

# On the adhesivity of EGGs

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## Abstract

a very nice abstract

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## 1 Introduction

A very nice introduction

- general on adhesive categories
- general on the use of eggs

In the original definition, e-graphs are defined as term graphs with an additional notion of equivalence on nodes. Adopting the more modern presentation via string diagrams, they are tree with sharing of subtrees, and with an equivalence  $\equiv$  on nodes that is closed under xxx. In plain words, if  $a$  and  $b$  are two constants such that  $a \equiv b$ , then  $f(a) \equiv f(b)$  for any unary operator  $f$ .

[9, Section 4.2] [21]

*Synopsis* The paper has the following structure. In Section 2 we briefly recall the theory of  $\mathcal{M}$ -adhesive categories and of kernel pairs. In Section 3 we present the graphical structures of our interest, (labelled, acyclic) hypergraphs and term graphs, and we provide a functorial characterisation, which allows for proving their adhesivity properties. This is expended in Section 4 for describing hypergraphs and term graphs with equivalence and in Section 5 for capturing their variants where the equivalences are closed with respect to operator application, thus subsuming EGGs. All the proofs have been moved to the appendices.

## 2 Facts about $\mathcal{M}$ -adhesive categories and kernel pairs

This section briefly recalls  $\mathcal{M}$ -adhesive categories [1, 11, 12, 17, 15]. Given a category  $\mathbf{X}$  we do not distinguish notationally between  $\mathbf{X}$  and its class of objects, so “ $X \in \mathbf{X}$ ” means that  $X$  is an object of  $\mathbf{X}$ . We let  $\text{Mor}(\mathbf{X})$ ,  $\text{Mono}(\mathbf{X})$  and  $\text{Reg}(\mathbf{X})$  denote the class of all arrows, monos and regular monos of  $\mathbf{X}$ , respectively. Given an object  $X$ , we denote by  $?_X$  the unique arrow from an initial object into  $X$  and by  $!_X$  that unique arrow from  $X$  into a terminal one.

### 2.1 $\mathcal{M}$ -adhesivity

The key property of  $\mathcal{M}$ -adhesive categories is the *Van Kampen condition* [4, 16, 17], and for defining it we need some notions. Let  $\mathbf{X}$  be a category. A subclass  $\mathcal{A}$  of  $\text{Mor}(\mathbf{X})$  is said to be

- *stable under pushouts (pullbacks)* if for every pushout (pullback) square as the one aside, if  $m \in \mathcal{A}$  ( $n \in \mathcal{A}$ ) then  $n \in \mathcal{A}$  ( $m \in \mathcal{A}$ );
- *closed under composition* if  $h, k \in \mathcal{A}$  implies  $h \circ k \in \mathcal{A}$  whenever  $h$  and  $k$  are composable.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ m \downarrow & & \downarrow n \\ C & \xrightarrow{g} & D \end{array}$$

► **Definition 2.1.** Let  $\mathcal{A} \subseteq \text{Mor}(\mathbf{X})$  be a class of arrows in a category  $\mathbf{X}$  and consider the cube below on the right.



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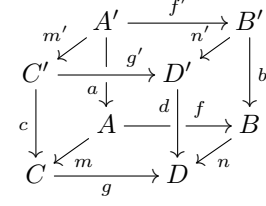
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## XX:2 On the adhesivity of EGGS

We say that the bottom square is an  $\mathcal{A}$ -Van Kampen square if

1. it is a pushout square;
2. whenever the cube above has pullbacks as back and left faces and the vertical arrows belong to  $\mathcal{A}$ , then its top face is a pushout if and only if the front and right faces are pullbacks.



Pushout squares that enjoy only the “if” half of item (2) above are called  $\mathcal{A}$ -stable. A  $\text{Mor}(\mathbf{X})$ -Van Kampen square is called Van Kampen and a  $\text{Mor}(\mathbf{X})$ -stable square stable.

We can now define  $\mathcal{M}$ -adhesive categories.

► **Definition 2.2.** Let  $\mathbf{X}$  be a category and  $\mathcal{M}$  a subclass of  $\text{Mono}(\mathbf{X})$  including all isomorphisms, closed under composition, and stable under pullbacks and pushouts. The category  $\mathbf{X}$  is said to be  $\mathcal{M}$ -adhesive if

1. it has  $\mathcal{M}$ -pullbacks, i.e. pullbacks along arrows of  $\mathcal{M}$ ;
2. it has  $\mathcal{M}$ -pushouts, i.e. pushouts along arrows of  $\mathcal{M}$ ;
3.  $\mathcal{M}$ -pushouts are  $\mathcal{M}$ -Van Kampen squares.

A category  $\mathbf{X}$  is said to be strictly  $\mathcal{M}$ -adhesive if  $\mathcal{M}$ -pushouts are Van Kampen. We write  $m: X \rightarrow Y$  to denote that an arrow  $m: X \rightarrow Y$  belongs to  $\mathcal{M}$ .

► **Remark 2.3.** Adhesivity and quasiadhesivity [17, 13] coincide with strict  $\text{Mono}(\mathbf{X})$ -adhesivity and strict  $\text{Reg}(\mathbf{X})$ -adhesivity, respectively.

$\mathcal{M}$ -adhesivity is well-behaved with respect to the construction of slice and functor categories [18], as shown by the following theorems [10, 17].

► **Proposition 2.4.** Let  $\mathbf{X}$  be an (strict)  $\mathcal{M}$ -adhesive category. Then it holds

1. if  $\mathbf{Y}$  is an (strict)  $\mathcal{N}$ -adhesive category  $L: \mathbf{Y} \rightarrow \mathbf{A}$  a functor preserving  $\mathcal{N}$ -pushouts and  $R: \mathbf{X} \rightarrow \mathbf{A}$  one preserving pullbacks, then  $L \downarrow R$  is (strictly)  $\mathcal{N} \downarrow \mathcal{M}$ -adhesive, where

$$\mathcal{N} \downarrow \mathcal{M} := \{(h, k) \in \text{Mor}(L \downarrow R) \mid h \in \mathcal{N}, k \in \mathcal{M}\}$$

2. for every object  $X$  the categories  $\mathbf{X}/X$  and  $X/\mathbf{X}$  are, respectively, (strictly)  $\mathcal{M}/X$ -adhesive and (strictly)  $X/\mathcal{M}$ -adhesive, where

$$\mathcal{M}/X := \{m \in \text{Mor}(\mathbf{X}/X) \mid m \in \mathcal{M}\} \quad X/\mathcal{M} := \{m \in \text{Mor}(X/\mathbf{X}) \mid m \in \mathcal{M}\}$$

3. for every small category  $\mathbf{Y}$ , the category  $\mathbf{X}^{\mathbf{Y}}$  of functors  $\mathbf{Y} \rightarrow \mathbf{X}$  is (strictly)  $\mathcal{M}^{\mathbf{Y}}$ -adhesive, where  $\mathcal{M}^{\mathbf{Y}} := \{\eta \in \text{Mor}(\mathbf{X}^{\mathbf{Y}}) \mid \eta_Y \in \mathcal{M} \text{ for every } Y \in \mathbf{Y}\}$ ;
4. if  $\mathbf{Y}$  is a full subcategory of  $\mathbf{X}$  closed in it under pullbacks and  $\mathcal{M}$ -pushouts, then  $\mathbf{Y}$  is (strictly)  $\mathcal{N}$ -adhesive for every class of arrows  $\mathcal{N}$  of  $\mathbf{Y}$  contained in  $\mathcal{M}$  that is stable under pullbacks and pushouts, contains all the isomorphisms, and is closed under composition and decomposition.

We will briefly list some examples of  $\mathcal{M}$ -adhesive categories.

► **Example 2.5.** **Set** is adhesive, and, more generally, every topos is adhesive [?]. By the closure properties above, every presheaf  $[\mathbf{X}, \mathbf{Set}]$  is adhesive, thus the category **Graph** =  $[E \rightrightarrows V, \mathbf{Set}]$  is adhesive where  $E \rightrightarrows V$  is the two objects category with two morphisms  $s, t: E \rightarrow V$ . Similarly, various categories of hypergraphs can be shown to be adhesive, such as term graphs and hierarchical graphs [7]. Note that the category **sGraphs** of simple graphs, i.e. graphs without parallel edges, is  $\text{Reg}(\mathbf{sGraphs})$ -adhesive [2] but not quasiadhesive.

73 We can state some useful properties of  $\mathcal{M}$ -adhesive category.

74 ► **Proposition 2.6.** *If  $\mathbf{X}$  is  $\mathcal{M}$ -adhesive then it holds*

- 75 1. *every  $\mathcal{M}$ -pushout square is also a pullback;*  
 76 2. *every arrow in  $\mathcal{M}$  is a regular mono.*

## 77 2.2 Kernel Pairs and Regular Epimorphisms

78 In this section we recall the definition and some properties of *kernel pairs*.

79 ► **Definition 2.7.** *A kernel pair for an arrow  $f: A \rightarrow B$  is an object  $K_f$  together with two arrows  $\pi_f^1, \pi_f^2: K_f \rightrightarrows A$ , denoted as  $(K_f, \pi_f^1, \pi_f^2)$ , such that the square aside is a pullback.*

$$\begin{array}{ccc} K_f & \xrightarrow{\quad} & A \\ \pi_f^1 \downarrow & & \downarrow \pi_f^2 \\ A & \xrightarrow{f} & B \end{array}$$

80 ► **Remark 2.8.** *If  $(K_f, \pi_f^1, \pi_f^2)$  is a kernel pair for  $f: X \rightarrow Y$  and a product of  $X$  with itself exists, then the canonical arrow  $\langle \pi_f^1, \pi_f^2 \rangle: K_f \rightarrow X \times X$  is a mono.*

82 ► **Remark 2.9.** *An arrow  $m: M \rightarrow X$  is a mono if and only if it admits  $(M, \text{id}_M, \text{id}_M)$  as a kernel pair.*

84 Together with Lemma A.1, the previous remarks allow us to prove the following result.

85 ► **Proposition 2.10.** *Let  $f: X \rightarrow Y$  be an arrow and  $m: Y \rightarrow Z$  a mono. If  $(K_f, \pi_f^1, \pi_f^2)$  is a kernel pair for  $f: X \rightarrow Y$ , then it is also a kernel pair for  $m \circ f$ .*

87 We explore the link between regular epis and kernel pairs.

88 ► **Proposition 2.11.** *Let  $e: X \rightarrow Y$  be a regular epi in a category  $\mathbf{X}$  with a kernel pair  $(K_e, \pi_e^1, \pi_e^2)$ . Then,  $e$  is the coequalizer of  $\pi_e^1$  and  $\pi_e^2$ .*

90 ► **Corollary 2.12.** *Let  $\mathbf{X}$  be a category with pullbacks and  $\phi: F \rightarrow G$  a natural transformation between functors  $F, G: \mathbf{D} \rightarrow \mathbf{X}$ . If  $\phi_d$  is a regular epi for every  $d$  in  $\mathbf{D}$ , then  $\phi$  is a regular epi.*

93 From the previous result we deduce that the class of regular epis is closed under colimits.

94 ► **Lemma 2.13.** *Let  $F, G: \mathbf{D} \rightarrow \mathbf{X}$  be two diagrams, and suppose that  $\mathbf{X}$  has all colimits of shape  $\mathbf{D}$ . Let  $(X, \{x_d\}_{d \in \mathbf{D}})$  and  $(Y, \{y_d\}_{d \in \mathbf{D}})$  be the colimits of  $F$  and  $G$ , respectively. If  $\phi: F \rightarrow G$  is a natural transformation whose components are regular epis, then the arrow induced by  $\phi$  from  $X$  to  $Y$  is a regular epi.*

98 We now explore some properties of kernel pairs and their link with adhesive categories.  
 99 The results below appear to be original, and we give their proofs in Appendix A.

100 ► **Lemma 2.14.** *Let  $f: X \rightarrow Y$  and  $g: Z \rightarrow W$  be two arrows admitting kernel pairs and suppose that the solid part of the three squares below is given. If the leftmost square is commutative, then there exists a unique arrow  $k_h: K_f \rightarrow K_g$  making the other two commutative. Moreover, if the leftmost is a pullback, then also the other two are so.*

$$\begin{array}{ccccc} X & \xrightarrow{h} & Z & & K_f & \xrightarrow{k_h} & K_g & & K_f & \xrightarrow{k_h} & K_g \\ f \downarrow & & \downarrow g & & \pi_f^1 \downarrow & & \downarrow \pi_g^1 & & \pi_f^2 \downarrow & & \downarrow \pi_g^2 \\ Y & \xrightarrow{t} & W & & X & \xrightarrow{h} & Z & & X & \xrightarrow{h} & Z \end{array}$$

## XX:4 On the adhesivity of EGGS

The previous result allows us to deduce the following lemma in an  $\mathcal{M}$ -adhesive context.

► **Proposition 2.15.** *Let  $\mathbf{X}$  be a strict  $\mathcal{M}$ -adhesive category with all pullbacks, and suppose that in the cube aside the top face is an  $\mathcal{M}$ -pushout. Then the right square is a pushout.*

$$\begin{array}{ccccc}
 & & A' & \xrightarrow{f'} & B' \\
 & m' \swarrow & \downarrow g' & \nwarrow n' & \\
 C' & \xrightarrow{a} & D' & & \\
 \downarrow c & & \downarrow d & \downarrow f & \\
 C & \xrightarrow{m} & A & \xrightarrow{f} & B \\
 & \searrow g & \downarrow n & & \\
 & & D & & 
 \end{array}
 \quad
 \begin{array}{ccc}
 K_a & \xrightarrow{k_{f'}} & K_b \\
 \downarrow k_{m'} & & \downarrow k_{n'} \\
 K_c & \xrightarrow{k_{g'}} & K_d
 \end{array}$$

## 3 Hypergraphical structures

In this section we briefly recall the notion  $\mathbf{X}$ -hypergraph. It is necessary to have a monad  $(-)^*: \mathbf{Set} \rightarrow \mathbf{Set}$ , also known as *list monad*, sending a set to the free monoid on it [19, 20] and playing a role analogous to the usual *Kleene star*. We recall some of its properties.

