Lecture 3: Linear Regression

STAT 1361/2360: Statistical Learning and Data Science

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Linear Models

- First supervised learning method: linear regression
  - **Read the Book (ISLR Ch. 3)**; slides are not meant to be a complete review
- Parametric statistical method
- Regression; we'll assume the response $Y \in \mathbb{R}$, though the method can be used for ordinal categorical responses in some cases (more on this in chapter 4)
- Amongst the oldest yet still most common modeling techniques
  - Regression and least-squares dating back to early Legendre and Gauss in early 1800s

Linear Models

Recall that in a standard regression framework, we assume

$$Y = f(X) + \epsilon$$

where X is simply shorthand for the predictors $(X_1, ..., X_p)$. A **linear model** is defined as one in which f is modeled as a linear function of parameters $\beta = (\beta_0, ..., \beta_p)$

$$\text{e.g.} \quad Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p + \epsilon$$

and the parameters correspond to coefficients

Linear Models

Note: The *linear* in linear models refers to the linearity in parameters:

$$\text{e.g.} \quad Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 (X_1 X_2) + \epsilon$$
$$Y = \beta_1 X_1 + \beta_2 X_1^2 + \beta_3 X_1^3 + \epsilon$$
$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2^2 + \beta_3 (X_1 e^{X_3}) + \epsilon$$

are all linear models.

Why so popular?

- Why is linear regression so popular?
  - Explicit and interpretable
  - Many useful corresponding statistics and inference procedures
  - Does not require a large sample to fit
  - Many natural relationships can be well-approximated by a suitable linear model
  - **Not** because we believe many natural relationships actually *are* linear

Simple Linear Regression

Simple Linear Regression

We begin with the simplest case, **simple linear regression**: only a single feature X_1 and we model

$$Y = \beta_0 + \beta_1 X_1 + \epsilon$$

- Given data $(x_{11}, y_1), \ldots, (x_{1n}, y_n)$, we need to estimate parameters $\beta = (\beta_0, \beta_1)$
  - $\beta_0 = y$-intercept (average y-value when $x = 0$)
  - $\beta_1 =$ slope (average change in y with 1 unit change in X_1)
- Given parameter estimates $\hat{\beta}_0, \hat{\beta}_1$, our model estimate becomes

$$\begin{aligned} \hat{y}^* &= \hat{f}(x_1^*) \\ &= \hat{\beta}_0 + \hat{\beta}_1 x_1^* \end{aligned}$$

Estimating the Parameters

How do we estimate the parameters β_0 and β_1?

- Standard statistical method: define a *loss function* and choose parameter estimates that minimize that loss
- Most common choice is squared error loss, or **Least Squares:** Choose $\hat{\beta}_0$ and $\hat{\beta}_1$ to minimize the residual sum of squares (RSS)

$$RSS = \sum_{i=1}^{n} \left(y_i - \hat{y}_i\right)^2 = \sum_{i=1}^{n} \left(y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_{1i})\right)^2$$

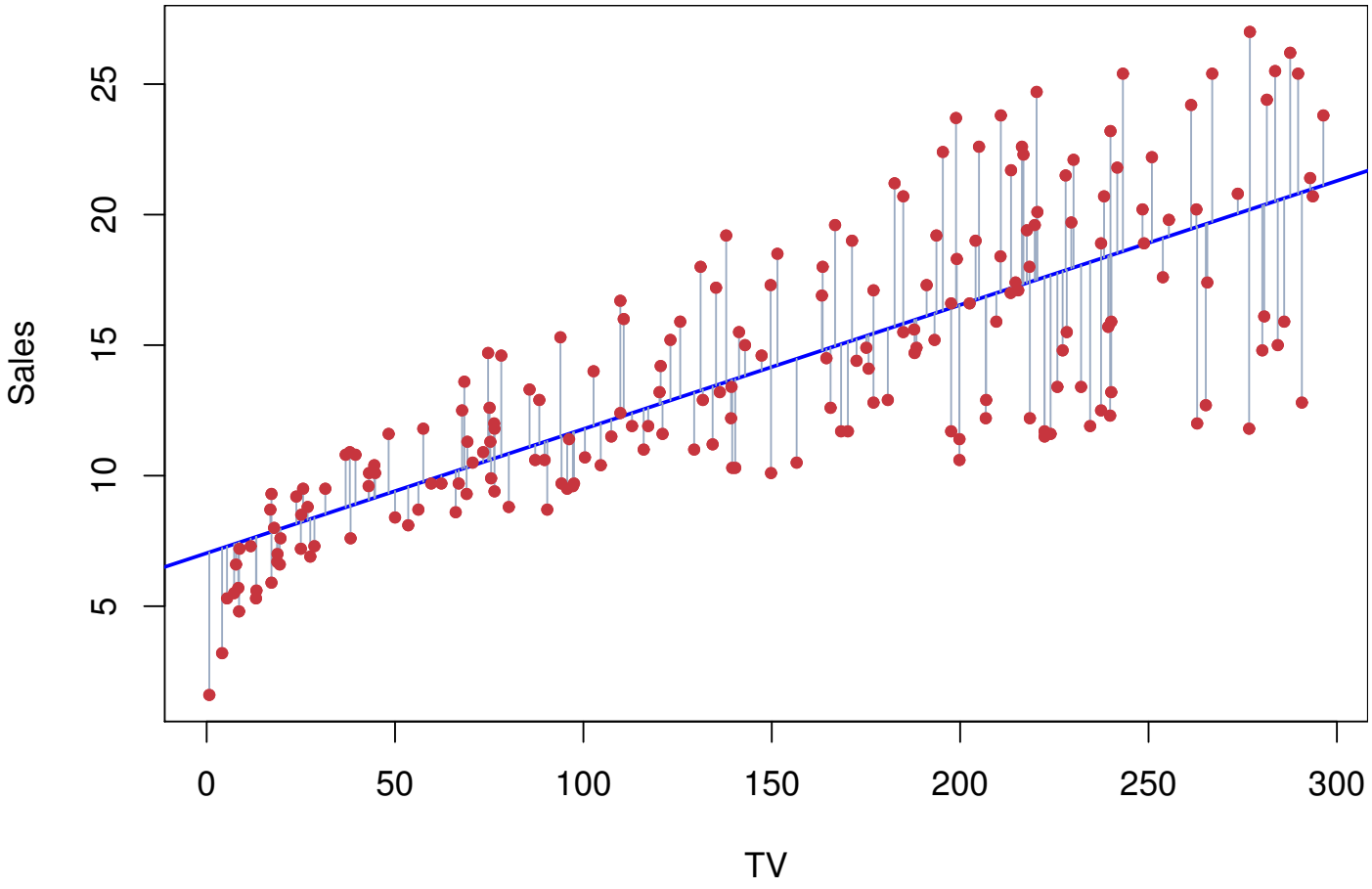
Minimizing gives *ordinary least squares* (OLS) estimates:

