The Grothendieck Spectral Sequence

For this lecture, we're going to discuss the Grothendieck spectral sequence for composition of derived functors. It is a bit more purely categorical than what we've done so far, but nevertheless has useful applications in algebraic geometry.

My main reference for this talk is the expository paper "Serre Duality and Applications" by Jun Hou Fung. You can supposedly also look at Grothendieck's famous Tohoku paper, but I can't read French and I didn't find the English translation to be that great of a read.

1 Derived Functors

First, let's quickly define derived functors. If you recall, an abelian category is one that is essentially the category of modules over a commutative ring.

In particular, a phenomenon that pops up with functors between abelian categories is they have different ways in which they interact with *exact sequences*. We can illustrate this by example.

Fix

$$0 \to M'' \to M \to M' \to 0$$

to be an exact sequence of modules over some commutative ring R.

If we pick some other module N, and take the direct sum, it is easy to verify that

$$0 \to M'' \oplus N \to M \oplus N \to M' \oplus N \to 0$$

is also exact.

On the other hand, if we instead take the *tensor product* with N, then we only get

$$M'' \otimes_R N \to M \otimes_R N \to M' \otimes_R N \to 0$$

is exact. One way to see this is to think about what a tensor product does to the kernel of a map of modules.

Finally, if we plug our exact sequence into the *Hom* functor, then

$$0 \to \mathrm{Hom}(N, M'') \to \mathrm{Hom}(N, M) \to \mathrm{Hom}(N, M')$$

is exact. You can verify this to yourself, another shortcut is the fact that the Hom and tensor product functors form an adjoint pair.

We say the first example is an *exact* functor. The latter two are *right exact* and *left exact* respectively.

Derived functors (from the classical viewpoint, I guess) provide a way to take the partial exact sequences given by left/right exact functors and extend them to *long exact sequences.*

We're just going to define and work with right derived functors, but flipping arrows gives you all the definitions for left derived functors.

Definition 1.1. *An object I in a category $\mathbf{C}$ is* injective *if it satisfies the following mapping property. For any objects $A, B \in \mathbf{C}$ along with a morphism $\alpha : A \to I$, and a monomorphism $\beta : A \to B$, there exists a morphism $f : B \to I$ extending α, i.e. $f\beta = \alpha$.*

Now let $F : \mathbf{C} \to \mathbf{D}$ be a left-exact functor between abelian categories.

We will fix a technical condition, namely that our category $\mathbf{C}$ *has enough injectives.* This means that any object $A \in \mathbf{C}$ has a monomorphism into an injective object.

Using this property, we may construct for any $A \in \mathbf{C}$ a *injective resolution*, which is an exact sequence of the form

$$0 \to A \to I^0 \to I^1 \to \dots$$

with all the I^* injective.

The sequence $F(I^*)$ in $\mathbf{D}$ is a chain complex. We define the *right-derived functor* $R^iF(A) = H^i(F(I^*))$. To show this is appropriately defined, we need to that this is both actually functorial and independent of the choice of injective resolution for every object in $\mathbf{C}$, but this is a good exercise.

We can make a couple of observations immediately.

Since F is left-exact, the sequence

$$0 \to F(A) \to F(I^0) \to F(I^1)$$

is exact. The zeroth homology of the chain complex $F(I^*)$ is just the kernel of the map $F(I^0) \to F(I^1)$, which is exactly $F(A)$. Therefore, $R^0F = F$.

Second, if F were actually exact and not just left-exact, then the chain complex $F(I^*)$ would be exact at everywhere except for the zeroth term. Therefore, the higher homology would vanish and we would have $R^iF(A) = 0$ for $i > 0$.

I also promised a long-exact sequence. This is the following theorem.

Theorem 1.2. *For any long exact sequence* $0 \to A \to B \to C \to 0$ *in* $\mathbf{C}$*, there is a long exact sequence*

$$\begin{gathered} 0 \to F(A) \to F(B) \to F(C) \to \\ \to R^1F(A) \to R^1F(B) \to R^1F(C) \to \ldots. \end{gathered}$$

The idea of the proof is that the exact sequence induces an exact sequence of injective resolutions. Applying the functor F to these injective resolutions gives us a left-exact sequence of chain complexes. Then we can apply the snake lemma.

