

Chapter 1

Introduction

[TODO]

Chapter 2

Background

[INTRO]

2.1 Categories

The very first notion to get familiar with is that of category. The entire work makes extensive use of the language of categories. To answer the question "what is a category?", here is a definition:

Definition 2.1.1 (Category). A *category* \mathcal{C} comprises:

1. A collection of *objects* $Ob(\mathcal{C})$;
2. A collection of *arrows* (or *morphisms*) $Hom(\mathcal{C})$.

For each morphism $f \in Hom(\mathcal{C})$, there are two operators dom and cod which map every morphism to two objects, respectively, its *domain* and its *codomain*. In case $dom f = A$ and $cod f = B$, we will write $f : A \rightarrow B$. The collection of all morphisms from an object A to an object B is denoted as $\mathcal{C}(A, B)$. An operator \circ of *composition* maps every couple of morphisms f, g with $cod f = dom g$ (in this case f and g are said to be composable) to a morphism $g \circ f : dom f \rightarrow cod g$. The composition operator is associative, i.e., for each composable arrows f, g and h , it holds that

$$h \circ (g \circ f) = (h \circ g) \circ f$$

For each object A , an *identity* morphism $id_A : A \rightarrow A$ (or, when it is clear from the context, just denoted A) such that, for each $f : A \rightarrow B$:

$$id_B \circ f = f = f \circ id_A$$

The most important thing here is not the structure of the objects, but instead how this structure is preserved by the morphisms.

To represent morphisms of a category \mathcal{C} it is possible to use **diagrams**, as the one below, in which the vertices are objects of \mathcal{C} , and the edges are morphisms of \mathcal{C} .

$$\begin{array}{ccc} X & \xrightarrow{f'} & Z \\ g' \downarrow & & \downarrow g \\ W & \xrightarrow{f} & Y \end{array}$$

The diagram is said to commute whenever $f \circ g' = g \circ f'$. Unique morphisms are represented with dashed arrows. A more rigorous definition of what a diagram is will be given later (Definition 2.3.2).

Example 2.1.2. It is easy to see that taking sets as objects and total functions as arrows, we obtain a category. In fact, given two functions $f : A \rightarrow B$ and $g : B \rightarrow C$, it is possible to compose them obtaining an arrow $g \circ f : A \rightarrow C$, and the composition is associative. For each set A there exists an identity function $id_A : A \rightarrow A$ such that $\forall a \in A : id_A(a) = a$. This category is usually denoted as **Set**.

Remark 2.1.3. It is important to note that the Definition 2.1.1 above does not specify what kind of collections are, for a category \mathcal{C} , $Ob(\mathcal{C})$ and $Hom(\mathcal{C})$. Taking **Set** as example, the collection $Ob(\mathbf{Set})$ cannot be a set itself, due to Russell's paradox. It would be more appropriate referring to a category \mathcal{C} which $Ob(\mathcal{C})$ and $Hom(\mathcal{C})$ are both sets as a *small category*, but it is assumed in this work, except where it is made explicit, for a category to be small.

Between all morphisms of a category, it is possible to distinguish some that have certain properties, as well as functions between sets can be surjective, injective or bijective.

Definition 2.1.4 (Monomorphism). An arrow $f : B \rightarrow C$ in a category \mathcal{C} is a *monomorphism* if, for any pair of arrows of \mathcal{C} $g : A \rightarrow B$, $h : A \rightarrow B$, the equality $f \circ g = f \circ h$ implies $g = h$.

Definition 2.1.5 (Epimorphism). An arrow $f : A \rightarrow B$ in a category \mathcal{C} is an *epimorphism* if, for any pair of arrows of \mathcal{C} $g : B \rightarrow C$, $h : B \rightarrow C$, the equality $g \circ f = h \circ f$ implies $g = h$.

Definition 2.1.6 (Isomorphism). An arrow $f : A \rightarrow B$ is an *isomorphism* if there is an arrow $f^{-1} : B \rightarrow A$, called the *inverse* of f , such that $f^{-1} \circ f = id_A$ and $f \circ f^{-1} = id_B$. Two objects are said to be *isomorphic* if there is an isomorphism between them.

Example 2.1.7. In **Set**, monomorphisms are injective functions, epimorphisms are surjective functions and isomorphisms are bijections.

[TODO: Capire quanto andare nel dettaglio su questo, in particolare che epi + mono non implica iso (dipende da se mi serve)]
 [TODO: Capire che esempi fare!]

