GLOBAL LE CALVEZ–YOCCOZ THEOREMS FOR AREA-PRESERVING SURFACE DIFFEOMORPHISMS AND THREE-DIMENSIONAL REEB FLOWS

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Abstract. We prove that for any monotone area-preserving diffeomorphism of a closed surface, or Reeb flow on a closed three-manifold with torsion Chern class, the complement of a compact invariant set is never minimal. As a corollary, we obtain that, under the same assumptions, there are infinitely many distinct proper compact invariant sets whose union is dense in the manifold. No genericity assumptions are required. The former class of systems includes all Hamiltonian diffeomorphisms of closed surfaces. We can view our results as generalizations to higher genus surfaces and three-manifolds, in the smooth symplectic setting, of results of Le Calvez–Yoccoz, Franks, and Salazar for homeomorphisms of the two-sphere.

Along the way, we prove a result, in any dimension, of potentially independent interest, detecting invariant sets via sequences of low-action pseudoholomorphic curves with controlled topology; as a corollary, this generalizes Ginzburg–Gürel's "crossing energy bound" for Floer cylinders to punctured holomorphic curves in symplectizations, resolving an open question posed by them in 2012. Another feature of the argument which also might be of independent interest is a probabilistic result controlling the average Euler characteristic of ECH/PFH U-map curves under the above assumptions.

Contents

1. Introduction

1.1. **Le Calvez-Yoccoz phenomena.** The detection and classification of compact invariant sets is a fundamental question in dynamical systems. For diffeomorphisms of the circle and smooth flows on the plane, the theorems of Denjoy [18] and Poincaré–Bendixson [5] give a nearly complete picture. The situation in higher-dimensions, on the other hand, is much more mysterious, with a far greater diversity in the possible behaviors. This paper is about the conservative setting, in one dimension higher than

Denjoy and Poincaré–Bendixson: area-preserving diffeomorphisms of 2-manifolds and volume-preserving flows on 3-manifolds. A first observation is that without further assumptions, there might be no interesting invariant sets at all:

Example 1.1. Let f be an irrational translation of a two-torus; then, this is area-preserving, but the only compact invariant set is the entire manifold. One can similarly consider an irrational volume-preserving flow on the three-torus.

The main theme of our paper is that when one adds some further natural conditions of a symplectic nature, the situation changes completely. Our inspiration comes from the following result of Le Calvez-Yoccoz. Recall that an invariant set U of a map or flow, closed or not, is called *minimal* if the orbit of each initial condition $p \in U$ is dense in U. A groundbreaking 1997 paper by Le Calvez–Yoccoz [38], improving on an earlier result of Handel [27], showed that for any homeomorphism of S^2 the complement of an invariant finite set of points is never minimal. Their result resolved the 2-dimensional case of an old question of Ulam from the Scottish Book [39, p. 208].

Our first results give a generalization of this, in the smooth symplectic setting, to higher-genus surfaces and 3-manifolds, with a finite invariant set of points replaced by any proper compact invariant subset:

Theorem 1. *Let Σ be a closed, oriented surface and let $\phi : \Sigma \to \Sigma$ be any monotone area-preserving diffeomorphism. Then for any proper compact invariant set $\Lambda \subset \Sigma$, the complement $\Sigma \setminus \Lambda$ is not minimal.*

Theorem 2. *Let Y be a closed, oriented 3-manifold equipped with a co-oriented contact structure ξ with torsion first Chern class. Let λ be any contact form defining ξ and let $\phi = \{\phi^t\}_{t\in\mathbb{R}}$ denote the Reeb flow of λ. Then for any proper compact invariant set $\Lambda \subset Y$, the complement $Y \setminus \Lambda$ is not minimal.*

The simplest kinds of compact invariant sets are *periodic orbits*: this means that the invariant set is homeomorphic to a circle (in the case of flows) or a finite set of points with a transitive action (in the discrete setting). There is a vast literature about periodic orbits in the setting we are considering here, but much less is known about compact invariant sets beyond periodic orbits. As we will discuss more below, our results can be used to detect many new compact invariant sets which are not periodic orbits.

What is novel about these theorems in the context of symplectic dynamics is the level of generality. Previous results have established theorems like the above under strong dynamical assumptions such as the existence of only finitely many periodic orbits [25, 9]; we discuss this further in Remark 1.5. That these theorems hold much more generally, at least in low-dimensions, seems to us to be a quite new and perhaps unexpected phenomenon. We also remark that the assumptions of these theorems apply to broad classes of systems in Hamiltonian dynamics. For example, any Hamiltonian diffeomorphism of a closed symplectic surface is monotone and any rational area-preserving diffeomorphism of the 2-torus is monotone (see Appendix A). Any Reeb flow on a rational homology 3-sphere has torsion Chern class, as does any Reeb flow on a closed 3-manifold with contact structure supporting a contact Anosov flow [28, Theorem 4.1].

For more about the terminology (e.g. the definition of a Reeb flow, the definition of a monotone map), see the beginning parts of Sections 3.1 and 4.1.

Theorem 1 and Theorem 2 guarantee the existence of an abundance of compact invariant sets. Moreover, the compact invariant sets are spread out in the manifold. For example, we obtain the following corollaries.

Corollary 1.1. *Under the assumptions of Theorem 1, the map ϕ has infinitely many distinct proper compact invariant sets whose union is dense in Σ.*

Corollary 1.2. *Under the assumptions of Theorem 2, the Reeb flow has infinitely many distinct proper compact invariant sets whose union is dense in Y.*

Corollary 1.1 and 1.2 are clearly false if one requires the compact invariant sets to be periodic orbits. For example, an irrational rotation of a two-sphere satisfies the assumptions of Theorem 1, but has just two periodic points.

Remark 1.3. The generality of the above theorems strongly precludes the space of possible improvements, without the imposition of additional restrictions. For example, one could hypothetically ask whether the invariant sets we detect support interesting invariant measures. However, Anosov–Katok famously constructed [2] an area-preserving diffeomorphism of S^2 whose invariant measures are as simple as possible: the only ergodic invariant measures are a pair of fixed points and the area measure. Corollary 1.1 applies to the Anosov–Katok example to produce many distinct proper compact invariant sets, but they all must therefore support essentially the same ergodic invariant measures.

Remark 1.4. The novelty of Theorems 1 and 2 is that they make no genericity assumptions at all. Indeed, prior results on the closing lemma (see [36, 3, 17, 19]) show that a C^∞-generic system of the type we consider has a dense set of periodic points.

1.2. **The work of Franks and Salazar.** In fact, Theorem 1 and Theorem 2 follow from slightly more general (but slightly harder to state) results, which we now explain. Shortly after the work by Le Calvez–Yoccoz, Franks discovered the following refinement of their theorem. Recall that a compact invariant set Λ of a homeomorphism or flow on a compact manifold is called *locally maximal* if any sufficiently C^0-close compact invariant set must be contained in Λ. If a compact invariant set Λ is *not* locally maximal, then any neighborhood U of Λ contains a point $z \notin U \setminus \Lambda$ with orbit closure contained in U. Franks [22] showed that for any homeomorphism of S^2, the union of periodic points is either infinite or not locally maximal. A subsequent refinement in the conservative case by Salazar [43] showed that for any area-preserving homeomorphism of S^2 and any compact invariant set $\Lambda \subseteq S^2$ containing all periodic points, either $\Lambda = S^2$ or Λ is not locally maximal. We are able to generalize these results in the smooth symplectic case as well:

Theorem 3. *Let Σ be a closed, oriented surface and let $\phi : \Sigma \to \Sigma$ be any monotone area-preserving diffeomorphism. Then for any compact invariant set $\Lambda \subseteq \Sigma$ containing all periodic orbits of ϕ, either $\Lambda = \Sigma$ or Λ is not locally maximal.*

Theorem 4. *Let Y be a closed, oriented 3-manifold equipped with a co-oriented contact structure ξ with torsion first Chern class. Let λ be any contact form defining ξ and let $\{\phi^t\}_{t\in\mathbb{R}}$ denote the Reeb flow of λ. Then for any compaact invariant set $\Lambda \subseteq Y$ containing all periodic orbits of $\{\phi^t\}_{t\in\mathbb{R}}$, either $\Lambda = Y$ or Λ is not locally maximal.*

As we will explain, Theorem 3 implies Theorem 1. The analogous chain of reasoning holds starting from Theorem 4. Thus, the above two theorems represent our most general results.

Remark 1.5. Related results for Hamiltonian diffeomorphisms of $\mathbb{CP}^n$ with finitely many periodic points and dynamically convex Reeb flows on S^{2n+1} with finitely many closed orbits were respectively proved by Ginzburg–Gürel [25] and Cineli–Ginzburg–Gürel–Mazzucchelli [9]. There is no dimensional restriction in these results, and we say a bit more about this in connection to our results in §1.7.

Remark 1.6. Le Calvez–Yoccoz, Franks, and Salazar used very different methods from ours. Le Calvez–Yoccoz and Franks argue by contradiction with the key technical step that after possibly iterating the map, the Lefschetz index of the fixed point set is ≤ 0, contradicting the condition that $\chi(S^2) > 0$; Salazar's argument also uses this. Since any oriented surface of higher genus does not have positive Euler characteristic, it seems unclear how to approach Theorem 3 using their methods; the case of flows on three-manifolds via their methods is also unclear.

1.3. **Invariant sets from low-action holomorphic curves.** We now explain a general theorem that we prove, applicable in any dimension and of independent interest, for extracting invariant sets; it is used to prove all of the above dynamical results. One can view this as a global detection method via "low-action" holomorphic curves.

1.3.1. *The setup.* We work in the general setting of punctured holomorphic curves in symplectizations $\mathbb{R}\times Y$ over framed Hamiltonian manifolds. A *framed Hamiltonian structure* on a smooth, oriented manifold Y of dimension $2n+1 \geq 3$ is a pair $\eta = (\lambda, \omega)$ of a 1-form λ and a closed 2-form ω such that $\lambda \wedge \omega^n > 0$. The *Hamiltonian vector field* R_η is defined implicitly by

$$\lambda(R_\eta) \equiv 1, \qquad \omega(R_\eta, -) \equiv 0.$$

The flow of R_η preserves ω and the volume form $\lambda \wedge \omega^n$. This setup is an abstraction of many important classes of systems in symplectic and conservative dynamics, including mapping torii of symplectic diffeomorphisms, Reeb and stable Hamiltonian flows, and volume-preserving flows on three-manifolds. For example, if $\omega = d\lambda$, then λ is a contact form and R_η is its Reeb vector field.

We follow the classical setup of holomorphic curve theory in symplectizations introduced by Hofer. Fix a Riemann surface (C, j). A *J-holomorphic curve* is a proper smooth map $u : C \to \mathbb{R}\times Y$ satisfying the Cauchy–Riemann equation

$$J \circ Du = Du \circ j$$

where J is an *η-adapted* almost-complex structure on $\mathbb{R}\times Y$. This is a translation-invariant almost-complex structure restricting to a compatible almost-complex structure on the symplectic bundle $(\ker(\lambda), \omega)$ and sending $-R_\eta$ to the vector field ∂_a defined by the $\mathbb{R}$-coordinate on $\mathbb{R} \times Y$. We say u is *standard* if the domain C is homeomorphic to the complement of a finite subset of a closed Riemann surface. The geometry of a J-holomorphic curve in $\mathbb{R} \times Y$ is controlled by the *action* and *Hofer energy*[1], defined respectively as

$$\mathcal{A}(u) := \int_C u^*\omega, \qquad \mathcal{E}(u) := \sup_{s\in\mathbb{R}} \int_{C\cap u^{-1}(\{s\}\times Y)} u^*\lambda.$$

The action controls how far on average the tangent planes of C, which are J-invariant, are from the vertical plane spanned by ∂_a and R_η. Therefore, a low-action holomorphic curve should approximate the vector field R_η very well. The Hofer energy is, informally, the maximum length of the level sets of C in $\mathbb{R} \times Y$.

1.3.2. *The limit set.* A key object in our method is the "limit set" of a sequence of holomorphic curves in a symplectization, which we now introduce. For any closed, smooth, odd-dimensional manifold Y, write $\mathcal{D}(Y)$ for the space of pairs (η, J) where η is a framed Hamiltonian structure and J is an η-adapted almost-complex structure. Equip it with the topology of C^∞ convergence in both η and J. Fix a pair $(\eta, J) \in \mathcal{D}(Y)$ and a sequence $\{(\eta_k, J_k)\}_{k\geq 1}$ in $\mathcal{D}(Y)$ converging to (η, J). Fix a sequence $\{u_k : C_k \to \mathbb{R} \times Y\}_{k\geq 1}$ where u_k is J_k-holomorphic for each k.

Define the *limit set* $\mathcal{X}$ of the sequence $\{u_k\}_{k\geq 1}$ to be the collection of all non-empty closed subsets $K \subset Y$ arising as Hausdorff limits of level sets in the sequence. That is, there exists a sequence $\{s_k\}$ of real numbers such that

$$u_k(C_k) \cap \{s_k\} \times Y$$

is non-empty for each k and a subsequence converges in the Hausdorff topology to K. The limit set $\mathcal{X}$ is a subset of $\mathcal{K}(Y)$, the space of all non-empty compact subsets of Y equipped with the topology of Hausdorff convergence.

The following proposition collects some basic useful properties of the limit set.

Proposition 1.7. *Fix a closed, smooth, oriented, odd-dimensional manifold Y and a sequence $\{(\eta_k, J_k)\}_{k\geq 1}$ converging in $\mathcal{D}(Y)$ to a pair (η, J). Let $\{u_k : C_k \to Y\}_{k\geq 1}$ denote a sequence where u_k is a standard J_k-holomorphic curve for each k and let $\mathcal{X} \subset \mathcal{K}(Y)$ denote their limit set. Then $\mathcal{X}$ is closed. Moreover, there exists a subsequence of $\{u_k\}$ whose limit set is connected with respect to the Hausdorff topology.*

A much deeper theorem, which we prove, is the following:

Theorem 5. *Fix a closed, smooth, oriented, odd-dimensional manifold Y and a sequence $\{(\eta_k, J_k)\}_{k\geq 1}$ converging in $\mathcal{D}(Y)$ to a pair (η, J). Let $\{u_k : C_k \to Y\}_{k\geq 1}$ denote a sequence where u_k is a standard J_k-holomorphic curve for each k and let $\mathcal{X} \subset \mathcal{K}(Y)$ denote their limit set. Assume in addition that*

$$\lim_{k\to\infty} \mathcal{A}(u_k) = 0 \quad \textit{and} \quad \inf_k \chi(C_k) > -\infty$$

. Then every set $\Lambda \in \mathcal{X}$ is invariant under the flow of the Hamiltonian vector field R_η.

[1] This is not Hofer's original definition. Finiteness of $\mathcal{E}(C)$, however, is equivalent to finiteness of the original Hofer energy.

Remark 1.8. The Euler characteristic $\chi(C_k)$ is finite for each k because we are assuming each curve is standard. The assumption of a finite k-independent lower bound on $\chi(C_k)$ is essential for our proof of Theorem 5. Removing this assumption would be of interest: it would shorten the proofs of our main dynamical Theorems 3 and 4 and extend them to any rational area-preserving diffeomorphism of a closed surface and any Reeb flow on a closed 3-manifold, respectively.

Remark 1.9. The novelty of Theorem 5 is that it extracts invariant sets without requiring that the Hofer energies $\{\mathcal{E}(u_k)\}_{k\geq 1}$ admit a finite k-independent upper bound, or even that any of the Hofer energies $\mathcal{E}(u_k)$ are finite. Bounds on Hofer energy are a standard assumption in the vast majority of the symplectic field theory literature; the only exceptions known to us are [21, 41]. In the case where $\sup_{k\geq 1}\mathcal{E}(u_k) \leq E$ for some finite E, Theorem 5 follows from the original work of Hofer, and moreover one obtains the stronger conclusion that any $K \in \mathcal{X}$ is a finite union of periodic orbits. The observation that the limit set is connected, despite being an elementary fact, also plays a key role in our proofs of Theorems 3 and 4 below.

1.4. **The crossing energy theorem in symplectizations.** To explain our final result, which is more technical, we need to recall the "crossing energy theorem" for Hamiltonian diffeomorphisms. The crossing energy theorem is a powerful tool introduced by Ginzburg–Gürel [23]. Recall that a neighborhood U of a locally maximal compact invariant set Λ of a homeomorphism or flow is *isolating* if any compact invariant set $\Lambda' \subset U$ is a subset of Λ. The crossing energy theorem asserts that if Λ is a locally maximal invariant set of a Hamiltonian diffeomorphism (for example, a hyperbolic fixed point), U is an isolating neighborhood, and $V \subset U$ is such that $\overline{V} \subset U$, then any "Floer cylinder" crossing the shell $U \setminus V$ must have a uniform lower bound on its Floer energy. Analogues have been established for generating functions [1], gradient flow lines of the energy functional on loop space [26], and Floer cylinders in symplectic homology [9]. It is central to many results, such as Conley conjecture type results on the multiplicity of periodic points [4, 23, 24], dynamics of Hamiltonian and Reeb pseudorotations [25, 9], and the study of topological entropy via barcode invariants [8, 26].

Thus, one would like to generalize it for Reeb flows. This was first posed as a question in 2012 by Ginzburg–Gürel, but prior to our work it had not been clear how to prove it; see e.g. the discussion in [26, p. 4]. In fact, Theorem 5, which uses new tools that did not exist at the time of [23], provides this theorem as a corollary, for any framed Hamiltonian flow and for holomorphic curves with domain any closed Riemann surface with finitely many punctures removed. Here is the precise statement:

Theorem 6. *Fix a closed framed Hamiltonian manifold (Y,η) and an η-adapted almost-complex structure J. Let Λ be a locally maximal R_η-invariant set, U an isolating neighborhood and $V \subset U$ such that $\overline{V} \subset \mathrm{Int}(U)$. Fix an integer $T > 0$ and let $u : C \to \mathbb{R} \times Y$ be any standard J-holomorphic curve with $\chi(C) \geq -T$. Then there is a constant $c = c(\eta, J, \Lambda, U, V, T) > 0$ such that*

$$\mathcal{A}(u) > c > 0$$

whenever there exists $s_-, s_+ \in \mathbb{R}$ *with*

$$u(C) \cap \{s_-\} \times Y \subset V, \quad u(C) \cap \{s_+\}. \times Y \not\subset U. \tag{1}$$

In analogy with the progress for Hamiltonian diffeomorphisms summarized above, one hopes that Theorem 6 has many potential applications concerning the dynamics of Reeb flows.

Remark 1.10. Theorem 6 directly generalizes the crossing energy theorem for Hamiltonian diffeomorphisms from [23, 25]. Given a Hamiltonian diffeomorphism ϕ, the mapping torus Y_ϕ carries a natural framed Hamiltonian structure η such that R_η generates the suspension flow. There is an explicit correspondence between Floer cylinders for a choice of Hamiltonian H generating ϕ and holomorphic cylinders in $\mathbb{R} \times Y_\phi$[2] The Floer energy of a Floer cylinder is equal to the action of its corresponding holomorphic cylinder. Thus, it suffices to apply Theorem 6 for $Y = Y_\phi$ and pass through this correspondence.

The crossing energy theorem proved in the recent work [9] applies to Floer cylinders in completed Liouville domains, for 1-periodic Hamiltonians that are linear on the symplectization end. We expect that this crossing energy theorem can also be proved using some extensions of our arguments, by working in the mapping torus as above and invoking some consequences of the maximum principle (which restricts level sets of holomorphic cylinders to a compact domain) and Bourgeois–Oancea monotonicity for Floer cylinders (see [9, Equation (3.6)]); however, this is not our focus here.

1.5. **Non-rational maps.** We note that, in the case of the torus, our results are close to being sharp. Indeed, any non-rational ϕ is either i) Hamiltonian isotopic to a translation $(x, y) \mapsto (x+a, y+b)$ where $(a, b) \notin \mathbb{Q}^2$ or ii) Hamiltonian isotopic to a smooth conjugate of an affine map

$$(x, y) \mapsto (x + ny, y + b),$$

where n is a nonzero integer and $b \notin \mathbb{Q}$. The examples in case i) have no proper compact invariant sets when both a and b are irrational and the examples in case ii) never have any proper compact invariant subsets.

It would be interesting to see whether our results hold for rational maps in higher genus.

1.6. **Three-dimensional energy surfaces.** Density results such as Corollary 1.1 and Corollary 1.2 also hold for compact regular Hamiltonian hypersurfaces in some symplectic 4-manifolds, including $\mathbb{R}^4$, without any contact type condition. This will be discussed in a future work [40].