► **Proposition 3.1.** *The following facts hold*

1. *for every set  $X$  and  $n \in \mathbb{N}$  there are arrows  $v_n: X^n \rightarrow X^*$  such that  $(X^*, \{v_n\}_{n \in \mathbb{N}})$  is a coproduct;*
2. *for every arrow  $f: X \rightarrow Y$ ,  $f^*$  is the coproduct of the family  $\{f^n\}_{n \in \mathbb{N}}$ ;*
3.  *$(-)^*$  preserves all connected limits [5], in particular it preserves pullbacks and equalizers.*

► **Remark 3.2.** Preservation of pullbacks implies that  $(-)^*$  sends monos to monos.

► **Remark 3.3.** Notice that  $1^*$  can be canonically identified with  $\mathbb{N}$ , thus for every set  $X$  the arrow  $!_X: X \rightarrow 1$  induces a *length function*  $!_X^*: X^* \rightarrow \mathbb{N}$ , which sends a word to its length.

### 3.1 The category of hypergraphs

We open this section with the definition of hypergraphs and we show how to label them with an algebraic signature.

► **Definition 3.4.** *An hypergraph is a 4-uple  $\mathcal{G} := (E_{\mathcal{G}}, V_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}})$  made by two sets  $E_{\mathcal{G}}$  and  $V_{\mathcal{G}}$ , called respectively the set of hyperedges and nodes, plus a pair of source and target arrows  $s_{\mathcal{G}}, t_{\mathcal{G}}: E_{\mathcal{G}} \rightrightarrows V_{\mathcal{G}}^*$ . A hypergraph morphism  $(E_{\mathcal{G}}, V_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}}) \rightarrow (E_{\mathcal{H}}, V_{\mathcal{H}}, s_{\mathcal{H}}, t_{\mathcal{H}})$  is a pair  $(h, k)$  of functions  $h: E_{\mathcal{G}} \rightarrow E_{\mathcal{H}}$ ,  $k: V_{\mathcal{G}} \rightarrow V_{\mathcal{H}}$  such that the following diagrams commute*

$$\begin{array}{ccc}
 E_{\mathcal{G}} & \xrightarrow{s_{\mathcal{G}}} & V_{\mathcal{G}}^* \\
 h \downarrow & & \downarrow k^* \\
 E_{\mathcal{H}} & \xrightarrow{s_{\mathcal{H}}} & V_{\mathcal{H}}^*
 \end{array}
 \quad
 \begin{array}{ccc}
 E_{\mathcal{G}} & \xrightarrow{t_{\mathcal{G}}} & V_{\mathcal{G}}^* \\
 h \downarrow & & \downarrow k^* \\
 E_{\mathcal{H}} & \xrightarrow{t_{\mathcal{H}}} & V_{\mathcal{H}}^*
 \end{array}$$

We define **Hyp** to be the resulting category.

Let  $\text{prod}^*$  be the functor sending  $X$  to  $X^* \times X^*$ : we can get **Hyp** as a comma category.

► **Proposition 3.5.** ***Hyp** is isomorphic to  $\text{id}_{\mathbf{Set}} \downarrow \text{prod}^*$*

► **Corollary 3.6.** ***Hyp** is an adhesive category.*

**Proof.** By hypothesis  $(-)^*$  preserves pullbacks, while  $\text{prod}$  is continuous by definition, thus the thesis follows from Proposition 2.4 and Proposition 3.5. ◀

Another useful corollary of Proposition 3.5 is the following one.

134 ► **Corollary 3.7.** *A morphism  $(h, k)$  is a mono in **Hyp** if and only if both its components*  
 135 *are injective functions.*

136 Propositions 3.5 and B.6 allow us to deduce immediately the following.

137 ► **Proposition 3.8.** *The forgetful functor  $U_{\mathbf{Hyp}}$  which sends an hypergraph  $\mathcal{G}$  to its object of*  
 138 *nodes has a left adjoint  $\Delta_{\mathbf{Hyp}}$ .*

139 ► **Example 3.9.** Since the initial object of **Set** is the empty set,  $\Delta_{\mathbf{Set}}(X)$  is the hypergraph  
 140 which has  $X$  as set of nodes,  $\emptyset$  as set of hyperedges, and  $?_X$  as source and target function.  
 142

143 We can represent hypergraphs graphically. We will use dots to denote nodes and squares to  
 144 denote hyperedges, the name of a node or of an hyperedge will be put near the corresponding  
 145 dot or square. Sources and targets are represented by lines between dots and squares: the  
 146 lines from the sources of an hyperedge will have an arrowhead in the middle pointing towards  
 147 the hyperedge, while the lines to the targets will have arrowheads pointing to the target  
 148 nodes. We will decorate the arrow corresponding to the  $i^{th}$  letter of a target or a source with  
 149 a label  $i$ .

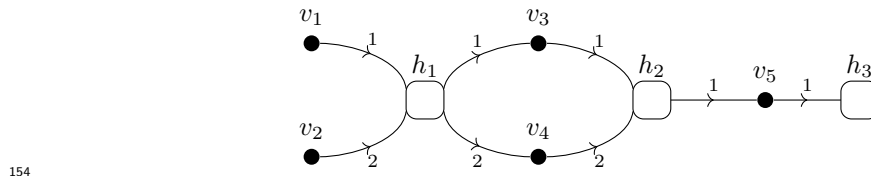
forse possiamo  
anche cancel-  
lare tutti gli  
esempi sotto

FG ne las-  
cerei uno, ri-  
preso magari  
dalla intro e  
presentato  
come string  
diagram

150 ► **Example 3.10.** Take  $V_{\mathcal{G}}$  to be  $\{v_1, v_2, v_3, v_4, v_5\}$  and  $E_{\mathcal{G}}$  to be  $\{h_1, h_2, h_3\}$ . Sources  
 151 and targets are given by:

$$\begin{array}{llll}
 s_{\mathcal{G}}(h_1): 2 \rightarrow V_{\mathcal{G}} & 0 \mapsto v_1 & s_{\mathcal{G}}(h_2): 2 \rightarrow V_{\mathcal{G}} & 0 \mapsto v_3 \\
 & 1 \mapsto v_2 & & 1 \mapsto v_4 \\
 t_{\mathcal{G}}(h_1): 2 \rightarrow V_{\mathcal{G}} & 0 \mapsto v_3 & t_{\mathcal{G}}(h_2): 2 \rightarrow V_{\mathcal{G}} & 0 \mapsto v_5 \\
 & 1 \mapsto v_4 & & t_{\mathcal{G}}(h_3): 0 \rightarrow V_{\mathcal{G}} \quad t_{\mathcal{G}}(h_3) = ?_{V_{\mathcal{G}}}
 \end{array}$$

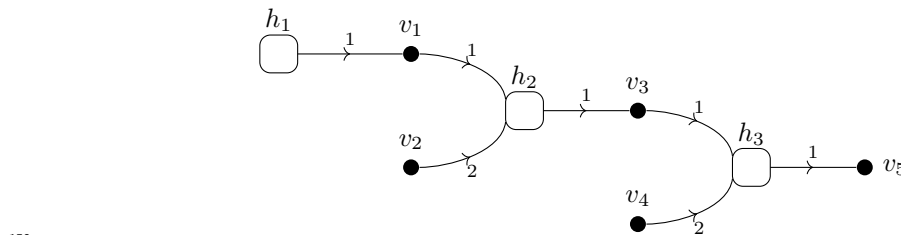
153 We can draw the resulting  $\mathcal{G}$  as follows:



155 ► **Example 3.11.** Let  $V_{\mathcal{G}}$  be as in the previous example and  $E_{\mathcal{G}} = \{h_1, h_2, h_3\}$ . Then we  
 156 define

$$\begin{array}{llll}
 s_{\mathcal{G}}(h_1): 0 \rightarrow V_{\mathcal{G}} & s_{\mathcal{G}}(h_1) = ?_{V_{\mathcal{G}}} & s_{\mathcal{G}}(h_2): 2 \rightarrow V_{\mathcal{G}} & 0 \mapsto v_1 \\
 & & & 1 \mapsto v_2 \\
 t_{\mathcal{G}}(h_1): 1 \rightarrow V_{\mathcal{G}} & 0 \mapsto v_1 & t_{\mathcal{G}}(h_2): 1 \rightarrow V_{\mathcal{G}} & 0 \mapsto v_3 \\
 & & & 1 \mapsto v_4 \\
 & & t_{\mathcal{G}}(h_3): 1 \rightarrow V_{\mathcal{G}} & 1 \mapsto v_5
 \end{array}$$

158 Now we can depict  $\mathcal{G}$  as



159

### 3.1.1 Hyp as a category of functors

Following [3], we can present **Hyp** as a category of functor over a suitable category.

► **Definition 3.12.** Let **H** be the category such that

- the set of objects is  $(\mathbb{N} \times \mathbb{N}) \cup \{\bullet\}$ ;
- arrows are given by the identities  $\text{id}_{k,l}$  and  $\text{id}_\bullet$  and exactly  $k + l$  arrows  $f_i: (k, l) \rightarrow \bullet$ , where  $i$  ranges from 0 to  $k + l - 1$ ;
- composition is defined by putting  $f_i = f_i \circ \text{id}_{k,l}$  and  $f_i = \text{id}_\bullet \circ f_i$  for every  $f_i: (k, l) \rightarrow \bullet$ .

The idea is that for every functor  $F: \mathbf{H} \rightarrow \mathbf{Set}$  we can define

$$E_F := \sum_{k,l \in \mathbb{N}} F(k, l)$$

Now, for every  $k, l, i$  and  $j$  in  $\mathbb{N}$  with  $i < k$  and  $j < l$  we define  $s_{k,l}^F: F(k, l) \rightarrow F(\bullet)^k$  and  $t_{k,l}^F: F(k, l) \rightarrow F(\bullet)^l$  as the unique arrows fitting in the diagrams below, where the vertical arrows are the projections

$$\begin{array}{ccc} F(k, l) & \xrightarrow{s_{k,l}^F} & F(\bullet)^k \\ & \searrow F(f_i) & \downarrow \pi_{k,i}^F \\ & & F(\bullet) \end{array} \quad \begin{array}{ccc} F(k, l) & \xrightarrow{t_{k,l}^F} & F(\bullet)^l \\ & \searrow F(f_{k+j}) & \downarrow \pi_{l,j}^F \\ & & F(\bullet) \end{array}$$

In turn, these arrows allow us to consider  $s_F, t_F: E_F \rightrightarrows F(\bullet)^*$  as the unique arrows fitting in the diagrams below, where the vertical arrows are coprojections

$$\begin{array}{ccc} F(k, l) & \xrightarrow{s_{k,l}^F} & F(\bullet)^k \\ a_{k,l}^F \downarrow & & \downarrow b_k^F \\ E_F & \xrightarrow{s_F} & F(\bullet)^* \end{array} \quad \begin{array}{ccc} F(k, l) & \xrightarrow{t_{k,l}^F} & F(\bullet)^l \\ a_{k,l}^F \downarrow & & \downarrow b_l^F \\ E_F & \xrightarrow{t_F} & F(\bullet)^* \end{array}$$

Let  $\mathcal{G}_F$  be the resulting hypergraph. One can now show that sending  $F$  to  $\mathcal{G}_F$  can be extended to an equivalence  $\mathcal{G}_-: \mathbf{Set}^{\mathbf{H}} \rightarrow \mathbf{Hyp}$  (see [6, 7] for details).

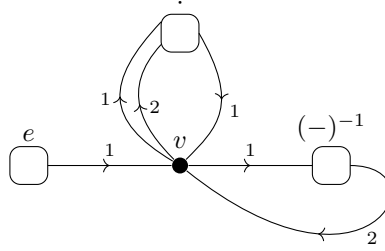
► **Proposition 3.13.** **Hyp** is equivalent to the category  $\mathbf{Set}^{\mathbf{H}}$ .

### 3.1.2 Labelling hypergraph with an algebraic signature

Our interest for hypergraphs stems from their use as a graphical representation of algebraic terms. We thus need a way to label hyperedges with symbols taken from a signature.

► **Definition 3.14.** An algebraic signature  $\Sigma$  is a pair  $(O_\Sigma, \text{ar}_\Sigma)$  given by a set of operations  $O_\Sigma$  and an arity function  $\text{ar}_\Sigma: O_\Sigma \rightarrow \mathbb{N}$ . We define the hypergraph  $\mathcal{G}_\Sigma$  associated with  $\Sigma$  taking  $O_\Sigma$  as set of hyperedges,  $1^*$  as set of nodes, so that  $1^*$  is  $\mathbb{N}$ ,  $\text{ar}_\sigma$  as the source function and  $\delta_1$  as target function, where  $\delta_1$  picks the element 1. The category  $\mathbf{Hyp}_\Sigma$  of algebraically labelled hypergraphs is the slice category  $\mathbf{Hyp}/\mathcal{G}_\Sigma$ .

► **Example 3.15.** Let  $\Sigma = (O_\Sigma, \text{ar}_\Sigma)$  be an algebraic signature in **Set**. This simply amount to a set of operations with an associated natural number, called *arity*. For instance let  $\Sigma_G$  be the signature of groups, then  $\mathcal{G}^{\Sigma_G}$  can be depicted as



190

191 Corollary B.5 and Proposition 2.4 give us immediately an adhesivity result for  $\mathbf{Hyp}_\Sigma$   
 192 and a characterisation of monos in it.

