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# Chapter 1

## Background

In this chapter the building blocks for this work, almost entirely based on categories, will be defined. The aim of what follows is not only to introduce concepts that will be used later, but also to understand how category theory is general enough to give the abstraction of known notions (mainly from set theory) to reuse them in different contexts. This is not a complete tutorial on categories, but instead a sufficient compendium of definitions to make clear what will be done in the next chapters.

### 1.1 Basic Notions

This section is all about basic definitions and examples, to get familiar with the formalism of categories.

#### 1.1.1 Categories

**Definition 1.1.1** (Category). A *category*  $\mathcal{C}$  comprises:

1. A collection of *objects*  $\mathcal{Ob}(\mathcal{C})$ ;
2. A collection of *arrows* (or *morphisms*)  $\mathcal{Hom}(\mathcal{C})$ , often called *homset*.

Two operators, *dom* and *cod*, that map every morphism  $f \in \mathcal{Hom}(\mathcal{C})$  to two objects, respectively, its *domain* and its *codomain*. In case  $\text{dom } f = A$  and  $\text{cod } f = B$ , we will write  $f : A \rightarrow B$ . The collection of morphisms from an object  $A$  to an object  $B$  is denoted as  $\mathcal{C}(A, B)$ . An operator  $\circ$  of *composition* maps every couple of morphisms  $f, g$  with  $\text{cod } f = \text{dom } g$  (in this case  $f$  and  $g$  are said to be

composable) to a morphism  $g \circ f : \text{dom } f \rightarrow \text{cod } g$ . The composition operator is associative, i.e., for each composable arrows  $f$ ,  $g$  and  $h$ , it holds that

$$h \circ (g \circ f) = (h \circ g) \circ f$$

For each object  $A$ , an *identity* morphism  $\text{id}_A : A \rightarrow A$  (or, when it is clear from the context, just denoted  $A$ ) such that, for each  $f : A \rightarrow B$ :

$$\text{id}_B \circ f = f = f \circ \text{id}_A$$

The most important thing here is not the structure of the objects, but instead how this structure is preserved by the morphisms.

**Example 1.1.2.** A trivial example of category is the one with no objects, and hence no morphisms. Such category is denoted by  $\mathbf{0}$  and is called *empty category*.

**Example 1.1.3.** The category with just one object and just one arrow, the identity arrow on that object, is denoted  $\mathbf{1}$ . In particular, the only object of this category is  $\bullet$ , and the only arrow is  $\text{id}_\bullet$ .

Given an arrow  $f : A \rightarrow B$  in a category  $\mathcal{C}$ , we say that  $f$  *factors through*  $g : C \rightarrow B$  if there exists an arrow  $h : A \rightarrow C$  such that  $f = h \circ g$ .

**Definition 1.1.4.** [Dual Category] Given a category  $\mathcal{C}$ , there exist a category  $\mathcal{C}^{op}$  such that:

- $\text{Ob}(\mathcal{C}^{op}) = \text{Ob}(\mathcal{C})$ ;
- if  $f : A \rightarrow B$  is a morphism in  $\mathcal{C}$ , then  $f : B \rightarrow A$  is a morphism in  $\mathcal{C}^{op}$ .

Hence, given  $f : A \rightarrow B$  and  $g : B \rightarrow C$  arrows in  $\mathcal{C}$ , as  $g \circ f : A \rightarrow C$  is an arrow in  $\mathcal{C}$ , then  $f \circ g : C \rightarrow A$  is an arrow in  $\mathcal{C}^{op}$ . Such category is called *dual category* or *opposite category*.

Duality is a concept that we will encounter most of the time. Given a property  $P$  valid for a category  $\mathcal{C}$ , we will refer to the same property in the opposite category  $\mathcal{C}^{op}$  as the *dual* of  $P$ , without explicitly constructing  $\mathcal{C}^{op}$ . There exist some properties that coincide exactly with their dual, and such properties are said to be *self dual* properties.

To represent morphisms of a category  $\mathcal{C}$  it is possible to use *diagrams*, as the one below, in which the vertices are objects of  $\mathcal{C}$ , and the edges are morphisms of  $\mathcal{C}$ .

$$\begin{array}{ccc} X & \xrightarrow{f'} & Z \\ g' \downarrow & & \downarrow g \\ W & \xrightarrow{f} & Y \end{array}$$

The diagram is said to commute whenever  $f \circ g' = g \circ f'$ . Unique morphisms are represented with dashed arrows. A more rigorous definition of what a diagram is will be given later (Definition 1.2.3).

**Example 1.1.5.** It is easy to see that taking sets as objects and (total) functions as arrows, we obtain a category. In fact, given two functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , it is possible to compose them obtaining an arrow  $g \circ f : A \rightarrow C$ , and the composition is associative. For each set  $A$  there exists an identity function  $id_A : A \rightarrow A$  such that  $id_A(a) = a$  for each  $a \in A$ . This category is denoted as **Set**.

**Remark 1.1.6.** It is important to note that the Definition 1.1.1 above does not specify what kind of collections  $\mathcal{Ob}(\mathcal{C})$  and  $\mathcal{Hom}(\mathcal{C})$  are. Taking **Set** as example, the collection  $\mathcal{Ob}(\mathbf{Set})$  cannot be a set itself, due to Russel's paradox. It would be more appropriate referring to a category  $\mathcal{C}$  which  $\mathcal{Ob}(\mathcal{C})$  and  $\mathcal{Hom}(\mathcal{C})$  are both sets as a *small category*, but it is assumed in this work, except where it is made explicit, for a category to be small. Another clarification must to be done, still considering **Set**. Given two sets  $A$  and  $B$ , it is possible to construct the set  $B^A$  of all functions from  $A$  to  $B$ . This is isomorphic to  $\mathbf{Set}(A, B)$ , for each pair of sets  $A$  and  $B$ . A category  $\mathcal{C}$  where, for each pair of objects  $A$  and  $B$ ,  $\mathcal{C}(A, B)$  is a set is said to be *locally small*.

### 1.1.2 Mono, Epi and Iso

Between the morphisms of a category, it is possible to distinguish some that have certain properties, as functions between sets can be surjective, injective or bijective.

**Definition 1.1.7** (Monomorphism). An arrow  $f : B \rightarrow C$  in a category  $\mathcal{C}$  is a *monomorphism* if, for any pair of arrows of  $\mathcal{C}$   $g : A \rightarrow B$ ,  $h : A \rightarrow B$ , the equality  $f \circ g = f \circ h$  implies  $g = h$ . The class of monomorphisms of  $\mathcal{C}$  is denoted  $\text{Mono}(\mathcal{C})$ .

**Remark 1.1.8.** For a morphism, from an algebraic point of view, being mono means being *left cancelable*. Let  $A$  be an object in a category  $\mathcal{C}$ . Given two monomorphism  $m : X \rightarrow A$  and  $n : Y \rightarrow A$ , then if  $h : X \rightarrow Y$  is a morphism such that  $m = n \circ h$ , then is the unique one: suppose  $k$  is another morphism such that  $m = n \circ k$ . We can conclude  $h = k$  observing that  $n \circ h = n \circ k$  implies  $h = k$  when  $n$  is mono, which is by hypothesis.

**Definition 1.1.9** (Subobject). Starting from this consideration, we can define a preorder on monomorphisms, placing  $m \leq n$  if  $m = n \circ h$  for some  $h$ . Such preorder induces an equivalence relation  $\equiv$  on monomorphisms with codomain  $A$ , where  $m \equiv n$  whenever  $m \leq n$  and  $n \leq m$ , and the corresponding equivalence class is called *subobject* of  $A$ .

**Definition 1.1.10** (Epimorphism). An arrow  $f : A \rightarrow B$  in a category  $\mathcal{C}$  is an *epimorphism* if, for any pair of arrows of  $\mathcal{C}$   $g : B \rightarrow C$ ,  $h : B \rightarrow C$ , the equality  $g \circ f = h \circ f$  implies  $g = h$ .

**Definition 1.1.11** (Isomorphism). An arrow  $f : A \rightarrow B$  is an *isomorphism* if there is an arrow  $f^{-1} : B \rightarrow A$ , called the *inverse* of  $f$ , such that  $f^{-1} \circ f = \text{id}_A$  and  $f \circ f^{-1} = \text{id}_B$ . Two objects are said to be *isomorphic* if there is an isomorphism between them.

**Example 1.1.12.** In **Set**, monomorphisms are injective functions, epimorphisms are surjective functions and isomorphisms are bijections.

**Remark 1.1.13.** Mono and epi are dual concepts. This fact is easily shown by considering how a monomorphism  $m$  in a category  $\mathcal{C}$  behaves in the dual category  $\mathcal{C}^{op}$ . In  $\mathcal{C}$  we have that  $m \circ f = m \circ g$  implies  $f = g$ . In  $\mathcal{C}^{op}$ , we can state that  $f \circ m = g \circ m$  implies  $f = g$ , obtaining the definition of epi.

**Proposition 1.1.14.** *The following statements hold for every pair of composable arrows  $f$  and  $g$  for any category  $\mathcal{C}$ :*

1. *if both  $f$  and  $g$  are mono, then  $g \circ f$  is mono;*
2. *if  $g \circ f$  is mono, then  $f$  is mono;*

3. if both  $f$  and  $g$  are epi, then  $g \circ f$  is epi;

4. if  $g \circ f$  is epi, then  $g$  is epi.

**Observation 1.1.15.** Consider two monomorphisms  $m : X \rightarrow A$  and  $n : Y \rightarrow A$ , and suppose  $m \equiv n$  (in the sense of Definition 1.1.9). This corresponds to having an isomorphism between  $X$  and  $Y$ . In fact, since  $m \leq n$ , there exists a unique  $h : Y \rightarrow X$  such that  $m = n \circ h$ , and, since  $n \leq m$ , there exists a unique  $k : X \rightarrow Y$  such that  $n = m \circ k$ . But then

$$\begin{aligned} m &= n \circ h & n &= m \circ k \\ &= m \circ h \circ k & &= n \circ h \circ k \end{aligned}$$

Since  $m$  is mono, we obtain  $id_X = h \circ k$ , and, since  $n$  is mono,  $id_Y = k \circ h$ , thus, an isomorphism.

The next proposition will be useful later.

**Proposition 1.1.16.** In **Set**, for every commutative square as the one below, if  $e : X \rightarrow Y$  is epi and  $m : M \rightarrow Z$  is mono, then there exists a unique morphism  $h : Y \rightarrow M$  making the whole diagram below commutative.

$$\begin{array}{ccc} X & \xrightarrow{f} & M \\ e \downarrow & \nearrow h & \downarrow m \\ Y & \xrightarrow{g} & Z \end{array}$$

*Proof.* Let  $y \in Y$ . Since  $e$  is epi (i.e., surjective), there exists  $x \in X$  such that  $e(x) = y$ . Then, we put  $h(y) = f(x)$ .  $h$  is well defined, in fact, let  $x' \in X$  be such that  $e(x') = y$ . In this case, we have  $h(x) = f(x')$ , and so

$$\begin{aligned} (m \circ f)(x') &= (g \circ e)(x') \\ &= g(y) \\ &= (g \circ e)(x) \\ &= (m \circ f)(x) \end{aligned}$$

since  $m$  is mono by hypothesis,  $f(x) = f(x')$ . □

### 1.1.3 Categories from other categories

Starting from a category, it is possible to construct other categories with some interesting properties, as the following examples show.

The first notion to introduce is the one of subcategory.

**Definition 1.1.17** (Subcategory). A category  $\mathcal{D}$  is a *subcategory* of a category  $\mathcal{C}$  if:

1. each object of  $\mathcal{D}$  is an object of  $\mathcal{C}$ ;
2. each morphism between two objects of  $\mathcal{D}$  is a morphism of  $\mathcal{C}$ ;  
and
3. composites and identities of  $\mathcal{D}$  are the same of  $\mathcal{C}$ .

If the inclusion at 2 is an equality (i.e.  $\mathcal{D}(A, B) = \mathcal{C}(A, B)$  for each couple of objects  $A, B$  of  $\mathcal{D}$ ), then  $\mathcal{D}$  is said to be a *full subcategory* of  $\mathcal{C}$ . Another way to express that composites are the same (point 3) is to say that if  $f, g \in \mathcal{H}om(\mathcal{D})$  are composable, then  $g \circ f \in \mathcal{H}om(\mathcal{D})$ , i.e.,  $\mathcal{H}om(\mathcal{D})$  is *closed under composition*.

An object of a category marks out a category itself. This is the case of slice (and coslice) categories.

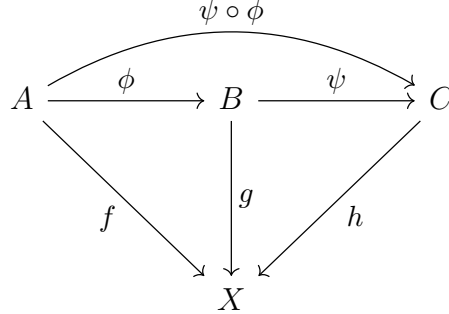
**Definition 1.1.18** (Slice Category). Given a category  $\mathcal{C}$  and an object  $X \in \mathcal{O}b(\mathcal{C})$ , the *slice category*  $\mathcal{C}/X$  is the category that has pairs  $(A, f)$  as objects, where  $A$  is an object of  $\mathcal{C}$  and  $f : A \rightarrow X$  is an arrow in  $\mathcal{C}$ , and arrows  $\phi : (A, f) \rightarrow (B, g)$  are given by a morphism  $\phi : A \rightarrow B$  of  $\mathcal{C}$  such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ & \searrow f & \downarrow g \\ & & X \end{array}$$

– i.e.,  $g \circ \phi = f$ . Composition between two arrows in  $\mathcal{C}/X$   $\phi : (A, f) \rightarrow (B, g)$  and  $\psi : (B, g) \rightarrow (C, h)$  is the arrow  $\psi \circ \phi : (A, f) \rightarrow$



$(C, h)$  obtained in the obvious way:



The dual definition of *coslice category*, noted  $X/\mathcal{C}$  (where  $X \in \mathcal{Ob}(\mathcal{C})$ ), is obtained by taking as objects the morphisms of  $\mathcal{C}$  with domain  $X$  and as arrows the morphisms  $\phi : (A, f) \rightarrow (B, g)$  such that  $f : X \rightarrow A, g : B \rightarrow X$  of  $\mathcal{C}$  and  $g = \phi \circ f$ .

Furthermore, it is possible to raise a new category from two old ones by taking their product, as the following definition shows.

**Definition 1.1.19** (Product category). Given two categories  $\mathcal{C}, \mathcal{D}$ , the *product category*  $\mathcal{C} \times \mathcal{D}$  has as objects pairs of objects  $(A, B)$ , where  $A \in \mathcal{Ob}(\mathcal{C}), B \in \mathcal{Ob}(\mathcal{D})$ , and as arrows pairs of morphisms  $(f, g)$ , where  $f$  is an arrow in  $\mathcal{C}$  and  $g$  is an arrow in  $\mathcal{D}$ . Composition and identities are defined pairwise:  $(f, g) \circ (h, k) = (f \circ h, g \circ k)$ , and  $id_{(A, B)} = (id_A, id_B)$ .

## 1.2 Functors, Natural Transformations, Adjoints

### 1.2.1 Functors

A functor is a structure preserving map between categories.

**Definition 1.2.1** (Functor). Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A *functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a map taking each object of  $A \in \mathcal{Ob}(\mathcal{C})$  to an object  $F(A) \in \mathcal{Ob}(\mathcal{D})$  and each arrow  $f : A \rightarrow B$  of  $\mathcal{C}$  to a arrow  $F(f) : F(A) \rightarrow F(B)$  of  $\mathcal{D}$ , such that, for all objects  $A \in \mathcal{Ob}(\mathcal{C})$  and composable arrows  $f$  and  $g$  of  $\mathcal{C}$ :

- $F(id_A) = id_{F(A)}$ ;

E cambierei anche l'intro a questa sezione DC

la 1.1 è un po' povera come sezione, se sposti questa sezione sui funtori prima, poi puoi arricchire la 1.1 con la sottosezione sulle comma

- $F(g \circ f) = F(g) \circ F(f)$ .

In this case,  $\mathcal{C}$  is called *domain* and  $\mathcal{D}$  is called *codomain* of the functor  $F$ .

**Example 1.2.2.** A first example of functor is the *identity functor*. Given a category  $\mathcal{C}$ , the identity functor  $Id_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$  is the functor that maps each object on itself and each arrow onto itself.

Once defined what a functor is, we can give a more rigorous definition of diagram. Although this may seem extremely technical, it will be useful, especially in the definition of limits (Definition 1.3.3).

**Definition 1.2.3** (Diagram). A *diagram in a category  $\mathcal{C}$  of shape  $\mathcal{I}$*  is a functor  $D : \mathcal{I} \rightarrow \mathcal{C}$ . The category  $\mathcal{I}$  can be considered as the category indexing the objects and the morphisms of  $\mathcal{C}$  shaped in  $\mathcal{I}$ .

**Example 1.2.4.** A diagram of shape  $\Lambda = (L \xleftarrow{l} X \xrightarrow{r} R)$  is said to be a *span*, and is denoted by  $(l, X, r) : L \rightarrow R$ . A span can be viewed as the generalization of relations between sets. In fact, in **Set**, a relation  $R \subseteq A \times B$  is a span, with the projections  $\pi_A : R \rightarrow A$  and  $\pi_B : R \rightarrow B$  as arrows.

The dual notion of span is a *cospan*, namely, a diagram of shape  $\Lambda^{op} = (L \xrightarrow{l} X \xleftarrow{r} R)$ , and is denoted by  $(l, X, r) : L \rightarrow R$ .

Functors are often used to generalize some structural behaviour that constructions in categories have. An important example of this fact is the universal property. The definition is not straightforward, but it gives the abstraction of a property that will be useful in further definitions

**Definition 1.2.5** (Universal property). Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor, and let  $B \in \mathcal{Ob}(\mathcal{D})$ . A pair  $(u, A)$ , with  $A \in \mathcal{Ob}(\mathcal{C})$  and  $u : B \rightarrow F(A)$  is said to be an *universal map for  $B$  with respect to  $F$*  if for each  $A' \in \mathcal{Ob}(\mathcal{C})$  and each  $f : B \rightarrow F(A')$  there exists a unique morphism  $h \in \mathcal{C}(A, A')$  such that the following triangle commute:

$$\begin{array}{ccc}
 B & \xrightarrow{u} & F(A) \\
 & \searrow f & \downarrow F(h) \\
 & & F(A')
 \end{array}
 \qquad
 \begin{array}{c}
 A \\
 \downarrow h \\
 A'
 \end{array}$$

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– i.e. there exists a unique  $h$  such that  $F(h) \circ u = f$ . In this case,  $(u, A)$  is said to have the *universal property*.

Dually, if  $G : \mathcal{C} \rightarrow \mathcal{D}$  is a functor and  $B \in \mathcal{Ob}(\mathcal{D})$ , then a pair  $(A, u)$  is a *co-universal map for  $B$  with respect to  $G$*  if  $u : G(A) \rightarrow B$  and for each  $A' \in \mathcal{Ob}(\mathcal{C})$  and each  $f : G(A') \rightarrow B$  there exists a unique morphism  $h \in \mathcal{C}(A', A)$  such that the following diagram commutes:

$$\begin{array}{ccc}
 A' & & G(A') \\
 \downarrow h & & \downarrow G(h) \\
 A & & G(A)
 \end{array}
 \quad
 \begin{array}{ccc}
 & & \\
 & \searrow f & \\
 G(A) & \xrightarrow{u} & B
 \end{array}$$

Some interesting properties of certain functors depend strictly on how they behave on the homsets of the domain and the codomain categories. The following definitions are about this particular type of functors.

**Definition 1.2.6** (Full functor, faithful functor, fully faithful functor). Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor, and consider the induced function

$$F_{A,B} : \mathcal{C}(A, B) \rightarrow \mathcal{D}(F(A), F(B))$$

If, for each  $A, B$  objects of  $\mathcal{C}$ ,  $F_{A,B}$  is surjective, then  $F$  is said to be *full*, if it is injective,  $F$  is said to be *faithful*, if it is both injective and surjective,  $F$  is said to be *fully faithful*.

**Observation 1.2.7.** Properties such as fullness and faithfulness are so called *self-dual*, because the dual notion coincide with the same notion. This fact can be advantageous because if for example the faithfulness implies the preservation of some property, then the dual property is implied at the same way.

**Example 1.2.8.** Let  $\mathcal{C}$  be a category and  $\mathcal{D}$  a subcategory. The inclusion functor  $I : \mathcal{D} \rightarrow \mathcal{C}$ , mapping each object and each arrow onto itself.  $I$  is a faithful functor, because, given any pair of objects  $A$  and  $B$  of  $\mathcal{D}$ ,  $I_{A,B}$  is injective. If  $\mathcal{D}$  is a full subcategory, then  $I$  is fully faithful.