1.7. **Higher dimensions.** We close with some speculations on the extension of our results to higher dimensions. As we mentioned in Remark 1.5, higher-dimensional versions of the theorems in this paper were proved in [25, 9], but with the dynamical assumption that the systems must have finitely many periodic orbits.

[2]An explicit derivation for the 2-disk, which generalizes to arbitrary symplectic manifolds, can be found in [6, Lemma 20].

It would be very interesting to find the weakest possible dynamical assumptions for which our theorems extend to higher-dimensional Hamiltonian diffeomorphisms and Reeb flows. Our Theorem 5 on the extraction of invariant sets from low-action holomorphic curves with bounded topology works in all dimensions, and as mentioned in Remark 1.10, implies a "crossing energy bound" which is a key technical ingredient (among many) in [25, 9]. Proposition 1.7 also clearly works in any dimension. However, our existence results for low-action holomorphic curves rely on deep properties of ECH/PFH, which are invariants defined for area-preserving surface diffeomorphisms and three-dimensional Reeb flows, respectively.

Finding low-action holomorphic curves with bounded topology in higher dimensions seems like it will require substantial new ideas. This is consistent with a more general theme in a range of problems in current symplectic research — ranging from the kind of questions considered in this paper to problems about the algebraic structure of certain homeomorphism groups to questions like symplectic packing stability [12, 11] — where one would like analogues of various properties related to ECH/PFH in higher dimensions.

On a more optimistic note, the fact that Theorem 5 works for *non-cylindrical* curves is an asset. It opens up for the first time the possibility of using powerful theories such as contact homology or SFT, which count non-cylindrical curves, to explore invariant sets of higher-dimensional Reeb flows. Indeed, as we have seen here, one needs to consider non-cylindrical curves in our arguments to get our results.

1.8. **Outline of article.** Section 2 proves Theorems 3, 4, and 6. The proofs of Theorems 3 and 4 require Theorem 5 and two propositions (Propositions 2.2 and 2.3). These respectively assert that monotone area-preserving maps and Reeb flows of torsion contact forms have many low-action holomorphic curves with controlled topology. Section 3 proves Proposition 2.3 using embedded contact homology. Section 4 proves Proposition 2.2 using periodic Floer homology. Section 5 proves Theorem 5.

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2. Proofs of main dynamical results

This section is primarily concerned with the proofs of Theorems 3 and 4. The proofs we give here rely on Theorem 5 and two propositions (Propositions 2.2 and 2.3), whose proofs are all deferred to subsequent sections. After these results are proved, we explain how they imply the other dynamical theorems stated in the introduction. We conclude the section with a proof of Theorem 6, which is a straightforward corollary of Theorem 5.

Before proceeding, let us explain the basic ideas behind the proofs. Proposition 2.3 asserts that any nondegenerate torsion contact form on a closed 3-manifold Y admits sequences of holomorphic curves in $\mathbb{R} \times Y$ with uniformly bounded topology, finite Hofer energy, and arbitrarily low action. Moreover, the curves in the sequence can be taken to pass through any point $(0, z)$ in $\mathbb{R} \times Y$ that is not on any closed Reeb orbit. The low action and bounded topology produce, via Theorem 5 and Proposition 1.7, a connected family $\mathcal{X}$ of compact invariant sets. The finite Hofer energy and point constraints are then exploited, via an elementary topological argument, to conclude Theorem 4. The theorem is proved for degenerate contact forms by approximating them by nondegenerate contact forms and using the resulting holomorphic curves; to make this work, we require a certain amount of quantitative control over the relevant curves, which is why Proposition 2.3 is quantitative in nature. The path from Proposition 2.2 to Theorem 3 is completely analogous.

Remark 2.1. Our style of argument is robust enough to generalize to other settings where low action holomorphic curves with bounded topology are present. For example, it follows from [25] that the mapping torus of any Hamiltonian pseudorotation of $\mathbb{CP}^n$ admits holomorphic cylinders of arbitrarily low action, with finite Hofer energy, passing through any point in the symplectization. Applying our argument proves the analogue of Theorems 3 and 4 for these maps.

2.1. Existence of low-action holomorphic curves with bounded topology.

The following two key propositions show that mapping torii of monotone area-preserving surface diffeomorphisms and Reeb flows of torsion 3-dimensional contact forms have, after possibly making small perturbations, many low-action holomorphic curves with bounded topology. We start with the statement for area-preserving surface diffeomorphisms. Relevant notations and definitions are found in Section 4.

Proposition 2.2. *Let ϕ be a monotone area-preserving diffeomorphism of a closed symplectic surface (Σ, ω). Let Y_ϕ denote the mapping torus of ϕ. There exists a positive integer $d_0 \geq 1$, depending only on the Hamiltonian isotopy class of ϕ, such that the following holds for all $d \geq d_0$ and any nondegenerate Hamiltonian perturbation ϕ' of ϕ: For any fixed $z' \in Y'_\phi$, not on any closed Reeb orbit, and generic choice of ϕ'-adapted J', there exists a standard J'-holomorphic curve $u : C \to \mathbb{R} \times Y_{\phi'}$ such that:*

(a) $(0, z') \in u(C)$;
(b) $\mathcal{E}(u) \leq d$;
(c) $\mathcal{A}(u) \leq d^{-1/2}$;
(d) $\chi(C) \geq -2$.

Next, the statement for Reeb flows. Relevant notations and definitions are found in Section 3.

Proposition 2.3. *Let λ be a torsion contact form on a closed 3-manifold Y. Then there exists a positive integer $k_0 \geq 0$ such that the following holds for any $k \geq k_0$ and any C^∞-small nondegenerate perturbation λ' of λ: For any fixed $z' \in Y$, not on any closed Reeb orbit, and generic choice of λ'-adapted J', there exists a standard J'-holomorphic curve $u : C \to \mathbb{R} \times Y$ such that:*

(a) $(0, z') \in u(C)$*;*
(b) $\mathcal{E}(u) \leq k^{3/4}$*;*
(c) $\mathcal{A}(u) \leq k^{-1/16}$*;*
(d) $\chi(C) \geq -2$.

Remark 2.4. In fact, we will see in the proofs of Proposition 2.2 and Proposition 2.3 that not only does there exist curves with the above properties, but that the ECH/PFH curves satisfy these properties under the assumptions of the proposition with probability 1. More precisely, as we will explain later, for k (resp. d) as in the statement of the propositions, the non-triviality of the "U"-map implies the existence of approximately k (resp. d) curves, and we can look at the proportion that satisfy the conclusions of the propositions. Our arguments imply that this number limits to 1.

2.2. **The Hausdorff topology.** Let Y be a compact smooth manifold. Recall that $\mathcal{K}(Y)$ denotes the space of all non-empty compact sets equipped with the topology of Hausdorff convergence. We collect some basic facts about the Hausdorff topology here.

2.2.1. *Hausdorff convergence.* A sequence $\{\Lambda_k\}_{k\geq 1}$ *converges* to some $\Lambda \in \mathcal{K}(Y)$ if:

(a) Every neighborhood of every $z \in \Lambda$ meets all but finitely many Λ_k;
(b) If every neighborhood of some $z \in Y$ meets infinitely many Λ_k then $z \in \Lambda$.

Note that $\mathcal{K}(Y)$ is sequentially compact with respect to the Hausdorff topology. For an arbitrary sequence $\{\Lambda_k\}_{k\geq 1}$, define

$$\limsup_{k\to\infty} \Lambda_k \in \mathcal{K}(Y)$$

as the set of all points satisfying (a) above. Note that if $\{\Lambda_k\}_{k\geq 1}$ converges to Λ, then $\Lambda = \limsup_{k\to\infty} \Lambda_k$.

2.2.2. *Invariant sets of flows.* Let R denote a vector field on Y. Let $\mathcal{K}(Y, R) \subseteq \mathcal{K}(Y)$ denote the subspace of all compact subsets invariant under the flow of R. This subspace is closed and sequentially compact, but it may not be connected. The following lemma states an important property of locally maximal $\Lambda \in \mathcal{K}(Y, R)$.

Lemma 2.5 ([10])**.** *If an element $\Lambda \in \mathcal{K}(Y, R)$ is locally maximal, then it is maximal with respect to inclusion in any connected subspace $\mathcal{X} \subseteq \mathcal{K}(Y, R)$ containing Λ.*

2.3. **Proofs of Theorem 3 and Theorem 4.** In this section, we prove Theorems 3 and Theorem 4. The proofs are virtually identical, so for brevity we will only give the proof of Theorem 3.

We prove Theorem 3 using Theorem 5 and Proposition 2.2. Choose any monotone area-preserving diffeomorphism ϕ of a closed symplectic surface (Σ, ω). Let Y_ϕ denote its mapping torus and $\eta = (dt, \omega_\phi)$ denote its associated framed Hamiltonian structure. We observe that Theorem 3 follows from proving its analogue in the mapping torus:

Proposition 2.6. *For any closed R_η-invariant set $\Lambda \subseteq Y_\phi$ containing all periodic orbits of R_η, either $\Lambda = Y_\phi$ or Λ is not locally maximal.*

Proof of Proposition 2.6. Let $\Lambda \subset Y_\phi$ be any compact invariant set. We assume without loss of generality that it is proper, since if $\Lambda = Y$ we are already done. Our argument will go via approximation to the nondegenerate case, so choose a sequence $\{H_k\}_{k\geq 1}$ of smooth functions $H_k : \mathbb{R}/\mathbb{Z} \times \Sigma \to \mathbb{R}$ converging in C^∞ to 0 such that for each k, the map $\phi_k := \phi \circ \psi^1_{H_k}$ is nondegenerate. For each k, the pair $\eta_k := (dt, \omega_\phi + dH_k \wedge dt)$ is a framed Hamiltonian structure, and the map

$$(t, p) \mapsto (t, (\psi^t_{H_k})^{-1}(p))$$

descends to an isomorphism

$$(Y_\phi, \eta_k) \to (Y_{\phi_k}, (dt, \omega_{\phi_k}))$$

of framed Hamiltonian manifolds. Since H_k converges to 0 in the C^∞ topology as $k \to \infty$, we have that η_k converges in C^∞ to η as $k \to \infty$. For each k we choose an η_k-adapted J_k such that the sequence $\{(\eta_k, J_k)\}_{k\geq 1}$ converges to a pair (η, J) in $\mathcal{D}(Y)$.

Now fix any point $z \in Y_\phi$ not in Λ. Then there is a sequence of points $\{z_k\}_{k\geq 1}$ converging to z, such that z_k is not on any closed Reeb orbit for ϕ_k; this follows from the fact that the ϕ_k are nondegenerate, hence the union of their closed Reeb orbits has measure zero. By Proposition 2.2, after possibly making an arbitrarily small perturbation to each J_k, for each sufficiently large d and each sufficiently large k there exists a J_k-holomorphic curve

$$u_{d,k} : C_{d,k} \to \mathbb{R} \times Y_\phi$$

such that:

(i) $(0, z_k) \in u_{d,k}(C_{d,k})$;
(ii) $\mathcal{E}(u_{d,k}) \leq d$;
(iii) $\mathcal{A}(u_{d,k}) \leq d^{-1/2}$;
(iv) $\chi(C_{d,k}) \geq -2$.

Write $\mathcal{P} \subseteq Y_\phi$ for the union of closed orbits of R_η and write $\overline{\mathcal{P}}$ for its closure. For each d and k, write $\mathcal{P}_d(k)$ for the union of closed orbits of R_{η_k} of period at most d. By the Hofer energy bound in (ii), we have that

$$\limsup_{s\to\infty} u_{d,k}(C_{d,k}) \cap \{s\} \times Y_\phi \subseteq \mathcal{P}_d(k),$$

that is the level sets concentrate around closed orbits of period $\leq d$ as $s \to \infty$. We also note that since $R_{\eta_k} \to R_\eta$, we have for each fixed d that

$$\limsup_{k\to\infty} \mathcal{P}_d(k) \subseteq \mathcal{P},$$

that is periodic orbits of R_{η_k} with bounded period converge to periodic orbits of R_η. It follows that for each d, we can choose some large $k_d \gg 1$ and some $s_d \in \mathbb{R}$ such that

$$\limsup_{d\to\infty} u_{d,k_d}(C_{d,k_d}) \cap \{s_d\} \times Y_\phi \subseteq \overline{\mathcal{P}}. \tag{2}$$

Write $\mathcal{X}$ for the limit set of the sequence $\{u_{d,k_d}\}_{d\geq 1}$; by passing to a subsequence, we can assume by Theorem 5 that this is connected. By (iii), (iv), and Theorem 5, we have that $\mathcal{X} \subseteq \mathcal{K}(Y_\phi, R_\eta)$. By (2), some $\Lambda' \in \mathcal{X}$ is contained entirely in $\overline{\mathcal{P}}$, and is therefore contained in Λ. By (i), some $\Lambda'' \in \mathcal{X}$ contains the point z. Now let $\mathcal{Y} \subset \mathcal{K}(Y_\phi, R_\eta)$ denote the collection of closed invariant sets equal to a union $K \cup \Lambda$ for some $K \in \mathcal{X}$. Then $\mathcal{Y}$ is a connected subset of the space $\mathcal{K}(Y_\phi, R_\eta)$. Moreover, $\Lambda \in \mathcal{Y}$, since $\Lambda = \Lambda' \cup \Lambda$, and $\Lambda'' \cup \Lambda \in \mathcal{Y}$ by definition. However, $\Lambda'' \cup \Lambda$ is not a subset of Λ, since it contains z. Hence, by Lemma 2.5, Λ is not locally maximal.

□

2.4. **Proofs of other dynamical results.**

2.4.1. *Proofs of Theorems 1 and 2.* We prove Theorem 1 using Theorem 3. Theorem 2, the version for three-dimensional Reeb flows, follows from the same formal argument using Theorem 4, and so its proof will be skipped for brevity.

Proof of Theorem 1. Let $\phi : \Sigma \to \Sigma$ be a monotone area-preserving diffeomorphism of a closed surface and let $\Lambda \subset \Sigma$ be a proper closed invariant set. Set $U := \Sigma \setminus \Lambda$. Let $\mathcal{P} \subseteq \Sigma$ denote the union of all periodic orbits. If $\mathcal{P} \not\subseteq \Lambda$, then there exists a periodic orbit contained in U. Therefore, U is not minimal. If $\mathcal{P} \subseteq \Lambda$, then by Theorem 3 there exists some proper closed invariant subset Λ' arbitrarily C^0-close to Λ but not equal to Λ. By taking Λ' sufficiently close to Λ, we can ensure that $\Lambda' \cap U$ is not equal to U. Then any point $z \in \Lambda' \cap U$ will not have dense orbit in U, so U is not minimal. □

2.4.2. *Proofs of Corollaries 1.1 and 1.2.* We prove Corollary 1.1; the same formal argument proves Corollary 1.2.

Proof of Corollary 1.1. Let $\phi : \Sigma \to \Sigma$ be a diffeomorphism satisfying the assumptions of Theorem 1. We construct a sequence $\{\Lambda_k\}_{k\geq 1}$ of distinct proper compact invariant subsets such that i) the union $\cup_{j=1}^{k}\Lambda_j$ is not dense in Σ for any finite k and ii) the full union $\cup_{k=1}^{\infty}\Lambda_k$ is dense in Σ. We define such a sequence inductively as follows. The base case $k = 1$ is straightforward: ϕ has at least one periodic orbit. A rather non-elementary way to prove this would be to use Proposition 4.1 below. Now, we assume that $\Lambda_1, \ldots, \Lambda_{k-1}$ are distinct proper compact invariant sets, with $k \geq 2$, whose union is not dense in Σ. Apply Theorem 1 with $\Lambda = \cup_{j=1}^{k-1}\Lambda_j$ to construct Λ_k. □

2.5. **Proof of crossing energy theorem.**

We prove Theorem 6 using Theorem 5, Proposition 1.7 and Lemma 2.5.

Proof of Theorem 6. Assume for the sake of contradiction that the corollary is false. Then there exists a sequence of J-holomorphic curves $\{u_k : C_k \to \mathbb{R} \times Y\}$ satisfying (1), $\chi(C_k) \geq -T$ and $\mathcal{A}(u_k) \leq 1/k$. By Theorem 5, every element of the limit set $\mathcal{X}$ is a closed invariant set. By Proposition 1.7, after replacing $\{u_k\}$ with a subsequence if necessary, the limit set is closed and connected.

Let $\mathcal{Y} \subseteq \mathcal{K}(Y)$ denote the collection of closed invariant sets which are a union $K \cup \Lambda$ for some $K \in \mathcal{X}$. The first item in (1) implies $\mathcal{X}$ contains an invariant set Λ' contained in U, and since U is an isolating neighborhood of Λ, this implies $\Lambda' \subseteq \Lambda$. It follows that $\Lambda \in \mathcal{Y}$. Since $\mathcal{X}$ is connected, $\mathcal{Y}$ is connected as well, so it follows from Lemma 2.5 that any $\Lambda'' \in \mathcal{Y}$ must be a subset of Λ. This implies that every element of $\mathcal{X}$ is a subset of Λ. This in turn implies that for sufficiently large k, every level set of $u_k(C_k)$ lies inside $\mathbb{R} \times \overline{V}$. We now arrive at a contradiction thanks to the second condition of (1). $\square$

3. Low-action curves from embedded contact homology

This section proves Proposition 2.3. Previously statements like this have been proved under the assumption of two periodic Reeb orbits [35], or later under the assumption of finitely many periodic Reeb orbits [14]; this section shows that this phenomenon in fact holds much more generally.

3.1. **Embedded contact homology.** We review the basic features of embedded contact homology [29, 31] here. Fix a closed, smooth, connected, oriented three-manifold Y and a contact structure ξ.

3.1.1. *Reeb flow basics.* Fix any contact form λ defining ξ, i.e. satisfying the identity $\ker(\lambda) = \xi$. Recall that the *Reeb vector field* R is the unique vector field solving the equations

$$\lambda(R) \equiv 1, \quad d\lambda(R, -) \equiv 0.$$

A *closed Reeb orbit* is a smooth map $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ for some $T > 0$ such that $\dot{\gamma}(t) = R(\gamma(t))$ for all t; as is standard we will make no distinction between two closed Reeb orbits that agree up to a reparameterization of the domain. A closed Reeb orbit is *simple* if γ is injective. For any closed Reeb orbit γ, we write $\gamma^k : \mathbb{R}/kT\mathbb{Z} \to Y$ for its k-th iteration. The number T is the *action* of γ, denoted by

$$\mathcal{A}(\gamma) := \int_\gamma \lambda = T.$$

The time T linearized flow of R determines a symplectic isomorphism $\xi_{\gamma(0)} \to \xi_{\gamma(0)}$ called the *Poincaré return map*. The orbit γ is *nondegenerate* if the return map does not have an eigenvalue equal to 1. A nondegenerate orbit γ is *hyperbolic* if P_γ has real eigenvalues and *elliptic* if P_γ has complex eigenvalues of unit length. We say λ is *nondegenerate* if all closed Reeb orbits are nondegenerate. For any fixed contact structure ξ, a generic defining contact form λ is nondegenerate.

3.1.2. *ECH generators.* A *Reeb orbit set* is a (possibly empty) finite set $\alpha = \{(\alpha_i, m_i)\}$ of pairs (α_i, m_i), where α_i is a simple closed Reeb orbit and $m_i \in \mathbb{N}$ is a positive integer multiplicity. An *ECH generator* is a Reeb orbit set $\alpha = \{(\alpha_i, m_i)\}$ such that i) each of the α_i are pairwise distinct and ii) $m_i = 1$ if α_i is hyperbolic. Denote by $\mathrm{ECC}(Y, \lambda)$ the $\mathbb{Z}/2$-vector space generated by the set of ECH generators. Any Reeb orbit set α has a homology class $[\alpha] := \sum_i m_i[\alpha_i] \in H_1(Y; \mathbb{Z})$. For each $\Gamma \in H_1(Y; \mathbb{Z})$, let $\mathrm{ECC}(Y, \lambda, \Gamma)$ denote the sub-module generated by ECH generators homologous to Γ.