2.2 Categories from other categories

Starting from a category \mathcal{C} , it is possible to construct some other categories with interesting property. An example is the *dual category* of a category \mathcal{C} , denoted \mathcal{C}^{op} , in which the objects are the same of \mathcal{C} , and the arrows are the opposite of the arrows in \mathcal{C} , i.e., if $f : A \rightarrow B$ is an arrow of \mathcal{C} , then $f : B \rightarrow A$ is an arrow of \mathcal{C}^{op} . Each definition in category theory has a dual form. In general, if a statement S is true in a category \mathcal{C} , then the opposite of the statement, S^{op} , obtained switching the words "domain" and "codomain" and replacing each composite $g \circ f$ into $f \circ g$, is still true in the category \mathcal{C}^{op} . Moreover, since every category is the opposite of its opposite, if a statement S is true for every category, then S^{op} is also true for every category [Pie91].

Another important notion is that of subcategory.

Definition 2.2.1 (Subcategory). A category \mathcal{D} is a *subcategory* of a category \mathcal{C} if:

1. each object of \mathcal{D} is an object of \mathcal{C} ;
2. each morphism between two objects of \mathcal{D} is a morphism of \mathcal{C} ; and
3. composites and identities of \mathcal{D} are the same of \mathcal{C}

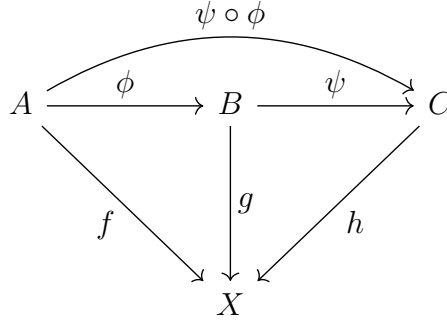
If the inclusion at 2 is an equality (i.e. $\mathcal{D}(A, B) = \mathcal{C}(A, B)$ for each couple of objects A, B of \mathcal{D}), then \mathcal{D} is said to be a *full subcategory* of \mathcal{C} .

An object of a category can raise a category itself. This is the case of slice (and coslice) categories.

Definition 2.2.2 (Slice Category). Given a category \mathcal{C} and an object $X \in Ob(\mathcal{C})$, the *slice category* \mathcal{C}/X is the category that has pairs (A, f) as objects, where A is an object of \mathcal{C} and $f : A \rightarrow X$ is an arrow in \mathcal{C} , and arrows $\phi : (A, f) \rightarrow (B, g)$ is given by a morphism $\phi : A \rightarrow B$ of \mathcal{C} such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ & \searrow f & \downarrow g \\ & & X \end{array}$$

– i.e., $g \circ \phi = f$. Composition between two arrows in \mathcal{C}/X $\phi : (A, f) \rightarrow (B, g)$ and $\psi : (B, g) \rightarrow (C, h)$ is the arrow $\psi \circ \phi : (A, f) \rightarrow (C, h)$ obtained in the obvious way:



The dual definition is *coslice category*, noted X/\mathcal{C} (where $X \in \text{Ob}(\mathcal{C})$), obtained by taking as objects all the morphisms of \mathcal{C} with domain X and as arrows the morphisms $\phi : (A, f) \rightarrow (B, g)$ such that $f : X \rightarrow A, g : B \rightarrow X$ of \mathcal{C} such that $g = \phi \circ f$.

Furthermore, it is possible to raise a new category from two old ones by taking their product, as the following definition shows.

Definition 2.2.3 (Product category). Given two categories \mathcal{C}, \mathcal{D} , the *product category* $\mathcal{C} \times \mathcal{D}$ has as objects pairs of objects (A, B) , where $A \in \text{Ob}(\mathcal{C}), B \in \text{Ob}(\mathcal{D})$, and as arrows pairs of morphisms (f, g) , where f is an arrow in \mathcal{C} and g is an arrow in \mathcal{D} . Composition and identities are defined pairwise: $(f, g) \circ (h, k) = (f \circ h, g \circ k)$, and $\text{id}_{(A, B)} = (\text{id}_A, \text{id}_B)$.

2.3 Functors, Natural transformations and Beyond

A functor is a structure preserving map between categories.

Definition 2.3.1 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ is a map taking each object of $A \in \text{Ob}(\mathcal{C})$ to an object $F(A) \in \text{Ob}(\mathcal{D})$ and each arrow $f : A \rightarrow B$ of \mathcal{C} to a arrow $F(f) : F(A) \rightarrow F(B)$ of \mathcal{D} , such that, for all objects $A \in \text{Ob}(\mathcal{C})$ and composable arrows f and g of \mathcal{C} :

- $F(\text{id}_A) = \text{id}_{F(A)}$;
- $F(g \circ f) = F(g) \circ F(f)$.

In this case, \mathcal{C} is called *domain* and \mathcal{D} is called *codomain* of the functor F .

