 : \mathbb{G} \to \mathbb{H}$, where $\mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G)$ and $\mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$. Then, we have the following situation

$$\begin{tikzcd}
V_G \arrow[r, "h_V"] \arrow[d, "q_G"'] & V_H \arrow[d, "q_H"] & V_G \arrow[l, "h_V"'] \arrow[d, "q_G"] \\
C_G \arrow[r, "\phi"'] & C_H & C_G \arrow[l, "\psi"]
\end{tikzcd}$$

Then, we have:

$$\begin{aligned} \psi \circ q_G &= q_H \circ h_V \\ &= \phi \circ q_G \end{aligned}$$

From the fact that q_G is epi, we can conclude $\phi = \psi$.

A graph with equivalence is then a graph with an extra structure, the quotient map. Hence, it is possible to get the underlying graph by forgetting it. Such action is described by the *forgetful functor* $U : \mathbf{EqGrph} \to \mathbf{Graph}$, that maps each graph with equivalence $\mathbb{G} = (E, V, C, s, t, q)$ onto $U(\mathbb{G}) = (E, V, s, t)$, and each morphisms $h = (h_E, h_V, h_C)$ onto $U(h) = (h_E, h_V)$. U is effectively a functor, since, on identities, $U((id_E, id_V, id_C)) = (id_E, id_V)$, and on compositions

$$\begin{aligned} U(h \circ k) &= U((h_E \circ k_E, h_V \circ k_V, h_C \circ k_C)) \\ &= (h_E \circ k_E, h_V \circ k_V) \\ &= (h_E, h_V) \circ (k_E \circ k_V) \\ &= U(h) \circ U(k) \end{aligned}$$

Proposition 2.1.6. *The forgetful functor* $U : \mathbf{EqGrph} \to \mathbf{Graph}$ *is faithful.*

Proof. Let $\mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G)$ and $\mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$ be two graphs with equivalences, and let $h, k : \mathbb{G} \to \mathbb{H}$. If $U(h) = U(k)$ (i.e., the first two component of h and k are the same), from Observation 2.1.5, we can conclude that $h = k$. Then, the restriction $U_{\mathbb{G},\mathbb{H}} : \mathbf{EqGrph}(\mathbb{G}, \mathbb{H}) \to \mathbf{Graph}(U(\mathbb{G}), U(\mathbb{H}))$ is injective, therefore U is faithful. $\square$

Another functor that will be useful later is $V : \mathbf{EqGrph} \to \mathbf{Set}$, sending $(E_G, V_G, C_G, s_G, t_G, q_G)$ to C_G and $h = (h_E, h_V, h_C)$ to h_C.

Proposition 2.1.7. EqGrph *has all limits, colimits and* U *preserves limits and colimits.*

Proof. Let $D : \mathscr{I} \to \mathbf{EqGrph}$ be a diagram. In the following, we will denote the graph with equivalence $D(i)$ as $(E_i, V_i, C_i, s_i, t_i, q_i)$. Let now be the graph (A, B, s, t) the limit of $U \circ D$, with projections $(\pi^i_E, \pi^i_V) : (A, B, s, t) \to (E_i, V_i, s_i, t_i)$. Notice now that $(B, (q_i \circ \pi^i_V)_{i \in \mathscr{I}})$ is a cone for $V \circ D$. To see this, let $\alpha : i \to j$ be an arrow of $\mathscr{I}$, $D(\alpha) = (h_E, h_V, h_C)$, $U \circ D(\alpha) = (h_E, h_V)$. From the definition of cone, we have that $U \circ D(\alpha) \circ (\pi^i_E, \pi^i_V) = (\pi^j_E, \pi^j_V)$, hence $h_V \circ \pi^i_V = \pi^j_V$. Consider now the following diagram in **Set**

$$\begin{tikzcd} & B \arrow[dl, "\pi^i_V"'] \arrow[dr, "\pi^j_V"] & \\ V_i \arrow[rr, "h_V"] \arrow[d, "q_i"'] & & V_j \arrow[d, "q_j"] \\ C_i \arrow[rr, "h_C"'] & & C_j \end{tikzcd}$$

So we have $q_j \circ h_V \circ \pi^i_V = q_j \circ \pi^j_V$, by definition of graph with equivalence, $h_C \circ q_i \circ \pi^i_V = q_j$, and, by definition of V, $V \circ D(\alpha) \circ q_i \circ \pi^i_V = q_j \circ \pi^j_V$. Suppose now $(L, (l_i)_{i \in \mathscr{I}})$ be a limit for $V \circ D$, so that we have an arrow $l : B \to L$. This arrow is not epi in general, so let Q be its image, $q : Q \to B$ be the resulting epi and $m : Q \to L$ the corresponding mono, as the diagram below shows. By definition, the external rectangle commutes, so, for each i object of $\mathscr{I}$, Proposition 1.1.17 yields the dotted arrow π^i_C.

$$\begin{tikzcd} B \arrow[r, "\pi^i_V"] \arrow[d, "q"'] & B_i \arrow[r, "q_i"] & Q_i \arrow[d, "id_{Q_i}"] \\ Q \arrow[r, "m"'] \arrow[urr, dashed, "\pi^i_C"] & L \arrow[r, "l_i"'] & Q_i \end{tikzcd}$$

We have to show that in this way we get a cone over the diagram D. Let $\alpha : i \to j$ be an arrow of $\mathscr{I}$, then we have:

$$\begin{aligned} U(D(\alpha) \circ (\pi^i_E, \pi^i_V, \pi^i_C)) &= U(D(\alpha)) \circ (\pi^i_E, \pi^i_V) \\ &= (\pi^j_E, \pi^j_V) \\ &= U(D(\alpha) \circ (\pi^j_E, \pi^j_V, \pi^j_C)) \end{aligned}$$

And faithfulness of U yields the thesis.

