Contents

Introduction

This work discusses the adhesivity of e-graphs, a data structure introduced in [**?**], giving them a structure of category and then working on its algebraic properties.

The first chapter is about basic concepts of category theory, such as categories, limits, constructions and adhesivity, that will be used to model the categories on which in the second chapter we will work on. The second chapter is entirely focused on graphs, starting from a general notion of graphs, that is, a two sorted algebra consisting of a set of edges and a set of vertices and two operators, source and target, which assign to each edge a source vertex and a target one. The structure we will use to model graphs are presheaves, which not only provides a straightforward formalization of them but also makes in evidence particular properties proper of graphs. Then, we endow graphs with an equivalence relation among verices, to end up with an adhesivity result. The last part is about e-graphs, that are nothing but graphs with a certain type of equivalence relation, inheriting then almost all the properties of the category of graphs with equivalences.

citations

Chapter 1

Background

In this chapter the building blocks for this work, almost entirely based on categories, will be defined. The aim of what follows is not only to introduce concepts that will be used later, but also to understand how category theory is general enough to give the abstraction of known notions (mainly from set theory) to reuse them in different contexts. This is not a complete tutorial on categories, but instead a sufficient compendium of definitions to make clear what will be done in the next chapters.

1.1 Basic Notions

This section is all about basic definitions and examples, to get familiar with the formalism of categories.

1.1.1 Categories

Definition 1.1.1 (Category). A *category* $\mathscr{C}$ comprises:

1. A collection of *objects* $\mathcal{O}b(\mathscr{C})$;

2. A collection of *arrows* (or *morphisms*) $\mathcal{H}om(\mathscr{C})$, often called *homset*.

Two operators, *dom* and *cod*, that map every morphism $f \in \mathcal{H}om(\mathscr{C})$ to two objects, respectively, its *domain* and its *codomain*. In case $dom\ f = A$ and $cod\ f = B$, we will write $f : A \to B$. The collection of morphisms from an object A to an object B is denoted as $\mathscr{C}(A, B)$. An operator $\circ$ of *composition* maps every couple of morphisms f, g with $cod\ f = dom\ g$ (in this case f and g are said to be

composable) to a morphism $g \circ f : dom\ f \to cod\ g$. The composition operator is associative, i.e., for each composable arrows f, g and h, it holds that

$$h \circ (g \circ f) = (h \circ g) \circ f$$

For each object A, an *identity* morphism $id_A : A \to A$ (or, when it is clear from the context, just denoted A) such that, for each $f : A \to B$:

$$id_B \circ f = f = f \circ id_A$$

The most important thing here is not the structure of the objects, but instead how this structure is preserved by the morphisms.

Example 1.1.2. A trivial example of category is the one with no objects, and hence no morphisms. Such category is denoted by **0** and is called *empty category.*

Example 1.1.3. The category with just one object and just one arrow, the identity arrow on that object, is denoted **1**. In particular, the only object of this category is •, and the only arrow is $id_\bullet$.

Given an arrow $f A \to B$ in a category $\mathscr{C}$, we say that f *factors through* $g : C \to B$ if there exists an arrow $h : A \to C$ such that $f = h \circ g$.

Definition 1.1.4. [Dual Category] Given a category $\mathscr{C}$, there exist a category $\mathscr{C}^{op}$ such that:

- $Ob(\mathscr{C}^{op}) = Ob(\mathscr{C})$;

- if $f : A \to B$ is a morphism in $\mathscr{C}$, then $f : B \to A$ is a morphism in $\mathscr{C}^{op}$.

Hence, given $f : A \to B$ and $g : B \to C$ arrows in $\mathscr{C}$, as $g \circ f : A \to C$ is an arrow in $\mathscr{C}$, then $f \circ g : C \to A$ is an arrow in $\mathscr{C}^{op}$. Such category is called *dual category* or *opposite category.*

Duality is a concept that we will encounter most of the time. Given a property P valid for a category $\mathscr{C}$, we will refer to the same property in the opposite category $\mathscr{C}^{op}$ as the *dual* of P, without explicitly constructing $\mathscr{C}^{op}$. There exist some properties that coincide exactly with their dual, and such properties are said to be *self dual* properties.

To represent morphisms of a category $\mathscr{C}$ it is possible to use *diagrams*, as the one below, in which the vertices are objects of $\mathscr{C}$, and the edges are morphisms of $\mathscr{C}$.

$$
\begin{array}{ccc}
X & \xrightarrow{\ f'\ } & Z \\
\downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\
W & \xrightarrow{\ f\ } & Y
\end{array}
$$

The diagram is said to commute whenever $f \circ g' = g \circ f'$. Unique morphisms are represented with dashed arrows. A more rigorous definition of what a diagram is will be given later (Definition 1.2.3).

Example 1.1.5. It is easy to see that taking sets as objects and (total) functions as arrows, we obtain a category. In fact, given two functions $f : A \to B$ and $g : B \to C$, it is possible to compose them obtaining an arrow $g \circ f : A \to C$, and the composition is associative. For each set A there exists an identity function $id_A : A \to A$ such that $id_A(a) = a$ for each $a \in A$. This category is denoted as **Set**.

Remark 1.1.6. It is important to note that the Definition 1.1.1 above does not specify what kind of collections $\mathcal{O}b(\mathscr{C})$ and $\mathcal{H}om(\mathscr{C})$ are. Taking **Set** as example, the collection $\mathcal{O}b(\textbf{Set})$ cannot be a set itself, due to Russel's paradox. It would be more appropriate referring to a category $\mathscr{C}$ which $\mathcal{O}b(\mathscr{C})$ and $\mathcal{H}om(\mathscr{C})$ are both sets as a *small category*, but it is assumed in this work, except where it is made explicit, for a category to be small. Another clarification must to be done, still considering **Set**. Given two sets A and B, it is possible to construct the set B^A of all functions from A to B. This is isomorphic to $\textbf{Set}(A, B)$, for each pair of sets A and B. A category $\mathscr{C}$ where, for each pair of objects A and B, $\mathscr{C}(A, B)$ is a set is said to be *locally small*.

1.1.2 Mono, Epi and Iso

Between the morphisms of a category, it is possible to distinguish some that have certain properties, as functions between sets can be surjective, injective or bijective.

Definition 1.1.7 (Monomorphism)**.** An arrow $f : B \to C$ in a category $\mathscr{C}$ is a *monomorphism* if, for any pair of arrows of $\mathscr{C}$ $g : A \to B$, $h : A \to B$, the equality $f \circ g = f \circ h$ implies $g = h$. The class of monomorphisms of $\mathscr{C}$ is denoted $\mathcal{M}ono(\mathscr{C})$.

Remark 1.1.8. For a morphism, from an algebraic point of view, being mono means being *left cancelable*. Let A be an object in a category $\mathscr{C}$. Given two monomorphism $m : X \to A$ and $n : Y \to A$, then if $h : X \to Y$ is a morphism such that $m = n \circ h$, then is the unique one: suppose k is another morphism such that $m = n \circ k$. We can conclude $h = k$ observing that $n \circ h = n \circ k$ implies $h = k$ when n is mono, which is by hypothesis.

Definition 1.1.9 (Subobject)**.** Starting from this consideration, we can define a preorder on monomorphisms, placing $m \leq n$ if $m = n \circ h$ for some h. Such preorder induces an equivalence relation $\equiv$ on monomorphisms with codomain A, where $m \equiv n$ whenever $m \leq n$ and $n \leq m$, and the corresponding equivalence class is called *subobject* of A.

Definition 1.1.10 (Epimorphism)**.** An arrow $f : A \to B$ in a category $\mathscr{C}$ is an *epimorphism* if, for any pair of arrows of $\mathscr{C}$ $g : B \to C$, $h : B \to C$, the equality $g \circ f = h \circ f$ implies $g = h$.

Definition 1.1.11 (Isomorphism)**.** An arrow $f : A \to B$ is an *isomorphism* if there is an arrow $f^{-1} : B \to A$, called the *inverse* of f, such that $f^{-1} \circ f = id_A$ and $f \circ f^{-1} = id_B$. Two objects are said to be *isomorphic* if there is an isomorphism between them.

Example 1.1.12. In **Set**, monomorphisms are injective functions, epimorphisms are surjective functions and isomorphisms are bijections.

Remark 1.1.13. Mono and epi are dual concepts. This fact is easily shown by considering how a monomorphism m in a category $\mathscr{C}$ behaves in the dual category $\mathscr{C}^{op}$. In $\mathscr{C}$ we have that $m \circ f = m \circ g$ implies $f = g$. In $\mathscr{C}^{op}$, the we can state that $f \circ m = g \circ m$ implies $f = g$, obtaining the definition of epi.

Proposition 1.1.14. *The following statements hold for every pair of composable arrows f and g for any category $\mathscr{C}$:*

1. *if both f and g are mono, then $g \circ f$ is mono;*

2. *if $g \circ f$ is mono, then f is mono;*

3. if both f and g are epi, then $g \circ f$ is epi;

4. if $g \circ f$ is epi, then g is epi.

Observation 1.1.15. Consider two monomorphisms $m : X \to A$ and $n : Y \to A$, and suppose $m \equiv n$ (in the sense of Definition 1.1.9). This corresponds to having a isomorphism between X and Y. in fact, since $m \leq n$, there exists s unique $h : Y \to X$ such that $m = n \circ h$, and, since $n \leq m$, there exists a unique $k : X \to Y$ such that $n = m \circ k$. But then

$$m = n \circ h \qquad\qquad n = m \circ k$$
$$= m \circ h \circ k \qquad\qquad = n \circ h \circ k$$

Since m is mono, we obtain $id_X = h \circ k$, and, since n is mono, $id_Y = k \circ h$, thus, an isomorphism.

The next proposition will be useful later.

Proposition 1.1.16. *In* **Set***, for every commutative square as the one below, if $e : X \to Y$ is epi and $m : M \to Z$ is mono, then there exists a unique morphism $h : Y \to M$ making the whole diagram below commutative.*

Proof. Let $y \in Y$. Since e is epi (i.e., surjective), there exists $x \in X$ such that $e(x) = y$. Then, we put $h(y) = f(x)$. h is well defined, in fact, let $x' \in X$ be such that $e(x') = y$. In this case, we have $h(x) = f(x')$, and so

$$(m \circ f)(x') = (g \circ e)(x')$$
$$= g(y)$$
$$= (g \circ e)(x)$$
$$= (m \circ f)(x)$$

since m is mono by hypothesis, $f(x) = f(x')$. $\qquad\square$

Corollary 1.1.17. *La fattorizzazione di un morfismo è unica a meno di isomorfismo*

Scrivilo bene (l'isomorfismo dovrà rispettare certe coerenze) e dimostralo

1.1.3   Categories from other categories

Starting from a category, it is possible to construct other categories with some interesting properties, as the following examples show.

The first notion to introduce is the one of subcategory.

Definition 1.1.18 (Subcategory). A category $\mathscr{D}$ is a *subcategory* of a category $\mathscr{C}$ if:

1. each object of $\mathscr{D}$ is an object of $\mathscr{C}$;

2. each morphism between two objects of $\mathscr{D}$ is a morphism of $\mathscr{C}$; and

3. composites and identities of $\mathscr{D}$ are the same of $\mathscr{C}$.

If the inclusion at 2 is an equality (i.e. $\mathscr{D}(A, B) = \mathscr{C}(A, B)$ for each couple of objects A, B of $\mathscr{D}$), then $\mathscr{D}$ is said to be a *full subcategory* of $\mathscr{C}$. Another way to express that composites are the same (point 3) is to say that if $f, g \in \mathcal{H}om(\mathscr{D})$ are composable, then $g \circ f \in \mathcal{H}om(\mathscr{D})$, i.e., $\mathcal{H}om(\mathscr{D})$ is *closed under composition*.

An object of a category marks out a category itself. This is the case of slice (and coslice) categories.

Definition 1.1.19 (Slice Category). Given a category $\mathscr{C}$ and an object $X \in \mathcal{O}b(\mathscr{C})$, the *slice category* $\mathscr{C}/X$ is the category that has pairs (A, f) as objects, where A is an object of C and $f : A \to X$ is an arrow in $\mathscr{C}$, and arrows $\phi : (A, f) \to (B, g)$ are given by a morphism $\phi : A \to B$ of $\mathscr{C}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ \phi\ \ } & B \\
& {\scriptstyle f}\searrow & \downarrow {\scriptstyle g} \\
& & X
\end{array}
$$

– i.e, $g \circ \phi = f$. Composition between two arrows in $\mathscr{C}/X$ $\phi : (A, f) \to (B, g)$ and $\psi : (B, g) \to (C, h)$ is the arrow $\psi \circ \phi : (A, f) \to$

(C, h) obtained in the obvious way:
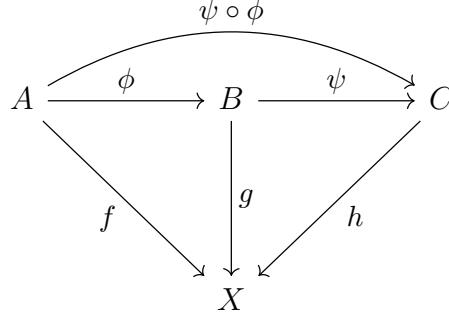
The dual definition of *coslice category*, noted $X/\mathscr{C}$ (where $X \in \mathcal{O}b(\mathscr{C})$), is obtained by taking as objects the morhisms of $\mathscr{C}$ with domain X and as arrows the morphisms $\phi : (A, f) \to (B, g)$ such that $f : X \to A, g : B \to X$ of $\mathscr{C}$ and $g = \phi \circ f$.

Furthermore, it is possible to raise a new category from two old ones by taking their product, as the following definition shows.

Definition 1.1.20 (Product category)**.** Given two categories $\mathscr{C}, \mathscr{D}$, the *product category* $\mathscr{C} \times \mathscr{D}$ has as objects pairs of objects (A, B), where $A \in \mathcal{O}b(\mathscr{C}), B \in \mathcal{O}b(\mathscr{D})$, and as arrows pairs of morphisms (f, g), where f is an arrow in $\mathscr{C}$ and g is an arrow in $\mathscr{D}$. Composition and identities are defined pairwise: $(f, g) \circ (h, k) = (f \circ h, g \circ k)$, and $id_{(A,B)} = (id_A, id_B)$.

1.2 Functors, Natural Transformations, Adjoints

1.2.1 Functors

A functor is a structure preserving map between categories.

Definition 1.2.1 (Functor)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *functor* $F : \mathscr{C} \to \mathscr{D}$ is a map taking each object of $A \in \mathcal{O}b(\mathscr{C})$ to an object $F(A) \in \mathcal{O}b(\mathscr{D})$ and each arrow $f : A \to B$ of $\mathscr{C}$ to a arrow $F(f) : F(A) \to F(B)$ of $\mathscr{D}$, such that, for all objects $A \in \mathcal{O}b(\mathscr{C})$ and composable arrows f and g of $\mathscr{C}$:

- $F(id_A) = id_{F(A)}$;

E cambierei anche l'intro a questa sezione DC

la 1.1 è un po' povera come sezione, se sposti questa sezione sui funtori prima, poi puoi arricchire la 1.1 con la sottosezione sulle comma

- $F(g \circ f) = F(g) \circ F(f)$.

In this case, $\mathscr{C}$ is called *domain* and $\mathscr{D}$ is called *codomain* of the functor F.

Example 1.2.2. A first example of functor is the *identity functor*. Given a category $\mathscr{C}$, the identity functor $Id_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ is the functor that maps each object on itself and each arrow onto itself.

Once defined what a functor is, we can give a more rigorous definition of diagram. Although this may seem extremely technical, it will be useful, especially in the definition of limits (Definition 1.3.3).

Definition 1.2.3 (Diagram). A *diagram in a category $\mathscr{C}$ of shape $\mathscr{I}$* is a functor $D : \mathscr{I} \to \mathscr{C}$. The category $\mathscr{I}$ can be considered as the category indexing the objects and the morphisms of $\mathscr{C}$ shaped in $\mathscr{I}$.

Example 1.2.4. A diagram of shape $\Lambda = (L \xleftarrow{l} X \xrightarrow{r} R)$ is said to be a *span*, and is denoted by $(l, X, r) : L \rightharpoonup R$. A span can be viewed as the generalization of relations between sets. In fact, in **Set**, a relation $R \subseteq A \times B$ is a span, with the projections $\pi_A : R \to A$ and $\pi_B : R \to B$ as arrows.

The dual notion of span is a *cospan*, namely, a diagram of shape $\Lambda^{op} = (L \xrightarrow{l} X \xleftarrow{r} R)$, and is denoted by $(l, X, r) : L \rightharpoonup R$.