Once defined what is a functor, we can give a more rigorous definition of diagram. Although this may seem extremely technical, it will be useful, especially in the definition of limits (Definition 2.4.11).

Definition 2.3.2 (Diagram). A *limit in a category \mathcal{C} of shape \mathcal{I}* is a functor $D : \mathcal{I} \rightarrow \mathcal{C}$. The category \mathcal{I} can be considered as the category indexing the objects and the morphisms of \mathcal{C} shaped in \mathcal{I} .

Functors are often used to generalize some structural behaviour that constructions in categories have. An important example of this fact is the universal property. The definition is not straightforward, but it gives the abstraction of a property that will be useful in further definitions. [HS79]

Definition 2.3.3 (Universal property). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor, and let $B \in \text{Ob}(\mathcal{D})$. A pair (u, A) , with $A \in \text{Ob}(\mathcal{C})$ and $u : B \rightarrow F(A)$ is said to be an *universal map for B with respect to F* if, for each $A' \in \text{Ob}(\mathcal{C})$ and each $f : B \rightarrow F(A')$ there exists a unique morphism $h \in \mathcal{C}(A, A')$ such that the following triangle commute:

$$\begin{array}{ccc}
 B & \xrightarrow{u} & F(A) \\
 & \searrow f & \downarrow F(h) \\
 & & F(A')
 \end{array}
 \qquad
 \begin{array}{c}
 A \\
 \vdots h \\
 \downarrow \\
 A'
 \end{array}$$

– i.e. there exists a unique h such that $F(h) \circ u = f$. In this case, (u, A) is said to have the *universal property*.

Dually, if $G : \mathcal{C} \rightarrow \mathcal{D}$ is a functor and $B \in \text{Ob}(\mathcal{D})$, then a pair (A, u) is a *co-universal map for B with respect to G* if $u : G(A) \rightarrow B$ and for each $A' \in \text{Ob}(\mathcal{C})$ and each $f : G(A') \rightarrow B$ there exists a unique morphism $h \in \mathcal{C}(A', A)$ such that the following diagram commutes:

$$\begin{array}{ccc}
 A' & & G(A') \\
 \vdots h & & \downarrow G(h) \\
 \downarrow & & \downarrow \\
 A & & G(A) \xrightarrow{u} B
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & \searrow f \\
 & & B
 \end{array}$$

Given two functors which share domain and codomain categories, it is possible to define a transformation between them, taking each object of the domain of the functors to an arrow in the codomain of the functors that represent the action of "changing the functor acting on that object".

Definition 2.3.4 (Natural transformation). Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be two functors. A *natural transformation* η between them, denoted $\eta : F \rightarrow G$ is a function $\eta : \text{Ob}(\mathcal{C}) \rightarrow \text{Hom}(\mathcal{D})$ taking each $A \in \text{Ob}(\mathcal{C})$ to a morphism $\eta_A : F(A) \rightarrow G(A)$ in \mathcal{D} , such that,

for each morphism $f : A \rightarrow B$ of \mathcal{C} , the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ F(f) \downarrow & & \downarrow G(f) \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array}$$

– i.e., such that $G(f) \circ \eta_A = \eta_B \circ F(f)$.

We say that $\eta : F \rightarrowtail G$ is a *natural isomorphism* if, for each $A \in \text{Ob}(\mathcal{C})$, η_A is an isomorphism in \mathcal{D} . In this case, F and G are said to be *naturally isomorphic*, and is denoted $F \cong G$.

Observation 2.3.5. It is easy to see that, given two natural transformation $\eta : F \rightarrowtail G$, $\theta : G \rightarrowtail H$, it is possible to compose them obtaining a new natural transformation $\xi = \theta \circ \eta : F \rightarrowtail H$. This follows by the fact that the diagram

$$\begin{array}{ccccc} F(A) & \xrightarrow{\eta_A} & G(A) & \xrightarrow{\theta_A} & H(A) \\ F(f) \downarrow & & \downarrow G(f) & & \downarrow H(f) \\ F(B) & \xrightarrow{\eta_B} & G(B) & \xrightarrow{\theta_B} & H(B) \end{array}$$

commutes because the two inner squares do. Sticking another diagram on the right of the one above, it is even possible to show associativity of composition of natural transformations.

The latter observation shows that natural transformations recreate on the functors the same structure that morphisms in a category have on objects. This led us to define a particular kind of category, in which objects are functors between two fixed categories, and arrow are natural transformations.

Definition 2.3.6 (Functor Category). Let \mathcal{C} and \mathcal{D} be two categories. The category whose objects are functors between \mathcal{C} and \mathcal{D} and whose arrows are natural transformation between them is said to be a *functor category*, and it is denoted by $[\mathcal{C}, \mathcal{D}]$.