193 ► **Proposition 3.16.** *Let  $\Sigma$  be an algebraic signature. Then it holds*

- 194 1. *a morphism  $(h, k)$  between two object of  $\mathbf{Hyp}_\Sigma$  is a mono if and only if  $h$  and  $k$  are*  
 195 *injective functions;*
- 196 2.  **$\mathbf{Hyp}_\Sigma$  is an adhesive category.**

197 ► **Remark 3.17.** Let  $\mathcal{H} = (E, V, s, t)$  be an hypergraph, by definition we know that  $U_{\mathbf{Hyp}}(\mathcal{G}^\Sigma)$   
 198 is the terminal object 1, so an arrow  $\mathcal{H} \rightarrow \mathcal{G}^\Sigma$ , is determined by a morphism  $h: E_{\mathcal{H}} \rightarrow O_\Sigma$   
 199 making the two squares below commute (cfr. Remark 3.3).

$$\begin{array}{ccc}
 E_{\mathcal{H}} & \xrightarrow{h} & O_\Sigma \\
 s_{\mathcal{H}} \downarrow & & \downarrow ar_\Sigma \\
 V_{\mathcal{H}}^* & \xrightarrow{!g_{V_{\mathcal{H}}}} & \mathbb{N}
 \end{array}
 \quad
 \begin{array}{ccc}
 E_{\mathcal{H}} & \xrightarrow{h} & O_\Sigma \\
 t_{\mathcal{H}} \downarrow & & \downarrow \delta_1 \\
 V_{\mathcal{H}}^* & \xrightarrow{!g_{V_{\mathcal{H}}}} & \mathbb{N}
 \end{array}$$

201 Let  $v_n: V_{\mathcal{H}}^n \rightarrow V_{\mathcal{H}}^*$  be a coprojection. The second diagram above entails that  $t_{\mathcal{H}}$  factors via  
 202 the inclusion  $v_1: V_{\mathcal{H}} \rightarrow V_{\mathcal{H}}^*$  of words of length 1, i.e.  $t_{\mathcal{H}} = v_1 \circ \tau_{\mathcal{H}}$  for some  $\tau_{\mathcal{H}}: E_{\mathcal{H}} \rightarrow \tau_{\mathcal{H}}$ .

203  $\mathbf{Hyp}_\Sigma$ , has a forgetful functor  $U_\Sigma: \mathbf{Hyp}_\Sigma \rightarrow \mathbf{X}$  which sends  $(h, k): \mathcal{H} \rightarrow \mathcal{G}^\Sigma$  to  $U_{\mathbf{X}}(\mathcal{H})$ .  
 204 Now,  $U_{\mathbf{X}}(\mathcal{G}^\Sigma) = 1$  thus, for every object  $X$ , there is only one arrow  $X \rightarrow U_{\mathbf{X}}(\mathcal{G}^\Sigma)$ . Define  
 205  $\Delta_\Sigma(X): \Delta_{\mathbf{X}}(X) \rightarrow \mathcal{G}^\Sigma$  as the transpose of this arrow. Explicitly,  $\Delta_{\mathbf{X}}(X) = (0, X, ?_{X^*}, ?_{X^*})$   
 206 and  $\Delta_\Sigma(X)$  is simply  $(?_{O_\Sigma}, !_X)$ .

207 ► **Proposition 3.18.**  *$U_\Sigma$  has a left adjoint  $\Delta_\Sigma$ .*

208 **Proof.** Let  $(h, !_{V_{\mathcal{H}}}): \mathcal{H} \rightarrow \mathcal{G}^\Sigma$  be an object of  $\mathbf{Hyp}_\Sigma$ , and suppose that there exists  $f: X \rightarrow$   
 209  $U_\Sigma(\mathcal{H})$ . Since,  $U_\Sigma(\mathcal{H}) = U_{\mathbf{X}}(\mathcal{H})$  and the identity is the unit of  $\Delta_{\mathbf{Hyp}} \dashv U_{\mathbf{Hyp}}$ , we get a  
 210 morphism  $(?_{E_{\mathcal{H}}}, f): \Delta_{\mathbf{X}}(X) \rightarrow \mathcal{H}$  of  $\mathbf{Hyp}$ . And then the thesis follows since we have

$$(h, !_{V_{\mathcal{H}}}) \circ (?_{E_{\mathcal{H}}}, f) = (h \circ ?_{E_{\mathcal{H}}}, !_{V_{\mathcal{H}}} \circ f) = (?_{O_\Sigma}, !_X) = \Delta_{\mathbf{Hyp}}(X)$$

212

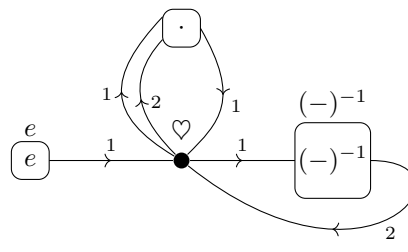
213

215 We will extend our graphical notation of hypergraphs to labeled ones putting the label of  
 216 an hyperedge  $h$  inside its corresponding square.

217 ► **Example 3.19.** The simplest example is given by the identity  $\text{id}_{\mathcal{G}^\Sigma}: \mathcal{G}^\Sigma \rightarrow \mathcal{G}^\Sigma$ . If  $\Sigma$  is the  
 218 signature of groups  $\Sigma_G$  we get

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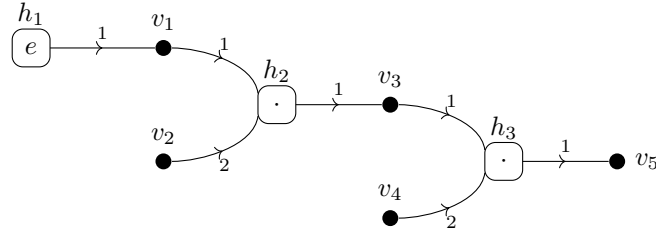


219

220 ► **Example 3.20.** Take again  $\Sigma_G$  the signature of groups, then the hypergraph  $\mathcal{G}$  of Ex-  
221 ample 3.11 can be labeled defining

$$222 \quad e = f(h_1) \quad \cdot = f(h_2) \quad \cdot = f(h_3)$$

223  
224 In this case we get the following picture



225

## 3.2 Term Graphs

a very nice  
introduction

Let us start using labelled hypergraphs to define term graphs.

229 ► **Definition 3.21.** Given an algebraic signature  $\Sigma$ , we say that a labelled hypergraph  
230  $(l, !_{V_{\mathcal{G}}}) : \mathcal{G} \rightarrow \mathcal{G}^{\Sigma}$  is a term graph if  $t_{\mathcal{G}}$  is mono. We define  $\mathbf{TG}_{\Sigma}$  to be the full subcat-  
231 egory of  $\mathbf{Hyp}_{\Sigma}$  and denote by  $I_{\Sigma}$  the inclusion. Restricting  $U_{\Sigma} : \mathbf{Hyp}_{\Sigma} \rightarrow \mathbf{Set}$  we get a  
232 forgetful functor  $U_{\mathbf{TG}_{\Sigma}} : \mathbf{TG}_{\Sigma} \rightarrow \mathbf{Set}$ .

233 ► **Remark 3.22.** By Remark 3.17, we know that if  $\mathcal{G}$  is a term graph then  $t_{\mathcal{G}} = v_1 \circ \tau_{\mathcal{G}}$ , where  
234  $v_1$  is the coprojection of  $V_{\mathcal{G}}$  into  $V_{\mathcal{G}}^*$ . Notice that since  $t_{\mathcal{G}}$  is mono then  $\tau_{\mathcal{G}}$  is mono too.

235 We now examine some properties of  $\mathbf{TG}_{\Sigma}$ , in order to study its adhesivity properties.

236 ► **Proposition 3.23.** The forgetful functor  $U_{\mathbf{TG}_{\Sigma}}$  has a left adjoint  $\Delta_{\mathbf{TG}_{\Sigma}}$ .

237 **Proof.** This follows noticing that  $\Delta_{\Sigma}(X)$  is a term graph for every object  $X$ . ◀

238 We can list some categorical properties of  $\mathbf{TG}_{\Sigma}$

239 ► **Proposition 3.24.** Let  $\Sigma$  be an algebraic signature. Then it holds

- 240 1. if  $(i, j) : \mathcal{H} \rightarrow \mathcal{G}$  is a mono between  $(l, !_{V_{\mathcal{G}}}) : \mathcal{G} \rightarrow \mathcal{G}^{\Sigma}$  and  $(l', !_{V_{\mathcal{H}}}) : \mathcal{H} \rightarrow \mathcal{G}^{\Sigma}$  in  $\mathbf{Hyp}_{\Sigma}$  and  
241 the latter is a term graph, then also the former is in  $\mathbf{TG}_{\Sigma}$
- 242 2.  $\mathbf{TG}_{\Sigma}$  has equalizers, binary products and pullbacks and they are created by  $I_{\Sigma}$ .

243 ► **Remark 3.25.**  $\mathbf{TG}_{\Sigma}$  in general does not have terminal objects. Since  $U_{\mathbf{TG}_{\Sigma}}$  preserves limits,  
244 if a terminal object exists it must have the singleton as set of nodes, therefore the set of  
245 hyperedges must be empty or a singleton. Hence, for a counterexample, it suffices to take as  
246 signature the one given by two operations  $a$  and  $b$ , both of arity 0.  $\mathbf{TG}_{\Sigma}$  is not an adhesive  
247 category, either. In particular, as noted in e.g. [7], it does not have pushouts along all monos.



► **Definition 3.26.** Let  $(l, !_{V_G}): \mathcal{G} \rightarrow \mathcal{G}^\Sigma$  be a term graph. A input node is an element of  $V_G$  not in the image of  $\tau_{\mathcal{H}}$ . A morphism  $(f, g)$  between  $(l, !_{V_G}): \mathcal{G} \rightarrow \mathcal{G}^\Sigma$  and  $(l', !_{V_{\mathcal{H}}}): \mathcal{H} \rightarrow \mathcal{G}^\Sigma$  in  $\mathbf{TG}_\Sigma$ , is said to preserve input nodes if  $g$  sends input nodes to input nodes.

► **Remark 3.27.** Suppose that  $(f, g): ((l, !_{V_G})) \rightarrow (l', !_{V_{\mathcal{H}}})$  preserves input nodes. Then if  $\tau_{\mathcal{H}}(h) = g(v)$  for some  $v \in V_G$  then  $h$  belongs to the image of  $f$ . Indeed, by hypothesis  $v$  must be in the image of  $\tau_{\mathcal{G}}$  and so there exists  $k$  such that  $\tau_{\mathcal{G}}(k) = v$ . But then  $\tau_{\mathcal{H}}(f(k)) = g(v)$  and we can conclude that  $f(k) = h$ .

Preservation of inputs characterizes regular monos in  $\mathbf{TG}_\Sigma$ .

► **Proposition 3.28.** Let  $(i, j)$  be a mon between two term graphs  $(l, !_{V_G}): \mathcal{G} \rightarrow \mathcal{G}^\Sigma$  and  $(l', !_{V_{\mathcal{H}}}): \mathcal{H} \rightarrow \mathcal{G}^\Sigma$ . Then it is a regular mono if and only if it preserves the input nodes.

This characterization, in turn, provides us with the following result [7, 6].

► **Lemma 3.29.** Consider three term graphs  $(l_0, !_{V_G}): \mathcal{G} \rightarrow \mathcal{G}^\Sigma$ ,  $(l_1, !_{V_{\mathcal{H}}}): \mathcal{H} \rightarrow \mathcal{G}^\Sigma$  and  $(l_2, !_{V_{\mathcal{K}}}): \mathcal{K} \rightarrow \mathcal{G}^\Sigma$ . Given  $(f_1, g_1): (l_0, !_{V_G}) \rightarrow (l_1, !_{V_{\mathcal{H}}})$ ,  $(f_2, g_2): (l_0, !_{V_G}) \rightarrow (l_2, !_{V_{\mathcal{K}}})$ , if  $(f_1, g_1)$  is a regular mono, then its pushout along  $(f_2, g_2)$ , then their pushout  $(p, !_{V_{\mathcal{P}}}): \mathcal{P} \rightarrow \mathcal{G}^\Sigma$  in  $\mathbf{Hyp}_\Sigma$  is a term graph too.

Proposition 2.4, Proposition 3.28 and Lemma 3.29 allow us to recover the following result, previously proved by direct computation in [8, Thm. 4.2].

► **Corollary 3.30.** The category  $\mathbf{TG}_\Sigma$  is quasiadhesive.

## 4 Hypergraphs and term graphs with equivalences

► **Definition 4.1.** A hypergraph with equivalence  $\mathcal{G} = (E_{\mathcal{G}}, V_{\mathcal{G}}, C_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}}, q_{\mathcal{G}})$  is a 6-tuple such that  $\mathcal{G} = (E_{\mathcal{G}}, V_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}})$  is a hypergraph,  $C_{\mathcal{G}}$  is a set and  $q_{\mathcal{G}}: V_{\mathcal{G}} \rightarrow C_{\mathcal{G}}$  is a regular epi called quotient map. A morphism  $h: \mathcal{G} \rightarrow \mathcal{H}$  is a triple  $(h_E, h_V, h_C)$  such that the following diagrams commute

$$\begin{array}{ccccc} E_{\mathcal{G}} & \xrightarrow{s_{\mathcal{G}}} & V_{\mathcal{G}}^* & & E_{\mathcal{G}} & \xrightarrow{t_{\mathcal{G}}} & V_{\mathcal{G}}^* & & V_{\mathcal{G}} & \xrightarrow{q_{\mathcal{G}}} & C_{\mathcal{G}} \\ h_E \downarrow & & \downarrow h_V^* & & h_E \downarrow & & \downarrow h_V^* & & h_V \downarrow & & \downarrow h_C \\ E_{\mathcal{H}} & \xrightarrow{s_{\mathcal{H}}} & V_{\mathcal{H}}^* & & E_{\mathcal{H}} & \xrightarrow{t_{\mathcal{H}}} & V_{\mathcal{H}}^* & & V_{\mathcal{H}} & \xrightarrow{q_{\mathcal{H}}} & C_{\mathcal{H}} \end{array}$$

The category of hypergraphs with equivalences and their morphisms is denoted  $\mathbf{EqHyp}$ .