To see that this cone is terminal, let $((E, F, G, a, b, c), (\tau_i)_{i\in\mathscr{I}})$, where $\tau_i = (\tau^i_E, \tau^i_V, \tau^i_C)$, be another cone. By construction, we have an arrow $(\tau_E, \tau_V) : (E, F, a, b) \to (A, B, s, t)$ such that

$$\begin{tikzcd} & E \arrow[dl, dashed, "\tau_E"'] \arrow[dr, "\tau^i_E"] & \\ A \arrow[rr, "\pi^i_E"'] & & A_i \end{tikzcd} \qquad \begin{tikzcd} & F \arrow[dl, dashed, "\tau_V"'] \arrow[dr, "\tau^i_V"] & \\ B \arrow[rr, "\pi^i_V"'] & & B_i \end{tikzcd}$$

For the same reason as before, $(G, (\tau^i_C)_{i\in\mathscr{I}})$ is a cone over $V \circ D$, thus there exists an arrow $\tau : G \to L$ such that $l_i \circ \tau = \tau^i_C$. At this point, we get

$$\begin{aligned} l_i \circ \tau \circ c &= \tau^i_C \circ c \\ &= q_i \circ \tau^i_V && \textit{τ_i is a morphism in } \mathbf{EqGrph} \\ &= q_i \circ \pi^i_V \circ \tau_V && \textit{Diagram above} \\ &= l_i \circ l \circ \tau_V && (B, (q_i \circ \pi^i_V)_{i\in\mathscr{I}}) \textit{ cone} \end{aligned}$$

Therefore, the outer part of the rectangle below commutes, and by Proposition 1.1.17 there exists a unique $\tau_C : G \to Q$

$$\begin{tikzcd} F \arrow[r, "\tau_V"] \arrow[d, "c"'] & B \arrow[r, "q"] & Q \arrow[d, "m"] \\ G \arrow[rr, "\tau"'] \arrow[urr, dashed, "\tau_C"] & & L \end{tikzcd}$$

Faithfulness of U and Observation 2.1.5 guarantees that (τ_E, τ_V, τ_C) is the unique arrow such that $(\pi^i_E, \pi^i_V, \pi^i_C) \circ (\tau_E, \tau_V, \tau_C) = (\tau^i_E, \tau^i_V, \tau^i_C)$.

For colimits, let (A, B, s, t) be the colimit of $U \circ D$, together with morphisms $(\kappa_i)_{i \in \mathscr{I}} = (\kappa^i_E, \kappa^i_V)_{i \in \mathscr{I}}$, and suppose $((c_i)_{i \in \mathscr{I}}, C)$ be the colimit of $V \circ D$. Then, we have the following situation

$$
\begin{tikzcd}
& B & \\
B_i \arrow[ur, "\kappa^i_V"] \arrow[rr, "h_V"] \arrow[d, "q_i"'] & & B_j \arrow[ul, "\kappa^j_V"'] \arrow[d, "q_j"] \\
C_i \arrow[rr, "h_C"'] \arrow[dr, "c_i"'] & & C_j \arrow[dl, "c_j"] \\
& C &
\end{tikzcd}
$$

Then, $((c_i \circ q_i)_{i \in \mathscr{I}}, C)$ is a cocone for all B_i, and, since $((\kappa_i)_{i \in \mathscr{I}}, B)$ is the colimit of all B_i (Lemma 1.3.27), there exists a unique arrow $q : B \to C$ such that, for each i, $q \circ \kappa^i_V = c_i \circ q_i$. Such q is epi, by application of Lemma 1.3.45, and thus (A, B, C, s, t, q), together with arrows $(\kappa^i_E, \kappa^i_V, c_i)_{i \in \mathscr{I}}$ is the colimit of D.

□

Corollary 2.1.8. *Let* $\mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G)$ *and* $\mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$ *be two graphs with equivalences. Then, an arrow* $h = (h_E, h_V, h_C) : \mathbb{G} \to \mathbb{H}$ *in* **EqGrph** *is mono if and only if* h_E *and* h_V *are mono in* **Set**.

Proof. The "if" part is given by the fact that U is faithful, and hence reflects monomorphisms. Since a morphism in a category of presheaves is mono if and only if it is injective on each component, we have that, if $U(h)$ is mono, that is, h_E and h_V are injective in **Set**, then h is mono. For the "only if" part, suppose $f = (f_E, f_V, f_C)$, $g = (g_E, g_V, g_C)$, $f, g : \mathbb{H} \to \mathbb{K}$, where $\mathbb{K} = (E_K, V_K, C_K, s_K, t_K, q_K)$, be such that $h \circ f = h \circ g$. Then, we have

$$
\begin{aligned}
h \circ f &= (h_E \circ f_E, h_V \circ f_V, h_C \circ f_C) \\
&= (h_E \circ f_E, h_V \circ f_V, h_V \circ f_V \circ q_K) \\
&= (h_E \circ g_E, h_V \circ g_V, h_V \circ g_V \circ q_K)
\end{aligned}
$$

Since q_K is epi, we have, on the third component, that $h_V \circ f_V \circ q_K = h_V \circ g_V \circ q_K$ implies $f_C = g_C$, and hence $f = g$ □

Corollary 2.1.9. *Let* $\mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G)$ *and* $\mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$ *be two graphs with equivalences, and let* $h = (h_E, h_V, h_C) : \mathbb{G} \to \mathbb{H}$ *be a regular monomorphism of* **EqGrph**, *then* h_E, h_V, h_C *are all monos.*

Proof. If h is mono, from Corollary 2.1.8 we have that h_E and h_V are monos. To derive h_C mono, suppose $f, g : \mathbb{H} \to \mathbb{K}$, where $\mathbb{K} = (E_K, V_K, C_K, s_K, t_K, q_K)$to be the arrows equalized by h. Then we have

$$\begin{aligned} f_C \circ h_C \circ q_G &= f_C \circ q_H \circ h_V \\ &= q_K \circ f_V \circ h_V \\ &= q_K \circ g_V \circ h_V \\ &= g_C \circ h_C \circ q_G \end{aligned}$$

since q_G is epi, we have that $f_C \circ h_C = g_C \circ h_C$, hence h_C is an equalizer for f_C and g_C, thus a monomorphism. □

Proposition 2.1.10. *Let $\mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G)$ and $\mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$ be two graphs with equivalences, and let $h = (h_E, h_V, h_C) : \mathbb{G} \to \mathbb{H}$ be a regular monomorphism of* **EqGrph**. *Then, h_E and h_V are mono and (K, π_1, π_2) is the kernel pair of $q_H \circ h_V$ if and only if (K, π_1, π_2) is the kernel pair of q_G.*

Proof. By Corollary 2.1.9, we have that h_E, h_V and h_E are all monos. Hence, by Corollary 1.3.41, (K, π_1, π_2) is the kernel pair of q_G if and only if it is the kernel pair also of $h_C \circ q_G$, since h_C is mono by hypothesis. The thesis follows from $h_C \circ q_G = q_H \circ h_V$, and from the hypothesis of h_E mono.

□

Remark 2.1.11. It is possible to restate the last proposition, by Example 1.3.37, as

> *h_E and h_V are mono and, for every $v, v' \in V_H$, $q_H(h_V(v)) = q_H(h_V(v'))$ if and only if $q_G(v) = q_G(v')$*

That is, a regular monomorphism in **EqGrph** is a morphism that reflects equivalences besides preserving them.