A functor with **Set** as codomain is said to be a *presheaf* on that category. Given a category \mathcal{C} , it is possible to construct the functor category of the presheaves on \mathcal{C} , i.e. $[\mathcal{C}, \mathbf{Set}]$.

Remark 2.3.7. What we are calling here a presheaf is not totally accurate, because technically a presheaf on a small category \mathcal{C} is a functor $F : \mathcal{C}^{op} \rightarrow \mathbf{Set}$. This technicality would bring more complexity, and it is far beyond the scope of this works, so we will continue adopting the definition we give above.

[introduction to comma categories]

Definition 2.3.8 (Comma category). Given two functors $F : \mathcal{C} \rightarrow \mathcal{E}$, $G : \mathcal{D} \rightarrow \mathcal{E}$, the *comma category* $(F \downarrow G)$ is the category whose objects are triples (A, f, B) , with $A \in \text{Ob}(\mathcal{C})$, $B \in \text{Ob}(\mathcal{D})$ and $f \in \mathcal{E}(F(A), G(B))$, and whose morphisms are the pairs $(a, b) : (A, f, B) \rightarrow (C, g, D)$ where $a : A \rightarrow C$, $b : B \rightarrow D$ and such that

$$\begin{array}{ccc} F(A) & \xrightarrow{f} & G(B) \\ F(a) \downarrow & & \downarrow G(b) \\ F(C) & \xrightarrow{g} & G(D) \end{array}$$

commutes; composition of morphisms is obtained via pairwise composition, i.e., $(a, b) \circ (c, d) = (a \circ c, b \circ d)$.

2.4 Universal Constructions

[prodotti, coprodotti, equalizzatori, coequalizzatori, limiti, colimiti]

The next definitions are about *universal constructions*. The simplest are the notion of initial and, dually, terminal objects.

Definition 2.4.1 (Initial and terminal objects). An object A of a category \mathcal{C} is said to be *initial* if, for each other object B of \mathcal{C} , there exists a unique morphism from A to B . Dually, an object Z is said to be a *terminal object* in a category \mathcal{C} if, for any other object X of \mathcal{C} , there exists a unique morphism from X to Z . An initial object of a category is indicated by the symbol $\mathbf{0}$, and a terminal object is indicated by the symbol $\mathbf{1}$.

Observation 2.4.2. It make sense to refer to an initial (and terminal) object as *the* initial (*the* terminal) object. Suppose indeed that $\mathbf{0}$ and $\mathbf{0}'$ are two distinct initial objects of a category \mathcal{C} . Then, there exists a unique morphism from $\mathbf{0}$ to $\mathbf{0}'$, say f . At the same way, it must exists a unique morphism from $\mathbf{0}'$ to $\mathbf{0}$, say g . Then, $g \circ f$ must be exactly $id_{\mathbf{0}}$, and $f \circ g = id_{\mathbf{0}'}$, and then they are isomorphic. The same argument works for the terminal object. The reason why this argument works is because initial and terminal objects have the **universal property** (Definition 2.3.3), [Integrare qui!]

We now illustrate a result on functor categories (Definition 2.3.6) that will be useful later.

Proposition 2.4.3. *Let \mathcal{D} be a category. If \mathcal{D} has an initial object, then, for any category \mathcal{C} , $[\mathcal{C}, \mathcal{D}]$ has an initial object. If \mathcal{D} has a terminal object, then, for any category \mathcal{C} , $[\mathcal{C}, \mathcal{D}]$ has a terminal object.*

Proof. Let $\mathbf{0}_{\mathcal{D}}$ be the initial object of \mathcal{D} , and consider the constant functor $I(f) = id_{\mathbf{0}_{\mathcal{D}}}$ for all $f \in Hom(\mathcal{C})$. Then, for any $G : \mathcal{C} \rightarrow \mathcal{D}$, $\eta : I \rightarrow G$, defining η_A as the *unique morphism from $\mathbf{0}_{\mathcal{D}}$ to $G(A)$* for each $A \in Ob(\mathcal{C})$, is a natural transformation $I \rightarrow G$, as the diagram below shows:

$$\begin{array}{ccc} I(A) = \mathbf{0}_{\mathcal{D}} & \xrightarrow{\eta_A} & G(A) \\ \downarrow I(f) = id_{\mathbf{0}_{\mathcal{D}}} & & \downarrow G(f) \\ I(A') = \mathbf{0}_{\mathcal{D}} & \xrightarrow{\eta_{A'}} & G(A') \end{array}$$

for each $f : A \rightarrow A'$, the square above must commute, since there is only one morphism from $\mathbf{0}_{\mathcal{D}}$ to $G(A')$. For the same reason, η is the only natural transformation from I to G , being indeed the initial object of $[\mathcal{C}, \mathcal{D}]$.