Having such classification among functors turns out to be useful in many contexts. For example, consider  $F(m) : F(B) \rightarrow F(C)$  be a monomorphism in a category  $\mathcal{D}$ , where  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a faithful functor. Then, if  $f, g : A \rightarrow B$  are two morphisms in  $\mathcal{C}$  such that  $m \circ f = m \circ g$ , then  $F(m \circ f) = F(m) \circ F(f) = F(m) \circ F(g) = F(m \circ g)$ . Since  $F(m)$  is mono, then  $F(f) = F(g)$ , and, since  $F_{A,B}$  is injective,  $f = g$ . Together with the fact that faithfulness is a self-dual concept, we have a proof for what follows [HS79].

**Proposition 1.2.9.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a faithful functor. Then  $F$  reflects monomorphisms and epimorphisms.*

### 1.2.2 Natural Transformations

Given two functors that share domain and codomain categories, it is possible to define a transformation between them, taking each object of the domain of the functors to an arrow in the codomain of the functors that represent the action of “changing the functor acting on that object”.

**Definition 1.2.10** (Natural transformation). Let  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  be two functors. A *natural transformation*  $\eta$  between them, denoted  $\eta : F \rightarrowtail G$ , is a function  $\eta : \mathcal{Ob}(\mathcal{C}) \rightarrow \mathcal{Hom}(\mathcal{D})$  taking each  $A \in \mathcal{Ob}(\mathcal{C})$  to a morphism  $\eta_A : F(A) \rightarrow G(A)$  in  $\mathcal{D}$ , such that, for each morphism  $f : A \rightarrow B$  of  $\mathcal{C}$ , the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ F(f) \downarrow & & \downarrow G(f) \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array}$$

– i.e., such that  $G(f) \circ \eta_A = \eta_B \circ F(f)$ .

We say that  $\eta : F \rightarrowtail G$  is a *natural isomorphism* if, for each  $A \in \mathcal{Ob}(\mathcal{C})$ ,  $\eta_A$  is an isomorphism in  $\mathcal{D}$ . In this case,  $F$  and  $G$  are said to be *naturally isomorphic*, and is denoted  $F \cong G$ .

**Observation 1.2.11.** It is easy to see that, given two natural transformations  $\eta : F \rightarrowtail G$ ,  $\theta : G \rightarrowtail H$ , it is possible to compose them obtaining a new natural transformation  $\xi = \theta \circ \eta : F \rightarrowtail H$ . This follows

by the fact that the diagram

$$\begin{array}{ccccc}
 F(A) & \xrightarrow{\eta_A} & G(A) & \xrightarrow{\theta_A} & H(A) \\
 \downarrow F(f) & & \downarrow G(f) & & \downarrow H(f) \\
 F(B) & \xrightarrow{\eta_B} & G(B) & \xrightarrow{\theta_B} & H(B)
 \end{array}$$

commutes because the two inner squares do. Sticking another diagram on the right of the one above, it is possible to show associativity of composition of natural transformations.

### 1.2.3 Functor Categories

The Observation 1.2.11 shows that natural transformations recreate on the functors the same structure that morphisms in a category have on objects. This leads us to define a particular kind of category, in which objects are functors between two categories, and arrow are natural transformations.

**Definition 1.2.12** (Functor Category). Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. The category whose objects are functors between  $\mathcal{C}$  and  $\mathcal{D}$  and whose arrows are natural transformations between them is said to be a *functor category*, and it is denoted by  $[\mathcal{C}, \mathcal{D}]$ .

**Lemma 1.2.13.** Let  $\mathcal{C}, \mathcal{D}, \mathcal{I}$  be categories. Then, it holds that

$$[\mathcal{I}, [\mathcal{C}, \mathcal{D}]] \cong [\mathcal{I} \times \mathcal{C}, \mathcal{D}]$$

A functor with a small category as domain (Remark 1.1.6) and **Set** as codomain is said to be a *presheaf* on that category. Given a category  $\mathcal{C}$ , it is possible to construct the functor category of the presheaves on  $\mathcal{C}$ , i.e.  $[\mathcal{C}, \mathbf{Set}]$ .

**Remark 1.2.14.** What we are calling here a presheaf is not totally accurate, because technically a presheaf on a small category  $\mathcal{C}$  is a functor  $F : \mathcal{C}^{op} \rightarrow \mathbf{Set}$ . This technicality would bring more complexity, and it is beyond the scope of this work, so we will continue adopting the definition given above.

**Lemma 1.2.15.** Let  $\mathcal{C}$  be a category, and let  $[\mathcal{C}, \mathbf{Set}]$  be the category of presheaves on  $\mathcal{C}$ . Let  $D : \mathcal{I} \rightarrow [\mathcal{C}, \mathbf{Set}]$  be a diagram of shape  $\mathcal{I}$  on the presheaves category early mentioned. Then,

1. The limit of  $D$  exists, and it is the presheaf  $L : \mathcal{C} \rightarrow \mathbf{Set}$  such that, for each object  $A$  of  $\mathcal{C}$ ,  $L(A)$  is the limit in  $\mathbf{Set}$  of the values of the presheaves  $D(i)(A)$  for each  $i$ .
2. The colimit of  $D$  exists, and it is the presheaf  $C : \mathcal{C} \rightarrow \mathbf{Set}$  such that, for each object  $A$  of  $\mathcal{C}$ ,  $C(A)$  is the colimit in  $\mathbf{Set}$  of the values of the presheaves  $D(i)(A)$ , for each  $i$ .

### 1.2.4 Comma Categories

Functor constructions allow us to generalise basic concepts already seen for categories. An important example of this fact are comma categories, a more general notion of slice categories (Definition 1.1.18).

**Definition 1.2.16** (Comma category). Let  $\mathcal{C}$ ,  $\mathcal{D}$  and  $\mathcal{E}$  be categories, and let  $S : \mathcal{C} \rightarrow \mathcal{E}$ ,  $T : \mathcal{D} \rightarrow \mathcal{E}$  be functors (source and target):

$$\mathcal{C} \xrightarrow{S} \mathcal{E} \xleftarrow{T} \mathcal{D}$$

Then, the *comma category*  $(S \downarrow T)$  is the category in which:

- the objects are triples  $(A, f, B)$ , where  $A \in \mathcal{Ob}(\mathcal{C})$ ,  $B \in \mathcal{Ob}(\mathcal{D})$  and  $f : S(A) \rightarrow T(B)$  is an arrow of  $\mathcal{E}$ ;
- the arrows are pairs  $(c, d) : (A, f, B) \rightarrow (C, g, D)$ , where  $c \in \mathcal{Hom}(\mathcal{C})$  and  $d \in \mathcal{Hom}(\mathcal{D})$ , such that the square below commutes;

$$\begin{array}{ccc} S(A) & \xrightarrow{f} & T(B) \\ S(c) \downarrow & & \downarrow T(d) \\ T(C) & \xrightarrow{g} & T(D) \end{array}$$

- composition of morphisms is obtained via pairwise composition, i.e.,  $(a, b) \circ (c, d) = (a \circ c, b \circ d)$ .

Thus, the slice category  $\mathcal{C}/X$  is the comma category given by the two functors  $Id_{\mathcal{C}}$  (the identity functor), and the functor  $!_X : \mathbf{1} \rightarrow \mathcal{C}$ , where  $\mathbf{1}$  is the one-object category defined in Example 1.1.3, and  $!_X$  sends the only object of  $\mathbf{1}$  to  $X$  (then the only morphism of  $\mathbf{1}$  to  $id_X$  of  $\mathcal{C}$ ):

$$\mathcal{C} \xrightarrow{Id_{\mathcal{C}}} \mathcal{C} \xleftarrow{!_X} \mathbf{1}$$

It is easy to see that  $(Id_{\mathcal{C}} \downarrow !_X)$  is exactly the same of  $\mathcal{C}/X$ .

In the same way, it is possible to define coslice categories in terms of comma categories: the category  $(!_X \downarrow Id_{\mathcal{C}})$  is exactly the coslice  $X/\mathcal{C}$ .

### 1.2.5 Adjoints

**Definition 1.2.17** (Right Adjoint). Let  $R : \mathcal{C} \rightarrow \mathcal{D}$  be a functor.  $R$  is said *right adjoint* if, for each object  $A$  of  $\mathcal{D}$ , there exists an object  $L(A)$  and an arrow  $\eta_A : A \rightarrow R(L(A))$  in  $\mathcal{C}$  such that, for each arrow  $f : A \rightarrow R(B)$  of  $\mathcal{D}$ , there is a unique arrow  $g : L(A) \rightarrow B$  such that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & R(L(A)) \\ & \searrow f & \downarrow R(g) \\ & & R(B) \end{array}$$

—i.e.,  $R(g) \circ \eta_A = f$ .

**Proposition 1.2.18.** *In Definition 1.2.17, the map that takes an object  $A$  to an object  $L(A)$  can be extended to a functor  $L : \mathcal{D} \rightarrow \mathcal{C}$ . Moreover, there exists a natural transformation  $id_{\mathcal{D}} \rightarrow R \circ L$ .*

*Proof.* Let  $R$  be the right adjoint as in Definition 1.2.17. Given  $f : X \rightarrow Y$ , we can define  $L(f)$  as the unique arrow  $L(X) \rightarrow L(Y)$  whose image through  $R$  fits in the diagram below.

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & R(L(X)) \\ \downarrow f & & \downarrow R(L(f)) \\ Y & \xrightarrow{\eta_Y} & R(L(Y)) \end{array}$$

To see that in this way we get a functor it is now enough to notice

the commutativity of the following diagrams.

$$\begin{array}{ccc}
 X & \xrightarrow{\eta_X} & R(L(X)) \\
 \downarrow id_X & & \downarrow R(id_{L(X)}) \\
 Y & \xrightarrow{\eta_Y} & R(L(Y))
 \end{array}$$
  

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
 \downarrow \eta_X & & \downarrow \eta_Y & & \downarrow \eta_Z \\
 R(L(X)) & \xrightarrow{R(L(f))} & R(L(Y)) & \xrightarrow{R(L(g))} & R(L(Z))
 \end{array}$$

Finally, by construction the family given by all the  $\eta_A: A \rightarrow R(L(A))$  is natural and we can conclude.  $\square$

**Remark 1.2.19.** The family above mentioned is called *unit* of the adjunction.

**Definition 1.2.20** (Left Adjoint). Let  $L: \mathcal{D} \rightarrow \mathcal{C}$  be a functor.  $L$  is a *left adjoint* if, for each object  $B$  of  $\mathcal{C}$ , there exists an object  $R(B)$  and an arrow  $\epsilon_B: L(R(B)) \rightarrow B$  in  $\mathcal{D}$  such that, for each arrow  $g: L(A) \rightarrow B$  of  $\mathcal{C}$ , there exists a unique arrow  $f: A \rightarrow R(B)$  such that the following diagram commutes.

$$\begin{array}{ccc}
 L(R(B)) & \xrightarrow{\epsilon_B} & B \\
 \uparrow L(f) & \nearrow g & \\
 L(A) & & 
 \end{array}$$

– i.e.,  $\epsilon_B \circ L(f) = g$ .

As we have shown before, it is possible to extend the mapping  $A \rightarrow R(B)$  to a functor  $R$ , whose functoriality follows placing  $\epsilon_X \circ L(R(f)) = f \circ \epsilon_Y$  for each  $f: X \rightarrow Y$ . The family  $\epsilon_B: L(R(B)) \rightarrow B$  is natural and it is called *counit* of the adjunction.



The connection between left and right adjoints is expressed in the following proposition.

**Proposition 1.2.21.** *Let  $L$  be the functor of Proposition 1.2.18. Then,  $L$  is a left adjoint.*

*Proof.* Given an object  $B$  in  $\mathcal{C}$ , we can consider the solid part of the diagram below. Since  $R$  is a right adjoint, we get a unique arrow whose image through  $R$  make the triangle commutative.

$$\begin{array}{ccc}
 R(B) & \xrightarrow{\eta_{R(B)}} & R(L(R(B))) \\
 & \searrow id_{R(B)} & \downarrow R(\epsilon_B) \\
 & & R(B)
 \end{array}$$

Let now  $A$  be an object of  $\mathcal{D}$  and  $g: L(A) \rightarrow B$  an arrow in  $\mathcal{C}$ . We can consider the composite  $R(g) \circ \eta_A: A \rightarrow R(B)$ . Then we have

$$\begin{aligned}
 R(\epsilon_B) \circ R(L(R(g))) \circ R(L(\eta_A)) \circ \eta_A &= R(\epsilon_B) \circ R(RL(R(g))) \circ \eta_{R(L(A))} \circ \eta_A \\
 &= R(\epsilon_B) \circ \eta_{R(B)} \circ R(g) \circ \eta_A \\
 &= R(g) \circ \eta_A
 \end{aligned}$$

Since  $R$  is a right adjoint and  $\eta$  its unit, it follows that  $\epsilon_B \circ L(R(g) \circ \eta_A)$  coincides with  $g$  as wanted.  $\square$

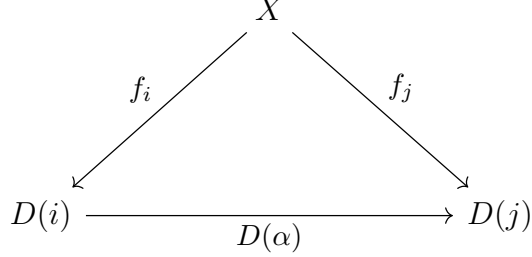
## 1.3 Limits and Universal Constructions

### 1.3.1 Limits and Colimits

**Definition 1.3.1** (Cones). Let  $D: \mathcal{I} \rightarrow \mathcal{C}$  be a diagram in  $\mathcal{C}$  of shape  $\mathcal{I}$ . A *cone* for  $D$  is an object  $X$  of  $\mathcal{C}$ , together with arrows  $f_i: X \rightarrow D(i)$  indexed by  $\mathcal{I}$  (i.e. one for each object  $i$  of  $\mathcal{I}$ ), such that, for each morphism  $\alpha: i \rightarrow j$  of  $\mathcal{I}$ , the following diagram

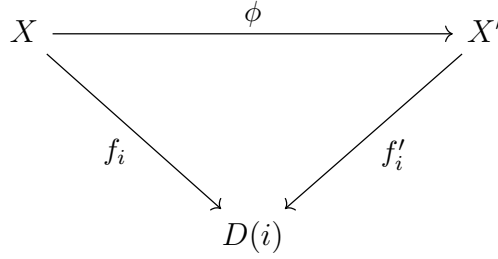
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commutes:

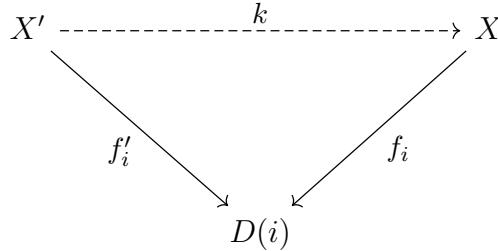


– i.e.,  $D(\alpha) \circ f_i = f_j$ . We denote such cone as  $\{f_i : X \rightarrow D(i)\}$ .

**Observation 1.3.2.** Given a diagram  $D$ , the category of the cones for  $D$ , denoted  $\mathbf{Cone}(D)$ , is defined to have cones for  $D$  as objects and cone morphisms as arrows, where a cone morphism  $\phi : C \rightarrow C'$  from  $C = \{f_i : X \rightarrow D(i)\}$  to  $C' = \{f'_i : X' \rightarrow D(i)\}$  is a morphism  $\phi : X \rightarrow X'$  such that the following diagram commutes for each  $i$ :



**Definition 1.3.3** (Limits). Let  $D : \mathcal{I} \rightarrow \mathcal{C}$  be a diagram in  $\mathcal{C}$  of shape  $\mathcal{I}$ . A cone  $\{f_i : X \rightarrow D(i)\}$  is a *limit* provided that, for any other cone  $\{f'_i : X' \rightarrow D(i)\}$  for  $D$ , then there exists a unique morphism  $k : X' \rightarrow X$  such that the following diagram commutes for each object  $i$  of  $\mathcal{I}$ :



– i.e.,  $f_i \circ k = f'_i$  for each object  $i$  of  $\mathcal{I}$ . Such limit is denoted as  $(X, f_i)_{i \in \mathcal{I}}$ . Sometimes we will refer to the unique arrow  $k$  as the *mediating arrow*.

**Observation 1.3.4.** Given a diagram  $D$ , a limit for  $D$  is exactly the terminal object of the category  $\mathbf{Cone}(D)$ , defined in Observation 1.3.2.

The dual notions of cones and limits are that of cocones and colimits.

**Definition 1.3.5.** (Cocones, Colimits) A *cocone* for a diagram  $D : \mathcal{I} \rightarrow \mathcal{C}$  is an object  $Y$  of  $\mathcal{C}$  together with arrows  $f_i : D(i) \rightarrow Y$  such that, for each  $g : D(i) \rightarrow D(j)$  of  $\mathcal{C}$ ,  $f_j \circ g = f_i$ . A cocone is denoted  $\{f_i : D(i) \rightarrow Y\}$ . A *colimit* for  $D$  is a cocone  $C = \{f_i : D(i) \rightarrow Y\}$  with the universal property – i.e., if  $C' = \{f'_i : D(i) \rightarrow Y'\}$  is another cone for  $D$ , then there exists a unique arrow  $h : Y \rightarrow Y'$  such that, for each  $i$ ,  $h \circ f_i = f'_i$ .

**Remark 1.3.6.** It makes sense to refer to a (co)limit as *the* (co)limit. Suppose  $(P, p_i)_{i \in \mathcal{I}}$  and  $(Q, q_i)_{i \in \mathcal{I}}$  be limits for a diagram  $D : \mathcal{I} \rightarrow \mathcal{C}$ . Then, there exists a unique morphism  $h : Q \rightarrow P$  such that  $p_i \circ k = q_i$  for each  $i$ . At the same way, there exists a unique morphisms  $k : P \rightarrow Q$  such that  $q_i \circ k = p_i$  for each  $i$ . From the existence of the identity, must be  $k \circ h = id_Q$  and  $h \circ k = id_P$ , that is,  $P$  and  $Q$  are isomorphic.

Notion such limits and colimits are generalization of more particular cases that will be now introduced, that we will often call *universal constructions*.

**Definition 1.3.7** (Initial Object, Terminal Object). Consider the empty diagram (i.e., a diagram  $D : \mathbf{0} \rightarrow \mathcal{C}$  where  $\mathbf{0}$  is the empty category Example 1.1.2). Then, the limit of  $D$  is called *terminal object* and the colimit of  $D$  is called *initial object*, denoted, respectively,  $\mathbb{1}_{\mathcal{C}}$  and  $\mathbb{0}_{\mathcal{C}}$ . (Subscripts are omitted when they are clear from the context).

**Example 1.3.8.** In **Set**, the initial object is the empty set  $\emptyset$ , because, for each set  $S$ , there exists the empty function from  $\emptyset$  to  $S$ . The terminal object of **Set** is the singleton  $\{\bullet\}$ , because there is exactly one function from a set  $S$  to  $\{\bullet\}$ , namely, the function which sends each  $s \in S$  to  $\bullet$ .

We now illustrate a result on functor categories (Definition 1.2.12) that will be useful later.

**Proposition 1.3.9.** *Let  $\mathcal{D}$  be a category. If  $\mathcal{D}$  has an initial object, then, for any category  $\mathcal{C}$ ,  $[\mathcal{C}, \mathcal{D}]$  has an initial object. If  $\mathcal{D}$  has a terminal object, then, for any category  $\mathcal{C}$ ,  $[\mathcal{C}, \mathcal{D}]$  has a terminal object.*

*Proof.* Let  $\mathbb{0}_{\mathcal{D}}$  be the initial object of  $\mathcal{D}$ , and consider the constant functor  $I(f) = id_{\mathbb{0}_{\mathcal{D}}}$  for all  $f \in \mathcal{H}om(\mathcal{C})$ . Then, for any  $G : \mathcal{C} \rightarrow \mathcal{D}$ ,  $\eta : I \rightarrow G$ , defining  $\eta_A$  as the *unique morphism from  $\mathbb{0}_{\mathcal{D}}$  to  $G(A)$*  for each  $A \in \mathcal{O}b(\mathcal{C})$ , is a natural transformation  $I \rightarrow G$ , as the diagram below shows:

$$\begin{array}{ccc} I(A) = \mathbb{0}_{\mathcal{D}} & \xrightarrow{\eta_A} & G(A) \\ \downarrow I(f) = id_{\mathbb{0}_{\mathcal{D}}} & & \downarrow G(f) \\ I(A') = \mathbb{0}_{\mathcal{D}} & \xrightarrow{\eta_{A'}} & G(A') \end{array}$$

for each  $f : A \rightarrow A'$ , the square above must commute, since there is only one morphism from  $\mathbb{0}_{\mathcal{D}}$  to  $G(A')$ . For the same reason,  $\eta$  is the only natural transformation from  $I$  to  $G$ , being indeed the initial object of  $[\mathcal{C}, \mathcal{D}]$ .