3.1.3. *ECH differential.* Assume that λ is nondegenerate. Choose a generic λ-adapted almost-complex structure J on $\mathbb{R} \times Y$. The ECH differential

$$\partial_J : \mathrm{ECC}(Y, \lambda) \to \mathrm{ECC}(Y, \lambda)$$

is defined by counting certain "J-holomorphic currents" which we now define. We say that a J-holomorphic curve $u : C \to \mathbb{R} \times Y$ is *somewhere injective* if there exists $\zeta \in C$ such that $u^{-1}(u(\zeta)) = \{\zeta\}$ and Du is injective at ζ. A *J-holomorphic current* is a finite set $\mathcal{C} = \{(C_k, d_k)\}$ of pairs where the C_k denote distinct standard, somewhere injective J-holomorphic curves with finite Hofer energy, and the d_k are positive integer multiplicities. We say that $\mathcal{C}$ is *somewhere injective* if $d_k = 1$ for each k and *embedded* if the C_k are pairwise disjoint and embedded. For any J-holomorphic current $\mathcal{C} = \{(C_k, d_k)\}$, the slices

$$\mathcal{C} \cap \{s\} \times Y = \{(C_k \cap \{s\} \times Y, d_k)\}$$

form for $|s| \gg 1$ a weighted collection of embedded loops in Y. The slices converge as 1-dimensional currents to Reeb orbit sets α and β as $s \to \infty$ and $s \to -\infty$, respectively. For any pair of Reeb orbit sets α and β with $[\alpha] = [\beta]$, we let $\mathcal{M}(\alpha, \beta)$ denote the moduli space of J-holomorphic curves with positive asymptotic limit at α and negative asymptotic limit at β.

Any $\mathcal{C} \in \mathcal{M}(\alpha, \beta)$ has an associated *ECH index* $I(\mathcal{C}) \in \mathbb{Z}$, defined below, and for each $k \in \mathbb{Z}$ we let $\mathcal{M}_k(\alpha, \beta)$ denote the subspace of curves of ECH index k. When J is sufficiently generic, the space $\mathcal{M}_1(\alpha, \beta)$ is a smooth 1-dimensional manifold. Moreover, it has a free $\mathbb{R}$-action given by the translation action on $\mathbb{R} \times Y$, and the quotient $\mathcal{M}_1(\alpha, \beta)/\mathbb{R}$ is a finite set of points. The matrix coefficient of the ECH differential with respect to a pair of ECH generators α and β is defined by the identity

$$\langle \partial_J \alpha, \beta \rangle := \#_2\, \mathcal{M}(\alpha, \beta)/\mathbb{R}$$

where $\#_2$ denotes the modulo 2 count of points. By [33, 34], $\partial_J^2 = 0$, and therefore $(\mathrm{ECC}(Y, \lambda), \partial_J)$ is a chain complex. The *embedded contact homology* $\mathrm{ECH}(Y, \xi)$ is its homology group. A consequence of Taubes' isomorphism of ECH with monopole Floer homology [45] is that $\mathrm{ECH}(Y, \xi)$ does not depend on the choice of contact form λ defining ξ or the choice of J used to define the ECH differential. The ECH differential preserves the homology class $[\alpha] \in H_1(Y; \mathbb{Z})$ of an orbit set α; for any $\Gamma \in H_1(Y; \mathbb{Z})$ we write $\mathrm{ECH}(Y, \xi, \Gamma)$ for the homology of $(\mathrm{ECC}(Y, \lambda, \Gamma), \partial_J)$.

3.1.4. *The U-map.* For a generic choice of λ-adapted almost-complex structure J and any point $z \in Y$ not on any closed Reeb orbit, there exists a chain map

$$U_{J,z} : \mathrm{ECC}(Y, \lambda, \Gamma) \to \mathrm{ECC}(Y, \lambda, \Gamma)$$

defined as follows. Write $\mathcal{M}_2(\alpha, \beta; z)$ for the space of all J-holomorphic currents with ECH index 2 whose support contains $(0, z) \in \mathbb{R} \times Y$. For a generic choice of J and any z not on any closed Reeb orbit, the space $\mathcal{M}_2(\alpha, \beta; z)$ is a finite set of points. The matrix coefficient of $U_{J,z}$ with respect to α and β is defined by the identity

$$\langle U_{J,z} \alpha, \beta \rangle = \#_2\, \mathcal{M}(\alpha, \beta; z).$$

The map $U_{J,z}$ descends to a map on homology that we call the *U-map*:

$$U : \mathrm{ECH}(Y, \xi, \Gamma) \to \mathrm{ECH}(Y, \xi, \Gamma).$$

The map $U_{J,z}$ may vary with different choices of J and z. However, the chain homotopy class of $U_{J,z}$ does not depend on the choice of J and z, so the induced map on homology does not depend on the choice of J and z.

3.1.5. *U-towers and the volume property.* Fix any $\Gamma \in H_1(Y;\mathbb{Z})$. A *$U$-tower* is a sequence of nonzero classes

$$\{\sigma_k\}_{k\geq 0} \subset \mathrm{ECH}(Y,\xi,\Gamma)$$

such that i) $U\sigma_k = \sigma_{k-1}$ for each $k > 0$ and ii) $U\sigma_0 = 0$. Taubes' isomorphism [45] and a computation by Kronheimer–Mrowka [37, Chapter 35] prove that $\mathrm{ECH}(Y,\xi,\Gamma)$ contains a U-tower whenever the class $c_1(\xi) + 2\,\mathrm{PD}(\Gamma) \in H^2(Y;\mathbb{Z})$ is torsion. Here $c_1(\xi)$ denotes the first Chern class of ξ with respect to any complex structure which rotates positively with respect to $d\lambda$.

For any nonzero class $\sigma \in \mathrm{ECH}(Y,\xi)$, define its *spectral invariant* $c_\sigma(\lambda) \in \mathbb{R}$ to be the infimum of all L such that σ is represented by a cycle in $\mathrm{ECC}(Y,\lambda)$ with all constituent generators having action $\leq L$. Here the action of an ECH generator $\alpha = \{(\alpha_i, m_i)\}$ is defined to be

$$\mathcal{A}(\alpha) := \sum_i m_i\,\mathcal{A}(\alpha).$$

A quantitative version of the proof that ECH is independent of the choice of J used to define the ECH differential shows that the spectral invariants do not depend on J either. However, the spectral invariants can and usually do vary with λ. Each spectral invariant $c_\sigma(\lambda)$ is C^0-continuous with respect to λ; this allows us to extend the definition of $c_\sigma(\lambda)$ to degenerate λ. The following lemma records some relevant chain-level information that we can extract from a U-tower.

Lemma 3.1. *Assume that there exists a U-tower $\{\sigma_k\}_{k\geq 0} \subset \mathrm{ECH}(Y,\xi,\Gamma)$ for some $\Gamma \in H_1(Y;\mathbb{Z})$. Assume that λ is nondegenerate and choose generic J, and $(0,z)$ not on any closed Reeb orbit, so that the ECH differential ∂_J and the chain map $U_{J,z}$ are well-defined. Then for each $\epsilon > 0$ and each $k \geq 1$, there exists an ECH generator α_k such that*

(a) $\mathcal{A}(\alpha_k) \leq c_{\sigma_k}(\lambda)$;
(b) $U^k_{J,z}(\alpha_k) \neq 0$.

Proof. For any fixed $\epsilon > 0$ and k, there exists a cycle $x \in \mathrm{ECC}(Y,\lambda,\Gamma)$ representing σ_k which splits as a sum of ECH generators $x = \sum_{i=1}^N x_i$ with action less than $c_{\sigma_k}(\lambda) + \epsilon$. Since there are only finitely many Reeb orbit sets of action $\leq c_{\sigma_k}(\lambda) + 1$, as λ is nondegenerate, we can therefore find a cycle x as above with action exactly $c_{\sigma_k}(\lambda)$. Since $U^k_{j,z}(x) \neq 0$, it follows that $U^k_{J,z}(x_i) \neq 0$ for some i. Set $\alpha_k := x_i$. □

One of the most powerful properties of the ECH spectral invariants is the *volume property* proved by Cristofaro-Gardiner–Hutchings–Ramos [15]. Their result, stated in the following proposition, shows that that the spectral invariants of a U-tower asymptotically recover the contact volume.

Theorem 7 (ECH volume property, [15])**.** *Assume that there exists a U-tower $\{\sigma_k\}_{k\geq 0} \subset$ $\mathrm{ECH}(Y,\xi,\Gamma)$ for some $\Gamma \in H_1(Y;\mathbb{Z})$. Then for any contact form λ we have*

$$\lim_{k\to\infty} c_{\sigma_k}(\lambda)^2/2k = \int_Y \lambda \wedge d\lambda. \tag{3}$$

3.1.6. *The ECH index.* We now give the previously deferred definition of the ECH index. Fix a nondegenerate contact form λ and a pair of ECH generators $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$. Let $H_2(Y,\alpha,\beta)$ denote the space of equivalence classes of integral 2-chains with boundary $\alpha - \beta$, where two such chains are equivalent if and only if they differ by a 2-boundary. The *ECH index* of a class $Z \in H_2(Y,\alpha,\beta)$ is an integer defined by the formula

$$I(Z) := c_\tau(Z) + Q_\tau(Z) + \sum_i \sum_{k=1}^{m_i} \mathrm{CZ}_\tau(\alpha_i^k) - \sum_j \sum_{l=1}^{n_j} \mathrm{CZ}_\tau(\beta_j^l). \tag{4}$$

Definitions of these terms can be found in [31]. We will narrow our discussion of the ECH index to exactly those terms which are useful for the proof of Proposition 2.3, namely the relative Chern class and the Conley-Zehnder index. The *relative Chern class* $c_\tau(Z)$ is defined as follows. We choose an oriented smooth surface $S \subset Y$ representing Z and choose a section $\psi : S \to \xi$, transverse to the zero section, such that ψ is a nonzero constant on each component of ∂S and we set

$$c_\tau(Z) := \#\psi^{-1}(0)$$

where $\#$ denotes the oriented count of points. As for the Conley-Zehnder index, we define it by

$$\mathrm{CZ}_\tau(\gamma) = \lceil \theta_\tau(\gamma) \rceil + \lfloor \theta_\tau(\gamma) \lfloor$$

where $\theta_\tau(\gamma)$ denotes the "monodromy number" of the linearized flow along γ in the trivialization τ. We refer the reader to [31] for a definition of the monodromy number. We define the ECH index of a curve to be the ECH index of its homology class, emphasizing that this is independent of the choice of τ.

3.1.7. *Topological complexity of U-map curves.* A variant of the ECH index called the J_0 *index* plays a key role in the proof of Propositions 2.2 and 2.3. In the notation of (4) we write

$$\begin{aligned} J_0(Z) &:= -c_\tau(Z) + Q_\tau(Z) + \sum_i \sum_{k=1}^{m_i - 1} \mathrm{CZ}_\tau(\alpha_i^k) - \sum_j \sum_{l=1}^{n_j - 1} \mathrm{CZ}_\tau(\beta_j^l) \\ &= I(Z) - 2c_\tau(Z) - (\sum_i \mathrm{CZ}_\tau(\alpha_i^{m_i}) - \sum_j \mathrm{CZ}_\tau(\beta_j^{n_j})). \end{aligned} \tag{5}$$

The J_0 index controls the topological complexity of a holomorphic current counted by the U-map. To state the bound precisely, we need to recall the following structural property of U-map currents, a proof of which is found in [31, Proposition 3.7]. Any current $\mathcal{C} \in \mathcal{M}_2(\alpha,\beta;z)$ counted by the U-map splits as a disjoint union $\mathcal{C}_0 \sqcup C_2$ where $\mathcal{C}_0$ is a union of trivial cylinders with multiplicities and C_2 is an embedded J-holomorphic curve with $I(C_2) = 2$.

Proposition 3.2 ([30, Section 6]). *Fix a generic J and point z so that the chain map $U_{J,z}$ is defined, and let $\mathcal{C} = \mathcal{C}_0 \sqcup C_2 \in \mathcal{M}_2(\alpha, \beta; z)$ be a J-holomorphic current counted by the U-map. Then*

$$J_0(\mathcal{C}) \geq -\chi(C_2). \tag{6}$$

3.2. **The based rotation number.** To prove what we need to know about the ECH curves, we will also need to recall some information about the "rotation number" of flows. Our treatment here is inspired by, and closely follows [7], though we handle a few points in a different way that is better suited for our purposes. For any closed Reeb orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ and any choice of (positive) symplectic trivialization τ there is a number $\rho_\tau(\gamma, \xi) \in \mathbb{R}$ called the *based rotation number* of γ, which measures how ξ rotates under the linearization of the Reeb flow. It is defined as follows. The linearized flow is symplectic; apply polar de-composition and take the unitary part. The unitary part descends to a flow on the oriented real projectivization $P(\xi)$. Pull back by γ and conjugate with the trivialization τ to define a flow

$$\bar{\Phi} : \mathbb{R} \times (\mathbb{R}/T\mathbb{Z} \times \mathbb{R}/\mathbb{Z}) \to \mathbb{R}/T\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$$

generated by a vector field $\overline{R}$. We write $\theta : \mathbb{R}/T\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ for the angular coordinate on the target. Let $\overline{\theta} : [0, T] \to \mathbb{R}$ be the unique real-valued lift of the circle-valued map $t \mapsto \theta(\bar{\Phi}_t(0, 0))$ satisfying the initial condition $\overline{\theta}(0) = 0$. Then the based rotation number is

$$\rho_\tau(\gamma, \xi) := \overline{\theta}(T).$$

This depends only on the homotopy class, rel endpoints, of the path of symplectic matrices arising from the linearized flow.

The next lemma relates the based rotation number for the contact structure to the Conley–Zehnder index.

Lemma 3.3. *For any closed Reeb orbit γ and any trivialization τ, we have the bound*

$$|\operatorname{CZ}_\tau(\gamma) - 2\rho_\tau(\gamma, \xi)| \leq 6. \tag{7}$$

Proof. The Conley–Zehnder index is defined by

$$\operatorname{CZ}_\tau(\gamma) = \lceil \theta_\tau(\gamma) \rceil + \lfloor \theta_\tau(\gamma) \lfloor$$

where $\theta_\tau(\gamma)$ denotes the monodromy number of γ in the trivialization τ. Define $\rho'_\tau(\gamma, \xi)$ analogously to $\rho_\tau(\gamma, \xi)$, but using the full linearized flow rather than just the unitary part. It is proved in [7, Lemma 2.6] that

$$|\theta_\tau(\gamma) - \rho'_\tau(\gamma, \xi)| \leq 1.$$

It remains to understand the relationship between ρ and ρ'. We claim that

$$|\rho_\tau(\gamma, \xi) - \rho'_\tau(\gamma, \xi)| \leq 1, \tag{8}$$

which implies the lemma in view of the above. To see why (8) holds, we first note that if we choose as our basepoint (i.e. our trivialization τ) an eigenvector of the positive-definite symmetric part of the polar decomposition of the time T linearized flow, then in fact $\rho'_\tau = \rho_\tau$. Indeed, the space of positive-define symmetric and symplectic matrices is contractible, and the rotation number only depends on the homotopy class, rel endpoints, so we can replace the positive-definite part of the path of matrices arising from the linearized flow by symmetric and symplectic positive-define matrices which all have τ as an eigenvector with positive eigenvalue. The claimed inequality (8) now follows from [7, Lemma 2.6], which bounds the difference between the based rotation number measured with respect to two different basepoints. □

3.3. **Proof of Proposition 2.3.** We can now give the proof of Proposition 2.3.

Proof. **Step 1:** To deal with the fact that we are considering contact structures that are torsion, but possibly non-trivial, we will need to work with an "n^{th}-power" construction. This step collects the results we will need about this.

Fix an integer $n \geq 1$ such that $n \cdot c_1(\xi) = 0$. Write $\xi_n = \xi \otimes \ldots \otimes \xi$ for the n-fold (complex) tensor product of ξ. This is a (trivial) complex line bundle. Choose a unitary trivialization τ of ξ over the simple closed Reeb orbits. This induces a trivialization τ_n of ξ_n. The line bundle ξ_n has a relative Chern class, defined analogously to the contact case. We first note that the relative Chern class of ξ_n with respect to τ_n is computed as follows:

$$c_{\tau_n}(Z, \xi_n) = n \cdot c_\tau(Z, \xi). \tag{9}$$

Next, it is useful to understand how the Chern class depends on the choice of trivialization. Fix any pair of unitary trivializations τ, τ' and any simple closed Reeb orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$. The trivializations define unitary bundle isomorphisms

$$\tau, \tau' : \gamma^* \xi_n \to \mathbb{R}/T\mathbb{Z} \times \mathbb{C};$$

the composition $\tau \circ (\tau')^{-1}$ defines a smooth map $\mathbb{R}/T\mathbb{Z} \to U(1)$. Denote the degree of this map by $\operatorname{wind}_\gamma(\tau, \tau'; \xi_n)$. Then we have the identity

$$c_\tau(Z, \xi_n) - c_{\tau'}(Z, \xi_n) = -\sum_i m_i \operatorname{wind}_{\alpha_i}(\tau, \tau'; \xi) + \sum_j n_j \operatorname{wind}_{\beta_j}(\tau, \tau'; \xi_n). \tag{10}$$

This is proved by the same argument as in the case of contact structures [29].

There is an analogous story for the based rotation number. The unitary part of the linearized flow, being complex linear, defines a map on the complex tensor product ξ_n and we can defined the based rotation number analogously, which we call the *induced based rotation number* (in the trivialization τ) on ξ_n, denoted $\rho_\tau(\gamma, \xi_n)$. We now prove some basic properties of the induced based rotation number analogous to the observed properties of the relative Chern class.

Lemma 3.4. *The unitary component of the based rotation number and the induced based rotation number satisfy the following basic properties:*

- *(Change of trivialization) For any pair τ, τ' of unitary trivializations we have*

$$\rho_\tau(\gamma, \xi_n) - \rho_{\tau'}(\gamma, \xi_n) = \operatorname{wind}_\gamma(\tau, \tau'; \xi_n). \tag{11}$$

- *(Additive under tensor product) For any $n \geq 1$ we have*

$$\rho_{\tau_n}(\gamma, \xi_n) = n \cdot \rho_\tau(\gamma, \xi). \tag{12}$$

Proof. Let $\bar{\Phi}$ and $\bar{\Phi}'$ be the respective flows on $\mathbb{R}/T\mathbb{Z}\times\mathbb{R}/\mathbb{Z}$ defined by τ and τ'. After applying a constant rotation to one of the trivializations, we may assume without loss of generality that $\tau^{-1}(0,0)=(\tau')^{-1}(0,0)$. It follows that

$$\bar{\Phi}_t(0,0)=(\tau\circ(\tau')^{-1})\circ\bar{\Phi}'_t(0,0)$$

for each $t\in\mathbb{R}$. The lifts $\overline{\theta}$ and $\overline{\theta}'$ corresponding to τ and τ' differ by the lift of the map $\mathbb{R}/T\mathbb{Z}\to U(1)$ defined by $\tau\circ(\tau')^{-1}$. The map $\mathbb{R}/T\mathbb{Z}\to U(1)$ has degree $\mathrm{wind}_\gamma(\tau,\tau';\Xi)$, so it follows that

$$\theta(T)-\overline{\theta}'(T)=\mathrm{wind}_\gamma(\tau,\tau';\Xi)$$

which proves (11).