The proof for the terminal object is similar, defining $T(f) = id_{\mathbf{1}_{\mathcal{D}}}$ for each $f \in Hom(\mathcal{C})$. Then, $\theta : F \rightarrow T$, for any $F : \mathcal{C} \rightarrow \mathcal{D}$, defining θ_A as the *unique morphism from $F(A)$ to $\mathbf{1}_{\mathcal{D}}$* is a natural transformation due to the commutativity of the following diagram for each $f : A \rightarrow A'$:

$$\begin{array}{ccc} F(A) & \xrightarrow{\theta_A} & G(A) = \mathbf{1}_{\mathcal{D}} \\ \downarrow F(f) & & \downarrow G(f) = id_{\mathbf{1}_{\mathcal{D}}} \\ F(A') & \xrightarrow{\theta_{A'}} & G(A') = \mathbf{1}_{\mathcal{D}} \end{array}$$

Hence, θ is the unique natural transformation from F to T , and T is the terminal object of $[\mathcal{C}, \mathcal{D}]$. \square

In particular, every presheaf has an initial and a terminal object.

More complex constructions are products (and, dually, coproducts).

Definition 2.4.4 (Product). A *product* of two objects A and B is an object $A \times B$ together with two *projection arrows* $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ such that, for every object C and pair of arrows $f : C \rightarrow A$, $g : C \rightarrow B$, there is exactly one arrow $\langle f, g \rangle : C \rightarrow A \times B$ making the diagram

$$\begin{array}{ccccc} & & C & & \\ & \swarrow f & \vdots \langle f, g \rangle & \searrow g & \\ A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \end{array}$$

commute – i.e., such that $\pi_1 \circ \langle f, g \rangle = f$ and $\pi_2 \circ \langle f, g \rangle = g$.

Definition 2.4.5 (Coproduct). The dual of the product is the *coproduct*. A coproduct of two objects A and B is an object $A + B$ together with two arrows $\iota_1 : A \rightarrow A + B$, $\iota_2 : B \rightarrow A + B$ such that, for every object C and pair of arrows $f : A \rightarrow C$, $g : B \rightarrow C$, there is a unique arrow $[f, g] : A + B \rightarrow C$ such that the diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{\iota_1} & A + B & \xleftarrow{\iota_2} & B \\
 & \searrow f & \downarrow [f, g] & \swarrow g & \\
 & & C & &
 \end{array}$$

commutes – i.e., such that $[f, g] \circ \iota_1 = f$ and $[f, g] \circ \iota_2 = g$.

Both the notion of product and coproduct can be easily generalized, extending the definition to the product (and coproduct) of a family of objects, together with appropriate arrows (e.g., the projection arrow for each object in the product). We will denote the product of a collection of objects indexed by a (finite) category \mathcal{I} as $(\prod_{i \in \text{Ob}(\mathcal{I})} X_i, (\pi_i)_{i \in \text{Ob}(\mathcal{I})})$, and the coproduct as $((\iota_i)_{i \in \text{Ob}(\mathcal{I})}, \coprod_{i \in \text{Ob}(\mathcal{I})} X_i)$.

Again, the definition of this constructions is divided into two parts: one stating what the construction is, and another stating that the construction owns the universal property.

The next notion is about a construction that make parallel arrows (i.e., two possibly distinct morphisms with the same domain and codomain) equal. In **Set**, given two functions $f, g : A \rightarrow B$, such construction corresponds exactly to the restriction of the domain to the set of elements on which f and g are equal. Specifically, if we take the set $E = \{x \in A \mid f(x) = g(x)\}$, and we take the function $e : E \rightarrow A$ defined by $e(x) = x$, we obtain what is called an *equalizer* for f and g . In fact, e is that function that make f and g be the same function – i.e., $f \circ e = g \circ e$. This concept can be generalized as it follows.

Definition 2.4.6 (Equalizer and Coequalizer). Let $f, g : A \rightarrow B$ be two arrows of a category \mathcal{C} . An *equalizer* for f and g is pair (E, e) , where E is an object and $e : E \rightarrow A$ is an arrow in \mathcal{C} such that:

1. $f \circ e = g \circ e$; and
2. if (E', e') is another pair that satisfies 1, then there exists a unique $h : E' \rightarrow E$ such that $e \circ h = e'$.