Functor are often used to generalize some structural behaviour that constructions in categories have. An important example of this fact is the universal property. The definition is not straightforward, but it gives the abstraction of a property that will be useful in further definitions

Definition 1.2.5 (Universal property). Let $F : \mathscr{C} \to \mathscr{D}$ be a functor, and let $B \in \mathcal{O}b(\mathscr{D})$. A pair (u, A), with $A \in \mathcal{O}b(\mathscr{C})$ and $u : B \to F(A)$ is said to be an *universal map for B with respect to F* if for each $A' \in \mathcal{O}b(\mathscr{C})$ and each $f : B \to F(A')$ there exists a unique morphism $h \in \mathscr{C}(A, A')$ such that the following triangle commute:

$$
\begin{array}{ccccc}
B & \xrightarrow{\;\;u\;\;} & F(A) & & A \\
 & {\scriptstyle f}\searrow & \big\downarrow{\scriptstyle F(h)} & & \big\downarrow{\scriptstyle h} \\
 & & F(A') & & A'
\end{array}
$$

– i.e. there exists a unique h such that $F(h) \circ u = f$. In this case, (u, A) is said to have the *universal property*.

Dually, if $G : \mathscr{C} \to \mathscr{D}$ is a functor and $B \in \mathcal{O}b(\mathscr{D})$, then a pair (A, u) is a *co-universal map for B with respect to G* if $u : G(A) \to B$ and for each $A' \in \mathcal{O}b(\mathscr{C})$ and each $f : G(A') \to B$ there exists a unique morphism $h \in \mathscr{C}(A', A)$ such that the following diagram commutes:

Some interesting properties of certain functors depend strictly on how they behave on the homsets of the domain and the codomain categories. The following definitions are about this particular type of functors.

Definition 1.2.6 (Full functor, faithful functor, fully faithful functor)**.** Let $F : \mathscr{C} \to \mathscr{D}$ be a functor, and consider the inducted function

$$F_{A,B} : \mathscr{C}(A, B) \to \mathscr{D}(F(A), F(B))$$

If, for each A, B objects of $\mathscr{C}$, $F_{A,B}$ is surjective, then F is said to be *full*, if it is injective, F is said to be *faithful*, if it is both injective and surjective, F is said to be *fully faithful*.

Observation 1.2.7. Properties such as fullness and faithfulness are so called *self-dual*, because the dual notion coincide with the same notion. This fact can be advantageous because if for example the faithfulness implies the preservation of some property, then the dual property is implied at the same way.

Example 1.2.8. Let $\mathscr{C}$ be a category and $\mathscr{D}$ a subcategory. The inclusion functor $I : \mathscr{D} \to \mathscr{C}$, mapping each object and each arrow onto itself. I is a faithful functor, because, given any pair of objects A and B of $\mathscr{D}$, $I_{A,B}$ is injective. If $\mathscr{D}$ is a full subcategory, then I is fully faithful.

Having such classification among functors turns out to be useful in many contexts. For example, consider $F(m) : F(B) \to F(C)$ be a monomorphism in a category $\mathscr{D}$, where $F : \mathscr{C} \to \mathscr{D}$ is a faithful functor. Then, if $f, g : A \to B$ are two morphisms in $\mathscr{C}$ such that $m \circ f = m \circ g$, then $F(m \circ f) = F(m) \circ F(f) = F(m) \circ F(g) = F(m \circ g)$. Since $F(m)$ is mono, then $F(f) = F(g)$, and, since $F_{A,B}$ is injective, $f = g$. Together with the fact that faithfulness is a self-dual concept, we have a proof for what follows [HS79].

Proposition 1.2.9. *Let $F : \mathscr{C} \to \mathscr{D}$ be a faithful functor. Then F reflects monomorphisms and epimorphisms.*

1.2.2   Natural Transformations

Given two functors that share domain and codomain categories, it is possible to define a transformation between them, taking each object of the domain of the functors to an arrow in the codomain of the functors that represent the action of "changing the functor acting on that object".

Definition 1.2.10 (Natural transformation)**.** Let $F, G : \mathscr{C} \to \mathscr{D}$ be two functors. A *natural transformation* η between them, denoted $\eta : F \dot{\to} G$, is a function $\eta : \mathcal{O}b(\mathscr{C}) \to \mathcal{H}om(\mathscr{D})$ taking each $A \in \mathcal{O}b(\mathscr{C})$ to a morphism $\eta_A : F(A) \to G(A)$ in $\mathscr{D}$, such that, for each morphism $f : A \to B$ of $\mathscr{C}$, the following diagram commutes:

$$
\begin{array}{ccc}
F(A) & \xrightarrow{\ \eta_A\ } & G(A) \\
\downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\
F(B) & \xrightarrow[\ \eta_B\ ]{} & G(B)
\end{array}
$$

– i.e., such that $G(f) \circ \eta_A = \eta_B \circ F(f)$.

We say that $\eta : F \dot{\to} G$ is a *natural isomorphism* if, for each $A \in \mathcal{O}b(\mathscr{C})$, η_A is an isomorphism in $\mathscr{D}$. In this case, F and G are said to be *naturally isomorphic*, and is denoted $F \cong G$.

Observation 1.2.11. It is easy to see that, given two natural transformations $\eta : F \dot{\to} G$, $\theta : G \dot{\to} H$, it is possible to compose them obtaining a new natural transformation $\xi = \theta \circ \eta : F \dot{\to} H$. This follows

by the fact that the diagram

$$\begin{array}{ccccc} F(A) & \xrightarrow{\eta_A} & G(A) & \xrightarrow{\theta_A} & H(A) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} & & \downarrow{\scriptstyle H(f)} \\ F(B) & \xrightarrow{\eta_B} & G(B) & \xrightarrow{\theta_B} & H(B) \end{array}$$

commutes because the two inner squares do. Sticking another diagram on the right of the one above, it is possible to show associativity of composition of natural transformations.

1.2.3 Functor Categories

The Observation 1.2.11 shows that natural transformations recreate on the functors the same structure that morphisms in a category have on objects. This leads us to define a particular kind of category, in which objects are functors between two categories, and arrow are natural transformations.

Definition 1.2.12 (Functor Category). Let $\mathscr{C}$ and $\mathscr{D}$ be categories. The category whose objects are functors between $\mathscr{C}$ and $\mathscr{D}$ and whose arrows are natural transformations between them is said to be a *functor category*, and it is denoted by $[\mathscr{C}, \mathscr{D}]$.

Lemma 1.2.13. *Let $\mathscr{C}, \mathscr{D}, \mathscr{I}$ be categories. Then, it holds that*

$$[\mathscr{I}, [\mathscr{C}, \mathscr{D}]] \cong [\mathscr{I} \times \mathscr{C}, \mathscr{D}]$$

A functor with a small category as domain (Remark 1.1.6) and **Set** as codomain is said to be a *presheaf* on that category. Given a category $\mathscr{C}$, it is possible to construct the functor category of the presheaves on $\mathscr{C}$, i.e. $[\mathscr{C}, \textbf{Set}]$.

Remark 1.2.14. What we are calling here a presheaf is not totally accurate, because technically a presheaf on a small category $\mathscr{C}$ is a functor $F : \mathscr{C}^{op} \to \textbf{Set}$. This technicality would bring more complexity, and it is beyond the scope of this work, so we will continue adopting the definition given above.

1.2.4   Comma Categories

Functor constructions allow us to generalise basic concepts already seen for categories. An important example of this fact are comma categories, a more general notion of slice categories (Definition 1.1.19).

Definition 1.2.15 (Comma category). Let $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{E}$ be categories, and let $S : \mathscr{C} \to \mathscr{E}$, $T : \mathscr{D} \to \mathscr{E}$ be functors (source and target):

$$\mathscr{C} \xrightarrow{\quad S \quad} \mathscr{E} \xleftarrow{\quad T \quad} \mathscr{D}$$

Then, the *comma category* $(S \downarrow T)$ is the category in which:

- the objects are triples (A, f, B), where $A \in \mathcal{O}b(\mathscr{C})$, $B \in \mathcal{O}b(\mathscr{D})$ and $f : S(A) \to T(B)$ is an arrow of $\mathscr{E}$;

- the arrows are pairs $(c, d) : (A, f, B) \to (C, g, D)$, where $c \in \mathcal{H}om(\mathscr{C})$ and $d \in \mathcal{H}om(\mathscr{D})$, such that the square below commutes;

$$
\begin{array}{ccc}
S(A) & \xrightarrow{\quad f \quad} & G(B) \\
S(c) \Big\downarrow & & \Big\downarrow T(d) \\
T(C) & \xrightarrow{\quad g \quad} & T(B)
\end{array}
$$

- composition of morphisms is obtained via pairwise composition, i.e., $(a, b) \circ (c, d) = (a \circ c, b \circ d)$.

Thus, the slice category $\mathscr{C}/X$ is the comma category given by the two functors $Id_{\mathscr{C}}$ (the identity functor), and the functor $!_X : \mathbf{1} \to \mathscr{C}$, where $\mathbf{1}$ is the one-object category defined in Example 1.1.3, and $!_X$ sends the only object of $\mathbf{1}$ to X (then the only morphism of $\mathbf{1}$ to id_X of $\mathscr{C}$):

$$\mathscr{C} \xrightarrow{\quad Id_{\mathscr{C}} \quad} \mathscr{C} \xleftarrow{\quad !_X \quad} \mathbf{1}$$

It is easy to see that $(Id_{\mathscr{C}} \downarrow !_X)$ is exactly the same of $\mathscr{C}/X$.

In the same way, it is possible to define coslice categories in terms of comma categories: the category $(!_X \downarrow Id_{\mathscr{C}})$ is exactly the coslice $X/\mathscr{C}$.

1.2.5 Adjoints

Definition 1.2.16 (Right Adjoint)**.** Let $R : \mathscr{C} \to \mathscr{D}$ be a functor. R is said *right adjoint* if, for each object A of $\mathscr{D}$, there exists an object $L(A)$ and an arrow $\eta_A : A \to R(L(A))$ in $\mathscr{C}$ such that, for each arrow $f : A \to R(B)$ of $\mathscr{D}$, there is a unique arrow $g : L(A) \to B$ such that the following diagram commutes.

$$
\begin{array}{ccc}
A & \xrightarrow{\ \eta_A\ } & R(L(A)) \\
 & \searrow^{f} & \downarrow{\scriptstyle R(g)} \\
 & & R(B)
\end{array}
$$

–i.e., $R(g) \circ \eta_A = f$.

Proposition 1.2.17. *In Definition 1.2.16, the map that takes an object A to an object $L(A)$ can be extended to a functor $L : \mathscr{D} \to \mathscr{C}$. Moreover, there exists a natural transformation $id_{\mathscr{D}} \to R \circ L$.*

Proof. Let R be the right adjoint as in Definition 1.2.16. Given $f : X \to Y$, we can define $L(f)$ as the unique arrow $L(X) \to L(Y)$ whose image through R fits in the diagram below.

$$
\begin{array}{ccc}
X & \xrightarrow{\ \eta_X\ } & R(L(X)) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle R(L(f))} \\
Y & \xrightarrow[\ \eta_Y\ ]{} & R(L(Y))
\end{array}
$$

To see that in this way we get a functor it is now enough to notice the commutativity of the following diagrams.

$$
\begin{array}{ccc}
X & \xrightarrow{\ \eta_X\ } & R(L(X)) \\
{\scriptstyle id_X}\downarrow & & \downarrow{\scriptstyle R(id_{L(X)})} \\
Y & \xrightarrow[\ \eta_Y\ ]{} & R(L(Y))
\end{array}
$$

$$
\begin{array}{ccccc}
X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & Z \\
\downarrow{\scriptstyle\eta_X} & & \downarrow{\scriptstyle\eta_Y} & & \downarrow{\scriptstyle\eta_Z} \\
R(L(X)) & \xrightarrow[R(L(f))]{} & R(L(Y)) & \xrightarrow[R(L(g))]{} & R(L(Z))
\end{array}
$$

Finally, by construction the family given by all the $\eta_A \colon A \to R(L(A))$ is natural and we can conclude. $\qquad\square$

Remark 1.2.18. The family above mentioned is called *unit* of the adjunction.

Definition 1.2.19 (Left Adjoint)**.** Let $L : \mathscr{D} \to \mathscr{C}$ be a functor. L is a *left adjoint* if, for each object B of $\mathscr{C}$, there exists an object $R(B)$ and an arrow $\epsilon_B : L(R(B)) \to B$ in $\mathscr{D}$ such that, for each arrow $g : L(A) \to B$ of $\mathscr{C}$, there exists a unique arrow $f : A \to R(B)$ such that the following diagram commutes.

$$
\begin{array}{ccc}
L(R(B)) & \xrightarrow{\ \epsilon_B\ } & B \\
\uparrow{\scriptstyle L(f)} & \nearrow{\scriptstyle g} & \\
L(A) & &
\end{array}
$$

– i.e., $\epsilon_B \circ L(f) = g$.

As we have shown before, it is possible to extend the mapping $A \to R(B)$ to a functor R, whose functoriality follows placing $\epsilon_X \circ L(R(f)) = f \circ \epsilon_Y$ for each $f : X \to Y$. The family $\epsilon_B : L(R(B)) \to B$ is natural and it is called *counit* of the adjunction.

The connection between left and right adjoints is expressed in the following proposition.

Proposition 1.2.20. *Let L be the functor of Proposition 1.2.17. Then, L is a left adjoint.*

Proof. Given an object B in $\mathscr{C}$, we can consider the solid part of the diagram below. Since R is a right adjoint, we get a unique arrow

whose image through R make the triangle commutative.

$$R(B) \xrightarrow{\eta_{R(B)}} R(L(R(B)))$$

$$id_{R(B)} \qquad R(\epsilon_B)$$

$$R(B)$$

Let now A be an object of $\mathscr{D}$ and $g \colon L(A) \to B$ an arrow in $\mathscr{C}$. We can consider the composite $R(g) \circ \eta_A \colon A \to R(B)$. Then we have

$$
\begin{aligned}
R(\epsilon_B) \circ R(L(R(g))) \circ R(L(\eta_A)) \circ \eta_A &= R(\epsilon_B) \circ R(RL(R(g))) \circ \eta_{R(L(A))} \circ \eta_A \\
&= R(\epsilon_B) \circ \eta_{R(B)} \circ R(g) \circ \eta_A \\
&= R(g) \circ \eta_A
\end{aligned}
$$

Since R is a right adjoint and η its unit, it follows that $\epsilon_B \circ L(R(g) \circ \eta_A)$ coincides with g as wanted. $\qquad \square$

1.3 Limits and Universal Constructions

aggiungere i diagrammi e sistemare per l'indice

1.3.1 Limits and Colimits

Definition 1.3.1 (Cones)**.** Let $D : \mathscr{I} \to \mathscr{C}$ be a diagram in $\mathscr{C}$ of shape $\mathscr{I}$. A *cone* for D is an object X of $\mathscr{C}$, together with arrows $f_i : X \to D(i)$ indexed by $\mathscr{I}$ (i.e. one for each object i of $\mathscr{I}$), such that, for each morphism $\alpha : i \to j$ of $\mathscr{I}$, the following diagram commutes:

$$X$$

$$f_i \qquad f_j$$

$$D(i) \xrightarrow{\quad D(\alpha) \quad} D(j)$$

– i.e., $D(\alpha) \circ f_i = f_j$. We denote such cone as $\{f_i : X \to D(i)\}$.

Observation 1.3.2. Given a diagram D, the category of the cones for D, denoted $\textbf{Cone}(D)$, is defined to have cones for D as objects and cone morphisms as arrows, where a cone morphism $\phi : C \to C'$ from $C = \{f_i : X \to D(i)\}$ to $C' = \{f'_i : X' \to D(i)\}$ is a morphism $\phi : X \to X'$ such that the following diagram commutes for each i:

$$
\begin{array}{ccc}
X & \xrightarrow{\quad\phi\quad} & X' \\
& \searrow f_i \quad\quad f'_i \swarrow & \\
& D(i) &
\end{array}
$$

Definition 1.3.3 (Limits). Let $D : \mathscr{I} \to \mathscr{C}$ be a diagram in $\mathscr{C}$ of shape $\mathscr{I}$. A cone $\{f_i : X \to D(i)\}$ is a *limit* provided that, for any other cone $\{f'_i : X' \to D(i)\}$ for D, then there exists a unique morphism $k : X' \to X$ such that the following diagram commutes for each object i of $\mathscr{I}$:

$$
\begin{array}{ccc}
X' & \dashrightarrow{\quad k \quad} & X \\
& \searrow f'_i \quad\quad f_i \swarrow & \\
& D(i) &
\end{array}
$$

– i.e., $f_i \circ k = f'_i$ for each object i of $\mathscr{I}$. Such limit is denoted as $(X, f_i)_{i \in \mathscr{I}}$. Sometimes we will refer to the unique arrow k as the *mediating arrow*.

Observation 1.3.4. Given a diagram D, a limit for D is exactly the terminal object of the category $\textbf{Cone}(D)$, defined in Observation 1.3.2.

The dual notions of cones and limits are that of cocones and colimits.