Let us turn to another functor **EqGrph** $\to$ **Graph**.

Definition 2.1.12. The *quotient functor* Q : **EqGrph** $\to$ **Graph** is defined as the one sending $(E_G, V_G, C_G, s_G, t_G, q_G)$ to $(E_G, C_G, q_G \circ s_G, q_G \circ t_G)$ and an arrow $(h_E, h_V, h_C)\colon (E_G, V_G, C_G, s_G, t_G, q_G) \to (E_H, V_H, C_H, s_H, t_H, q_H)$ to (h_E, h_C).

Remark 2.1.13. The action of the functor on a morphism of graphs with equivalences gives a morphism of graphs, in fact $q_H \circ s_H \circ h_E = q_H \circ h_V \circ s_G = h_C \circ q_G \circ s_G$. The same is valid for t_H and t_G.

Lemma 2.1.14. *Q is a left adjoint.*

Proof. Let $R((A, B, s, t))$ be (A, B, B, s, t, id_B), so that $Q(R((A, B, s, t))) = (A, B, s, t)$. Now, suppose that $h = (h_E, h_V) : Q((E, V, C, s', t', q)) \to (A, B, s, t)$ is an arrow in **Graph**, and consider the triple $(h_E, h_V, h_V \circ q)$. Since h is a morphism of **Graph**,

$$h_V \circ q \circ s' = s \circ h_E \qquad h_V \circ q \circ t' = t \circ h_E$$

Then we have the following squares:

$$
\begin{tikzcd}
E \arrow[r, "h_E"] \arrow[d, "s_G"'] & A \arrow[d, "s"] \\
V \arrow[r, "h_V \circ q"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "h_E"] \arrow[d, "t_G"'] & A \arrow[d, "t"] \\
V \arrow[r, "h_V \circ q"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "h_V \circ q"] \arrow[d, "q"'] & B \arrow[d, "id_B"] \\
C \arrow[r, "h_V"'] & B
\end{tikzcd}
$$

We have therefore found a morphism $(E, V, C, s', t', q) \to R((A, B, s, t))$ whose image through Q fits in the diagram below.

$$
\begin{tikzcd}
(A, B, s, t) \arrow[r, "{id_A, id_B}"] & (A, B, s, t) \\
(E, C, q \circ s', q \circ t') \arrow[u, "{Q((h_E, h_V \circ q, h_V))}"] \arrow[ur, "{(h_E, h_V)}"'] &
\end{tikzcd}
$$

Such arrow is unique. Suppose $f = (f_E, f_V, f_C)$ to be another arrow wit such property. Then, it must be $(id_A, id_B) \circ Q(f) = (f_E, f_C) = (h_E, h_C)$. Finally, $f_C = f_V \circ q = h_V \circ q$. $\square$

Proposition 2.1.15. *Q creates colimits.*

Proof. Preserve from Theorem 1.3.32. Remain to see Reflect. Let $D : \mathscr{I} \to$ **EqGrph** be a diagram, and let $((c_i)_{i\in\mathscr{I}}, \mathbb{C})$ be the colimit of $Q \circ D$, where $\mathbb{C} = (A, C, q \circ s, q \circ t)$, and $D(i)$ is $(A_i, B_i, C_i, s_i, t_i, q_i)$. Let now $T :$ **EqGrph** $\to$ **Set** be the functor mapping each graph with equivalence onto its second component, $T((X, Y, Z, x, y, z)) = Y$, and each morphims onto its second component. Let then $((b_i)_{i\in\mathscr{I}}, B)$ be the colimit of $T \circ D$. Consider the following situation.

$$
\begin{tikzcd}
& B & \\
B_i \arrow[ur, "b_i"] \arrow[rr, "h_V"] \arrow[d, "q_i"] & & B_j \arrow[ul, "b_j"'] \arrow[d, "q_j"'] \\
C_i \arrow[rr, "h_C"'] \arrow[dr, "c^i_C"'] & & C_j \arrow[dl, "c^j_C"] \\
& C &
\end{tikzcd}
$$

Now, since $((c^i_C \circ q_i)_{i\in\mathscr{I}}, C)$ is a cocone for $T \circ D$, there exists a unique $q : B \to C$, which is epi by Lemma 1.3.45. Consider now the functor $W :$ **EqGrph** $\to$ **Set** mapping each (X, Y, Z, x, y, z) onto X, and each morphism on its first component.

By Proposition 2.1.7 and Lemma 1.3.27, we have that $((c^i_E)_{i\in\mathscr{I}}, A)$ is the colimit of $W \circ D$. Notice that $((b_i \circ s_i)_{i\in\mathscr{I}}, B)$ and $((b_i \circ t_i)_{i\in\mathscr{I}}, B)$ are cocones for $W \circ D$, so let s and t be, respectively, the mediating arrow for the first one and the mediating arrow for the second one. It remains now to show that (A, B, C, s, t, q), together with $(c^i_E, b_i, c^i_C)_{i\in\mathscr{I}}$, is a colimit for D, but this follows by the proof of Proposition 2.1.7. $\square$

Example 2.1.16. Q does not preserve limits. Indeed, let $\mathbb{G}_1 = (E_1, A, A, s_1, t_1, id_A)$, $\mathbb{G}_2 = (E_2, B, B, s_2, t_2, id_B)$ and $\mathbb{G}_3 = (E_3, A+B, \mathbb{1}, s_3, t_3, !_{A+B})$, and let $h = (h_E, \iota_A, !_A) : \mathbb{G}_1 \to \mathbb{G}_3$, $k = (k_E, \iota_B, !_B) : \mathbb{G}_2 \to \mathbb{G}_3$, where $(\iota_A, \iota_B, A + B)$ is the coproduct of A and B, $\mathbb{1}$ is the intial object (in **Set**, the singleton set as shown in Example 1.3.9), and $!_X$ the unique arrow $X \to \mathbb{1}$. The following two diagrams show the pullback of h and k and the pullback of $Q(h)$ and $Q(k)$, on the second component (the vertices of the graphs)

$$\begin{tikzcd}
\mathbb{0} \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "\iota_A"] \\
B \arrow[r, "\iota_B"'] & A + B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times B \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \arrow[d, "!_A"] \\
B \arrow[r, "!_B"'] & \mathbb{1}
\end{tikzcd}$$

But the arrow $\mathbb{0} \to A \times B$ is not epi in general (this is easy to see taking **Set** as example), hence such pullback is not preserved by Q.

