# Spheres On Spheres: Adams Operations and the Hopf Invariant

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## 1 Introduction

The Hopf fibration $S^3 \to S^2$ is well-known as one of the first classical examples of a nontrivial fiber bundle.

Remarkably, not only are the total and base spaces spheres, but the fiber of each point is homeomorphic to the sphere $S^1$.

A natural question to ask upon observing this is whether this is the only such fibration consisting entirely of spheres. Namely, for which triples of integers $(l, m, n)$ do we have a fibration of spheres $S^m \to S^n$ with fiber $S^l$?

While this seems to be a rather superficial question, the answer is quite interesting. There are exactly *four* such fibrations.

**Theorem 1.1.** *The only fibrations between spheres are:*

- $S^1 \to S^1$ *with fiber* $S^0$
- $S^3 \to S^2$ *with fiber* $S^1$ *(Hopf fibration)*
- $S^7 \to S^4$ *with fiber* $S^3$
- $S^{15} \to S^8$ *with fiber* $S^7$

**Proposition 1.2.** *These fibrations exist.*

*Proof.* Constructing these is actually quite simple. Let $\mathbb{A}$ be equal to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (the quaternions), or $\mathbb{O}$ (the octonions).

Note that $\mathbb{A}$ is a real division algebra. We can construct the projective line $\mathbb{AP}^1$ over $\mathbb{A}$ by taking the quotient of $\mathbb{A}^2 - \{0\}$ by the equivalence relation $(a_0, a_1) \sim (\lambda a_0, \lambda a_1)$ for any $\lambda \in \mathbb{A}$. Furthermore, $\mathbb{AP}^1$ inherits the topological structure given by viewing $\mathbb{A}$ as a real vector space.

Letting $\dim_{\mathbb{R}} \mathbb{A} = k$, we have that $\mathbb{AP}^1$ is homeomorphic to the sphere $S^k$. This is clear for the real and complex cases and is not difficult to show for the others.

Next, observe that we can canonically embed $S^{2k-1}$ into $\mathbb{A}^2$ as the set of vectors of unit norm.

Our fibration is then given by the restriction of the quotient map $\mathbb{A}^2 - \{0\} \to \mathbb{AP}^1$. The fiber of a point $(a_0, a_1) \in \mathbb{AP}^1$ is the set of vectors of unit norm in the line $\{(\lambda a_0, \lambda a_1) \mid \lambda \in \mathbb{A}\}$. By definition, this is a copy of $S^{k-1}$.

Our four algebras have $k = 1, 2, 4, 8$, giving us our four fibrations between spheres. ∎

The more difficult part, of course, is proving that no other fibrations exist. We can turn to the most basic tool of homotopy theory, the long exact sequence of a fibration, to establish a preliminary result.

**Proposition 1.3.** *Any fibration $S^m \to S^n$ with fiber $S^l$ must satisfy $l = n - 1$.*

*Proof.* Apply the long exact sequence of a fibration. The sequence

$$\pi_n(S^m) \to \pi_n(S^n) \to \pi_{n-1}(S^l) \to \pi_{n-1}(S^m)$$

is exact.

The leftmost and rightmost groups vanish since $m > n$. As a result, we find $\pi_{n-1}(S^l) \simeq \pi_n(S^n) = \mathbb{Z}$, so we must have $l \leq n - 1$.

Furthermore, we have for any $k > 1$ that

$$\pi_{n-k+1}(S^n) \to \pi_{n-k}(S^l) \to \pi_{n-k}(S^m)$$

is exact, so it follows that $\pi_{n-k}(S^l) = 0$ for all $k > 1$.

From this, we immediately find that $l = n - 1$. ∎

Determining which triples form fibrations from this set onwards is significantly more difficult and follows from a much deeper theorem known as Adam's theorem or the Hopf Invariant One Theorem. While there are a couple of methods of proving Adam's theorem, we will discuss Atiyah's beautiful K-theoretic proof in this paper.

Before diving straight into the proof, we spend a couple of sections developing the necessary theory. In Section 2, we will construct and prove some basic properties of the

Adams operations on K-theory. Then, in Section 3, we will construct the Atiyah-Hirzebruch spectral sequence for generalized cohomology theories.