Defining  $T(f) = id_{\mathbb{1}_{\mathcal{D}}}$  for each  $f \in \mathcal{H}om(\mathcal{C})$ . Then,  $\theta : F \rightarrow T$ , for any  $F : \mathcal{C} \rightarrow \mathcal{D}$ , defining  $\theta_A$  as the *unique morphism from  $F(A)$  to  $\mathbb{1}_{\mathcal{D}}$*  is a natural transformation due to the commutativity of the following diagram for each  $f : A \rightarrow A'$ :

$$\begin{array}{ccc} F(A) & \xrightarrow{\theta_A} & T(A) = \mathbb{1}_{\mathcal{D}} \\ \downarrow F(f) & & \downarrow T(f) = id_{\mathbb{1}_{\mathcal{D}}} \\ F(A') & \xrightarrow{\theta_{A'}} & T(A') = \mathbb{1}_{\mathcal{D}} \end{array}$$

Hence,  $\theta$  is the unique natural transformation from  $F$  to  $T$ , and  $T$  is the terminal object of  $[\mathcal{C}, \mathcal{D}]$ .  $\square$

In particular, every presheaf has an initial and a terminal object, because **Set** does (Example 1.3.8).

**Definition 1.3.10** (Product, Coproduct). Let  $D$  be the following diagram:

$$\begin{array}{ccc} A & & B \end{array}$$

Then, a cone for  $D$  is an object  $X$  and two arrows  $f : X \rightarrow A$ ,  $g : X \rightarrow B$  (i.e., a span, defined in Example 1.2.4):

$$A \xleftarrow{f} X \xrightarrow{g} B$$

If it exists, a limit for  $D$  is called *product* of  $A$  and  $B$ , usually denoted as  $(A \times B, \pi_A, \pi_B)$ , while whose arrows are called *projections*. The colimit of  $D$  is called *coproduct* of  $A$  and  $B$ , usually denoted as  $(\iota_A, \iota_B, A + B)$ .

**Example 1.3.11.** **Set** has both products and coproducts. Given two sets  $A$  and  $B$ , the categorical product is the set-theoretic cartesian product  $A \times B$ , together with the two projections  $\pi_A$  and  $\pi_B$ , while the coproduct is the disjoint sum  $A \amalg B = \{(x, 0) \mid x \in A\} \cup \{(y, 1) \mid y \in B\}$ , together with the two canonical injections  $\iota_A$  and  $\iota_B$ , where  $\iota_A(a) = (a, 0)$  and  $\iota_B(b) = (b, 1)$ .

The notions of product and coproduct can be easily generalized, extending the definition to the product (and coproduct) of a family of objects, together with appropriate arrows (e.g., the projection arrows for each object in the product). We will denote the product of a collection of objects indexed by a (finite) category  $\mathcal{I}$  as  $(\prod_{i \in \text{Ob}(\mathcal{I})} X_i, (\pi_i)_{i \in \text{Ob}(\mathcal{I})})$ , and the coproduct as  $((\iota_i)_{i \in \text{Ob}(\mathcal{I})}, \coprod_{i \in \text{Ob}(\mathcal{I})} X_i)$ .

**Definition 1.3.12** (Equalizer, Coequalizer). Let  $D$  be the diagram below.

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

The limit of  $D$  is called *equalizer*, and its colimit is called *coequalizer*.

**Proposition 1.3.13.** Let  $e : E \rightarrow A$  be the arrow that equalizes  $f, g : A \rightarrow B$  in a category  $\mathcal{C}$ . Then,  $e$  is a monomorphism.

*Proof.* Suppose  $X$  be an object and  $x, y : X \rightarrow E$  be two morphisms in  $\mathcal{C}$  such that  $e \circ x = e \circ y$ , and let  $z = e \circ x$ . Then, since  $e$  is an equalizer,  $f \circ e = g \circ e$ , and  $f \circ z = g \circ z$ . But, for the universal property of limits, there must be exactly one  $u : X \rightarrow E$  such that  $z = e \circ u$ . It follows that  $x = u$  and  $y = u$ , hence  $x = y$ .  $\square$

The dual of the proposition above states that a coequalizer is an epimorphism.

Of all monomorphisms, an interesting subclass of them is the one that contains only the equalizers.

**Definition 1.3.14** (Regular Monomorphism, Regular Epimorphism). A monomorphism that is an equalizer for a pair of arrows is said *regular monomorphism*. The class of all regular monomorphisms of a category  $\mathcal{C}$  is denoted  $\mathcal{R}eg(\mathcal{C})$ . An epimorphism that is a coequalizer for a pair of arrows is said *regular epimorphism*.

**Definition 1.3.15** (Pullback, Pushout). Let  $D$  be the cospan  $(f, C, g) : A \rightarrow B$ . A cone for  $D$  is an object  $P$  and three arrows  $\phi : P \rightarrow A$ ,  $\psi : P \rightarrow B$ , and  $h : P \rightarrow C$ , but the latter is uniquely determined by the other ones ( $f \circ \phi = h = g \circ \psi$ ). Thus, the following diagram is a cone:

$$\begin{array}{ccc} P & \xrightarrow{\psi} & B \\ \downarrow \phi & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

Then, the limit of  $D$  is called *pullback* of  $f$  and  $g$ . Given a span  $S = (l, X, r) : L \rightarrow R$ , shown in the diagram below,

$$L \xleftarrow{l} X \xrightarrow{r} R$$

a cocone for  $S$  is any commutative square of the form

$$\begin{array}{ccccc} & & C & & \\ & \nearrow f & & \nwarrow g & \\ L & \xleftarrow{l} & X & \xrightarrow{r} & R \end{array}$$

(the morphism  $X \rightarrow C$  is uniquely determined by the relation  $f \circ l = g \circ r$ ). The colimit for  $S$  is called *pushout* of  $l$  and  $r$ .

**Example 1.3.16.** In **Set**, given two functions  $f : A \rightarrow C$  and  $g : B \rightarrow C$ , a pullback of  $f$  and  $g$  exists and is exactly the set  $P =$

$\{(x, y) \in A \times B \mid f(x) = g(y)\}$ , with  $\pi_f : P \rightarrow B$  and  $\pi_g : P \rightarrow C$  defined, respectively, by  $\pi_f((x, y)) = y$  and  $\pi_g((x, y)) = x$ . In this way, we have then,  $\forall (x, y) \in P$ :

$$\begin{aligned}
 (f \circ \pi_g)((x, y)) &= f(\pi_g((x, y))) \\
 &= f(x) && \text{Definition of } \pi_g \\
 &= g(y) && (x, y) \in P \\
 &= g(\pi_f((x, y))) && \text{Definition of } \pi_f \\
 &= (g \circ \pi_f)((x, y))
 \end{aligned}$$

thus,  $f \circ \pi_g = g \circ \pi_f$ .

Another important example to our aims is a concrete definition of what is a pushout in the category of sets, and why morally we can regard a pushout as *the way to identify part of an object with a part of another* [BW95].

**Example 1.3.17.** In **Set**, given two functions  $f : A \rightarrow B$  and  $g : A \rightarrow C$ , the pushout of them is the set  $X = (B \amalg C)/\sim$ , where  $\sim$  is the least equivalence relation such that  $f(a) \sim g(a)$  for each  $a \in A$ , with  $\iota_g : B \rightarrow X$  and  $\iota_f : C \rightarrow X$  as arrows, sending each element of the domain in the corresponding equivalence class in  $X$ . In particular, for each  $a \in A$ :

$$\begin{aligned}
 (\iota_g \circ f)(a) &= \iota_g(f(a)) \\
 &= [(f(a), 0)] && \text{Definition of } \iota_g \\
 &= [(g(a), 1)] && f(a) \sim g(a) \\
 &= \iota_f(g(a)) && \text{Definition of } \iota_f \\
 &= (\iota_f \circ g)(a)
 \end{aligned}$$

When both  $f$  and  $g$  are monos (that is, injections), then we can construct the pushout in the same way we have done above, with  $(f(a), 0) \sim (g(a), 1)$  when such  $a$  exists and  $(b, 0) \sim (c, 1)$  on each  $b$  and  $c$  with no preimage in  $A$ , with  $\iota_f$  and  $\iota_g$  injective. An easy way to see this fact is considering the following situation: let  $f : A \rightarrow A \cup B$  and  $g : A \rightarrow A \cup C$ , with  $A$  disjoint from  $B$  and  $C$ ,  $f(a) = a$  and  $g(a) = a$ . Then the pushout is the object  $A \cup B \cup C$ , with the inclusions as arrows, that are also injective. A more general case is what happens considering functions  $f : A \rightarrow B$  and  $g : A \rightarrow C$  injective. Differently from the previous example, in this case is not possible to take just the union of codomains as the pushout, but

rather the disjoint union of them and then identify the elements  $f(a)$  with  $g(a)$ , as we have done above. In the category of sets and functions, we have the certainty that the pullback arrows are injective. In fact, taking the equivalence relation  $\sim$ , we have that  $f(a) \sim f(a')$  if and only if  $a = a'$  by hypothesis, and then  $x \sim x'$  if and only if  $x = x'$ , then the pushout morphisms sends each element in an equivalence class composed only by the element itself, thus are injective. This is an interesting property that in other categories may do not hold, and will be recalled later.

Given a subclass of morphisms of a category, an important property is *stability* under certain type of constructions. In our case, we are interested in stability under pullbacks and under pushouts.

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 m \downarrow & & \downarrow n \\
 C & \xrightarrow{g} & D
 \end{array} \quad (*)$$

**Definition 1.3.18** (Stability under pullbacks, pushouts). Given a category  $\mathcal{C}$ , a subclass  $\mathcal{A} \subseteq \mathcal{H}om(\mathcal{C})$  is said to be *stable under pullbacks* if, for every pullback square as the one in (\*), if  $n \in \mathcal{A}$ , then  $m \in \mathcal{A}$ .  $\mathcal{A}$  is said to be *stable under pushouts* if, for every pushout square as the one in (\*), if  $m \in \mathcal{A}$ , then  $n \in \mathcal{A}$ .

**Proposition 1.3.19.** Let  $f : A \rightarrow C$ ,  $g : B \rightarrow C$  be arrows in any category  $\mathcal{C}$ , and consider the following pullback square:

$$\begin{array}{ccc}
 P & \xrightarrow{\pi_f} & B \\
 \pi_g \downarrow & & \downarrow g \\
 A & \xrightarrow{f} & C
 \end{array}$$

If  $g$  is mono, then so is  $\pi_g$ .

The proposition above can be dualised stating that pushouts preserves epimorphisms.

The following lemma is a classical result, its proof is in the appendix.



**Lemma 1.3.20** (Pullback Lemma). *Suppose that the following diagram is given and its right half is a pullback. Then the whole rectangle is a pullback if and only if its left half is a pullback.*

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
 \downarrow t & & \downarrow k & & \downarrow h \\
 A & \xrightarrow{a} & B & \xrightarrow{b} & C
 \end{array}$$

**Corollary 1.3.21.** *Let  $\mathcal{C}$  be a category and suppose that the solid part of the following cube is given*

$$\begin{array}{ccccc}
 & & Y' & \xrightarrow{g'} & Z' \\
 & \swarrow q' & \downarrow & \swarrow r' & \downarrow z \\
 B' & \xrightarrow{\quad} & C' & & \\
 \downarrow b & \swarrow q & \downarrow y & \xrightarrow{g} & \downarrow c \\
 B & \xrightarrow{k} & C & & Z
 \end{array}$$

*If the front face is a pullback then there is a unique  $q': Y' \rightarrow B'$  filling the diagram. If, moreover, the other two vertical faces are also pullbacks, then the following square is a pullback too.*

$$\begin{array}{ccc}
 Y' & \xrightarrow{q'} & B' \\
 \downarrow y & & \downarrow b \\
 Y & \xrightarrow{q} & B
 \end{array}$$

*Proof.* Let us compute:

$$\begin{aligned}
 c \circ r' \circ g' &= r \circ z \circ g' \\
 &= r \circ g \circ y \\
 &= k \circ q \circ y
 \end{aligned}$$

Since the front face is a pullback, this guarantees the existence of  $q'$ . The second half of the thesis follows applying Lemma 1.3.20 to the following rectangle.

$$\begin{array}{ccccc}
 & & r' \circ g' & & \\
 & \nearrow & & \searrow & \\
 Y' & \xrightarrow{q'} & B' & \xrightarrow{k'} & C' \\
 \downarrow y & & \downarrow b & & \downarrow c \\
 Y & \xrightarrow{q} & B & \xrightarrow{k} & C \\
 & \searrow & & \nearrow & \\
 & & r \circ g & & 
 \end{array}$$

□

The connection between constructions as products and equalizers and limits is made clear by the following theorem. The idea behind the proof is the fact that, given a diagram  $D : \mathcal{J} \rightarrow \mathcal{C}$ , if each subset of objects  $X = \{D(i) \mid i \in \mathcal{Ob}(\mathcal{J})\} \subseteq \mathcal{Ob}(\mathcal{C})$  has a product  $(\prod_{i \in I} D(i), (\pi_i)_{i \in \mathcal{Ob}(\mathcal{J})})$  and each pair of arrows  $f, g \in \mathcal{C}(D(i), D(j))$  has an equalizer  $Eq(f, g)$ , then one can construct the cone taking the equalizer of the arrows that has as domain the product of the objects of the diagram, and as codomain the product of the codomains of the arrows of the diagram. This construction has the universal property because equalizers and products do. A detailed proof is in the appendix.

**Theorem 1.3.22** (Limit theorem). *Let  $\mathcal{C}$  be a category. Then  $\mathcal{C}$  has all finite limits if and only if  $\mathcal{C}$  has all finite products and all finite equalizers.*

**Remark 1.3.23.** The theorem above (and its relative proof) can be stated in its dual form leading to a theorem on existence of colimits, and a relative criterion to calculate them (taking the dual of the proof).

**Example 1.3.24.** Limit theorem gives us an easy way to calculate limits. An example of this fact is how limits are computed in **Set**. Given a diagram  $D : \mathcal{J} \rightarrow \mathbf{Set}$ , where  $\mathcal{J}$  is a small category and  $I = \mathcal{Ob}(\mathcal{J})$ , its limit is the set  $L$  defined as follows:

$$L = \{(d_i)_{i \in I} \in \prod_{i \in I} D(i) \mid \forall \phi \in \mathcal{J}(i, i'), D(\phi)(d_i) = d_{i'}\}$$

with projections as arrows.

**Example 1.3.25.** As we have done in Example 1.3.24, we illustrate how to construct colimits in the category of sets. Given a small category  $\mathcal{J}$ ,  $I = \text{Ob}(\mathcal{J})$ , and a diagram  $D : \mathcal{J} \rightarrow \mathbf{Set}$ , consider the equivalence relation  $\sim$  defined on  $\coprod_{i \in I} D(i)$  such that  $d_i \sim d_{i'}$  if  $d_i \in D(i)$ ,  $d_{i'} \in D(i')$  and there exists some  $\phi \in \mathcal{J}(i, i')$  such that  $D(\phi)(d_i) = d_{i'}$ . Then, a colimit for  $D$  is the set

$$C = (\coprod_{i \in I} D(i)) / \sim$$

with the inclusions as arrows.

**Remark 1.3.26.** Since a diagram is nothing more than a functor from a “shape” category to another, it makes sense to talk about limits of functors in general, even when they are not intended to be diagrams.

In the next sections, we will work on a special kind of diagrams with certain properties. In particular, we are interested in how a functor behaves with respect to the constructions defined so far.

**Definition 1.3.27.** Let  $D : \mathcal{J} \rightarrow \mathcal{C}$  be a diagram, and  $F : \mathcal{C} \rightarrow \mathcal{D}$  a functor. We say that  $F$ :

1. *preserves limits* of  $D$  if, given a limit  $(L, l_i)_{i \in \mathcal{J}}$  for  $D$ , then  $(F(L), F(l_i))_{i \in \mathcal{J}}$  is a limit for  $F \circ D$ .
2. *reflects limits* of  $D$  if a cone  $(L, l_i)_{i \in \mathcal{J}}$  is a limit for  $D$  whenever  $(F(L), F(l_i))_{i \in \mathcal{J}}$  is a limit for  $F \circ D$ .
3. *lifts limits (uniquely)* of  $D$  if, given a limit  $(L, l_i)_{i \in \mathcal{J}}$  for  $F \circ D$ , there exists a (unique) limit  $(L', l'_i)_{i \in \mathcal{J}}$  for  $D$  such that  $(F(L'), F(l'_i))_{i \in \mathcal{J}} = (L, l_i)_{i \in \mathcal{J}}$ .
4. *creates limits* of  $D$  if  $D$  has a limit and  $F$  preserves and reflects limits along it.

The dual notions are obtained in the obvious way, namely, substituting the words “limits” and “cones” with “colimits” and “cocones”, respectively

**Observation 1.3.28.** It holds that if a functor creates limits, then lifts uniquely limits [AHS09].

**Proposition 1.3.29.** A fully faithful functor reflects all limits and colimits.

The next theorem is about a particular property that adjoint functors have.

**Theorem 1.3.30.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor, and  $G : \mathcal{D} \rightarrow \mathcal{C}$  its right adjoint. Then,  $G$  preserves limits.*

**Remark 1.3.31.** The dual of the theorem above states that, if  $G$  is a functor and  $F$  is a left adjoint, then  $F$  preserves colimits.

**Proposition 1.3.32.** *Let  $D, D' : \mathcal{J} \rightarrow \mathcal{C}$  be two functors, and let  $((c_i)_{i \in \mathcal{J}}, C)$  and  $((c'_i)_{i \in \mathcal{J}}, C')$  be, respectively, the colimit of  $D$  and  $D'$ . Then, a natural transformation  $\phi : D \rightarrow D'$  induces a unique arrow  $c : C \rightarrow C'$ .*

*Proof.* Consider the following situation.

$$\begin{array}{ccc}
 & C & \\
 c_i \nearrow & & \nwarrow c_j \\
 D(i) & \xrightarrow{D(\alpha)} & D(j) \\
 \phi_i \downarrow & & \downarrow \phi_j \\
 D'(i) & \xrightarrow{D'(\alpha)} & D'(j) \\
 c'_i \searrow & & \swarrow c'_j \\
 & C' &
 \end{array}$$

To prove the statement, we note that  $((c'_i \circ \phi_i)_{i \in \mathcal{J}}, C')$  is a cocone for  $D$ . Computing, we have, for each  $i, j$  and  $\alpha : i \rightarrow j$

$$\begin{aligned}
 c'_j \circ \phi_j \circ D(\alpha) &= c'_j \circ D'(\alpha) \circ \phi_i && \text{Naturality of } \phi \\
 &= c'_i \circ \phi_i && ((c'_i)_{i \in \mathcal{J}}, C') \text{ is a colimit for } D'
 \end{aligned}$$

Since  $((c'_i)_{i \in \mathcal{J}}, C')$  is a limit by hypothesis, then must exists a unique arrow  $c : C \rightarrow C'$  such that  $c \circ c_i = \phi_i \circ c'_i$ , having the thesis.  $\square$

### 1.3.2 Kernel Pairs and Regular Epimorphisms

**Definition 1.3.33** (Kernel Pair). A *kernel pair* for an arrow  $f : A \rightarrow B$  is an object  $K_f$  together with two arrows  $\pi_f^1, \pi_f^2 : K_f \rightarrow A$ ,

denoted as  $(K_f, \pi_f^1, \pi_f^2)$ , such that the following square is a pullback.

$$\begin{array}{ccc}
 K_f & \xrightarrow{\pi_f^1} & A \\
 \pi_f^2 \downarrow & & \downarrow f \\
 A & \xrightarrow{f} & B
 \end{array}$$

**Remark 1.3.34.** If a category  $\mathcal{C}$  has pullbacks then every arrow has a kernel pair.

**Remark 1.3.35.** Since a kernel pair is nothing more than a pullback, that is, a limit, by Remark 1.3.6, it makes sense to refer to it as *the* kernel pair for a morphism  $f$ .

**Example 1.3.36.** In **Set**, a kernel pair for a function  $f : A \rightarrow B$  is the set

$$K_f = \{(x, y) \in A \times A \mid f(x) = f(y)\}$$

together with the canonical projection on the first and the second component of the pairs.

**Proposition 1.3.37.** Let  $(K, p_1, p_2)$  be the kernel pair of  $f : X \rightarrow Y$ , and let  $(X \times X, \pi_1, \pi_2)$  be the product of  $X$  with itself. Then, the mediating arrow  $\langle p_1, p_2 \rangle : K \rightarrow X \times X$  is mono.