2.1.1 Adhesivity of EqGrph

Lemma 2.1.17. *In* **EqGrph**, *pushouts along regular monos are stable.*

Proof. Let $\mathcal{G}_i = (A_i, B_i, C_i, s_i, t_i, q_i)$, $\mathcal{G}' = (A'_i, B'_i, C'_i, s'_i, t'_i, q'_i)$, for $i = 1, 2, 3, 4$, and, in the diagram above, suppose all the vertical faces are pullbacks, the bottom face is a pushout and h is regular mono.

$$\begin{tikzcd}
& \mathcal{G}'_1 \arrow[rr, "h'"] \arrow[dl, "k'"'] \arrow[dd, "b" near end] & & \mathcal{G}'_2 \arrow[dl, "t'"] \arrow[dd, "c"] \\
\mathcal{G}'_3 \arrow[rr, "p'" near end] \arrow[dd, "a"'] & & \mathcal{G}'_4 \arrow[dd, "d" near start] & \\
& \mathcal{G}_1 \arrow[rr, "h" near end] \arrow[dl, "k"'] & & \mathcal{G}_2 \arrow[dl, "t"] \\
\mathcal{G}_3 \arrow[rr, "p"'] & & \mathcal{G}_4 &
\end{tikzcd}$$

By Proposition 2.1.7 and Corollary 2.1.8, the following two cubes have $\mathcal{M}$-pushouts as bottom faces and pullbacks as vertical faces, thus their top faces are $\mathcal{M}$-pushouts.

$$\begin{tikzcd}
& A'_1 \arrow[rr, "h'_V"] \arrow[dl, "k'_V"'] \arrow[dd, "b_V" near end] & & A'_2 \arrow[dl, "t'_V"] \arrow[dd, "c_V"] \\
A'_3 \arrow[rr, "p'_V" near end] \arrow[dd, "a_V"'] & & A'_4 \arrow[dd, "d_V" near start] & \\
& A_1 \arrow[rr, "h_V" near end] \arrow[dl, "k_V"'] & & A_2 \arrow[dl, "t_V"] \\
A_3 \arrow[rr, "p_V"'] & & A_4 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& B'_1 \arrow[rr, "h'_E"] \arrow[dl, "k'_E"'] \arrow[dd, "b_E" near end] & & B'_2 \arrow[dl, "t'_E"] \arrow[dd, "c_E"] \\
B'_3 \arrow[rr, "p'_E" near end] \arrow[dd, "a_E"'] & & B'_4 \arrow[dd, "d_E" near start] & \\
& B_1 \arrow[rr, "h_E" near end] \arrow[dl, "k_E"'] & & B_2 \arrow[dl, "t_E"] \\
B_3 \arrow[rr, "p_E"'] & & B_4 &
\end{tikzcd}$$

Now, using Corollary 1.3.21, we can consider a third cube, which, by Proposition 2.1.15, has a $\mathcal{M}$-pushout as bottom face and pullbacks as vertical faces, so that its top face is a $\mathcal{M}$-pushout too.

$$\begin{tikzcd}
& T \arrow[rr, "x_1"] \arrow[dl, "w"'] \arrow[dd, "x_2" near end] & & U \arrow[dl, "u_1"] \arrow[dd, "u_2"] \\
Y \arrow[rr, "y_1" near end] \arrow[dd, "y_2"'] & & C'_4 \arrow[dd, "d_C" near start] & \\
& C_1 \arrow[rr, "h_C" near end] \arrow[dl, "k_C"'] & & C_2 \arrow[dl, "t_C"] \\
C_3 \arrow[rr, "p_C"'] & & C_4 &
\end{tikzcd}$$

Moreover, by the proof of Proposition 2.1.7, we know that there are monos $m_2 : C'_2 \to U$ and $m_3 : C'_3 \to Y$ fitting in the diagrams

$$\begin{tikzcd}
B'_3 \arrow[r, "p'_V"] \arrow[d, "q'_3"] \arrow[dd, bend right, "c_V"'] & B'_4 \arrow[dr, bend left, "q'_4"] & \\
C'_3 \arrow[r, "m_3"] & Y \arrow[r, "y_1"] \arrow[d, "y_2"'] & C'_4 \arrow[d, "d_C"] \\
B_3 \arrow[r, "q_3"'] & C_3 \arrow[r, "p_C"'] & C_4
\end{tikzcd}
\qquad
\begin{tikzcd}
B'_2 \arrow[r, "t'_V"] \arrow[d, "q'_2"] \arrow[dd, bend right, "b_V"'] & B'_4 \arrow[dr, bend left, "q'_4"] & \\
C'_2 \arrow[r, "m_2"] & U \arrow[r, "u_1"] \arrow[d, "u_2"'] & C'_4 \arrow[d, "d_C"] \\
B_2 \arrow[r, "q_2"'] & C_2 \arrow[r, "t_C"'] & C_4
\end{tikzcd}$$

For C'_1, we can make a similar argument, let S be the pullback of m_2 along x_1, using Lemma 1.3.20 and, again, the proof of Proposition 2.1.7 we know that q'_1 arise as the factorization of the arrow $B'_1 \to S$ induced by $q'_2 \circ h'_2$ and a_2 so that we have a diagram.

$$
\begin{tikzcd}
B'_1 \arrow[r, "h'_V"] \arrow[d, "q'_1"] \arrow[dd, bend right, "a_V"'] & B'_2 \arrow[dr, bend left, "q'_2"] & \\
C'_1 \arrow[r, "m_1"] & S \arrow[r, "s_1"] \arrow[d, "s_2"'] & C'_2 \arrow[d, "m_2"] \\
B_1 \arrow[dr, bend right, "q_1"'] & T \arrow[r, "x_1"] \arrow[d, "x_2"'] & U \arrow[d, "u_2"] \\
& C_1 \arrow[r, "h_C"'] & C_2
\end{tikzcd}
$$

Moreover, we have that

$$
\begin{aligned}
s_1 \circ m_1 \circ q_1 &= q'_2 \circ h'_V & \qquad q'_4 \circ t'_V &= t'_C \circ q'_2 \\
&= h'_C \circ q'_1 & &= u_1 \circ m_2 \circ q'_2
\end{aligned}
$$

$$
\begin{aligned}
y_1 \circ m_3 \circ q'_3 &= p'_V \circ q'_4 \\
&= p'_C \circ q'_3
\end{aligned}
$$