As is the issue with many of these complicated homological algebra devices, it's often unclear how to actually calculate any of them.

A common way to calculate derived functors is to use *acyclic resolutions.*

Lemma 1.3. *An object* $C \in \mathbf{C}$ *is called* F-acyclic *if* $R^iF(C) = 0$ *for any* $i > 0$*. If*

$A \in \mathbf{C}$ and there is an exact sequence

$$0 \to A \to C^0 \to C^1 \to \dots$$

where all the C^i are F-acyclic, then this is an acyclic resolution *of A. If C^* is such an acyclic resolution, then for any $i \geq 0$, $R^iF(A) = H^i(F(C^*))$.*

Proof. Let K^0 be the cokernel of the map $A \to C^0$, and let K^i be the cokernel of the map $C^{i-1} \to C^i$ for $i \geq 1$.

There are exact sequences

$$0 \to A \to C^0 \to K^0 \to 0$$

and

$$0 \to K^{i-1} \to C^i \to K^i \to 0.$$

Now we apply the long exact sequence of derived functors. Since the C^i are F-acyclic, the first sequence tells us that $R^1F(A)$ is the cokernel of the map $F(C^0) \to F(K^0)$ and $R^iF(A) = R^{i-1}F(K^0)$ for all $i > 1$.

Similarly, the other sequences tell us for $j \geq 1$ that $R^1F(K^{j-1})$ is the cokernel of the map $F(C^j) \to F(K^j)$, and $R^iF(K^{j-1}) = R^{i-1}F(K^j)$ for $i > 1$.

Chaining these together, we get for $i > 1$ that $R^iF(A) = R^1F(K^{i-2})$.

Now it suffices to calculate $R^1F(K^i)$ for every $i \geq 0$. Note that there is a unique map $F(K^{i+1}) \to F(C^{i+2})$ that factors through the projection $F(C^{i+1}) \to F(K^{i+1})$.

Therefore, we have the diagram

$$\begin{tikzcd}
 & F(C^{i+1}) \arrow[r, equal] \arrow[d] & F(C^{i+1}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F(K^{i+1}) \arrow[r] & F(C^{i+2}) \arrow[r] & F(C^{i+3}) &
\end{tikzcd}$$

Recall that $R^1F(K^i)$ is the cokernel of the left vertical map. Applying the snake lemma, we get the exact sequence

$$0 \to R^1F(K^i) \to \operatorname{coker}(F(C^{i+1}) \to F(C^{i+2})) \to F(C^{i+3}).$$

This immediately tells us $R^1F(K^i) \simeq H^{i+2}(F(C^*))$. The same thing works for calculating $R^1F(A) \simeq H^1(F(C^*))$. ■

2 The Grothendieck Spectral Sequence

Now we are ready to construct the Grothendieck spectral sequence. Say we have two left-exact functors $G : \mathbf{C} \to \mathbf{D}$, $F : \mathbf{D} \to \mathbf{E}$.

Theorem 2.1. *If G maps injective objects to F-acyclic objects, then for any object $A \in \mathbf{C}$ there is a spectral sequence starting on the E^2 page given by*

$$E_2^{p,q} = (R^qF)(R^pG(A)) \Rightarrow R^{p+q}(F \circ G)(A).$$

For a general complex C^* with an increasing filtration $F_0C^* \subseteq F_1C^* \subseteq F_2C^* \ldots$, set $\operatorname{Gr}_p C^* = F_pC^*/F_{p-1}C^*$. Then, we have the spectral sequence of a filtration:

$$E_0^{p,q} = \operatorname{Gr}_p C^{p+q} \Rightarrow H^{p+q}(C^*).$$

The Grothendieck spectral sequence is a special case of this, arising as the spectral sequence of a *double complex.* The double complex we will use in question is called a Cartan-Eilenberg resolution.