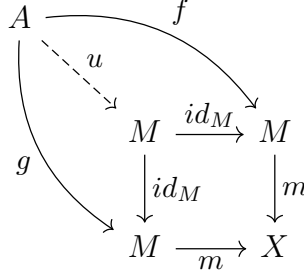
*Proof.* Suppose  $\langle p_1, p_2 \rangle \circ f = \langle p_1, p_2 \rangle \circ g$  for  $f, g : Z \rightarrow K$ . Then, we have

$$\begin{array}{ll}
 \langle p_1, p_2 \rangle \circ f = \langle p_1, p_2 \rangle \circ g & \langle p_1, p_2 \rangle \circ f = \langle p_1, p_2 \rangle \circ g \\
 \pi_1 \circ \langle p_1, p_2 \rangle \circ f = \pi_1 \circ \langle p_1, p_2 \rangle \circ g & \pi_2 \circ \langle p_1, p_2 \rangle \circ f = \pi_2 \circ \langle p_1, p_2 \rangle \circ g \\
 p_1 \circ f = p_1 \circ g & p_2 \circ f = p_2 \circ g
 \end{array}$$

Thus, from the universal property of the pullback,  $f = g$ .  $\square$

**Proposition 1.3.38.** An arrow  $m : M \rightarrow X$  is mono if and only if  $(M, id_M, id_M)$  is a kernel pair for it.

*Proof.* To prove the “if” part of the statement, let  $f, g : A \rightarrow M$  be such that  $m \circ f = m \circ g$ , and consider the following situation.



For the universal property of pullbacks, we have that

$$f = id_M \circ u = g$$

Hence,  $m$  is mono.

Conversely, if  $m$  is mono, then, we have that

$$\begin{aligned} m \circ f &= m \circ g \Rightarrow f = g \\ &\Rightarrow f \circ id_M = g \circ id_M \end{aligned}$$

Hence,  $f$  is the unique arrow that makes the commutative square illustrated above a pushout.  $\square$

**Remark 1.3.39.** From characterization of monos via pullbacks in Proposition 2.2.5 and Lemma 1.2.15, we have that a mono in a category of presheaves is a natural transformation of which each component is mono.

**Corollary 1.3.40.**  $(K_f, \pi_f^1, \pi_f^2)$  is a kernel pair for  $f : X \rightarrow Y$  if and only if, for each mono  $m : Y \rightarrow Z$ ,  $(K_f, \pi_f^1, \pi_f^2)$  is a kernel pair also for  $m \circ f$ .

*Proof.* It is enough to see that, by Lemma 1.3.20 and Proposition 2.2.5 the outer boundary of the following square is a pullback.

$$\begin{array}{ccccc} K_f & \xrightarrow{\pi_f^2} & X & \xrightarrow{id_X} & X \\ \pi_f^2 \downarrow & & \downarrow f & & \downarrow f \\ X & \xrightarrow{f} & Y & \xrightarrow{id_Y} & Y \\ id_X \downarrow & & \downarrow id_Y & & \downarrow m \\ X & \xrightarrow{f} & Y & \xrightarrow{m} & Z \end{array}$$

The leftward part of the statement follows by definition of monomorphism and Lemma 1.3.20.  $\square$

**Lemma 1.3.41.** *Suppose the following situation, and that  $f : X \rightarrow Y$  and  $g : Z \rightarrow W$  have kernel pairs.*

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{t} & W \end{array}$$

*Then, there exists a unique arrow  $k_h : K_f \rightarrow K_g$  making the squares below commute.*

$$\begin{array}{ccc} K_f & \xrightarrow{k_h} & K_g \\ \pi_f^1 \downarrow & & \downarrow \pi_g^1 \\ X & \xrightarrow{h} & Z \end{array} \quad \begin{array}{ccc} K_f & \xrightarrow{k_h} & K_g \\ \pi_f^2 \downarrow & & \downarrow \pi_g^2 \\ X & \xrightarrow{h} & Z \end{array}$$

*Moreover, if the beginning square is a pullback, then also the preceding ones are so.*

*Proof.* Computing, we have

$$\begin{aligned} g \circ h \circ \pi_f^1 &= t \circ f \circ \pi_f^1 \\ &= t \circ f \circ \pi_f^2 \\ &= g \circ h \circ \pi_f^2 \end{aligned}$$

By the universal property of  $K_g$  as the pullback of  $g$  along itself, such  $k_h$  exists and it is unique.

To prove the second half of the thesis, let us consider the two rectangles below, which, by Lemma 1.3.20 are pullbacks.

$$\begin{array}{ccccc} K_f & \xrightarrow{\pi_f^1} & X & \xrightarrow{h} & Z \\ \pi_f^2 \downarrow & & \downarrow f & & \downarrow g \\ X & \xrightarrow{f} & Y & \xrightarrow{t} & W \end{array} \quad \begin{array}{ccccc} K_f & \xrightarrow{\pi_f^2} & X & \xrightarrow{h} & Z \\ \pi_f^1 \downarrow & & \downarrow f & & \downarrow g \\ X & \xrightarrow{f} & Y & \xrightarrow{t} & W \end{array}$$

But then the following ones are pullbacks too.

$$\begin{array}{ccc}
 & \xrightarrow{h \circ \pi_f^2} & \\
 K_f & \xrightarrow{k_h} K_g & \xrightarrow{\pi_g^2} Z \\
 \pi_f^1 \downarrow & & \downarrow \pi_g^1 \\
 X & \xrightarrow{h} Y & \xrightarrow{g} W \\
 & \xleftarrow{t \circ f} &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \xrightarrow{h \circ \pi_f^1} & \\
 K_f & \xrightarrow{k_h} K_g & \xrightarrow{\pi_g^1} Z \\
 \pi_f^2 \downarrow & & \downarrow \pi_g^2 \\
 X & \xrightarrow{h} Y & \xrightarrow{g} W \\
 & \xleftarrow{t \circ f} &
 \end{array}$$

The thesis follows again by Lemma 1.3.20.  $\square$

**Proposition 1.3.42.** *Let  $e : X \rightarrow Y$  be a regular epimorphism in a category  $\mathcal{C}$  with a kernel pair  $(K, \pi_1, \pi_2)$ . Then,  $e$  is the coequalizer of  $\pi_1$  and  $\pi_2$ .*

*Proof.* By hypothesis, there exists a pair  $f, g : Z \rightarrow X$  of which  $e$  is the coequalizer. Since  $e \circ f = e \circ g$ , we have

$$\begin{array}{ccc}
 Z & \xrightarrow{f} & X \\
 \searrow k & & \downarrow \pi_1 \\
 & K & \xrightarrow{\pi_1} X \\
 \downarrow g & \pi_2 \downarrow & \downarrow e \\
 & X & \xrightarrow{e} Y
 \end{array}$$

thus there exists the unique  $k : Z \rightarrow K$ . Let now  $h : Z \rightarrow Y$  be an arrow such that  $h \circ \pi_1 = h \circ \pi_2$ , then

$$\begin{aligned}
 h \circ f &= h \circ \pi_1 \circ k \\
 &= h \circ \pi_2 \circ k \\
 &= h \circ g
 \end{aligned}$$

and thus there exists a unique  $l : Y \rightarrow V$  such that  $l \circ e = h$ .  $\square$

**Corollary 1.3.43.** *Let  $\mathcal{C}$  be a category with pullbacks and  $\phi : D \rightarrow D'$  be a natural transformation between two functors  $D, D' : \mathcal{I} \rightarrow \mathcal{C}$ . If  $\phi_i$  is a regular epi for every  $i$ , then  $\phi$  is a regular epi.*

*Proof.* Let  $(K_i, \pi_i^1, \pi_i^2)$  be the kernel pair of  $\phi_i$  for each  $i$ . Given an arrow  $\alpha : i \rightarrow j$  of  $\mathcal{I}$ , we have

$$\begin{aligned}
 \phi_j \circ D(\alpha) \circ \pi_i^1 &= D'(\alpha) \circ \phi_i \circ \pi_i^1 \\
 &= D'(\alpha) \circ \phi_i \circ \pi_i^2 \\
 &= \phi_j \circ D(\alpha) \circ \pi_i^2
 \end{aligned}$$



Thus, the outer boundary of the diagram below commutes, yielding the arrow  $K(\alpha)$

$$\begin{array}{ccccc}
 K_i & \xrightarrow{\pi_i^1} & D(i) & & \\
 \pi_i^2 \downarrow & \searrow K(\alpha) & \searrow D(\alpha) & & \\
 D(i) & & K_j & \xrightarrow{\pi_j^1} & D(j) \\
 & \searrow D(\alpha) & \pi_j^2 \downarrow & & \downarrow \phi_j \\
 & & D(j) & \xrightarrow{\phi_j} & D'(j)
 \end{array}$$

In this way, we get a functor  $K : \mathcal{J} \rightarrow \mathcal{C}$ , which maps each  $i$  onto  $K_i$  and each arrow  $\alpha$  onto  $K(\alpha)$ . We have in fact  $K(id_i) : K_i \rightarrow K_i$  is the arrow such that

$$\begin{aligned}
 D(id_i) \circ \pi_i^1 &= \pi_i^1 \circ K(id_i) & D(id_i) \circ \pi_i^2 &= \pi_i^2 \circ K(id_i) \\
 \pi_i^1 &= \pi_i^1 \circ K(id_i) & \pi_i^2 &= \pi_i^2 \circ K(id_i)
 \end{aligned}$$

Thus, for the universal property of pullbacks,  $K(id_i) = id_{K_i}$ .

Suppose now  $\alpha : i \rightarrow j$  and  $\beta : j \rightarrow k$ . Computing, we have

$$\begin{aligned}
 \pi_k^1 \circ K(\beta \circ \alpha) &= D(\beta \circ \alpha) \circ \pi_i^1 & \pi_k^2 \circ K(\beta \circ \alpha) &= D(\beta \circ \alpha) \circ \pi_i^2 \\
 &= D(\beta) \circ D(\alpha) \circ \pi_i^1 & &= D(\beta) \circ D(\alpha) \circ \pi_i^2 \\
 &= D(\beta) \circ \pi_j^1 \circ K(\alpha) & &= D(\beta) \circ \pi_j^2 \circ K(\alpha) \\
 &= \pi_k^1 \circ K(\beta) \circ K(\alpha) & &= \pi_k^2 \circ K(\beta) \circ K(\alpha)
 \end{aligned}$$

Again, for universal property of pullbacks, necessarily we have  $K(\beta \circ \alpha) = K(\beta) \circ K(\alpha)$ , proving functoriality of  $K$ .

Hence, we have two natural transformations  $\pi^1, \pi^2 : E \rightarrow D$ . By Proposition 1.3.42, every component  $\phi_i$  is the coequalizer of  $\pi_i^1, \pi_i^2 : E \rightarrow D$ , and so  $\phi$  is the coequalizer of  $\pi^1$  and  $\pi^2$ .  $\square$

**Lemma 1.3.44.** *Let  $D, D' : \mathcal{J} \rightarrow \mathcal{C}$  be two diagrams, and let  $((c_i)_{i \in \mathcal{J}}, C)$  and  $((c'_i)_{i \in \mathcal{J}}, C')$  be, respectively, the colimit of  $D$  and  $D'$ . If  $\mathcal{C}$  has all colimits, for diagrams of shape  $\mathcal{J}$  and  $\phi : D \rightarrow D'$  is a natural transformation in which all components are regular epimorphisms, then, the arrow induced by  $\phi$  from  $C$  to  $C'$  (Proposition 1.3.32) is a regular epimorphism too.*

*Proof.* By Corollary 1.3.43, we know that  $\phi : D \rightarrow D'$  is a regular epimorphism, so that there is a functor  $E : \mathcal{J} \rightarrow \mathcal{C}$  and  $\eta, \theta : E \rightarrow D$

such that  $\phi$  is the coequalizer of  $\eta$  and  $\theta$ . Let now  $((p_i)_{i \in \mathcal{I}}, P)$  be the colimit of  $E$ , by Proposition 1.3.32, we have  $a, b : P \rightarrow C$  fitting in the diagram below.

$$\begin{array}{ccc} E(i) & \xrightarrow{p_i} & P \\ \eta_i \downarrow & & \downarrow a \\ D(i) & \xrightarrow{c_i} & C \end{array} \quad \begin{array}{ccc} E(i) & \xrightarrow{p_i} & P \\ \theta_i \downarrow & & \downarrow b \\ D(i) & \xrightarrow{c_i} & C \end{array}$$

We want to show that  $c$  coequalizes  $\eta$  and  $\theta$ . Let thus  $t : C \rightarrow T$  be an arrow such that  $t \circ a = t \circ b$ . Then, for every  $i$ , we have

$$\begin{aligned} t \circ c_i \circ \eta_i &= t \circ a \circ p_i \\ &= t \circ b \circ p_i \\ &= t \circ c_i \circ \theta_i \end{aligned}$$

Thus, there is  $t_i : D(i) \rightarrow T$  such that  $t \circ c_i = t_i \circ \phi_i$ . It is now easy to see that  $((t_i)_{i \in \mathcal{I}}, T)$  is a cocone of  $D'$ : suppose  $\alpha : i \rightarrow j$  be an arrow of  $\mathcal{I}$ , obtaining

$$\begin{aligned} t_i \circ \phi_i &= t \circ c_i \\ &= t \circ c_j \circ D(\alpha) \\ &= t_j \circ \phi_j \circ D(\alpha) \\ &= t_j \circ D'(\alpha) \circ \phi_i \end{aligned}$$

By the hypothesis that  $\phi_i$  is regular epi for each  $i$ , therefore epi (by the dual of Proposition 1.3.13), we can conclude  $t_i = t_j \circ D'(\alpha)$ .

Hence, we have an arrow  $k : C' \rightarrow T$  such that  $k \circ c'_i = t_i$ . But then

$$\begin{aligned} c \circ c \circ c_i &= k \circ c'_i \circ \phi_i \\ &= t_i \circ \phi \\ &= t \circ c_i \end{aligned}$$

Showing that  $k \circ c = t$ .

For the uniqueness, let  $k' : C' \rightarrow T$  be another arrow such that  $k' \circ c = t$ . Then we have

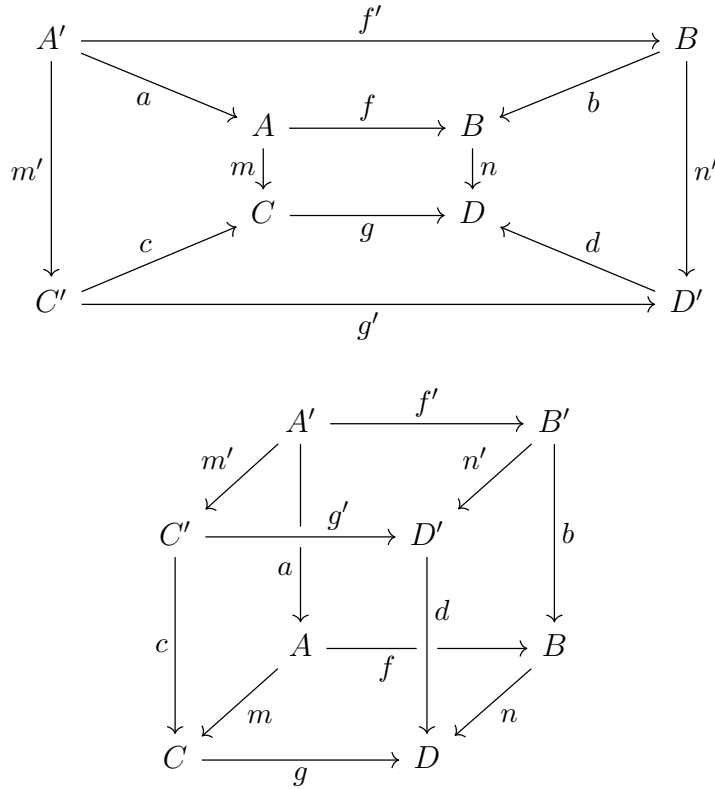
$$\begin{aligned} k' \circ c'_i \circ \phi_i &= k' \circ c \circ c_i \\ &= t \circ c_i \\ &= t_i \circ \phi_i \end{aligned}$$

Since  $\phi_i$  is a regular epimorphism, we have  $k' \circ c'_i = t_i$ , and, because  $k \circ c'_i = t_i$  by construction, we can conclude that  $k' = k$  since  $((c'_i)_{i \in \mathcal{I}}, C')$  is a colimit.  $\square$

## 1.4 Adhesivity

The next section is about adhesivity. An adhesive category is intuitively a category in which pushouts of (some) monomorphisms exist and they behave more or less as they do among sets.

**Definition 1.4.1.** (Van Kampen property) Let  $\mathcal{A}$  be a subclass of  $\mathcal{H}om(\mathcal{C})$ , and consider the diagram below:



we say that the inner square is an  $\mathcal{A}$ -Van Kampen square if:

- it is a pushout;
- $a, b, c, d \in \mathcal{A}$ ;
- whenever the top and the left squares are pullbacks then the outer square is a pushout if and only if the right and the bottom squares are pullbacks.

We are now ready to give the notion of  $\mathcal{M}$ -adhesivity.

**Definition 1.4.2** ( $\mathcal{M}$ -adhesivity). Let  $\mathcal{C}$  be a category and  $\mathcal{M} \subseteq \text{Mono}(\mathcal{C})$  containing all isomorphisms, closed under composition and stable under pullbacks and pushouts (Definition 1.3.18). Then  $\mathcal{C}$  is  $\mathcal{M}$ -adhesive if

1. every cospan  $C \xrightarrow{g} D \xleftarrow{m} B$  with  $m \in \mathcal{M}$  can be completed to a pullback (such pullbacks are called  $\mathcal{M}$ -pullbacks);
2. every span  $C \xleftarrow{m} A \xrightarrow{f} B$  with  $m \in \mathcal{M}$  can be completed to a pushout (such pushouts are called  $\mathcal{M}$ -pushouts);
3. pushouts along  $\mathcal{M}$ -arrows are  $\mathcal{M}$ -Van Kampen squares.

We also say that  $\mathcal{C}$  is *adhesive* when it is  $\text{Mono}(\mathcal{C})$ -adhesive, and *quasiadhesive* when it is  $\text{Reg}(\mathcal{C})$ -adhesive.

**Observation 1.4.3.** **Set** is adhesive.

Here it follows an interesting property of adhesive categories [Lac11].

**Proposition 1.4.4.** *In any adhesive category, the pushout of a monomorphism along any morphism is a monomorphism, and the resulting square is also a pullback.*

Verifying  $\mathcal{M}$ -adhesivity using the definition above may turn out to be very complex, so we can make use of the following result [CGM22].

**Theorem 1.4.5.** *Let  $\mathcal{C}$  be a category,  $\mathcal{M} \subseteq \text{Mono}(\mathcal{C})$  containing all isomorphisms, closed under composition and stable under pullbacks and pushouts. Let now  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor with  $\mathcal{D}$   $\mathcal{N}$ -adhesive for some  $\mathcal{N} \subseteq \text{Mono}(\mathcal{D})$ . If  $F$  is such that  $F(\mathcal{M}) \subseteq \mathcal{N}$  and creates pullbacks and  $\mathcal{M}$ -pushout, then  $\mathcal{C}$  is  $\mathcal{M}$ -adhesive.*

The idea behind this theorem is to simplify calculations to show that a certain category is adhesive for some subclass of monomorphisms, considering a functor from the category of which we want to prove adhesivity to a category we know it is adhesive, requiring that such functor has some properties.

*Proof.* In order to prove  $\mathcal{M}$ -adhesivity of  $\mathcal{C}$ , we have to verify the condition in Definition 1.4.2.

- Let  $C \xrightarrow{g} D \xleftarrow{m} B$  with  $m \in \mathcal{M}$  be a cospan in  $\mathcal{C}$ . Applying  $F$ , we obtain  $F(C) \xrightarrow{F(g)} F(D) \xleftarrow{F(m)} B$ , with  $F(m) \in \mathcal{N}$  by hypothesis. Then, there exists a pullback  $(P_F, p_{F(B)}, p_{F(D)})$  in  $\mathcal{D}$ , which is an  $\mathcal{N}$ -pullback (Definition 1.3.15). Since  $F$  creates pullbacks, hence lifts them (Observation 1.3.28), there exist a pullback  $(P, p_B, p_D)$  in  $\mathcal{C}$ .

- Let  $C \xleftarrow{m} A \xrightarrow{f} B$  with  $m \in \mathcal{M}$  be a cospan in  $\mathcal{C}$ . Analogously to the previous point, applying the functor  $F$  we obtain  $F(C) \xleftarrow{F(m)} F(A) \xrightarrow{F(f)} F(B)$  with  $F(m) \in \mathcal{N}$ , and there exists a  $\mathcal{N}$ -pushout  $(q_{F(C)}, q_{F(B)}, F(Q))$  in  $\mathcal{D}$ . Since  $F$  reflects pushouts,  $(q_C, q_B, Q)$  is a  $\mathcal{M}$ -pushout in  $\mathcal{C}$ .
- the Van Kampen property of  $\mathcal{M}$ -pullbacks follows from the closure under pullbacks and pushouts of  $\mathcal{M}$  and from the fact that  $F$  reflects pullbacks.