Hence, $s_1 \circ m_1 = h'_C$, $w \circ s_2 \circ m_1 = m_3 \circ k'_C$, $t'_C = u_1 \circ m_2$ and $p'_C = y_1 \circ m_3$. Let now $z : \mathcal{G}'_2 \to \mathcal{H} = (E, V, Q, s, t, q)$, $w : \mathcal{G}^1_3 \to \mathcal{H}$ be two morphisms such that $z \circ h' = w \circ k'$, and let $\phi : B'_4 \to V$ be the arrow induced by z_V and w_V. We want to construct the arrow $v : C'_4 \to Q$ in the diagram below.

$$
\begin{tikzcd}
& B'_1 \arrow[rr, "h'_V"] \arrow[dl, "k'_V"'] \arrow[dd, "s_2 \circ m_1 \circ q'_1" near end] & & B'_2 \arrow[dl, "t'_V"] \arrow[dd, "m_2 \circ q'_2" near end] \arrow[dr, "z_V"] & \\
B'_3 \arrow[rr, "p'_V"] \arrow[dd, "m_3 \circ q'_3"'] & & B'_4 \arrow[rr, "\phi"] \arrow[dd, "q'_4"' near start] & & V \arrow[dd, "q"] \\
& T \arrow[rr, "x_1"] \arrow[dl, "w"'] \arrow[dd, "x_2"' near start] & & U \arrow[dl, "u_1"] \arrow[dd, "u_2" near start] & \\
Y \arrow[rr, "y_1"] \arrow[dd, "y_2"'] & & C'_4 \arrow[rr, dashed, "v"] \arrow[dd, "d_C"' near start] & & Q \\
& C_1 \arrow[rr, "h_C"] \arrow[dl, "k_C"] & & C_2 \arrow[dl, "t_C"] & \\
C_3 \arrow[rr, "p_C"'] & & C_4 & &
\end{tikzcd}
$$

By Proposition 1.3.43, d_C is the coequalizer of its kernel pair. On the other hand, by Lemma 1.4.8, we know that the top face of the cube below is a pushout.

$$
\begin{tikzcd}
& K_{s_2\circ m_1\circ q_1'} \arrow[rr, "k_{h_V'}"] \arrow[dl, "k_{k_V'}"'] \arrow[dd, "\pi^1_{s_2\circ m_1\circ q_1'}" near end] & & K_{m_2\circ q_2'} \arrow[dl, "k_{t_V'}"] \arrow[dd, "\pi^1_{m_2\circ q_2'}"] \\
K_{m_3\circ q_3'} \arrow[rr, "k_{p_V'}" near end] \arrow[dd, "\pi^1_{m_3\circ q_3'}"'] & & K_{q_4'} \arrow[dd, "\pi^1_{q_4'}" near start] & \\
& B_1' \arrow[rr, "h_V'" near end] \arrow[dl, "k_V'"'] & & B_2' \arrow[dl, "t_V'"] \\
B_3' \arrow[rr, "p_V'"'] & & B_4' &
\end{tikzcd}
$$

And, since m_3 and m_2 are monos,

$$q_3' \circ \pi^1_{m_3\circ q_3'} = q_3' \circ \pi^2_{m_3\circ q_3'} \qquad q_2' \circ \pi^1_{m_2\circ q_2'} = q_2' \circ \pi^2_{m_2\circ q_2'}$$

Computing, we obtain

$$
\begin{aligned}
q\circ\phi\circ\pi^1_{q_4'}\circ k_{p_V'} &= q\circ\phi\circ p_V'\circ\pi^1_{m_3\circ q_3'} \\
&= q\circ w_V\circ\pi^1_{m_3\circ q_3'} \\
&= w_C\circ q_3'\circ\pi^1_{m_3\circ q_3'} \\
&= w_C\circ q_3'\circ\pi^2_{m_3\circ q_3} \\
&= q\circ w_V\circ\pi_{m_3\circ q_3'} \\
&= q\circ\phi\circ p_V'\circ\pi^2_{m_3\circ q_3'} \\
&= q\circ\phi\circ\pi^2_{q_4'}\circ k_{p_V'}
\end{aligned}
\qquad
\begin{aligned}
q\circ\phi\circ\pi^1_{q_4'}\circ k_{t_V'} &= q\circ\phi\circ t_V'\circ\pi^1_{m_2\circ q_2'} \\
&= q\circ z_V\circ\pi^1_{m_2\circ q_2'} \\
&= z_C\circ q_2'\circ\pi^1_{m_2\circ q_2'} \\
&= z_C\circ q_2'\circ\pi^2_{m_2\circ q_2} \\
&= q\circ z_V\circ\pi_{m_2\circ q_2'} \\
&= q\circ\phi\circ t_V'\circ\pi^2_{m_2\circ q_2'} \\
&= q\circ\phi\circ\pi^2_{q_4'}\circ k_{t_V'}
\end{aligned}
$$

Since the previous cube has a pushout as top face, by universal property, we have

$$q\circ\phi\circ\pi^1_{q_4'} = q\circ\phi\circ\pi^2_{q_4'}$$

hence, v is the mediating arrow.

$$v\circ q_4'\circ\pi^1_{q_4'} = v\circ q_4'\circ\pi^2_{q_4'}$$

□

Lemma 2.1.18. *In* **EqGrph**, *pushouts along regular monos are* $\mathcal{R}eg$(**EqGrph**)*-Van Kampen.*

Proof. In lieu of Lemma 2.1.17, it is enough to proof that, given a cube as the one below, with pullbacks as back faces, pushouts as bottom and top faces and such that h is a regular mono, the front faces are pullbacks too, where $\mathcal{G}_i = (A_i, B_i, C_i, s_i, t_i, q_i)$,

$\mathcal{G}' = (A'_i, B'_i, C'_i, s'_i, t'_i, q'_i)$, for $i = 1, 2, 3, 4$.

$$\begin{tikzcd}
& \mathcal{G}'_1 \arrow[rr, "h'"] \arrow[dl, "k'"'] \arrow[dd, "b" near end] & & \mathcal{G}'_2 \arrow[dl, "t'"] \arrow[dd, "c"] \\
\mathcal{G}'_3 \arrow[rr, "p'" near end] \arrow[dd, "a"'] & & \mathcal{G}'_4 \arrow[dd, "d" near start] & \\
& \mathcal{G}_1 \arrow[rr, "h" near end] \arrow[dl, "k"'] & & \mathcal{G}_2 \arrow[dl, "t"] \\
\mathcal{G}_3 \arrow[rr, "p"'] & & \mathcal{G}_4 &
\end{tikzcd}$$

By Proposition 2.1.7 and Corollary 2.1.8, the following two cubes have $\mathcal{M}$-pushouts as bottom faces and pullbacks as back faces, thus their front faces are pullbacks too.