Now fix any $n\geq 1$ and let $\bar{\Phi}$ and $\bar{\Phi}_n$ be the respective flows on $\mathbb{R}/T\mathbb{Z}\times\mathbb{R}/\mathbb{Z}$ defined by τ and τ_n. We observe that

$$(\bar{\Phi}_n)_t=n\cdot\bar{\Phi}_t$$

for each $t\in\mathbb{R}$. This implies that $\overline{\theta}_n=n\cdot\overline{\theta}$ where $\overline{\theta}$ and $\overline{\theta}_n$ are the lifts corresponding to τ and τ_n. Evaluating both sides at T yields (12). □

Step 2: Recall that the tensor power ξ_n is a trivial complex line bundle. We fix a *global* unitary trivialization $\mathfrak{t}$ of ξ_n over the entire manifold Y. Let λ' be any nondegenerate contact form. Let $\alpha=\{(\alpha_i,m_i)\}$ and $\beta=\{(\beta_j,n_j)\}$ be any pair of homologous ECH generators such that $\mathcal{A}(\beta)\leq\mathcal{A}(\alpha)$. This step proves that there exists a constant $\delta>0$ depending only on the background metric, the C^2 norm of λ', and the trivialization $\mathfrak{t}$ such that for any $Z\in H_2(Y,\alpha,\beta)$ we have

$$|I(Z)-J_0(Z)|\leq\delta\,\mathcal{A}(\alpha). \tag{13}$$

Write $K(Z):=I(Z)-J_0(Z)$. Choose any symplectic trivialization τ of ξ over the simple closed Reeb orbits. Then $K(Z)$ expands as

$$K(Z)=2c_\tau(Z)+\sum_i\mathrm{CZ}_\tau(\alpha_i^{m_i})-\sum_j\mathrm{CZ}_\tau(\beta_j^{m_j}).$$

We define

$$K_{\mathrm{approx}}(Z,\xi):=2c_\tau(Z,\xi)+2\sum_i\rho_\tau(\alpha_i^{m_i},\xi)-2\sum_j\rho_\tau(\beta_j^{m_j},\xi)$$

and a corresponding version

$$K_{\mathrm{approx}}(Z,\xi_n):=2c_{\tau'}(Z,\xi_n)+2\sum_i\rho_{\tau'}(\alpha_i^{m_i},\xi_n)-2\sum_j\rho_{\tau'}(\beta_j^{m_j},\xi_n) \tag{14}$$

for ξ_n, where τ' denotes a unitary trivialization of ξ_n over the simple closed Reeb orbits. It follows from (10) and (11) that the definition of $K_{\mathrm{approx}}(Z,\xi_n)$ does not depend on the choice of τ'. It follows from (7) that

$$|K(Z)-K_{\mathrm{approx}}(Z,\xi)|\leq 6\sum_i m_i+6\sum_j m_j\leq 12T_{\min}(\lambda')^{-1}\,\mathcal{A}(\alpha) \tag{15}$$

where $T_{\min}(\lambda')$ denotes the minimal period of a closed Reeb orbit of λ'. Note that $T_{\min}(\lambda')$ admits a positive lower bound depending only on the C^2 norm of λ'.

We now bound $K_{\text{approx}}(Z, \xi)$. Since the left-hand side in (14) does not depend on the choice of τ' on the right-hand side, we set $\tau' = \tau_n$ and use (9) and (12) to show that

$$K_{\text{approx}}(Z, \xi) = n^{-1} \cdot K_{\text{approx}}(Z, \xi_n). \tag{16}$$

Now we set $\tau' = \mathfrak{t}$ and expand

$$K_{\text{approx}}(Z, \xi_n) := 2c_{\mathfrak{t}}(Z, \xi_n) + 2\sum_i \rho_{\mathfrak{t}}(\alpha_i^{m_i}, \xi_n) - 2\sum_j \rho_{\mathfrak{t}}(\beta_j^{m_j}, \xi_n).$$

It follows that immediately that $c_{\mathfrak{t}}(Z, \xi_n) = 0$. It remains to bound $\rho_{\mathfrak{t}}(\gamma, \xi_n)$ for any closed Reeb orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$. To do so, it is convenient to observe that the based rotation number along γ can be computed by integrating the "rotation density" of the flow with respect to $\mathfrak{t}$. To be precise, the global trivialization $\mathfrak{t}$ and the action of the unitary part of the linearized flow on ξ_n define a flow

$$\bar{\Phi}_t : \mathbb{R} \times Y \times \mathbb{R}/\mathbb{Z} \to Y \times \mathbb{R}/\mathbb{Z}$$

generated by a vector field $\bar{R}$. The Lie derivative of the $\mathbb{R}/\mathbb{Z}$-coordinate on the target is a smooth function $r_{\mathfrak{t}} : Y \to \mathbb{R}$, which restricts to the rotation density on any simple closed Reeb orbit. The function $r_{\mathfrak{t}}$ depends only on $\mathfrak{t}$ and the linearized Reeb flow, so $|r_{\mathfrak{t}}|$ admits a finite upper bound $c > 0$ depending only on $\mathfrak{t}$ and the C^2 norm of λ'. We conclude that

$$|\rho_{\mathfrak{t}}(\gamma, \xi_n)| \leq \sup |r_{\mathfrak{t}}| \cdot \mathcal{A}(\gamma) \leq \delta_1 \cdot \mathcal{A}(\gamma).$$

It follows from the above bound that

$$|K_{\text{approx}}(Z, \xi_n)| \leq 2\delta_1 (\sum_i m_i\, \mathcal{A}(\alpha_i) + \sum_j n_j\, \mathcal{A}(\beta_j)) \leq 4\delta_1\, \mathcal{A}(\alpha). \tag{17}$$

Combine (15), (16), and (17) to show

$$|K(Z)| \leq (12 T_{\min}(\lambda')^{-1} + 4\delta_1)\, \mathcal{A}(\alpha)$$

which proves (13) with $\delta := 12 T_{\min}(\lambda')^{-1} + 4\delta_1$.

Step 3: To simplify the notation, write $c'_k = c_{\sigma_k}(\lambda')$. Choose generic J and z not on a closed Reeb orbit such that the chain map $U_{J,z}$ is well-defined. By Lemma 3.1, for any $k \geq 1$, there exists an ECH generator α_k such that

(i) $\mathcal{A}(\alpha_k) \leq c'_k$;
(ii) $U^k_{J,z}(\alpha_k) \neq 0$.

It follows that there exists a sequence of ECH generators $\{\beta_j\}_{j=0}^k$, each with $\mathcal{A}(\beta_j) \leq c'_k$, and J-holomorphic currents $\{\mathcal{C}_j\}_{j=1}^k$ such that $\mathcal{C}_j \in \mathcal{M}(\beta_j, \beta_{j-1})$, $I(\mathcal{C}_j) = 2$, and $(0, z) \in \operatorname{supp}(\mathcal{C}_j)$ for each j. Now set $Z := \sum_{j=1}^k [\mathcal{C}_j]$. Using the fact that $I(\mathcal{C}_j) = 2$ for each j and the bound (13) we derive the bound

$$\sum_{j=1}^k J_0(\mathcal{C}_j) = J_0(Z) \leq 2k + \delta\, \mathcal{A}(\alpha_k) \leq 2k + 2\delta c'_k. \tag{18}$$

It is an immediate consequence of (6) that $J_0(\mathcal{C}_j) \geq -1$ for each j. Write S_1 for the set of indices j such that

$$J_0(\mathcal{C}_j) \geq 3.$$

It follows that $3S_1 - (k - S_1) \leq 2k + \delta c'_k$, hence

$$\#S_1 \leq \frac{3}{4}k + \frac{\delta}{4}c'_k. \tag{19}$$

Write S_2 for the set of all indices j such that $\mathcal{A}(\mathcal{C}_j) \geq k^{-1/16}$. Since $\sum_{j=1}^{k} \mathcal{A}(\mathcal{C}_j) = \mathcal{A}(\alpha_k) - \mathcal{A}(\beta_0) \leq c'_k$, and the action is nonnegative, it follows that

$$\#S_2 \leq c'_k k^{1/8}. \tag{20}$$

The quantity $c_k(\lambda)$ is $O(k^{1/2})$, in view of (3), and for λ' sufficiently close to λ $c'_k \leq 2c_k(\lambda)$.

Thus, by (19) and (20) there exists an index $0 \leq j \leq k$ in neither S_1 nor S_2. Take C_k to be any component of $\mathcal{C}_j$ passing through $(0, z)$. By (6) it follows that $\chi(C_k) \geq -J_0(\mathcal{C}_j)$. Thus, C_k satisfies the requirements of Proposition 2.3. □

4. Low-action curves from periodic Floer homology

This section proves Proposition 2.2.

4.1. **Periodic Floer homology.**

We review the theory of periodic Floer homology (PFH), a cousin of ECH defined in [29, 32]. We will discuss both the basics and some key new developments in the theory. Fix a closed, oriented surface Σ of genus g, an area form ω, and a diffeomorphism $\phi : \Sigma \to \Sigma$ preserving the area form.

4.1.1. *Basics.* The *mapping torus* of ϕ is the 3-manifold

$$Y_\phi := [0,1] \times \Sigma \,/\, (1,p) \sim (0, \phi(p)).$$

Write t for the coordinate on the interval $[0,1]$. The one-form dt on $[0,1] \times \Sigma$ descends to a closed 1-form, also denoted by dt, on Y_ϕ. The area form ω defines a closed two-form ω_ϕ on Y_ϕ. The pair $\eta = (dt, \omega_\phi)$ is a framed Hamiltonian structure and the Reeb vector field $R_\phi := R_\eta$ generates the suspension flow of ϕ. The mapping torus Y_ϕ fibers over the circle; write $V_\phi \to Y_\phi$ for the vertical tangent bundle.

Several key definitions carry over to this setting from ECH. In analogy with ECH, we will call periodic orbits of R_ϕ *closed Reeb orbits.* The definitions of elliptic/hyperbolic orbits from ECH have analogues here, replacing the bundle ξ with the bundle V_ϕ. Moreover, the ECH and J_0 indices are also defined in this setting, again replacing ξ with V_ϕ.

4.1.2. *Rationality and monotonicity.* We say ϕ is *rational* if the cohomology class $[\omega_\phi]$ is a real multiple of a rational class. We say that ϕ is *monotone* if it is rational and we have $c_1(V_\phi) = c[\omega_\phi]$ for some constant $c \in \mathbb{R}$. When $g \neq 1$, our monotonicity condition coincides with the monotonicity condition introduced by Seidel [44]. When $g = 1$, we show in Lemma A.3 that $c_1(V_\phi)$ always vanishes, so any rational area-preserving diffeomorphism is monotone in this case.

4.1.3. *Definition of PFH.* The definition of the version of PFH that we will use requires that ϕ is rational and also *nondegenerate*, meaning every closed Reeb orbit is either elliptic or hyperbolic. Choose a generic η-adapted almost-complex structure J on $\mathbb{R} \times Y_\phi$. Choose a union γ of embedded loops, transverse to Y_ϕ, called a *reference cycle*. Let Σ denote the homology class of a fiber of the map $Y_\phi \to S^1$. The *degree* $d(\gamma)$ of γ is the oriented intersection number of γ with $[\Sigma] \in H_2(Y_\phi;\mathbb{Z})$. We assume that $d(\gamma) > \max(0, g-1)$ and that that γ is *monotone*. This means that the homology class $\Gamma := [\gamma] \in H_1(Y_\phi;\mathbb{Z})$ satisfies the identity

$$c_1(V_\phi) + 2\,\mathrm{PD}(\Gamma) = c \cdot [\omega_\phi] \tag{21}$$

for some constant $c \neq 0$. The constant c is explicitly computable: pairing both sides of (21) with $[\Sigma]$ shows that the constant $c = 2A^{-1}(d-g+1)$, where $A := \int_\Sigma \omega$. We note that (21) has a solution if and only if ϕ is rational, and that if (21) has a solution, it has solutions of arbitrarily high degree. Finally, let $K_\phi := \ker(\omega_\phi)$ denote the subgroup of all integral homology classes on which ω_ϕ integrates to 0.

The PFH chain complex $\mathrm{PFC}_*(\phi,\gamma)$ is defined to be the vector space over $\mathbb{Z}/2$ freely generated by pairs $\Theta = (\alpha, Z)$ that we call *anchored ECH generators*. Here $\alpha = \{(\alpha_i, m_i)\}$ is an ECH generator such that $[\alpha] = [\gamma]$ and Z is an element of $H_2(Y_\phi,\alpha,\gamma)/K_\phi$ (recall that $H_2(Y_\phi,\alpha,\gamma)$ is an affine space over $H_2(Y_\phi;\mathbb{Z})$).

The differential ∂_J is defined similarly to the ECH differential, although now we take the relative homology classes of the holomorphic curves into account. Write $\mathcal{M}(\alpha,\beta,W)$ for the moduli space of holomorphic currents from α to β that represent the class $W \in H_2(Y,\alpha,\beta)$; let $\mathcal{M}_k(\alpha,\beta,W)$ denote the subspace of currents with ECH index k. Fix a pair of anchored ECH generators $\Theta = (\alpha, Z)$, $\Theta' = (\beta, Z')$. The matrix coefficient of ∂_J with respect to Θ and Θ' is defined by the formula

$$\langle \partial\Theta, \Theta'\rangle := \#_2\,\mathcal{M}_1(\alpha,\beta,Z-Z')/\,\mathbb{R}\,.$$

Write $\mathrm{PFH}_*(\phi,\gamma)$ for the homology of the complex $(\mathrm{PFC}_*(\phi,\gamma),\partial_J)$. The PFH chain complex and homology group carry some additional basic features that we now review. There is a natural action of $H_2(Y_\phi;\mathbb{Z})$ on $\mathrm{PFC}_*(\phi,\gamma,J)$; a class $W \in H_2(Y_\phi;\mathbb{Z})$ acts on a generator (α, Z) by sending it to $(\alpha, Z+W)$. This action commutes with the differential and so descends to an action on $\mathrm{PFH}_*(\phi,\gamma)$ as well. The U-map on PFH is also defined analogously to the U-map for ECH.

After choosing a framing of V_ϕ over γ, the PFH complex also comes equipped with a $\mathbb{Z}$-grading, which is defined for each anchored ECH generator by the formula

$$I(\Theta) := c_\tau(Z) + Q_\tau(Z) + \sum_i \sum_{k=1}^{m_i} \mathrm{CZ}_\tau(\alpha_i^k). \tag{22}$$

The differential and U-map have degree -1 and -2 with respect to this grading. The H_2-action shifts the grading as follows:

$$I(W\cdot\Theta) = I(\Theta) + \langle c_1(V_\phi) + 2\,\mathrm{PD}(\Gamma), W\rangle = I(\Theta) + 2A^{-1}(d-g+1)\int_W \omega_\phi \tag{23}$$

for any anchored ECH generator Θ and any $W \in H_2(Y_\phi;\mathbb{Z})$. The last line uses (21) and our computation of the monotonicity constant above. The identity (23) also shows that the $\mathbb{Z}$-grading is well-defined.

4.1.4. *The U-cycle property.* The analogue of a U-tower in ECH is a *U-cycle*. Assume that ϕ is nondegenerate and rational and choose a monotone reference cycle γ so that PFH is well-defined. A nonzero element $\sigma \in \mathrm{PFH}_*(\phi,\gamma,G)$ is *U-cyclic of order m* for some integer $m \geq 1$ if

$$U^{m(d(\gamma)-g+1)}\sigma = (-m[\Sigma]) \cdot \sigma.$$

It is known that every nonzero element of PFH is U-cyclic as long as γ has sufficiently high degree.

Proposition 4.1 (Existence of U-cyclic elements, [16])**.** *Assume that ϕ is nondegenerate and rational and fix a monotone reference cycle γ. There exists an integer $d_0 > \max(0, g-1)$, depending only on the Hamiltonian isotopy class of ϕ, such that if $d(\gamma) \geq d_0$, then $\mathrm{PFH}_*(\phi,\gamma) \neq 0$ and every nonzero class is U-cyclic.*

The following lemma is a "chain-level" version of Proposition 4.1.

Lemma 4.2. *Assume that ϕ is nondegenerate and rational and fix a monotone reference cycle γ. There exists an integer $d_0 > \max(0, g-1)$, depending only on the Hamiltonian isotopy class of ϕ, such that the following holds. Choose any monotone reference cycle γ such that $d(\gamma) \geq d_0$. Choose generic J and $z \in Y_\phi$ so that the chain-level map $U_{J,z}$ is well-defined. Then there exist positive integers m_0 and m_1 and a sequence $\{\Theta_j\}_{j=1}^{m_1}$ of nonzero generators of $\mathrm{PFC}_*(\phi,\gamma)$ such that*

$$\langle U_{J,z}^{m_0(d(\gamma)-g+1)}\Theta_j, \Theta_{j+1}\rangle \neq 0$$

for each $j \in \{1,\dots,m_1-1\}$ and

$$\langle U_{J,z}^{m_0(d(\gamma)-g+1)}\Theta_{m_1}, m_0m_1[\Sigma]\cdot\Theta_1\rangle \neq 0.$$

Proof. Suppose γ has sufficiently high degree so that Proposition 4.1 holds. Choose a trivialization of the restriction of V_ϕ to γ and use this to define a $\mathbb{Z}$-grading on PFH. Fix a grading k for which $\mathrm{PFH}_k(\phi,\gamma) \neq 0$. Write $Z_k \subset \mathrm{PFC}_k(\phi,\gamma)$ for the space of cycles of degree k, and B_k for the space of boundaries of degree k. The proof will take 3 steps.

Step 1: This step shows that Z_k and B_k have finite dimension over $\mathbb{Z}/2$. By the change of grading formula (23), it follows that for each ECH generator α there exists at most one anchored ECH generator $\Theta = (\alpha, Z)$ such that $I(\Theta) = k$. Since ϕ is nondegenerate, it has finitely many ECH generators representing any given homology class in $H_1(Y_\phi;\mathbb{Z})$. This implies that $\mathrm{PFC}_k(\phi,\gamma)$ contains finitely many anchored ECH generators, so it is a finite-dimensional $\mathbb{Z}/2$-vector space. This implies that Z_k and B_k have finite dimension as well.

Step 2: This step uses Proposition 4.1 to show that there is a nonzero cycle $x \in Z_k$ fixed up to a shift by an iterate of the U-map. Fix generic J and z so that the chain-level map $U_{J,z}$ is well defined. Proposition 4.1 implies that there exists an integer $m_0 \geq 1$ such that for any nonzero cycle $x \in Z_k$, there exists a chain z such that

$$m_0[\Sigma] \cdot U_{J,z}^{m_0(d(\gamma)-g+1)} x = x + \partial z. \tag{24}$$

Let T be the restriction of $m_0[\Sigma] \cdot U_{J,z}^{m_0(d(\gamma)-g+1)}$ to Z_k. Then (24) implies that $\mathrm{Im}(T - 1) \subseteq B_k$. Since $\mathrm{PFH}_k(\phi, \gamma) \neq 0$, it follows that $\dim(Z_k) > \dim(B_k)$. This implies that the operator $T - 1$ has nonzero kernel and therefore there exists some nonzero $x \in Z_k$ such that $Tx = x$.

Step 3: This step completes the proof. Expand the element x from the previous step into a sum $\sum_{i=1}^{N} x_i$ where each x_i is an anchored ECH generator. The desired cyclic sequence $\{\Theta_j\}_{j=1}^{m_1}$ of anchored ECH generators will be picked out from the x_i using a short combinatorial argument. Define a directed graph G as follows. The vertex set of G is $\{1, \ldots, N\}$ and there is an edge from i to j if and only if $\langle Tx_i, x_j \rangle \neq 0$. We allow edges to start and end at the same vertex. It is well-known that any directed graph with no *sources*, i.e. vertices which have no incoming edges, has a directed cycle. Now, G has no sources: this follows because $Tx = x$ implies that for each j, the identity $\langle Tx, x_j \rangle \neq 0$ holds, which in turn implies that there exists some i such that G has an edge from i to j. Thus, G has a cycle. Thus, there exists a set $\{x'_j\}_{j=1}^{m_1}$ of anchored ECH generataors such that

$$\langle Tx'_j, x'_{j+1} \rangle \neq 0, \quad \langle Tx'_{m_1}, x'_1 \rangle \neq 0, \tag{25}$$

for each $j \in \{1, \ldots, m_1 - 1\}$. For each j, set $\Theta_j := -(j-1)m_0[\Sigma] \cdot x'_j$. Then, the Θ_j satisfy the conditions of the lemma by (25), since $T = m_0[\Sigma] \cdot U_{J,z}^{m_0(d(\gamma)-g+1)}$ by definition. $\square$

4.2. **Proof of Proposition 2.2.** We now suppose that ϕ is *monotone*, which we recall means $c_1(V_\phi) = c[\omega_\phi]$ for some constant $c \in \mathbb{R}$. The proof of Proposition 2.2 is an immediate consequence of the following result, since the monotonicity condition is preserved under Hamiltonian isotopy.

Proposition 4.3. *Assume that ϕ is nondegenerate and monotone. There exists an integer $d_0 \geq \max(0, g-1)$ depending only on g and the Hamiltonian isotopy class such that for any $z \in Y_\phi$ not on any closed Reeb orbit, and generic J, there exists a standard J-holomorphic curve $u_d : C_d \to \mathbb{R} \times Y_\phi$ satisfying the following properties:*

(a) $(0, z) \in u_d(C_d)$.
(b) $\mathcal{E}(u_d) \leq d$.
(c) $\mathcal{A}(u_d) \leq d^{-1/2}$.
(d) $\chi(C_d) \geq -2$.

Proof. Fix $d_0 > 0$ so that Lemma 4.2 holds and fix any monotone reference cycle γ with degree $d := d(\gamma) \geq d_0$. The proof will take 2 steps.