Definition 1.3.5. (Cocones, Colimits) A *cocone* for a diagram $D : \mathscr{I} \to \mathscr{C}$ is an object Y of $\mathscr{C}$ together with arrows $f_i : D(i) \to Y$ such that, for each $g : D(i) \to D(j)$ of $\mathscr{C}$, $f_j \circ g = f_i$. A cocone is denoted $\{f_i : D(i) \to Y\}$. A *colimit* for D is a cocone $C = \{f_i : D(i) \to Y\}$ with the universal property – i.e., if $C' = \{f'_i : D(i) \to Y'\}$ is

another cone for D, then there exists a unique arrow $h : Y \to Y'$ such that, for each i, $h \circ f_i = f_i'$.

Remark 1.3.6. It makes sense to refer to a (co)limit as *the* (co)limit. Suppose $(P, p_i)_{i \in \mathscr{I}}$ and $(Q, q_i)_{i \in \mathscr{I}}$ be limits for a diagram $D : \mathscr{I} \to \mathscr{C}$. Then, there exists a unique morphism $h : Q \to P$ such that $p_i \circ k = q_i$ for each i. At the same way, there exists a unique morphisms $k : P \to Q$ such that $q_i \circ k = p_i$ for each i. From the existence of the identity, must be $k \circ h = id_Q$ and $h \circ k = id_P$, that is, P and Q are isomorphic.

Notion such limits and colimits are generalization of more particular cases that will be now introduced, that we will often call *universal constructions*.

Definition 1.3.7 (Initial Object, Terminal Object)**.** Consider the empty diagram (i.e., a diagram $D : \mathbf{0} \to \mathscr{C}$ where $\mathbf{0}$ is the empty category Example 1.1.2). Then, the limit of D is called *terminal object* and the colimit of D is called *initial object*, denoted, respectively, $\mathbb{1}_{\mathscr{C}}$ and $\mathbb{0}_{\mathscr{C}}$. (Subscripts are omitted when they are clear from the context).

Example 1.3.8. In **Set**, the initial object is the empty set $\varnothing$, because, for each set S, there exists the empty function from $\varnothing$ to S. The terminal object of **Set** is the singleton $\{\bullet\}$, because there is exactly one function from a set S to $\{\bullet\}$, namely, the function which sends each $s \in S$ to $\bullet$.

We now illustrate a result on functor categories (Definition 1.2.12) that will be useful later.

Proposition 1.3.9. *Let $\mathscr{D}$ be a category. If $\mathscr{D}$ has an initial object, then, for any category $\mathscr{C}$, $[\mathscr{C}, \mathscr{D}]$ has an initial object. If $\mathscr{D}$ has a terminal object, then, for any category $\mathscr{C}$, $[\mathscr{C}, \mathscr{D}]$ has a terminal object.*

Proof. Let $\mathbb{0}_{\mathscr{D}}$ be the initial object of $\mathscr{D}$, and consider the constant functor $I(f) = id_{\mathbb{0}_{\mathscr{D}}}$ for all $f \in \mathcal{H}om(\mathscr{C})$. Then, for any $G : \mathscr{C} \to \mathscr{D}$, $\eta : I \to G$, defining η_A as the *unique morphism from $\mathbb{0}_{\mathscr{D}}$ to $G(A)$* for each $A \in \mathcal{O}b(\mathscr{C})$, is a natural transformation $I \dashrightarrow G$, as the diagram

below shows:

$$I(A) = \mathbb{0}_{\mathscr{D}} \xrightarrow{\eta_A} G(A)$$

$$I(f) = id_{\mathbb{0}_{\mathscr{D}}} \Big\downarrow \qquad\qquad \Big\downarrow G(f)$$

$$I(A') = \mathbb{0}_{\mathscr{D}} \xrightarrow{\eta_{A'}} G(A')$$

for each $f : A \to A'$, the square above must commute, since there is only one morphism from $\mathbb{0}_{\mathscr{D}}$ to $G(A')$. For the same reason, η is the only natural transformation from I to G, being indeed the initial object of $[\mathscr{C}, \mathscr{D}]$.

Defining $T(f) = id_{\mathbb{1}_{\mathscr{D}}}$ for each $f \in \mathcal{H}om(\mathscr{C})$. Then, $\theta : F \to T$, for any $F : \mathscr{C} \to \mathscr{D}$, defining θ_A as the *unique morphism from $F(A)$ to $\mathbb{1}_{\mathscr{D}}$* is a natural transformation due to the commutativity of the following diagram for each $f : A \to A'$:

$$F(A) \xrightarrow{\theta_A} T(A) = \mathbb{1}_{\mathscr{D}}$$

$$F(f) \Big\downarrow \qquad\qquad \Big\downarrow T(f) = id_{\mathbb{1}_{\mathscr{D}}}$$

$$F(A') \xrightarrow{\theta_{A'}} T(A') = \mathbb{1}_{\mathscr{D}}$$

Hence, θ is the unique natural transformation from F to T, and T is the terminal object of $[\mathscr{C}, \mathscr{D}]$. $\qquad\qquad\square$

In particular, every presheaf has an initial and a terminal object, because **Set** does (Example 1.3.8).

Definition 1.3.10 (Product, Coproduct)**.** Let D be the following diagram:

$$A \qquad\qquad\qquad B$$

Then, a cone for D is an object X and two arrows $f : X \to A$, $g : X \to B$ (i.e., a span, defined in Example 1.2.4):

$$A \xleftarrow{\quad f \quad} X \xrightarrow{\quad g \quad} B$$

If it exists, a limit for D is called *product* of A and B, usually denoted as $(A \times B, \pi_A, \pi_B)$, while whose arrows are called *projections*. The

colimit of D is called *coproduct* of A and B, usually denoted as $(\iota_A, \iota_B, A + B)$.

Example 1.3.11. Set has both products and coproduts. Given two sets A and B, the categorical product is the set-theoretic cartesian product $A \times B$, together with the two projections π_A and π_B, while the coproduct is the disjoint sum $A \amalg B = \{(x, 0) \mid x \in A\} \cup \{(y, 1) \mid y \in B\}$, together with the two canonical injections ι_A and ιB, where $\iota_A(a) = (a, 0)$ and $\iota_B(b) = (b, 1)$.

The notions of product and coproduct can be easily generalized, extending the definition to the product (and coproduct) of a family of objects, together with appropriate arrows (e.g., the projection arrows for each object in the product). We will denote the product of a collection of objects indexed by a (finite) category $\mathscr{I}$ as $\left(\prod_{i \in \mathcal{O}b(\mathscr{I})} X_i, (\pi_i)_{i \in \mathcal{O}b(\mathscr{I})} \right)$, and the coproduct as $\left((\iota_i)_{i \in \mathcal{O}b(\mathscr{I})}, \coprod_{i \in \mathcal{O}b(\mathscr{I})} X_i \right)$.

Definition 1.3.12 (Equalizer, Coequalizer)**.** Let D be the diagram below.

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

The limit of D is called *equalizer*, and its colimit is called *coequalizer*.

Proposition 1.3.13. *Let $e : E \to A$ be the arrow that equalizes $f, g : A \to B$ in a category $\mathscr{C}$. Then, e is a monomorphism.*

Proof. Suppose X be an object and $x, y : X \to E$ be two morphisms in $\mathscr{C}$ such that $e \circ x = e \circ y$, and let $z = e \circ x$. Then, since e is an equalizer, $f \circ e = g \circ e$, and $f \circ z = g \circ z$. But, for the universal property of limits, there must be exactly one $u : Z \to E$ such that $z = e \circ u$. It follow that $x = u$ and $y = u$, hence $x = y$. $\square$

The dual of the proposition above states that a coequalizer is an epimorphism.

Of all monomorphisms, an interesting subclass of them is the one that contains only the equalizers.

Definition 1.3.14 (Regular Monomorphism, Regular Epimorphism)**.** A monomorphism that is an equalizer for a pair of arrows is said *regular monomorphism*. The class of all regular monomorphisms of a category $\mathscr{C}$ is denoted $\mathcal{R}eg(\mathscr{C})$. An epimorphism that is a coequalizer for a pair of arrows is said *regular epimorphism*.

Definition 1.3.15 (Pullback, Pushout)**.** Let D be the cospan (f, C, g) : $A \to B$. A cone for D is an object P and three arrows $\phi : P \to A$, $\psi : P \to B$, and $h : P \to C$, but the latter is uniquely determined by the other ones ($f \circ \phi = h = g \circ \psi$). Thus, the following diagram is a cone:

$$
\begin{array}{ccc}
P & \xrightarrow{\ \psi\ } & B \\
\phi \downarrow & & \downarrow g \\
A & \xrightarrow{\ f\ } & C
\end{array}
$$

Then, the limit of D is called *pullback* of f and g. Given a span $S = (l, X, r) : L \rightharpoonup R$, shown in the diagram below,

$$
L \xleftarrow{\ \ l\ \ } X \xrightarrow{\ \ r\ \ } R
$$

a cocone for S is any commutative square of the form

$$
\begin{array}{ccc}
 & C & \\
f \nearrow & & \nwarrow g \\
L \xleftarrow{\ \ l\ \ } & X & \xrightarrow{\ \ r\ \ } R
\end{array}
$$

(the morphism $X \to C$ is uniquely determined by the relation $f \circ l = g \circ r$). The colimit for S is called *pushout* of l and r.

Example 1.3.16. In **Set**, given two functions $f : A \to C$ and $g : B \to C$, a pullback of f and g exists and is exactly the set $P = \{(x, y) \in A \times B \mid f(x) = g(y)\}$, with $\pi_f : P \to B$ and $\pi_g : P \to C$ defined, respectively, by $\pi_f((x, y)) = y$ and $\pi_g((x, y)) = x$. In this way, we have then, $\forall (x, y) \in P$:

$$
\begin{aligned}
(f \circ \pi_g)((x, y)) &= f(\pi_g((x, y))) & \\
&= f(x) & \text{\textit{Definition of} } \pi_g \\
&= g(y) & (x, y) \in P \\
&= g(\pi_f((x, y))) & \text{\textit{Definition of} } \pi_f \\
&= (g \circ \pi_f)((x, y)) &
\end{aligned}
$$

thus, $f \circ \pi_g = g \circ \pi_f$.

Another important example to our aims is a concrete definition of what is a pushout in the category of sets, and why morally we can regard a pushout as *the way to identify part of an object with a part of another* [BW95].

Example 1.3.17. In **Set**, given two functions $f : A \to B$ and $g : A \to C$, the pushout of them is the set $X = (B \amalg C)/_\sim$, where $\sim$ is the least equivalence relation such that $f(a) \sim g(a)$ for each $a \in A$, with $\iota_g : B \to X$ and $\iota_f : C \to X$ as arrows, sending each element of the domain in the corresponding equivalence class in X. In particular, for each $a \in A$:

$$
\begin{aligned}
(\iota_g \circ f)(a) &= \iota_g(f(a)) \\
&= [(f(a), 0)] && \textit{Definition of } \iota_g \\
&= [(g(a), 1)] && f(a) \sim g(a) \\
&= \iota_f(g(a)) && \textit{Definition of } \iota_f \\
&= (\iota_f \circ g)(a)
\end{aligned}
$$

When both f and g are monos (that is, injections), then we can construct the pushout in the same way we have done above, with $(f(a), 0) \sim (g(a), 1)$ when such a exists and $(b, 0) \sim (c, 1)$ on each b and c with no preimage in A, with ι_f and ι_g injective. An easy way to see this fact is considering the following situation: let $f : A \to A \cup B$ and $g : A \to A \cup C$, with A disjoint from B and C, $f(a) = a$ and $g(a) = a$. Then the pushout is the object $A \cup B \cup C$, with the inclusions as arrows, that are also injective. A more general case is what happens considering functions $f : A \to B$ and $g : A \to C$ injective. Differently from the previous example, in this case is not possible to take just the union of codomains as the pushout, but rather the disjoint union of them and then identify the elements $f(a)$ with $g(a)$, as we have done above. In the category of sets and functions, we have the certainty that the pullback arrows are injective. In fact, taking the equivalence relation $\sim$, we have that $f(a) \sim f(a')$ if and only if $a = a'$ by hypothesis, and then $x \sim x'$ if and only if $x = x'$, then the pushout morphisms sends each element in an equivalence class composed only by the element itself, thus are injective. This is an interesting property that in other categories may do not hold, and will be recalled later.

Given a subclass of morphisms of a category, an important property is *stability* under certain type of constructions. In our case, we

are interested in stability under pullbacks and under pushouts.

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} \\
C & \xrightarrow{\ g\ } & D
\end{array}
\qquad (*)
$$

Definition 1.3.18 (Stability under pullbacks, pushouts)**.** Given a category $\mathscr{C}$, a subclass $\mathcal{A} \subseteq \mathcal{H}om(\mathscr{C})$ is said to be *stable under pullbacks* if, for every pullback square as the one in $(*)$, if $n \in \mathcal{A}$, then $m \in \mathcal{A}$. $\mathcal{A}$ is said to be *stable under pushouts* if, for every pushout square as the one in $(*)$, if $m \in \mathcal{A}$, then $n \in \mathcal{A}$.

Proposition 1.3.19. *Let $f : A \to C$, $g : B \to C$ be arrows in any category $\mathscr{C}$, and consider the following pullback square:*

$$
\begin{array}{ccc}
P & \xrightarrow{\ \pi_f\ } & B \\
\downarrow{\scriptstyle \pi_g} & & \downarrow{\scriptstyle g} \\
A & \xrightarrow{\ f\ } & C
\end{array}
$$

If g is mono, then so is π_g.

The proposition above can be dualised stating that pushouts preserves epimorphisms.

The following lemma is a classical result, its proof is in the appendix.

Lemma 1.3.20 (Pullback Lemma)**.** *Suppose that the following diagram is given and its right half is a pullback. Then the whole rectangle is a pullback if and only if its left half is a pullback.*

$$
\begin{array}{ccccc}
X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & Z \\
\downarrow{\scriptstyle t} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle h} \\
A & \xrightarrow{\ a\ } & B & \xrightarrow{\ b\ } & C
\end{array}
$$

Corollary 1.3.21. *Let $\mathscr{C}$ be a category and suppose that the solid part of the following cube is given*

If the front face is a pullback then there is a unique $q' \colon Y' \to B'$ filling the diagram. If, moreover, the other two vertical faces are also pullbacks, then the following square is a pullback too.

Proof. Let us compute:

$$c \circ r' \circ g' = r \circ z \circ g'$$
$$= r \circ g \circ y$$
$$= k \circ q \circ y$$

Since the front face is a pullback, this guarantees the existence of q'. The second half of the thesis follows applying Lemma 1.3.20 to the following rectangle.

$\square$

The connection between constructions as products and equalizers and limits is made clear by the following theorem. The idea behind the proof is the fact that, given a diagram $D : \mathscr{I} \to \mathscr{C}$, if each subset of objects $X = \{D(i) \mid i \in \mathcal{O}b(\mathscr{I})\} \subseteq \mathcal{O}b(\mathscr{C})$ has a product $(\prod_{i \in I} D(i), (\pi_i)_{i \in \mathcal{O}b(\mathscr{I})})$ and each pair of arrows $f, g \in \mathscr{C}(D(i), D(j))$ has an equalizer $Eq(f, g)$, then one can construct the cone taking the equalizer of the arrows that has as domain the product of the objects of the diagram, and as codomain the product of the codomains of the arrows of the diagram. This construction has the universal property because equalizers and products do. A detailed proof is in the appendix.

Theorem 1.3.22 (Limit theorem). *Let $\mathscr{C}$ be a category. Then $\mathscr{C}$ has all finite limits if and only if $\mathscr{C}$ has all finite products and all finite equalizers.*

Remark 1.3.23. The theorem above (and its relative proof) can be stated in its dual form leading to a theorem on existence of colimits, and a relative criterion to calculate them (taking the dual of the proof).

Example 1.3.24. Limit theorem gives us an easy way to calculate limits. An example of this fact is how limits are computed in **Set**. Given a diagram $D : \mathscr{I} \to \mathbf{Set}$, where $\mathscr{I}$ is a small category and $I = Ob(\mathscr{I})$, its limit is the set L defined as follows:

$$L = \{(d_i)_{i \in I} \in \prod_{i \in I} D(i) \mid \forall \phi \in \mathscr{I}(i, i'), D(\phi)(d_i) = d_{i'}\}$$

with projections as arrows.

Example 1.3.25. As we have done in Example 1.3.24, we illustrate how to construct colimits in the category of sets. Given a small category $\mathscr{I}$, $I = Ob(\mathscr{I})$, and a diagram $D : \mathscr{I} \to \mathbf{Set}$, consider the equivalence relation $\sim$ defined on $\coprod_{i \in I} D(i)$ such that $d_i \sim d_{i'}$ if $d_i \in D(i), d_{i'} \in D(i')$ and there exists some $\phi \in \mathscr{I}(i, i')$ such that $D(\phi)(d_i) = d_{i'}$. Then, a colimit for D is the set

$$C = \big(\coprod_{i \in I} D(i)\big) / \sim$$

with the inclusions as arrows.