$$\begin{tikzcd}
& A'_1 \arrow[rr, "h'_V"] \arrow[dl, "k'_V"'] \arrow[dd, "b_V" near end] & & A'_2 \arrow[dl, "t'_V"] \arrow[dd, "c_V"] \\
A'_3 \arrow[rr, "p'_V" near end] \arrow[dd, "a_V"'] & & A'_4 \arrow[dd, "d_V" near start] & \\
& A_1 \arrow[rr, "h_V" near end] \arrow[dl, "k_V"'] & & A_2 \arrow[dl, "t_V"] \\
A_3 \arrow[rr, "p_V"'] & & A_4 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& B'_1 \arrow[rr, "h'_E"] \arrow[dl, "k'_E"'] \arrow[dd, "b_E" near end] & & B'_2 \arrow[dl, "t'_E"] \arrow[dd, "c_E"] \\
B'_3 \arrow[rr, "p'_E" near end] \arrow[dd, "a_E"'] & & B'_4 \arrow[dd, "d_E" near start] & \\
& B_1 \arrow[rr, "h_E" near end] \arrow[dl, "k_E"'] & & B_2 \arrow[dl, "t_E"] \\
B_3 \arrow[rr, "p_E"'] & & B_4 &
\end{tikzcd}$$

On the other hand we can consider the diagrams below, in which the inner squares are pullbacks. Since the outer diagrams commute, by definition of porphism of **EqGrph**, then we have the existence of $m_2 \colon C'_2 \to U$, $m_3 \colon C'_3 \to Y$, $a_3 \colon B'_3 \to Y$ and $a_2 \colon B'_2 \to Y$.

$$\begin{tikzcd}
C'_3 \arrow[dr, dashed, "m_3"] \arrow[drr, bend left, "p'_C"] \arrow[ddr, bend right, "c_C"'] & & \\
& Y \arrow[r, "y_1"] \arrow[d, "y_2"'] & C'_4 \arrow[d, "d_C"] \\
& C_3 \arrow[r, "p_C"'] & C_4
\end{tikzcd}
\qquad
\begin{tikzcd}
C'_2 \arrow[dr, dashed, "m_2"] \arrow[drr, bend left, "t'_C"] \arrow[ddr, bend right, "b_C"'] & & \\
& U \arrow[r, "u_1"] \arrow[d, "u_2"'] & C'_4 \arrow[d, "d_C"] \\
& C_2 \arrow[r, "t_C"'] & C_4
\end{tikzcd}$$

$$\begin{tikzcd}
B'_3 \arrow[r, "p'_V"] \arrow[d, "q'_3"'] \arrow[drr, dashed, "f"] & B'_4 \arrow[drr, "q'_4"] & & \\
C'_3 \arrow[drr, "c_C"'] & & Y \arrow[r, "y_1"] \arrow[d, "y_2"'] & C'_4 \arrow[d, "d_C"] \\
& & C_3 \arrow[r, "p_C"'] & C_4
\end{tikzcd}
\qquad
\begin{tikzcd}
B'_2 \arrow[r, "t'_V"] \arrow[d, "q'_2"'] \arrow[drr, dashed, "g"] & B'_4 \arrow[drr, "q'_4"] & & \\
C'_2 \arrow[drr, "b_C"'] & & U \arrow[r, "u_1"] \arrow[d, "u_2"'] & C'_4 \arrow[d, "d_C"] \\
& & C_2 \arrow[r, "t_C"'] & C_4
\end{tikzcd}$$

Now, notice that m_3 and m_2 are monos because c_C and b_C are monos (Corollary 2.1.9). By the proof of Proposition 2.1.7, to conclude it is enough to show that

$$m_3 \circ q_3' = f \qquad m_2 \circ q_2' = g$$

Indeed, if the previous equations hold, then C_3' and C_2' are epi-mono factorizations of f and g, and the thesis follows from Corollary 1.1.18 and the proof of Proposition 2.1.7.

Now, if we compute, we obtain:

$$\begin{aligned} y_1 \circ f &= q_4' \circ p_V' & \qquad u_1 \circ g &= q_4' \circ t_V' \\ &= p_C' \circ q_3' & &= t_C' \circ q_3' \\ &= y_1 \circ m_3 \circ q_3' & &= u_1 \circ m_2 \circ q_2' \end{aligned}$$

$$\begin{aligned} y_2 \circ f &= d_C \circ q_3' & \qquad u_2 \circ g &= d_V \circ q_2' \\ &= y_2 \circ m_3 \circ q_3' & &= u_2 \circ m_2 \circ q_2' \end{aligned}$$

Concluding the proof. □

From Proposition 2.1.7 and Lemmas 2.1.17 and 2.1.18, by Theorem 1.4.5, we can deduce at once the following.

Corollary 2.1.19. **EqGrph** *is* $\mathcal{R}eg(\mathbf{EqGrph})$*-adhesive.*

2.2 E-Graphs

E-Graphs are a particular type of graphs with equivalences.

Definition 2.2.1 (E-Graph)**.** Given a graph with equivalence $\mathbb{G} = (E, V, C, s, t, q)$, let (S, p_1, p_2) be the kernel pair of $q \circ s$. Then, $\mathbb{G}$ is an *e-graph* if it holds that

$$q \circ t \circ p_1 = q \circ t \circ p_2$$

Remark 2.2.2. From a set-theoretic point of view, $\mathbb{G} = (\mathcal{G}, \sim)$ is a graph with equivalence, where $\mathcal{G} = (E, V, s, t)$ in which holds that, for each pair of edges e, e',

$$\frac{s(e) \sim s(e')}{t(e) \sim t(e')}$$

As we have considered in the section on graphs with equivalences, we can formalize the equivalence relation with a surjective function $q : V \to V/_\sim = C$, rewriting the inference rule above as an inclusion. Let $S = \{(e, e') \in E \times E \mid q(s(e)) = q(s(e'))\}$ and $T = \{(e, e') \in E \times E \mid q(t(e)) = q(t(e'))\}$. Then, $\mathbb{G}$ is an e-graph if $S \subseteq T$. As we noted in Example 1.3.37, S is the kernel pair of $q \circ s$ and T is the kernel pair of $q \circ t$, if endowed with canonical projections. Moreover, we notice that the pairing of such projections yields a subobject of the product $E \times E$ Proposition 1.3.38. Hence, by Remark 1.1.9, the Definition 2.2.1 express this fact using categorial constructions.

Definition 2.2.3 (Category of E-Graphs)**. EGG** is the full subcategory of **EqGrph** in which objects are e-graphs.

Since **EGG** is a full subcategory of **EqGrph**, there exists a fully faithful functor $I : \mathbf{EGG} \to \mathbf{EqGrph}$ (Example 1.2.8).