$$\hat{\beta}_{1,OLS} = \frac{\sum_{i=1}^{n}(x_{1i} - \bar{x}_1)(y_i - \bar{y})}{\sum_{i=1}^{n}(x_{1i} - \bar{x}_1)^2}$$

$$\hat{\beta}_{0,OLS} = \bar{y} - \hat{\beta}_1 \bar{x}$$

where $\bar{x}_1 = \frac{1}{n}\sum_{i=1}^{n} x_{1i}$ and $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ are the sample means of x_1 and y.

Ordinary Least Squares (OLS)



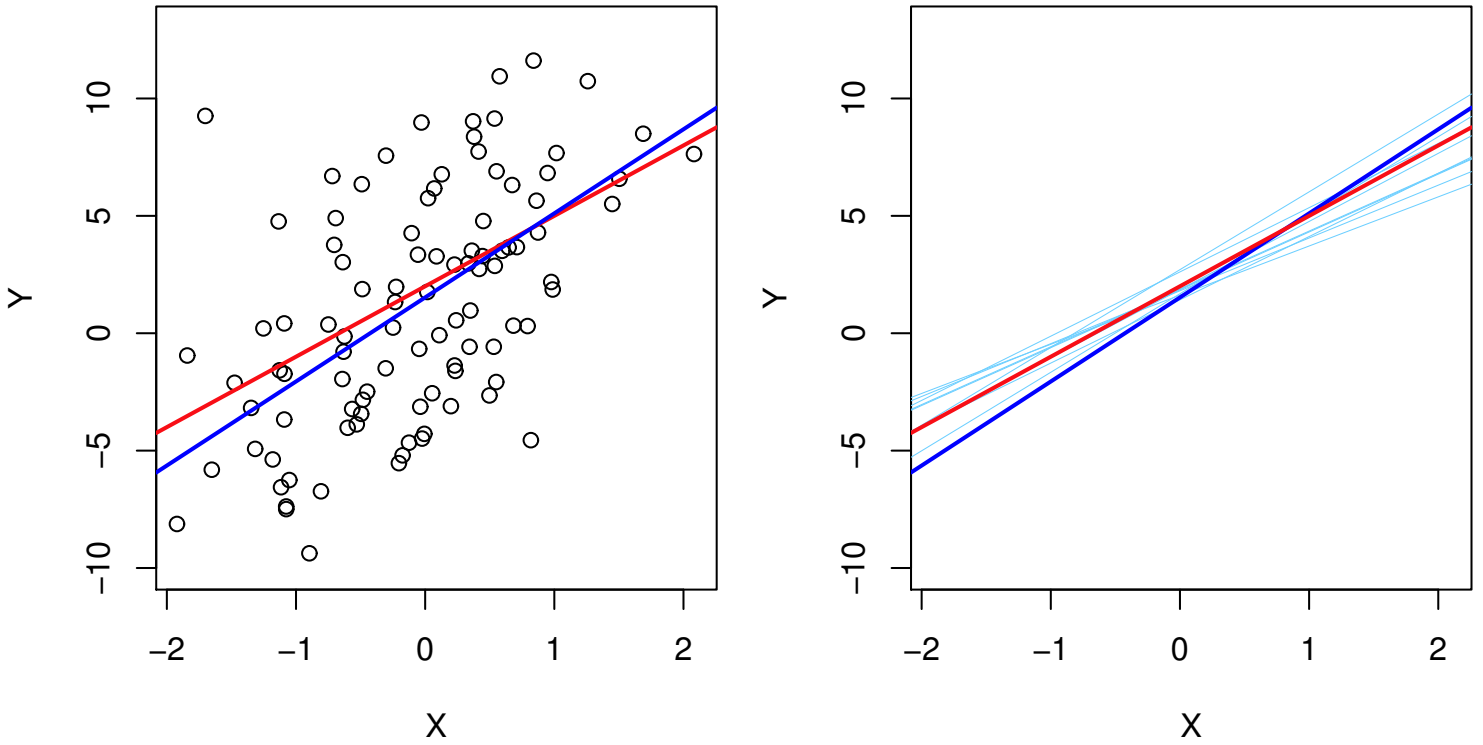
ISLR Fig. 3.1: OLS Linear regression of TV advertisting (X_1) on Sales (Y). Note that this is not the *only* way to choose the parameter estimates, but this method provides nice, useful propterties.

$\hat{\beta}$ vs β

Note very importantly:

- We think of some "true"/"best" regression line corresponding to the true values β_0 and β_1
  - This can be the true relationship between X and Y or simply the best linear approximation – in other words, we think of $\hat{\beta}_0$ and $\hat{\beta}_1$ as being estimates of some true values β_0 and β_1, though these need not describe the true underlying relationship
- However, we *never* observe this: we see only a finite dataset from which we calculate estimates $\hat{\beta}_0$ and $\hat{\beta}_1$
  - If we got a new dataset, we would get a different estimates of $\hat{\beta}_0$ and $\hat{\beta}_1$ and hence a different estimate of the regression line

$\hat{\beta}$ vs β



ISLR Fig. 3.3: **Left:** True regression line (red) and estimate of that line based on the data (blue). **Right:** additional possible estimated regression lines based on different observed datasets (light blue).