Remark 1.3.26. Since a diagram is nothing more than a functor from a "shape" category to another, it makes sense to talk about limits of functors in general, even when they are not intended to be diagrams.

Lemma 1.3.27. *Let $\mathscr{C}$ be a category, and let $[\mathscr{C}, \mathbf{Set}]$ be the category of presheaves on $\mathscr{C}$. Let $D : \mathscr{I} \to [\mathscr{C}, \mathbf{Set}]$ be a diagram of shape $\mathscr{I}$ on the presheaves category early mentioned. Then,*

1. *The limit of D exists, and it is the presheaf $L : \mathscr{C} \to \mathbf{Set}$ such that, for each object A of $\mathscr{C}$, $L(A)$ is the limit in $\mathbf{Set}$ of the values of the presheaves $D(i)(A)$ for each i.*

2. *The colimit of D exists, and it is the presheaf $C : \mathscr{C} \to \mathbf{Set}$ such that, for each object A of $\mathscr{C}$, $C(A)$ is the colimit in $\mathbf{Set}$ of the values of the presheaves $D(i)(A)$, for each i.*

In the next sections, we will work on a special kind of diagrams with certain properties. In particular, we are interested in how a functor behaves with respect to the constructions defined so far.

Definition 1.3.28. Let $D : \mathscr{I} \to \mathscr{C}$ be a diagram, and $F : \mathscr{C} \to \mathscr{D}$ a functor. We say that F:

1. *preserves limits* of D if, given a limit $(L, l_i)_{i \in \mathscr{I}}$ for D, then $(F(L), F(l_i))_{i \in \mathscr{I}}$ is a limit for $F \circ D$.

2. *reflects limits* of D if a cone $(L, l_i)_{i \in \mathscr{I}}$ is a limit for D whenever $(F(L), F(l_i))_{i \in \mathscr{I}}$ is a limit for $F \circ D$.

3. *lifts limits (uniquely)* of D if, given a limit $(L, l_i)_{i \in \mathscr{I}}$ for $F \circ D$, there exists a (unique) limit $(L', l'_i)_{i \in \mathscr{I}}$ for D such that $(F(L'), F(l'_i))_{i \in \mathscr{I}} = (L, l_i)_{i \in \mathscr{I}}$.

4. *creates limits* of D if D has a limit and F preserves and reflects limits along it.

The dual notions are obtained in the obvious way, namely, substituting the words "limits" and "cones" with "colimits" and "cocones", respectively

Observation 1.3.29. It holds that if a functor creates limits, then lifts uniquely limits [AHS09].

Proposition 1.3.30. *A fully faithful functor reflects all limits and colimits.*

The next theorem is about a particular property that adjoint functors have.

Theorem 1.3.31. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor, and $G : \mathscr{D} \to \mathscr{C}$ its right adjoint. Then, G preserves limits.*

Remark 1.3.32. The dual of the theorem above states that, if G is a functor and F is a left adjoint, then F preserves colimits.

Proposition 1.3.33. *Let $D, D' : \mathscr{I} \to \mathscr{C}$ be two functors, and let $((c_i)_{i \in \mathscr{I}}, C)$ and $((c'_i)_{i \in \mathscr{I}}, C')$ be, respectively, the colimit of D and D'. Then, a natural transformation $\phi : D \dot\to D'$ induces a unique arrow $c : C \to C'$.*

Proof. Consider the following situation.

$$
\begin{array}{ccc}
 & C & \\
c_i \nearrow & & \nwarrow c_j \\
D(i) \xrightarrow{D(\alpha)} & & D(j) \\
\phi_i \downarrow & & \downarrow \phi_j \\
D'(i) \xrightarrow[D'(\alpha)]{} & & D'(j) \\
c'_i \searrow & & \swarrow c'_j \\
 & C' &
\end{array}
$$

To prove the statement, we note that $((c'_i \circ \phi_i)_{i \in \mathscr{I}}, C')$ is a cocone for D. Computing, we have, for each i, j and $\alpha : i \to j$

$$
\begin{aligned}
c'_j \circ \phi_j \circ D(\alpha) &= c'_j \circ D'(\alpha) \circ \phi_i \quad \textit{Naturality of } \phi \\
&= c'_i \circ \phi_i \qquad\qquad ((c'_i)_{i \in \mathscr{I}}, C') \textit{ is a colimit for } D'
\end{aligned}
$$

Since $((c'_i)_{i \in \mathscr{I}}, C)$ is a colimit by hypothesis, then must exists a unique arrow $c : C \to C'$ such that $c \circ c_i = \phi_i \circ c'_i$, having the thesis. $\square$

1.3.2   Kernel Pairs and Regular Epimorphisms

Definition 1.3.34 (Kernel Pair)**.** A *kernel pair* for an arrow $f : A \to B$ is an object K_f together with two arrows $\pi_f^1, \pi_f^2 : K_f \to A$,

denoted as (K_f, π_f^1, π_f^2), such that the following square is a pullback.

$$
\begin{array}{ccc}
K_f & \xrightarrow{\ \pi_f^1\ } & A \\
\Big\downarrow{\scriptstyle \pi_f^2} & & \Big\downarrow{\scriptstyle f} \\
A & \xrightarrow{\ f\ } & B
\end{array}
$$

Remark 1.3.35. If a category $\mathscr{C}$ has pullbacks then every arrow has a kernel pair.

Remark 1.3.36. Since a kernel pair is nothing more that a pullback, that is, a limit, by Remark 1.3.6, it make sense to refer to it as *the* kernel pair for a morphism f.

Example 1.3.37. In **Set**, a kernel pair for a function $f : A \to B$ is the set

$$K_f = \{(x, y) \in A \times A \mid f(x) = f(y)\}$$

together with the canonical projection on the first and the second component of the pairs.

Proposition 1.3.38. *Let (K, p_1, p_2) be the kernel pair of $f : X \to Y$, and let $(X \times X, \pi_1, \pi_2)$ be the product of X with itself. Then, the mediating arrow $\langle p_1, p_2 \rangle : K \to X \times X$ is mono.*

Proof. Suppose $\langle p_1, p_2 \rangle \circ f = \langle p_1, p_2 \rangle \circ g$ for $f, g : Z \to K$. Then, we have

$$
\begin{aligned}
\langle p_1, p_2 \rangle \circ f &= \langle p_1, p_2 \rangle \circ g & \langle p_1, p_2 \rangle \circ f &= \langle p_1, p_2 \rangle \circ g \\
\pi_1 \circ \langle p_1, p_2 \rangle \circ f &= \pi_1 \circ \langle p_1, p_2 \rangle \circ g & \pi_2 \circ \langle p_1, p_2 \rangle \circ f &= \pi_2 \circ \langle p_1, p_2 \rangle \circ g \\
p_1 \circ f &= p_1 \circ g & p_2 \circ f &= p_2 \circ g
\end{aligned}
$$

Thus, from the universal property of the pullback, $f = g$. $\qquad\square$

Proposition 1.3.39. *An arrow $m \colon M \to X$ is mono if and only if (M, id_M, id_M) is a kernel pair for it.*

Proof. To prove the "if" part of the statement, let $f, g : A \to M$ be such that $m \circ f = m \circ g$, and consider the following situation.

$$
\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & \\
 & \searrow^{u} & \\
g & M \xrightarrow{id_M} M & \\
 & \downarrow{id_M} \quad \downarrow{m} & \\
 & M \xrightarrow{\quad m \quad} X &
\end{array}
$$
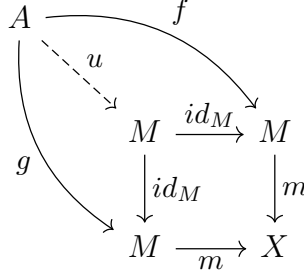
For the universal property of pullbacks, we have that

$$f = id_M \circ u = g$$

Hence, m is mono.

Conversely, if m is mono, then, we have that

$$m \circ f = m \circ g \Rightarrow f = g$$
$$\Rightarrow f \circ id_M = g \circ id_M$$

Hence, f is the unique arrow that makes the commutative square illustrated above a pushout. $\square$

Remark 1.3.40. From characterization of monos via pullbacks in Proposition 1.3.39 and Lemma 1.3.27, we have that a mono in a category of presheaves is a natural transformation of which each component is mono.

Corollary 1.3.41. (K_f, π_f^1, π_f^2) *is a kernel pair for* $f \colon X \to Y$ *if and only if, for each mono* $m \colon Y \to Z$, (K_f, π_f^1, π_f^2) *is a kernel pair also for* $m \circ f$.

Proof. It is enough to see that, by Lemma 1.3.20 and Proposition 1.3.39 the outer boundary of the following square is a pullback.

$$
\begin{array}{ccccc}
K_f & \xrightarrow{\pi_f^2} & X & \xrightarrow{id_X} & X \\
\pi_f^2 \downarrow & & \downarrow{f} & & \downarrow{f} \\
X & \xrightarrow{f} & Y & \xrightarrow{id_Y} & Y \\
id_X \downarrow & & \downarrow{id_Y} & & \downarrow{m} \\
X & \xrightarrow{f} & Y & \xrightarrow{m} & Z
\end{array}
$$

The leftward part of the statement follows by definition of monomorphism an Lemma 1.3.20. $\square$

Lemma 1.3.42. *Suppose the following situation, and that $f : X \to Y$ and $g : Z \to W$ have kernel pairs.*

$$
\begin{array}{ccc}
X & \xrightarrow{\;h\;} & Z \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
Y & \xrightarrow{\;t\;} & W
\end{array}
$$

Then, there exists a unique arrow $k_h : K_f \to K_g$ making the squares below commute.

$$
\begin{array}{ccc}
K_f & \dashrightarrow^{k_h} & K_g \\
{\scriptstyle \pi_f^1}\downarrow & & \downarrow{\scriptstyle \pi_g^1} \\
X & \xrightarrow{\;h\;} & Z
\end{array}
\qquad
\begin{array}{ccc}
K_f & \dashrightarrow^{k_h} & K_g \\
{\scriptstyle \pi_f^2}\downarrow & & \downarrow{\scriptstyle \pi_g^2} \\
X & \xrightarrow{\;h\;} & Z
\end{array}
$$

Moreover, if the beginning square is a pullback, then also the preceding ones are so.

Proof. Computing, we have

$$
\begin{aligned}
g \circ h \circ \pi_f^1 &= t \circ f \circ \pi_f^1 \\
&= t \circ f \circ \pi_f^2 \\
&= g \circ h \circ \pi_f^2
\end{aligned}
$$

By the universal property of K_g as the pullback of g along itself, such k_h exists and it is unique.

To prove the second half of the thesis, let us consider the two rectangles below, which, by Lemma 1.3.20 are pullbacks.

$$
\begin{array}{ccccc}
K_f & \xrightarrow{\pi_f^1} & X & \xrightarrow{\;h\;} & Z \\
{\scriptstyle \pi_f^2}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{\;f\;} & Y & \xrightarrow{\;t\;} & W
\end{array}
\qquad
\begin{array}{ccccc}
K_f & \xrightarrow{\pi_f^2} & X & \xrightarrow{\;h\;} & Z \\
{\scriptstyle \pi_f^1}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{\;f\;} & Y & \xrightarrow{\;t\;} & W
\end{array}
$$

But then the following ones are pullbacks too.

The thesis follows again by Lemma 1.3.20.   □

Proposition 1.3.43. *Let $e : X \to Y$ be a regular epimorphism in a category $\mathscr{C}$ with a kernel pair (K, π_1, π_2). Then, e is the coequalizer of π_1 and π_2.*

Proof. By hypothesis, there exists a pair $f, g : Z \to X$ of which e is the coequalizer. Since $e \circ f = e \circ g$, we have

thus there exists the unique $k : Z \to K$. Let now $h : Z \to V$ be an arrow such that $h \circ \pi_1 = h \circ \pi_2$, then

$$h \circ f = h \circ \pi_1 \circ k$$
$$= h \circ \pi_2 \circ k$$
$$= h \circ g$$

and thus there exists a unique $l : Y \to V$ such that $l \circ e = h$.   □

Corollary 1.3.44. *Let $\mathscr{C}$ be a category with pullbacks and $\phi : D \dot\to D'$ be a natural transformation between two functors $D, D' : \mathscr{I} \to \mathscr{C}$. If ϕ_i is a regular epi for every i, then ϕ is a regular epi.*

Proof. Let (K_i, π_i^1, π_i^2) be the kernel pair of ϕ_i for each i. Given an arrow $\alpha : i \to j$ of $\mathscr{I}$, we have

$$\phi_j \circ D(\alpha) \circ \pi_i^1 = D'(\alpha) \circ \phi_i \circ \pi_i^1$$
$$= D'(\alpha) \circ \phi_i \circ \pi_i^2$$
$$= \phi_j \circ D(\alpha) \circ \pi_i^2$$

Thus, the outer boundary of the diagram below commutes, yielding the arrow $K(\alpha)$

In this way, we get a functor $K : \mathscr{I} \to \mathscr{C}$, which maps each i onto K_i and each arrow α onto $K(\alpha)$. We have in fact $K(id_i) : K_i \to K_i$ is the arrow such that

$$D(id_i) \circ \pi_i^1 = \pi_i^1 \circ K(id_i) \qquad D(id_i) \circ \pi_i^2 = \pi_i^2 \circ K(id_i)$$
$$\pi_i^1 = \pi_i^1 \circ K(id_i) \qquad\qquad \pi_i^2 = \pi_i^2 \circ K(id_i)$$

Thus, for the universal property of pullbacks, $K(id_i) = id_{K_i}$.

Suppose now $\alpha : i \to j$ and $\beta : j \to k$. Computing, we have

$$\pi_k^1 \circ K(\beta \circ \alpha) = D(\beta \circ \alpha) \circ \pi_i^1 \qquad\qquad \pi_k^2 \circ K(\beta \circ \alpha) = D(\beta \circ \alpha) \circ \pi_i^2$$
$$= D(\beta) \circ D(\alpha) \circ \pi_i^1 \qquad\qquad\qquad = D(\beta) \circ D(\alpha) \circ \pi_i^2$$
$$= D(\beta) \circ \pi_j^1 \circ K(\alpha) \qquad\qquad\qquad = D(\beta) \circ \pi_j^2 \circ K(\alpha)$$
$$= \pi_k^1 \circ K(\beta) \circ K(\alpha) \qquad\qquad\qquad = \pi_k^2 \circ K(\beta) \circ K(\alpha)$$

Again, for universal property of pullbacks, necessarily we have $K(\beta \circ \alpha) = K(\beta) \circ K(\alpha)$, proving functoriality of K.

Hence, we have two natural transformations $\pi^1, \pi^2 : E \dot{\to} D$. By Proposition 1.3.43, every component ϕ_i is the coequalizer of $\pi_i^1, \pi_i^2 : E \to D$, and so ϕ is the coequalizer of π^1 and π^2. $\qquad\square$

Lemma 1.3.45. *Let $D, D' : \mathscr{I} \to \mathscr{C}$ be two diagrams, and let $((c_i)_{i \in \mathscr{I}}, C)$ and $((c'_i)_{i \in \mathscr{I}}, C')$ be, respectively, the colimit of D and D'. If $\mathscr{C}$ has all colimits, for diagrams of shape $\mathscr{I}$ and $\phi : D \dot{\to} D'$ is a natural transformation in which all components are regular epimorphisms, then, the arrow induced by ϕ from C to C' (Proposition 1.3.33) is a regular epimorphism too.*

Proof. By Corollary 1.3.44, we know that $\phi : D \dot{\to} D'$ is a regular epimorphism, so that there is a functor $E : \mathscr{I} \to \mathscr{C}$ and $\eta, \theta : E \dot{\to} D$

such that ϕ is the coequalizer of η and θ. Let now $((p_i)_{i \in \mathscr{I}}, P)$ be the colimit of E, by Proposition 1.3.33, we have $a, b : P \to C$ fitting in the diagram below.

$$
\begin{array}{ccc}
E(i) \xrightarrow{\ p_i\ } P & \qquad & E(i) \xrightarrow{\ p_i\ } P \\
\eta_i \downarrow \qquad \downarrow a & & \theta_i \downarrow \qquad \downarrow b \\
D(i) \xrightarrow[\ c_i\ ]{} C & & D(i) \xrightarrow[\ c_i\ ]{} C
\end{array}
$$

We want to show that c coequalizes η and θ. Let thus $t : C \to T$ be an arrow such that $t \circ a = t \circ b$. Then, for every i, we have

$$
\begin{aligned}
t \circ c_i \circ \eta_i &= t \circ a \circ p_i \\
&= t \circ b \circ p_i \\
&= t \circ c_i \circ \theta_i
\end{aligned}
$$

Thus, there is $t_i : D(i) \to T$ such that $t \circ c_i = t_i \circ \phi_i$. It is now easy to see that $((t_i)_{i \in \mathscr{I}}, T)$ is a cocone of D': suppose $\alpha : i \to j$ be an arrow of $\mathscr{I}$, obtaining

$$
\begin{aligned}
t_i \circ \phi_i &= t \circ c_i \\
&= t \circ c_j \circ D(\alpha) \\
&= t_j \circ \phi_j \circ D(\alpha) \\
&= t_j \circ D'(\alpha) \circ \phi_i
\end{aligned}
$$

By the hypothesis that ϕ_i is regular epi for each i, therefore epi (by the dual of Proposition 1.3.13), we can conclude $t_i = t_j \circ D'(\alpha)$.