Lemma 2.2.4. EGG *has all limits and* I *preserves them.*

Proof. Let $D : \mathscr{I} \to \mathbf{EGG}$ be a diagram, with $D(i) = (A_i, B_i, C_i, s_i, t_i, q_i)$, let (U, u_1^i, u_2^i) be the kernel pair of $q_i \circ s_i$. Let now be (A, B, C, s, t, q), togehter with projections $(\pi_E^i, \pi_V^i, \pi_C^i)_{i \in \mathscr{I}}$ the limit of $I \circ D$, let (U, u_1, u_2) be the kernel pair of $q \circ s$ and let $(L, (l_i)_{i \in \mathscr{I}})$ be the limit of $V \circ I \circ D$. By construction (proof of Proposition 2.1.7), there exists a monomorphism $m : Q \to L$ such that $\pi_C^i = l_i \circ m$. Notice that

$$
\begin{aligned}
q_i \circ s_i \circ \pi_E^i \circ u_1 &= q_i \circ \pi_V^i \circ s \circ u_1 \\
&= \pi_C^i \circ q \circ s \circ u_1 \\
&= \pi_C^i \circ q \circ s \circ u_2 \\
&= q_i \circ s_i \circ \pi_E^i \circ u_2
\end{aligned}
$$

Then, for each i, there exists an arrow $a_i : U \to U_i$ making the following diagram to commute

$$
\begin{tikzcd}
U \arrow[r, "u_1"] \arrow[d, "u_2"'] \arrow[dr, dashed, "a_i"] & A \arrow[drr, "\pi_E^i"] & & \\
A \arrow[ddr, "\pi_E^i"'] & U_i \arrow[rr, "u_1^i"] \arrow[dd, "u_2^i"] & & A_i \arrow[d, "s_i"] \\
& & & B_i \arrow[d, "q_i"] \\
& A_i \arrow[r, "s_i"'] & B_i \arrow[r, "q_i"'] & C_i
\end{tikzcd}
$$

We have then

$$
\begin{aligned}
l_i \circ m \circ q \circ t \circ u_1 &= q_i \circ \pi_V^i \circ t \circ u_1 \\
&= q_i \circ t_i \circ \pi_E^i \circ u_1 \\
&= q_i \circ t_i \circ u_1^i \circ a_i \\
&= q_i \circ t_i \circ u_2^i \circ a_i \\
&= q_i \circ t_i \circ \pi_E^i \circ u_2 \\
&= q_i \circ \pi_V^i \circ t \circ u_2 \\
&= l_i \circ m \circ q \circ t \circ u_2
\end{aligned}
$$

By universal property of limits, we have that $m \circ q \circ t \circ u_1 = m \circ q \circ t \circ u_2$, and, since m is mono, $q \circ t \circ u_1 = q \circ t \circ u_2$, hence the thesis. □

By the previous results, we can deduce what follows from Proposition 1.3.31.

Corollary 2.2.5. *I creates limits.*

Corollary 2.2.6. *$h : \mathbb{G} \to \mathbb{H}$ is a regular mono in* **EGG** *if and only if it is a regular mono in* **EqGrph**.

Lemma 2.2.7. *Let $\mathbb{G}_1 = (A_1, B_1, C_1, s_1, t_1, q_1)$, $\mathbb{G}_2 = (A_2, B_2, C_2, s_2, t_2, q_2)$ and $\mathbb{G}_3 = (A_3, B_3, C_3, s_3, t_3, q_3)$ be e-graphs, $h : \mathbb{G}_1 \to \mathbb{G}_2$ and $m : \mathbb{G}_1 \to \mathbb{G}_3$ be two morphisms with m mono. Then, the pushout of h and m is again an e-graph.*

Proof. Let $\mathbb{P} = (A, B, C, s, t, q)$, together with $k : \mathbb{G}_3 \to \mathbb{P}$ and $n : \mathbb{G}_2 \to \mathbb{P}$, the pushout of h and m. Let (K_i, π_i^1, π_i^2) the kernel pair of $q_i \circ s_i$, for $i = 1, 2, 3$, and let (U, u_1, u_2) be the kernel pair of $q \circ s$. Consider the following cube in **Set**

$$\begin{tikzcd}
& A_1 \arrow[rr, "h_E"] \arrow[dl, "m_E"'] \arrow[dd, "q_1 \circ s_1" near end] & & A_2 \arrow[dl, "n_E"] \arrow[dd, "q_2 \circ s_2"] \\
A_3 \arrow[rr, "k_E" near start] \arrow[dd, "q_3 \circ s_3"'] & & A \arrow[dd, "q \circ s" near start] & \\
& C_1 \arrow[rr, "h_C" near end] \arrow[dl, "m_C"'] & & C_2 \arrow[dl, "n_C"] \\
C_3 \arrow[rr, "k_C"'] & & C &
\end{tikzcd}$$

Since m is a regular mono then m_C is mono by Corollary 2.2.6 and Corollary 2.1.9. Then, by adhesivity of **Set**, the bottom face of the cube above is a Van Kampen square and thus a pullback. Therefore by Lemma 1.4.8, the square below is a pushout.

$$\begin{tikzcd}
K_1 \arrow[r, "f_h"] \arrow[d, "f_m"'] & K_2 \arrow[d, "f_n"] \\
K_3 \arrow[r, "f_k"'] & U
\end{tikzcd}$$

Hence, we end up with the following situation

$$\begin{tikzcd}
& K_1 \arrow[rr, "f_h"] \arrow[dl, "f_m"'] \arrow[dd, "\pi_1^1"'] & & K_2 \arrow[dl, "f_n"] \arrow[dd, "\pi_2^1"] \\
K_3 \arrow[rr, "f_k"] \arrow[dd, "\pi_3^1"'] & & U \arrow[dd, "u_1"'] & \\
& A_1 \arrow[rr, "h_E"] \arrow[dl, "m_E"'] \arrow[dd, "q_1 \circ s_1"'] & & A_2 \arrow[dl, "n_E"] \arrow[dd, "q_2 \circ s_2"] \\
A_3 \arrow[rr, "k_E"] \arrow[dd, "q_3 \circ s_3"'] & & A \arrow[dd, "q \circ s"'] & \\
& C_1 \arrow[rr, "h_C"] \arrow[dl, "m_C"'] & & C_2 \arrow[dl, "n_C"] \\
C_3 \arrow[rr, "k_C"'] & & C &
\end{tikzcd}
\qquad
\begin{tikzcd}
& K_1 \arrow[rr, "f_h"] \arrow[dl, "f_m"'] \arrow[dd, "\pi_1^2"'] & & K_2 \arrow[dl, "f_n"] \arrow[dd, "\pi_2^2"] \\
K_3 \arrow[rr, "f_k"] \arrow[dd, "\pi_3^2"'] & & U \arrow[dd, "u_2"'] & \\
& A_1 \arrow[rr, "h_E"] \arrow[dl, "m_E"'] \arrow[dd, "q_1 \circ s_1"'] & & A_2 \arrow[dl, "n_E"] \arrow[dd, "q_2 \circ s_2"] \\
A_3 \arrow[rr, "k_E"] \arrow[dd, "q_3 \circ s_3"'] & & A \arrow[dd, "q \circ s"'] & \\
& C_1 \arrow[rr, "h_C"] \arrow[dl, "m_C"'] & & C_2 \arrow[dl, "n_C"] \\
C_3 \arrow[rr, "k_C"'] & & C &
\end{tikzcd}$$