Following this, we are ready to present the main result. In Section 4, we introduce the Hopf invariant and prove Adam's theorem.

## 2 Adams Operations

Complex K-theory, like other cohomology theories, is functorial. First, it sends a topological space $X$ to the ring $K(X)$. Next, given any continuous map $f : X \to Y$, there is an induced pullback map $f^* : K(Y) \to K(X)$. Therefore, K-theory satisfies the conditions to be a **functor** from the category of topological spaces to the category of rings.

The Adams operations on complex K-theory are a special case of a **cohomology operation**, which are simply natural transformations from the K-theory functor to itself.

We first define these operations in the following existence theorem.

**Theorem 2.1.** *There exist cohomology operations called **Adams operations** $\psi^k$ that satisfy $\psi^k(L_1 \oplus \cdots \oplus L_n) = L_1^k \oplus \cdots \oplus L_n^k$ for line bundles $L_1, L_2, \cdots, L_n$.*

Before we can construct the Adams operations, we must make use of a quick lemma, which follows by simple algebra.

**Lemma 2.2.** *Let $\sigma_k(x_1, x_2, \cdots, x_n)$ be the $k$th symmetric sum of the variables $x_i$.*

*Then there exists a unique polynomial $s_k$ such that $s_k(\sigma_1(x_1, \cdots, x_n), \cdots, \sigma_n(x_1, \cdots, x_n)) = x_1^k + \cdots + x_n^k$.*

Now we can prove our theorem.

*Proof.* For a line bundle $L$, we have $\Lambda^k(L) = L$ if $k = 1$ and 0 if $k > 1$. Note that the exterior power $\Lambda^k$ of $L_1 \oplus \cdots \oplus L_n$ is equal to

$$\oplus_{\sum e_i = k} \Lambda^{e_1}(L_1) \otimes \Lambda^{e_2}(L_2) \otimes \cdots \otimes \Lambda^{e_n}(L_n)$$

If any of the $e_i$ are greater than 1, then that term in the direct sequence cancels out. We are therefore left with the $k$th symmetric sum of the $L_i$. As a result, we can immediately apply our lemma and define

$$\psi^k(L_1 \oplus \cdots \oplus L_n) = s_k(\Lambda^1(L_1 \oplus \cdot \oplus L_n), \cdots, \Lambda^n(L_1 \oplus \cdot \oplus L_n))$$

We can then extend this to general $E$ by setting $\psi^k(E) = s_k(\Lambda^1(E), \cdots, \Lambda^n(E))$. ■

Next, we prove some properties of these Adams operations.

**Lemma 2.3.** *The Adams operations satisfy the following identities:*

- $\psi^k \psi^l = \psi^{kl}$
- $\psi^p(E) = E^p \mod p$ *for prime* $p$
- $\psi^k : \widetilde{K}(S^{2n}) \to \widetilde{K}(S^{2n})$ *is multiplication by* $k^n$.

*Proof.* The first identity is an immediate consequence of the splitting principle.

To prove the next identity, we again apply the splitting principle and notice that $(L_1 \oplus \cdots \oplus L_n)^p$ is equal to $\oplus L_i^p$ plus $p$ times several other terms. Therefore, taking this mod $p$ gives us $\oplus L_i^p$ as desired.

For the final identity, first consider $n = 1$. We have that the Bott generator is $b_2 = [H] - 1$, where $H$ is the hyperplane bundle. Then, applying $\psi^k$ gives us $[H]^k - 1 = (1 + b_2)^k - 1$. Since $b_2^i = 0$ for any $i > 1$, we have this is equal to $1 + kb_2 - 1 = kb_2$.

Next, we induct using the external tensor product. We have the Bott generator $b_{2n}$ is equal to $b_{2n-2} \otimes b_2$. Since $\psi^k$ is a ring homomorphism, we find that $\psi^k(b_{2n}) = \psi^k(b_{2n-2}) \otimes \psi^k(b_2) = k^{n-1} b_{2n-2} \otimes kb_2 = k^n b_{2n}$. ■

# 3 The Atiyah-Hirzebruch Spectral Sequence

This section assumes familiarity with the basic definitions regarding spectral sequences. Interested readers can find the necessary material in [Beh].