OLS Properties

- Least squares (OLS) is not the only way we can estimate $\hat{\beta}_0$ and $\hat{\beta}_1$, but it does have several nice properties
- One very nice property: unbiasedness
  - $\hat{\beta}_0$ is an *unbiased* estimate of β_0; we do not systematically over/under-estimate the parameter:

$$\mathbb{E}(\hat{\beta}_{0,OLS}) = \beta_0$$

  - The same holds for β_1

$$\mathbb{E}(\hat{\beta}_{1,OLS}) = \beta_1$$

Inference Procedures: Standard Error

OLS also allows for straightforward calculations of standard errors:

$$SE(\hat{\beta}_0)^2 = \sigma^2 \left[\frac{1}{n} + \frac{\bar{x}_1^2}{\sum_{i=1}^{n}(x_{1i} - \bar{x}_1)^2} \right]$$

$$SE(\hat{\beta}_1)^2 = \frac{\sigma^2}{\sum_{i=1}^{n}(x_{1i} - \bar{x}_1)^2}$$

Note: $\sigma^2 = \text{var}(\epsilon)$ which, in practice, we won't know. Instead, we estimate it with the residual standard error (RSE):

$$RSE = \hat{\sigma} = \sqrt{RSS/(n-2)}$$

and write $\hat{SE}(\hat{\beta}_0)^2$ and $\hat{SE}(\hat{\beta}_1)^2$ to indicate the estimate of σ

Inference Procedures: Confidence Intervals

- The standard errors tell us something about how much variability there is in the estimates $\hat{\beta}_0$ and $\hat{\beta}_1$
- It is common to want to characterize this variability in the form of a *confidence interval* (CI):

$$95\% \text{ CI for } \beta_0 = \hat{\beta}_0 \pm 2SE(\hat{\beta}_0)$$
$$95\% \text{ CI for } \beta_1 = \hat{\beta}_1 \pm 2SE(\hat{\beta}_1)$$

- **Note:** Book uses a factor of 2, but it should really be ...

Inference Procedures: Hypothesis Testing

- Finally, we'll often want to perform hypothesis tests on β_0 and β_1; most commonly

  H_0: No *linear* relationship between X_1 and $Y \iff \beta_1 = 0$

  H_1: Some *linear* relationship between X_1 and $Y \iff \beta_1 \neq 0$

- Under H_0,

$$t = \frac{\hat{\beta}_1 - 0}{SE(\hat{\beta}_1)} \sim t_{n-2}$$

  where $\sim$ means "is distributed as" and t_{n-2} denotes the t-distribution with $n - 2$ degrees of freedom

  - We can use this result to provide a p-value; small values mean we reject H_0 and conclude that there is some linear relationship between X_1 and Y

Model Fit: RSE

- **Big Question:** At the end of the day, we still need to ask the question "How well does the model actually fit our data?"
- A few different ways to assess model fit:

  1. Residual Standard Error (RSE):

$$RSE = \sqrt{\frac{1}{n-2}RSS} = \sqrt{\frac{1}{n-2}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}$$

- Large values mean y_i and $\hat{y}_i$ far apart for at least some data $\implies$ bad model fit
- Downside:

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- Large values mean y_i and $\hat{y}_i$ far apart for at least some data $\implies$ bad model fit
- Downside: Difficult to determine a "good" RSE value because it depends on the scale of Y

Model Fit: R^2

2. $R^2 = \frac{TSS - RSS}{TSS} = 1 - \frac{RSS}{TSS}$

where TSS = total sum of squares = $\sum_{i=1}^{n}(y_i - \bar{y})^2$

- R^2 = "proportion of variability in Y explained by X_1 in this model"

$$\implies 0 \leq R^2 \leq 1; \text{ larger values better}$$

- In this simple linear regression setting, $R^2 = r^2$ where $r = \text{corr}(X_1, Y)$
- Standardizing by TSS means that the measure is invariant to the scale of Y

Multiple Linear Regression

Multiple Linear Regression

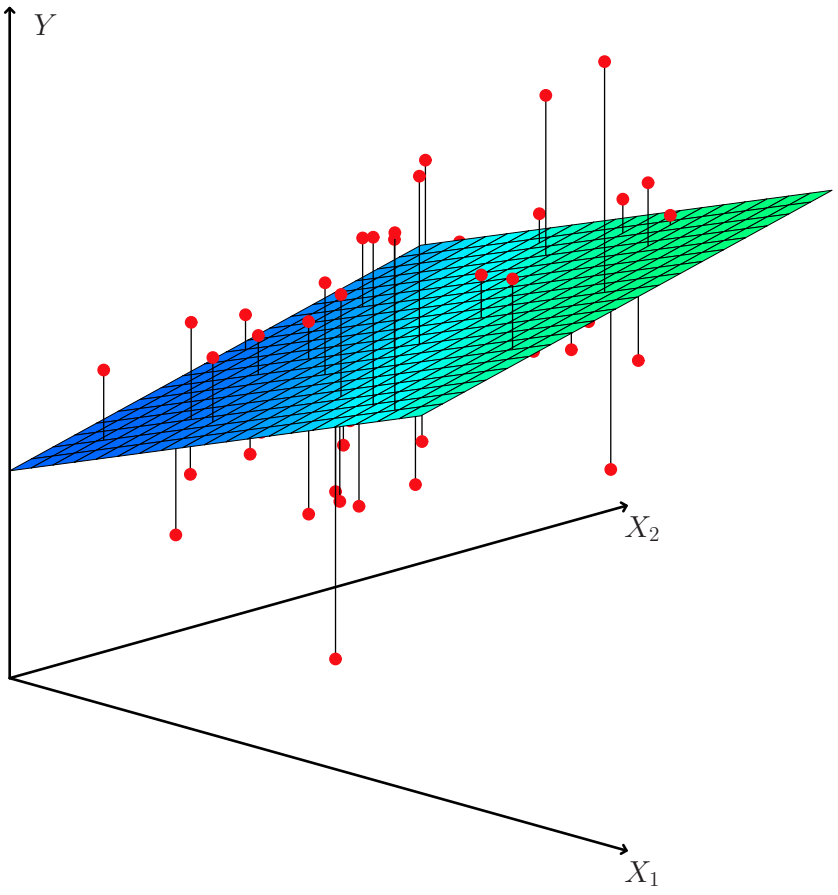
- Now suppose that instead of only 1 predictor, we have p predictors $X_1, \ldots, X_p$
- The "standard" multiple linear regression (MLR) model is

$$Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p + \epsilon$$

- β_j = average change in Y with 1 unit change in X_j **with all other predictors held fixed**
- Coefficients estimates $\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_p$ are estimated in same way, by minimizing residual sum of squares:

$$RSS = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} (y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_{1i} + \cdots + \hat{\beta}_p x_{pi}))^2$$