Step 1: Fix generic J and z so that the map $U_{J,z}$ is well-defined on $\mathrm{PFC}_*(\phi, \gamma)$. Let $\{\Theta_j\}_{j=1}^{m_1}$ denote the sequence of generators provided by Lemma 4.2. Then, by Lemma 4.2, we obtain a sequence $\mathcal{C}_1, \ldots, \mathcal{C}_{m_0 m_1 (d-g+1)}$ of J-holomorphic currents counted by the U-map such that

$$\sum_{i=1}^{m_1 m_0(d(\gamma)-g+1)} [\mathcal{C}_i] = m_0 m_1[\Sigma] \in H_2(Y_\phi, \alpha_1, \alpha_1) = H_2(Y_\phi; \mathbb{Z})/K_\phi.$$

By additivity of the action and of J_0 we therefore obtain

$$\sum_{i=1}^{m_1 m_0(d(\gamma)-g+1)} \mathcal{A}(\mathcal{C}_i) = m_0 m_1 A, \qquad \sum_{i=1}^{m_1 m_0(d(\gamma)-g+1)} J_0(\mathcal{C}_i) = 2m_0 m_1(d(\gamma)+g-1). \tag{26}$$

In the equality for J_0, we have used the fact that ϕ is monotone, which implies that $c_1(V_\phi)$ has zero pairing with K_ϕ, together with the fact that $J_0([\Sigma]) = 2(d(\gamma)+g-1)$.
Step 2: This step finishes the proof of the proposition. Write S_1 for the set of i such that $\mathcal{A}(\mathcal{C}_i) > d(\gamma)^{-1/2}$ and write S_2 for the set of i such that $J_0(\mathcal{C}_i) \geq 3$. Then, by nonnegativity of the action of pseudoholomorphic curves, and (26), we have

$$\#S_1 \leq A m_0 m_1 (d(\gamma))^{1/2}. \tag{27}$$

Since the J_0 index is bounded below by -1, the bound (26) implies

$$\#S_2 \leq m_0 m_1 (3d(\gamma)+g-1)/4. \tag{28}$$

Thus, after possibly increasing d_0, we have the strict inequality

$$\#(S_1 \cup S_2) < m_0 m_1 (d(\gamma)-g+1).$$

This implies that there exists some i such that $\mathcal{A}(\mathcal{C}_i) \leq (d(\gamma))^{-1/2}$ and $J_0(\mathcal{C}_i) \leq 2$. Thus, the component $u_d : C_d \to \mathbb{R} \times Y_\phi$ of $\mathcal{C}_i$ containing $(0,z)$ has $\mathcal{A}(u_d) \leq (d(\gamma))^{1/2}$ and $\chi(C_d) \geq -2$, by (6). It remains to show that $\mathcal{E}(u_d) \leq d(\gamma)$: this follows since, as dt is closed, the integral of dt over any level set of $\mathcal{C}_i$ is equal to the pairing $\langle dt, [\gamma]\rangle = d(\gamma)$. □

5. Invariant sets from low-action holomorphic curves

The purpose of this section is to prove Theorem 5. For the remainder of the section, we fix a closed, smooth, connected, oriented manifold Y of odd dimension $2n+1 \geq 3$.

5.1. Notational preliminaries.

Let us begin by reviewing the setup.

5.1.1. *Stable constants.* The statements and proofs below will involve several constants which depend on Y, η, and J, where η is a framed Hamiltonian structure on Y and J is an η-adapted almost-complex structure. We say that such a constant is *stable* if it can be taken to be invariant under C^∞-small perturbations of η and J.

5.1.2. *Geometry of symplectizations.* Let $\mathcal{D}(Y)$ be the space of pairs (η, J) where η is a framed Hamiltonian structure and J is an η-adapted almost-complex structure of $\mathbb{R} \times Y$. We equip $\mathcal{D}(Y)$ with the topology of C^∞-convergence. That is, a sequence $\{(\eta_k = (\lambda_k, \omega_k), J_k)\}_{k\geq 1}$ in $\mathcal{D}(Y)$ converges to $(\eta = (\lambda, \omega), J)$ if and only if the sequences

$\{\lambda_k\}_{k\geq 1}$, $\{\omega_k\}_{k\geq 1}$, and $\{J_k\}_{k\geq 1}$ converge in the C^∞-topology to λ, ω, and J, respectively. Choose a pair $(\eta = (\lambda, \omega), J) \in \mathcal{D}(Y)$. To this pair we associate the following translation-invariant and J-invariant Riemannian metric on $\mathbb{R} \times Y$:

$$g := da \otimes da + \lambda \otimes \lambda + \omega(-, J-).$$

We fix notation for norms of tensors with respect to g. For any smooth tensor $\mathcal{T}$ on $\mathbb{R} \times Y$, write $|\mathcal{T}|_g$ for its pointwise g-norm, which is a smooth function on $\mathbb{R} \times Y$. Write $\|\mathcal{T}\|_g := \sup_{z\in\mathbb{R}\times Y} |\mathcal{T}|_g(z)$ for the C^0 norm of $\mathcal{T}$ with respect to g. We fix notation for the metric balls of g. Let dist_g denote the distance function of g. Omitting the dependence on g for brevity, we let

$$\overline{B}_r(z) := \{w \in \mathbb{R} \times Y \mid \mathrm{dist}_g(z, w) \leq r\}$$

denote the closed metric ball of radius $r > 0$ centered at $z \in \mathbb{R} \times Y$.

5.1.3. *Geometry of J-holomorphic curves.* Fix a J-holomorphic curve $u : C \to \mathbb{R} \times Y$. We say u is *compact* and *connected* if the domain C is respectively compact and connected. We say u is *generally immersed* if the critical point set $\mathrm{Crit}(u)$ is discrete. This is always true if C is connected and u is not a constant map. We say u is *boundary immersed* if the restriction of u to ∂C is an immersion. We assume for the sake of convenience that any J-holomorphic curves is generally immersed and boundary immersed unless stated otherwise.

We let $\gamma := u^*g$ denote the pullback metric on C, which is defined at any point $z \in C$ such that $du(z) \neq 0$. Let $\alpha := u^*\lambda$ denote the pullback of λ. Let $|\mathcal{T}|_\gamma$ and $\|\mathcal{T}\|_\gamma$ denote the pointwise and C^0 norms of a tensor $\mathcal{T}$ with respect to γ.

5.2. **The connected-local area bound and its significance.** The main estimate required for the proof of Theorem 5 is a so-called "connected-local area bound". In this section we state this estimate, deferring the proof to Section 5.5, and then use it to prove Theorem 5.

Given a J-holomorphic curve $u : C \to \mathbb{R} \times Y$, any point $\zeta \in C$, and any $r > 0$, let $S_r(\zeta)$ denote the connected component of $u^{-1}(\overline{B}_r(u(\zeta)))$ containing ζ. Our estimate gives an a priori bound on the area of $S_r(\zeta)$ assuming that $\mathcal{A}(u)$ is small and r is small. The bound depends on the Euler characteristic of C, which is the primary reason why Euler characteristic bounds are assumed in Theorem 5.

Proposition 5.1 (Connected-local area bound for low-action curves)**.** *Fix $(\eta, J) \in \mathcal{D}(Y)$. There exists stable constants $\epsilon_7 = \epsilon_7(\eta, J) > 0$ and $\epsilon_8 = \epsilon_8(\eta, J) > 0$ such that the following holds. Let $u : C \to \mathbb{R} \times Y$ be a standard J-holomorphic curve such that $\mathcal{A}(u) \leq \epsilon_7$. Then for any point $\zeta \in C$, we have the bound*

$$\mathrm{Area}_\gamma(S_{\epsilon_8}(\zeta)) \leq \epsilon_8^{-1}(\chi(C)^2 + 1). \tag{29}$$

We now prove Theorem 5 assuming Proposition 5.1.

Proof of Theorem 5. Fix $(\eta, J) \in \mathcal{D}(Y)$ and a sequence $\{(\eta_k, J_k)\}_{k\geq 1}$ in $\mathcal{D}(Y)$ converging to it. Fix a sequence $\{u_k : C_k \to Y\}_{k\geq 1}$ of standard J_k-holomorphic curves such that $\lim_{k\to\infty} \mathcal{A}(u_k) = 0$ and $\inf_{k\geq 1} \chi(C_k) > -\infty$. Let $\mathcal{X} \subset \mathcal{K}(Y)$ denote the limit set.

Pick any $\Lambda \in \mathcal{X}$. Write $\{\phi^t\}_{t\in\mathbb{R}}$ for the flow of R_η. To show that Λ is R_η-invariant, it suffices to show that for any $z \in \Lambda$, there exists some $\epsilon > 0$ such that $\phi^t(z) \in \Lambda$ for any $t \in (-\epsilon, \epsilon)$. After passing to a subsequence and translating the J-holomorphic curves, we may assume without loss of generality that

$$\lim_{k\to\infty} u_k(C_k) \cap \{0\} \times Y = \Lambda.$$

Fix $z \in \Lambda$. After passing to a further subsequence, there exists a sequence $\{z_k\}_{k\geq 1}$ in Y, converging to z, and for each k a point $\zeta_k \in C_k$ such that $u_k(\zeta_k) = (0, z_k)$. Set $S_k := S_{\epsilon_8}(\zeta_k)$ for each k, where $\epsilon_8 > 0$ is the stable constant introduced in Proposition 5.1. Since $\mathcal{A}(u_k) \to 0$, $\{\chi(C_k)\}_{k\geq 1}$ is uniformly bounded, and all constants are stable, we deduce from Proposition 5.1 a k-independent upper bound on $\mathrm{Area}_\gamma(S_k)$ for each sufficiently large k. The Euler characteristic bound implies a k-independent upper bound on $\mathrm{Genus}(S_k)$ as well. We use Fish's target-local Gromov compactness theorem [20] to deduce the following. There exist compact surfaces $\widetilde{S}_k \subset S_k \setminus \partial S_k$ such that

$$u_k(\partial\widetilde{S}_k) \cap \overline{B}_{\epsilon_8/2}(0, z) = \emptyset$$

for each k and the restrictions $v_k := u_k|_{\widetilde{S}_k}$ converge in the Gromov topology after passing to a subsequence to a compact and boundary-immersed J-holomorphic curve $v : \widetilde{S} \to \mathbb{R} \times Y$. We remark that v may not be generally immersed. It follows from the Gromov convergence that

(a) $\mathcal{A}(v) = \lim_{k\to\infty} \mathcal{A}(v_k) = 0$;
(b) the images $v_k(\widetilde{S}_k)$ converge to $v(\widetilde{S})$ in the Hausdorff topology as $k \to \infty$;
(c) $v(\partial\widetilde{S}) \cap \overline{B}_{\epsilon_8/2}(0, z) = \emptyset$.

Let $\gamma_z : \mathbb{R} \to Y$ denote the unique trajectory of R_η such that $\gamma_z(0) = z$. It follows from (a) that the image $v(\widetilde{S})$ is contained inside the immersed surface $\mathbb{R} \times \gamma_z(\mathbb{R}) \subset \mathbb{R} \times Y$. It follows from (b) that $(0, z) \in v(\widetilde{S})$. We use (b) and (c) to show that $v(\widetilde{S})$ contains $\{0\} \times \gamma_z((-\epsilon, \epsilon))$ for some $\epsilon > 0$.

The surface $\mathbb{R} \times \gamma_z(\mathbb{R})$ is parameterized by a J-holomorphic map

$$\phi : \mathbb{C} \to \mathbb{R} \times Y$$

sending $s + it \mapsto (s, \gamma_z(t))$. Assume that γ is not periodic, so the map ϕ is injective. It follows that the function $f := \phi^{-1} \circ v : \widetilde{S} \to \mathbb{C}$ is a non-constant holomorphic function and sends the point ζ to $0 \in \mathbb{C}$. Using (c) we observe that f does not have any zeroes on $\partial\widetilde{S}$. By the open mapping theorem, we conclude that the image of f contains an open neighborhood of the origin. It follows that there exists some $\epsilon > 0$ such that $v(\widetilde{S})$ contains $\{0\} \times \gamma_z((-\epsilon, \epsilon))$. If γ is periodic, then the function f may instead be defined as a holomorphic map from f onto an open annulus, and the same argument using the open mapping theorem still applies.

It follows from (b) that $v_k(\widetilde{S}_k) \cap \{0\} \times Y$ converges to $v(\widetilde{S}) \cap \{0\} \times Y$ in the Hausdorff topology as $k \to \infty$. By definition, the set Λ contains $v(\widetilde{S}) \cap \{0\} \times Y$, and therefore contains $\gamma_z((-\epsilon, \epsilon))$. Since this property holds for any $z \in \Lambda$, we have proved that Λ is R_η-invariant. $\square$

5.3. **Properties of the limit set.** Before diving into the proof of the connected-local area bound, we provide a proof of Proposition 1.7 from the introduction.

Proof of Proposition 1.7. Fix $(\eta, J) \in \mathcal{D}(Y)$ and a sequence $\{(\eta_k, J_k)\}_{k\geq 1}$ in $\mathcal{D}(Y)$ converging to it. Fix a sequence $\{u_k : C_k \to \mathbb{R} \times Y\}_{k\geq 1}$ of standard J_k-holomorphic curves. The set $\mathcal{I}_k := (a \circ u_k)(C_k) \subseteq \mathbb{R}$ is closed and connected for each k. Each curve u_k defines a continuous map $L_k : \mathcal{I}_k \to \mathcal{K}(Y)$ by

$$s \mapsto u_k(C_k) \cap \{s\} \times Y.$$

After translating, we may assume without loss of generality that $0 \in \mathcal{I}_k$ for each k. We choose $k_j \to \infty$ such that $\lim_{j\to\infty} L_{k_j}(0) = \Lambda$ for some $\Lambda \in \mathcal{K}(Y)$. Write $\mathcal{X}$ for the limit set of $\{u_k\}$ and $\mathcal{X}'$ for the limit set of the subsequence $\{u_{k_j}\}$. The proposition will follow by showing that $\mathcal{X}$ and $\mathcal{X}'$ are closed and that $\mathcal{X}'$ is connected. This will take 2 steps.

Step 1: We show that $\mathcal{X}$ and $\mathcal{X}'$ are closed. It follows from the definition that

$$\mathcal{X} = \bigcap_{k=1}^{\infty} \overline{\bigcup_{m=k}^{\infty} L_m(\mathcal{I}_m)}, \quad \mathcal{X}' = \bigcap_{j=1}^{\infty} \overline{\bigcup_{m=j}^{\infty} L_{k_m}(\mathcal{I}_{k_m})}.$$

Both are countable intersections of closed sets, so they are themselves closed.

Step 2: We show that $\mathcal{X}'$ is connected. More precisely, we show that for any pair of disjoint open subsets $\mathcal{U}$ and $\mathcal{U}'$ of $\mathcal{K}(Y)$ such that $\mathcal{X}' \subseteq \mathcal{U} \sqcup \mathcal{U}'$, we have that $\mathcal{X}'$ is contained in one of them. Pick any such pair of open sets. Assume without loss of generality that

$$\Lambda = \lim_{j\to\infty} L_{k_j}(0) \in \mathcal{U}\,.$$

We will show that $\mathcal{X}' \subseteq \mathcal{U}$. For each j, set $U_j := L_{k_j}^{-1}(\mathcal{U})$ and $U_j' := L_{k_j}^{-1}(\mathcal{U}')$. It suffices to show that U_j' is empty for all sufficiently large j. It follows from our setup that, after taking j to be sufficiently large, the sets U_j and U_j' are disjoint open subsets of $\mathcal{I}_{k_j}$, and moreover that U_j is non-empty. For each j, write $Z_j := L_{k_j}^{-1}(\mathcal{K}(Y) \setminus (\mathcal{U} \sqcup \mathcal{U}'))$ for the complement of $U_j \sqcup U_j'$. Since $\mathcal{X}'$ lies inside the open set $\mathcal{U} \sqcup \mathcal{U}'$, it follows that for each sequence $\{s_j\}_{k\geq 1}$ and each sufficiently large j that $L_{k_j}(s_j) \in \mathcal{U} \sqcup \mathcal{U}'$. This implies that Z_j can only be non-empty for finitely many j, so it follows that $U_j \sqcup U_j' = \mathcal{I}_{k_j}$ for sufficiently large j. For any such j, we have that U_j is non-empty and $\mathcal{I}_{k_j}$ is connected for each j, so we conclude that U_j' is empty. □

5.4. **Preliminaries from feral curve theory.** It remains to prove Proposition 5.1, which will take up the remainder of the paper. To do this, we need to first collect some more preliminaries from the work of Fish-Hofer [21], which is the purpose of this section.

5.4.1. *Perturbed holomorphic curves.* A *perturbed J-holomorphic curve* is a pair (u, f) where $u : C \to \mathbb{R} \times Y$ is a J-holomorphic curve and $f : C \to \mathbb{R}$ is a smooth function which is compactly supported in the open subset $C \setminus (\partial C \cap \mathrm{Crit}(u))$.

Perturbing u in the vertical direction by f defines a new map

$$\widetilde{u} : \zeta \mapsto \exp^g_{u(\zeta)}(f(\zeta)\partial_a).$$

Write $\widetilde{\gamma} := \widetilde{u}^* g$ for the induced pullback metric. Define an almost-complex structure $\widetilde{j}$ on C as the unique almost-complex structure which is a $\widetilde{\gamma}$-isometry and coincides with j on the complement of $\mathrm{supp}(f)$. We then define a one-form

$$\widetilde{\alpha} := -(\widetilde{u}^* da \circ \widetilde{j})$$

on C. This should be thought of as a perturbation of $\alpha = u^*\lambda = -(u^* da \circ j)$. We work with perturbed J-holomorphic curves in order to make the height function $a \circ u$ a Morse function outside of small neighborhoods of the critical points. More precisely, we require several quantitative conditions to hold that we list below. Given positive constants $\delta > 0$ and $\epsilon > 0$, a (δ, ϵ)*-tame perturbed J-holomorphic curve* is a perturbed J-holomorphic curve (u, f) for which the following conditions are satisfied:

(i) The constant δ satisfies the bound

$$10\delta < \min\Big\{ \mathrm{dist}_\gamma(\mathrm{Crit}(a \circ u), \partial C), \inf_{\substack{\zeta_0, \zeta_1 \in \mathrm{Crit}(u) \\ \zeta_0 \neq \zeta_1}} \mathrm{dist}_\gamma(\zeta_0, \zeta_1), \mathrm{dist}_\gamma(\mathrm{Crit}(u), \partial C) \Big\};$$

(ii) The function f is compactly supported in

$$C \setminus \{\zeta \in C \mid \mathrm{dist}_\gamma(\zeta, \mathrm{Crit}(u)) \leq \delta/2\}.$$

(iii) The restricted function $f : C \setminus \{\zeta \in C \mid \mathrm{dist}_\gamma(\zeta, \mathrm{Crit}(u)) < \delta\} \to \mathbb{R}$ is a Morse function.

(iv) We have $\epsilon < 2^{-24} c(\eta, J, \delta, u)$ where $c(\eta, J, \delta, u) \in (0, 1)$ depends only on η, J, δ and the extrinsic curvature of the map u.

(v) The C^2-norm of f with respect to γ is less than ϵ^2.

A precise statement of (iv) is found in [21, Definition 4.24]. The following technical result shows that any J-holomorphic curve has a (δ, ϵ)-tame perturbation for any sufficiently small δ and ϵ.

Lemma 5.2 ([21, Lemma 4.26])**.** *Fix $(\eta, J) \in \mathcal{D}(Y)$. Fix a J-holomorphic curve $u : C \to \mathbb{R} \times Y$ and let $\delta > 0$ be any constant satisfying the bound in (i) above. Then for any sufficiently small $\epsilon > 0$, there exists a smooth map $f : C \to \mathbb{R}$ such that (u, f) is (δ, ϵ)-tame.*

5.4.2. *Tracts and strips.* Fix a perturbed J-holomorphic curve (u, f). Tracts and strips, introduced in [21], are highly structured compact portions of the domain C. They are important because they satisfy several geometric estimates. A *tract* in (u, f) is a connected, compact embedded surface $\widetilde{C} \subset C$, possibly with boundary and corners, such that:

(i) The boundary $\partial\widetilde{C}$ is disjoint from $\mathrm{Crit}(a \circ \widetilde{u})$;

(ii) The boundary of $\widetilde{C}$ decomposes as $\partial_0\widetilde{C} \cup \partial_1\widetilde{C}$ where
   - (a) $\partial_0\widetilde{C} \cap \partial_1\widetilde{C}$ is a finite set of corners;
   - (b) If it is non-empty, each component of $\partial_1\widetilde{C}$ is tangent to the gradient vector field $\mathrm{grad}_{\widetilde{\gamma}}(a \circ \widetilde{u})$;
   - (c) The function $a \circ \widetilde{u}$ is constant on each component of $\partial_0\widetilde{C}$.