Hence, we have an arrow $k : C' \to T$ such that $k \circ c'_i = t_i$. But then

$$
\begin{aligned}
c \circ c \circ c_i &= k \circ c'_i \circ \phi_i \\
&= t_i \circ \phi \\
&= t \circ c_i
\end{aligned}
$$

Showing that $k \circ c = t$.

For the uniqueness, let $k' : C' \to T$ be another arrow such that $k' \circ c = t$. Then we have

$$
\begin{aligned}
k' \circ c'_i \circ \phi_i &= k' \circ c \circ c_i \\
&= t \circ c_i \\
&= t_i \circ \phi_i
\end{aligned}
$$

Since ϕ_i is a regular epimorphism, we have $k' \circ c'_i = t_i$, and, because $k \circ c'_i = t_i$ by construction, we can conclude that $k' = k$ since $((c'_i)_{i \in \mathscr{I}}, C')$ is a colimit. $\square$

1.4 Adhesivity

The next section is about adhesivity. An adhesive category is intuitively a category in which pushouts of (some) monomorphisms exist and they behave more or less as they do among sets.

Definition 1.4.1. (Van Kampen property) Let $\mathcal{A}$ be a subclass of $\mathcal{H}om(\mathscr{C})$, and consider the diagram below:
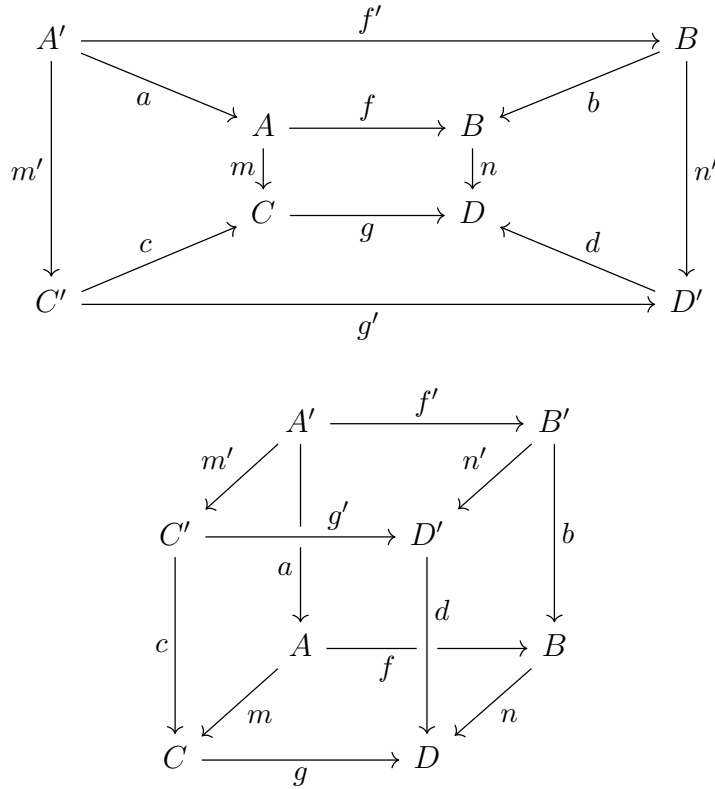
we say that the inner square is an $\mathcal{A}$-*Van Kampen* square if:

- it is a pushout;

- $a, b, c, d \in \mathcal{A}$;

- whenever the top and the left squares are pullbacks then the outer square is a pushout if and only of the right and the bottom squares are pullbacks.

We are now ready to give the notion of $\mathcal{M}$-adhesivity.

Definition 1.4.2 ($\mathcal{M}$-adhesivity)**.** Let $\mathscr{C}$ be a category and $\mathcal{M} \subseteq \mathcal{M}ono(\mathscr{C})$ containing all isomorphisms, closed under composition and stable under pullbacks and pushouts (Definition 1.3.18). Then $\mathscr{C}$ is $\mathcal{M}$-*adhesive* if

1. every cospan $C \xrightarrow{g} D \xleftarrow{m} B$ with $m \in \mathcal{M}$ can be completed to a pullback (such pullbacks are called $\mathcal{M}$-pullbacks);

2. every span $C \xleftarrow{m} A \xrightarrow{f} B$ with $m \in \mathcal{M}$ can be completed to a pushout (such pushouts are called $\mathcal{M}$-pushouts);

3. pushouts along $\mathcal{M}$-arrows are $\mathcal{M}$-Van Kampen squares.

We also say that $\mathscr{C}$ is *adhesive* when it is $\mathcal{M}ono(\mathscr{C})$-adhesive, and *quasiadhesive* when it is $\mathcal{R}eg(\mathscr{C})$-adhesive.

> NO, questo è sbagliato: bisogna introdurre la nozione di adesività forte (cioè VK vale per tutti i cubi)

Observation 1.4.3. Set is adhesive.

Here it follows an interesting property of adhesive categories [Lac11].

> almeno la seconda parte devi provarla o citarla per le M-adesive

Proposition 1.4.4. *In any adhesive category, the pushout of a monomorphism along any morphism is a monomorphism, and the resulting square is also a pullback.*

Proof. Let the following square be a pushout, with m mono.

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\
C & \xrightarrow{\ g\ } & D
\end{array}
$$

Consider now the following cube.

We have then that the top face is a pullback, and by Proposition 1.3.39, the left face is a pullback. Since the bottom face is a pushout by hypothesis, we have that the front face and the right one are pullbacks (by adhesivity). Hence, we can conclude that the starting square (which is the front and the bottom face of the cube) is a pushout, and by Proposition 1.3.39, since the left face is a pullback, n is mono. $\square$

Verifying $\mathcal{M}$-adhesivity using the definition above may turn out to be very complex, so we can make use of the following result [CGM22].

Theorem 1.4.5. *Let $\mathscr{C}$ be a category, $\mathcal{M} \subseteq \mathcal{M}ono(\mathscr{C})$ containing all isomorphisms, closed under composition and stable under pullbacks and pushouts. Let now $F : \mathscr{C} \to \mathscr{D}$ be a functor with $\mathscr{D}$ $\mathcal{N}$-adhesive for some $\mathcal{N} \subseteq \mathcal{M}ono(\mathscr{D})$. If F is such that $F(\mathcal{M}) \subseteq \mathcal{N}$ and creates pullbacks and $\mathcal{M}$-pushout, then $\mathscr{C}$ is $\mathcal{M}$-adhesive.*

The idea behind this theorem is to simplify calculations to show that a certain category is adhesive for some subclass of monomorphisms, considering a functor from the category of which we want to prove adhesivity to a category we know it is adhesive, requiring that such functor has some properties.

Proof. In order to prove $\mathcal{M}$-adhesivity of $\mathscr{C}$, we have to verify the condition in Definition 1.4.2.

- Let $C \xrightarrow{g} D \xleftarrow{m} B$ with $m \in \mathcal{M}$ be a cospan in $\mathscr{C}$. Applying F, we obtain $F(C) \xrightarrow{F(g)} F(D) \xleftarrow{F(m)} B$, with $F(m) \in \mathcal{N}$ by hypothesis. Then, there exists a pullback $(P_F, p_{F(B)}, p_{(F(D))})$ in $\mathscr{D}$, which is an $\mathcal{N}$-pullback (Definition 1.3.15). Since F creates pullbacks, hence lifts them (Observation 1.3.29), there exist a pullback (P, p_B, p_D) in $\mathscr{C}$.

- Let $C \xleftarrow{m} A \xrightarrow{f} B$ with $m \in \mathcal{M}$ be a cospan in $\mathscr{C}$. Analogously to the previous point, applying the functor F we obtain $F(C) \xleftarrow{F(m)} F(A) \xrightarrow{F(f)} F(B)$ with $F(m) \in \mathcal{N}$, and there exists a $\mathcal{N}$-pushout $(q_{F(C)}, q_{F(B)}, F(Q))$ in $\mathscr{D}$. Since F reflects pushouts, (q_C, q_B, Q) is a $\mathcal{M}$-pushout in $\mathscr{C}$.

- the Van Kampen property of $\mathcal{M}$-pullbacks follows from the closure under pullbacks and pushouts of $\mathcal{M}$ and from the fact that F reflects pullbacks.

$\square$

Corollary 1.4.6. *Let $\mathscr{A}$ be a $\mathcal{M}$-adhesive category for some $\mathcal{M} \subseteq$ $\mathcal{H}om(\mathscr{A})$. Then, for every other category $\mathscr{C}$, the functor category $[\mathscr{C}, \mathscr{A}]$ is $\mathcal{M}^{\mathscr{C}}$-adhesive, where*

$$\mathcal{M}^{\mathscr{C}} = \{\eta \in \mathcal{H}om([\mathscr{C}, \mathscr{A}]) \mid \eta_C \in \mathcal{M} \text{ for each object } C \text{ of } \mathscr{C} \}$$
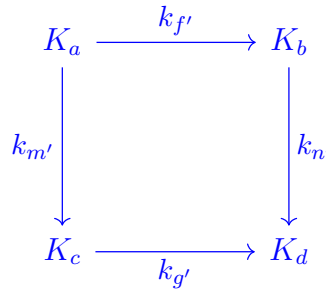
Since **Set** is adhesive, we can conclude what follows.

Corollary 1.4.7. *Every category of presheaves is adhesive.*

Lemma 1.4.8. *Let $\mathscr{C}$ be an $\mathcal{M}$-adhesive category with all pullbacks, and suppose that in the cube below the top face is an $\mathcal{M}$-pushout.*
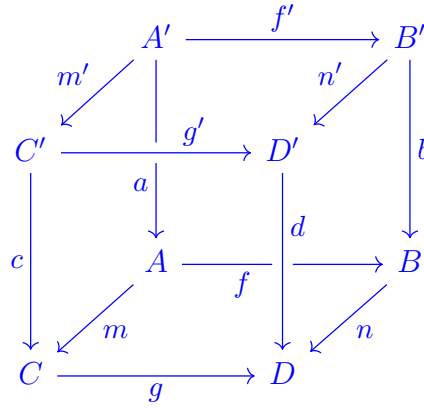
Then, the square below is a pushout.

Proof. Since f', being the pullback of f, is in $\mathcal{M}$ then by Proposition 1.4.4 the bottom face of the cube is a pullback. Thus Lemma 1.3.42 entails that in the following cube the vertical faces are pullbacks.

Now the thesis follows from strong $\mathcal{M}$-adhesivity. $\qquad\qquad\square$

Lemma 1.4.9. *Let $\mathscr{C}$ be an $\mathcal{M}$-adhesive category with all pullbacks, and suppose that the cube below is given, in which every face is a pullback and the bottom one is a $\mathcal{M}$-pushout.*

Qua serve adesività forte

Then, the square below is a pushout.

Proof. By Lemma 1.3.42, in the following cube the vertical faces are

pullbacks.

$$
\begin{array}{ccc}
& K_a \xrightarrow{\ k_{f'}\ } K_b & \\
k_{m'} \swarrow \quad \Big\downarrow k_{g'} \quad k_{n'} \swarrow \quad \Big\downarrow \pi_b^1 & & \\
K_c \xrightarrow{\ \ \ } K_d & & \\
\pi_c^1 \Big\downarrow \quad \pi_a^1 \Big\downarrow & & \\
A' \xrightarrow{\ f'\ } B' & & \\
m' \swarrow \quad \pi_d^1 \Big\downarrow \quad n' \swarrow & & \\
C' \xrightarrow{\ g'\ } D' & &
\end{array}
$$

f' is in $\mathcal{M}$ as it is the pullback of $\mathcal{M}$, thus the bottom face of the cube is a Van Kampen pushout and the thesis follows.

□

Chapter 2

Categories of Graphs

This chapter is about graphs, and how it is possible to formalize them using categories in order to point out their properties from an abstract point of view. Starting from the set-theoretical definition of graphs, we will give an abstraction via functor categories, in which a graph is nothing but a functor between a category to another.

2.1   Graphs

A (directed graph) $\mathcal{G}$ is a mathematical structure consisting of a set of edge, a set of nodes and two functions, one assigning a *source* node and one assigning a *target* node to an edge. Formally, $\mathcal{G}$ is a quadruple $(V_{\mathcal{G}}, E_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}})$, where $V_{\mathcal{G}}$ is the set of nodes, $E_{\mathcal{G}}$ is the set of edges, and $s_{\mathcal{G}}, t_{\mathcal{G}} : E_{\mathcal{G}} \to V_{\mathcal{G}}$ are the source and the target functions.

A *graph homomorphism* $h : \mathcal{G} \to \mathcal{H}$ is then a pair of functions $h = (h_V : V_{\mathcal{G}} \to V_{\mathcal{H}}, h_E : E_{\mathcal{G}} \to E_{\mathcal{H}})$ such that

$$h_V \circ s_{\mathcal{G}} = s_{\mathcal{H}} \circ h_E$$

and

$$h_V \circ t_{\mathcal{G}} = t_{\mathcal{H}} \circ h_E$$

that is, a structure preserving map.

We can then generalize such notion to something more abstract, considering a graph to be nothing more than a presheaf from the category $(E \rightrightarrows V)$ to the category of sets. Having two of such presheaves, a natural transformation from one to another encapsulates the behavior of a graph morphism due to naturality. We can now define the category of graphs.

Definition 2.1.1 (Category of Graphs). We denote as **Graph** the category

$$[E \underset{t}{\overset{s}{\rightrightarrows}} V, \mathbf{Set}]$$

Since **Graph** is a category of presheaves, Lemma 1.3.27 guarantees the existence of limits and colimits, and gives us an easy way to compute them.

Corollary 2.1.2. Graph *has all limits and colimits.*

Example 2.1.3. The initial object in **Graph** is the empty graph, i.e., the graph with an empty set of vertices and an empty set of edges. The initial object instead is the graph with exactly one node and a single edge from that node to itself.

Example 2.1.4. Given two graphs $G = (V_G, E_G, s_G, t_G)$ and $H = (V_H, E_H, s_H, t_H)$, the graph $G \times H = (V_G \times V_H, E_G \times E_H, (s_G, s_H), (t_G, t_H))$, where $(s_G, s_H), (t_G, t_H) : V_G \times V_H \to E_G \times E_H$ are the pairwise sources and targets, is the categorical product in **Graph**, together with the two projections $\pi_G : G \times H \to G$, $\pi_H : G \times H \to H$ defined in the obvious way.

Example 2.1.5. The equalizer of two morphisms $h, k : G \to H$ in **Graph** is defined as in **Set**, that is, a graph Q together with a graph morphism q that is the restriction of G to all the vertices and all the arcs that are mapped on the same vertices and edges both from h and k. Formally, (Q, q) is an equalizer for $h, k : G \to H$, $h = (h_V, h_E), k = (k_V, k_E)$ where $V(Q) = \{n \in V(G) \mid h_V(n) = k_V(n)\}$, $E(Q) = \{e \in E(G) \mid h_E(e) = k_E(e)\}$, $s_Q = s_G \mid_{V(Q)}$, $t_E = t_G \mid_{V(Q)}$.

Remark 2.1.6. TODO: Si può generalizzare a tutte le categorie regolari per evitare di perdere le proprietà che usiamo (da eq.rel. a quot.).

2.2 Graphs with Equivalences

A graph with equivalence is a 6-tuple $\mathbb{G} = (E, V, C, s, t, q)$, where E and V are, respectively, the edges and the vertices sets, and C is the set of the equivalence classes among vertices, $s, t : E \to V$ are the source and target functions and $q : V \to C$ is the *quotient* function, that is, the map from a vertex to its equivalence class. For this definition to make sense, q needs to be surjective. A morphisms

h from a graph with equivalence $\mathbb{G} = (E, V, C, s, t, q)$ to another $\mathbb{H} = (E', V', C', s', t', q')$ is a triple $h = (h_E, h_V, h_C)$ of functions $h_V : V \to V'$, $h_E : E \to E'$ and $h_C : C \to C'$ such that:

- $h_E \circ s = s' \circ h_V$;

- $h_E \circ t = t' \circ h_V$;

- $h_C \circ q = q' \circ h_V$.