Multivariate OLS



ISLR Fig. 3.4: Visualization of multiple linear regression with two predictors, X_1 and X_2 with the coefficients fit via ordinary least squares (OLS).

Inference Procedures: Hypothesis Testing

- With MLR, we're interested in many of the same kinds of questions as with SLR
- Most simply:

  H_0: There is no linear relationship between Y and **any** of the predictors

  H_1: There is a linear relationship between Y and **at least one** of the predictors

- Often referred to as the **overall F-test**, since we're testing whether **any** of the predictors are significant

Inference Procedures: Hypothesis Testing

- Equivalent hypotheses:

$$H_0: \ \beta_1 = \beta_2 = \cdots = \beta_p = 0$$
$$H_1: \ \beta_j \neq 0 \text{ for some } j \in \{1, \ldots, p\}$$

- Under H_0,

$$F = \frac{(TSS - RSS)/p}{RSS/(n-p-1)} \sim F_{p,n-p-1}$$

where again, $\sim$ means "is distributed as" and $F_{p-1,n-p}$ denotes the F-distribution with $p-1$ and $n-p$ degrees of freedom (often called the numerator and denominator degrees of freedom, respectively)

Inference Procedures: Hypothesis Testing

- More often we're interested in testing whether a particular *subset* of the predictors are significant
  - Let $X_1^*, \ldots, X_m^*$ denote the subset of predictors we're interested in and $\beta_1^*, \ldots, \beta_m^*$ the corresponding coefficients

Hypothesis test becomes

$$H_0: \ \beta_1^* = \beta_2^* = \cdots = \beta_m^* = 0$$

$$H_1: \ \beta_j^* \neq 0 \text{ for some } j \in \{1, \ldots, m\}$$

and under H_0,

$$F = \frac{(RSS^* - RSS)/m}{RSS/(n-p-1)} \sim F_{m,n-p-1}$$

where RSS^* is the residual sum of squares from the linear model that **does not** include $X_1^*, \ldots, X_m^*$

t-test vs F-test

- Recall that when we had only a single predictor (SLR), we could do a t-test for significance of X_1
  - In fact, even when you perform MLR, statistical software will report a t-test statistic and p-value for each term in the model
- So if we conduct the previous hypothesis test and we're only interested in testing whether one predictor X_1^* is significant, is this the same as just doing the t-test?

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  - Yes!

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  - Yes!
- So then why can't we just look at each individual t-test, keep the ones that are significant, and throw the rest away?

R Output

```
> str(freeny)
'data.frame':    39 obs. of  5 variables:
 $ y                   : Time-Series  from 1962 to 1972: 8.79 8.79 8.81 8.81 8.91 ...
 $ lag.quarterly.revenue: num  8.8 8.79 8.79 8.81 8.81 ...
 $ price.index         : num  4.71 4.7 4.69 4.69 4.64 ...
 $ income.level        : num  5.82 5.83 5.83 5.84 5.85 ...
 $ market.potential    : num  13 13 13 13 13 ...
> lm1 <- lm(y~.,data=freeny)
> summary(lm1)

Call:
lm(formula = y ~ ., data = freeny)

Residuals:
       Min         1Q     Median         3Q        Max
-0.0259426 -0.0101033  0.0003824  0.0103236  0.0267124

Coefficients:
                    Estimate Std. Error t value Pr(>|t|)
(Intercept)          -10.4726     6.0217  -1.739   0.0911 .
lag.quarterly.revenue   0.1239     0.1424   0.870   0.3904
price.index            -0.7542     0.1607  -4.693 4.28e-05 ***
income.level            0.7675     0.1339   5.730 1.93e-06 ***
market.potential        1.3306     0.5093   2.613   0.0133 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.01473 on 34 degrees of freedom
Multiple R-squared:  0.9981,  Adjusted R-squared:  0.9978
F-statistic:  4354 on 4 and 34 DF,  p-value: < 2.2e-16

>
```

Example of R output from a linear model.

t-test vs F-test

Two big reasons:

1. Results (coefficient estimates, test statistics, p-values) are all *with respect to all other terms in the model*
    - If some (at least one) term(s) are removed from the model, the rest changes

2. If each t-test has 5% false positive rate, then about 1 in 20 coefficients will be significant by random chance
    - If we have 100 predictors, on average, 5 would be significant by chance
    - F-test adjusts for the number of predictors; doesn't have this problem

Subset Selection

Big question then: How do we find the 'best' model (i.e. the model with all significant predictors and no 'extra' redundant information)

- Still very much an ongoing area of research in high-dimensional problems (big data)
- Once we have a given model, many standard statistical models can be used to assess how well that model fits (more on this later in the course)
  - Akaike Information Criterion (AIC)
  - Bayesian Information Criterion (BIC)
  - Mallow's C_p
  - Adjusted R^2

Subset Selection

How do we find good 'candidate' models?