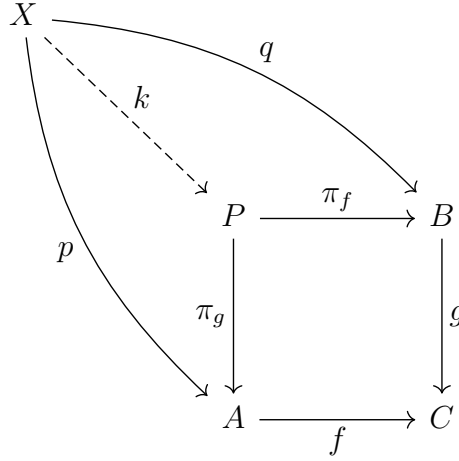
$$\begin{array}{ccccc}
 E & \xrightarrow{e} & A & \xrightleftharpoons[f]{g} & B \\
 & \nearrow e' & \uparrow k & & \\
 E' & & & &
 \end{array}$$

A *coequalizer* of f and g , dually, is a pair (c, C) , where C is an object and $c : B \rightarrow C$ such that $c \circ f = c \circ g$, with the universal property.

Another (pair of) construction(s) one can find in a category, given two arrows, is the pullback (and the pushout).

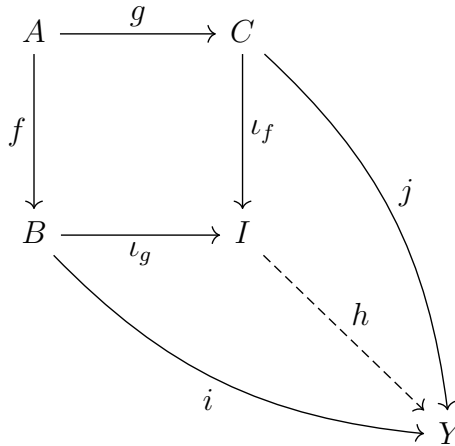
Definition 2.4.7 (Pullback, Pushout). A *pullback* of two arrows $f : A \rightarrow C$ and $g : B \rightarrow C$ is a triple (P, π_g, π_f) containing an object P and a pair of arrows $\pi_g : P \rightarrow A$, $\pi_f : P \rightarrow B$ such that:

1. $f \circ \pi_g = g \circ \pi_f$; and
2. if (X, p, q) , where $p : X \rightarrow A$ and $q : X \rightarrow B$, satisfies 1, then there is a unique $k : X \rightarrow P$ such that $p = \pi_g \circ k$ and $q = \pi_f \circ k$.



Dually, the *pushout* of two arrows $f : A \rightarrow B$ and $g : A \rightarrow C$ is a triple (ι_g, ι_f, I) , where I is an object and $\iota_g : B \rightarrow I$ and $\iota_f : C \rightarrow I$ are morphisms such that:

1. $\iota_g \circ f = \iota_f \circ g$; and
2. if (i, j, Y) , where $i : B \rightarrow Y$ and $j : C \rightarrow Y$, satisfies 1, then there is a unique $h : I \rightarrow Y$ such that $i = h \circ \iota_g$ and $j = h \circ \iota_f$.



Pullbacks (and, dually pushouts) are construction slightly more general than products and equalizers. An intuition of what they represent is given by considering what is concretely a pullback in the category of sets.

Example 2.4.8. In **Set**, given two functions $f : A \rightarrow C$ and $g : B \rightarrow C$, a pullback of f and g exists and is exactly the set $P = \{(x, y) \in A \times B \mid f(x) = g(y)\}$, with $\pi_f : P \rightarrow B$ and $\pi_g : P \rightarrow C$ defined, respectively, by $\pi_f((x, y)) = y$ and $\pi_g((x, y)) = x$. In this way, we have then, $\forall (x, y) \in P$:

$$\begin{aligned}
 (f \circ \pi_g)((x, y)) &= f(\pi_g((x, y))) \\
 &= f(x) && \text{Definition of } \pi_g \\
 &= g(y) && (x, y) \in P \\
 &= g(\pi_f((x, y))) && \text{Definition of } \pi_f \\
 &= (g \circ \pi_f)((x, y))
 \end{aligned}$$

thus, $f \circ \pi_g = g \circ \pi_f$.

We now introduce the notion that generalize all the universal constructions defined above. In fact, initial objects, products, equalizers and pullback (dually, terminal objects, coproducts, coequalizers and pushouts) can be seen as the particular case of a certain type of construction, called limit.

To define what a limit is, we first need to define cones

Definition 2.4.9 (Cones). Let $D : \mathcal{I} \rightarrow \mathcal{C}$ be a diagram in \mathcal{C} of shape \mathcal{I} . A *cone* for D is an object X of \mathcal{C} , together with arrows $f_i : X \rightarrow D(i)$ indexed by \mathcal{I} (i.e. one for each object i of \mathcal{I}), such that, for each morphism $g : D(i) \rightarrow D(j)$ of \mathcal{I} , the following diagram commutes:

$$\begin{array}{ccc}
 & X & \\
 f_i \swarrow & & \searrow f_j \\
 D(i) & \xrightarrow{g} & D(j)
 \end{array}$$

– i.e., $D(g) \circ f_i = f_j$. We denote such cone as $\{f_i : X \rightarrow D(i)\}$.