Remark 2.2.1. A graph with equivalence can be viewed as a graph endowed with an equivalence relation over its set of vertices, $(\mathcal{G}, \sim_{\mathcal{G}})$. An homomorphism between two graphs with equivalences $h : \mathbb{G} = (\mathcal{G}, \sim_{\mathcal{G}}) \to \mathbb{H} = (\mathcal{H}, \sim_{\mathcal{H}})$ is a graph homomorphism $h = (h_V, h_E) : \mathcal{G} \to \mathcal{H}$ such that if $v_1 \sim_{\mathcal{G}} v_2$ then $h_V(v_1) \sim_{\mathcal{H}} h_V(v_2)$. In **Set**, it is possible to formalize an equivalence relation $\sim$ over X as a surjective function sending each element x on its equivalence class $[x]_\sim$, and this justify our formalization via surjective functions (i.e., epimorphisms).

As we have done in Section 2.1, we can think to a graph with equivalence as a presheaf, this time from a category $E \rightrightarrows V \to C$, where the image of C along the presheaf is the set of the equivalence classes, requiring that the morphism $V \to C$ is an epimorphism (that is, a surjective function).

Definition 2.2.2 (Category of Graphs with Equivalences)**.** The category **EqGrph** is the subcategory of

$$[E \overset{s}{\underset{t}{\rightrightarrows}} V \overset{q}{\to} C, \mathbf{Set}]$$

such that, for each $\mathbb{G} \in \mathcal{O}b(\mathbf{EqGrph})$, $\mathbb{G}(q)$ is an epimorphism.

Observation 2.2.3. Morphisms of graphs with equivalences are uniquely determined by the first two component. That is, if $h_1 = (h_E, h_V, \phi)$ and $h_2 = (h_E, h_V, \psi)$, then $\phi = \psi$.

Proof. Let $h_1, h_2 : \mathbb{G} \to \mathbb{H}$, where $\mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G)$ and $\mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$. Then, we have the following situation

$$
\begin{array}{ccccc}
V_G & \xrightarrow{\ h_V\ } & V_H & \xleftarrow{\ h_V\ } & V_G \\
\downarrow{\scriptstyle q_G} & & \downarrow{\scriptstyle q_H} & & \downarrow{\scriptstyle q_G} \\
C_G & \xrightarrow[\ \phi\ ]{} & C_H & \xleftarrow[\ \psi\ ]{} & C_G
\end{array}
$$

Then, we have:

$$\psi \circ q_G = q_H \circ h_V$$
$$= \phi \circ q_G$$

From the fact that q_G is epi, we can conclude $\phi = \psi$. $\square$

A graph with equivalence is then a graph with an extra structure, the quotient map. Hence, it is possible to get the underlying graph by forgetting it. Such action is described by the *forgetful functor* $U :$ **EqGrph** $\to$ **Graph**, that maps each graph with equivalence $\mathbb{G} = (E, V, C, s, t, q)$ onto $U(\mathbb{G}) = (E, V, s, t)$, and each morphisms $h = (h_E, h_V, h_C)$ onto $U(h) = (h_E, h_V)$. U is effectively a functor, since, on identities, $U((id_E, id_V, id_C)) = (id_E, id_V)$, and on compositions

$$\begin{aligned} U(h \circ k) &= U((h_E \circ k_E, h_V \circ k_V, h_C \circ k_C)) \\ &= (h_E \circ k_E, h_V \circ k_V) \\ &= (h_E, h_V) \circ (k_E \circ k_V) \\ &= U(h) \circ U(k) \end{aligned}$$

Proposition 2.2.4. *The forgetful functor $U :$ **EqGrph** $\to$ **Graph** is faithful.*

Proof. Let $\mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G)$ and $\mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$ be two graphs with equivalences, and let $h, k : \mathbb{G} \to \mathbb{H}$. If $U(h) = U(k)$ (i.e., the first two component of h and k are the same), from Observation 2.2.3, we can conclude that $h = k$. Then, the restriction $U_{\mathbb{G}, \mathbb{H}} :$ **EqGrph**$(\mathbb{G}, \mathbb{H}) \to$ **Graph**$(U(\mathbb{G}), U(\mathbb{H}))$ is injective, therefore U is faithful. $\square$

Another functor that will be useful later is $V :$ **EqGrph** $\to$ **Set**, that sends $(E_G, V_G, C_G, s_G, t_G, q_G)$ to C_G, and $h = (h_E, h_V, h_C)$ to h_C.

Proposition 2.2.5. **EqGrph** *has all limits, colimits and U preserves limits and colimits.*

Proof. Let $D : \mathscr{I} \to$ **EqGrph** be a diagram. In the following, we will denote the graph with equivalence $D(i)$ as $(E_i, V_i, C_i, s_i, t_i, q_i)$. Let now be the graph (A, B, s, t) the limit of $U \circ D$, with projections $(\pi_E^i, \pi_V^i) : (A, B, s, t) \to (E_i, V_i, s_i, t_i)$. Notice now that $(B, (q_i \circ \pi_V^i)_{i \in \mathscr{I}})$ is a cone for $V \circ D$. To see this, let $\alpha : i \to j$ be an arrow of $\mathscr{I}$, $D(\alpha) = (h_E, h_V, h_C)$, $U \circ D(\alpha) = (h_E, h_V)$. From the

definition of cone, we have that $U \circ D(\alpha) \circ (\pi_E^i, \pi_V^i) = (\pi_E^j, \pi_V^j)$, hence $h_V \circ \pi_V^i = \pi_V^j$. Consider now the following diagram in **Set**

$$
\begin{array}{ccc}
& B & \\
\pi_V^i \swarrow & & \searrow \pi_V^j \\
V_i \xrightarrow{\quad h_V \quad} & & V_j \\
q_i \downarrow & & \downarrow q_j \\
C_i \xrightarrow{\quad h_C \quad} & & C_j
\end{array}
$$

So we have $q_j \circ h_V \circ \pi_V^i = q_j \circ \pi_V^j$, by definition of graph with equivalence, $h_C \circ q_i \circ \pi_V^i = q_j$, and, by definition of V, $V \circ D(\alpha) \circ q_i \circ \pi_V^i = q_j \circ \pi_V^j$. Suppose now $(L, (l_i)_{i \in \mathscr{I}})$ be a limit for $V \circ D$, so that we have an arrow $l : B \to L$. This arrow is not epi in general, so let Q be its image, $q : Q \to B$ be the resulting epi and $m : Q \to L$ the corresponding mono, as the diagram below shows. By definition, the external rectangle commutes, so, for each i object of $\mathscr{I}$, Proposition 1.1.16 yields the dotted arrow π_C^i.

$$
\begin{array}{ccccc}
B & \xrightarrow{\pi_V^i} & B_i & \xrightarrow{q_i} & Q_i \\
q \downarrow & & & \nearrow \pi_C^i & \downarrow id_{Q_i} \\
Q & \xrightarrow{\quad m \quad} & L & \xrightarrow{\quad l_i \quad} & Q_i
\end{array}
$$

We have to show that in this way we get a cone over the diagram D. Let $\alpha : i \to j$ be an arrow of $\mathscr{I}$, then we have:

$$
\begin{aligned}
U(D(\alpha) \circ (\pi_E^i, \pi_V^i, \pi_C^i)) &= U(D(\alpha)) \circ (\pi_E^i, \pi_V^i) \\
&= (\pi_E^j, \pi_V^j) \\
&= U(D(\alpha) \circ (\pi_E^j, \pi_V^j, \pi_C^j))
\end{aligned}
$$

And faithfulness of U yields the thesis.

To see that this cone is terminal, let $((E, F, G, a, b, c), (\tau_i)_{i \in \mathscr{I}})$, where $\tau_i = (\tau_E^i, \tau_V^i, \tau_C^i)$, be another cone. By construction, we have

an arrow $(\tau_E, \tau_V) : (E, F, a, b) \to (A, B, s, t)$ such that
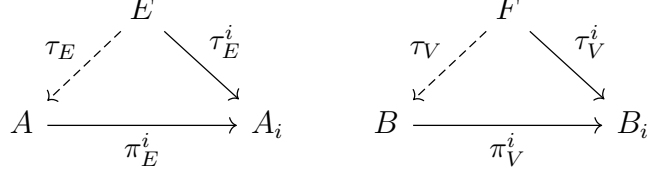
For the same reason as before, $(G, (\tau_C^i)_{i \in \mathscr{I}})$ is a cone over $V \circ D$, thus there exists an arrow $\tau : G \to L$ such that $l_i \circ \tau = \tau_C^i$. At this point, we get

$$
\begin{aligned}
l_i \circ \tau \circ c &= \tau_C^i \circ c \\
&= q_i \circ \tau_V^i && \tau_i \text{ is a morphism in } \mathbf{EqGrph} \\
&= q_i \circ \pi_V^i \circ \tau_V && \text{Diagram above} \\
&= l_i \circ l \circ \tau_V && (B, (q_i \circ \pi_V^i)_{i \in \mathscr{I}}) \text{ cone}
\end{aligned}
$$

Therefore, the outer part of the rectangle below commutes, and by Proposition 1.1.16 there exists a unique $\tau_C : G \to Q$
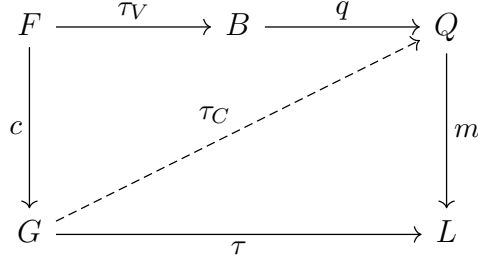
Faithfulness of U and Observation 2.2.3 guarantees that (τ_E, τ_V, τ_C) is the unique arrow such that $(\pi_E^i, \pi_V^i, \pi_C^i) \circ (\tau_E, \tau_V, \tau_C) = (\tau_E^i, \tau_V^i, \tau_C^i)$.

For colimits, let (A, B, s, t) be the colimit of $U \circ D$, together with morphisms $(\kappa_i)_{i \in \mathscr{I}} = (\kappa_E^i, \kappa_V^i)_{i \in \mathscr{I}}$, and suppose $((c_i)_{i \in \mathscr{I}}, C)$ be the colimit of $V \circ D$. Then, we have the following situation
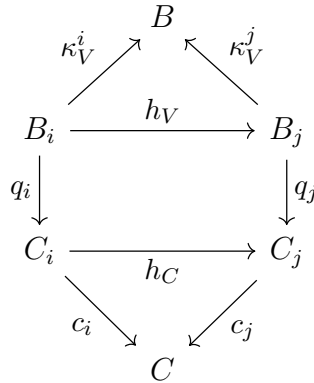
Then, $((c_i \circ q_i)_{i \in \mathscr{I}}, C)$ is a cocone for all B_i, and, since $((\kappa_i)_{i \in \mathscr{I}}, B)$ is the colimit of all B_i (Lemma 1.3.27), there exists a unique arrow $q : B \to C$ such that, for each i, $q \circ \kappa_V^i = c_i \circ q_i$. Such q is epi, by application of Lemma 1.3.45, and thus (A, B, C, s, t, q), together with arrows $(\kappa_E^i, \kappa_V^i, c_i)_{i \in \mathscr{I}}$ is the colimit of D.

$\square$

Corollary 2.2.6. *An arrow* $h = (h_E, h_V, h_C) : \mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G) \to \mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$ *in* **EqGrph** *is mono if and only if* h_E *and* h_V *are mono in* **Set**.

Proof. The "if" part is given by the fact that U is faithful, and hence reflects monomorphisms. Since a morphism in a category of presheaves is mono if and only if it is injective on each component, we have that, if $U(h)$ is mono, that is, h_E and h_V are injective in **Set**, then h is mono. For the "only if" part, suppose $f = (f_E, f_V, f_C)$, $g = (g_E, g_V, g_C)$, $f, g : \mathbb{H} \to \mathbb{K}$ be such that $h \circ f = h \circ g$. Then, we have

$$\begin{aligned} h \circ f &= (h_E \circ f_E, h_V \circ f_V, h_C \circ f_C) \\ &= (h_E \circ f_E, h_V \circ f_V, h_V \circ f_V \circ \mathbb{K}(q)) \\ &= (h_E \circ g_E, h_V \circ g_V, h_V \circ g_V \circ \mathbb{K}(q)) \end{aligned}$$

Since $\mathbb{K}(q)$ is epi, we have, on the third component, that $h_V \circ f_V \circ \mathbb{K}(q) = h_V \circ g_V \circ \mathbb{K}(q))$ implies $f_C = g_C$, and hence $f = g$ $\square$

Corollary 2.2.7. *Let* $h = (h_E, h_V, h_C) : \mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G) \to \mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$ *be a morphism of* **EqGrph***, then the following are equivalent:*

1. h *is a regular mono;*

2. h_E, h_V, h_C *are all monos;*

3. h_E *and* h_V *are mono and* (K, π_1, π_2) *is the kernel pair of* $q_H \circ h_V$ *if and only if* (K, π_1, π_2) *is the kernel pair of* q_G.

Proof. $1 \Rightarrow 2$. If h is mono, from Corollary 2.2.6 we have that h_E and h_V are monos. To derive h_C mono, suppose $f, g : \mathbb{H} \to \mathbb{K}$ to be the arrows equalized by h. Then we have

$$\begin{aligned} f_C \circ h_C \circ \mathbb{G}(q) &= f_C \circ \mathbb{H}(q) \circ h_V \\ &= \mathbb{K}(q) \circ f_V \circ h_V \\ &= \mathbb{K}(q) \circ g_V \circ h_V \\ &= g_C \circ h_C \circ \mathbb{G}(q) \end{aligned}$$

since $\mathbb{G}(q)$ is epi, we have that $f_C \circ h_C = g_C \circ h_C$, hence h_C is an equalizer for f_C and g_C, thus a monomorphism.

$2 \Rightarrow 3$. We note that, by Corollary 1.3.41, (K, π_1, π_2) is the kernel pair of q_G if and only if it is the kernel pair also of $h_C \circ q_G$, since h_C is mono by hypothesis. The thesis follows from $h_C \circ q_G = q_H \circ h_V$, and from the hypothesis of h_E mono.

Esercizio

$3 \Rightarrow 1.$ idea: force the comm. of the diagram on the last two components to obtain the two arrows that are equalized, and show that the condition in 3 is sufficient to conclude reg. mono ☐

Remark 2.2.8. It is possible to restate the third point of the Corollary 2.2.7, by Example 1.3.37, as

> h_E and h_V are mono and, for every $v, v' \in V_H$, $q_H(h_V(v)) = q_H(h_V(v'))$ if and only if $q_G(v) = q_G(v')$

That is, a regular monomorphism in **EqGrph** is a morphism that reflects equivalences besides preserving them.

Let us turn to another functor **EqGrph** $\to$ **Graph**.

Definition 2.2.9. The *quotient functor* $Q : $ **EqGrph** $\to$ **Graph** sends $(E_G, V_G, C_G, s_G, t_G, q_G)$ to $(E_G, C_G, q_G \circ s_G, q_G \circ t_G)$ and an arrow $(h_E, h_V, h_C) : (E_G, V_G, C_G, s_G, t_G, q_G) \to (E_H, V_H, C_H, s_H, t_H, q_H)$ to (h_E, h_C).

Remark 2.2.10. The action of the functor on a morphism of graphs with equivalences gives a morphism of graphs, in fact $q_H \circ s_H \circ h_E = q_H \circ h_V \circ s_G = h_C \circ q_G \circ s_G$. The same is valid for t_H and t_G.