- Given p (potential) predictors, there are 2^p possible models; where do we begin?
  - **Forward Selection:** Begin with the null model ($Y = \hat{\beta}_0$), add the variable that gives lowest RSS ($Y = \hat{\beta}_0 + \hat{\beta}_1^* X_1^*$), then add the next variable with lowest RSS ($Y = \hat{\beta}_0 + \hat{\beta}_1^* X_1^* + \hat{\beta}_2^* X_2^*$), etc.
  - **Backward Selection:** Begin with full model ($Y = \hat{\beta}_0 + \hat{\beta}_1 X_1 + \cdots + \hat{\beta}_p X_p$) and eliminate variables one at a time that have the largest p-value

Inference Procedures: Confidence Intervals

Can create confidence intervals for individual β_i's, or for $f(X)$, or prediction intervals:

- **95% CI's for β_i:** 95% of these kinds of intervals should cover the true β_i (assuming the truth is really a linear model)
- **95% CI's for $f(X)$:** 95% of these kinds of intervals should cover the true value of $f(X)$ (assuming the truth is really a linear model)
- **95% Prediction Intervals:** 95% of these kinds of intervals should cover the true value of $Y = f(X) + \epsilon$ (assuming the truth is really a linear model)

Linear Model Extensions

Linear Model Extensions

Thus far we have assumed that the p terms in our model correspond to p continuous predictors $X_1, \ldots, X_p$; many possible extensions of this setup that still yield linear models:

1. Predictors can be categorical
2. Terms can be functions of individual predictors, e.g.

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2$$

3. Terms can be functions of multiple predictors, e.g.

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2$$

Categorical Predictors

1. How do we set up and interpret things if we have a categorical predictor X_i? Two options:

   I. **Dummy Coding:** Treat one level of X_i as the baseline/reference and estimate the change from this baseline for the other levels:

   e.g. Let's say we have one continuous predictor X_1 and one categorical predictor X_2 with three levels $a_1, a_2,$ and a_3 and we treat a_1 as the baseline level:

$$y_i = \beta_0 + \beta_1 x_{1,i} + \beta_2 1_{x_{2,i}=a_2} + \beta_3 1_{x_{2,i}=a_3} + \epsilon_i$$

$$= \begin{cases} \beta_0 + \beta_1 x_{1,i} + \epsilon_i & \text{if } x_{2,i} = a_1 \\ \beta_0 + \beta_1 x_{1,i} + \beta_2 + \epsilon_i & \text{if } x_{2,i} = a_2 \\ \beta_0 + \beta_1 x_{1,i} + \beta_3 + \epsilon_i & \text{if } x_{2,i} = a_3 \end{cases}$$

Categorical Predictors

Note that with dummy coding, information about the baseline level is incorporated into the intercept term β_0

II. **Effect Coding:** Intercept β_0 is interpreted as the overall mean across all levels of X_2; coefficients represent change from this overall mean

e.g. Under the same setup:

$$y_i = \beta_0 + \beta_1 x_{1,i} + \beta_2 1_{x_{2,i} \neq a_2} + \beta_3 1_{x_{2,i} \neq a_1} + \epsilon_i$$

$$= \begin{cases} \beta_0 + \beta_1 x_{1,i} + \beta_2 + \epsilon_i & \text{if } x_{2,i} = a_1 \\ \beta_0 + \beta_1 x_{1,i} + \beta_3 + \epsilon_i & \text{if } x_{2,i} = a_2 \\ \beta_0 + \beta_1 x_{1,i} + \beta_2 + \beta_3 + \epsilon_i & \text{if } x_{2,i} = a_3 \end{cases}$$

Categorical Predictors

- **Dummy Coding:** β_0 corresponds to the baseline effect of one predictor level
- **Effect Coding:** β_0 corresponds to the overall mean across all predictor levels
- Most software will do dummy coding by default; be careful with this, sometimes it is more intuitive to use one level or another as the baseline
- See ISLR Section 3.3.1 for a more thorough overview on dummy coding; see here for a good comparison of the methods

Higher Order Terms

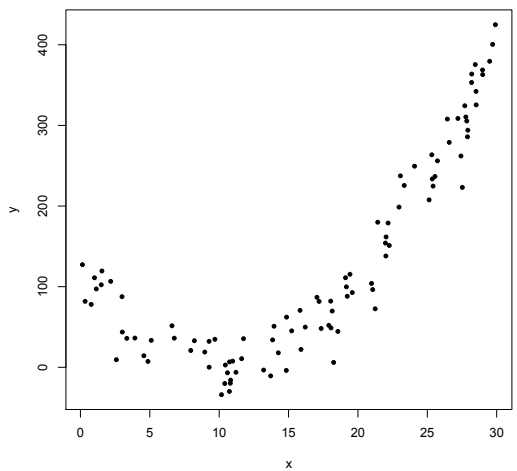
2. Terms can be functions of individual predictors
   - Usually this takes the form of higher order polynomial terms for one or more of the predictors, e.g.

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2$$
$$Y = \beta_0 + \beta_1 X_1^2 + \beta_2 X_2 + \beta_3 X_3^5$$

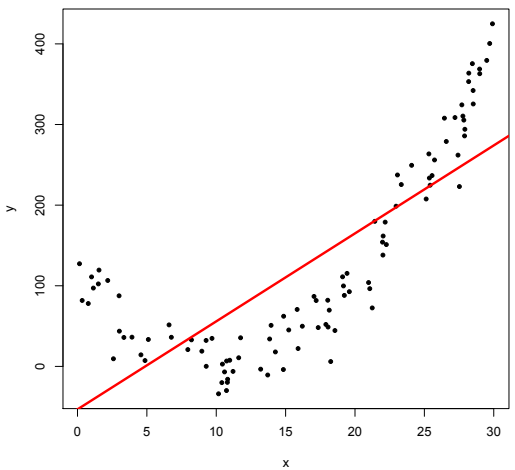
   - Remember, each of these is still linear **in the parameters** so these are still linear models
   - In many situations, relationships are clearly not (first-order) linear, but linear models with higher-order polynomial terms may provide a very good fit

Higher Order Term Example

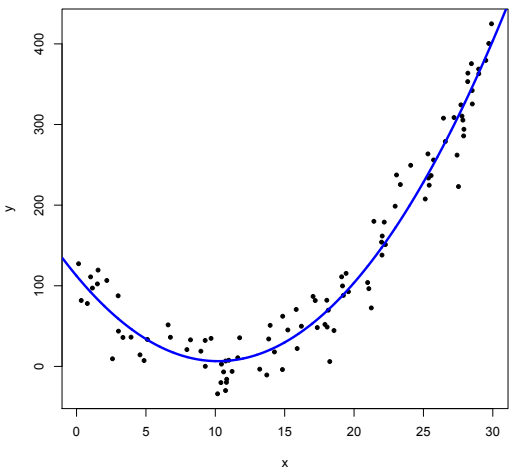
Example of (first order) linear model vs (second order linear) quadratic model



Raw Data



Model Fit: $Y = \beta_0 + \beta_1 X$



Model Fit: $Y = \beta_0 + \beta_1 X + \beta_2 X^2$

Interaction Terms

3. Terms can be functions of multiple predictors

- Usually this takes the form of products of predictors, e.g.