A nice property of K-theory is that it satisfies the conditions to be a *generalized cohomology theory.*

**Definition 3.1.** A **generalized cohomology theory** is a set of functors $\{h^i\}_{i \in \mathbb{Z}}$ from the category of CW-pairs to the category of abelian groups with natural transformations (or "boundary maps") $\delta^i : h^i \to h^{i+1}$ that satisfy the following axioms:

- Homotopy: $h^i(f) = h^i(g)$ for any two homotopic maps $f, g : (X, A) \to (Y, B)$

- Long Exact Sequence: The inclusions $i : (A, \emptyset) \to (X, \emptyset)$ and $j : (X, \emptyset) \to (X, A)$ induce the long exact sequence

$$\cdots \to h^i(X, A) \xrightarrow{h^i(j)} h^i(X, \emptyset) \xrightarrow{h^i(i)} h^i(A, \emptyset) \xrightarrow{\delta} h^{i+1}(X, A) \to \cdots$$

- Excision: If $X = A \cup B$ is a CW-complex with subcomplexes $A$ and $B$, then the inclusion $i : (A, A \cap B) \to (X, B)$ induces an isomorphism $h^i(i) : h^i(X, B) \to h^i(A, A \cap B)$.

- Additivity: If $(X, A) = \cup_\alpha (X_\alpha, A_\alpha)$ then the inclusions induce an isomorphism $h^i(X, A) \to \prod_\alpha h^i(X_\alpha, A_\alpha)$.

Note that these are exactly the Eilenberg-Steenrod axioms for regular cohomology, minus the "dimension axiom" which states that the cohomology of a point is 0. Due to the lack of this dimension axiom, generalized cohomology theories are more flexible and can often give more information about a topological space. However, they are often much more difficult to calculate.

This is where the Atiyah-Hirzebruch spectral sequence comes in. We use the shorthand $h^i(X)$ for $h^I(X, \emptyset)$.

**Theorem 3.2.** *Given a generalized cohomology theory $h$ and singular cohomology $H$, the* ***Atiyah-Hirzebruch spectral sequence*** *is a spectral sequence with $E_2$ page satisfying $E_2^{p,q} = H^p(X, h^q(*))$ abutting to $h^{p+q}(X)$.*

The Atiyah-Hirzebruch spectral sequence can be constructed using the method of exact couples. We give the outline of the construction below.

*Proof.* Given a CW-complex $X$, we have a natural filtration

$$X^{(0)} \hookrightarrow X^{(1)} \hookrightarrow X^{(2)} \hookrightarrow \cdots$$

where $X^{(p)}$ is the $p$-skeleton of $X$.

For every $p$, we have the long exact sequence of a pair

$$\cdots \to h^i(X^{(p)}, X^{(p-1)}) \xrightarrow{h^i(j)} h^i(X^{(p)})) \xrightarrow{h^i(i)} h^i(X^{(p-1)}) \xrightarrow{\delta} h^{i+1}(X^{(p)}, X^{(p-1)}) \to \cdots$$

We can put these all together to create an exact couple

$$A \xrightarrow{i_*} A, A \xrightarrow{\delta} E, E \xrightarrow{j_*} A$$

where $A = \oplus_{p,q} h^{p+q}(X^{(p)})$ and $E = \oplus_{p,q} h^{p+q}(X^{(p)}, X^{(p-1)})$ are bigraded over $p, q$.

This yields a derived exact couple

$$A' \xrightarrow{i'} A', A' \xrightarrow{\delta'} E', E' \xrightarrow{j'} A'$$

where $E' = \ker(\delta j_*)/\operatorname{im}(\delta j_*)$ and $A' = i_*(A)$. The map $i'$ is defined as the restriction of $i_*$ to $A'$ and $j'$ is induced by $j$ canonically. To define $\delta'$, pick $a' \in A'$ and $a$ such that $i_*(a) = a'$. Then we have that $\delta(a)$ is in the kernel of $\delta j_*$, so we can define $\delta'(a')$ to be the image of $\delta_*(a)$ in $E'$.