Lemma 2.2.11. Q *is a left adjoint.*

Proof. Let $R((A, B, s, t))$ be (A, B, B, s, t, id_B), so that $Q(R((A, B, s, t))) = (A, B, s, t)$. Now, suppose that $h = (h_E, h_V) : Q((E, V, C, s', t', q)) \to (A, B, s, t)$ is an arrow in **Graph**, and consider the triple $(h_E, h_V, h_V \circ q)$. Since h is a morphism of **Graph**,

$$h_V \circ q \circ s' = s \circ h_E \qquad h_V \circ q \circ t' = t \circ h_E$$

Then we have the following squares:

$$
\begin{array}{ccc}
E \xrightarrow{h_E} A & E \xrightarrow{h_E} A & V \xrightarrow{h_V \circ q} B \\
\downarrow{s_G} \quad \downarrow{s} & \downarrow{t_G} \quad \downarrow{t} & \downarrow{q} \quad \downarrow{id_B} \\
V \xrightarrow[h_V \circ q]{} B & V \xrightarrow[h_V \circ q]{} B & C \xrightarrow[h_V]{} B
\end{array}
$$

We have therefore found a morphism $(E, V, C, s', t', q) \to R((A, B, s, t))$ whose image through Q fits in the diagram below.

$$
\begin{array}{ccc}
(A, B, s, t) & \xrightarrow{\ id_A, id_B\ } & (A, B, s, t) \\
\Big\uparrow\scriptstyle{Q((h_E, h_V \circ q, h_V))} & \nearrow\scriptstyle{(h_E, h_V)} & \\
(E, C, q \circ s', q \circ t') & &
\end{array}
$$

Such arrow is unique. Suppose $f = (f_E, f_V, f_C)$ to be another arrow wit such property. Then, it must be $(id_A, id_B) \circ Q(f) = (f_E, f_C) = (h_E, h_C)$. Finally, $f_C = f_V \circ q = h_V \circ q$. $\qquad\square$

Proposition 2.2.12. *Q creates colimits.*

Proof. Preserve from Theorem 1.3.31. Remain to see Reflect. $\qquad\square$

Example 2.2.13. Q does not preserve limits. Indeed, let $\mathbb{G}_1 = (E_1, A, A, s_1, t_1, id_A)$, $\mathbb{G}_2 = (E_2, B, B, s_2, t_2, id_B)$ and $\mathbb{G}_3 = (E_3, A + B, \mathbb{1}, s_3, t_3, !_{A+B})$, and let $h = (h_E, \iota_A, !_A) : \mathbb{G}_1 \to \mathbb{G}_3$, $k = (k_E, \iota_B, !_B) : \mathbb{G}_2 \to \mathbb{G}_3$, where $(\iota_A, \iota_B, A+B)$ is the coproduct of A and B, $\mathbb{1}$ is the intial object (in **Set**, the singleton set as shown in Example 1.3.8), and $!_X$ the unique arrow $X \to \mathbb{1}$. The following two diagrams show the pullback of h and k and the pullback of $Q(h)$ and $Q(k)$, on the second component (the vertices of the graphs)

$$
\begin{array}{ccc}
\mathbb{0} & \xrightarrow{\ p_1\ } & A \\
\scriptstyle{p_2}\Big\downarrow & & \Big\downarrow\scriptstyle{\iota_A} \\
B & \xrightarrow[\ \iota_B\ ]{} & A + B
\end{array}
\qquad\qquad
\begin{array}{ccc}
A \times B & \xrightarrow{\ \pi_A\ } & A \\
\scriptstyle{\pi_B}\Big\downarrow & & \Big\downarrow\scriptstyle{!_A} \\
B & \xrightarrow[\ !_B\ ]{} & \mathbb{1}
\end{array}
$$

But the arrow $\mathbb{0} \to A \times B$ is not epi in general (this is easy to see taking **Set** as example), hence such pullback is not preserved by Q.

2.2.1   Adhesivity of EqGrph

Lemma 2.2.14. *In* **EqGrph**, *pushouts along regular monos are stable.*

Proof. Let $\mathcal{G}_i = (A_i, B_i, C_i, s_i, t_i, q_i)$, $\mathcal{G}' = (A'_i, B'_i, C'_i, s'_i, t'_i, q'_i)$, for $i = 1, 2, 3, 4$, and, in the diagram above, suppose all the vertical faces are pullbacks, the bottom face is a pushout and h is regular mono.

$$
\begin{array}{ccc}
\mathcal{G}'_1 & \xrightarrow{\ h'\ } & \mathcal{G}'_2 \\
 & & \\
\mathcal{G}'_3 & \xrightarrow{\ p'\ } & \mathcal{G}'_4 \\
 & & \\
 & \mathcal{G}_1 & \xrightarrow{\ h\ } \mathcal{G}_2 \\
 & & \\
\mathcal{G}_3 & \xrightarrow{\ p\ } & \mathcal{G}_4
\end{array}
$$

By Proposition 2.2.5 and Corollary 2.2.6, the following two cubes have $\mathcal{M}$-pushouts as bottom faces and pullbacks as vertical faces, thus their top faces are $\mathcal{M}$-pushouts.

$$
\begin{array}{ccc}
A'_1 & \xrightarrow{\ h'_V\ } & A'_2 \\
A'_3 & \xrightarrow{\ p'_V\ } & A'_4 \\
 & A_1 & \xrightarrow{\ h_V\ } A_2 \\
A_3 & \xrightarrow{\ p_V\ } & A_4
\end{array}
\qquad
\begin{array}{ccc}
B'_1 & \xrightarrow{\ h'_E\ } & B'_2 \\
B'_3 & \xrightarrow{\ p'_E\ } & B'_4 \\
 & B_1 & \xrightarrow{\ h_E\ } B_2 \\
B_3 & \xrightarrow{\ p_E\ } & B_4
\end{array}
$$

Now, using Corollary 1.3.21, we can consider a third cube, which, by Proposition 2.2.12, has a $\mathcal{M}$-pushout as bottom face and pullbacks

as vertical faces, so that its top face is a $\mathcal{M}$-pushout too.

$$
\begin{array}{ccc}
T \xrightarrow{\;x_1\;} U \\
\end{array}
$$

Moreover, by the proof of Proposition 2.2.5, we know that there are monos $m_2 : C_2' \to U$ and $m_3 : C_3' \to Y$ fitting in the diagrams

For C_1', we can make a similar argument, let S be the pullback of m_2 along x_1, using Lemma 1.3.20 and, again, the proof of Proposition 2.2.5 we know that q_1' arise as the factorization of the arrow

$B_1' \to S$ induced by $q_2' \circ h_2'$ and a_2 so that we have a diagram.

$$
\begin{array}{ccc}
B_1' & \xrightarrow{\;h_V'\;} & B_2' \\
 & & \searrow{\scriptstyle q_2'} \\
\downarrow{\scriptstyle q_1'} & & \\
C_1' & \xrightarrow{\;m_1\;} S \xrightarrow{\;s_1\;} & C_2' \\
 & \downarrow{\scriptstyle s_2} & \downarrow{\scriptstyle m_2} \\
B_1 & T \xrightarrow{\;x_1\;} & U \\
 & \downarrow{\scriptstyle x_2} & \downarrow{\scriptstyle u_2} \\
 & C_1 \xrightarrow{\;h_C\;} & C_2
\end{array}
$$
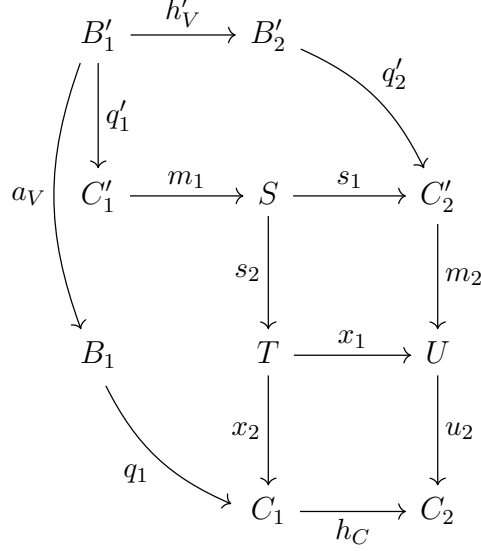
Moreover, we have that

$$
\begin{aligned}
s_1 \circ m_1 \circ q_1 &= q_2' \circ h_V' & q_4' \circ t_V' &= t_C' \circ q_2' \\
&= h_C' \circ q_1' & &= u_1 \circ m_2 \circ q_2'
\end{aligned}
$$

$$
\begin{aligned}
y_1 \circ m_3 \circ q_3' &= p_V' \circ q_4' \\
&= p_C' \circ q_3'
\end{aligned}
\qquad TODO: \; w \circ s_2 \circ m_1 = m_3 \circ k_C'
$$

Hence, $s_1 \circ m_1 = h_C'$, $w \circ s_2 \circ m_1 = m_3 \circ k_C'$, $t_C' = u_1 \circ m_2$ and $p_C' = y_1 \circ m_3$. Let now $z : \mathcal{G}_2' \to \mathcal{H} = (E, V, Q, s, t, q)$, $w : \mathcal{G}_3^1 \to \mathcal{H}$ be two morphisms such that $z \circ h' = w \circ k'$, and let $\phi : B_4' \to V$ be the arrow induced by z_V and w_V. We want to construct the arrow

$v : C_4' \to Q$ in the diagram below.

$$
\begin{array}{c}
\text{(commutative diagram)}
\end{array}
$$
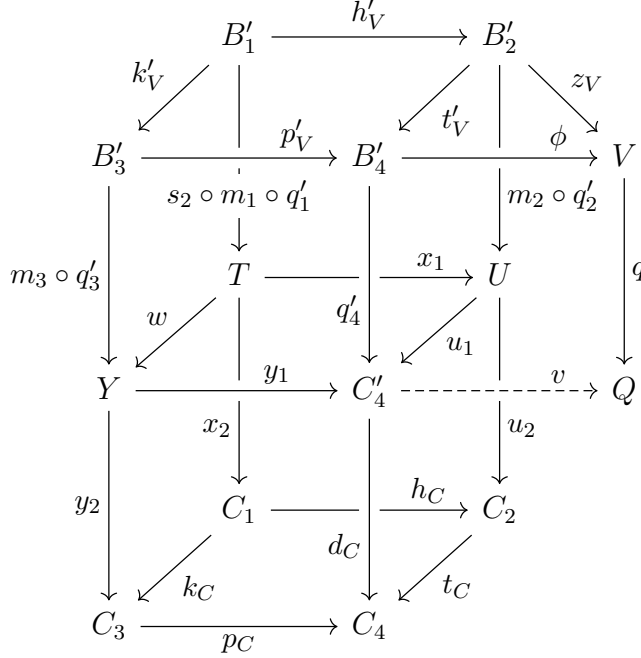
By Proposition 1.3.43, d_C is the coequalizer of its kernel pair. On the other hand, by Lemma 1.4.9, we know that the top face of the cube below is a pushout.

$$
\begin{array}{c}
\text{(commutative cube diagram)}
\end{array}
$$
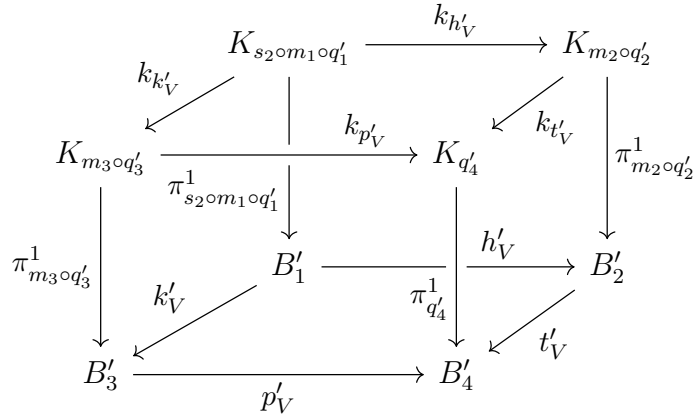
And, since m_3 and m_2 are monos,

$$
q_3' \circ \pi^1_{m_3 \circ q_3'} = q_3' \circ \pi^2_{m_3 \circ q_3'} \qquad q_2' \circ \pi^1_{m_2 \circ q_2'} = q_2' \circ \pi^2_{m_2 \circ q_2'}
$$

Computing, we obtain

$$q \circ \phi \circ \pi^1_{q'_4} \circ k_{p'_V} = q \circ \phi \circ p'_V \circ \pi^1_{m_3 \circ q'_3} \qquad q \circ \phi \circ \pi^1_{q'_4} \circ k_{t'_V} = q \circ \phi \circ t'_V \circ \pi^1_{m_2 \circ q'_2}$$

$$= q \circ w_V \circ \pi^1_{m_3 \circ q'_3} \qquad\qquad = q \circ z_V \circ \pi^1_{m_2 \circ q'_2}$$

$$= w_C \circ q'_3 \circ \pi^1_{m_3 \circ q'_3} \qquad\qquad = z_C \circ q'_2 \circ \pi^1_{m_2 \circ q'_2}$$

$$= w_C \circ q'_3 \circ \pi^2_{m_3 \circ q_3} \qquad\qquad = z_C \circ q'_2 \circ \pi^2_{m_2 \circ q_2}$$

$$= q \circ w_V \circ \pi_{m_3 \circ q'_3} \qquad\qquad = q \circ z_V \circ \pi_{m_2 \circ q'_2}$$

$$= q \circ \phi \circ p'_V \circ \pi^2_{m_3 \circ q'_3} \qquad\qquad = q \circ \phi \circ t'_V \circ \pi^2_{m_2 \circ q'_2}$$

$$= q \circ \phi \circ \pi^2_{q'_4} \circ k_{p'_V} \qquad\qquad = q \circ \phi \circ \pi^2_{q'_4} \circ k_{t'_V}$$

Since the previous cube has a pushout as top face, by universal property, we have

$$q \circ \phi \circ \pi^1_{q'4} = q \circ \phi \circ \pi^2_{q'_4}$$

hence

$$v \circ q'_4 \circ \pi^1_{q'_4} = v \circ q'_4 \circ \pi^2_{q'_4}$$

rimane unicità

$\square$

Lemma 2.2.15. *In* **EqGrph**, *pushouts along regular monos are* $\mathcal{R}eg(\textbf{EqGrph})$-*Van Kampen.*

Proof. In lieu of Lemma 2.2.14, it is enough to proof that, given a cube as the one below, with pullbacks as back faces, pushouts as bottom and top faces and such that h is a regular mono, the front faces are pullbacks too, where $\mathcal{G}_i = (A_i, B_i, C_i, s_i, t_i, q_i)$, $\mathcal{G}' = (A'_i, B'_i, C'_i, s'_i, t'_i, q'_i)$, for $i = 1, 2, 3, 4$.


By Proposition 2.2.5 and Corollary 2.2.6, the following two cubes have $\mathcal{M}$-pushouts as bottom faces and pullbacks as back faces, thus

their front faces are pullbacks too.
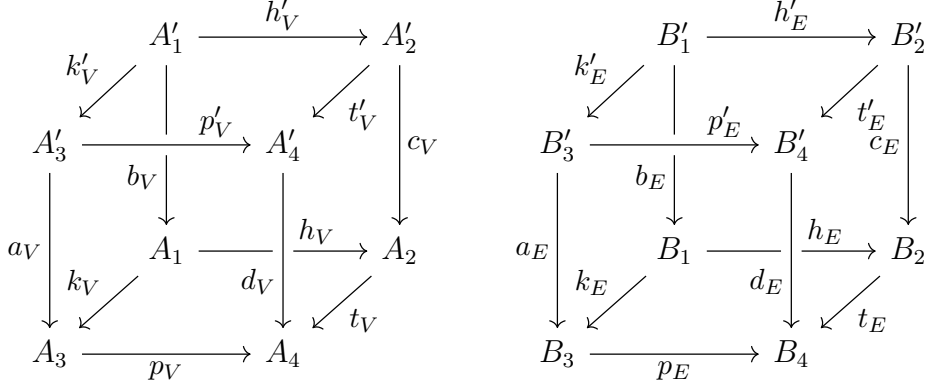
On the other hand we can consider the diagrams below, in which the inner squares are pullbacks. Since the outer diagrams commute, by definition of porphism of **EqGrph**, then we have the existence of $m_2\colon C_2' \to U$, $m_3\colon C_3' \to Y$, $a_3\colon B_3' \to Y$ and $a_2\colon B_2' \to Y$.
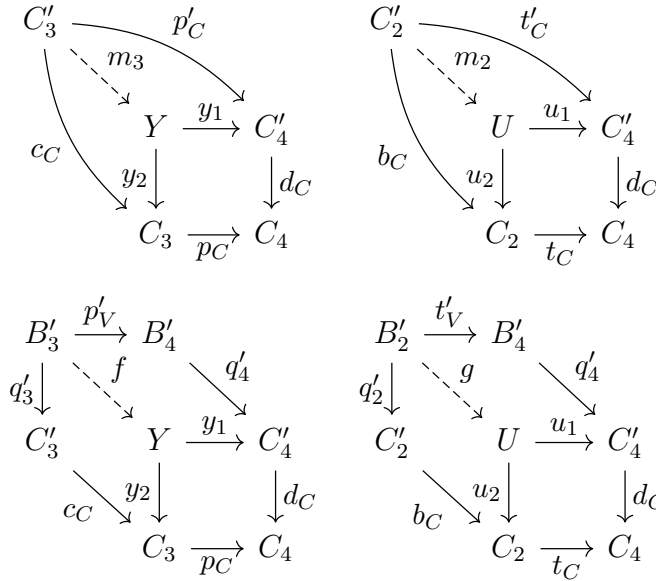
Now, notice that m_3 and m_2 are monos because c_C and b_C are monos (Corollary 2.2.7). By the proof of Proposition 2.2.5, to conclude it is enough to show that

$$m_3 \circ q_3' = f \qquad m_2 \circ q_2' = g$$

Indeed, if the previous equations hold, then C_3' and C_2' are epi-mono factorizations of f and g, and the thesis follows from Corollary 1.1.17 and the proof of Proposition 2.2.5.

Now, if we compute, we obtain:

$$
\begin{aligned}
y_1 \circ f &= q'_4 \circ p'_V & u_1 \circ g &= q'_4 \circ t'_V \\
&= p'_C \circ q'_3 & &= t'_C \circ q'_3 \\
&= y_1 \circ m_3 \circ q'_3 & &= u_1 \circ m_2 \circ q'_2
\end{aligned}
$$

$$
\begin{aligned}
y_2 \circ f &= d_C \circ q'_3 & u_2 \circ g &= d_V \circ q'_2 \\
&= y_2 \circ m_3 \circ q'_3 & &= u_2 \circ m_2 \circ q'_2
\end{aligned}
$$

Concluding the proof. $\qquad\square$

From Proposition 2.2.5 and Lemmas 2.2.14 and 2.2.15, by Theorem 1.4.5, we can deduce at once the following.