► **Remark 4.2.** Morphisms of hypergraphs with equivalences are uniquely determined by the first two components. That is, if  $h_1 = (h_E, h_V, f)$  and  $h_2 = (h_E, h_V, g)$  are two morphisms  $\mathcal{G} \rightarrow \mathcal{H}$ , then we have

$$\begin{array}{ccccc} V_{\mathcal{G}} & \xrightarrow{h_V} & V_{\mathcal{H}} & \xleftarrow{h_V} & V_{\mathcal{G}} \\ q_{\mathcal{G}} \downarrow & & \downarrow q_{\mathcal{H}} & & \downarrow q_{\mathcal{G}} \\ C_{\mathcal{G}} & \xrightarrow{f} & C_{\mathcal{H}} & \xleftarrow{g} & C_{\mathcal{G}} \end{array}$$

Hence  $f \circ q_{\mathcal{G}} = q_{\mathcal{H}} \circ h_V = g \circ q_{\mathcal{G}}$ , and since  $q_{\mathcal{G}}$  is epi, we obtain  $f = g$ .

$\mathbf{EqHyp}$  has a forgetful functor  $U_{\mathbf{EqHyp}}: \mathbf{EqHyp} \rightarrow \mathbf{Set}$ , which sends each  $\mathcal{G} = (E_{\mathcal{G}}, V_{\mathcal{G}}, C_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}}, q_{\mathcal{G}})$  into  $V_{\mathcal{G}}$ , and each  $h = (h_E, h_V, h_C)$  onto  $h_V$ .

Se questo remark sotto non serve nel pezzo sulle equivalenze possiamo toglierlo

## XX:10 On the adhesivity of EGGS

281 ► **Proposition 4.3.**  $U_{\mathbf{EqHyp}}$  has a left adjoint  $\Delta_{\mathbf{EqHyp}} : \mathbf{Set} \rightarrow \mathbf{EqHyp}$ .

282 **Proof.** For each set  $X$ , define  $\Delta_{\mathbf{EqHyp}}(X) := (\emptyset, X, \{\bullet\}, ?_X, ?_X, !_X)$ . Consider now  $h :$   
 283  $\Delta_{\mathbf{EqHyp}}(X) \rightarrow \mathcal{H}$ .

$$\begin{array}{ccc}
 & \Delta_{\mathbf{EqHyp}}(X) & \\
 \Delta_{\mathbf{EqHyp}}(f) \downarrow & \searrow h & \\
 \Delta_{\mathbf{EqHyp}}(U_{\mathbf{EqHyp}}(\mathcal{H})) & \xrightarrow{\epsilon_{\mathcal{H}}} & \mathcal{H}
 \end{array}$$

285 Where  $\Delta_{\mathbf{EqHyp}}(U_{\mathbf{EqHyp}}(\mathcal{H})) = (\emptyset, V_{\mathcal{H}}, \{\bullet\}, ?_{V_{\mathcal{H}}}, ?_{V_{\mathcal{H}}}, !_V)$  and  $\epsilon_{\mathcal{H}} = (?_{E_{\mathcal{H}}}, \text{id}_{V_{\mathcal{H}}}, g)$ . Note  
 286 that, since  $\Delta_{\mathbf{EqHyp}}(X)$  has the empty set as object of edges,  $h_E = ?_{E_{\mathcal{H}}}$ , then, the unique  
 287 arrow that fits in the diagram is  $\Delta_{\mathbf{EqHyp}}(f) = (?_{E_{\mathcal{H}}}, h_V, \text{id}_{\{\bullet\}})$ . ◀

288 We now define another functor  $T : \mathbf{EqHyp} \rightarrow \mathbf{Hyp}$ , which “forgets” the quotient  
 289 part, mapping each hypergraph with equivalence  $\mathcal{G} = (E_{\mathcal{G}}, V_{\mathcal{G}}, C_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}}, q_{\mathcal{G}})$  onto  $T(\mathcal{G}) =$   
 290  $(E_{\mathcal{G}}, V_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}})$ . Then, we have the following result.

291 ► **Proposition 4.4.**  $T$  has a left adjoint  $L : \mathbf{Hyp} \rightarrow \mathbf{EqHyp}$ .

292 **Proof.** Let  $\mathcal{G}$  be a hypergraph, and define  $L(\mathcal{G}) := (E_{\mathcal{G}}, V_{\mathcal{G}}, \{\bullet\}, s_{\mathcal{G}}, t_{\mathcal{G}}, !_V)$ . Let now  $h :$   
 293  $L(\mathcal{G}) \rightarrow \mathcal{H}$  be a morphism in  $\mathbf{EqHyp}$ , and consider the following situation

$$\begin{array}{ccc}
 & L(\mathcal{G}) & \\
 L(f) \downarrow & \searrow h & \\
 L(T(\mathcal{H})) & \xrightarrow{\epsilon_{\mathcal{H}}} & \mathcal{H}
 \end{array}$$

295 for  $L(T(\mathcal{H})) = (E_{\mathcal{H}}, V_{\mathcal{H}}, \{\bullet\}, s_{\mathcal{H}}, t_{\mathcal{H}}, !_V)$ . Then,  $\epsilon_{\mathcal{H}} = (\text{id}_{E_{\mathcal{H}}}, \text{id}_{V_{\mathcal{H}}}, h_C)$  (by Remark 4.2, the  
 296 last component is uniquely determined by the first two), and  $L(f)$  must be  $(h_E, h_V, \text{id}_{\{\bullet\}})$ . ◀

297 ► **Remark 4.5.**  $T$  is faithful. Indeed, consider two morphisms  $h = (h_E, h_V, h_C)$  and  $k =$   
 298  $(k_E, k_V, k_C)$ , and suppose  $T(h) = T(k)$ , that is,  $(h_E, h_V) = (k_E, k_V)$ . By Remark 4.2, we  
 299 can conclude also  $h_C = k_C$ , and hence the faithfulness of  $T$ .

300 Let now  $K : \mathbf{EqHyp} \rightarrow \mathbf{Set}$  be the functor which sends each hypergraph with equivalence  
 301  $\mathcal{G} = (E, V, C, s, t, q)$  onto  $K(\mathcal{G}) = C$ , and each morphism  $(h_E, h_V, h_C)$  to  $h_C$ .

302 ► **Proposition 4.6.**  $\mathbf{EqHyp}$  is complete and cocomplete, and  $T$  preserves limits and colimits.

303 **Proof.** Let  $D : \mathbf{I} \rightarrow \mathbf{EqHyp}$  be a diagram, and, for each  $i \in \mathbf{I}$ ,  $D(i) = (E_i, V_i, C_i, s_i, t_i, q_i)$ .  
 304 Suppose now  $(E, V, s, t)$ , together with morphisms  $(\pi_i^E, \pi_i^V)$ , be the limit of  $T \circ D$ . Then,  
 305  $V$ , together with  $(q_i \circ \pi_i^V)_{i \in \mathbf{I}}$ , is a cone for  $K \circ D$ . Indeed, let  $\alpha : i \rightarrow j$  be an arrow of  $\mathbf{I}$ ,  
 306  $D(\alpha) = (h_E, h_V, h_C)$ . By definition of  $T$ ,  $(T \circ D)(\alpha) = (h_E, h_V)$ , hence we have

$$\begin{array}{ccccc}
 & & V & & \\
 & \pi_V^i \swarrow & & \searrow \pi_V^j & \\
 V_i & \xrightarrow{h_V} & V_j & & \\
 q_i \downarrow & & & & \downarrow q_j \\
 C_i & \xrightarrow{h_C} & C_j & & 
 \end{array}$$

308 Suppose now that  $L$ , with morphisms  $(l_i)_{i \in \mathbf{I}}$  be the limit of  $K \circ D$ . Hence, we have an arrow  
 309  $l : V \rightarrow L$ , which is not epi in general. Let then  $l = m \circ q$  be the epi-mono factorization of it.  
 310 Consider the following situation, where the outer rectangle commutes by definition, and the  
 311 dotted arrow is yielded by (cite left lifting prop).

$$\begin{array}{ccccc}
 V & \xrightarrow{\pi_V^i} & V_i & \xrightarrow{q_i} & C_i \\
 q \downarrow & & \searrow \pi_C^i & & \downarrow \text{id}_{C_i} \\
 C & \xrightarrow{m} & L & \xrightarrow{l_i} & C_i
 \end{array}$$

313 Thus,  $(E, V, C, s, t, q)$ , together with  $(\pi_E^i, \pi_V^i, \pi_C^i)$  is a cone over  $D$ . remain to show that this  
 314 cone is terminal

315 Suppose now  $(E', V', s', t')$ , together with  $(\kappa_E^i, \kappa_V^i)_{i \in \mathbf{I}}$ , be the colimit of  $T \circ D$ , and  $C'$ ,  
 316 with  $(c_i)_{i \in \mathbf{I}}$  be the colimit of  $K \circ D$ . Then, we have the following situation.

$$\begin{array}{ccccc}
 & & V' & & \\
 & \swarrow \kappa_i & & \nwarrow \kappa_j & \\
 V_i & \xrightarrow{h_V} & V_j & & \\
 q_i \downarrow & & & & \downarrow q_j \\
 C_i & \xrightarrow{h_C} & C_j & & \\
 c_i \searrow & & \swarrow c_j & & \\
 & & C' & & 
 \end{array}$$

318 Then,  $C'$  with morphisms  $(c_i \circ q_i)_{i \in \mathbf{I}}$  is a conone for  $U \circ D$ . Then, there exists a unique  
 319 morphism  $q' : V' \rightarrow C'$  such that  $q' \circ \kappa_V^i = c_i \circ q_i$ . Such morphism is epi (cite Lemma 1.3.45  
 320 of the thesis), and thus  $(E', V', C', s', t', q')$ , together with  $(\kappa_E^i, \kappa_V^i, c_i)_{i \in \mathbf{I}}$  is the colimit of  $D$ .  
 321  $\blacktriangleleft$

322 ► **Corollary 4.7.** Let  $h = (h_E, h_V, h_C) : \mathcal{G} \rightarrow \mathcal{H}$  be an arrow in **EqHyp**. Then it is a mono  
 323 if and only if  $T(h)$  is a mono.

324 **Proof.** The “if” part is given by the faithfulness of  $T$ . The “only if” part is given by  
 325 Remark 4.2.  $\blacktriangleleft$

326 ► **Corollary 4.8.** If  $h = (h_E, h_V, h_C) : \mathcal{G} \rightarrow \mathcal{H}$  is a regular mono in **EqHyp**, then  $h_E, h_V$   
 327 and  $h_C$  are all monos.

328 **Proof.** If  $h$  is mono, from Corollary 4.7 we have that  $h_E$  and  $h_V$  are monos. Suppose now  
 329  $f, g : \mathcal{H} \rightrightarrows \mathcal{K}$  be the arrows equalized by  $h$ . Then, we have:

$$\begin{aligned}
 330 \quad f_C \circ h_C \circ q_G &= f_C \circ q_{\mathcal{H}} \circ h_V \\
 331 \quad &= q_{\mathcal{K}} \circ g_V \circ h_V \\
 332 \quad &= q_{\mathcal{K}} \circ f_V \circ h_V \\
 333 \quad &= g_C \circ h_C \circ q_G \\
 334
 \end{aligned}$$

335 Since  $q_{\mathcal{G}}$  is epi, we have  $f_C \circ h_C = g_C \circ h_C$ , hence  $h_C$  is an equalizer for  $f_C$  and  $g_C$ , and thus  
 336 a mono.  $\blacktriangleleft$

## XX:12 On the adhesivity of EGGS

337 ► **Proposition 4.9.** *Let  $h = (h_E, h_V, h_C) : \mathcal{G} \rightarrow \mathcal{H}$  be a regular mono in **EqHyp**. Then,  $h_E$*   
 338 *and  $h_V$  are monos and  $(K, \pi_1, \pi_2)$  is the kernel pair of  $q_{\mathcal{H}} \circ h_V$  if and only if  $(K, \pi_1, \pi_2)$  is*  
 339 *the kernel pair of  $q_{\mathcal{G}}$ .*

340 **Proof.** By Corollary 4.8, we have that  $h_E, h_V$  and  $h_C$  are all monos. Hence, by Proposi-  
 341 tion 2.10,  $(K, \pi_1, \pi_2)$  is the kernel pair of  $q_{\mathcal{G}}$  if and only if it is the kernel pair also of  $h_C \circ q_{\mathcal{G}}$ ,  
 342 since  $h_C$  is mono by hypothesis. The thesis follows from  $h_C \circ q_{\mathcal{G}} = q_{\mathcal{H}} \circ h_V$ , and from the  
 343 hypothesis of  $h_E$  mono. ◀

344 ► **Remark 4.10.** It is possible to restate the last proposition, by ??, as

345  *$h_E$  and  $h_V$  are mono and, for every  $v, v' \in V_H$ ,  $q_H(h_V(v)) = q_H(h_V(v'))$  if and only*  
 346 *if  $q_G(v) = q_G(v')$*

347 That is, a regular mono in **EqHyp** is a morphism that both reflects and preserves equivalences.

348 Let us turn to another functor **EqHyp**  $\rightarrow$  **Hyp**.

349 ► **Definition 4.11.** *The quotient functor  $Q : \mathbf{EqHyp} \rightarrow \mathbf{Hyp}$  is defined as the one sending*  
 350  *$(E, V, C, s, t, q)$  to  $(E, C, q^* \circ s, q^* \circ t)$  and an arrow  $(h_E, h_V, h_C)$  to  $(h_E, h_C)$ .*

351 ► **Remark 4.12.** The action of the functor on a morphism of hypergraphs with equivalences  
 352 gives a morphism of hypergraphs, in fact  $q_{\mathcal{H}}^* \circ s_{\mathcal{H}} \circ h_E = q_{\mathcal{H}}^* \circ h_V^* \circ s_{\mathcal{G}} = h_C^* \circ q_{\mathcal{G}}^* \circ s_{\mathcal{G}}$ . The  
 353 same is valid for  $t_{\mathcal{H}}$  and  $t_{\mathcal{G}}$ .