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2$$
$$Y = \beta_0 + \beta_1 X_1^2 + \beta_2 X_2 + \beta_3 X_3^5 + \beta_4 X_2 X_3^5$$

- Terms in a linear model that involve multiple predictors are called **interaction terms**. Usually these take the form $X_1^* \cdots X_m^*$ and if the term involves m predictors, we refer to it as an m-way interaction, e.g.

$X_1 X_2$ = 2-way interaction $\quad X_1 X_2 X_3 X_4$ = 4-way interaction

Terms that involve only 1 predictor are called **main effects**

Interaction Terms

What is implied when we include an interaction term?

- When an interaction term is included and is significant, we say that an **interaction exists** between the predictors involved in that term
- If, for example, the (coefficient corresponding to the) X_1X_2 interaction term is significant, it means the effect/impact of one predictor (say X_1) is different depending on the value of the other (X_2)
- **Note:** Whenever an interaction term is significant, we almost always want to include the individual main effects in the model as well, even if they are not significant

Interaction Terms

- When an interaction term is included, we still have a linear model but we no longer have an additive model; Additive Models are of the form:

$$Y = f(X) = f_1(X_1) + f_2(X_2) + \cdots + f_p(X_p) + \epsilon$$

- Models that include only main effects and/or higher order polynomial terms are both additive and linear

- **Note:** In practice, it is rare to include many-way interactions (more than 3-way or 4-way); There are some special circumstances, but generally we only think about 2-way and 3-way interactions

What can go wrong?

What can go wrong?

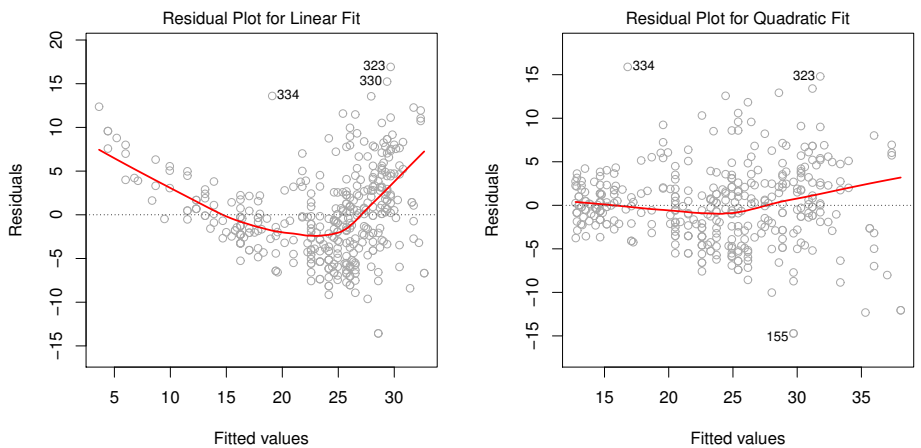
There are a multitude of issues commonly encountered when trying to fit a linear model (see ISLR Section 3.3.3 for more thorough review):

- The true underlying relationship is not linear
- Correlation of error terms
- Non-constant variance of error terms (Heteroscedasticity)
- Outliers and/or high leverage points
- Collinearity/Multicollinearity

Nonlinearity

What if the true underlying relationship just isn't linear?

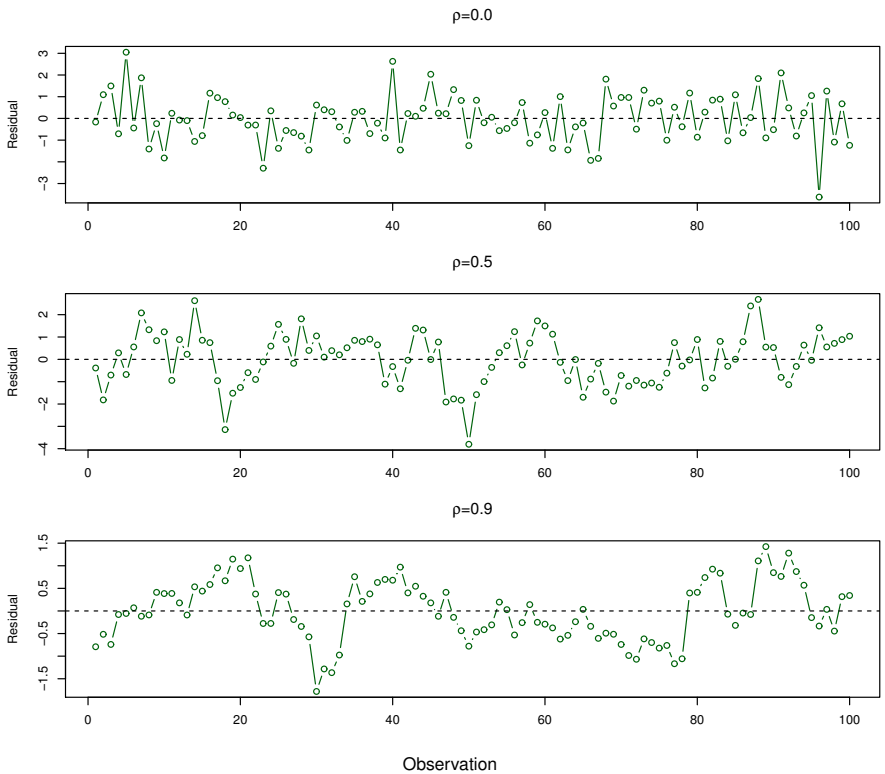
- Sometimes including higher order polynomial terms will help; sometimes not. Transforming the response in such a way that makes the transformed relationship more linear can also sometimes help.
- Plots of residuals vs fitted values can help assess non-linearity



ISLR Fig. 3.9: Example Fitted vs Residual plots. Ideally we want no discernable patterns here.

