Chapter 1

Introduction

[TODO]

Chapter 2

Background

[INTRO]

2.1 Categories

Definition 2.1.1 (Category)**.** A *category* $\mathscr{C}$ comprises:

1. A collection of *objects* $Ob(\mathscr{C})$;
2. A collection of *arrows* (or *morphisms*) $Hom(\mathscr{C})$.

For each morphism $f \in Hom(\mathscr{C})$, there are two operators *dom* and *cod* which map every morphism to two objects, respectively, its *domain* and its *codomain.* In case $dom\ f = A$ and $cod\ f = B$, we will write $f : A \to B$. The collection of all morphisms from an object A to an object B is denoted as $\mathscr{C}(A, B)$. An operator $\circ$ of *composition* maps every couple of morphisms f, g with $cod\ f = dom\ g$ (in this case f and g are said to be composable) to a morphism $g \circ f : dom\ f \to cod\ g$. The composition operator is associative, i.e., for each composable arrows f, g and h, it holds that

$$h \circ (g \circ f) = (h \circ g) \circ f$$

For each object A, an *identity* morphism $id_A : A \to A$ (or, when it is clear from the context, just denoted A) such that, for each $f : A \to B$:

$$id_B \circ f = f = f \circ id_A$$

The most important thing here is not the structure of the objects, but instead how this structure is preserved by the morphisms.

To represent morphisms of a category $\mathscr{C}$ it is possible to use **diagrams**, as the one below, in wich the vertices are objects of $\mathscr{C}$, and the edges are morphisms of $\mathscr{C}$.

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\
W \arrow[r, "f"] & Y
\end{tikzcd}$$

The diagram is said to commute whenever $f \circ g' = g \circ f'$.

Example 2.1.2. It is easy to see that taking sets as objects and total functions as arrows, we obtain a category. In fact, given two functions $f : A \to B$ and $g : B \to C$, it is possible to compose them obtaining an arrow $g \circ f : A \to C$, and the composition is associative. Moreover, for each set A there exists an identity function $id_A : A \to A$ such that $\forall a \in A : id_A(a) = a$. This category is usually denoted as **Set**.

Remark 2.1.3. It is important to note that the Definition 2.1.1 above does not specify what kind of collections are, for a category $\mathscr{C}$, $Ob(\mathscr{C})$ and $Hom(\mathscr{C})$. Taking **Set** as example, the collection $Ob(\mathbf{Set})$ cannot be a set itself, due to Russel's paradox. It would be more appropriate referring to a category $\mathscr{C}$ which $Ob(\mathscr{C})$ and $Hom(\mathscr{C})$ are both sets as a *small category*, but it is assumed in this work, except where it is made explicit, for a category to be small.

Between all morphisms of a category, it is possible to distinguish some that have certain properties, as well as functions between sets can be surjective, injective or bijective.

Definition 2.1.4 (Monomorphism)**.** An arrow $f : B \to C$ in a category $\mathscr{C}$ is a *monomorphism* if, for any pair of arrows of $\mathscr{C}$ $g : A \to B$, $h : A \to B$, the equality $f \circ g = f \circ h$ implies $g = h$.

Definition 2.1.5 (Epimorphism)**.** An arrow $f : A \to B$ in a category $\mathscr{C}$ is an *epimorphism* if, for any pair of arrows of $\mathscr{C}$ $g : B \to C$, $h : B \to C$, the equality $g \circ f = h \circ f$ implies $g = h$.

Definition 2.1.6 (Isomorphism)**.** An arrow $f : A \to B$ is an *isomorphism* if there is an arrow $f^{-1} : B \to A$, called the *inverse* of f, such that $f^{-1} \circ f = id_A$ and $f \circ f^{-1} = id_B$. Two objects are said to be *isomorphic* if there is an isomorphism between them.

Example 2.1.7. In **Set**, monomorphisms are injective functions, epimorphism are surjective functions and ismorphisms are bijections.

[TODO: Capire quanto andare nel dettaglio so questo, in particolare che epi + mono non implica iso (dipende da se mi serve)]

[TODO: Capire che esempi fare!]