This is an exact couple itself, so we can continue to take successive derived couples. It is then immediate that the sequence $E, E', E'', \cdots$ forms the pages of a spectral sequence.

Diagram chasing through the differentials which are defined as the composition of $\delta j$ at every exact couple, we see that they map $E_r^{p,q}$ to $E_r^{p+r,q-r+1}$.

Therefore, eventually we can see that these differentials stabilize since $E_r^{p,q} = 0$ for any $p, q < 0$. We call the stabilized groups $E_\infty^{p,q}$ We omit the proof, which can be found in [Adams], that $E_\infty^{p,q} = h^{p+q}(X)$.

Going back to our original $E$, we have that

$$\begin{aligned}
E^{p,q} &= h^{p+q}(X^{(p)}, X^{(p-1)}) \\
&= \widetilde{h}^{p+q}(X^{(p)}/X^{(p-1)}) \\
&= \widetilde{h}^{p+q}(\bigvee S^p) \\
&= \prod \widetilde{h}^{p+q}(S^p) \\
&= \prod h^q(*) \\
&= C^p_{cell}(X, h^q(*))
\end{aligned}$$

The identity $\widetilde{h}^{p+q}(S^p) = h^q(*)$ is immediate from applying excision to the pair $(D^p, S^{p-1})$ to get that $\widetilde{h}^{p+q}(S^p) = \widetilde{h}^{p+q-1}(S^{p-1})$ and the inducting.

It follows immediately that the $E_2$ page is equal to $H^p(X, h^q(*))$ as desired. ■

## 4 Adams' Theorem

Finally, we arrive at the proof of our main theorem. Before we proceed, we introduce the Hopf invariant.

**Definition 4.1.** Let $f$ be a map $S^{2n-1} \to S^n$. Using this, we can form a CW complex $C(f)$ given by attaching a $2n$-cell to $S^n$ via $f$. By cellular cohomology, it is immediate that $H^i(C(f)) = \mathbb{Z}$ for $i = 0, n, 2n$ and 0 elsewhere.

We define the **Hopf invariant** of $f$ to be the image of the generator of $H^n(C(f))$ under the cup square $H^n(C(f)) \to H^{2n}(C(f))$.

**Lemma 4.2.** *The Hopf invariant is always* 0 *for odd* $n$.

*Proof.* This is immediate by graded commutativity of the cup product. If $\alpha$ generates $H^n(C(f))$, then $\alpha \smile \alpha = (-1)^{n^2}(\alpha \smile \alpha) = -(\alpha \smile \alpha)$. Therefore, it must be equal to 0. ■

Consider our problem of classifying fibrations $S^m \to S^n$. We know that the fiber must be $S^{n-1}$ from our derivation with the homotopy long exact sequence. From here, it is easy to deduce that $m = 2n - 1$ by dimensional considerations.

To go further, we must use the following lemma, which is proved in [Price].

**Lemma 4.3.** *The map* $f$ *can be a fibration for* $n > 1$ *with fiber* $S^{n-1}$ *only if it has Hopf invariant one.*

*Proof.* Consider the mapping cylinder $M_f$ of $f$, which by definition is a disk bundle with fiber $D^n$ over $S^n$.

We have a fiber bundle of pairs $(D^n, S^{n-1}) \to (M_f, S^{2n-1}) \to (S^n, S^n)$.

Note that the space $C(f)$ is just $M_f/S^{2n-1}$.

By the Leray-Hirsch theorem, the Thom class $\tau \in H^n(M_f, S^{2n-1})$ is a generator of $H^n(C(f))$. However, if $\alpha$ is the generator of $H^n(S^n)$, then $f^*\alpha$ also generates $H^n(C(f))$. It follows immediately that $\tau = \pm f^*\alpha$.