□

**Corollary 1.4.6.** *Let  $\mathcal{A}$  be a  $\mathcal{M}$ -adhesive category for some  $\mathcal{M} \subseteq \mathcal{H}om(\mathcal{A})$ . Then, for every other category  $\mathcal{C}$ , the functor category  $[\mathcal{C}, \mathcal{A}]$  is  $\mathcal{M}^{\mathcal{C}}$ -adhesive, where*

$$\mathcal{M}^{\mathcal{C}} = \{ \eta \in \mathcal{H}om([\mathcal{C}, \mathcal{A}]) \mid \eta_C \in \mathcal{M} \text{ for each object } C \text{ of } \mathcal{C} \}$$

Since **Set** is adhesive, we can conclude what follows.

**Corollary 1.4.7.** *Every category of presheaves is adhesive.*

**Lemma 1.4.8.** *Let  $\mathcal{C}$  be an  $\mathcal{M}$ -adhesive category with all pullbacks, and suppose that the cube below is given, in which every face is a pullback and the bottom one is a  $\mathcal{M}$ -pushout.*

$$\begin{array}{ccccc}
 & & A' & \xrightarrow{f'} & B' \\
 & m' \swarrow & \downarrow & & \searrow n' \\
 C' & \xrightarrow{g'} & D' & & \downarrow b \\
 & \downarrow a & \downarrow d & & \\
 & A & \xrightarrow{f} & B & \\
 c \downarrow & \swarrow m & \downarrow & \searrow n & \\
 C & \xrightarrow{g} & D & & 
 \end{array}$$

Then, the square below is a pushout.

$$\begin{array}{ccc}
 K_a & \xrightarrow{k_{f'}} & K_b \\
 k_{m'} \downarrow & & \downarrow k_{n'} \\
 K_c & \xrightarrow{k_{g'}} & K_d
 \end{array}$$

*Proof.* By Lemma 1.3.41, in the following cube the vertical faces are pullbacks.

$$\begin{array}{ccccc}
 & K_a & \xrightarrow{k_{f'}} & K_b & \\
 k_{m'} \swarrow & \downarrow k_{g'} & & \swarrow k_{n'} & \downarrow \pi_b^1 \\
 K_c & \xrightarrow{\quad} & K_d & & \\
 \downarrow \pi_c^1 & & \downarrow \pi_a^1 & & \\
 & A' & \xrightarrow{f'} & B' & \\
 m' \swarrow & \downarrow \pi_d^1 & & \swarrow n' & \\
 C' & \xrightarrow{g'} & D' & & 
 \end{array}$$

$f'$  is in  $\mathcal{M}$  as it is the pullback of  $\mathcal{M}$ , thus the bottom face of the cube is a Van Kampen pushout and the thesis follows.  $\square$

# Chapter 2

## Categories of Graphs

This chapter is about graphs, and how it is possible to formalize them using categories in order to point out their properties from an abstract point of view. Starting from the set-theoretical definition of graphs, we will give an abstraction via functor categories, in which a graph is nothing but a functor between a category to another.

### 2.1 Graphs

A (directed graph)  $\mathcal{G}$  is a mathematical structure consisting of a set of edge, a set of nodes and two functions, one assigning a *source* node and one assigning a *target* node to an edge. Formally,  $\mathcal{G}$  is a quadruple  $(V_{\mathcal{G}}, E_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}})$ , where  $V_{\mathcal{G}}$  is the set of nodes,  $E_{\mathcal{G}}$  is the set of edges, and  $s_{\mathcal{G}}, t_{\mathcal{G}} : E_{\mathcal{G}} \rightarrow V_{\mathcal{G}}$  are the source and the target functions.

A *graph homomorphism*  $h : \mathcal{G} \rightarrow \mathcal{H}$  is then a pair of functions  $h = (h_V : V_{\mathcal{G}} \rightarrow V_{\mathcal{H}}, h_E : E_{\mathcal{G}} \rightarrow E_{\mathcal{H}})$  such that

$$h_V \circ s_{\mathcal{G}} = s_{\mathcal{H}} \circ h_E$$

and

$$h_V \circ t_{\mathcal{G}} = t_{\mathcal{H}} \circ h_E$$

that is, a structure preserving map.

We can then generalize such notion to something more abstract, considering a graph to be nothing more than a presheaf from the category  $(E \rightrightarrows V)$  to the category of sets. Having two of such presheaves, a natural transformation from one to another encapsulates the behavior of a graph morphism due to naturality. We can now define the category of graphs.

**Definition 2.1.1** (Category of Graphs). We denote as **Graph** the category

$$[E \rightrightarrows^s_V \mathbf{Set}]$$

Since **Graph** is a category of presheaves, Lemma 1.2.15 guarantees the existence of limits and colimits, and gives us an easy way to compute them.

**Corollary 2.1.2.** **Graph** has all limits and colimits.

**Example 2.1.3.** The initial object in **Graph** is the empty graph, i.e., the graph with an empty set of vertices and an empty set of edges. The initial object instead is the graph with exactly one node and a single edge from that node to itself.

**Example 2.1.4.** Given two graphs  $G = (V_G, E_G, s_G, t_G)$  and  $H = (V_H, E_H, s_H, t_H)$ , the graph  $G \times H = (V_G \times V_H, E_G \times E_H, (s_G, s_H), (t_G, t_H))$ , where  $(s_G, s_H), (t_G, t_H) : V_G \times V_H \rightarrow E_G \times E_H$  are the pairwise sources and targets, is the categorical product in **Graph**, together with the two projections  $\pi_G : G \times H \rightarrow G$ ,  $\pi_H : G \times H \rightarrow H$  defined in the obvious way.

**Example 2.1.5.** The equalizer of two morphisms  $h, k : G \rightarrow H$  in **Graph** is defined as in **Set**, that is, a graph  $Q$  together with a graph morphism  $q$  that is the restriction of  $G$  to all the vertices and all the arcs that are mapped on the same vertices and edges both from  $h$  and  $k$ . Formally,  $(Q, q)$  is an equalizer for  $h, k : G \rightarrow H$ ,  $h = (h_V, h_E), k = (k_V, k_E)$  where  $V(Q) = \{n \in V(G) \mid h_V(n) = k_V(n)\}$ ,  $E(Q) = \{e \in E(G) \mid h_E(e) = k_E(e)\}$ ,  $s_Q = s_G \upharpoonright_{V(Q)}$ ,  $t_Q = t_G \upharpoonright_{V(Q)}$ .

**Remark 2.1.6.** TODO: Si può generalizzare a tutte le categorie regolari per evitare di perdere le proprietà che usiamo (da eq.rel. a quot.).

## 2.2 Graphs with equivalences - notes and results

### 2.2.1 Some results on kernel pair and regular epis

Questi risultati vanno completati e distribuiti lungo la tesi: quello sui kernel pair nella parte sui pullback, quello sugli epi regolari nella parte sui coequalizzatori, quelli che riguardano l'adesività nella relativa sezione



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**Lemma 2.2.1.** *Suppose that the following diagram is given and its right half is a pullback. Then the whole rectangle is a pullback if and only if its left half is a pullback.*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow t & & \downarrow k & & \downarrow h \\ A & \xrightarrow{a} & B & \xrightarrow{b} & C \end{array}$$

*Proof.*



Esercizio per te

**Corollary 2.2.2.** *Let  $\mathcal{C}$  be a category and suppose that the solid part of the following cube is given*

$$\begin{array}{ccccc} & & Y' & \xrightarrow{g'} & Z' \\ & \swarrow q' & \downarrow k' & \swarrow r' & \downarrow z \\ B' & \xrightarrow{y} & C' & & \\ \downarrow b & & \downarrow c & & \downarrow g \\ & Y & \xrightarrow{g} & Z \\ & \swarrow q & \downarrow r & & \\ B & \xrightarrow{k} & C & & \end{array}$$

*If the front face is a pullback then there is a unique  $q': Y' \rightarrow B'$  filling the diagram. If, moreover, the other two vertical faces are also pullbacks, then the following square is a pullback too.*

$$\begin{array}{ccc} Y' & \xrightarrow{q'} & B' \\ \downarrow y & & \downarrow b \\ Y & \xrightarrow{q} & B \end{array}$$

*Proof.* Let us compute:

$$\begin{aligned} c \circ r' \circ g' &= r \circ z \circ g' \\ &= r \circ g \circ y \\ &= k \circ q \circ y \end{aligned}$$

Since the front face is a pullback, this guarantees the existence of  $q'$ . The second half of the thesis follows applying lemma 2.2.1 to the following rectangle.

$$\begin{array}{ccccc} Y' & \xrightarrow{q'} & B' & \xrightarrow{k'} & C' \\ \downarrow y & & \downarrow b & & \downarrow c \\ Y & \xrightarrow{q} & B & \xrightarrow{k} & C \end{array}$$

$\xrightarrow{r' \circ g'}$  (top curved arrow)  $\xrightarrow{r \circ g}$  (bottom curved arrow)

□

**Definition 2.2.3.** A *kernel pair*  $(K_f, p_{f,1}, p_{f,2})$  for an arrow  $f: X \rightarrow Y$  is an object  $K_f$  with two arrows  $p_{f,1}, p_{f,2}: K_f \rightarrow X$  making the following square a pullback.

$$\begin{array}{ccc} K_f & \xrightarrow{p_{f,1}} & X \\ p_{f,2} \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

**Remark 2.2.4.** If a category  $\mathcal{C}$  has pullbacks then every arrow has a kernel pair.

**Proposition 2.2.5.** An arrow  $m: M \rightarrow X$  is mono if and only if  $(M, id_M, id_M)$  is a kernel pair for it.

esercizio

Proof.

□

**Corollary 2.2.6.** Let  $(K_f, p_{f,1}, p_{f,2})$  be a kernel pair for  $f: X \rightarrow Y$ . Then for every mono  $m: Y \rightarrow Z$ ,  $(K_f, p_{f,1}, p_{f,2})$  is a kernel pair also for  $m \circ f$ .

*Proof.* It is enough to see that, by Lemma 2.2.1 and proposition 2.2.5 the outer boundary of the following square is a pullback.

$$\begin{array}{ccccc} K_f & \xrightarrow{p_{f,1}} & X & \xrightarrow{id_X} & X \\ p_{f,2} \downarrow & & \downarrow f & & \downarrow f \\ X & \xrightarrow{f} & Y & \xrightarrow{id_Y} & Y \\ id_X \downarrow & & id_Y \downarrow & & \downarrow m \\ X & \xrightarrow{f} & Y & \xrightarrow{m} & Z \end{array}$$

□

**Lemma 2.2.7.** Suppose that the following square is given and that  $f: X \rightarrow Y$  and  $g: Z \rightarrow W$  have kernel pairs.

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{t} & W \end{array}$$

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Then there exists a unique arrow  $k_h: K_f \rightarrow K_g$  making the squares below commutes.

$$\begin{array}{ccc} K_f & \xrightarrow{\quad k_h \quad} & K_g \\ p_{f,1} \downarrow & & \downarrow p_{g,1} \\ X & \xrightarrow{\quad h \quad} & Z \end{array} \quad \begin{array}{ccc} K_f & \xrightarrow{\quad k_h \quad} & K_g \\ p_{f,2} \downarrow & & \downarrow p_{g,2} \\ X & \xrightarrow{\quad h \quad} & Z \end{array}$$

Moreover, if the beginning square is a pullback, then also the preceding ones are so.

*Proof.* Computing we have

$$\begin{aligned} g \circ h \circ p_{f,1} &= t \circ f \circ p_{f,1} \\ &= t \circ f \circ p_{f,2} \\ &= g \circ h \circ p_{f,2} \end{aligned}$$

So that the wanted  $k_h$  exists, and it is unique, by the universal property of  $K_g$  as the pullback of  $g$  along itself.

To prove the second half of the thesis, let us consider the two rectangles below, which, by Lemma 2.2.1 are pullbacks.

$$\begin{array}{ccccc} K_f & \xrightarrow{p_{f,1}} & X & \xrightarrow{h} & Z \\ p_{f,2} \downarrow & & \downarrow f & & \downarrow g \\ X & \xrightarrow{f} & Y & \xrightarrow{t} & W \end{array} \quad \begin{array}{ccccc} K_f & \xrightarrow{p_{f,2}} & X & \xrightarrow{h} & Z \\ p_{f,1} \downarrow & & \downarrow f & & \downarrow g \\ X & \xrightarrow{f} & Y & \xrightarrow{t} & W \end{array}$$

But then the following ones are pullbacks too.

$$\begin{array}{ccccc} & & \xrightarrow{h \circ p_{f,2}} & & \\ K_f & \xrightarrow{\quad k_h \quad} & K_g & \xrightarrow{\quad p_{g,2} \quad} & Z \\ p_{f,1} \downarrow & & \downarrow p_{g,1} & & \downarrow g \\ X & \xrightarrow{\quad h \quad} & Z & \xrightarrow{\quad g \quad} & W \\ & & \xleftarrow{t \circ f} & & \end{array} \quad \begin{array}{ccccc} & & \xrightarrow{h \circ p_{f,1}} & & \\ K_f & \xrightarrow{\quad k_h \quad} & K_g & \xrightarrow{\quad p_{g,1} \quad} & Z \\ p_{f,2} \downarrow & & \downarrow p_{g,2} & & \downarrow g \\ X & \xrightarrow{\quad h \quad} & Z & \xrightarrow{\quad g \quad} & W \\ & & \xleftarrow{t \circ f} & & \end{array}$$

The thesis now follows again by Lemma 2.2.1.  $\square$

**Lemma 2.2.8.** *Let  $\mathcal{C}$  be an  $\mathcal{M}$ -adhesive category with all pullbacks and suppose that the cube below is given, in which every face is a pullback and the bottom one is an  $\mathcal{M}$ -pushout.*

$$\begin{array}{ccccc}
& & A' & \xrightarrow{f'} & B' \\
& m' \swarrow & \downarrow & \nwarrow n' & \downarrow b \\
C' & \xrightarrow{g'} & D' & & \\
\downarrow c & \searrow a & \downarrow d & \searrow f & \downarrow \\
C & \xrightarrow{g} & D & & \\
& \swarrow m & & \nwarrow n & 
\end{array}$$

Then the square below is a pushout.

$$\begin{array}{ccc}
K_a & \xrightarrow{k_{f'}} & K_b \\
k_{m'} \downarrow & & \downarrow k_{n'} \\
K_c & \xrightarrow{k_{g'}} & K_d
\end{array}$$

*Proof.* By Lemma 2.2.7 in the following cube the vertical faces are all pullbacks.

$$\begin{array}{ccccc}
& & K_a & \xrightarrow{k_{f'}} & K_b \\
& k_{m'} \swarrow & \downarrow & \nwarrow k_{n'} & \downarrow p_{b,1} \\
K_c & \xrightarrow{p_{a,1}} & K_d & & \\
\downarrow p_{c,1} & \searrow & \downarrow p_{d,1} & \searrow f' & \downarrow \\
& C' & \xrightarrow{g'} & D' & \\
& \swarrow m' & & \nwarrow n' & 
\end{array}$$

$f'$  is in  $\mathcal{M}$  as it is the pullback of  $\mathcal{M}$ , thus the bottom face of the cube is a Van Kampen pushout and the thesis follows.  $\square$

**Proposition 2.2.9.** *Let  $e: X \rightarrow Y$  be a regular epi in a category  $\mathcal{C}$  with a kernel pair  $p_1, p_2: P \rightrightarrows X$ , then  $e$  is the coequalizer of  $p_1$  and  $p_2$ .*

*Proof.* By hypothesis there exists a pair  $f, g: Z \rightrightarrows X$  of which  $e$  is the coequalizer, since  $e \circ f = e \circ g$  we have a diagram

$$\begin{array}{ccccc}
Z & \xrightarrow{f} & & & \\
\downarrow g & \searrow k & P & \xrightarrow{p_1} & X \\
& & \downarrow p_2 & & \downarrow e \\
& & X & \xrightarrow{e} & Y
\end{array}$$

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and thus there exists the dotted  $k: Z \rightarrow P$ . Let  $h: Z \rightarrow V$  be an arrow such that  $h \circ p_1 = h \circ p_2$ , then

$$\begin{aligned} h \circ f &= h \circ p_1 \circ k \\ &= h \circ p_2 \circ k \\ &= h \circ g \end{aligned}$$

and thus there exists a unique  $l: Y \rightarrow V$  such that  $l \circ e = h$ .  $\square$

**Corollary 2.2.10.** *Let  $\mathcal{C}$  be a category with pullbacks and  $\phi: D \rightarrow D'$  be a natural transformation between two functors  $D, D': \mathcal{I} \rightarrow \mathcal{C}$ . If  $\phi_i$  is a regular epi for every  $i$ , then  $\phi$  is a regular epi.*

*Proof.* Let  $K_i$  be the kernel pair of  $\phi_i$ , with projections  $p_{1,i}, p_{2,i}: K_i \rightrightarrows D(i)$ . Given an arrow  $f: i \rightarrow j$  in  $\mathcal{I}$ , we have

$$\begin{aligned} \phi_j \circ D(f) \circ p_{1,i} &= D'(f) \circ \phi_i \circ p_{1,i} \\ &= D'(f) \circ \phi_i \circ p_{2,i} \\ &= \phi_j \circ D(f) \circ p_{2,i} \end{aligned}$$

Thus the outer boundary of the diagram below commutes, yielding the dotted arrows  $K(f)$ .

$$\begin{array}{ccccc} K_i & \xrightarrow{p_{1,i}} & D(i) & & \\ p_{2,i} \downarrow & \swarrow K(f) & \searrow D(f) & & \\ D(i) & & K_j & \xrightarrow{p_{1,j}} & D(j) \\ & \searrow D(f) & \downarrow p_{2,j} & & \downarrow \phi_j \\ & D(j) & \xrightarrow{\phi_j} & D'(j) & \end{array}$$

In this way we get a functor  $E: \mathcal{I} \rightarrow \mathcal{C}$  with two natural transformations  $p_1, p_2: E \rightrightarrows D$ . By Proposition 2.2.9 every component  $\phi_i$  of  $\phi$  is the coequalizer of  $p_{1,i}, p_{2,i}: E \rightrightarrows D$  and so  $\phi$  is the coequalizer of  $p_1$  and  $p_2$ .  $\square$

**Lemma 2.2.11.** *Let  $D, D': \mathcal{I} \rightrightarrows \mathcal{C}$  be two diagrams with colimiting cocone  $(C, \{c_i\}_{i \in \mathcal{I}})$  and  $(Q, \{q_i\}_{i \in \mathcal{I}})$ . If  $\mathcal{C}$  has all colimits for diagrams of shape  $\mathcal{I}$  and  $\phi: D \rightarrow D'$  is a natural transformation in which all components are regular epis, then the canonical arrow  $c: C \rightarrow Q$  is a regular epi to.*

esercizio per te

**COSE DA FARE:**  
nel capitolo 1 metti una proposizione in cui mostri che da ogni trasf naturale  $D \rightarrow D'$  puoi ricavare una freccia tra i colimiti, così in questa e nelle altre proposizioni puoi citarla.

richiama prop precedente

*Proof.* By Corollary 2.2.10 we know that  $\phi: D \rightarrow D'$  is a regular epi, so that there is a functor  $E: \mathcal{J} \rightarrow \mathcal{C}$  and  $\alpha, \beta: E \rightrightarrows D$  such that  $\phi$  is a coequalizer for  $\alpha$  and  $\beta$ . Let  $(P, \{p_i\}_{i \in \mathcal{J}})$  be the colimit of  $E$ , by ??? we have arrows  $a, b: P \rightrightarrows C$  fitting in the diagram belows:

$$\begin{array}{ccc} E(i) & \xrightarrow{p_i} & P \\ \alpha_i \downarrow & & \downarrow a \\ D(i) & \xrightarrow{c_i} & C \end{array} \quad \begin{array}{ccc} E(i) & \xrightarrow{p_i} & P \\ \alpha_i \downarrow & & \downarrow b \\ D(i) & \xrightarrow{c_i} & C \end{array}$$

richiama prop sull  
freccia tra i colimi

We want to show that  $c$  coequalizes  $a$  and  $b$ . Let thus  $t: C \rightarrow T$  be an arrow such that  $t \circ a = t \circ b$ . Then for every  $i \in I$  we have

$$\begin{aligned} t \circ c_i \circ \alpha_i &= t \circ a \circ p_i \\ &= t \circ b \circ p_i \\ &= t \circ c_i \circ \beta_i \end{aligned}$$

Thus there is  $t_i: D'(i) \rightarrow T$  such that  $t \circ c_i = t_i \circ \phi_i$ . It is now easy to see that  $(T, \{t_i\}_{i \in \mathcal{J}})$  is a cocone on  $D'$ . Thus we have an arrow  $k: Q \rightarrow T$  such that  $k \circ q_i = t_i$ . But then we have

Verificalo!

$$\begin{aligned} k \circ c \circ c_i &= k \circ q_i \circ \phi_i \\ &= t_i \circ \phi \\ &= t \circ c_i \end{aligned}$$

Showing that  $k \circ c = t$ .