354 ► **Lemma 4.13.**  *$Q$  is a left adjoint.*

355 **Proof.** Let  $R((A, B, s, t))$  be  $(A, B, B, s, t, \text{id}_B)$ , so that  $Q(R((A, B, s, t))) = (A, B, s, t)$ . Now,  
 356 suppose that  $h = (h_E, h_V) : Q((E, V, C, s', t', q)) \rightarrow (A, B, s, t)$  is an arrow in **Hyp**, and  
 357 consider the triple  $(h_E, h_V, h_V \circ q)$ . Since  $h$  is a morphism of **Hyp**, we have  $h_V^* \circ q^* \circ s' = s \circ h_E$   
 358 and  $h_V^* \circ q^* \circ t' = t \circ h_E$ . Then we have the following squares

$$\begin{array}{ccc}
 \begin{array}{ccc} E & \xrightarrow{h_E} & A \\ s' \downarrow & & \downarrow s \\ V^* & \xrightarrow{h_V^* \circ q^*} & B^* \end{array} & 
 \begin{array}{ccc} E & \xrightarrow{h_E} & A \\ t' \downarrow & & \downarrow t \\ V^* & \xrightarrow{h_V^* \circ q^*} & B^* \end{array} & 
 \begin{array}{ccc} V & \xrightarrow{h_V \circ q} & B \\ q \downarrow & & \downarrow \text{id}_B \\ C & \xrightarrow{h_V} & B \end{array}
 \end{array}$$

360 We have therefore found a morphism  $(E, V, C, s', t', q) \rightarrow R((A, B, s, t))$  whose image  
 361 through  $Q$  fits in the diagram below

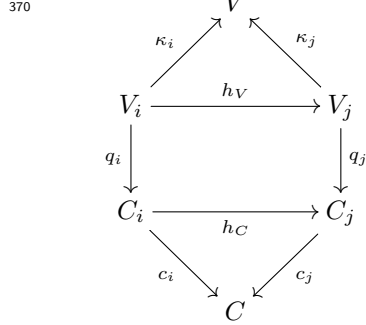
$$\begin{array}{ccc}
 & (A, B, s, t) & \xrightarrow{(\text{id}_A, \text{id}_B)} (A, B, s, t) \\
 Q(h_E, h_V \circ q, h_V) \uparrow & & \nearrow (h_E, h_V) \\
 (E, C, q^* \circ s', q^* \circ t') & & 
 \end{array}$$

363 Such arrow is unique. Suppose  $f = (f_E, f_V, f_C)$  to be another arrow with such property.  
 364 Then, it must be  $(\text{id}_A, \text{id}_B) \circ Q(f) = (f_E, f_C) = (h_E, h_C)$ . Finally,  $f_C = f_V \circ q = h_V \circ q$ . ◀

365 ► **Proposition 4.14.**  *$Q$  creates colimits.*

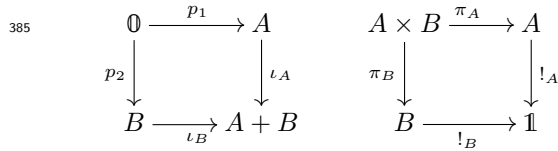
366 **Proof.** Since  $Q$  is a left adjoint, it preserves colimits. Let  $D : \mathbf{I} \rightarrow \mathbf{EqHyp}$  be a diagram,  
 367 and let  $\mathcal{C}$ , together with  $(c_i)_{i \in \mathbf{I}}$  be the colimit of  $Q \circ D$ , where  $\mathcal{C} = (A, C, q \circ s, q \circ t)$ , and

368  $D(i)$  is  $(A_i, B_i, C_i, s_i, t_i, q_i)$ . Let  $((\kappa_i)_{i \in \mathbf{I}}, V)$  be the colimit of  $U_{\mathbf{EqHyp}} \circ D$ . Consider the  
 369 following situation



371 Now, since  $((c_C^i \circ q_i)_{i \in \mathbf{I}}, C)$  is a cocone for  $U_{\mathbf{EqHyp}} \circ D$ , there exists a unique  $q : V \rightarrow C$ ,  
 372 which is epi by Lemma 2.13. Consider now the functor  $W : \mathbf{EqHyp} \rightarrow \mathbf{Set}$  mapping each  
 373  $(X, Y, Z, x, y, z)$  onto  $X$ , and each morphism on its first component. By Proposition 4.6  
 374 and ??, we have that  $((c_E^i)_{i \in \mathbf{I}}, E)$  is the colimit of  $W \circ D$ . Notice that  $((\kappa_i \circ s_i)_{i \in \mathbf{I}}, B)$  and  
 375  $((\kappa_i \circ t_i)_{i \in \mathbf{I}}, B)$  are cocones for  $W \circ D$ , so let  $s$  and  $t$  be, respectively, the mediating arrow  
 376 for the first one and the mediating arrow for the second one. It remains now to show that  
 377  $(E, V, C, s, t, q)$ , together with  $(c_E^i, \kappa_i, c_C^i)_{i \in \mathbf{I}}$ , is a colimit for  $D$ , but this follows by the proof  
 378 of ??.

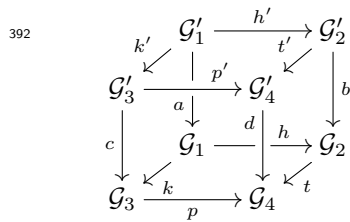
379 ► **Example 4.15.**  $Q$  does not preserve limits. Indeed, let  $\mathcal{G}_1 = (E_1, A, A, s_1, t_1, id_A)$ ,  $\mathcal{G}_2 =$   
 380  $(E_2, B, B, s_2, t_2, id_B)$  and  $\mathcal{G}_3 = (E_3, A + B, \mathbb{1}, s_3, t_3, !_{A+B})$ , and let  $h = (h_E, \iota_A, !_A) : \mathcal{G}_1 \rightarrow \mathcal{G}_3$ ,  
 381  $k = (k_E, \iota_B, !_B) : \mathcal{G}_2 \rightarrow \mathcal{G}_3$ , where  $(\iota_A, \iota_B, A + B)$  is the coproduct of  $A$  and  $B$ ,  $\mathbb{1}$  is the initial  
 382 object (in  $\mathbf{Set}$ , the singleton set as shown in ??), and  $!_X$  the unique arrow  $X \rightarrow \mathbb{1}$ . The  
 383 following two diagrams show the pullback of  $h$  and  $k$  and the pullback of  $Q(h)$  and  $Q(k)$ , on  
 384 the second component (the vertices of the graphs)



386 But the arrow  $0 \rightarrow A \times B$  is not an epi in general (this is easy to see taking  $\mathbf{Set}$  as  
 387 example), hence such pullback is not preserved by  $Q$ .

388 ► **Lemma 4.16.** In  $\mathbf{EqHyp}$ , pushouts along regular monos are stable.

389 **Proof.** Let  $\mathcal{G}_i = (A_i, B_i, C_i, s_i, t_i, q_i)$ ,  $\mathcal{G}'_i = (A'_i, B'_i, C'_i, s'_i, t'_i, q'_i)$ , for  $i \in \{1, 2, 3, 4\}$ , be hy-  
 390 pergraphs with equivalence, and, in the diagram below, suppose all the vertical faces are  
 391 pullbacks, the bottom face is a pushout and  $h$  is regular mono.



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By Proposition 4.6 and Corollary 4.7, the following cubes in **Set** have pushouts as bottom faces and pullbacks as vertical faces, hence their top faces are pushouts.

$$\begin{array}{ccc}
 & A'_1 & \xrightarrow{h'_E} A'_2 \\
 k'_E \swarrow & \downarrow p'_E & \swarrow t'_E \\
 A'_3 & \xrightarrow{a_E} A'_4 & \\
 c_E \downarrow & \downarrow d_E & \downarrow b_E \\
 A_3 & \xrightarrow{p_E} A_4 & \\
 & \downarrow h_E & \\
 & A_2 &
 \end{array}
 \quad
 \begin{array}{ccc}
 & B'_1 & \xrightarrow{h'_V} B'_2 \\
 k'_V \swarrow & \downarrow p'_V & \swarrow t'_V \\
 B'_3 & \xrightarrow{a_V} B'_4 & \\
 c_V \downarrow & \downarrow d_V & \downarrow b_V \\
 B_3 & \xrightarrow{p_V} B_4 & \\
 & \downarrow h_V & \\
 & B_2 &
 \end{array}$$

Consider now the following pullbacks.

$$\begin{array}{ccc}
 Y & \xrightarrow{y_1} & C'_4 \\
 y_2 \downarrow & & \downarrow d_C \\
 C_3 & \xrightarrow{p_C} & C_4
 \end{array}
 \quad
 \begin{array}{ccc}
 U & \xrightarrow{u_1} & C'_4 \\
 u_2 \downarrow & & \downarrow d_C \\
 C_2 & \xrightarrow{t_C} & C_4
 \end{array}
 \quad
 \begin{array}{ccc}
 T & \xrightarrow{x_1} & U \\
 x_2 \downarrow & & \downarrow u_2 \\
 C_1 & \xrightarrow{h_C} & C_2
 \end{array}$$

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proposizione

Thus, [???] yields the following situation, in which the bottom face is a pushout, and the vertical faces are pullbacks, hence the top face is a pushout too.

$$\begin{array}{ccc}
 & T & \xrightarrow{x_1} U \\
 w \swarrow & \downarrow y_1 & \swarrow u_1 \\
 Y & \xrightarrow{x_2} C'_4 & \\
 y_2 \downarrow & \downarrow d_C & \downarrow u_2 \\
 C_3 & \xrightarrow{p_C} C_4 & \\
 & \downarrow h_C & \\
 & C_2 &
 \end{array}$$

By the proof of Proposition 4.6, we have that  $m_2 \circ q'_3 : B'_3 \rightarrow Y$  and  $m_3 \circ q'_2 : B'_2 \rightarrow U$  are two epi-mono factorizations, with  $m_2$  and  $m_3$  monos. At the same way, let the following square to be a pullback.

$$\begin{array}{ccc}
 S & \xrightarrow{s_1} & C'_2 \\
 s_2 \downarrow & & \downarrow m_3 \\
 T & \xrightarrow{x_1} & U
 \end{array}$$

Hence, in the following diagram, the outer rectangle is a pullback.

$$\begin{array}{ccc}
 S & \xrightarrow{s_1} & C'_2 \\
 s_2 \downarrow & & \downarrow m_3 \\
 T & \xrightarrow{x_1} & U \\
 x_2 \downarrow & & \downarrow u_2 \\
 C_1 & \xrightarrow{h_C} & C_2
 \end{array}$$

By the same argument as before, there exists a mono  $m_1$  such that  $m_1 \circ q'_1 : B'_1 \rightarrow S$ .

We have to show that the top face of the cube at the beginning of the proof is a pushout.

Suppose then that  $z : \mathcal{G}'_2 \rightarrow \mathcal{H}$  and  $w : \mathcal{G}'_3 \rightarrow \mathcal{H}$ , with  $\mathcal{H} = (E, V, C, s, t, q)$ , are two morphisms

such that  $z \circ h' = w \circ h'$ , and let  $v_V : B'_4 \rightarrow V$  the arrow induced by  $z_V$  and  $w_V$ . We want to construct the dotted arrow  $v_C$  which fits in the diagram below.

$$\begin{array}{ccccc}
 & B'_1 & \xrightarrow{h'_V} & B'_2 & \\
 & \swarrow k'_V & \downarrow p'_V & \swarrow k'_V & \downarrow z_V \\
 B'_3 & \xrightarrow{q'_1} & B'_4 & \xrightarrow{q'_2} & V \\
 \downarrow q'_3 & \downarrow k'_C & \downarrow q'_4 & \downarrow h'_C & \downarrow z_C \\
 C'_3 & \xrightarrow{p'_C} & C'_4 & \xrightarrow{t'_C} & C \\
 & & & \swarrow v_C & \\
 & & & & q
 \end{array}$$

By Proposition 2.15, we know that the top face of the cube below is a pushout.

$$\begin{array}{ccccc}
 & K_{s_2 \circ m_1 \circ q'_1} & \xrightarrow{k_{h'_2}} & K_{m_2 \circ q'_2} & \\
 & \swarrow k_{k'_2} & \downarrow k_{p'_2} & \swarrow k_{t'_2} & \downarrow \pi^1_{m_2 \circ q'_2} \\
 K_{m_3 \circ q'_3} & \xrightarrow{\pi^1_{s_2 \circ m_1 \circ q'_1}} & K_{q'_4} & \xrightarrow{\pi^1_{q'_4}} & B'_2 \\
 \downarrow \pi^1_{m_3 \circ q'_3} & \downarrow \pi^1_{q'_4} & \downarrow h'_2 & \downarrow t'_2 & \\
 B'_3 & \xrightarrow{p'_2} & C_4 & \xrightarrow{t'_2} & \\
 & & & \swarrow k'_2 & \\
 & & & & B'_1
 \end{array}$$

And, since  $m_3$  and  $m_2$  are monos,

$$q'_3 \circ \pi^1_{m_3 \circ q'_3} = q'_3 \circ \pi^2_{m_3 \circ q'_3} \quad q'_2 \circ \pi^1_{m_2 \circ q'_2} = q'_2 \circ \pi^2_{m_2 \circ q'_2}$$

Computing, we obtain

$$\begin{aligned}
 q \circ v_V \circ \pi^1_{q'_4} \circ k_{p'_V} &= q \circ v_V \circ p'_V \circ \pi^1_{m_3 \circ q'_3} & q \circ v_V \circ \pi^1_{q'_4} \circ k_{t'_V} &= q \circ v_V \circ t'_V \circ \pi^1_{m_2 \circ q'_2} \\
 &= q \circ w_V \circ \pi^1_{m_3 \circ q'_3} & &= q \circ z_V \circ \pi^1_{m_2 \circ q'_2} \\
 &= w_C \circ q'_3 \circ \pi^1_{m_3 \circ q'_3} & &= z_C \circ q'_2 \circ \pi^1_{m_2 \circ q'_2} \\
 &= w_C \circ q'_3 \circ \pi^2_{m_3 \circ q'_3} & &= z_C \circ q'_2 \circ \pi^2_{m_2 \circ q'_2} \\
 &= q \circ w_V \circ \pi^2_{m_3 \circ q'_3} & &= q \circ z_V \circ \pi^2_{m_2 \circ q'_2} \\
 &= q \circ v_V \circ p'_V \circ \pi^2_{m_3 \circ q'_3} & &= q \circ v_V \circ t'_V \circ \pi^2_{m_2 \circ q'_2} \\
 &= q \circ v_V \circ \pi^2_{q'_4} \circ k_{p'_V} & &= q \circ v_V \circ \pi^2_{q'_4} \circ k_{t'_V}
 \end{aligned}$$