Corollary 2.2.16. EqGrph *is* $\mathcal{R}eg(\textbf{EqGrph})$-*adhesive.*

2.3   E-Graphs

E-Graphs are a particular type of graphs with equivalences.

Definition 2.3.1 (E-Graph)**.** Given a graph with equivalence $\mathbb{G} = (E, V, C, s, t, q)$, let (S, p_1, p_2) be the kernel pair of $q \circ s$. Then, $\mathbb{G}$ is an *e-graph* if it holds that

$$
q \circ t \circ p_1 = q \circ t \circ p_2
$$

Remark 2.3.2. From a set-theoretic point of view, $\mathbb{G} = (\mathcal{G}, \sim)$ is a graph with equivalence, where $\mathcal{G} = (E, V, s, t)$ in which holds that, for each pair of edges e, e',

$$
\frac{s(e) \sim s(e')}{t(e) \sim t(e')}
$$

As we have considered in the section on graphs with equivalences, we can formalize the equivalence relation with a surjective function $q : V \to V/_\sim = C$, rewriting the inference rule above as an inclusion. Let $S = \{(e, e') \in E \times E \mid q(s(e)) = q(s(e'))\}$ and $T = \{(e, e') \in E \times E \mid q(t(e)) = q(t(e'))\}$. Then, $\mathbb{G}$ is an e-graph if $S \subseteq T$. As we noted in Example 1.3.37, S is the kernel pair of $q \circ s$ and T is the kernel pair of $q \circ t$, if endowed with canonical projections. Moreover, we notice that the pairing of such projections yields a subobject of the product $E \times E$ Proposition 1.3.38. Hence, by Remark 1.1.8, the Definition 2.3.1 express this fact using categorial constructions.

Definition 2.3.3 (Category of E-Graphs). **EGG** is the subcategory of **EqGrph** in which objects are e-graphs, and, given two e-graphs $\mathbb{G}, \mathbb{H}$, **EGG**$(\mathbb{G}, \mathbb{H}) = $ **EqGrph**$(\mathbb{G}, \mathbb{H})$.

Since **EGG** is a full subcategory of **EqGrph**, there exists a fully faithful functor $I : $ **EGG** $\rightarrow$ **EqGrph** (Example 1.2.8).

Lemma 2.3.4. EGG *has all limits and I preserves them.*

Proof. Let $D : \mathscr{I} \rightarrow $ **EGG** be a diagram, with $D(i) = (A_i, B_i, C_i, s_i, t_i, q_i)$, let (U, u_1^i, u_2^i) be the kernel pair of $q_i \circ s_i$. Let now be (A, B, C, s, t, q), togehter with projections $(\pi_E^i, \pi_V^i, \pi_C^i)_{i \in \mathscr{I}}$ the limit of $I \circ D$, let (U, u_1, u_2) be the kernel pair of $q \circ s$ and let $(L, (l_i)_{i \in \mathscr{I}})$ be the limit of $V \circ I \circ D$. By construction (proof of Proposition 2.2.5), there exists a monomorphism $m : Q \rightarrow L$ such that $\pi_C^i = l_i \circ m$. Notice that

$$
\begin{aligned}
q_i \circ s_i \circ \pi_E^i \circ u_1 &= q_i \circ \pi_V^i \circ s \circ u_1 \\
&= \pi_C^i \circ q \circ s \circ u_1 \\
&= \pi_C^i \circ q \circ s \circ u_2 \\
&= q_i \circ s_i \circ \pi_E^i \circ u_2
\end{aligned}
$$

Then, for each i, there exists an arrow $a_i : U \rightarrow U_i$ making the following diagram to commute

We have then

$$l_i \circ m \circ q \circ t \circ u_1 = q_i \circ \pi_V^i \circ t \circ u_1$$
$$= q_i \circ t_i \circ \pi_E^i \circ u_1$$
$$= q_i \circ t_i \circ u_1^i \circ a_i$$
$$= q_i \circ t_i \circ u_2^i \circ a_i$$
$$= q_i \circ t_i \circ \pi_E^i \circ u_2$$
$$= q_i \circ \pi_V^i \circ t \circ u_2$$
$$= l_i \circ m \circ q \circ t \circ u_2$$

By universal property of limits, we have that $m \circ q \circ t \circ u_1 = m \circ q \circ t \circ u_2$, and, since m is mono, $q \circ t \circ u_1 = q \circ t \circ u_2$, hence the thesis. $\qquad\square$

Corollary 2.3.5. *I creates limits.*

Corollary 2.3.6. $h : \mathbb{G} \to \mathbb{H}$ *is a regular mono in EGG if and only if it is a regular mono in* **EqGrph**.

Lemma 2.3.7. *I preserves pushouts along regular monomorphisms.*

Proof. Let $\mathbb{G}_1 = (A_1, B_1, C_1, s_1, t_1, q_1)$, $\mathbb{G}_2 = (A_2, B_2, C_2, s_2, t_2, q_2)$ and $\mathbb{G}_3 = (A_3, B_3, C_3, s_3, t_3, q_3)$ be e-graphs, $h : \mathbb{G}_1 \to \mathbb{G}_2$ and $m : \mathbb{G}_1 \to \mathbb{G}_3$ be two morphisms with m mono. Let now $\mathbb{P} = (A, B, C, s, t, q)$, together with $k : \mathbb{G}_3 \to \mathbb{P}$ and $n : \mathbb{G}_2 \to \mathbb{P}$, the pushout of h and m. Let (K_i, π_i^1, π_i^2) the kernel pair of $q_i \circ s_i$, for $i = 1, 2, 3$, and let (U, u_1, u_2) be the kernel pair of $q \circ s$. Consider the following cube in **Set**

By adhesivity of **Set**, and since m is regular mono, the cube above is Van Kampen ,

Non esisono i cubi di Van Kampen, a essere Van Kampen sono certi quadrati

Since m is a regular mono then m_C is mono by ???. Then, by adhesivity of **Set**, the bottom face of the cube above is a Van Kampen square and thus a pullback. Therefore by Lemma 1.4.9, the square below is a pushout.

and, by Proposition 1.4.4, all the faces are pullbacks. Then, [vertical] applying Lemma 1.4.9, the square below is a pushout

$$
\begin{array}{ccc}
K_1 & \xrightarrow{f_h} & K_2 \\
\downarrow{\scriptstyle f_m} & & \downarrow{\scriptstyle f_n} \\
K_3 & \xrightarrow{f_k} & U
\end{array}
$$

Hence, we end up with the following situation

Computing, we have

$$q \circ t \circ u_1 \circ f_n = q \circ t \circ n_E \circ \pi_2^1 \qquad q \circ t \circ u_1 \circ f_k = q \circ t \circ k_E \circ \pi_3^1$$
$$= n_C \circ q_2 \circ s_2 \circ \pi_2^1 \qquad\qquad = k_C \circ q_3 \circ s_3 \circ \pi_3^1$$
$$= n_C \circ q_2 \circ s_2 \circ \pi_2^2 \qquad\qquad = k_C \circ q_3 \circ s_3 \circ \pi_3^2$$
$$= q \circ t \circ u_2 \circ f_n \qquad\qquad = q \circ t \circ u_2 \circ f_k$$

By universal property of pushouts, we can conclude $q \circ t \circ u_1 = q \circ t \circ u_2$, hence the thesis. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

[Qua estendi un pochino: da questo si deduce che esiste una freccia blah blah]

Corollary 2.3.8. *I creates pushouts along regular monos.*

By direct application of Theorem 1.4.5, we can conclude what follows.

Corollary 2.3.9. EGG *is* $\mathcal{R}eg(\mathbf{EGG}) - adhesive.$

Appendix A

Omitted Proofs

Lemma 1.3.20. Suppose that the following diagram is given and its right half is a pullback. Then the whole rectangle is a pullback if and only if its left half is a pullback.

$$
\begin{array}{ccccc}
X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & Z \\
\downarrow{\scriptstyle t} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle h} \\
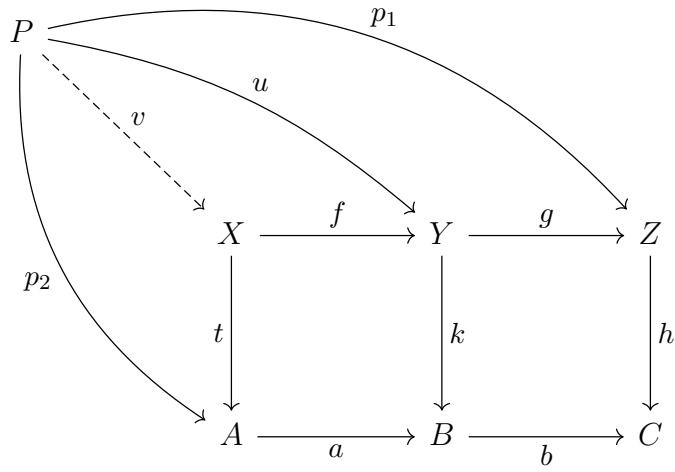A & \xrightarrow{\ a\ } & B & \xrightarrow{\ b\ } & C
\end{array}
$$

Proof. For the "only if" part, suppose the left square to be a pullback. To verify the outer rectangle is a pullback, consider the following situation:

$$
\begin{array}{ccc}
P & \xrightarrow{\ p_1\ } & \\
\downarrow{\scriptstyle p_2} & & Z \\
 & & \downarrow{\scriptstyle h} \\
A \xrightarrow{\ a\ } B & \xrightarrow{\ b\ } & C
\end{array}
$$

But the right square is a pullback implies that there exists a unique u such that
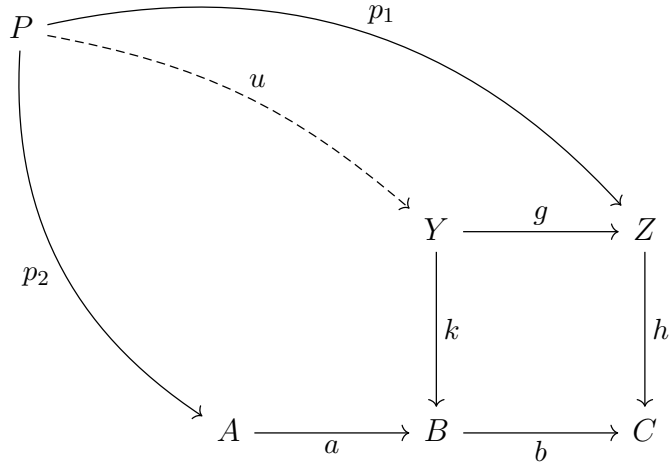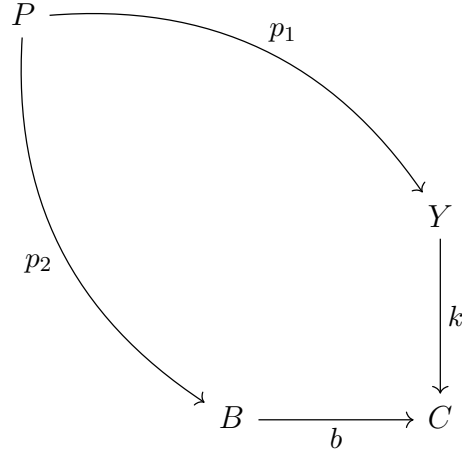

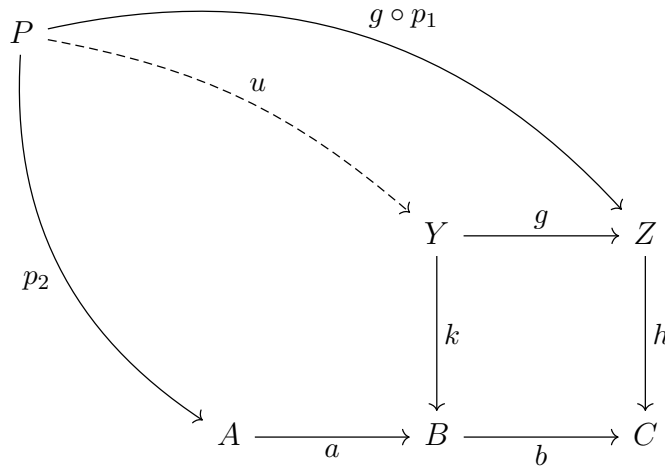And, since the left square is a pullback, there exists a unique v such that

Hence, the whole rectangle is a pullback.

For the "if" part, consider the following situation.

$$P \xrightarrow{p_1} Y$$

We have now to show that the unique arrow $v : P \to X$ (of the outer rectangle) is such that $f \circ v = p_1$, but this follows from the fact that the right square is a pullback, having the following situation.

$$P \xrightarrow{g \circ p_1} Z$$

$\square$

Rivedere questa dimostrazione

Theorem 1.3.22. Let $\mathscr{C}$ be a category. Then $\mathscr{C}$ has all finite limits if and only if $\mathscr{C}$ has all finite products and all equalizers.

Proof. Let $D : \mathscr{I} \to \mathscr{C}$ a diagram, with $\mathscr{I}$ finite.

The *if* statement follows from definitions of products and equalizers (Definition 1.3.15, Definition 1.3.12)

To satisfy the *only if* statement, we want an object L together with morphisms $p_i : L \to D(j)$ such that:

1. $\{p_i : L \to D(i)\}$ is a cone – i.e., for each morphism of $\mathscr{I}$ $\alpha : i \to j$, $D(\alpha) \circ p_i = p_j$; and

2. for each E and $q_i : E \to D(j)$ in $\mathscr{C}$, with $D(\alpha) \circ q_i = q_j$ for each $\alpha : i \to j$ of $\mathscr{I}$, there exists a unique $f : E \to L$ such that $q_i = p_i \circ f$ for each $i \in \mathcal{O}b(\mathscr{I})$.

Consider the two products (which exist by hypothesis) $\prod_{j \in \mathcal{O}b(\mathscr{I})} D(j)$, the product of the objects of the diagram, and $\prod_{\alpha \in \mathcal{H}om(\mathscr{I})} D(cod\ \alpha)$, the product of the codomains of the morphisms in D, where π_x is the x-th projection of the product. Let now:

$$\gamma, \varepsilon : \prod_{j \in \mathcal{O}b(\mathscr{I})} D(j) \longrightarrow \prod_{\alpha \in \mathcal{H}om(\mathscr{I})} D(cod\ \alpha)$$

be defined by $\gamma_\alpha = \pi_{D(cod\ \alpha)}$ (the projection on the codomain of α) and $\varepsilon_\alpha = D(\alpha) \circ \pi_{D(dom\ \alpha)}$. Let now $e : L \to \prod_{j \in \mathcal{O}b(\mathscr{I})} D(j)$ the equalizer of γ and ε (which exists by hypothesis), and, for each $j \in \mathcal{O}b(\mathscr{I})$, $p_j : L \to D(j)$, defined by $p_j = \pi_{D(j)} \circ e$.

What we want now is to show that $(L, (p_i))_{i \in \mathscr{I}}$ is the limit of D, namely, to prove that the conditions given at the beginning are valid.

For condition 1, we have to show that, for each $\alpha : i \to j$ of $\mathscr{I}$, we have $D(\alpha) \circ p_i = p_j$:

$$
\begin{aligned}
D(\alpha) \circ p_i &= D(\alpha) \circ \pi_{D(i)} \circ e & &\textit{Definition of } p_j \\
&= \varepsilon_\alpha \circ e & &\textit{Definition of } \varepsilon \\
&= \gamma_\alpha \circ e & &\textit{e is an equalizer of } \pi, \varepsilon \\
&= \pi_{D(j)} \circ e & &\textit{Definition of } \pi \\
&= p_j & &\textit{Definition of } p_j
\end{aligned}
$$

For condition 2, suppose that $(E, (q_i)_{i \in \mathcal{O}b(\mathscr{I})})$ has the properties stated. By definition of product, there exists a (unique) arrow $q : E \to \prod_{j \in \mathcal{O}b(\mathscr{I})} D(j)$. For each arrow $\alpha : i \to j$, we have:

$$
\begin{aligned}
\gamma_\alpha \circ q &= \pi_{D(j)} \circ q & &\textit{Definition of } \pi \\
&= q_j & &\textit{Definition of } q_j \\
&= D(\alpha) \circ q_i & &\textit{Assumption on } q_j \\
&= D(\alpha) \circ \pi_{D(j)} \circ q & &\textit{Definition of } q_i \\
&= \varepsilon_\alpha \circ q & &\textit{Definition of } \varepsilon
\end{aligned}
$$

Since e equalizes π and ε, there exists a unique $f : E \to L$ in $\mathscr{C}$ such that $q = e \circ f$. Then, for each $j \in \mathcal{O}b(\mathscr{I})$, we have $\pi_{D(j)} \circ q = \pi_{D(j)} \circ e \circ f$, hence, $q_i = p_i \circ f$. $\qquad\square$

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