For uniqueness, let  $k'$  be another arrow  $Q \rightarrow T$  such that  $k' \circ c = t$ , then we have

$$\begin{aligned} k' \circ q_i \circ \phi_i &= k' \circ c \circ c_i \\ &= t \circ c_i \\ &= t_i \circ \phi_i \end{aligned}$$

mettere nel capitolo  
1 una proposizione  
che mostra che epi  
regolari sono epi

Since  $\phi_i$  is a regular epi, by ??? this entails  $k' \circ q_i = t_i$ . By construction  $k \circ q_i = t_i$  and so  $k = k'$  since  $(Q, \{q_i\}_{i \in \mathcal{J}})$  is a colimiting cocone.  $\square$

**Definition 2.2.12.** A graph with equivalence is a 6-uple  $(A, B, C, s, t, q)$  where  $A, B$  and  $C$  are set,  $s, t: A \rightrightarrows B$  are functions and  $q: B \rightarrow C$  is another surjective function.

A morphism  $(A, B, C, s, t, q) \rightarrow (A', B', C', s', t', q')$  is a triple  $(h_1, h_2, h_3)$  of functions  $h_1: A \rightarrow A'$ ,  $h_2: B \rightarrow B'$  and  $h_3: C \rightarrow C'$  making the following diagrams commute.

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$$\begin{array}{ccc}
 A \xrightarrow{s} B & A \xrightarrow{s} B & B \xrightarrow{q} C \\
 h_1 \downarrow & h_1 \downarrow & h_2 \downarrow \\
 A' \xrightarrow{s'} B' & A' \xrightarrow{t'} B' & B' \xrightarrow{q'} C'
 \end{array}$$

In this way, defining the composition componentwise, we get a category **EqGrph**.

**Remark 2.2.13.** There is a faithful functor  $U: \mathbf{EqGrph} \rightarrow \mathbf{Graph}$ , forgetting the quotient part.

**Remark 2.2.14.** There is another functor  $V: \mathbf{EqGrph} \rightarrow \mathbf{Set}$  sending  $(A, B, C, s, t, q)$  to  $C$  and a morphism to its last component.

**Proposition 2.2.15.** *EqGrph is complete, cocomplete and  $U$  preserves limits and colimits.*

**Remark 2.2.16.** In **Set** we have the following property: for every square as the one below, if  $e: X \rightarrow Y$  is epi and  $m: M \rightarrow Z$  is mono, then there exists a unique dotted arrow  $Y \rightarrow M$  making the diagram below commutative.

$$\begin{array}{ccc}
 X & \xrightarrow{f} & M \\
 e \downarrow & \nearrow h & \downarrow m \\
 Y & \xrightarrow{g} & Z
 \end{array}$$

*Proof.* NOTATION:  $D(i)$  is  $(A_i, B_i, Q_i, s_i, t_i, q_i)$ . Let  $(A, B, s, t)$  be the limit of  $U \circ D$ , with projections  $(p_{1,i}, p_{2,i}): (A, B, s, t) \rightarrow (A_i, B_i, s_i, t_i)$ . Let  $(L, \{l_i\}_{i \in \mathcal{I}})$  be a limiting cone for  $V \circ D$ .

Now, notice that  $(B, \{q_i \circ p_{2,i}\}_{i \in \mathcal{I}})$  is a cone over  $V \circ D$ , so that we have an arrow  $l: B \rightarrow L$ . This arrow is not epi in general, let  $Q$  be its image and  $q: B \rightarrow Q$  be the resulting epi and  $m: Q \rightarrow L$  the corresponding mono. By definition the external square in the diagram below commutes, so for every  $i \in \mathcal{I}$ , Remark 2.2.16 yields the dotted arrow  $p_{3,i}$ .

$$\begin{array}{ccccc}
 B & \xrightarrow{p_{2,i}} & B_i & \xrightarrow{q_i} & Q_i \\
 q \downarrow & & & \nearrow p_{3,i} & \downarrow id_{Q_i} \\
 Q & \xrightarrow{m} & L & \xrightarrow{l_i} & Q_i
 \end{array}$$

Questo è per te da dimostrare (forse meglio come proposizione che come remark).

Dimostrala

Per generalizzare ad altre categorie: serve poter fattorizzare con un epi regolare.

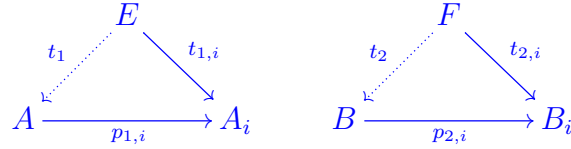
Dimostralo

We have to show that in this way we get a cone over the diagram  $D$ . Let  $f: i \rightarrow j$  be an arrow of  $\mathcal{J}$ , then we have:

$$\begin{aligned} U(D(f) \circ (p_{1,i}, p_{2,i}, p_{3,i})) &= U(D(f)) \circ (p_{1,i}, p_{2,i}) \\ &= (p_{1,j}, p_{2,j}) \\ &= U(D(f) \circ (p_{1,j}, p_{2,j}, p_{3,j})) \end{aligned}$$

And faithfulness of  $U$  yields the thesis.

To see that this cone is terminal, let  $(E, F, G, a, b, c)$  be another graph with the vertex of a cone with sides  $(t_{1,i}, t_{2,i}, t_{3,i})$ . By construction, we have an arrow  $(t_1, t_2): (E, F, a, b) \rightarrow (A, B, s, t)$  such that

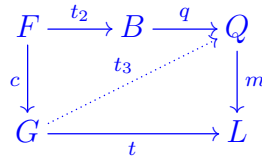


Verificalo

Moreover  $(G, \{t_{3,i}\}_{i \in \mathcal{J}})$  is a cone over  $V \circ D$ , thus there exists an arrow  $t: G \rightarrow L$  such that  $l_i \circ t = t_{3,i}$ . Now, precomposing with  $c$  we get

$$\begin{aligned} l_i \circ t \circ c &= t_{3,i} \circ c \\ &= q_i \circ t_{2,i} \\ &= q_i \circ p_{2,i} \circ t_2 \\ &= l_i \circ l \circ t_2 \end{aligned}$$

Therefore the solid part of the diagram below commutes and Remark 2.2.16 yields the dotted arrow  $t_3: G \rightarrow Q$ .



esercizio per te: scrivere la dimostrazione di queste ultime due righe

Faithfulness of  $U$  now guarantees that  $(t_1, t_2, t_3)$  is the unique arrow such that  $(p_{1,i}, p_{2,i}, p_{3,i}) \circ (t_1, t_2, t_3) = (t_{1,i}, t_{2,i}, t_{3,i})$ .  $\square$

**Corollary 2.2.17.** *An arrow  $(h_1, h_2, h_3): (A, B, C, s, t, q) \rightarrow (E, F, G, a, b, c)$  in **EqGrph** is mono if and only if  $h_1$  and  $h_2$  are mono in **Set**.*

*Proof.*  $\square$

**Corollary 2.2.18.** *Let  $(h_1, h_2, h_3): (A, B, C, s, t, q) \rightarrow (E, F, G, a, b, c)$  be a morphism of **EqGrph**, then the following are equivalent:*

esercizio per te: fare i colimiti. Hint: la dimostrazione è diversa ma più semplice. Se  $(A, B, s, t)$  è il colimite di  $U \circ D$  puoi considerare il colimite  $Q$  dei vari  $Q_i$ . Per la proprietà del colimite hai una freccia  $B \rightarrow Q$ , mostra che è epi (segue in una riga



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1.  $(h_1, h_2, h_3)$  is a regular mono;
2.  $h_1, h_2, h_3$  are all monos;
3.  $h_1$  and  $h_2$  are mono and for every  $f, f' \in F$ ,  $c(h_2(f)) = c(h_2(f'))$  if and only if  $q(f) = q(f')$ .

*Proof.*  $1 \Rightarrow 2$ .

Esercizio

$2 \Rightarrow 3$ .

Esercizio

$3 \Rightarrow 1$ .

□

Esercizio

Let us turn to another functor  $\mathbf{EqGrph} \rightarrow \mathbf{Graph}$ .

**Definition 2.2.19.** The *quotient functor*  $Q : \mathbf{EqGrph} \rightarrow \mathbf{Graph}$

sends  $(A, B, C, s, t, q)$  to  $(A, C, q \circ s, q \circ t)$  and an arrow  $(h_1, h_2, h_3) : (A, B, C, s, t, q) \rightarrow (E, F, G, a, b, c)$  to  $(h_1, h_3)$ .

Verifica che questa cosa funziona  
 $(h_1, h_3)$  è un morfismo di grafi?

**Lemma 2.2.20.**  $Q$  is a left adjoint.

*Proof.* Let us start proving that  $Q$  is a left adjoint. Let  $R(A, B, s, t)$  be  $(A, B, B, s, t, id_B)$ , so that  $Q(R(A, B, s, t)) = (A, B, s, t)$ . Now, suppose that an arrow  $(h_1, h_2) : Q(E, F, G, a, b, c) \rightarrow (A, B, s, t)$  is given. Consider the triple  $(h_1, h_2, h_2 \circ c)$ . Notice that, since  $(h_1, h_2)$  is an arrow in  $\mathbf{Graph}$ :

$$h_2 \circ c \circ a = s \circ h_1 \quad h_2 \circ c \circ b = t \circ h_1$$

Then we have three squares:

$$\begin{array}{ccccc} E & \xrightarrow{a} & F & & E & \xrightarrow{b} & F & & B & \xrightarrow{c} & C \\ h_1 \downarrow & & \downarrow h_2 \circ c & h_1 \downarrow & \downarrow h_2 \circ c & h_2 \circ c \downarrow & \downarrow h_2 & & \downarrow h_2 \\ A & \xrightarrow{s} & B & & A & \xrightarrow{t} & B & & B & \xrightarrow{id_B} & B \end{array}$$

We have therefore found a morphism  $(E, F, G, a, b, c) \rightarrow R(A, B, s, t)$  whose image through  $Q$  fits in the diagram below.

$$\begin{array}{ccc} (A, B, s, t) & \xrightarrow{(id_A, id_B)} & (A, B, s, t) \\ \uparrow Q(h_1, h_2 \circ c, h_2) & \nearrow (h_1, h_2) & \\ (E, G, q \circ s, q \circ t) & & \end{array}$$

Esercizio: prova unicità (è facile)

□

### 2.2.2 Rimanenze da integrare

**Proposition 2.2.21.**  *$Q$  creates colimits.*

*Proof.* □

Q non preserva i limiti

**Example 2.2.22.**

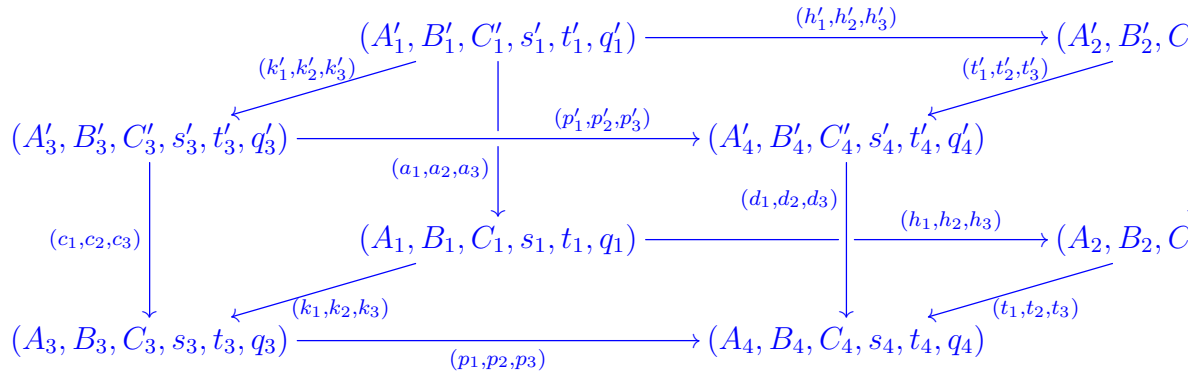
esercizio per te (da verificare solo la relazione: ma dato un diagramma in  $\mathcal{G}$  fatto di quozienti, come costruisci un insieme di vertici per il colimite?)

### 2.2.3 Adhesivity of EqGrph

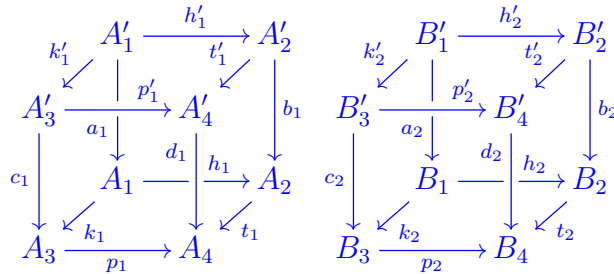
**Lemma 2.2.23.** *In EqGrph pushouts along regular monos are stable.*

*Proof.* Suppose that the cube below is given, in which all the vertical faces are pullbacks and the bottom face is a pushout, with  $(h_1, h_2, h_3): (A_1, B_1, C_1, s_1, t_1, q_1) \rightarrow (A_2, B_2, C_2, s_2, t_2, q_2)$  a regular mono.

Questo diagramma va sistemato per farlo stare nella pagina (val la pena magari dire "sia  $\mathcal{G}_1$  il grafo....")

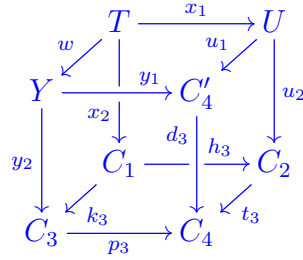


By Proposition 2.2.15 and corollary 2.3.7 the following two cubes have  $\mathcal{M}$ -pushouts as bottom faces and pullbacks as vertical faces, thus their top faces are  $\mathcal{M}$ -pushouts.

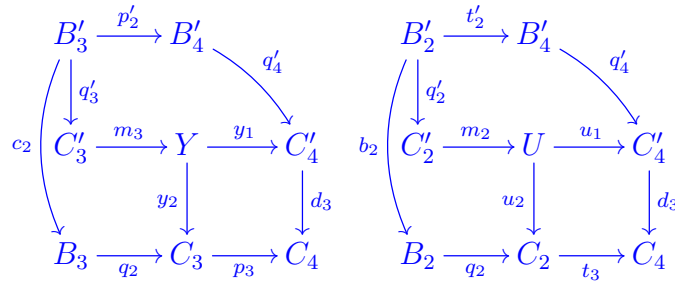


Now, using Corollary 2.2.2, we can consider a third cube, which, by Proposition 2.2.21, has a bottom face an  $\mathcal{M}$ -pushout and pullbacks as vertical faces, so that its top face is an  $\mathcal{M}$ -pushout too.

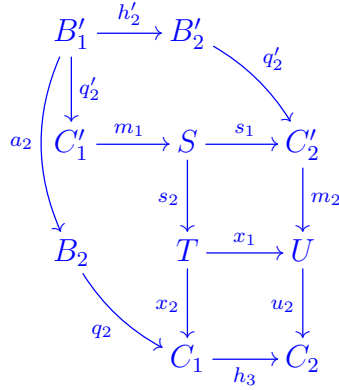
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Moreover, by the proof of Proposition 2.2.15 we know that there are monos  $m_2: C'_2 \rightarrow U$  and  $m_3: C'_3 \rightarrow Y$  fitting in the diagrams



For  $C'_1$ , the we can make a similar argument, let  $S$  be the pull-back of  $m_2$  along  $x_1$ , using Lemma 2.2.1 and, again, the proof of Proposition 2.2.15 we know that  $q'_1$  arise as the factorization of the arrow  $B'_1 \rightarrow S$  induced by  $q'_2 \circ h'_2$  and  $a_2$  so that we have a diagram.



Moreover, notice that

$$s_1 \circ m_1 = h'_3 \quad w \circ s_2 \circ m_1 = m_3 \circ k'_3 \quad t'_3 = u_1 \circ m_2 \quad p_3 = y_1 \circ m_3$$

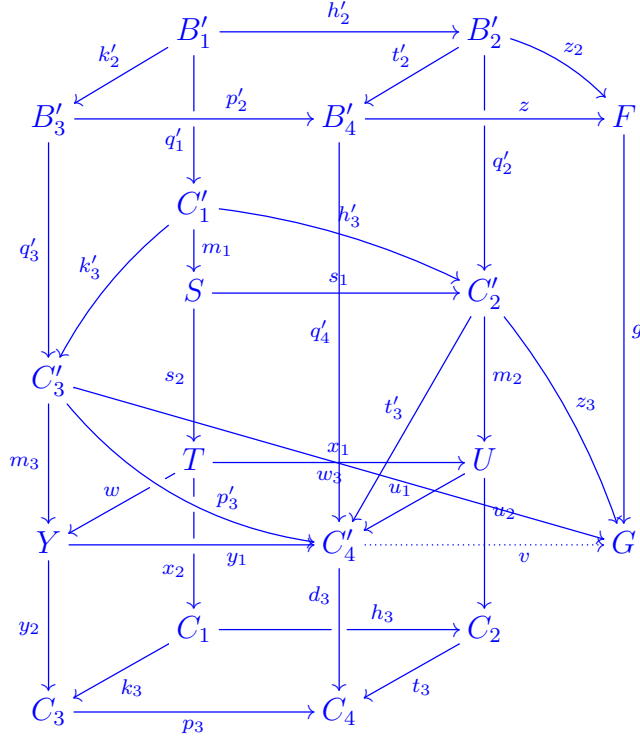
Let now  $(z_1, z_2, z_3): (A'_2, B'_2, C'_2) \rightarrow (E, F, G, e, f, g)$  and  $(w_1, w_2, w_3): (A'_3, B'_3, C'_3) \rightarrow (E, F, G, e, f, g)$  be two morphisms such that

$$(z_1, z_2, z_3) \circ (h'_1, h'_2, h'_3) = (w_1, w_2, w_3) \circ (k'_1, k'_2, k'_3)$$

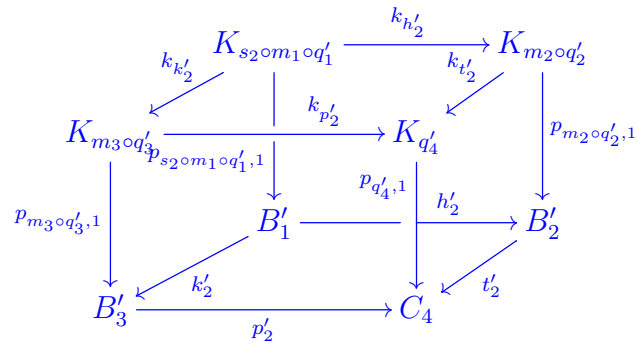
Esercizio (basta comporre prima e dopo con gli opportuni epi e mono)

Sistema il diagramma mettendo gli opportuni buchi

Let  $z: B'_4 \rightarrow F$  be the arrow induced by  $z_2$  and  $w_2$ , we want to construct the dotted arrow  $v: C'_4 \rightarrow G$  in the diagram below



Now by Proposition 2.2.9  $d_3$  is the coequalizer of its kernel pair. On the other hand, by Lemma 2.2.8 we know that the top face of the cube below is a pushout.