Since the previous cube has a pushout as top face, by universal property, we have

$$q \circ v_V \circ \pi^1_{q'_4} = q \circ v_V \circ \pi^2_{q'_4}$$

hence,  $v_C$  is the mediating arrow.

$$v_C \circ q'_4 \circ \pi^1_{q'_4} = v_C \circ q'_4 \circ \pi^2_{q'_4}$$

423

► **Lemma 4.17.** *In EqHyp, pushouts along regular monos are Reg(EqHyp)-Van Kampen.*

424

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**Proof.** In lieu of Lemma 4.16, it is enough to proof that, given a cube as the one below, with pullbacks as back faces, pushouts as bottom and top faces and such that  $h$  is a regular mono, the front faces are pullbacks too, where  $\mathcal{G}_i = (A_i, B_i, C_i, s_i, t_i, q_i)$ ,  $\mathcal{G}' = (A'_i, B'_i, C'_i, s'_i, t'_i, q'_i)$ , for  $i = 1, 2, 3, 4$ .

$$\begin{array}{ccccc}
 & & \mathcal{G}'_1 & \xrightarrow{h'} & \mathcal{G}'_2 \\
 & \swarrow k' & \downarrow p' & \swarrow t' & \downarrow b \\
 \mathcal{G}'_3 & \xrightarrow{a} & \mathcal{G}'_4 & & \\
 \downarrow c & & \downarrow d & \xrightarrow{h} & \mathcal{G}_2 \\
 & \swarrow k & \downarrow t & \swarrow t & \\
 \mathcal{G}_3 & \xrightarrow{p} & \mathcal{G}_4 & & 
 \end{array}$$

By Proposition 4.6 and ??, the following two cubes have  $\mathcal{M}$ -pushouts as bottom faces and pullbacks as back faces, thus their front faces are pullbacks too.

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 & & A'_1 & \xrightarrow{h'_E} & A'_2 \\
 & \swarrow k'_E & \downarrow p'_E & \swarrow t'_E & \downarrow b_E \\
 A'_3 & \xrightarrow{a_E} & A'_4 & & \\
 \downarrow c_E & & \downarrow d_E & \xrightarrow{h_E} & A_2 \\
 & \swarrow k_E & \downarrow t_E & \swarrow t_E & \\
 A_3 & \xrightarrow{p_E} & A_4 & & 
 \end{array} & & 
 \begin{array}{ccccc}
 & & B'_1 & \xrightarrow{h'_V} & B'_2 \\
 & \swarrow k'_V & \downarrow p'_V & \swarrow t'_V & \downarrow b_V \\
 B'_3 & \xrightarrow{a_V} & B'_4 & & \\
 \downarrow c_V & & \downarrow d_V & \xrightarrow{h_V} & B_2 \\
 & \swarrow k_V & \downarrow t_V & \swarrow t_V & \\
 B_3 & \xrightarrow{p_V} & B_4 & & 
 \end{array}
 \end{array}$$

On the other hand we can consider the diagrams below, in which the inner squares are pullbacks. Since the outer diagrams commute, by definition of morphism of **EqHyp**, then we have the existence of  $m_2: C'_2 \rightarrow U$ ,  $m_3: C'_3 \rightarrow Y$ ,  $a_3: B'_3 \rightarrow Y$  and  $a_2: B'_2 \rightarrow Y$ .

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 C'_3 & & & & \\
 \downarrow d_3 & \searrow m_3 & \searrow p'_3 & & \\
 & Y & \xrightarrow{y_1} & C'_4 & \\
 & \downarrow y_2 & & \downarrow d_3 & \\
 & C_3 & \xrightarrow{p_3} & C_4 & 
 \end{array} & & 
 \begin{array}{ccccc}
 C'_2 & & & & \\
 \downarrow d_2 & \searrow m_2 & \searrow t'_3 & & \\
 & U & \xrightarrow{u_1} & C'_4 & \\
 & \downarrow u_2 & & \downarrow d_3 & \\
 & C_2 & \xrightarrow{t_3} & C_4 & 
 \end{array} \\
 \\
 \begin{array}{ccccc}
 B'_3 & \xrightarrow{p'_2} & B'_4 & & \\
 \downarrow q'_3 & \searrow a_3 & \searrow q'_4 & & \\
 & Y & \xrightarrow{y_1} & C'_4 & \\
 & \downarrow y_2 & & \downarrow d_3 & \\
 & C_3 & \xrightarrow{p_3} & C_4 & 
 \end{array} & & 
 \begin{array}{ccccc}
 B'_2 & \xrightarrow{t'_2} & B'_4 & & \\
 \downarrow q'_2 & \searrow a_2 & \searrow q'_4 & & \\
 & U & \xrightarrow{u_1} & C'_4 & \\
 & \downarrow u_2 & & \downarrow d_3 & \\
 & C_2 & \xrightarrow{t_3} & C_4 & 
 \end{array}
 \end{array}$$

Now, notice that  $m_3$  and  $m_2$  are monos because  $d_3$  and  $d_2$  are regular monos. By the proof of Proposition 4.6, to conclude it is enough to show that

$$m_3 \circ q'_3 = a_3 \quad m_2 \circ q'_2 = a_2$$

Indeed, if the previous equations hold, then  $C'_3$  and  $C'_2$  are epi-mono factorizations of  $a_3$  and  $a_2$  and the thesis follows from ?? and the proof of Proposition 4.6.



443 No if we compute we have:

$$\begin{aligned}
 y_1 \circ a_3 &= q'_4 \circ p'_2 & u_1 \circ a_2 &= q'_4 \circ t'_2 \\
 &= p'_3 \circ q'_3 & &= t'_3 \circ q'_3 \\
 &= y_1 \circ m_3 \circ q'_3 & &= u_1 \circ m_2 \circ q'_2 \\
 y_2 \circ a_3 &= d_3 \circ q'_3 & u_2 \circ a_2 &= d_2 \circ q'_2 \\
 &= y_2 \circ m_3 \circ q'_3 & &= u_2 \circ m_2 \circ q'_2
 \end{aligned}$$

447 And we have done. ◀

## 4.1 Labeled Hypergraphs with Equivalences

449 As we have done in Section 3.1.2, we can define the category of hypergraphs with equivalence  
450 labeled over an algebraic signature.

451 ► **Definition 4.18.** Let  $\Sigma = (O_\Sigma, \text{ar}_\Sigma)$  be an algebraic signature and  $\mathcal{G}^\Sigma$  the hypergraph  
452 associated to  $\Sigma$ . Then the hypergraph with equivalence associated to  $\Sigma$  is  $L(\mathcal{G}^\Sigma)$  and the  
453 category of hypergraphs with equivalence labeled over  $\Sigma$  is the slice category  $\mathbf{EqHyp}_\Sigma =$   
454  $\mathbf{EqHyp}/L(\mathcal{G}^\Sigma)$ .

455 By ??, we can deduce the following.

456 ► **Proposition 4.19.**  $\mathbf{EqHyp}_\Sigma$  is  $\text{Reg}(\mathbf{EqHyp}_\Sigma)$ -adhesive.

457 We can lift the adjunction given by  $T$  and  $L$  to  $\text{EqHyp}_\Sigma$  and  $\mathbf{Hyp}_\Sigma$ .

458 By Corollary 4.7, we can deduce what follows.

459 ► **Proposition 4.20.** A morphism  $h$  between two objects of  $\mathbf{EqHyp}_\Sigma$  is mono if and only if  
460  $T(h)$  is mono.

Non capisco se  
è uno svarione  
mio o ha senso

### 4.1.1 Term Graphs with Equivalences

462 ► **Definition 4.21.** Let  $\Sigma$  be an algebraic signature. A labeled hypergraph with equivalence  
463  $l : \mathcal{G} \rightarrow L(\mathcal{G}^\Sigma)$  is a term graph with equivalence if  $t_{\mathcal{G}}$  is mono. We define category of term  
464 graphs with equivalence over  $\Sigma$ , denoted  $\mathbf{EqTG}_\Sigma$ , as the full subcategory of  $\mathbf{EqHyp}_\Sigma$ , and  
465 the corresponding inclusion functor  $I_{\mathbf{EqTG}_\Sigma}$ .

466 ► **Proposition 4.22.** If  $l : \mathcal{G} \rightarrow \mathcal{G}^\Sigma$  is a term graph, then  $L(l)$  is a term graph with equivalence.

## 5 EGGs

introduction

469 ► **Definition 5.1.** Let  $\mathcal{G} = (E, V, C, s, t, q)$  be a hypergraph with equivalence and  $(S, \pi_1, \pi_2)$  a  
470 kernel pair of  $q^* \circ s$ . Then,  $\mathcal{G}$  is an e-hypergraph whenever  $q^* \circ t \circ \pi_1 = q^* \circ t \circ \pi_2$ . **EGG** is  
471 the full subcategory of  $\mathbf{EqHyp}$  whose objects are e-hypergraphs, and  $I : \mathbf{EGG} \rightarrow \mathbf{EqHyp}$  is  
472 the inclusion functor.

473 ► **Lemma 5.2.** **EGG** has all limits and  $I$  preserves them.

474 **Proof.** Let  $D : \mathbf{I} \rightarrow \mathbf{EGG}$  be a diagram, with  $D(i) = (A_i, B_i, C_i, s_i, t_i, q_i)$ , let  $(U_i, u_1^i, u_2^i)$  be  
475 the kernel pair of  $q_i \circ s_i$ . Let now be  $(A, B, C, s, t, q)$ , together with projections  $(\pi_E^i, \pi_V^i, \pi_C^i)_{i \in \mathbf{I}}$   
476 the limit of  $I \circ D$ , let  $(U, u_1, u_2)$  be the kernel pair of  $q \circ s$  and let  $(L, (l_i)_{i \in \mathbf{I}})$  be the limit of

## XX:18 On the adhesivity of EGGS

477  $K \circ I \circ D$ . By construction (proof of Proposition 4.6), there exists a mono  $m : C \rightarrow L$  such  
 478 that  $\pi_C^i = l_i \circ m$ . Notice that

$$\begin{aligned}
 479 \quad q_i^* \circ s_i \circ \pi_E^i \circ u_1 &= q_i^* \circ (\pi_V^i)^* \circ s \circ u_1 \\
 480 &= (\pi_C^i)^* \circ q^* \circ s \circ u_1 \\
 481 &= (\pi_C^i)^* \circ q^* \circ s \circ u_2 \\
 482 &= q_i^* \circ s_i \circ \pi_E^i \circ u_2 \\
 483
 \end{aligned}$$

484 Then, for each  $i$ , there exists an arrow  $a_i : U \rightarrow U_i$  making the following diagram to commute

$$\begin{array}{ccccc}
 485 & U & \xrightarrow{u_1} & A & \\
 & \downarrow u_2 & \searrow a_i & \swarrow \pi_E^i & \\
 & A & & U_i & \xrightarrow{u_1^i} & A_i \\
 & \searrow \pi_E^i & & \downarrow u_2^i & & \downarrow q_i^* \circ s_i \\
 & & & A_i & \xrightarrow{q_i^* \circ s_i} & C_i^*
 \end{array}$$

486 We have then

$$\begin{aligned}
 487 \quad l_i^* \circ m^* \circ q^* \circ t \circ u_1 &= q_i^* \circ (\pi_V^i)^* \circ t \circ u_1 \\
 488 &= q_i^* \circ t_i \circ \pi_E^i \circ u_1 \\
 489 &= q_i^* \circ t_i \circ u_1^i \circ a_i \\
 490 &= q_i^* \circ t_i \circ u_2^i \circ a_i \\
 491 &= q_i^* \circ t_i \circ \pi_E^i \circ u_2 \\
 492 &= q_i^* \circ (\pi_V^i)^* \circ t \circ u_2 \\
 493 &= l_i^* \circ m^* \circ q^* \circ t \circ u_2 \\
 494
 \end{aligned}$$

495 By universal property of limits, we have that  $m^* \circ q^* \circ t \circ u_1 = m^* \circ q^* \circ t \circ u_2$ , and, since  $m$   
 496 is mono,  $q^* \circ t \circ u_1 = q^* \circ t \circ u_2$ , hence the thesis.  $\blacktriangleleft$

497 **► Corollary 5.3.**  *$I$  creates limits.*

498 **► Corollary 5.4.** *Let  $h : \mathcal{G} \rightarrow \mathcal{H}$  be an arrow in **EGG**. Then it is a regular mono if and only*  
 499 *if  $I(h)$  is a regular mono.*

500 **► Lemma 5.5.** *Consider the following pushout square in **EqHyp**.*

$$\begin{array}{ccc}
 501 & \mathcal{G}_1 & \xrightarrow{h} \mathcal{G}_2 \\
 & \downarrow m & \downarrow n \\
 & \mathcal{G}_3 & \xrightarrow{k} \mathcal{P}
 \end{array}$$

502 *with  $m$  regular mono. If  $\mathcal{G}_1$ ,  $\mathcal{G}_2$  and  $\mathcal{G}_3$  are  $e$ -hypergraphs, then  $\mathcal{P}$  is an  $e$ -hypergraph too,*  
 503 *and  $n$  is regular mono.*