Strips are variants of tracts which are homeomorphic to compact disks but have more complicated boundaries. A *strip* in (u, f) is a datum $(\widetilde{C}, \mathcal{I}, p, h^-, h^+)$ satisfying the following properties:

(i) $\widetilde{C} \subset C$ is a compact surface with boundary and corners and is is homeomorphic to a disk;
(ii) $\mathcal{I} \subset \mathbb{R}$ is a closed interval of finite length equipped with a coordinate t;
(iii) $p : \mathcal{I} \to C$ is a smooth map such that $\widetilde{u} \circ p : \mathcal{I} \to \mathbb{R} \times Y$ is an embedding, $a \circ \widetilde{u} \circ p : \mathcal{I} \to \mathbb{R}$ is a constant map, and $p^* \widetilde{\lambda} = dt$;
(iv) $h^\pm : \mathcal{I} \to \mathbb{R}$ are C^1 functions such that $h^- < h^+$, and for each $t \in \mathcal{I}$ there exists a flow line
$$q^t : [\min(0, h^-(t)), \max(0, h^+(t))] \to C$$
for the normalized gradient vector field $|\operatorname{grad}_{\widetilde{\gamma}}(a \circ \widetilde{u})|_{\widetilde{\gamma}}^{-2} \cdot \operatorname{grad}_{\widetilde{\gamma}}(a \circ \widetilde{u})$ satisfying the initial condition $q^t(0) = p(t)$.
(v) The surface $\widetilde{C}$ equals the union of the flow lines:
$$\widetilde{C} = \bigcup_{t \in \mathcal{I}} q^t([h^-(t), h^+(t)]).$$

The boundary of a strip $\widetilde{C}$ also decomposes as a union $\partial \widetilde{C} = \partial_0 \widetilde{C} \cup \partial_1 \widetilde{C}$ of a "horizontal" boundary $\partial_0 \widetilde{C}$ and a "vertical" boundary $\partial_1 \widetilde{C}$. The set $\partial_0 \widetilde{C}$ decomposes further into "top" and "bottom" boundary components
$$\partial_0^+ \widetilde{C} := \bigcup_{t \in \mathcal{I}} q^t(h^+(t)), \quad \partial_0^- \widetilde{C} := \bigcup_{t \in \mathcal{I}} q^t(h^-(t)).$$

A strip is *rectangular* if h^- and h^+ are constant functions. Note that a strip is rectangular if and only if $\partial_0^\pm \widetilde{C}$ are mapped into level sets of $\mathbb{R} \times Y$.

5.4.3. *Exponential area bound for tracts.* In the classical theory of J-holomorphic curves in symplectizations, area bounds are deduced from Hofer energy bounds. The following result provides an alternative in the absence of Hofer energy bounds.

Proposition 5.3 ([21, Theorem 9])**.** *Fix $(\eta, J) \in \mathcal{D}(Y)$. Fix any constants $a_+ > a_- \in \mathbb{R}$ and $A > 0$. Let (u, f) be a (δ, ϵ)-tame perturbed J-holomorphic curve with domain C and let $\widetilde{C} \subset C$ be a tract of (u, f). Suppose that the following conditions are satisfied:*

(i) $(a \circ \widetilde{u})(\widetilde{C}) \subset [a_-, a_+]$;
(ii) $(a \circ \widetilde{u})(\partial_0 \widetilde{C}) \cap (a_-, a_+) = \emptyset$;
(iii) $\int_{\widetilde{C}} \widetilde{u}^* \omega \leq A$;
(iv) a_+ *and* a_- *are regular values of* $a \circ \widetilde{u}$.

Then there exists a stable constant $c_1 = c_1(\eta, J) > 0$ such that

$$\operatorname{Area}_{\widetilde{\gamma}}(\widetilde{C}) \leq \Big(c_1 \min\Big\{ \int_{(a \circ u)^{-1}(a_+)} \widetilde{\alpha}, \int_{(a \circ u)^{-1}(a_-)} \widetilde{\alpha} \Big\} + A\Big)(e^{c_1(a_+ - a_-)} - 1) + A. \tag{30}$$

5.4.4. *Strip estimates.* We now state two technical estimates for strips of perturbed J-holomorphic curves. Lemma 5.4 shows that the length of the bottom boundary of a strip is controlled by the length of the top boundary, provided that the strip is not too tall.

Lemma 5.4 ([21, Lemma 4.21])**.** *Fix $(\eta, J) \in \mathcal{D}(Y)$. Then there exists a stable constant $c_2 = c_2(\eta, J) \geq 1$ such that the following holds. Let $(\widetilde{C}, \mathcal{I}, p, h^-, h^+)$ be a strip of a (δ, ϵ)-tame perturbed J-holomorphic curve (u, f), such that*

$$-c_2^{-1} \leq h^- \leq 0 \leq h^+ \leq c_2^{-1}.$$

Then we have the bound

$$\int_{\partial_0^- \widetilde{C}} \widetilde{\alpha} \leq 2 \int_{\partial_0^+ \widetilde{C}} \widetilde{\alpha} + 2c_2 \int_{\widetilde{C}} u^* \omega.$$

The next lemma asserts that for any finite collection of strips which are not too tall or too short, there exists a gradient flow line of $a \circ \widetilde{u}$ of controlled length running from the bottom horizontal boundary component to the top horizontal boundary component.

Lemma 5.5 ([21, Lemma 4.23])**.** *Fix $(\eta, J) \in \mathcal{D}(Y)$. There exists a stable constant $c_3 = c_3(\eta, J) \geq 1$ such that the following holds. Let (u, f) be a (δ, ϵ)-tame perturbed J-holomorphic curve. Let $\{\widetilde{C}_k\}_{k=1}^n$ be a finite set of rectangular strips of (u, f) satisfying the following properties:*

(i) $a_0 = \inf_{\zeta \in \widetilde{C}_k} (a \circ \widetilde{u})(\zeta)$ *is independent of k;*
(ii) $a_1 = \sup_{\zeta \in \widetilde{C}_k} (a \circ \widetilde{u})(\zeta)$ *is independent of k;*
(iii) $a_1 - a_0 \leq c_3^{-1}$*;*
(iv) $\sum_{k=1}^n \int_{\widetilde{C}_k} u^* \omega \leq (a_1 - a_0) \sum_{k=1}^n \int_{\partial_0^- \widetilde{C}_k} \widetilde{\alpha}$*;*
(v) Each of the strips $\{\widetilde{C}_k\}_{k=1}^n$ are pairwise disjoint.

Then there exists $k \in \{1, \ldots, n\}$ and a smooth map $q : [0, s] \to \widetilde{C}_k$ such that

$$\dot{q}(s) = \mathrm{grad}_{\widetilde{\gamma}}(a \circ \widetilde{u})(q(s)), \quad (a \circ \widetilde{u} \circ q)(0) = a_0, \quad (a \circ \widetilde{u} \circ q)(s) = a_1$$

and

$$\mathrm{length}_{\widetilde{\gamma}}(q([0, s])) \leq c_3(a_1 - a_0).$$

5.4.5. *Action quantization.* Fish–Hofer [21, Theorem 4] proved that a holomorphic curve $u : C \to \mathbb{R} \times Y$ has a positive lower bound on its action near any interior global maximum/minimum of the function $a \circ u$. Their lower bound depends on the genus of C, which suffices for our intended applications. For the sake of a cleaner statement, we note that the bound can be made genus-independent using the compactness theory of J-holomorphic currents [42, Remark 5.20]. We now state the precise quantization result.

Proposition 5.6. *Fix $(\eta, J) \in \mathcal{D}(Y)$. Fix any real number $s > 0$. There exists a stable constant $\hbar = \hbar(\eta, J, s) > 0$ such that, for any compact, connected J-holomorphic curve $u : C \to \mathbb{R} \times Y$, we have*

$$\mathcal{A}(u) \geq \hbar > 0$$

provided that the following properties are satisfied for some $a_0 \in \mathbb{R}$:

(i) Either $\inf_{\zeta \in C}(a \circ u)(\zeta)$ *or* $\sup_{\zeta \in C}(a \circ u)(\zeta)$ *is equal to* a_0*;*
(ii) $(a \circ u)(\partial C) \cap [a_0 - s, a_0 + s] = \emptyset$*.*

Proposition 5.6 is proved via a contradiction argument using Proposition 5.3 and Fish's target-local Gromov compactness theorem [20].

5.4.6. *Geodesic distance lemma.* The following elementary lemma is used in the proof of Lemma 5.14 below.

Lemma 5.7 ([21, Lemma 4.29])**.** *Fix* $(\eta, J) \in \mathcal{D}(Y)$*. There exists a stable constant* $\epsilon_4 = \epsilon_4(\eta, J) \geq 100$ *such that the following holds. Each smooth unit-speed immersion* $q : [0, T] \to \mathbb{R} \times Y$ *such that*

(i) $\lambda(\dot{q}(t)) > 0$ *for each* $t \in [0, T]$*;*
(ii) $\epsilon_4 \leq \int_q \lambda \leq 10\epsilon_4$*;*
(iii) the set of $t \in [0, T]$ *such that* $\lambda(\dot{q}(t)) < 1/2$ *has Lebesgue measure at most* ϵ_4^{-1}

satisfies the bound

$$\mathrm{dist}_g(q(0), q(T)) > \epsilon_4/2.$$

We need to state Lemma 5.7 here, instead of alongside Lemma 5.14, because the constant ϵ_4 appears in the statement of Proposition 5.8 below.

5.5. **Proof of the connected local area bound.** We now prove Proposition 5.1. In fact, this will be deduced from a more technical bound, Proposition 5.8, which we now state. The statement requires defining a stable constant

$$\epsilon_5 := 2^{-24} \min(c_2^{-1}, c_3^{-4}, \epsilon_4) \tag{31}$$

where c_2, c_3, ϵ_4 are the stable constants from Lemmas 5.4, 5.5, and 5.7, respectively.

Proposition 5.8. *Fix* $(\eta, J) \in \mathcal{D}(Y)$*. Let* ϵ_4 *and* ϵ_5 *denote the stable constants from Lemma 5.7 and* (31)*, respectively. There exists a stable constant* $c_6 = c_6(\eta, J) \geq 1$ *such that the following holds. Let* $u : C \to \mathbb{R} \times Y$ *be a compact, connected* J*-holomorphic curve satisfying the following properties:*

(L1) $(a \circ u)(\partial C) = \{a_0, a_1\}$ *where* $a_1 > a_0$*;*
(L2) a_0 *and* a_1 *are regular values of the projection* $a \circ u : C \to \mathbb{R}$*;*
(L3) $a_1 - a_0 \geq \epsilon_5/8$*;*
(L4) $\sup_{\zeta \in C}(a \circ u)(\zeta) - \inf_{\zeta \in C}(a \circ u)(\zeta) \leq \epsilon_5$*;*
(L5) $\mathcal{A}(u) \leq 2^{-48}\epsilon_4\epsilon_5$*;*
(L6) The set of all $\zeta \in (a \circ u)^{-1}(a_0)$ *such that* $|u^*\lambda|_\gamma(\zeta) < 1/2$ *has Lebesgue measure at most* ϵ_4 *in* $(a \circ u)^{-1}(a_0)$*, where Lebesgue measure is defined using the pullback metric* γ*.*

Then for each $\zeta \in C$*, we have the bound*

$$\mathrm{Area}_\gamma(S_{\epsilon_5}(\zeta)) \leq c_6(\chi(C)^2 + 1).$$

Proposition 5.8 generalizes an estimate proved by Fish–Hofer (stated in [21, Proposition 4.30]). They made the additional assumption that the domain C is homeomorphic to a compact annulus. The main novelty in the proof of Proposition 5.8 is the introduction of combinatorial and topological arguments to deal with non-annular holomorphic curves. Let us defer the proof for the moment and first prove Proposition 5.1 assuming that Proposition 5.8 holds.

Proof of Proposition 5.1. Let $u : C \to \mathbb{R} \times Y$ be a standard J-holomorphic curve. Pick any point $\zeta \in C$. We will show that, when $\mathcal{A}(u)$ is sufficiently small, there exists a compact surface $\widetilde{C} \subset C$ containing ζ in the middle which satisfies (L1)–(L6) and has Euler characteristic comparable to C. Proposition 5.8 then implies the area bound. The proof will take 6 steps.

Step 1: This step defines the surface $\widetilde{C}$. Choose a positive parameter $r \in (\epsilon_5/4, \epsilon_5/2)$ and set $a_0 := (a \circ u)(\zeta) - r$ and $a_1 := (a \circ u)(\zeta) + r$. We fix r so that both a_0 and a_1 are regular values of $a \circ u$. Set $C_0 := (a \circ u)^{-1}([a_0, a_1]) \subset C$. Write Δ for the disjoint union of compact components of $C \setminus \mathrm{Int}(C_0)$ and set $C_1 := C_0 \cup \Delta$. Let $\widetilde{C}$ be the connected component of C_1 containing ζ.

Step 2: This step proves the two-sided bound

$$2 \geq \chi(\widetilde{C}) \geq \chi(C). \tag{32}$$

The upper bound follows from the fact that $\widetilde{C}$ is connected. To prove the lower bound, we must show $\chi(\Sigma) \leq 0$ where $\Sigma := C \setminus \mathrm{Int}(\widetilde{C})$. Since C is connected, it follows that each connected component of Σ must be non-compact and have at least one boundary component. This proves that Σ has non-positive Euler characteristic.

Step 3: Write $\widetilde{u}$ for the restriction of u to $\widetilde{C}$. This step verifies that $\widetilde{u}$ satisfies (L1)–(L3). Condition (L1) follows from the fact that the whole surface C has empty boundary. Condition (L2) is satisfied becaause a_0 and a_1 are regular values of $a \circ u$. Condition (L3) is satisfied because $r \in (\epsilon_5/4, \epsilon_5/2)$.

Step 4: This step verifies that $\widetilde{u}$ satisfies (L4) if $\mathcal{A}(u)$ is smaller than a stable constant which we specify below. Write

$$L := \sup_{z \in \widetilde{C}} (a \circ u)(z) - \inf_{z \in \widetilde{C}} (a \circ u)(z)$$

and write $L' := L - (a_1 - a_0)$. Since $a_1 - a_0 < \epsilon_5/2$, it suffices to show $L' \leq \epsilon_5/2$. We may assume that L' is strictly positive, since otherwise we are done. Recall the definitions of the surfaces C_0 and Δ from Step 1. If L' is positive, then Δ is non-empty and there exists a connected component $\Sigma \in \pi_0(\Delta)$ such that

$$\sup_{z \in \Sigma} (a \circ u)(z) - \inf_{z \in \Sigma} (a \circ u)(z) \geq L'/2.$$

We further note that Σ must only have boundary components on one side, that is $(a \circ u)(\partial\Sigma)$ is either equal to $\sup_{z \in \partial\Sigma} (a \circ u)(z)$ or $\inf_{z \in \partial\Sigma} (a \circ u)(z)$. Assume that $\mathcal{A}(u) < \hbar(\eta, J, \epsilon_5/2)$ where $\hbar$ denotes the constant from Proposition 5.6. It follows from the contrapositive of Proposition 5.6 that $L' < \epsilon_5/2$. This shows that (L4) is satisfied.

Step 5: This step verifies that $\widetilde{u}$ satisfies (L5) and (L6) if $\mathcal{A}(u)$ is sufficiently small. This is close to a tautology in the case of (L5) Condition (L6), using [21, Lemma 4.27], is satisfied after taking $\mathcal{A}(u)$ sufficiently small and slightly increasing r. The cited lemma shows, in a precise quantitative sense, that if $\mathcal{A}(u)$ is small, then most tangent planes of $u(C)$ are close to $\mathrm{Span}(\partial_a, R_\eta)$. This shows in particular that for any given level set,

as long as $\mathcal{A}(u)$ is sufficiently small, we can find an arbitrarily close level set such that most of its tangent vectors are close to R_η.
Step 6: By Proposition 5.8, we have the bound

$$\mathrm{Area}_\gamma(S_{\epsilon_5/16}(\zeta) \cap \widetilde{C}) = \mathrm{Area}_\gamma(S_{\epsilon_5/16}(\zeta)) \leq c_0(\chi(\widetilde{C})^2 + 1) \tag{33}$$

where $c_0 = c_0(\eta, J) > 0$ is a stable constant. The area bound then follows from (32). □

It remains to prove Proposition 5.8 and this will take up the remainder of the paper. As the proof gets rather technical, let us start by providing a sketch of the argument.

Outline of the proof of Proposition 5.8. Let $u : C \to \mathbb{R} \times Y$ denote a compact, connected J-holomorphic curve satisfying (L1)–(L6). We begin by constructing a *tract decomposition* of u (Proposition 5.9). Recall that a *tract* is a compact embedded surface with corners in C, with *horizontal* boundary components mapping into level sets and *vertical* boundary components mapping into gradient flow lines of the function $a \circ u$. We show that, after perturbing u slightly, the domain C can be cut up into tracts with each boundary component having controlled length. The existence of such a decomposition is itself implicit in the first five steps of the proof of [21, Proposition 4.30]. Our proof follows these steps.

The construction of the tract decomposition relies on a careful analysis of the gradient flow of the height function $a \circ u$. When the action $\mathcal{A}(u)$ is small, most of the tangent planes of $u(C)$ are close to $\mathrm{Span}(\partial_a, R_\eta)$. One can then show that most gradient flow lines of $a \circ u$ starting at the bottom boundary will terminate at the top boundary and have an a priori upper bound on their length. Choosing many of these gradient flow lines and cutting along them produces the tract decomposition. Significant technical difficulties arise because $a \circ u$ is not guaranteed to be a Morse function and because the gradient flow is not well-defined at critical points of u. This is why it is necessary to perturb u before constructing the tract decomposition.

The next step after construction of the tract decomposition is to bound the area of each tract. We show that the number of vertical and horizontal boundary components are each controlled by $\chi(C)$ (Lemma 5.10) as is the total Euler characteristic of all the tracts (Lemma 5.11). Recalling that each horizontal boundary component has controlled length, we conclude that each tract has a uniform bound on the length of its entire bottom boundary, depending only on $\chi(C)$ and ambient geometry. Applying Proposition 5.3 bounds the area of each tract by a constant depending only on $\chi(C)$ and ambient geometry.

The final step is to cover $S_{\epsilon_5}(\zeta)$ by a controlled number of tracts. This gives a bound on its area since we have already bounded the area of each tract. Call a tract *rectangular* if it has zero genus, two horizontal boundary components, and two vertical boundary components. A geodesic distance argument that we learned from [21] implies that $S_{\epsilon_5}(\zeta)$ cannot intersect both vertical boundary components of a rectangular tract. The topological lemmas mentioned above imply that most tracts are rectangular. These results are combined with a graph-theoretic argument to prove the desired covering bound.

5.5.1. *Statement of tract decomposition.* We now begin the process of making the above outline rigorous. The first part is the statement of the tract decomposition.

Proposition 5.9 (Tract decomposition)**.** *Fix $(\eta, J) \in \mathcal{D}(Y)$. Let $u : C \to \mathbb{R} \times Y$ be a compact, connected J-holomorphic curve satisfying (L1)–(L6). Then, for any sufficiently small $\epsilon > 0$, there exists a (δ, ϵ)-tame perturbation (u, f) and a finite set of tracts $\{\widetilde{C}_k\}_{k=1}^N$ satisfying the following properties:*

(a) $C = \bigcup_{k=1}^N \widetilde{C}_k$.

(b) For each $k \neq k'$, the intersection $\widetilde{C}_k \cap \widetilde{C}_{k'}$ is either empty or equal to a disjoint union of components of $\partial_1 \widetilde{C}_k$.

(c) For each k, we have $(a \circ \widetilde{u})(\widetilde{C}_k) = \{a_0, a_1\}$.

(d) For each k and each component $L \in \pi_0(\partial_0 \widetilde{C}_k)$ such that $(a \circ \widetilde{u})(L) = a_0$, we have $\int_L \widetilde{\alpha} < 16\epsilon_4$. Moreover, if L is not a circle, then $\int_L \widetilde{\alpha} > \epsilon_4$.

(e) For each k and each component $L \in \pi_0(\partial_1 \widetilde{C}_k)$, we have
$$\mathrm{length}_{\widetilde{\gamma}}(L) \leq c_3(a_1 - a_0).$$

We defer the proof for the moment, collecting some useful lemmas about the asserted tract decomposition first.