2.2 Categories from other categories

Starting from a category $\mathscr{C}$, it is possible to construct some other categories with interesting property. An example is the *dual category* of a category $\mathscr{C}$, denoted $\mathscr{C}^{op}$, in which the objects are the same of $\mathscr{C}$, and the arrows are the opposite of the arrows in $\mathscr{C}$, i.e., if $f : A \to B$ is an arrow of $\mathscr{C}$, then $f : B \to A$ is an arrow of $\mathscr{C}^{op}$. Each definition in category theory has a dual form. In general, if a statement S is true in a category $\mathscr{C}$, then the opposite of the statement, S^{op}, obtained switching the words "domain" and "codomain" and replacing each composite $g \circ f$ into $f \circ g$, is still true in the category $\mathscr{C}^{op}$. Moreover, since every category is the opposite of its opposite, if a statement S is true for every category, then S^{op} is also true for every category [Pie91].

Another important notion is that of subcategory.

Definition 2.2.1 (Subcategory)**.** A category $\mathscr{D}$ is a *subcategory* of a category $\mathscr{C}$ if:

1. each object of $\mathscr{D}$ is an object of $\mathscr{C}$;

2. each morphism between two objects of $\mathscr{D}$ is a morphism of $\mathscr{C}$; and

3. composites and identities of $\mathscr{D}$ are the same of $\mathscr{C}$

If the inclusion at 2 is an equality (i.e. $\mathscr{D}(A, B) = \mathscr{C}(A, B)$ for each couple of objects A, B of $\mathscr{D}$), then $\mathscr{D}$ is said to be a *full subcategory* of $\mathscr{C}$.

An object of a category can raise a category itself. This is the case of slice (and coslice) categories.

Definition 2.2.2 (Slice Category)**.** Given a category $\mathscr{C}$ and an object $X \in Ob(\mathscr{C})$, the *slice category* $\mathscr{C}/X$ is the category that has pairs (A, f) as objects, where A is an object of C and $f : A \to X$ is an arrow in $\mathscr{C}$, and arrows $\phi : (A, f) \to (B, g)$ is given by a morphism $\phi : A \to B$ of $\mathscr{C}$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, "f"'] & B \arrow[d, "g"] \\ & X \end{tikzcd}$$

– i.e, $g \circ \phi = f$. Composition between two arrows in $\mathscr{C}/X$ $\phi : (A, f) \to (B, g)$ and

$\psi : (B, g) \to (C, h)$ is the arrow $\psi \circ \phi : (A, f) \to (C, h)$ obtained in the obvious way:

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[rr, bend left, "\psi\circ\phi"] \arrow[dr, "f"'] & B \arrow[r, "\psi"] \arrow[d, "g"] & C \arrow[dl, "h"] \\
& X &
\end{tikzcd}$$

The dual definition is *coslice category*, noted $X/\mathscr{C}$ (where $X \in Ob(\mathscr{C})$), obtained by taking as objects all the morhisms of $\mathscr{C}$ with domain X and as arrows the morphisms $\phi : (A, f) \to (B, g)$ such that $f : X \to A, g : B \to X$ of $\mathscr{C}$ such that $g = \phi \circ f$.

Moreover, it is possible to raise a new category from two old ones by taking their product, as the following definition shows.

Definition 2.2.3 (Product category)**.** Given two categories $\mathscr{C}, \mathscr{D}$, the *product category* $\mathscr{C} \times \mathscr{D}$ has as objects pairs of objects (A, B), where $A \in Ob(\mathscr{C}), B \in Ob(\mathscr{D})$, and as arrows pairs of arrows (f, g), where f is a morphism in $\mathscr{C}$ and g is a morphism in $\mathscr{D}$. Composition and identities are defined pairwise: $(f, g) \circ (h, k) = (f \circ h, g \circ k)$, and $id_{(A,B)} = (id_A, id_B)$.

2.3 Functors, Natural transformations and beyond

A functor is a structure preserving map between categories.

Definition 2.3.1 (Functor)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *functor* $F : \mathscr{C} \to \mathscr{D}$ is a map taking each object of $A \in Ob(\mathscr{C})$ to an object $F(A) \in Ob(\mathscr{D})$ and each arrow $f : A \to B$ of $\mathscr{C}$ to a arrow $F(f) : F(A) \to F(B)$ of $\mathscr{D}$, such that, for all objects $A \in Ob(\mathscr{C})$ and composable arrows f and g of $\mathscr{C}$:

- $F(id_A) = id_{F(A)}$;
- $F(g \circ f) = F(g) \circ F(f)$.