Observation 2.4.10. Given a diagram D , the category of the cones for D , denoted $\mathbf{Cone}(D)$ is defined to have cones for D as objects and cone morphisms as arrows, where a cone morphism $\phi : C \rightarrow C'$ from a cone $C = \{f_i : X \rightarrow D(i)\}$ to $C' = \{f'_i : X' \rightarrow D(i)\}$ is a morphism $\phi : X \rightarrow X'$ such that the following diagram commutes

for each i :

$$\begin{array}{ccc}
 X & \xrightarrow{\phi} & X' \\
 & \searrow f_i & \swarrow f'_i \\
 & D(i) &
 \end{array}$$

Definition 2.4.11 (Limits). Let $D : \mathcal{I} \rightarrow \mathcal{C}$ be a diagram in \mathcal{C} of shape \mathcal{I} . A cone $\{f_i : X \rightarrow D(i)\}$ is a *limit* provided that, for any other cone $\{f'_i : X' \rightarrow D(i)\}$ for D , then there exists a unique morphism $k : X' \rightarrow X$ such that the following diagram commutes for each object i of \mathcal{I} :

$$\begin{array}{ccc}
 X' & \xrightarrow{\quad k \quad} & X \\
 & \searrow f'_i & \swarrow f_i \\
 & D(i) &
 \end{array}$$

– i.e., $f_i \circ k = f'_i$ for each object i of \mathcal{I} .

Observation 2.4.12. Given a diagram D , a limit for D is exactly the terminal object of the category $\mathbf{Cone}(D)$, defined in Observation 2.4.10.

The following examples show how limits are general concepts for the constructions defined above.

Example 2.4.13. Let D be the empty diagram in the category \mathcal{C} . Then, a cone for D is any object of \mathcal{C} , while a limit is the terminal object of \mathcal{C} .

Example 2.4.14. Let D be the following diagram:

$$A \qquad B$$

Then, a cone for D is an object X and two arrows $f : X \rightarrow A$, $g : X \rightarrow B$:

$$A \xleftarrow{f} X \xrightarrow{g} B$$

A limit for D is, if it exists, the product of A and B .

Example 2.4.15. A pullback is the limit for the diagram:

$$\begin{array}{ccc}
 & B & \\
 & \downarrow g & \\
 A & \xrightarrow{f} & C
 \end{array}$$

In fact, a cone for the diagram above is an object P and three arrows $\phi : P \rightarrow A$, $\psi : P \rightarrow B$ and $h : P \rightarrow C$, but the latter is uniquely determined by the other ones ($f \circ \phi = h = g \circ \psi$). Thus, the following diagram is a cone:

$$\begin{array}{ccc} P & \xrightarrow{\psi} & B \\ \phi \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

For (P, ϕ, ψ) to be a pullback, it must have the universal property. In other words, it has to be a limit.

Example 2.4.16. A limit for the diagram below is an equalizer for f and g .

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

The connection between constructions as products and equalizers and limits is made clear by the following theorem. The idea behind the proof is the fact that, given a diagram $D : \mathcal{J} \rightarrow \mathcal{C}$, if each subset of objects $X = \{D(i) \mid i \in \text{Ob}(\mathcal{J})\} \subseteq \text{Ob}(\mathcal{C})$ has a product $(\prod_{i \in I} D(i), (\pi_i)_{i \in \text{Ob}(\mathcal{J})})$ and each pair of arrows $f, g \in \mathcal{C}(D(i), D(j))$ has an equalizer $Eq(f, g)$, then one can construct the cone taking the equalizer of the arrows that has as domain the product of the objects of the diagram, and as codomain the product of the codomains of the arrows of the diagram. This construction has the universal property because equalizers and product do.

Theorem 2.4.17 (Limit theorem). *Let \mathcal{C} be a category. Then \mathcal{C} has all finite limits if and only if \mathcal{C} has all finite products and all finite equalizers.*

Proof. Let $D : \mathcal{J} \rightarrow \mathcal{C}$ a diagram, with \mathcal{J} finite.

The *if* statement is easily satisfied as shown in Example 2.4.16 and Example 2.4.14.