Moreover, since  $m_3$  and  $m_2$  are monos, or by Corollary 2.2.6 we also know that

$$q'_3 \circ p_{m_3 \circ q'_3, 1} = q'_3 \circ p_{m_3 \circ q'_3, 2} \quad q'_2 \circ p_{m_2 \circ q'_2, 1} = q'_2 \circ p_{m_2 \circ q'_2, 2}$$

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Now, we have

$$\begin{aligned}
 g \circ z \circ p_{q'_4,1} \circ k_{p'_2} &= g \circ z \circ p'_2 \circ p_{m_3 \circ q'_3,1} & g \circ z \circ p_{q'_4,1} \circ k_{t'_2} &= g \circ z \circ t'_2 \circ p_{m_2 \circ q'_2,1} \\
 &= g \circ w_2 \circ p_{m_3 \circ q'_3,1} & &= g \circ z_2 \circ p_{m_2 \circ q'_2,1} \\
 &= w_3 \circ q'_3 \circ p_{m_3 \circ q'_3,1} & &= z_3 \circ q'_2 \circ p_{m_2 \circ q'_2,1} \\
 &= w_3 \circ q'_3 \circ p_{m_3 \circ q'_3,2} & &= z_3 \circ q'_2 \circ p_{m_2 \circ q'_2,2} \\
 &= g \circ w_2 \circ p_{m_3 \circ q'_3,2} & &= g \circ z_2 \circ p_{m_2 \circ q'_2,2} \\
 &= g \circ z \circ p'_2 \circ p_{m_3 \circ q'_3,2} & &= g \circ z \circ t'_2 \circ p_{m_2 \circ q'_2,2} \\
 &= g \circ z \circ p_{q'_4,2} \circ k_{p'_2} & &= g \circ z \circ p_{q'_4,2} \circ k_{t'_2}
 \end{aligned}$$

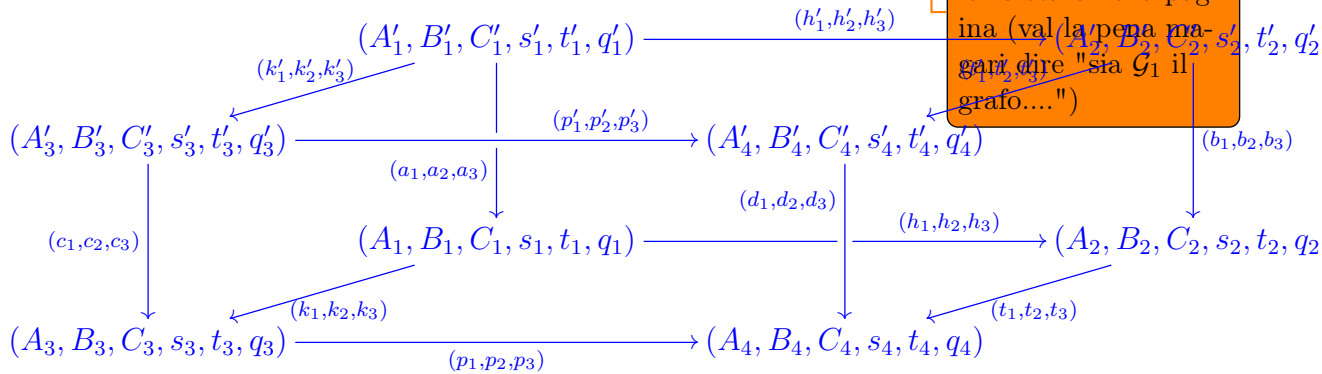
The thesis now follows

Scrivi perché

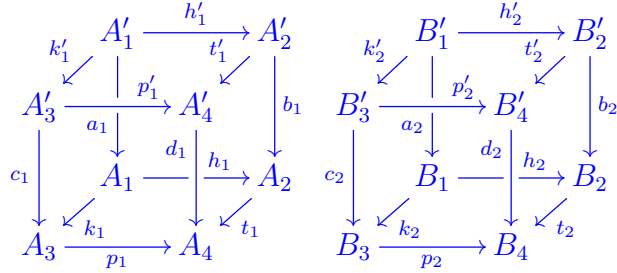
Esercizio: prova  
unicità

**Lemma 2.2.24.** *In  $\mathbf{EqGrph}$  pushouts along regular monos are Reg-Van Kampen.*

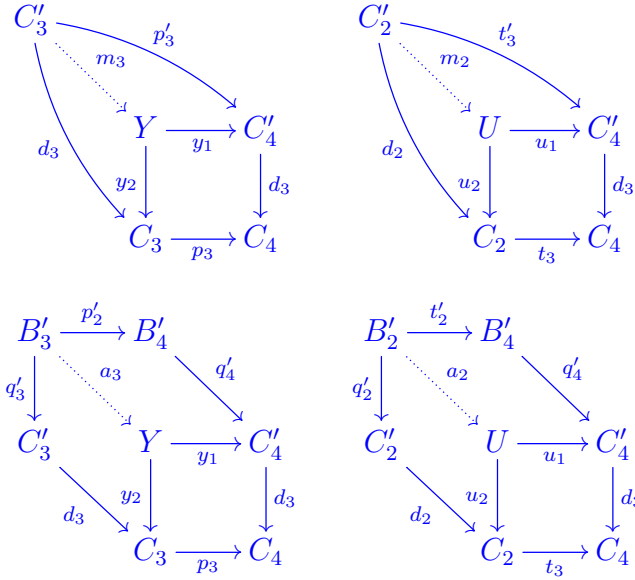
*Proof.* In lieu of Lemma 2.2.23, it is enough to proof that, given a cube as the one below, with pullbacks as back faces, pushouts as bottom and top faces and such that  $(h_1, h_2, h_3): (A_1, B_1, C_1, s_1, t_1, q_1) \rightarrow (A_2, B_2, C_2, s_2, t_2, q_2)$  is a regular mono, the front faces are pullbacks too.



By Proposition 2.2.15 and corollary 2.3.7 the following two cubes have  $\mathcal{M}$ -pushouts as bottom and top faces and pullbacks as back faces, thus their front faces are pullbacks too.



On the other hand we can consider the diagrams below, in which the inner squares are pullbacks. Since the outer diagrams commute, by definition of porphism of **EqGrph**, then we have the existence of the dotted  $m_2: C'_2 \rightarrow U$ ,  $m_3: C'_3 \rightarrow Y$ ,  $a_3: B'_3 \rightarrow Y$  and  $a_2: B'_2 \rightarrow Y$ .



Now, notice that  $m_3$  and  $m_2$  are monos because  $d_3$  and  $d_2$  are regular monos. By the proof of Proposition 2.2.15, to conclude it is enough to show that

$$m_3 \circ q'_3 = a_3 \quad m_2 \circ q'_2 = a_2$$

Indeed, if the previous equations hold, then  $C'_3$  and  $C'_2$  are epi-mono factorizations of  $a_3$  and  $a_2$  and the thesis follows from ?? and the proof of Proposition 2.2.15.

No if we compute we have:

$$\begin{aligned}
 y_1 \circ a_3 &= q'_4 \circ p'_2 & u_1 \circ a_2 &= q'_4 \circ t'_2 \\
 &= p'_3 \circ q'_3 & &= t'_3 \circ q'_3 \\
 &= y_1 \circ m_3 \circ q'_3 & &= u_1 \circ m_2 \circ q'_2 \\
 \\ 
 y_2 \circ a_3 &= d_3 \circ q'_3 & u_2 \circ a_2 &= d_2 \circ q'_2 \\
 &= y_2 \circ m_3 \circ q'_3 & &= u_2 \circ m_2 \circ q'_2
 \end{aligned}$$

And we have done.  $\square$

From Proposition 2.2.15 and Lemmas 2.2.23 and 2.2.24 we deduce at once the following.

**Corollary 2.2.25.** *EqGrph is Reg-adhesive.*

## 2.3 Graphs with Equivalences

A graph with equivalence is a 6-tuple  $\mathbb{G} = (E, V, C, s, t, q)$ , where  $E$  and  $V$  are, respectively, the edges and the vertices sets, and  $C$  is the set of the equivalence classes among vertices,  $s, t : E \rightarrow V$  are the source and target functions and  $q : V \rightarrow C$  is the *quotient* function, that is, the map from a vertex to its equivalence class. For this definition to make sense,  $q$  needs to be surjective. A morphisms  $h$  from a graph with equivalence  $\mathbb{G} = (E, V, C, s, t, q)$  to another  $\mathbb{H} = (E', V', C', s', t', q')$  is a triple  $h = (h_E, h_V, h_C)$  of functions  $h_V : V \rightarrow V'$ ,  $h_E : E \rightarrow E'$  and  $h_C : C \rightarrow C'$  such that:

- $h_E \circ s = s' \circ h_V$ ;
- $h_E \circ t = t' \circ h_V$ ;
- $h_C \circ q = q' \circ h_V$ .

**Remark 2.3.1.** A graph with equivalence can be viewed as a graph endowed with an equivalence relation over its set of vertices,  $(\mathcal{G}, \sim_{\mathcal{G}})$ . An homomorphism between two graphs with equivalences  $h : \mathbb{G} = (\mathcal{G}, \sim_{\mathcal{G}}) \rightarrow \mathbb{H} = (\mathcal{H}, \sim_{\mathcal{H}})$  is a graph homomorphism  $h = (h_V, h_E) : \mathcal{G} \rightarrow \mathcal{H}$  such that if  $v_1 \sim_{\mathcal{G}} v_2$  then  $h_V(v_1) \sim_{\mathcal{H}} h_V(v_2)$ . In **Set**, it is possible to formalize an equivalence relation  $\sim$  over  $X$  as a surjective function sending each element  $x$  on its equivalence class  $[x]_{\sim}$ , and this justify our formalization via surjective functions (i.e., epimorphisms).

Una cosa che manca nei due lemmi precedenti è la seguente: bisognerebbe controllare che lse mappe source e target che appaiano tra i vari A e B sono esattamente quelle che mi danno la struttura di pushout. Questo si può fare a mano, ma sarebbe meglio inserire un lemma da qualche parte che dice: i due funtori  $\text{Graph} \rightarrow \text{Set}$  jointly creano tutto (questo segue immediatamente dal fatto che  $\text{Graph}$  è una categoria di prefasci) Però va scritto e citato nelle prove di entrambi.

As we have done in Section 2.1, we can think to a graph with equivalence as a presheaf, this time from a category  $E \rightrightarrows V \rightarrow C$ , where the image of  $C$  along the presheaf is the set of the equivalence classes, requiring that the morphism  $V \rightarrow C$  is an epimorphism (that is, a surjective function).

**Definition 2.3.2** (Category of Graphs with Equivalences). The category **EqGrph** is the subcategory of

$$[E \begin{smallmatrix} \xrightarrow{s} \\ \xrightarrow[t]{} \end{smallmatrix} V \xrightarrow{q} C, \mathbf{Set}]$$

such that, for each  $\mathbb{G} \in \mathcal{Ob}(\mathbf{EqGrph})$ ,  $\mathbb{G}(q)$  is an epimorphism.

**Observation 2.3.3.** Morphisms of graphs with equivalence are uniquely determined by the first two component. That is, if  $h_1 = (h_E, h_V, \phi)$  and  $h_2 = (h_E, h_V, \psi)$ , then  $\phi = \psi$ .

*Proof.* Let  $h_1, h_2 : \mathbb{G} \rightarrow \mathbb{H}$ , where  $\mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G)$  and  $\mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$ . Then, we have the following situation

$$\begin{array}{ccccc} V_G & \xrightarrow{h_V} & V_H & \xleftarrow{h_V} & V_G \\ \downarrow q_G & & \downarrow q_H & & \downarrow q_G \\ C_G & \xrightarrow{\phi} & C_H & \xleftarrow{\psi} & C_G \end{array}$$

Then, we have:

$$\begin{aligned} \psi \circ q_G &= q_H \circ h_V \\ &= \phi \circ q_G \end{aligned}$$

From the fact that  $q_G$  is epi, we can conclude  $\phi = \psi$ .  $\square$

A graph with equivalence is then a graph with an extra structure, the quotient map. Hence, it is possible to get the underlying graph by forgetting it. Such action is described by the *forgetful functor*  $U : \mathbf{EqGrph} \rightarrow \mathbf{Grph}$ , that maps each graph with equivalence  $\mathbb{G} = (E, V, C, s, t, q)$  onto  $U(\mathbb{G}) = (E, V, s, t)$ , and each morphisms  $h = (h_E, h_V, h_C)$  onto  $U(h) = (h_E, h_V)$ .  $U$  is effectively a functor, since, on identities,  $U((id_E, id_V, id_C)) = (id_E, id_V)$ , and on compositions  $U(h \circ k) = U((h_E \circ k_E, h_V \circ k_V, h_C \circ k_C)) = (h_E \circ k_E, h_V \circ k_V) = (h_E, h_V) \circ (k_E, k_V) = U(h) \circ U(k)$ .



**Proposition 2.3.4.** *The forgetful functor  $U : \mathbf{EqGrph} \rightarrow \mathbf{Graph}$  is faithful.*

*Proof.* Let  $\mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G)$  and  $\mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$  be two graphs with equivalences, and let  $h, k : \mathbb{G} \rightarrow \mathbb{H}$ . If  $U(h) = U(k)$  (i.e., the first two component of  $h$  and  $k$  are the same), from Observation 2.3.3, we can conclude that  $h = k$ . Then, the restriction  $U_{\mathbb{G}, \mathbb{H}} : \mathbf{EqGrph}(\mathbb{G}, \mathbb{H}) \rightarrow \mathbf{Graph}(U(\mathbb{G}), U(\mathbb{H}))$  is injective, therefore  $U$  is faithful.  $\square$

Another functor that will be useful later is  $V : \mathbf{EqGrph} \rightarrow \mathbf{Set}$ , that sends  $(E_G, V_G, C_G, s_G, t_G, q_G)$  to  $C_G$ , and  $h = (h_E, h_V, h_C)$  to  $h_C$ .

**Proposition 2.3.5.**  *$\mathbf{EqGrph}$  has all limits, colimits and  $U$  preserves limits and colimits.*

*Proof.* Let  $D : \mathcal{J} \rightarrow \mathbf{EqGrph}$  be a diagram. In the following, we will denote the graph with equivalence  $D(i)$  as  $(E_i, V_i, C_i, s_i, t_i, q_i)$ . Let now be the graph  $(A, B, s, t)$  the limit of  $U \circ D$ , with projections  $(\pi_E^i, \pi_V^i) : (A, B, s, t) \rightarrow (E_i, V_i, C_i, s_i, t_i)$ . Notice now that  $(B, (q_i \circ \pi_V^i)_{i \in \mathcal{J}})$  is a cone for  $V \circ D$ . To see this, let  $\alpha : i \rightarrow j$  be an arrow of  $\mathcal{J}$ ,  $D(\alpha) = (h_E, h_V, h_C)$ ,  $U \circ D(\alpha) = (h_E, h_V)$ . From the definition of cone, we have that  $U \circ D(\alpha) \circ (\pi_E^i, \pi_V^i) = (\pi_E^j, \pi_V^j)$ , hence  $h_V \circ \pi_V^i = \pi_V^j$ . Consider now the following diagram in  $\mathbf{Set}$

$$\begin{array}{ccc}
 & B & \\
 \pi_V^i \swarrow & & \searrow \pi_V^j \\
 V_i & \xrightarrow{h_V} & V_j \\
 q_i \downarrow & & \downarrow q_j \\
 C_i & \xrightarrow{h_C} & C_j
 \end{array}$$

So we have  $q_j \circ h_V \circ \pi_V^i = q_j \circ \pi_V^j$ , by definition of graph with equivalence,  $h_C \circ q_i \circ \pi_V^i = q_j$ , and, by definition of  $V$ ,  $V \circ D(\alpha) \circ q_i \circ \pi_V^i = q_j \circ \pi_V^j$ . Suppose now  $(L, (l_i)_{i \in \mathcal{J}})$  be a limit for  $V \circ D$ , so that we have an arrow  $l : B \rightarrow L$ . This arrow is not epi in general, so let  $Q$  be its image,  $q : Q \rightarrow B$  be the resulting epi and  $m : Q \rightarrow L$  the corresponding mono, as the diagram below shows. By definition, the external rectangle commutes, so, for each  $i$  object

of  $\mathcal{J}$ , Proposition 1.1.16 yields the dotted arrow  $\pi_C^i$ .

$$\begin{array}{ccccc}
 B & \xrightarrow{\pi_V^i} & B_i & \xrightarrow{q_i} & Q_i \\
 \downarrow q & & & \nearrow \pi_C^i & \downarrow id_{Q_i} \\
 Q & \xrightarrow{m} & L & \xrightarrow{l_i} & Q_i
 \end{array}$$

We have to show that in this way we get a cone over the diagram  $D$ . Let  $\alpha : i \rightarrow j$  be an arrow of  $\mathcal{J}$ , then we have:

$$\begin{aligned}
 U(D(\alpha) \circ (\pi_E^i, \pi_V^i, \pi_C^i)) &= U(D(\alpha)) \circ (\pi_E^i, \pi_V^i) \\
 &= (\pi_E^j, \pi_V^j) \\
 &= U(D(\alpha) \circ (\pi_E^j, \pi_V^j, \pi_C^j))
 \end{aligned}$$

And faithfulness of  $U$  yields the thesis.

To see that this cone is terminal, let  $((E, F, G, a, b, c), (\tau_i)_{i \in \mathcal{J}})$ , where  $\tau_i = (\tau_E^i, \tau_V^i, \tau_C^i)$ , be another cone. By construction, we have an arrow  $(\tau_E, \tau_V) : (E, F, a, b) \rightarrow (A, B, s, t)$  such that

$$\begin{array}{ccc}
 & E & \\
 \tau_E \swarrow & & \searrow \tau_E^i \\
 A & \xrightarrow{\pi_E^i} & A_i
 \end{array}
 \qquad
 \begin{array}{ccc}
 & F & \\
 \tau_V \swarrow & & \searrow \tau_V^i \\
 B & \xrightarrow{\pi_V^i} & B_i
 \end{array}$$

For the same reason as before,  $(G, (\tau_C^i)_{i \in \mathcal{J}})$  is a cone over  $V \circ D$ , thus there exists an arrow  $\tau : G \rightarrow L$  such that  $l_i \circ \tau = \tau_C^i$ . At this point, we get

$$\begin{aligned}
 l_i \circ \tau \circ c &= \tau_C^i \circ c \\
 &= q_i \circ \tau_V^i && \tau_i \text{ is a morphism in } \mathbf{EqGrph} \\
 &= q_i \circ \pi_V^i \circ \tau_V && \text{Diagram above} \\
 &= l_i \circ l \circ \tau_V && (B, (q_i \circ \pi_V^i)_{i \in \mathcal{J}}) \text{ cone}
 \end{aligned}$$

Therefore, the outer part of the rectangle below commutes, and

by Proposition 1.1.16 there exists a unique  $\tau_C : G \rightarrow Q$

$$\begin{array}{ccccc}
 F & \xrightarrow{\tau_V} & B & \xrightarrow{q} & Q \\
 \downarrow c & & & \nearrow \tau_C & \downarrow m \\
 G & & & & L \\
 & & \xrightarrow{\tau} & &
 \end{array}$$

Faithfulness of  $U$  guarantees that  $(\tau_E, \tau_V, \tau_C)$  is the unique arrow such that  $(\pi_E^i, \pi_V^i, \pi_C^i) \circ (\tau_E, \tau_V, \tau_C) = (\tau_E^i, \tau_V^i, \tau_C^i)$ .

For colimits, let  $(A, B, s, t)$  be the colimit of  $U \circ D$ , together with morphisms  $(\kappa_i)_{i \in \mathcal{I}} = (\kappa_E^i, \kappa_V^i)_{i \in \mathcal{I}}$ , and suppose  $((c_i)_{i \in \mathcal{I}}, C)$  be the colimit of  $V \circ D$ . Then, we have the following situation

Dimostrazione dello statement di sopra, e colimiti

$$\begin{array}{ccc}
 & B & \\
 \kappa_V^i \nearrow & & \nwarrow \kappa_V^j \\
 B_i & \xrightarrow{h_V} & B_j \\
 q_i \downarrow & & \downarrow q_j \\
 C_i & \xrightarrow{h_C} & C_j \\
 c_i \searrow & & \swarrow c_j \\
 & C &
 \end{array}$$

Then,  $((c_i \circ q_i)_{i \in \mathcal{I}}, C)$  is a cocone for all  $B_i$ , and, since  $((\kappa_i)_{i \in \mathcal{I}}, B)$  is the colimit of all  $B_i$  (Lemma 1.2.15), there exists a unique arrow  $q : B \rightarrow C$  such that, for each  $i$ ,  $q \circ \kappa_V^i = c_i \circ q_i$ . Such  $q$  is epi **dimostrare**, and thus  $(A, B, C, s, t, q)$  is the colimit of  $D$ . **rifinire, dimostrare unicit .**

□

**Corollary 2.3.6.** *An arrow  $h = (h_E, h_V, h_C) : \mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G) \rightarrow \mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$  in **EqGrph** is mono if and only if  $h_E$  and  $h_V$  are mono in **Set**.*

*Proof.* The “if” part is given by the fact that  $U$  is faithful, and hence reflects monomorphisms. Since a morphism in a category of presheaves is mono if and only if it is injective on each component, we have that, if  $U(h)$  is mono, that is,  $h_E$  and  $h_V$  are injective in **Set**, then  $h$  is mono. For the “only if” part, suppose  $f = (f_E, f_V, f_C)$ ,

$g = (g_E, g_V, g_C)$ ,  $f, g : \mathbb{H} \rightarrow \mathbb{K}$  be such that  $h \circ f = h \circ g$ . Then, we have

$$\begin{aligned} h \circ f &= (h_E \circ f_E, h_V \circ f_V, h_C \circ f_C) \\ &= (h_E \circ f_E, h_V \circ f_V, h_V \circ f_V \circ \mathbb{K}(q)) \\ &= (h_E \circ g_E, h_V \circ g_V, h_V \circ g_V \circ \mathbb{K}(q)) \end{aligned}$$

Since  $\mathbb{K}(q)$  is epi, we have, on the third component, that  $h_V \circ f_V \circ \mathbb{K}(q) = h_V \circ g_V \circ \mathbb{K}(q)$  implies  $f_C = g_C$ , and hence  $f = g$   $\square$

**Corollary 2.3.7.** *Let  $h = (h_E, h_V, h_C) : \mathbb{G} = (E_G, V_G, C_G, s_G, t_G, q_G) \rightarrow \mathbb{H} = (E_H, V_H, C_H, s_H, t_H, q_H)$  be a morphism of **EqGrph**, then the following are equivalent:*

1.  $h$  is a regular mono;
2.  $h_E, h_V, h_C$  are all monos;
3.  $h_E$  and  $h_V$  are mono and  $(K, \pi_1, \pi_2)$  is the kernel pair of  $q_H \circ h_V$  if and only if  $(K, \pi_1, \pi_2)$  is the kernel pair of  $q_G$ .