Correlated Error Terms

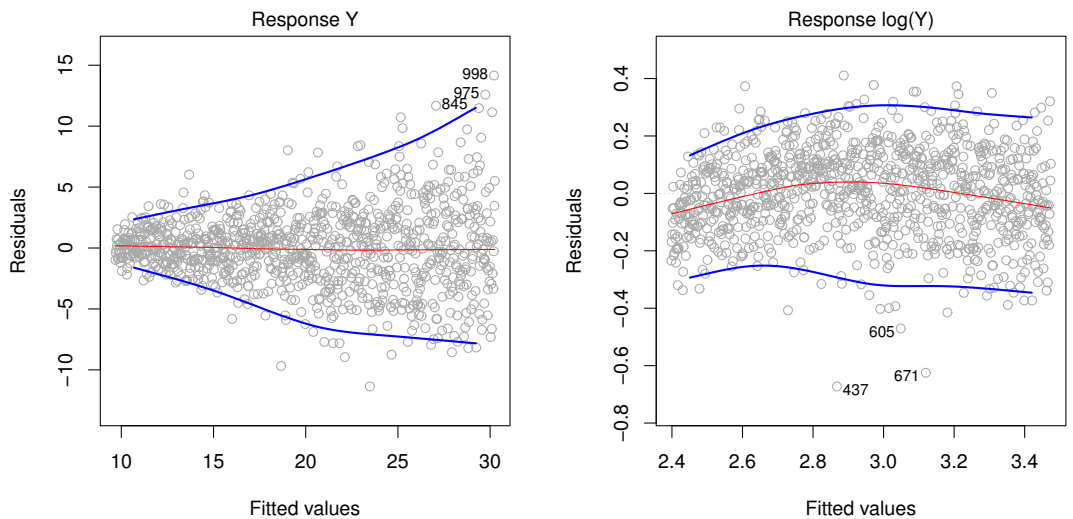
- Linear models assume that the error terms (the ϵ_i) are independent, or at least uncorrelated
- Sometimes this is not the case, usually with time series data



ISLR Fig. 3.10: Residual plots from time series data with varying degrees of correlation ρ.

Heteroscedasticity

- Linear models assume that the error terms (the ϵ_i) have constant variance: $\text{var}(\epsilon_i) = \sigma^2$; note that σ^2 does not depend on i
- Often this is not the case; it is common to see larger error variances for larger predictor values. Transforming the response to $\log(Y)$ or $\sqrt{Y}$ usually helps:



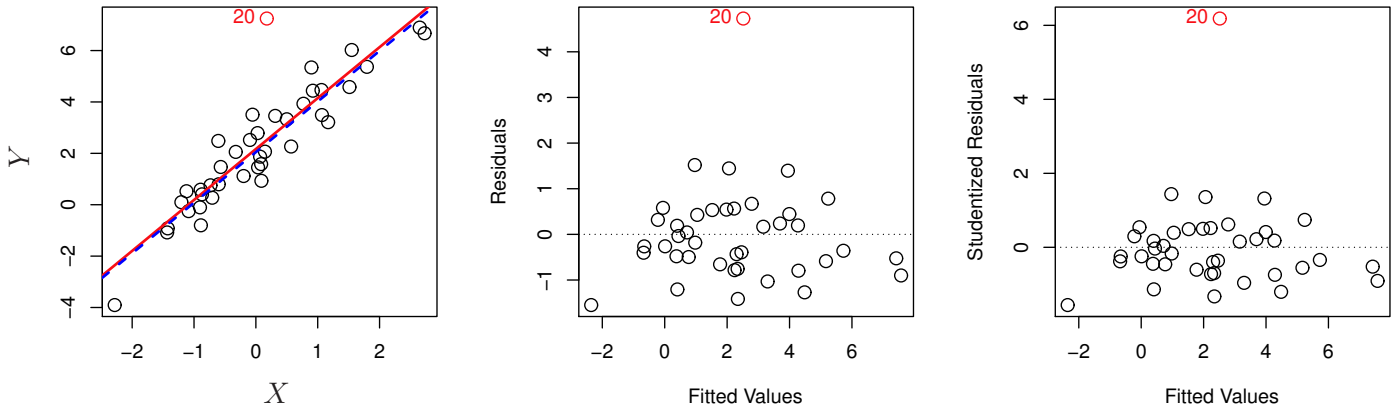
ISLR Fig. 3.11: (Left) Heteroscedastic residual plots using Y as response. (Right) Residual plot using $\log(Y)$ as the response.