In this case, $\mathscr{C}$ is called *domain* and $\mathscr{D}$ is called *codomain* of the functor F.

Functor are often used to generalize some structural behaviour that constructions in categories have. An important example of this fact is the universal property. The definition is not straightforward, but it gives the abstraction of a property that will be useful in further definitions. [HS79]

Definition 2.3.2 (Universal property)**.** Let $F : \mathscr{C} \to \mathscr{D}$ be a functor, and let $B \in Ob(\mathscr{D})$. A pair (u, A), with $A \in Ob(\mathscr{C})$ and $u : B \to F(A)$ is said to be an *universal* *map for B with respect to F* if, for each $A' \in Ob(\mathscr{C})$ and each $f : B \to F(A')$ there exists a unique morphism $h \in \mathscr{C}(A, A')$ such that the following triangle commute:

$$\begin{tikzcd}
B \arrow[r, "u"] \arrow[dr, "f"'] & F(A) \arrow[d, dashed, "F(h)"] & A \arrow[d, dashed, "h"] \\
& F(A') & A'
\end{tikzcd}$$

– i.e. there exists a unique h such that $F(h) \circ u = f$. In this case, (u, A) is said to have the *universal property.*

Dually, if $G : \mathscr{C} \to \mathscr{D}$ is a functor and $B \in Ob(\mathscr{D})$, then a pair (A, u) is a *co-universal map for B with respect to G* if $u : G(A) \to B$ and for each $A' \in Ob(\mathscr{C})$ and each $f : G(A') \to B$ there exists a unique morphism $h \in \mathscr{C}(A', A)$ such that the following diagram commutes:

$$\begin{tikzcd}
A' \arrow[d, dashed, "h"'] & G(A') \arrow[d, dashed, "G(h)"'] \arrow[dr, "f"] & \\
A & G(A) \arrow[r, "u"] & B
\end{tikzcd}$$

Given two functors which share domain and codimain categories, it is possible to define a transformation between them, taking each object of the domain of the functors to an arrow in the codomain of the functors that represent the action of "changing the functor acting on that object".

Definition 2.3.3 (Natural transformation)**.** Let $F, G : \mathscr{C} \to \mathscr{D}$ be two functors. A *natural transformation* η between them, denoted $\eta : F \dot{\to} G$ is a function $\eta : Ob(\mathscr{C}) \to Hom(\mathscr{D})$ taking each $A \in Ob(\mathscr{C})$ to a morphism $\eta_A : F(A) \to G(A)$ in $\mathscr{D}$, such that, for each morphism $f : A \to B$ of $\mathscr{C}$, the following diagram commutes:

$$\begin{tikzcd}
F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\eta_B"'] & G(B)
\end{tikzcd}$$

– i.e., such that $G(f) \circ \eta_A = \eta_B \circ F(f)$.

We say that $\eta : F \dot{\to} G$ is a *natural isomorphism* if, for each $A \in Ob(\mathscr{C})$, η_A is an isomorphism in $\mathscr{D}$. In this case, F and G are said to be *naturally isomorphic*, and is denoted $F \cong G$.

Observation 2.3.4. It is easy to see that, given two natural transformation $\eta : F \dot{\to} G$, $\theta : G \dot{\to} H$, it is possible to compose them obtaining a new natural transformation $\xi = \theta \circ \eta : F \dot{\to} H$. This follows by the fact that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[r, "\theta_A"] \arrow[d, "G(f)"] & H(A) \arrow[d, "H(f)"] \\ F(B) \arrow[r, "\eta_B"'] & G(B) \arrow[r, "\theta_B"'] & H(B) \end{tikzcd}$$

commutes because the two inner squares do. Sticking another diagram on the right of the one above, it is even possible to show associativity of composition of natural transformations.

The latter observation shows that natural transformations recreate on the functors the same structure that morphisms in a category have on objects. This led us to define a particular kind of category, in which objects are functors between two fixed categories, and arrow are natural transformations.

Definition 2.3.5 (Functor Category)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. The category whose objects are functors between $\mathscr{C}$ and $\mathscr{D}$ and whose arrows are natural transformation between them is said to be a *functor category*, and it is denoted by $[\mathscr{C}, \mathscr{D}]$.