Computing, we have

$$\begin{aligned}
q \circ t \circ u_1 \circ f_n &= q \circ t \circ n_E \circ \pi_2^1 & \qquad q \circ t \circ u_1 \circ f_k &= q \circ t \circ k_E \circ \pi_3^1 \\
&= n_C \circ q_2 \circ s_2 \circ \pi_2^1 & &= k_C \circ q_3 \circ s_3 \circ \pi_3^1 \\
&= n_C \circ q_2 \circ s_2 \circ \pi_2^2 & &= k_C \circ q_3 \circ s_3 \circ \pi_3^2 \\
&= q \circ t \circ u_2 \circ f_n & &= q \circ t \circ u_2 \circ f_k
\end{aligned}$$

By universal property of pushouts, we deduce $q \circ t \circ u_1 = q \circ t \circ u_2$, and the thesis follows. $\square$

Corollary 2.2.8. *I creates pushouts along regular monos.*

By direct application of Theorem 1.4.5, we can conclude what follows.

Corollary 2.2.9. **EGG** *is* $\mathcal{R}eg(\mathbf{EGG}) -$ *adhesive.*

Conclusions

In this work, we have defined graphs with equivalences and e-graphs in an abstract manner, making it independent of specific implementations such as in [WNW+21]. This allows to analyse the categorical properties of these structures. Building upon this framework, we have proved, in particular, the adhesivity of graphs with equivalences and e-graphs with respect to the class of regular monomorphisms. This result is potentially useful for advancing state-of-the-art techniques, such as the DPO approach for concurrent updates to the structure, which has not yet been explored.

Both graphs with equivalences and e-graphs can be realized as functors with values in **Set**. One might ask whether it is strictly necessary to use **Set** as the codomain of these functors. Our framework is developed from a more general perspective, using minimal set-theoretic concepts and instead relying on categorical constructions. It only requires $\mathcal{M}$-adhesivity and exactness [Bar71]. These properties certainly holdy in **Set** and more generally in every elementary topos [MM12]. Roughly speaking, this is the property that enabled the equivalence noted in Remark 2.1.3. Therefore, a potential direction for future work will be to provide a more general presentation of these structures, replacing **Set** with any adhesive and exact category.

Bibliography

[ACR19] Guilherme Grochau Azzi, Andrea Corradini, and Leila Ribeiro. On the essence and initiality of conflicts in $\mathcal{M}$-adhesive transformation systems. *Journal of Logical and Algebraic Methods in Programming*, 109:100482, 2019.

[AHS09] Jiri Adámek, Horst Herrlich, and George E. Strecker. *Abstract and concrete categories: The joy of cats*. Dover Publications, 2009.

[Bar71] Michael Barr. Exact categories. In Michael Barr, Pierre Grillet, and Donovan van Osdol, editors, *Exact Categories and Categories of Sheaves*, volume 236 of *Lecture Notes in Mathematics*, pages 1–120. Springer Science & Business Media, 1971.

[BCE+99] Paolo Baldan, Andrea Corradini, Hartmut Ehrig, Michael Löwe, Ugo Montanari, and Francesca Rossi. Concurrent semantics of algebraic graph transformation. *Handbook of Graph Grammars and Computing by Graph Transformation*, 3:107–187, 1999.

[BHK22] Nicolas Behr, Russ Harmer, and Jean Krivine. Fundamentals of compositional rewriting theory. *arXiv preprint arXiv:2204.07175*, 2022.

[BW95] Michael Barr and Charles Wells. *Category theory for computing science*. Prentice Hall, 2 edition, 1995.

[CGM22] Davide Castelnovo, Fabio Gadducci, and Marino Miculan. A new criterion for $\mathcal{M}, \mathcal{N}$-adhesivity, with an application to hierarchical graphs, 2022.

[Che21] Alessandro Cheli. Automated code optimization with e-graphs, 2021.

[EEPT06] Harmut Ehrig, Karsten Ehrig, Ulrike Prange, and Gabriele Taentzer. *Fundamentals of Algebraic Graph Transformation*. Monographs in Theoretical Computer Science. Springer Science & Business Media, 2006.

[EG06] Hartmut Ehrig and Ulrike Golas. Weak adhesive high-level replacement categories and systems: A unifying framework for graph and petri net transformations, 01 2006.

[ERMK99] Hartmut Ehrig, Grzegorz Rozenberg, Ugo Montanari, and Hans-Jorg Kreowski. *Handbook of graph grammars and computing by graph transformation*, volume 3. world Scientific, 1999.

[HS79] Horst Herrlich and George E. Strecker. *Category Theory*, volume 1 of *Sigma Series in Pure Mathematics*. Heldermann Verlag Berlin, 2 edition, 1979.

[Lac11] Stephen Lack. An embedding theorem for adhesive categories, 2011.

[LS05] Sthephen Lack and Pawel Sobociński. Adhesive and quasiadhesive categories. *RAIRO-Theoretical Informatics and Applications*, 39(3):511–545, 2005.

[MM12] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012.

[Nel80] Charles Gregory Nelson. *Techniques for program verification*. PhD thesis, Stanford, CA, USA, 1980.

[TSTL11] Ross Tate, Michael Stepp, Zachary Tatlock, and Sorin Lerner. Equality saturation: A new approach to optimization. *Logical Methods in Computer Science*, Volume 7, Issue 1, March 2011.

[WNW+21] Max Willsey, Chandrakana Nandi, Yisu Remy Wang, Oliver Flatt, Zachary Tatlock, and Pavel Panchekha. egg: Fast and extensible equality saturation. *Proc. ACM Program. Lang.*, 5(POPL), January 2021.