504 **Proof.** Let  $\mathcal{P} = (A, B, C, s, t, q)$ ,  $(K_i, \pi_i^1, \pi_i^2)$  the kernel pair of  $q_i^* \circ s_i$ , and let  $(U, u_1, u_2)$  the

kernel pair of  $q^* \circ s$ . Consider then the following situation.

$$\begin{array}{ccccc}
 & & A_1 & \xrightarrow{h_E} & A_2 \\
 & \swarrow m_E & \downarrow k_E & \swarrow k_E & \downarrow q_2^* \circ s_2 \\
 A_3 & \xrightarrow{q_1^* \circ s_1} & A & \xrightarrow{q^* \circ s} & C_2^* \\
 \downarrow q_3^* \circ s_3 & & \downarrow q_1^* \circ s_1 & & \downarrow q_2^* \circ s_2 \\
 C_3^* & \xrightarrow{m_C^*} & C^* & \xrightarrow{n_C^*} & C^*
 \end{array}$$

Since  $m$  is regular mono,  $m_E$  is mono (inserire citazione). Then, by construction, the top face is a pushout, and since **Set** is adhesive, by Proposition 2.15, the square below is a pushout.

$$\begin{array}{ccc}
 K_1 & \xrightarrow{f_k} & K_2 \\
 \downarrow f_m & & \downarrow f_n \\
 K_3 & \xrightarrow{f_k} & U
 \end{array}$$

Computing, we have

$$\begin{aligned}
 q^* \circ t \circ u_1 \circ f_n &= q^* \circ t \circ n_E \circ \pi_2^1 & q^* \circ t \circ u_1 \circ f_k &= q^* \circ t \circ k_E \circ \pi_3^1 \\
 &= n_C^* \circ q_2^* \circ s_2 \circ \pi_2^1 & &= k_C^* \circ q_3^* \circ s_3 \circ \pi_3^1 \\
 &= n_C^* \circ q_2^* \circ s_2 \circ \pi_2^2 & &= k_C^* \circ q_3^* \circ s_3 \circ \pi_3^2 \\
 &= q^* \circ t \circ u_2 \circ f_n & &= q^* \circ t \circ u_2 \circ f_k
 \end{aligned}$$

By universal property of pushouts, we deduce  $q^* \circ t \circ u_1 = q^* \circ t \circ u_2$ , and the thesis follows. ◀

By direct application of Proposition 2.4, we can conclude what follows.

► **Corollary 5.6.** *EGG is Reg(EGG)-adhesive.*

## 6 Conclusions and further works

The aim of our paper was to extend the theory of adhesive categories in order to include EGGs, an up-and-coming formalism for program optimisation and synthesis via a compact representation and efficient implementation of equality saturation. To do so, we revisited and generalised the notions of hyper-graph and term graph with equivalence, and we extended it in order to capture EGGs as term graphs satisfying a suitable closure property.

Our result opens two threads of research. The first is to use the quasi-adhesivity of EGGs to model their rewriting via the double-pushout (DPO) approach. This seems now easy, since the rules adopted in the literature of EGGs appears to be span of regular monos, and such rules perfectly fit the mold of rewriting on  $\mathcal{M}$ -adhesive categories. For example, the equivalence  $x \div x = 1$ , from the introductory example in [?], can be modelled as the rule

*todrawDPOrule*

It still needs to be investigated what parallelism and termination, the key properties for DPO rewriting on adhesive categories, mean in the context of EGGs. More interestingly, another venue for development is using the adhesive machinery to extend the EGGs formalism. In fact, most of the results presented here for hyper-graphs can be generalised to hierarchical

hyper-graphs, that is, hypergraphs with a hierarchy (a partial order) among edges that is useful for adding structural information, such as encapsulation and sandboxing [?].

Finally, we need to draw a comparison with an alternative categorical presentation for EGGS advanced in [14]. The proposal is quite different from our own. Simplifying, the key is to equip categories of trees with a lattice on hom-sets. It seems that such proposal generalises our own, even if at the expenses of a more complex machinery.

## References

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## A Omitted proofs

584 This section contains the proofs which are omitted from the main body of the paper. We  
 585 begin recalling a well-known fact about composition and decomposition of pullbacks [18].

586 ► **Lemma A.1.** *Let  $\mathbf{X}$  be a category, and consider the diagram*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ a \downarrow & & \downarrow b & & \downarrow c \\ A & \xrightarrow{h} & B & \xrightarrow{k} & C \end{array}$$
*aside, in which the right square is a pullback. Then the whole rectangle is a pullback if and only if the left square is one.*

587 ► **Lemma 2.14.** *Let  $f: X \rightarrow Y$  and  $g: Z \rightarrow W$  be two arrows admitting kernel pairs*  
 588 *and suppose that the solid part of the three squares below is given. If the leftmost square*  
 589 *is commutative, then there exists a unique arrow  $k_h: K_f \rightarrow K_g$  making the other two*  
 590 *commutative. Moreover, if the leftmost is a pullback, then also the other two are so.*

$$\begin{array}{ccc} \begin{array}{ccc} X & \xrightarrow{h} & Z \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{t} & W \end{array} & \begin{array}{ccc} K_f & \xrightarrow{k_h} & K_g \\ \pi_f^1 \downarrow & & \downarrow \pi_g^1 \\ X & \xrightarrow{h} & Z \end{array} & \begin{array}{ccc} K_f & \xrightarrow{k_h} & K_g \\ \pi_f^2 \downarrow & & \downarrow \pi_g^2 \\ X & \xrightarrow{h} & Z \end{array} \end{array}$$

592 **Proof.** Computing, we have

$$593 \quad g \circ h \circ \pi_f^1 = t \circ f \circ \pi_f^1 = t \circ f \circ \pi_f^2 = g \circ h \circ \pi_f^2$$

594 Therefore the existence and uniqueness of the wanted  $k_h$  follows at once from the the  
 595 universal property of  $K_g$  as the pullback of  $g$  along itself.

596 To prove the second half of the thesis, we can notice that, by Lemma A.1, two rectangles  
 597 below are pullbacks.

$$\begin{array}{ccc} \begin{array}{ccccc} K_f & \xrightarrow{\pi_f^2} & X & \xrightarrow{h} & Z \\ \pi_f^1 \downarrow & & f \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y & \xrightarrow{t} & W \end{array} & \begin{array}{ccccc} K_f & \xrightarrow{\pi_f^1} & X & \xrightarrow{h} & Z \\ \pi_f^2 \downarrow & & f \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y & \xrightarrow{t} & W \end{array} \end{array}$$

599 But then the following ones are pullbacks too.

$$\begin{array}{ccc} \begin{array}{ccccc} & \xrightarrow{h \circ \pi_f^2} & & & \\ K_f & \xrightarrow{k_h} & K_g & \xrightarrow{\pi_g^2} & Z \\ \pi_f^1 \downarrow & & \pi_g^1 \downarrow & & \downarrow g \\ X & \xrightarrow{h} & Y & \xrightarrow{g} & W \\ & \xleftarrow{t \circ f} & & & \end{array} & \begin{array}{ccccc} & \xrightarrow{h \circ \pi_f^2} & & & \\ K_f & \xrightarrow{k_h} & K_g & \xrightarrow{\pi_g^2} & Z \\ \pi_f^1 \downarrow & & \pi_g^1 \downarrow & & \downarrow g \\ X & \xrightarrow{h} & Y & \xrightarrow{g} & W \\ & \xleftarrow{t \circ f} & & & \end{array} \end{array}$$

601 The thesis now follows again by Lemma A.1. ◀

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► **Proposition 2.15.** *Let  $\mathbf{X}$  be a strict  $\mathcal{M}$ -adhesive category with all pullbacks, and suppose that in the cube aside the top face is an  $\mathcal{M}$ -pushout. Then the right square is a pushout.*

$$\begin{array}{ccccc}
 & & A' & \xrightarrow{f'} & B' \\
 & m' \swarrow & \downarrow g' & \nwarrow n' & \downarrow b \\
 C' & \xrightarrow{a} & D' & & \\
 \downarrow c & & \downarrow d & \searrow f & \downarrow \\
 C & \xrightarrow{g} & D & & 
 \end{array}
 \quad
 \begin{array}{ccc}
 K_a & \xrightarrow{k_{f'}} & K_b \\
 \downarrow k_{m'} & & \downarrow k_{n'} \\
 K_c & \xrightarrow{k_{g'}} & K_d
 \end{array}$$

$$\begin{array}{ccccc}
 & & K_a & \xrightarrow{k_{f'}} & K_n \\
 & k_{m'} \swarrow & \downarrow k_{g'} & \nwarrow k_{n'} & \downarrow \pi_b^1 \\
 K_c & \xrightarrow{\pi_a^1} & K_d & & \\
 \downarrow \pi_c^1 & & \downarrow \pi_d^1 & \searrow f' & \downarrow \\
 C' & \xrightarrow{g'} & D' & & 
 \end{array}$$

**Proof.** By Proposition 2.6 we know that the top face of the original cube is a pullback. Thus Lemma 2.14 entails that in the following cube the vertical faces are pullbacks. The thesis now follows from strict  $\mathcal{M}$ -adhesivity. ◀

## B Some properties of comma categories

In this section we will briefly recall the definition of the comma category [18] associated to two functors and some of its properties.

► **Definition B.1.** *Let  $L: \mathbf{A} \rightarrow \mathbf{X}$  and  $R: \mathbf{B} \rightarrow \mathbf{X}$  be two functors with the same codomain, the comma category  $L \downarrow R$  is the category in which*

- *objects are triples  $(A, B, f)$  with  $A \in \mathbf{A}$ ,  $B \in \mathbf{B}$ , and  $f: L(A) \rightarrow R(B)$ ;*
- *a morphism  $(A, B, f) \rightarrow (A', B', g)$  is a pair  $(h, k)$  with  $h: A \rightarrow A'$  in  $\mathbf{A}$  and  $k: B \rightarrow B'$  in  $\mathbf{B}$  such that the following diagram commutes*

$$\begin{array}{ccc}
 L(A) & \xrightarrow{L(h)} & L(A') \\
 f \downarrow & & \downarrow g \\
 R(B) & \xrightarrow{R(k)} & R(B')
 \end{array}$$

We have two forgetful functors  $U_L: L \downarrow R \rightarrow \mathbf{A}$  and  $U_R: L \downarrow R \rightarrow \mathbf{B}$  given, respectively by

$$\begin{array}{ccc}
 (A, B, f) & \mapsto & A \\
 (h, k) \downarrow & & \downarrow h \\
 (A', B', g) & \mapsto & A'
 \end{array}
 \quad
 \begin{array}{ccc}
 (A, B, f) & \mapsto & B \\
 (h, k) \downarrow & & \downarrow k \\
 (A', B', g) & \mapsto & B'
 \end{array}$$

Given  $L: \mathbf{A} \rightarrow \mathbf{X}$  and  $R: \mathbf{B} \rightarrow \mathbf{X}$ , we can also consider their duals  $L^{op}: \mathbf{A}^{op} \rightarrow \mathbf{X}^{op}$  and  $R^{op}: \mathbf{B}^{op} \rightarrow \mathbf{X}^{op}$ . An arrow  $f: L(A) \rightarrow R(B)$  in  $\mathbf{X}$  is the same thing as an arrow  $f: R^{op}(B) \rightarrow L^{op}(A)$  in  $\mathbf{X}^{op}$ , thus  $(L \downarrow R)$  and  $R^{op} \downarrow L^{op}$  have the same objects. Moreover, the commutativity in  $\mathbf{X}$  of the square

$$\begin{array}{ccc}
 L(A) & \xrightarrow{L(h)} & L(A') \\
 f \downarrow & & \downarrow g \\
 R(B) & \xrightarrow{R(k)} & R(B')
 \end{array}$$

is tantamount to the commutativity in  $\mathbf{X}^{op}$  of the square

$$\begin{array}{ccc} R(B') & \xrightarrow{R(k)} & R(B) \\ g \downarrow & & \downarrow f \\ L(A') & \xrightarrow{L(h)} & L(A) \end{array}$$

Summing up we have just proved the following fact.

► **Proposition B.2.**  $(L \downarrow R)^{op}$  is equal to  $R^{op} \downarrow L^{op}$ , moreover  $U_L^{op} = U_{L^{op}}$  and  $U_R^{op} = U_{R^{op}}$ .

► **Lemma B.3.** Let  $L: \mathbf{A} \rightarrow \mathbf{X}$  and  $R: \mathbf{B} \rightarrow \mathbf{X}$  be functors and  $F: \mathbf{D} \rightarrow L \downarrow R$  be a diagram such that  $L$  preserves colimits along  $U_L \circ F$ . Then the family  $\{U_L, U_R\}$  jointly creates colimits of  $F$  (see [6, 7]).

**Proof.** Suppose that  $U_L \circ F$  and  $U_R \circ F$  have colimiting cocones  $(A, \{a_D\}_{D \in \mathbf{D}})$  and  $(B, \{b_D\}_{D \in \mathbf{D}})$  respectively. By hypothesis  $(L(A), \{L(a_D)\}_{D \in \mathbf{D}})$  is colimiting for  $L \circ U_L \circ F$ . Now, if we define

$$F(D) := (A_D, B_D, f_D)$$

then we have arrows  $R(a_i) \circ f_D: L(A_D) \rightarrow R(B)$  that forms a cocone on  $L \circ U_L \circ F$ : if  $d: D \rightarrow D'$  is an arrow in  $\mathbf{D}$  then  $F(d)$  is an arrow in  $L \downarrow R$  and so

$$\begin{aligned} R(b_{D'}) \circ f_{D'} \circ L(U_L(F(d))) &= R(b_{D'}) \circ R(U_R(F(d))) \circ f_D \\ &= R(b_{D'} \circ U_R(F(d))) \circ f_D \\ &= R(b_D) \circ f_D \end{aligned}$$

Thus there exists  $f: L(A) \rightarrow R(B)$  such that

$$\begin{array}{ccc} L(A_D) & \xrightarrow{L(a_D)} & L(A) \\ f_D \downarrow & & \downarrow f \\ R(B_D) & \xrightarrow{R(b_D)} & R(B) \end{array}$$

Notice that  $f$  is the unique arrow in  $\mathbf{X}$  which makes  $(a_D, b_D)$  an arrow  $(A_D, B_D, f_D) \rightarrow (A, B, f)$  of  $L \downarrow R$ . If we show that  $((A, B, f), \{(a_D, b_D)\}_{D \in \mathbf{D}})$  is colimiting for  $F$  we are done.