A functor with **Set** as codomain is said to be a *presheaf* on that category. Given a category $\mathscr{C}$, it is possible to construct the functor category of the presheaves on $\mathscr{C}$, i.e. $[\mathscr{C}, \mathbf{Set}]$.

Remark 2.3.6. What we are calling here a presheaf is not totally accurate, because technically a presheaf on a small category $\mathscr{C}$ is a functor $F : \mathscr{C}^{op} \to \mathbf{Set}$. This technicality would bring more complexity, and it is far beyond the scope of this works, so we will continue adopting the definition we give above.

[introduction to comma categories]

Definition 2.3.7 (Comma category)**.** Given two functors $F : \mathscr{C} \to \mathscr{E}$, $G : \mathscr{D} \to \mathscr{E}$, the *comma category* $(F \downarrow G)$ is the category whose objects are triples (A, f, B), with $A \in Ob(\mathscr{C})$, $B \in Ob(\mathscr{D})$ and $f \in \mathscr{E}(F(A), G(B))$, and whose morphisms are the pairs $(a, b) : (A, f, B) \to (C, g, D)$ where $a : A \to C$, $b : B \to D$ and such that

$$\begin{tikzcd} F(A) \arrow[r, "f"] \arrow[d, "F(a)"'] & G(B) \arrow[d, "G(b)"] \\ F(C) \arrow[r, "g"] & G(B) \end{tikzcd}$$

commutes; composition of morphisms is obtained via pairwise composition, i.e., $(a, b) \circ (c, d) = (a \circ c, b \circ d)$.

2.4 Universal Constructions

[prodotti, coprodotti, equalizzatori, coequalizzatori, limiti, colimiti]

The next definitions are about *universal constructions*. The simplest are the notion of initial and, dually, terminal objects.

Definition 2.4.1 (Initial and terminal objects)**.** An object A of a category $\mathscr{C}$ is said to be *initial* if, for each other object B of $\mathscr{C}$, there exists a unique morphism from A to B. Dually, an object Z is said to be a terminal object in a category $\mathscr{C}$ if, for any other object X of $\mathscr{C}$, there exists a unique morphism from X to Z. An initial object of a category is indicated by the symbol $\mathbf{0}$, and a terminal object is indicated by the symbol $\mathbb{1}$

Observation 2.4.2. It make sense to refer to an initial (and terminal) object as *the* initial (*the* terminal) object. Suppose indeed that $\mathbf{0}$ and $\mathbf{0}'$ are two distinct initial objects of a category $\mathscr{C}$. Then, there exists a unique morphism from $\mathbf{0}$ to $\mathbf{0}'$, say f. At the same way, it must exists a unique morphism from $\mathbf{0}'$ to $\mathbf{0}$, say g. Then, $g \circ f$ must be exactly $id_{\mathbf{0}}$, and $f \circ g = id_{\mathbf{0}'}$, and then they are isomorphic. The same argument works for the terminal object. The reason why this argument works is because initial and terminal objects have the **universal property** (Definition 2.3.2), [Integrare qui!]

More complex constructions are products (and, dually, coproducts)

Definition 2.4.3 (Product)**.** A *product* of two objects A an B is an object $A \times B$ together with two *projection arrows* $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that, for every object C and pair of arrows $f : C \to A$, $g : C \to B$, there is exactly one arrow $\langle f, g\rangle : c \to A \times B$ making the diagram

$$\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g\rangle"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

commute – i.e., such that $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$.

Definition 2.4.4 (Coproduct)**.** The dual of the product is the *coproduct*. A coproduct of two objects A and B is an object $A + B$ together with two arrows $\iota_1 : A \to A + B$, $\iota_2 : B \to A + B$ such that, for every object C and pair of arrows

$f : A \to C$, $g : B \to C$, there is a unique arrow $[f, g] : A + B \to C$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\iota_1"] \arrow[dr, "f"'] & A + B \arrow[d, dashed, "{[f,g]}"] & B \arrow[l, "\iota_2"'] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

commutes – i.e., such that $[f, g] \circ \iota_1 = f$ and $[f, g] \circ \iota_2 = g$.

Both the notion of product and coproduct can be easily generalized, extending the definition to the product (and coproduct) of a family of objects, together with appropriate arrows (e.g., the projection arrow for each object in the product).

Again, the definition of this constructions is divided into two parts: one stating what the construction is, and another stating that the construction owns the universal property.

Bibliography

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