*Proof.*  $1 \Rightarrow 2$ . If  $h$  is mono, from Corollary 2.3.6 we have that  $h_E$  and  $h_V$  are monos. To derive  $h_C$  mono, suppose  $f, g : \mathbb{H} \rightarrow \mathbb{K}$  to be the arrows equalized by  $h$ . Then we have

$$\begin{aligned} f_C \circ h_C \circ \mathbb{G}(q) &= f_C \circ \mathbb{H}(q) \circ h_V \\ &= \mathbb{K}(q) \circ f_V \circ h_V \\ &= \mathbb{K}(q) \circ g_V \circ h_V \\ &= g_C \circ h_C \circ \mathbb{G}(q) \end{aligned}$$

since  $\mathbb{G}(q)$  is epi, we have that  $f_C \circ h_C = g_C \circ h_C$ , hence  $h_C$  is an equalizer for  $f_C$  and  $g_C$ , thus a monomorphism.

$2 \Rightarrow 3$ . We note that, by Corollary 2.2.6,  $(K, \pi_1, \pi_2)$  is the kernel pair of  $q_G$  if and only if it is the kernel pair also of  $h_C \circ q_G$ , since  $h_C$  is mono by hypothesis. The thesis follows from  $h_C \circ q_G = q_H \circ h_V$ , and from the hypothesis of  $h_E$  mono.

Esercizio

$3 \Rightarrow 1$ . idea: force the comm. of the diagram on the last two components to obtain the two arrows that are equalized, and show that the condition in 3 is sufficient to conclude reg. mono  $\square$

**Remark 2.3.8.** It is possible to restate the third point of the Corollary 2.3.7, by Example 1.3.36, as

$h_E$  and  $h_V$  are mono and, for every  $v, v' \in V_H$ ,  $q_H(h_V(v)) = q_H(h_V(v'))$  if and only if  $q_G(v) = q_G(v')$

That is, a regular monomorphism in **EqGrph** is a morphism that reflects equivalences besides preserving them.

Let us turn to another functor **EqGrph**  $\rightarrow$  **Graph**.

**Definition 2.3.9.** The *quotient functor*  $Q : \mathbf{EqGrph} \rightarrow \mathbf{Graph}$  sends  $(E_G, V_G, C_G, s_G, t_G, q_G)$  to  $(E_G, C_G, q_G \circ s_G, q_G \circ t_G)$  and an arrow  $(h_E, h_V, h_C) : (E_G, V_G, C_G, s_G, t_G, q_G) \rightarrow (E_H, V_H, C_H, s_H, t_H, q_H)$  to  $(h_E, h_C)$ .

**Remark 2.3.10.** The action of the functor on a morphism of graphs with equivalences gives a morphism of graphs, in fact  $q_H \circ s_H \circ h_E = q_H \circ h_V \circ s_G = h_C \circ q_G \circ s_G$ . The same is valid for  $t_H$  and  $t_G$ .

**Lemma 2.3.11.**  $Q$  is a left adjoint.

*Proof.* Let  $R((A, B, s, t))$  be  $(A, B, B, s, t, id_B)$ , so that  $Q(R((A, B, s, t))) = (A, B, s, t)$ . Now, suppose that  $h = (h_E, h_V) : Q((E, V, C, s', t', q)) \rightarrow (A, B, s, t)$  is an arrow in **Graph**, and consider the triple  $(h_E, h_V, h_V \circ q)$ . Since  $h$  is a morphism of **Graph**,

$$h_V \circ q \circ s' = s \circ h_E \quad h_V \circ q \circ t' = t \circ h_E$$

Then we have the following squares:

$$\begin{array}{ccccc} E & \xrightarrow{h_E} & A & & E & \xrightarrow{h_E} & A & & V & \xrightarrow{h_V \circ q} & B \\ \downarrow s_G & & \downarrow s & & \downarrow t_G & & \downarrow t & & \downarrow q & & \downarrow id_B \\ V & \xrightarrow{h_V \circ q} & B & & V & \xrightarrow{h_V \circ q} & B & & C & \xrightarrow{h_V} & B \end{array}$$

We have therefore found a morphism  $(E, V, C, s', t', q) \rightarrow R((A, B, s, t))$  whose image through  $Q$  fits in the diagram below.

$$\begin{array}{ccc} (A, B, s, t) & \xrightarrow{id_A, id_B} & (A, B, s, t) \\ \uparrow Q((h_E, h_V \circ q, h_V)) & \nearrow (h_E, h_V) & \\ (E, C, q \circ s', q \circ t') & & \end{array}$$

Such arrow is unique. Suppose  $f = (f_E, f_V, f_C)$  to be another arrow with such property. Then, it must be  $(id_A, id_B) \circ Q(f) = (f_E, f_C) = (h_E, h_C)$ . Finally,  $f_C = f_V \circ q = h_V \circ q$ .  $\square$

**Proposition 2.3.12.** *Q creates colimits.*

*Proof.* Preserve from Theorem 1.3.30. Remain to see Reflect.  $\square$

## 2.4 E-Graphs

### COSE DA FARE:

Questa sezione ha più o meno gli stessi problemi della precedente. L'ordine da rispettare inho è il seguente:

>Definizione >Calcolo dei limiti e certi colimiti (si fanno come in EqGrph) >cor del calcolo dei limiti: caratterizzare mono regolari >I crea limiti e i giusti pushout > e-graph sono quasiadesivi

E-Graphs are a particular type of graphs with equivalences, in which holds that

$$\frac{s(e) \sim s(e')}{t(e) \sim t(e')}$$

for each pair of edges  $e, e'$  of  $\mathbb{G} = (G, \sim)$ . In a more general case, considering a graph with equivalence as a functor  $\mathbb{G} : (E \rightrightarrows V \rightarrow Q) \rightarrow \mathbf{Set}$ , the inference rule above rewrites as

$$\frac{\mathbb{G}(q \circ s)(e) = \mathbb{G}(q \circ s)(e')}{\mathbb{G}(q \circ t)(e) = \mathbb{G}(q \circ t)(e')}$$

for each  $e, e' \in \mathbb{G}(E)$ . But this is to say that, given the two set  $S = \{(e, e') \in \mathbb{G}(E) \times \mathbb{G}(E) \mid \mathbb{G}(q \circ s)(e) = \mathbb{G}(q \circ s)(e')\}$  and  $T = \{(e, e') \in \mathbb{G}(E) \times \mathbb{G}(E) \mid \mathbb{G}(q \circ t)(e) = \mathbb{G}(q \circ t)(e')\}$ ,  $S \subseteq T$ . But  $S$  (with the projection arrows  $p_1$  and  $p_2$ ) is exactly the kernel pair of  $\mathbb{G}(q \circ s)$ , and  $T$  (together with the projections  $q_1, q_2$ ) is the kernel pair of  $\mathbb{G}(q \circ t)$  (Example 1.3.36). Then, a more general way to express that  $\mathbb{G}$  is an e-graph is by saying that  $\mathbb{G}$  is such that there exists a monomorphism, which is the canonical inclusion, in  $\mathbf{Set}$  from  $S$  to  $T$ . To find a structure to express this fact, we have to consider a more general case.

Since both  $S$  and  $T$  are subobjects of  $\mathbb{G}(E) \times \mathbb{G}(E)$ , we can make use of the  $??$ . Specifically, we want, in the following situation,  $i$  to be mono and unique.

$$\begin{array}{ccc} S & \xrightarrow{i} & T \\ & \searrow \langle p_1, p_2 \rangle & \swarrow \langle q_1, q_2 \rangle \\ & \mathbb{G}(E) \times \mathbb{G}(E) & \end{array}$$

**Definition 2.4.1** (Category of E-Graphs). **EGG** is the subcategory of **EqGrph** such that

- $\mathbb{G} \in \mathbf{EGG}$  if, being  $(S, p_1, p_2)$  and  $(T, q_1, q_2)$ , respectively, the kernel pair of  $\mathbb{G}(q \circ s)$  and the kernel pair of  $\mathbb{G}(q \circ t)$ , it holds that  $\langle p_1, p_2 \rangle \leq \langle q_1, q_2 \rangle$  (in the sense of Definition 1.1.9);
- given two e-graphs  $\mathbb{G}, \mathbb{H}$ ,  $\mathbf{EGG}(\mathbb{G}, \mathbb{H}) = \mathbf{EqGrph}(\mathbb{G}, \mathbb{H})$  (i.e., a full subcategory).

Since **EGG** is a full subcategory of **EqGrph**, there exists a fully faithful functor  $I : \mathbf{EGG} \rightarrow \mathbf{EqGrph}$  (Example 1.2.8).

**Lemma 2.4.2.** ***EGG** has all limits and colimits, and  $I$  preserves them.*

*Proof.* Let  $D : \mathcal{J} \rightarrow \mathbf{EGG}$  be a diagram, and suppose  $(\mathbb{L}, (l_i)_{i \in \mathcal{J}})$  be a limit for  $I \circ D$ , where  $l_i = (l_E^i, l_V^i, l_C^i)$ . Then, it is sufficient to show that  $(\mathbb{L}, (l_i)_{i \in \mathcal{J}})$  is a limit for  $D$  (i.e.,  $\mathbb{L}$  is an e-graph). To see this, consider  $e, e' \in \mathbb{L}(E)$ , and suppose  $\mathbb{L}(q \circ s)(e) = \mathbb{L}(q \circ s)(e')$ . Then, we have  $(l_C^i \circ \mathbb{L}(q \circ s))(e) = (l_C^i \circ \mathbb{L}(q \circ s))(e')$ , but, since

$$\begin{aligned} l_C^i \circ \mathbb{L}(q \circ s) &= l_C^i \circ \mathbb{L}(q) \circ \mathbb{L}(s) \\ &= D(i)(q) \circ l_V^i \circ \mathbb{L}(s) \\ &= D(i)(q \circ s) \circ l_E^i \end{aligned}$$

the equation above rewrites as

$$(D(i)(q \circ s))(l_E^i(e)) = (D(i)(q \circ s))(l_E^i(e'))$$

Since  $D(i)$  is an e-graph, we have then

$$(D(i)(q \circ t))(l_E^i(e)) = (D(i)(q \circ t))(l_E^i(e'))$$

and, retracing the previous steps,

$$(l_C^i \circ \mathbb{L}(q \circ t))(e) = (l_C^i \circ \mathbb{L}(q \circ t))(e')$$

□

finire qui.





# Appendix A

## Omitted Proofs

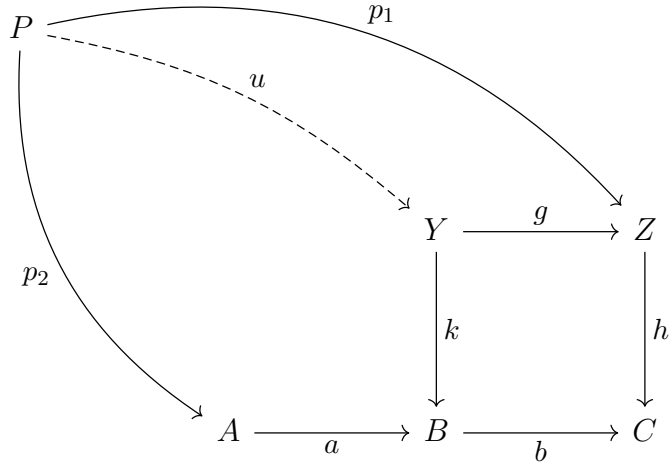
**Lemma 1.3.20.** Suppose that the following diagram is given and its right half is a pullback. Then the whole rectangle is a pullback if and only if its left half is a pullback.

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
 \downarrow t & & \downarrow k & & \downarrow h \\
 A & \xrightarrow{a} & B & \xrightarrow{b} & C
 \end{array}$$

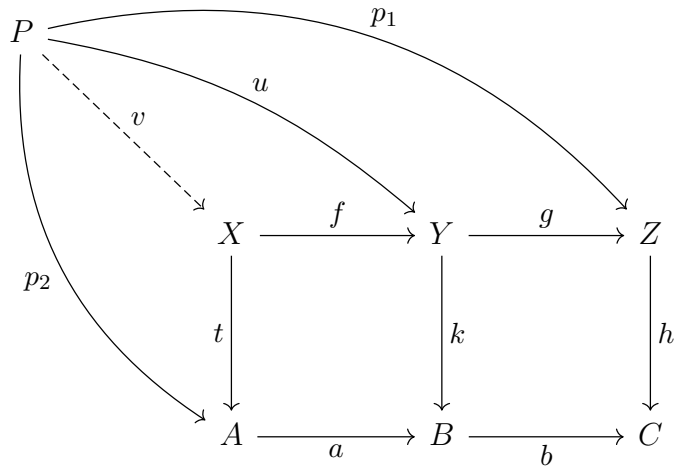
*Proof.* For the “only if” part, suppose the left square to be a pullback. To verify the outer rectangle is a pullback, consider the following situation:

$$\begin{array}{ccccccc}
 P & \xrightarrow{p_1} & & & & & Z \\
 \downarrow p_2 & & & & & & \downarrow h \\
 & & A & \xrightarrow{a} & B & \xrightarrow{b} & C
 \end{array}$$

But the right square is a pullback implies that there exists a unique  $u$  such that

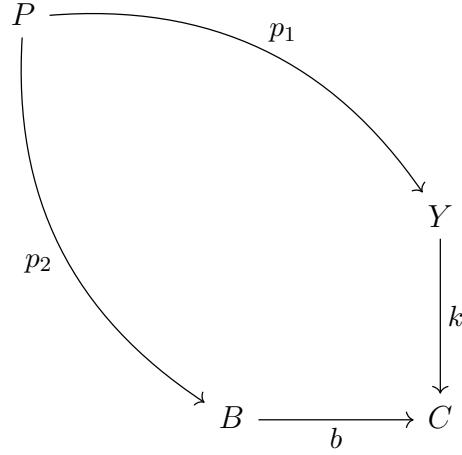


And, since the left square is a pullback, there exists a unique  $v$  such that

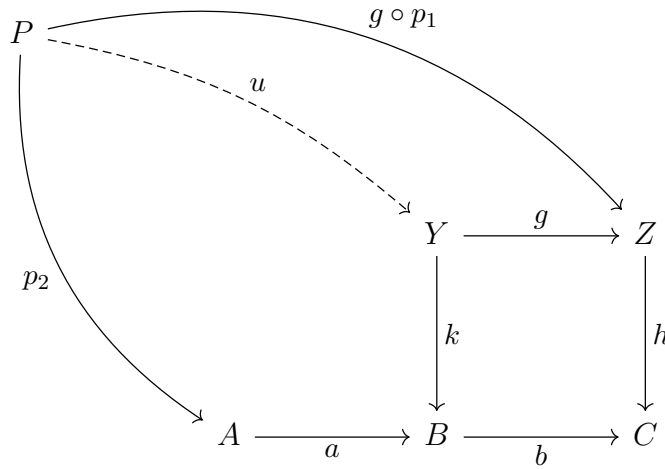


Hence, the whole rectangle is a pullback.

For the “if” part, consider the following situation.



We have now to show that the unique arrow  $v : P \rightarrow X$  (of the outer rectangle) is such that  $f \circ v = p_1$ , but this follows from the fact that the right square is a pullback, having the following situation.



□

Rivedere questa dimostrazione

**Theorem 1.3.22.** Let  $\mathcal{C}$  be a category. Then  $\mathcal{C}$  has all finite limits if and only if  $\mathcal{C}$  has all finite products and all equalizers.

*Proof.* Let  $D : \mathcal{I} \rightarrow \mathcal{C}$  a diagram, with  $\mathcal{I}$  finite.

The *if* statement follows from definitions of products and equalizers (Definition 1.3.15, Definition 1.3.12)

To satisfy the *only if* statement, we want an object  $L$  together with morphisms  $p_i : L \rightarrow D(j)$  such that:

1.  $\{p_i : L \rightarrow D(i)\}$  is a cone – i.e., for each morphism of  $\mathcal{J}$   $\alpha : i \rightarrow j$ ,  $D(\alpha) \circ p_i = p_j$ ; and
2. for each  $E$  and  $q_i : E \rightarrow D(j)$  in  $\mathcal{C}$ , with  $D(\alpha) \circ q_i = q_j$  for each  $\alpha : i \rightarrow j$  of  $\mathcal{J}$ , there exists a unique  $f : E \rightarrow L$  such that  $q_i = p_i \circ f$  for each  $i \in \mathcal{Ob}(\mathcal{J})$ .

Consider the two products (which exist by hypothesis)  $\prod_{j \in \mathcal{Ob}(\mathcal{J})} D(j)$ , the product of the objects of the diagram, and  $\prod_{\alpha \in \mathcal{Hom}(\mathcal{J})} D(\text{cod } \alpha)$ , the product of the codomains of the morphisms in  $D$ , where  $\pi_x$  is the  $x$ -th projection of the product. Let now:

$$\gamma, \varepsilon : \prod_{j \in \mathcal{Ob}(\mathcal{J})} D(j) \longrightarrow \prod_{\alpha \in \mathcal{Hom}(\mathcal{J})} D(\text{cod } \alpha)$$

be defined by  $\gamma_\alpha = \pi_{D(\text{cod } \alpha)}$  (the projection on the codomain of  $\alpha$ ) and  $\varepsilon_\alpha = D(\alpha) \circ \pi_{D(\text{dom } \alpha)}$ . Let now  $e : L \rightarrow \prod_{j \in \mathcal{Ob}(\mathcal{J})} D(j)$  the equalizer of  $\gamma$  and  $\varepsilon$  (which exists by hypothesis), and, for each  $j \in \mathcal{Ob}(\mathcal{J})$ ,  $p_j : L \rightarrow D(j)$ , defined by  $p_j = \pi_{D(j)} \circ e$ .

What we want now is to show that  $(L, (p_i)_{i \in \mathcal{J}})$  is the limit of  $D$ , namely, to prove that the conditions given at the beginning are valid.

For condition 1, we have to show that, for each  $\alpha : i \rightarrow j$  of  $\mathcal{J}$ , we have  $D(\alpha) \circ p_i = p_j$ :

$$\begin{aligned} D(\alpha) \circ p_i &= D(\alpha) \circ \pi_{D(i)} \circ e && \text{Definition of } p_j \\ &= \varepsilon_\alpha \circ e && \text{Definition of } \varepsilon \\ &= \gamma_\alpha \circ e && e \text{ is an equalizer of } \pi, \varepsilon \\ &= \pi_{D(j)} \circ e && \text{Definition of } \pi \\ &= p_j && \text{Definition of } p_j \end{aligned}$$

For condition 2, suppose that  $(E, (q_i)_{i \in \mathcal{Ob}(\mathcal{J})})$  has the properties stated. By definition of product, there exists a (unique) arrow  $q : E \rightarrow \prod_{j \in \mathcal{Ob}(\mathcal{J})} D(j)$ . For each arrow  $\alpha : i \rightarrow j$ , we have:

$$\begin{aligned} \gamma_\alpha \circ q &= \pi_{D(j)} \circ q && \text{Definition of } \pi \\ &= q_j && \text{Definition of } q_j \\ &= D(\alpha) \circ q_i && \text{Assumption on } q_j \\ &= D(\alpha) \circ \pi_{D(i)} \circ q && \text{Definition of } q_i \\ &= \varepsilon_\alpha \circ q && \text{Definition of } \varepsilon \end{aligned}$$

Since  $e$  equalizes  $\pi$  and  $\varepsilon$ , there exists a unique  $f : E \rightarrow L$  in  $\mathcal{C}$  such that  $q = e \circ f$ . Then, for each  $j \in \mathcal{O}b(\mathcal{J})$ , we have  $\pi_{D(j)} \circ q = \pi_{D(j)} \circ e \circ f$ , hence,  $q_i = p_i \circ f$ .  $\square$



# Bibliography

- [AHS09] J. Adámek, Horst Herrlich, and George E. Strecker. *Abstract and concrete categories: The joy of cats*. Dover Publications, 2009.
- [BW95] Michael Barr and Charles Wells. *Category theory for computing science*. Prentice Hall, 2 edition, 1995.
- [CGM22] Davide Castelnovo, Fabio Gadducci, and Marino Miculan. A new criterion for  $\mathcal{M}, \mathcal{N}$ -adhesivity, with an application to hierarchical graphs, 2022.
- [HS79] Horst Herrlich and George E. Strecker. *Category Theory*, volume 1 of *Sigma Series in Pure Mathematics*. Heldermann Verlag Berlin, 2 edition, 1979.
- [Lac11] Stephen Lack. An embedding theorem for adhesive categories, 2011.