First of all, let us show that it is a cocone. Given  $d: D \rightarrow D'$  in  $\mathbf{D}$  we have:

$$\begin{aligned} (a_{D'}, b_{D'}) \circ F(d) &= (a_{D'}, b_{D'}) \circ (U_L(F(d)), U_R(F(d))) \\ &= (a_{D'} \circ U_L(F(d)), b_{D'} \circ U_R(F(d))) \\ &= (a_D, b_D) \end{aligned}$$

For the colimiting property, let  $((X, Y, g), \{(x_D, y_D)\}_{D \in \mathbf{D}})$  be another cocone on  $F$ . In particular  $(X, \{x_D\}_{D \in \mathbf{D}})$  and  $(Y, \{y_D\}_{D \in \mathbf{D}})$  are cocones on  $U_L \circ F$  and  $U_R \circ F$  respectively, so we have uniquely determined arrows  $x: A \rightarrow X$  and  $y: B \rightarrow Y$  such that

$$x \circ a_D = x_D \quad y \circ b_D = y_D$$

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Let us show that  $(x, y)$  is an arrow of  $L \downarrow R$ . Given  $D \in \mathbf{D}$  we have

$$\begin{aligned}
 R(y) \circ f \circ L(a_D) &= R(y) \circ R(b_D) \circ f_D \\
 &= R(y \circ b_D) \circ f_D \\
 &= R(y_D) \circ f_D \\
 &= g \circ L(x_D) \\
 &= g \circ L(x \circ a_D) \\
 &= g \circ L(x) \circ L(a_D)
 \end{aligned}$$

from which it follows that the following diagram commutes.

$$\begin{array}{ccc}
 L(A) & \xrightarrow{L(x)} & X \\
 f \downarrow & & \downarrow g \\
 R(B) & \xrightarrow{R(y)} & Y
 \end{array}$$

This shows that  $((A, B, f), \{(a_D, b_D)\}_{D \in \mathbf{D}})$  is colimiting for  $F$  and the thesis follows.  $\blacktriangleleft$

Proposition B.2 and Lemma B.3 now yields the following.

► **Corollary B.4.** *The family  $\{U_L, U_R\}$  jointly creates limits along every diagram  $F: \mathbf{D} \rightarrow L \downarrow R$  such that  $R$  preserves the limit of  $U_R \circ I$ .*

We can use Corollary B.4 to characterize monos in comma categories.

► **Corollary B.5.** *If  $R$  preserves pullbacks then an arrow  $(h, k)$  in  $L \downarrow R$  is mono if and only if both  $h$  and  $k$  are monos.*

**Proof.**  $(\Rightarrow)$  If  $(h, k): (A, B, f) \rightarrow (A', B', g)$  is a mono then the following square is a pullback in  $L \downarrow R$

$$\begin{array}{ccc}
 (A, B, f) & \xrightarrow{\text{id}_{(A, B, f)}} & (A, B, f) \\
 \text{id}_{(A, B, f)} \downarrow & & \downarrow (h, k) \\
 (A, B, f) & \xrightarrow{(h, k)} & (A', B', g)
 \end{array}$$

Using Corollary B.4 we deduce that the following two squares are pullbacks in  $\mathbf{A}$  and  $\mathbf{B}$ .

$$\begin{array}{ccc}
 A & \xrightarrow{\text{id}_A} & A \\
 \text{id}_A \downarrow & & \downarrow h \\
 A & \xrightarrow{h} & A'
 \end{array}
 \quad
 \begin{array}{ccc}
 B & \xrightarrow{\text{id}_B} & B \\
 \text{id}_B \downarrow & & \downarrow k \\
 B & \xrightarrow{k} & B'
 \end{array}$$

From which it follows that  $h$  and  $k$  are monos.

$(\Leftarrow)$  Since  $h$  and  $k$  are monos then we have two pullback squares

$$\begin{array}{ccc}
 A & \xrightarrow{\text{id}_A} & A \\
 \text{id}_A \downarrow & & \downarrow h \\
 A & \xrightarrow{h} & A'
 \end{array}
 \quad
 \begin{array}{ccc}
 B & \xrightarrow{\text{id}_B} & B \\
 \text{id}_B \downarrow & & \downarrow k \\
 B & \xrightarrow{k} & B'
 \end{array}$$



By Corollary B.4 this implies that

$$\begin{array}{ccc}
 (A, B, f) & \xrightarrow{\text{id}_{(A, B, f)}} & (A, B, f) \\
 \text{id}_{(A, B, f)} \downarrow & & \downarrow (h, k) \\
 (A, B, f) & \xrightarrow{(h, k)} & (A', B', g)
 \end{array}$$

is a pullback in  $L \downarrow R$  and we are done.  $\blacktriangleleft$

We end this section pointing out another useful fact, showing that in some cases we can guarantee the existence of a left adjoint to  $U_R$ .

► **Proposition B.6.** *If  $\mathbf{A}$  has initial objects and  $L$  preserves them then the forgetful functor  $U_R: L \downarrow R \rightarrow \mathbf{B}$  has a left adjoint  $\Delta$ .*

**Proof.** For an object  $B \in \mathbf{B}$  we can define  $\Delta(B)$  as  $(0, B, ?_B)$ , where  $0$  is an initial object in  $\mathbf{A}$  and  $?_B: L(0) \rightarrow R(B)$ . Consider  $\text{id}_B: B \rightarrow U_R(\Delta(B))$  be the identity, and suppose that a  $k: B \rightarrow U_R(A, B', f)$  in  $\mathbf{B}$  is given. By initiality of  $0$ , there is only one arrow  $?_A: 0 \rightarrow A$  in  $\mathbf{A}$  and, since  $L$  preserves initial objects, the following square commutes.

$$\begin{array}{ccc}
 L(0) & \xrightarrow{L(?_A)} & L(A) \\
 ?_{R(B)} \downarrow & & \downarrow f \\
 R(B) & \xrightarrow{R(k)} & R(B')
 \end{array}$$

Thus  $(h, k)$  is the unique morphism  $\Delta(B) \rightarrow (A, B', f)$  such that  $U_R(h, k) = k$ .  $\blacktriangleleft$

Dualizing we get immediately the following.

► **Corollary B.7.** *If  $\mathbf{B}$  has terminal objects preserved by  $R$  then  $U_L: L \downarrow R \rightarrow \mathbf{A}$  has a right adjoint.*

## B.1 Slice categories

This section is devoted to recall some basic facts about the so called *slice categories*.

► **Definition B.8.** *Let  $X$  be an object of a category  $\mathbf{X}$ , we will define the following two categories.*

■ *The slice category over  $X$  is the category  $\mathbf{X}/X$  which has as objects arrows  $f: Y \rightarrow X$  and in which an arrow  $h: f \rightarrow g$  is  $h: Y \rightarrow Y'$  in  $\mathbf{X}$  such that the following triangle commutes.*

$$\begin{array}{ccc}
 Y & \xrightarrow{h} & Y' \\
 f \searrow & & \swarrow g \\
 & X &
 \end{array}$$

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701 ■ Dually, the slice category under  $X$  is the category  $X/\mathbf{X}$  in which objects are arrows  
 702  $f: X \rightarrow Y$  with domain  $X$  and a morphism  $h: f \rightarrow g$  is an arrow of  $\mathbf{X}$  fitting in a  
 703 triangle as the one below.

$$704 \quad \begin{array}{ccc} & X & \\ f \swarrow & & \searrow g \\ Y & \xrightarrow{h} & Y' \end{array}$$

705 ► Remark B.9. For every  $X \in \mathbf{X}$  we have forgetful functors

$$706 \quad \begin{array}{ccc} \text{dom}_X: \mathbf{X}/X \rightarrow \mathbf{X} & \text{cod}_X: X/\mathbf{X} \rightarrow \mathbf{X} \\ f \mapsto \text{dom}(f) & f \mapsto \text{cod}(f) \\ h \downarrow \quad \downarrow h & h \downarrow \quad \downarrow h \\ g \mapsto \text{dom}(g) & g \mapsto \text{cod}(g) \end{array}$$

707 We can realize the slice over and under an object  $X \in \mathbf{X}$  as comma categories.

708 ► **Proposition B.10.** For every object  $X$  in a category  $\mathbf{X}$ , if  $\delta_X: \mathbf{1} \rightarrow \mathbf{X}$  is the constant  
 709 functor of value  $X$  from the category with only one object  $*$ , then  $\mathbf{X}/X$  and  $X/\mathbf{X}$  are  
 710 isomorphic to, respectively,  $\text{id}_X \downarrow \delta_X$  and  $\delta_X \downarrow \text{id}_X$ .

711 **Proof.** Define functors  $F_1: \text{id}_X \downarrow \delta_X \rightarrow \mathbf{X}/X$  and  $G_1: \mathbf{X}/X \rightarrow \text{id}_X \downarrow \delta_X$  as follows

$$712 \quad \begin{array}{ccc} (Y, *, f) \mapsto f & f \mapsto (Y, *, f) \\ (h, \text{id}_*) \downarrow \quad \downarrow h & h \downarrow \quad \downarrow (h, \text{id}_*) \\ (Y', *, g) \mapsto g & g \mapsto (Y', *, g) \end{array}$$

713 Similarly, we have  $F_2: \delta_X \downarrow \text{id}_X \rightarrow X/\mathbf{X}$  and  $G_2: X/\mathbf{X} \rightarrow \delta_X \downarrow \text{id}_X$

$$714 \quad \begin{array}{ccc} (*, Y, f) \mapsto f & f \mapsto (*, Y, f) \\ (\text{id}_*, h) \downarrow \quad \downarrow h & h \downarrow \quad \downarrow (\text{id}_*, h) \\ (*, Y', g) \mapsto g & g \mapsto (*, Y', g) \end{array}$$

715 It is now obvious to see that  $F_1, G_1$  and  $F_2, G_2$  are pairs of inverses. ◀

716 A straightforward application of Corollary B.4 now yields the following.

717 ► **Corollary B.11.** If  $\mathbf{X}$  has pullbacks, then for every object  $X$ , the slice  $\mathbf{X}/X$  has pullbacks  
 718 too.

719 In a category  $\mathbf{X}$  with pullbacks, each  $f: X \rightarrow Y$  induces a functor  $f^*: \mathbf{X}/Y \rightarrow \mathbf{X}/X$ ,  
 720 which sends each morphism  $a: A \rightarrow Y$  of  $\mathbf{X}$  onto its pullback along  $f$ ,  $p_a$ , and each morphism  
 721  $h: a \rightarrow b$  onto the unique arrow from the pullback of  $a$  along  $f$  to the pullback of  $b$  along  $f$ .

722 Then, we have the following result.

723 ► **Proposition B.12.** Let  $\mathbf{X}$  be a category with pullbacks,  $R: \mathbf{Y} \rightarrow \mathbf{X}$  be a functor and  
 724  $L: \mathbf{X} \rightarrow \mathbf{Y}$  be its left adjoint, and  $\eta$  the unit of the adjunction. Then, each object  $X$  of  $\mathbf{X}$   
 725 induces an adjoint pair of functors  $L_X: \mathbf{X}/X \rightarrow \mathbf{Y}/L(X)$ ,  $R_X: \mathbf{Y}/L(X) \rightarrow \mathbf{X}/X$ , where  
 726  $L_X$  is the obvious functor, and  $R_X$  is the composite  $(\eta_X)^* \circ R$ .

727 **Proof.** Let  $f : L(l) \rightarrow b$  be a morphism of  $\mathbf{Y}/L(X)$ , where  $l : A \rightarrow X$  in  $\mathbf{X}$  and  $b : B \rightarrow L(X)$   
 728 in  $\mathbf{Y}$ , as shown below.

$$729 \quad \begin{array}{ccc} L(A) & \xrightarrow{f} & B \\ & \searrow L(l) \quad \swarrow b & \\ & L(X) & \end{array}$$

730 Then, we have the following situation in  $\mathbf{X}$ .

$$731 \quad \begin{array}{ccccc} & & g & & \\ & \nearrow & & \searrow & \\ A & \xrightarrow{\eta_A} & R(L(A)) & \xrightarrow{R(f)} & R(B) \\ & \searrow l & \searrow R(L(l)) & \swarrow R(b) & \\ & X & \xrightarrow{\eta_X} & R(L(X)) & \end{array}$$

732 Where  $g$  is the adjunct of  $f$ . Consider now the pullback  $P$  of  $R(p)$  along  $\eta_X$ , as shown below.

$$733 \quad \begin{array}{ccc} A & \xrightarrow{g} & R(B) \\ \vdots v & & \downarrow R(b) \\ & P \xrightarrow{q} & R(B) \\ & \downarrow \eta_X^*(R(b)) & \downarrow R(b) \\ & X \xrightarrow{\eta_X} & R(L(X)) \end{array}$$

734 By universal property of pullbacks, the two diagrams express the same morphism  $g$ . Hence,  
 735 we can rewrite the first diagram as follows.

$$736 \quad \begin{array}{ccccc} & & f & & \\ & \nearrow & & \searrow & \\ L(A) & \xrightarrow{L(g)} & (L(R(B))) & \xrightarrow{\epsilon_B} & B \\ \downarrow L(l) & & \downarrow L(R(b)) & & \downarrow b \\ L(X) & \xrightarrow{L(\eta_X)} & L(R(L(X))) & \xrightarrow{\epsilon_{L(X)}} & L(X) \\ & & \text{id}_{L(X)} & & \end{array}$$

737 where  $\epsilon$  is the counit of the adjunction. This describes the action of functors and thus the  
 738 adjunction, obtained considering the isomorphism on hom-sets of the categories.

739