5.5.2. *Tract topology bounds.* Fix $\eta \in \mathcal{D}(Y)$ and a compact, connected J-holomorphic curve satisfying (L1)–(L6). Use Proposition 5.9 to construct a perturbation (u, f) and tract decomposition $\{\widetilde{C}_k\}_{k=1}^N$. The next two lemmas provide bounds on the topology of the tracts. Lemma 5.10 bounds the number of horizontal and vertical boundary components of each tract in terms of $\chi(C)$.

Lemma 5.10. *For each k, we have the bounds*
$$\#\pi_0(\partial_1 \widetilde{C}_k) \leq 2 - 2\chi(C), \quad \#\pi_0(\partial_0^- \widetilde{C}_k) \leq 3 - 2\chi(C). \tag{34}$$

Proof. Fix any k. Write $M := \#\pi_0(\partial_1 \widetilde{C}_k)$ for the number of vertical boundary components of $\widetilde{C}_k$. Let $\widehat{C}$ denote the closure of $C \setminus \widetilde{C}_k$. The proof will take 2 steps.

Step 1: This step proves the first bound in (34). It follows from the Mayer–Vietoris sequence that
$$M = \chi(\widetilde{C}_k) + \chi(\widehat{C}) - \chi(C).$$

We show $\chi(\widetilde{C}_k) \leq 1$ and $\chi(\widehat{C}) \leq M/2$. The upper bound on $\chi(\widetilde{C}_k)$ follows from the fact that $\chi(\widetilde{C}_k)$ is connected and has non-empty boundary. The upper bound on $\chi(\widehat{C})$ is deduced as follows. Each connected component of $\widehat{C}$ is a tract which shares at least at least two vertical boundary components with $\widetilde{C}_k$. Therefore, $\widehat{C}$ has at most $M/2$ connected components. Each connected component of $\widehat{C}$ has non-empty boundary and therefore has Euler characteristic ≤ 1. Combine both of these upper bounds with the identity for M and re-arrange to get the first bound in (34).

Step 2: This step proves the second bound in (34). Each connected component of $\partial_0^- \widetilde{C}_k$ is either i) a compact interval which intersects two components of $\partial_1 \widetilde{C}_k$ or ii) a circle which is connected component of $(a \circ u)^{-1}(a_0) \subset \partial C$. Any component of $\partial_1 \widetilde{C}_k$ intersects a unique component of $\partial_0^- \widetilde{C}_k$, so there are at most $M/2$ components of the former type, which we showed in Step 1 is bounded above by $1 - \chi(C)$. There are at most $\#\pi_0(\partial C)$ components of the latter type. This gives the bound

$$\#\pi_0(\partial_0^- \widetilde{C}_k) \leq 1 + \#\pi_0(\partial C) - \chi(C).$$

The second bound in (34) now follows from plugging in the inequality $\#\pi_0(\partial C) \leq 2 - \chi(C)$. □

Lemma 5.11 provides an elementary identity used in the proof of Lemma 5.13 below.

Lemma 5.11. *The Euler characteristic of the domain C satisfies the following identity:*

$$2\chi(C) = \sum_k (2\chi(\widetilde{C}_k) - \#\pi_0(\partial_1 \widetilde{C}_k)). \tag{35}$$

Proof. For each k, write $M_k := \#\pi_0(\partial_1 \widetilde{C}_k)$ for the number of vertical boundary components of $\widetilde{C}_k$. Write M for the number of gradient trajectories of $a \circ \widetilde{u}$ which are vertical boundary components of some $\widetilde{C}_k$. It follows that $2M = \sum_k M_k$ because each gradient trajectory is a vertical boundary component of exactly two tracts. The Mayer–Vietoris sequence then implies

$$2\chi(C) = \sum_k 2\chi(C_k) - 2M = \sum_k (2\chi(C_k) - M_k).$$

□

Remark 5.12. The primary implication of the identity (35) is that the tract $\widetilde{C}_k$ is rectangular for all but at most $-\chi(C)$ indices k. Each term on the right-hand side of (35) is bounded above by 0 and equality is achieved if and only if $\widetilde{C}_k$ is rectangular.

5.5.3. *Tract coverings of controlled size.* As above, fix a perturbed J-holomorphic curve (u, f) and a tract decomposition $\{\widetilde{C}_k\}_{k=1}^N$. The following lemma asserts that local connected components can be covered by a controlled number of tracts.

Lemma 5.13. *For any point $\zeta \in C$, there exists a covering of the surface $\widetilde{S}_{4\epsilon_5}(\zeta) := \widetilde{u}^{-1}(B_{4\epsilon_5}(\widetilde{u}(\zeta)))$ by $2 - 3\chi(C)$ tracts.*

Call a tract $\widetilde{C}$ *rectangular* if $\chi(\widetilde{C}) = 1$ and $\#\pi_0(\partial_1 \widetilde{C}) = 2$. The proof of Lemma 5.13 requires the following technical lemma, which asserts that $\widetilde{S}_{4\epsilon_5}(\zeta)$ cannot intersect both vertical boundary components of a rectangular tract. The proof of this lemma is similar to the proof of [21, Lemma 4.35].

Lemma 5.14. *For any point $\zeta \in C$ and any k such that $\widetilde{C}_k$ is rectangular, the surface $\widetilde{S}_{4\epsilon_5}(\zeta)$ does not intersect both vertical boundary components of $\widetilde{C}_k$.*

Proof. Let $\widetilde{C} := \widetilde{C}_k$ denote any rectangular tract. Observe that $\partial_0^- \widetilde{C}$, the bottom part of the horizontal boundary, must be a compact interval connecting the two vertical boundary components. Write $L := \partial_0^- \widetilde{C}$ and write $\partial L = \zeta_+ - \zeta_-$ where $\zeta_\pm$ are distinct points in $\widetilde{C}$. Write $\gamma_\pm$ for the vertical boundary components intersecting L at $\zeta_\pm$ respectively. Write

$$d := \inf_{\substack{z_+ \in \gamma_+ \\ z_- \in \gamma_-}} \mathrm{dist}_g(\widetilde{u}(z_+), \widetilde{u}(z_-))$$

for the extrinsic distance between γ_+ and γ_-. Since both γ_+ and γ_- have length at most $c_3(a_1 - a_0)$ by Proposition 5.9(e), it follows from the triangle inequality that

$$d \geq \operatorname{dist}_g(\widetilde{u}(\zeta_+), \widetilde{u}(\zeta_-)) - 2c_3(a_1 - a_0).$$

Let $q : [0, T] \to C$ denote the unique unit-speed parameterization of L such that $q(0) = \zeta_-$ and $q_+ = \zeta_+$. Apply (L6) and Lemma 5.7 to the curve $\widetilde{q} := \widetilde{u} \circ q$ to bound $\operatorname{dist}_g(\widetilde{u}(\zeta_+), \widetilde{u}(\zeta_-))$ from below. We deduce the bound

$$d \geq \epsilon_4/2 - 2c_3(a_1 - a_0).$$

The right-hand side is seen to be strictly greater than $8\epsilon_5$ using (L4) and the bound $\epsilon_5 \leq 2^{-24}\min(c_3^{-3}, \epsilon_4)$. We conclude that γ_+ and γ_- have extrinsic distance greater than $8\epsilon_5$ from each other. By the triangle inequality, they cannot both intersect $\widetilde{S}_{4\epsilon_5}(\zeta)$ for any choice of $\zeta \in C$. □

Lemma 5.13 is proved by combining Lemmas 5.11 and 5.14 with a combinatorial argument.

Proof of Lemma 5.13. Let $\widehat{C}_1, \ldots, \widehat{C}_D$ denote a minimal-size cover of $\widetilde{S}_{4\epsilon_5}(\zeta)$ by tracts. Our goal is to prove the bound $D \leq 2 - 3\chi(C)$. We assume without loss of generality that $D \geq 2$. The proof will take 4 steps.

Step 1: Write Z for the number of indices i such that $\widehat{C}_i$ is rectangular. This step uses Lemma 5.11 to bound Z from below. Recall from Remark 5.12 that each term on the right-hand side of (35) is non-positive, with equality if and only if $\widetilde{C}_k$ is rectangular. It follows that at most $-\chi(C)$ tracts are non-rectangular, which implies $Z \geq D + \chi(C)$.

Step 2: This step defines a connected graph G as follows. The vertices are $\{1, \ldots, D\}$. An edge between i and j exists if and only if $i \neq j$ and the tracts $\widehat{C}_i$ and $\widehat{C}_j$ share a vertical boundary component. The connected surface $\widetilde{S}_{4\epsilon_5}(\zeta)$ intersects each of the tracts $\widehat{C}_i$ since they form a cover of minimal size. This implies that $\bigcup_{i=1}^{d} \widehat{C}_i$ is connected, which in turn implies that G is connected.

Step 3: Let i be any vertex such that $\widehat{C}_i$ is rectangular. This step shows that i is a vertex of degree 1, which is equivalent to the assertion that $\widetilde{S}_{4\epsilon_5}(\zeta)$ intersects exactly one vertical boundary component of $\widehat{C}_i$. This assertion follows from Lemma 5.14 and the fact that G is connected.

Step 4: This step uses Lemma 5.11 and some basic graph theory to prove that $D \leq 2 - 3\chi(C)$. For each $i \in \{1, \ldots, D\}$, let N_i denote the degree of the vertex i and M_i denote the number of vertical boundary components of $\widehat{C}_i$. Note that $N_i \leq M_i$ for any i. By Step 3, we have $N_i = 1$ and $M_i = 2$ when $\widehat{C}_i$ is rectangular, so we get the improved bound $N_i \leq M_i - 1$ in this case. We deduce the inequality

$$2\chi(C) \leq \sum_{i=1}^{D}(2\chi(\widehat{C}_i) - M_i) \leq 2D - Z - \sum_{i=1}^{D} N_i. \tag{36}$$

The first inequality uses (35), multipled through by 2, and the fact that every term on its right-hand side is ≤ 0. The second inequality uses the observed bounds for each M_i above and the bound $\chi(\widehat{C}_i) \leq 1$. The last two terms on the right are controlled by D and $\chi(C)$. We proved that $Z \geq D + \chi(C)$ in Step 1. The sum $\sum_{i=1}^{D} N_i$ is twice the number of edges of G, which is $\geq 2D-2$ since G is connected. Plug these bounds into (36) to get an upper bound on D:

$$2\chi(C) \leq 2D - (D+\chi(C)) - (2D-2) \quad \Rightarrow \quad D \leq 2 - 3\chi(C).$$

□

5.5.4. *Proof of Proposition 5.8.* We can now explain the proof of the technical bound Proposition 5.8, contingent on Proposition 5.9.

Proof of Proposition 5.8. Use Proposition 5.9 to construct a (δ,ϵ)-tame perturbation (u,f) and tract decomposition $\{\widetilde{C}_k\}_{k=1}^N$. By making ϵ sufficiently small, we ensure that $S_{\epsilon_5}(\zeta) \subset \widetilde{S}_{4\epsilon_5}(\zeta)$. We can also ensure that $\widetilde{\gamma} \leq 2\gamma$ (see [21, Lemma 4.5]). By Lemma 5.10, each tract has at most $3-2\chi(C)$ bottom boundary components, and by Proposition 5.9(d) it follows that

$$\int_{\partial_0^- \widetilde{C}_k} \widetilde{\alpha} \leq 8\epsilon_4(3-2\chi(C))$$

for each k. The area bound in Proposition 5.3 shows for each k the bound

$$\mathrm{Area}_\gamma(\widetilde{C}_k) \leq 2\,\mathrm{Area}_{\widetilde{\gamma}}(\widetilde{C}_k) \leq c_0(1-\chi(C))$$

where $c_0(\eta,J) > 0$ is stable and k-independent. By Lemma 5.13, $S_{\epsilon_5}(\zeta)$ is covered by $2-3\chi(C)$ tracts, and the desired area bound follows. □

5.5.5. *Proof of tract decomposition.* To conclude, we need to provide the promised proof of Proposition 5.9, which will take up the remainder of the paper.

Proof of Proposition 5.9. The proof of Proposition 5.9 is simple when $\mathcal{A}(u) = 0$, and we begin by explaining this: in this case, the map u is a branched covering map from C onto $[a_0,a_1]\times\gamma$, where γ is a closed orbit of R_η. Cut up γ into intervals $\{\mathcal{I}_\ell\}_{\ell=1}^M$ with length in $(3\epsilon_4, 4\epsilon_4)$ (or leave it be if it is shorter than that) such that the segments $[a_0,a_1]\times\{z\}$ do not intersect a critical value of u for any ℓ and any endpoint $z \in \partial\mathcal{I}_\ell$. For each ℓ, define $\widetilde{C}_\ell := u^{-1}([a_0,a_1]\times\mathcal{I}_\ell)$. The set $\{\widetilde{C}_\ell\}_{\ell=1}^M$ is the desired tract decomposition.

Thus, we can assume $\mathcal{A}(u) > 0$, which will be our standing assumption for the rest of the proof. Fix $(\eta,J) \in \mathcal{D}(Y)$ and a compact, connected J-holomorphic curve $u : C \to \mathbb{R}\times Y$ satisfying (L1)–(L6). The proof takes 6 steps. They closely follow the first five steps in the proof of [21, Proposition 4.30]. The stable constants c_2, c_3, ϵ_4 from Lemmas 5.4, 5.5, 5.7, respectively, and the stable constant ϵ_5 from (31) will appear frequently.

Step 1: Set $\mathcal{Z} := \mathrm{Crit}(u) \subset C \setminus \partial C$. This step defines certain neighborhoods of $\mathcal{Z}$ and performs a (δ,ϵ)-tame perturbation so that $a\circ u$ is a Morse function outside of these neighborhoods. Define

$$\partial_0^- C := (a\circ u)^{-1}(a_0), \quad \partial_0^+ C := (a\circ u)^{-1}(a_1).$$

Fix a constant

$$0 < \delta_2 < 2^{-24}\min(\mathrm{dist}_\gamma(\mathrm{Crit}(a\circ u),\partial C), \min_{\substack{z,z'\in\mathcal{Z}\\ z\neq z'}} \mathrm{dist}_\gamma(z,z'), \mathrm{dist}_\gamma(\mathcal{Z},\partial C))$$

where we recall that $\gamma := u^*g$. Assumptions (L1) and (L2) ensure that the right-hand side is positive. Fix a radial coordinate around each $z \in \mathcal{Z}$ as follows. By standard local theory (see [46, Theorem 2.114]), there exists a local holomorphic chart $\phi_z : \mathcal{O}(z) \to \mathcal{O}(0) \subset \mathbb{C}$, geodesic normal coordinates $\Phi_z : \mathcal{O}(z) \to \mathcal{O}(0) \subset \mathbb{C}^{n+1}$, and an integer $k_z \geq 2$ such that $\phi_z(z) = 0$, $\Phi_z(z) = 0$, and

$$\Phi_z \circ u \circ \phi_z^{-1}(w) = (w^{k_z}, 0, \dots, 0) + F_z(w)$$

where $F_z(w) = O(|w|^{k_z+1})$ and $dF_z(w) = O(|w|^{k_z})$. After possibly shrinking $\mathcal{O}(z)$, define a function $r_z : \mathcal{O}(z) \to \mathbb{R}$ by setting $r_z \circ \phi_z^{-1}(w) = |w|^{k_z}$. Fix a small constant $\delta_1 \in (0, \delta_2)$ and define

$$\mathcal{V}_z := \{\zeta \in \mathcal{O}(z) \,|\, r_z(\zeta) < \delta_1\}.$$

We assume that δ_1 is sufficiently small so that for each $\delta \in (0, \delta_1]$, we have

(i) $\{\zeta \in \mathcal{V}_z \mid r_z = \delta\}$ is homeomorphic to a circle;
(ii) $\{\zeta \in \mathcal{V}_z \mid r_z < \delta\} \cap \partial C = \emptyset$;
(iii) $\text{length}_\gamma(\{\zeta \in \mathcal{V}_z \mid r_z = \delta\}) \leq 4\pi\delta k_z$;
(iv) $\mathcal{V}_z \cap \mathcal{V}_{z'} = \emptyset$ for each $z, z' \in \mathcal{Z}$ such that $z \neq z'$;
(v) $\delta(\sum_{z\in\mathcal{Z}} k_z) \leq c_2\,\mathcal{A}(u)/8\pi$.

Note that (v) is only possible because we are assuming $\mathcal{A}(u) > 0$. Define $\mathcal{V} := \cup_{z\in\mathcal{Z}}\,\mathcal{V}_z$. Collect the radial coordinates r_z together to define a radial coordinate $r : \mathcal{V} \to \mathbb{R}$. Choose $\delta \in (0, \delta_1/10)$ such that

$$\{\zeta \in C \,|\, \text{dist}_\gamma(\zeta, \mathcal{Z}) < \delta\} \subset \bigcup_{z\in\mathcal{Z}} \{\zeta \in \mathcal{O}(z) \,|\, r_z(\zeta) < \delta_1/10\}$$

. Since $\delta < \delta_1$, it satisfies the required bounds in Lemma 5.2. The above condition ensures that the δ-neighborhood of $\mathcal{Z}$ lies well within the disjoint union of the neighborhoods $\mathcal{V}_z$. Choose any sufficiently small $\epsilon \in (0, \epsilon_5)$. By Lemma 5.2, there exists a (δ, ϵ)-tame perturbation which we denote by (u, f).

Step 2: Our goal is to show that for a large measure set of initial conditions $\zeta \in \partial_0^- C$, there exists a solution of the gradient flow equation

$$q : [0,T] \to C, \quad q'(s) = \text{grad}_{\widetilde{\gamma}}(a \circ \widetilde{u})(q(s), \quad q(0) = \zeta \tag{37}$$

terminating on $\partial_0^+ C$. We define several relevant sets:

$$\begin{aligned}
&\mathcal{A} := \{\zeta \in C \,|\, \exists\, T > 0 \text{ and a solution to (37) with } q(T) = \zeta\},\\
&\mathcal{C} := \{\zeta \in \mathcal{V} \mid r(\zeta) = \delta_1/2\}, \quad \mathcal{C}' := \mathcal{C} \cap \mathcal{A},\\
&\mathcal{D} := \{\zeta \in \mathcal{V} \mid r(\zeta) < \delta_1/2\}, \quad \mathcal{D}' := \mathcal{D} \cap \mathcal{A}\,.
\end{aligned} \tag{38}$$

The goal of this step is to show that solutions to (37) only pass through $\mathcal{D}$ for a small measure set of initial conditions. By existence, uniqueness and continuous dependence on initial conditions for solutions of ODEs, there exists a smooth map $\pi : \mathcal{A} \to \partial_0^- C$ defined by sending $\zeta \in \mathcal{A}$ to $q(0)$, where q is the unique solution to (37) such that $q(T) = \zeta$. We will prove the bound

$$|\int_{\pi(\mathcal{D}')} \widetilde{\alpha}| \leq 4c_2\,\mathcal{A}(u). \tag{39}$$

Set $\mathcal{C}'' := \mathcal{C}' \setminus \mathrm{Crit}(\pi)$. The set $\mathcal{C}''$ is open in $\mathcal{C}$ and $\mathcal{C}$ is homeomorphic to a disjoint union of circles, so $\mathcal{C}'$ is homeomorphic to a countable disjoint union of open intervals and circles, written as $\sqcup_k \mathcal{I}_k$. For each k, the map $\pi : \mathcal{I}_k \to \pi(\mathcal{I}_k) \subset \partial_0^- C$ is a diffeomorphism. We note that the one-form $\widetilde{\alpha}$ restricts to a volume form on $\mathcal{I}_k$ for each k because $\mathcal{I}_k$ is disjoint from $\mathrm{Crit}(\pi)$. The one-form $\pi^*\widetilde{\alpha}$ is also a volume form, but its induced orientation does not need to agree with $\widetilde{\alpha}$ on any given $\mathcal{I}_k$. We write $\mathcal{C}''_+ \subset \mathcal{C}''$ for the disjoint union of $\mathcal{I}_k$ for which $\widetilde{\alpha}$ and $\pi^*\widetilde{\alpha}$ induce the same orientation. The following elementary lemma has a very similar proof to [21, Lemma 4.32].