Outliers and High Leverage Points

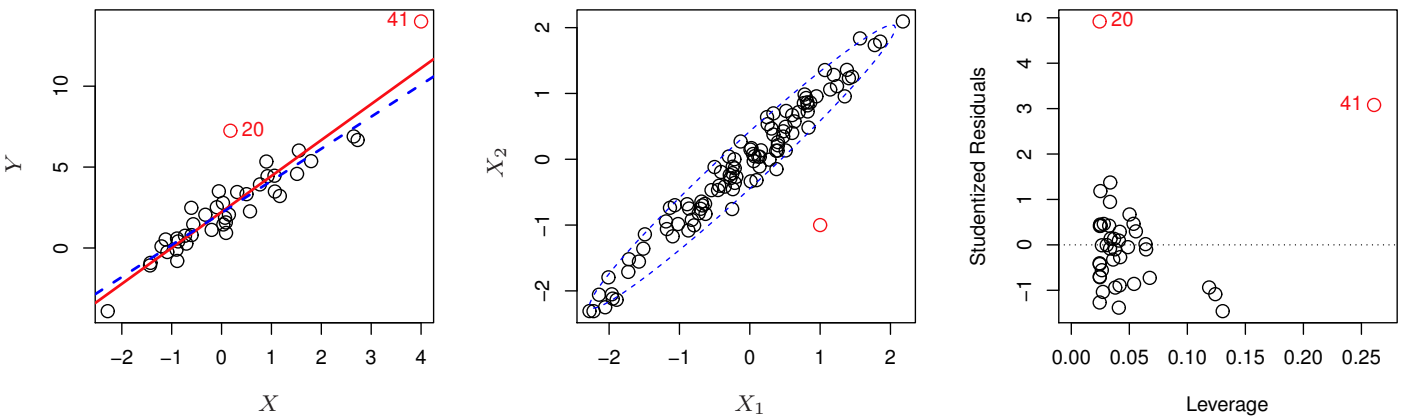
Outliers and high leverage points are often confused, but are not the same:

- **Outliers:** Points for which the true value y_i lies far from the predicted value $\hat{y}_i$
  - Response y_i is unusual given the predictor value(s) x_i
- **High Leverage Points:** Points for which the value of the predictor(s) x_i lies far from the rest of the data
  - Often have a substantial effect on model fit
  - Removing a high leverage point results in a bigger change in $\hat{f}$ than removing an outlier

Outliers and High Leverage Points



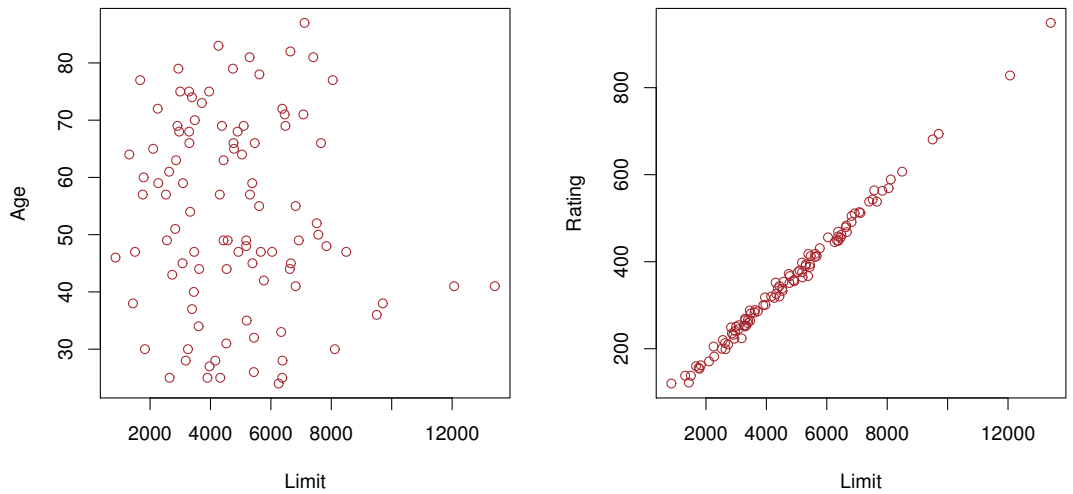
ISLR Fig. 3.12: Linear model fit and residual plots. Point 20 is an outlier.



ISLR Fig. 3.13: (Left) Linear model fit; point 20 is an outlier and point 41 is a high leverage point. (Center) Red point indicates an outlier in the case of two predictors X_1 and X_2. (Right) Outlier and high leverage point in residual plot.

Collinearity and Multicollinearity

- Very common issue; almost always present when many predictors are involved
- Collinearity: very high correlation between predictors to the point where one often explains the other. With only two variables (predictors), it's not *exactly* the same as correlation, but similar (specific meaning in the context of linear regression).



ISLR Fig. 3.14: (Left) `Limit` and `Age` are not collinear. (Right) `Limit` and `Rating` are collinear.

Collinearity

Effects of collinearity:

- Significant additional uncertainty in estimating the coefficients; significant increase in standard error of the estimates
  - If we refit the model on new data, we may see very different coefficient estimates

- Suppose X_i and X_j are highly correlated:
  - When fitting simple linear regression model with either X_i or X_j, both terms look very important
  - When fitting a multiple linear regression model with *both* terms present, often both look less important (in extreme cases, not even significant)

Multicollinearity

- Easy enough to spot if we're only dealing with two predictors
  - Can check before fitting the model by simply looking at covariance matrix
- With more than two predictors, we have to worry about **multicollinearity**
  - Collinearity that exists between three or more predictors even though pairwise correlations are small
  - Most common way to check for this is to compute the *variance inflation factor* (VIF) – see book for formula and details

Takeaway

Advantages:

- Simple and explicit model; easy to interpret
- Parameters (β's) are computationally fast and easy to fit
- Many inference procedures (hypothesis tests and confidence intervals) readily available if you're willing to make certain assumptions

Disadvantages:

- If true underlying relationship between predictors and response is not at least approximately linear, can be some difficulties in interpretation/inference
- Even though a good linear model approximation often exists, it can be *very* difficult to find the 'best' one or even a relatively good one, especially in large dimensions (many predictors)

Bias vs Variance

- Linear models, at least in low-dimensional settings ($p < n$) are generally considered high bias, low variance (not very flexible)
- With linear models, the model “complexity" is determined by the number of terms in the model: more terms $\implies$ more flexible/complex
- Adding more terms to a linear model will improve the error on the training set, but adding useless terms will increase the test error (some very minor exceptions – will discuss in the last lecture of the course)

Final Food for Thought

1. **Linear regression can be performed without the strong assumptions generally given in textbooks**

   - Very few assumptions are needed to fit actually a linear model; stronger assumptions (true underlying linear model, independent Normal errors) are needed for the kinds of inference and interpretations described in the previous slides

2. **In practice, there is rarely one “good" linear model**

   - There will almost always be many different linear models (with different covariates included) with similarly high accuracy – be very careful about reading too much into the particular model you estimated with the data at hand