By the Thom isomorphism theorem, $H^n(S^n) \simeq H^{2n}(M_f, S^{2n-1})$ by taking the cup product with $\tau$. It follows that the generator of $H^{2n}(M_f, S^{2n-1})$ is $f^*\alpha \smile \tau$, so it generates $H^{2n}(C(f))$ as well. This is equal to $\pm(f^*\alpha)^2$, so the Hopf invariant is 1. ■

Therefore, it now suffices to establish when a map can have Hopf invariant one, which we prove in the style of [Sia].

**Theorem 4.4.** *(Adams Theorem) Any map $f : S^{2n-1} \to S^n$ for $n > 1$ has Hopf invariant one iff $n = 2, 4, 8$.*

*Proof.* First, we apply the Atiyah-Hirzebruch spectral sequence to calculate $K(C(f))$. The K-theory of a point is just $\mathbb{Z}$, so the $E_2$ page is given by $E_2^{p,q} = \mathbb{Z}$ for $p = 0, n, 2n$.

The differentials are 0 up to the $E_n$ page, where we have maps $E_n^{p,q} \to E_n^{p+n,q-n+1}$. Unrolling the definition of the Atiyah-Hirzebruch sequence leads us to see that these differentials are 0 as well, so the sequence collapses and the $K$-theory of $C(f)$ is exactly the ordinary cohomology ring.

As a result, we have $\widetilde{K}(C(f))$ is free with two generators in degrees $n$ and $2n$.

We pick the generators $\alpha_n$, $\alpha_{2n}$ such that $\alpha_n$ restricts to a generator of $\widetilde{K}(S^n)$ and $\alpha_{2n}$ is the pullback of a generator of $\widetilde{K}(S^{2n})$ under the map $C(f) \to C(f)/S^n = S^{2n}$.

Recall that $\psi^k$ is multiplication by $k^n$ on $\widetilde{K}(S^{2n})$. Since the Hopf invariant is 0 when $n$ is odd, we can restrict to when $n$ is even. As a result, we find that by construction of our generators

$$\psi^2(\alpha_n) = 2^{n/2}\alpha_n + \mu\alpha_{2n}$$

for some integer $\mu$. We also have however that $\psi^2(\alpha_n) = \alpha_n^2 = H(f)\alpha_{2n}$ modulo 2, so it follows that $\mu$ mod 2 is the Hopf invariant mod 2. Now it suffices to find when $\mu$ is equal to 1.

We have $\psi^k(\alpha_n) = k^{n/2}\alpha_n + \mu_k\alpha_{2n}$. We have $\psi^k\psi^2 = \psi^{k+2} = \psi^2\psi^k$.

Evaluating both of these on $\alpha_n$, we get

$$\psi^k\psi^2(\alpha_n) = k^{n/2}(2^{n/2}\alpha_n + \mu\alpha_{2n}) + \mu_k\psi^k(\alpha_{2n}) = k^{n/2}(2^{n/2}\alpha_n + \mu\alpha_{2n}) + 2^n\mu_k\alpha_{2n}$$

and

$$\psi^2\psi^k(\alpha_n) = 2^{n/2}(k^{n/2}\alpha_n + \mu_k\alpha_{2n}) + k^n\mu\alpha_{2n}$$

Therefore, we can compare coefficients to get

$$2^{n/2}\mu_k + k^n\mu = k^{n/2}\mu + 2^n\mu_k$$

which implies

$$k^{n/2}(k^{n/2} - 1)\mu = 2^{n/2}(2^{n/2} - 1)\mu_k$$

If $\mu$ is odd, then we must have $2^{n/2}$ divides $k^{n/2} - 1$ for any odd $k$. If $n > 2$, then $k^{n/2} - 1$ is only equal to 0 modulo 4 if $n/2$ is even.

Then, we can take $k = 1 + 2^{n/4}$ to get that $2^{n/2}$ divides $(1 + 2^{n/4})^{n/2} - 1$, which implies that $2^{n/2}$ divides $\frac{n}{2} 2^{n/4}$, which is equivalent to $2^{n/4}$ divides $n/2$.

We have $2^{2.5} > 5$ for, so we can only consider $1 < n/2 < 4$. A quick check tells us that $n = 4, 8$ are the only possibilities for $n > 2$.

We also have $n = 2$ clearly works, which concludes our proof. ■

## 5 Bibliography


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