Lemma 5.15. *The following equality of sets holds:*

$$\pi(\mathrm{Crit}(\pi)) \cup \pi(\mathcal{C}''_+) = \pi(\mathcal{C}').$$

Proof. The only nontrivial part of the proof is proving the inclusion

$$\pi(\mathcal{C}') \subseteq \pi(\mathrm{Crit}(\pi)) \cup \pi(\mathcal{C}''_+).$$

Choose any point $\zeta \in \pi(\mathcal{C}')$. Let $q : [0,T] \to C$ be a gradient flow trajectory such that $q(0) = \zeta$ and $q((0,T])$ intersects $\mathcal{C}'$ at least once. If q is tangent to $\mathcal{C}'$, then $\zeta \in \pi(\mathrm{Crit}(\pi))$. If q intersects $\mathcal{C}'$ transversely, then there exists some intersection point $\zeta' \in q((0,T]) \cap \mathcal{C}'$ at which $\dot{q}$ points inwards relative to $\mathcal{D}$. Write $\mathcal{I}_k$ for the unique connected component of $\mathcal{C}''$ containing ζ'. It follows that $\mathrm{grad}_{\widetilde{\gamma}}(a \circ \widetilde{u})$ points inward with respect to $\mathcal{D}$ along $\mathcal{I}_k$, which implies that $\widetilde{\alpha}$ and $\pi^*\widetilde{\alpha}$ induce the same orientation. This shows that

$$\zeta = \pi(\zeta') \in \pi(\mathcal{I}_k) \subset \pi(\mathcal{C}''_+)$$

in this case. This concludes the proof of the lemma. □

The next elementary lemma has a very similar proof to [21, Lemma 4.33].

Lemma 5.16. *The following equality of integrals holds:*

$$\int_{\pi(\mathcal{D}')} \widetilde{\alpha} = \int_{\pi(\mathcal{C}')} \widetilde{\alpha}.$$

Proof. Any gradient flow trajectory starting at $\partial_0^- C$ and ending at a point in $\mathcal{D}'$ must intersect $\mathcal{C}'$. This implies

$$\pi(\mathcal{D}') \subseteq \pi(\mathcal{C}').$$

Any gradient flow trajectory starting at $\partial_0^- C$ and intersecting $\mathcal{C}'$ transversely must intersect $\mathcal{D}'$. This implies

$$\pi(\mathcal{C}'') \subset \pi(\mathcal{D}').$$

The set $\pi(\mathcal{C}'')$ contains $\pi(\mathcal{C}') \setminus \pi(\mathrm{Crit}(\pi))$. By Sard's theorem, the set $\pi(\mathrm{Crit}(\pi))$ has measure zero, so $\pi(\mathcal{C}'')$ has full measure in $\pi(\mathcal{C}')$. This shows that $\pi(\mathcal{D}')$ has full measure in $\pi(\mathcal{C}')$. Since $\widetilde{\alpha}$ is a smooth volume form on $\partial_0^- C$, it must have the same integral on both $\pi(\mathcal{D}')$ and $\pi(\mathcal{C}')$. □

Choose a finite collection of sets $\{\mathcal{J}_\ell\}_{\ell=1}^N$ in $\mathcal{A}$ with the following properties:

(1) Each $\mathcal{J}_\ell$ is diffeomorphic to a compact interval,
(2) $\pi(\mathcal{J}_\ell) \cap \pi(\mathcal{J}_{\ell'}) = \emptyset$ if $\ell \neq \ell'$,
(3) For each ℓ, there exists some k such that $\mathcal{I}_k \subset \mathcal{C}''_+$ and $\mathcal{J}_\ell \subset \mathcal{I}_k$.

(4) The map $\pi : \mathcal{J}_\ell \to \pi(\mathcal{J}_\ell) \subset \partial_0^- C$ is a diffeomorphism.
(5) We have the bound

$$|\sum_\ell \int_{\pi(\mathcal{J}_\ell)} \widetilde{\alpha} - \int_{\pi(\mathcal{C}_+'')} \widetilde{\alpha}| \leq c_2\, \mathcal{A}(u). \tag{40}$$

For each $\ell \in \{1, \dots, N\}$, there exists a strip of (u, f) with domain $\widetilde{C}_\ell$ such that

$$\partial_0^+ \widetilde{C}_\ell = \mathcal{J}_\ell, \quad \partial_0^- \widetilde{C}_\ell = \pi(\mathcal{J}_\ell).$$

We are now prepared to prove (39). Combining (40) with Lemmas 5.15 and 5.16, we observe that

$$|\int_{\pi(\mathcal{D}')} \widetilde{\alpha}| = \int_{\pi(\mathcal{C}_+'')} \widetilde{\alpha} \leq \sum_\ell \int_{\pi(\mathcal{J}_\ell)} \widetilde{\alpha} + c_2\, \mathcal{A}(u).$$

Use the height bound (L4) and the strip estimate (Lemma 5.4) to show that

$$\sum_\ell \int_{\pi(\mathcal{J}_\ell)} \widetilde{\alpha} \leq 2 \sum_\ell \int_{\mathcal{J}_\ell} \widetilde{\alpha} + 2c_2\, \mathcal{A}(u).$$

Next, we observe

$$\sum_\ell \int_{\mathcal{J}_\ell} \widetilde{\alpha} \leq \|\widetilde{\alpha}\|_{\widetilde{\gamma}}\, \mathrm{length}_{\widetilde{\gamma}}(\mathcal{C}).$$

Using (iii) and (v) in Step 1, we observe that $\mathrm{length}_\gamma(\mathcal{C}) \leq c_2\, \mathcal{A}(u)/2$. Since $\widetilde{\gamma} \leq 2\gamma$ for (δ, ϵ)-tame perturbations (see [21, Lemma 4.5]) we find $\mathrm{length}_{\widetilde{\gamma}}(\mathcal{C}) \leq c_2\, \mathcal{A}(u)$. We also have $\|\widetilde{\alpha}\|_{\widetilde{\gamma}} \leq 1$. Combining these inequalities proves (39).

Step 3: Observe that, by the definition of the perturbation in Step 1, the function $a \circ \widetilde{u}$ is Morse on $C \setminus \mathcal{D}$. It follows that $a \circ \widetilde{u}$ has finitely many critical points in $C \setminus \mathcal{D}$ and each one is non-degenerate. For each $k \in \{0, 1, 2\}$, write $\mathcal{M}_k$ for the set of index-k critical points of $a \circ \widetilde{u}$ in $C \setminus \mathcal{D}$. The goal of this step is to show, for each k, that the set of initial conditions in $\partial_0^- C$ whose gradient flow lines limit to a point in $\mathcal{M}_0$, $\mathcal{M}_1$, or $\mathcal{M}_2$ is small.

The set $\mathcal{M}_0$ consists of local minima of $a \circ \widetilde{u}$. Therefore, there are no gradient flow trajectories starting at $\partial_0^- C$ and limiting to a point in $\mathcal{M}_0$. The set $\mathcal{M}_1$ consists of non-degenerate saddle points. Therefore, there are finitely many gradient flow trajectories starting at $\partial_0^- C$ and limiting to a point in $\mathcal{M}_1$. The set of initial conditions of these trajectories is a finite subset $\mathcal{F} \subset \partial_0^- C$.

Write $\mathcal{E}$ for the set of points in $\partial_0^- C$ such that the gradient flow trajectory starting at $\mathcal{E}$ exists for all positive time and limits to a point in $\mathcal{M}_2$. This is an open subset of $\partial_0^- C$. We write it as a disjoint union $\mathcal{E} = \sqcup_k \mathcal{I}_k$ of open intervals and circles. For each $z \in \mathcal{M}_2$, we choose a small embedded circle $\mathcal{C}_z \subset C$, such that for any $\zeta \in \mathcal{E}$ the gradient flow trajectory starting at ζ intersects $\cup_{z \in \mathcal{M}_2} \mathcal{C}_z$ transversely and exactly once. We are free to make the circles as small as we would like, so we choose them such that

$$\sum_{z \in \mathcal{M}_2} \mathrm{length}_{\widetilde{\gamma}}(\mathcal{C}_z) \leq c_2\, \mathcal{A}(u)/2.$$

The gradient flow defines a smooth embedding

$$\tau : \mathcal{E} \hookrightarrow \bigcup_{z \in \mathcal{M}_2} \mathcal{C}_z\,.$$

Choose a finite set $\{\mathcal{K}_\ell\}_{\ell=1}^N$ of compact, pairwise disjoint intervals in $\partial_0^- C$ such that

$$\Big|\int_{\mathcal{E}} \widetilde{\alpha} - \sum_\ell \int_{\mathcal{K}_\ell} \widetilde{\alpha}\Big| \leq c_2\, \mathcal{A}(u).$$

For each $\ell \in \{1, \ldots, N\}$, the gradient flow defines a strip of (u, f) with domain $\widetilde{C}_\ell$ such that

$$\partial_0^- \widetilde{C}_\ell = \mathcal{K}_\ell, \quad \partial_0^+ \widetilde{C}_\ell = \tau(\mathcal{K}_\ell) \subset \bigcup_{z \in \mathcal{M}_2} \mathcal{C}_z\,.$$

The height bound (L4) and the strip estimate (Lemma 5.4) imply that

$$\sum_\ell \int_{\mathcal{K}_\ell} \widetilde{\alpha} \leq 2 \sum_{z \in \mathcal{M}_2} \int_{\mathcal{C}_z} \widetilde{\alpha} + 2c_2\, \mathcal{A}(u).$$

The right-hand side is bounded by

$$2\, \mathrm{length}_{\widetilde{\gamma}}\Big(\bigcup_{z \in \mathcal{M}_2} \mathcal{C}_z\Big) + 2c_2\, \mathcal{A}(u) \leq 3c_2\, \mathcal{A}(u)$$

Putting these inequalities together shows

$$\Big|\int_{\mathcal{E}} \widetilde{\alpha}\Big| \leq 4c_2\, \mathcal{A}(u). \tag{41}$$

Step 4: Define $\mathcal{T} \subset \partial_0^- C$ to be the set of points $\zeta \in \partial_0^- C$ such that there exists a solution q to (37) such that $q(0) = \zeta$ and $q(T) \in \partial_0^+ C$. This step shows that $\mathcal{T}$ has large $\widetilde{\alpha}$-measure.

It follows from (39) and (41) that

$$\int_{\partial_0^- C \setminus \mathcal{T}} \widetilde{\alpha} \leq \int_{\pi(\mathcal{D}') \cup \mathcal{E} \cup \mathcal{F}} \widetilde{\alpha} \leq 8c_2\, \mathcal{A}(u).$$

It follows that there exists a (possibly empty) finite set $\{\mathcal{T}_k\}_{k=1}^N$ of pairwise-disjoint intervals, each contained in $\mathcal{T}$, such that

$$\sum_{k=1}^N \int_{\mathcal{T}_k} \widetilde{\alpha} \geq \int_{\partial_0^- C} \widetilde{\alpha} - 10c_2\, \mathcal{A}(u). \tag{42}$$

Step 5: This step proves a technical lemma. The lemma asserts that any interval in $\partial_0^- C$ with sufficiently large $\widetilde{\alpha}$-measure contains an initial condition for a gradient flow line of $a \circ u$, terminating at $\partial_0^+ C$ and having length at most $2^7(a_1 - a_0)$. The lemma is very similar to [21, Lemma 4.34].

Lemma 5.17. *For each closed interval $\mathcal{I} \subset \partial_0^- C$ satisfying*

$$\int_{\mathcal{I}} \widetilde{\alpha} \geq ((a_1 - a_0)^{-1} + 10c_2)\, \mathcal{A}(u)$$

there exists a solution $q : [0,T] \to C$ *to the equation*

$$q'(s) = \mathrm{grad}_{\widetilde{\gamma}}(a \circ \widetilde{u})(q(s))$$

such that $q(0) \in \mathcal{I}$, $q(T) \in \partial_0^+ C$, *and*

$$\mathrm{length}_{\widetilde{\gamma}}(q([0,T])) \le c_3(a_1 - a_0).$$

Proof. Write $\mathcal{I}' := \mathcal{I} \cap \cup_{k=1}^N \mathcal{T}_k$. It follows from (42) that

$$\int_{\mathcal{I}'} \widetilde{\alpha} \ge \int_{\mathcal{I}} \widetilde{\alpha} - 10c_2\,\mathcal{A}(u) \ge (a_1 - a_0)^{-1}\,\mathcal{A}(u).$$

The strips associated to the intervals $\mathcal{T}_k \cap \mathcal{I}'$ satisfy the assumptions of Lemma 5.5. The only nontrivial assumptions to check are (iii) and (iv). Assumption (iii) follows from (L4) and Assumption (iv) follows from the inequality above. Applying Lemma 5.5 produces the desired gradient flow line. □

Step 6**:** This step completes the proof of the proposition. Choose a finite cover $\{\mathcal{I}_\ell\}_{\ell=1}^M$ of $\partial_0^- C$ satisfying the following properties:

(1) Each $\mathcal{I}_\ell$ is homeomorphic to a closed interval or a circle.
(2) $\int_{\mathcal{I}_\ell} \widetilde{\alpha} < 7\epsilon_4$.
(3) If $\mathcal{I}_\ell$ is homeomorphic to a closed interval, then $\int_{\mathcal{I}_\ell} \widetilde{\alpha} > 3\epsilon_4$.
(4) For any $\ell \neq \ell'$, the interiors of $\mathcal{I}_\ell$ and $\mathcal{I}_{\ell'}$ (relative to $\partial_0^- C$) are disjoint.

For any ℓ such that $\mathcal{I}_\ell$ is homeomorphic to an interval, we define a sub-interval $\widetilde{\mathcal{I}}_\ell \subset \mathcal{I}_\ell$ as follows. Write ζ_ℓ^- and ζ_ℓ^+ for the left and right endpoints of $\mathcal{I}_\ell$ with respect to the orientation defined by $\widetilde{\alpha}$. Fix interior points $\zeta_\ell^0, \zeta_\ell^1 \in \mathcal{I}_\ell$ such that ζ_ℓ^0 is to the left of ζ_ℓ^1 and the following holds. Write $\mathcal{I}_\ell^0$, $\mathcal{I}_\ell^1$, and $\mathcal{I}_\ell^2$ for the sub-intervals with oriented boundaries $\zeta_\ell^0 - \zeta_\ell^-$, $\zeta_\ell^1 - \zeta_\ell^0$, and $\zeta_\ell^+ - \zeta_\ell^1$, respectively. Then we require

$$\int_{\mathcal{I}_\ell^j} \widetilde{\alpha} \in (\epsilon_4, 3\epsilon_4)$$

for each $j \in \{0,1,2\}$, and set $\widetilde{\mathcal{I}}_\ell := \mathcal{I}_\ell^1$.

Note that by (L3), (L5), and the bound $c_2 \le 2^{24}\epsilon_5^{-1}$, we have the bound $((a_1 - a_0)^{-1} + 10c_2)\,\mathcal{A}(u) \le \epsilon_4$. Therefore, the interval $\widetilde{\mathcal{I}}_\ell$ satisfies the required length lower bound in Lemma 5.17. It follows from Lemma 5.17 that for each $\ell \in \{1,\dots,M\}$ such that $\mathcal{I}_\ell$ is homeomorphic to an interval, there exists a point $\zeta_\ell \in \widetilde{\mathcal{I}}_\ell$ and a gradient flow trajectory $q_\ell : [0, T_\ell] \to C$ such that $q_\ell(0) = \zeta_\ell$, $q_\ell(T_\ell) \in \partial_0^+ C$, and the length of $q_\ell(T_\ell)$ is at most $c_3(a_1 - a_0)$.

Set

$$\dot{C} := C \setminus \cup_\ell q_\ell([0, T_\ell])$$

and write $\{\dot{C}_k\}_{k=1}^N$ for the connected components of $\dot{C}$. For each $k \in \{1,\dots,N\}$, the closure $\widetilde{C}_k$ of $\dot{C}_k$ relative to C is a tract. We verify that $\{\widetilde{C}_k\}_{k=1}^N$ satisfy the properties of Proposition 5.9. Proposition 5.9(a–c) are evident from the construction. The upper bound in Proposition 5.9(d) follows from the fact that any $L \in \pi_0(\partial_0^- \widetilde{C}_k)$ is contained in the union of at most two of the sets $\mathcal{I}_\ell$. The lower bound in Proposition 5.9(d) follows from the fact that if L is not a circle, then L must contain either $\mathcal{I}_\ell^0$ or $\mathcal{I}_\ell^2$ for some ℓ. Proposition 5.9(e) follows from the fact that for each k and each component $L' \in \pi_0(\partial_1 \widetilde{C}_k)$, there exists some ℓ such that $L' = q_\ell([0, T_\ell])$, and therefore L' has length at most $c_3(a_1 - a_0)$.

□

Appendix A. Verifying monotonicity

This short appendix contains some elementary arguments verifying that Hamiltonian surface diffeomorphisms and rational area-preserving 2-torus diffeomorphisms are monotone. We start with Hamiltonian diffeomorphisms.

Lemma A.1. *Any Hamiltonian diffeomorphism of a closed, oriented surface Σ is monotone.*

Lemma A.1 follows immediately from the next lemma and the easily verified fact that the identity map is monotone.

Lemma A.2. *Let ϕ and ϕ' be a pair of Hamiltonian isotopic area-preserving diffeomorphisms of a closed, oriented surface Σ equipped with an area form ω. Then ϕ' is monotone if and only ϕ is.*

Proof. Choose a Hamiltonian function $H : \mathbb{R}/\mathbb{Z} \times \Sigma \to \mathbb{R}$ whose time-one Hamiltonian flow is $\phi^{-1}\phi'$. Write $\{\psi^t\}_{t\in\mathbb{R}}$ for the Hamiltonian flow of H. We use this choice to identify the mapping torii of ϕ and ϕ':

$$\begin{gathered} f_H : Y_\phi \simeq Y_{\phi'} \\ (t, p) \mapsto (t, (\psi^t)^{-1}(p)). \end{gathered}$$

We compute

$$f_H^* c_1(V_{\phi'}) = c_1(V_\phi), \quad f_H^*[\omega_{\phi'}] = [f_H^*\omega_{\phi'}] = [\omega_\phi + dH \wedge dt] = [\omega_\phi].$$

It follows from this computation that ϕ' is monotone if and only if ϕ is. □

Next, we show that any rational area-preserving torus diffeomorphism is monotone.

Lemma A.3. *Write $\mathbb{T}^2 := (\mathbb{R}/\mathbb{Z})^2$ and let $\omega := dx \wedge dy$ denote the standard area form. Any area-preserving diffeomorphism $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ has $c_1(V_\phi) = 0$. Therefore, if ϕ is rational, then it is monotone.*

Proof. The proof of the lemma will take 3 steps. Fix any area-preserving diffeomorphism $\phi : \mathbb{T}^2 \to \mathbb{T}^2$.

Step 1: For any matrix $B \in \mathrm{SL}(2,\mathbb{Z})$, write ϕ_B for the map $w \mapsto Bw$. This step recalls a well-known fact about the symplectic mapping class group of $\mathbb{T}^2$. Identify $H_1(\mathbb{T}^2;\mathbb{Z})$ with $\mathbb{Z}^2$ using the simple closed curves $\{x = 0\}$, $\{y = 0\}$ as a basis. Then the action of ϕ on $H_1(\mathbb{T}^2;\mathbb{Z})$ is identified with a matrix $A \in \mathrm{SL}(2,\mathbb{Z})$ and ϕ is symplectically isotopic to ϕ_A.

Step 2: Write $r(A) := \mathrm{rank}(\ker(A - \mathrm{Id}))$. This is an integer between 0 and 2, inclusive, and the remaining steps prove the corollary case-by-case depending on the value of $r(A)$. This step proves that $c_1(V_\phi) = 0$ when $r(A) = 0$. Since ϕ acts by the matrix A on the first homology group, it follows from the Mayer–Vietoris sequence that $b_2(Y_\phi) =$

$1 + r(A) = 1$ and that the second homology group of Y_ϕ is generated by a torus fiber. The class $c_1(V_\phi)$ has zero pairing with a torus fiber, so $c_1(V_\phi) = 0$.

Step 3: This step proves the corollary when $r(A) \geq 1$. The mapping torus Y_ϕ is homeomorphic as a fibered 3-manifold to the mapping torus $Y_A := Y_{\phi_A}$. Since $r(A) \geq 1$, the matrix A fixes some nonzero vector $v \in \mathbb{R}^2$. Therefore, the differential of ϕ_A fixes the constant vector field v on $\mathbb{T}^2$. This vector field defines a non-vanishing section of the vertical tangent bundle $V_A := V_{\phi_A}$. We conclude that $c_1(V_\phi) = c_1(V_A) = 0$. □

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