To satisfy the *only if* statement, we want an object L together with morphisms $p_i : L \rightarrow D(i)$ such that:

1. $\{p_i : L \rightarrow D(i)\}$ is a cone – i.e., for each morphism of \mathcal{J} $\alpha : i \rightarrow j$, $D(\alpha) \circ p_i = p_j$; and
2. for each E and $q_i : E \rightarrow D(i)$ in \mathcal{C} , with $D(\alpha) \circ q_i = q_j$ for each $\alpha : i \rightarrow j$ of \mathcal{J} , there exists a unique $f : E \rightarrow L$ such that $q_i = p_i \circ f$ for each $i \in \text{Ob}(\mathcal{J})$.

Consider the two products (which exist by hypothesis) $\prod_{j \in \text{Ob}(\mathcal{J})} D(j)$, the product of the objects of the diagram, and $\prod_{\alpha \in \text{Hom}(\mathcal{J})} D(\text{cod } \alpha)$, the product of the

codomains of the morphisms in D , where π_x is the x -th projection of the product. Let now:

$$\gamma, \varepsilon : \prod_{j \in \text{Ob}(\mathcal{J})} D(j) \longrightarrow \prod_{\alpha \in \text{Hom}(\mathcal{J})} D(\text{cod } \alpha)$$

defined by $\gamma_\alpha = \pi_{D(\text{cod } \alpha)}$ (the projection on the codomain of α) and $\varepsilon_\alpha = D(\alpha) \circ \pi_{D(\text{dom } \alpha)}$. Let now $e : L \rightarrow \prod_{j \in \text{Ob}(\mathcal{J})} D(j)$ the equalizer of γ and ε (which exists by hypothesis), and, for each $j \in \text{Ob}(\mathcal{J})$, $p_j : L \rightarrow D(j)$, defined by $p_j = \pi_{D(j)} \circ e$.

What we want now is to show that $(L, (p_i)_{i \in \text{Ob}(\mathcal{J})})$ is the limit of D , namely, show that conditions given at the beginning are valid.

For point 1, we have to show that, for each $\alpha : i \rightarrow j$ of \mathcal{J} , we have $D(\alpha) \circ p_i = p_j$:

$$\begin{aligned} D(\alpha) \circ p_i &= D(\alpha) \circ \pi_{D(i)} \circ e && \text{Definition of } p_j \\ &= \varepsilon_\alpha \circ e && \text{Definition of } \varepsilon \\ &= \gamma_\alpha \circ e && e \text{ is an equalizer of } \pi, \varepsilon \\ &= \pi_{D(j)} \circ e && \text{Definition of } \pi \\ &= p_j && \text{Definition of } p_j \end{aligned}$$

For point 2, suppose that $(E, (q_i)_{i \in \text{Ob}(\mathcal{J})})$ has the properties stated. then, by definition of product, there exists a (unique) arrow $q : E \rightarrow \prod_{j \in \text{Ob}(\mathcal{J})} D(j)$. For each arrow $\alpha : i \rightarrow j$, we have:

$$\begin{aligned} \gamma_\alpha \circ q &= \pi_{D(j)} \circ q && \text{Definition of } \pi \\ &= q_j && \text{Definition of } q_j \\ &= D(\alpha) \circ q_i && \text{Assumption on } q_j \\ &= D(\alpha) \circ \pi_{D(i)} \circ q && \text{Definition of } q_i \\ &= \varepsilon_\alpha \circ q && \text{Definition of } \varepsilon \end{aligned}$$

Since e equalizes π and ε , there exists a unique $f : E \rightarrow L$ in \mathcal{C} such that $q = e \circ f$. Then, for each $j \in \text{Ob}(\mathcal{J})$, we have $\pi_{D(j)} \circ q = \pi_{D(j)} \circ e \circ f$, hence, $q_j = p_j \circ f$. \square

The dual notions of cones and limits are that of cocones and colimits.

Definition 2.4.18. (Cocones, Colimits) A *cocone* for a diagram $D : \mathcal{J} \rightarrow \mathcal{C}$ is an object Y of \mathcal{C} together with arrows $f_i : D(i) \rightarrow Y$ such that, for each $g : D(i) \rightarrow D(j)$ of \mathcal{C} , $f_j \circ g = f_i$. A cocone is denoted $\{f_i : D(i) \rightarrow Y\}$. A *colimit* for D is a cocone $C = \{f_i : D(i) \rightarrow Y\}$ with the universal property – i.e., if $C' = \{f'_i : D(i) \rightarrow Y'\}$ is another cone for D , then there exists a unique arrow $h : Y \rightarrow Y'$ such that, for each i , $h \circ f_i = f'_i$.

Of all examples done about limits are still valid, in their dual form, for what regard colimits. Infact, initial objects, coequalizer, coproducts and pullbacks are

particular cases of limits, and the way to see this fact is the same seen in previous examples. The limit theorem is still valid, considering coequalizers and coproducts, and the proof is much the same.

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