Definition 2.2. *Let A^* be a chain complex of objects in an abelian category $\mathbf{C}$. A* Cartan-Eilenberg resolution *is a bigraded object $I^{*,*}$ consisting of injective objects and differentials $\partial_{hor}, \partial_{vert} : I^{*,*} \to I^{*,*}$ such that:*

- *$\partial_{hor}, \partial_{vert}$ have bidegrees $(1, 0)$ and $(0, 1)$ respectively.*
- *$\partial_{hor}\partial_{vert} = \partial_{vert}\partial_{hor}$.*

- *The columns* $(I^{p,*}, \partial_{vert})$ *are injective resolutions of* A^p*.*
- *The induced complexes* $B(I^{p,*})$*,* $H(I^{p,*})$ *induced by* ∂_{hor} *are injective resolutions of* $B(A^p)$*,* $H(A^p)$*.*

Lemma 2.3. *If* $\mathbf{C}$ *has enough injectives, a Cartan-Eilenberg resolution always exists.*

Proof. We have two exact sequences $0 \to B(A^p) \to Z(A^p) \to H(A^p) \to 0$ and $0 \to Z(A^p) \to A^p \to B(A^{p+1}) \to 0$ for every p.

Using the "horseshoe lemma", we can pick injective resolutions for $B(A^p), H(A^p)$ and add them to get an injective resolution for $Z(A^p)$.

Then, we can do the same thing for the second exact sequence to get an injective resolution for A^p. The map between injective resolutions of A^p and A^{p+1} is induced by the composition $A^p \to B(A^{p+1}) \hookrightarrow Z(A^{p+1}) \hookrightarrow A^{p+1}$. ■

To construct the Grothendieck spectral sequence, take functors F, G satisfying the hypotheses of our theorem.

Then for any object A, pick an injective resolution I^*. Then, we can take the complex $G(I^*)$ and pick a Cartan-Eilenberg resolution, which by abuse of notation we write as $I^{*,*}$.

Then, $G(I^{*,*})$ is a double complex. There is an associated *total complex* given by $C^k_{tot} = \oplus_{p+q=k} I^{p,q}$ with differential $d : C^k_{tot} \to C^{k+1}_{tot}$ given by the map $\partial_{hor} + (-1)^p \partial_{vert} : I^{p,q} \to I^{p+1,q} \oplus I^{q,p+1}$.

There are two ways to filter C_{tot}. The first way, the "column filtration" is to take $F^p C^k_{tot} = \oplus_{i+j=k, i \leq p} I^{i,j}$.

Plugging this into the spectral sequence of the filtered complex, we get

$$I^{p,q} \Rightarrow H^{p+q}(C^*_{tot}).$$

However, if instead we filter by *rows*, say by setting $F^q C^k_{tot} = \oplus_{i+j=k, j \leq q} I^{i,j}$ then we get a spectral sequence

$$I^{q,p} \Rightarrow H^{p+q}(C^*_{tot}).$$

This is the exciting part of the spectral sequence of a double complex. We get two distinct spectral sequences that we know converge to the same thing. Therefore, we can play them off of each other and use one to gain information about the other.

Going back to the Grothendieck spectral sequence, let's see how these two spectral sequences work.

First let's consider the column filtration. Recall the columns $I^{p,*}$ are injective resolutions of $G(I^p)$. Therefore, calculating their homology gives us exactly $E_1^{p,q} = (R^qF)(G(I^*))$.

The horizontal differential is much more mysterious though. This is where our technical assumption comes into play. Since G maps injectives to F-acyclics, we actually have $E_1^{p,q} = 0$ for $q \neq 0$.

Therefore, all the nonzero objects in the second page lie on the zeroth row as well, and the spectral sequence collapses after that. The spectral sequence abuts to the top-graded part of the $p+q$th homology of the total complex. By definition of the total complex, this is $H^{p+q}(FG(I^*)) = R^{p+q}(FG)(A)$.

Next, we will do the row filtration. We get $E_1^{p,q} = H^p_{hor}(F(I^{*,q}))$.

Recall the Cartan-Eilenberg resolution has induced resolutions on the cycles, boundaries and homology of $G(I^*)$. Write these as $I_B^{*,*}, I_Z^{*,*}, I_H^{*,*}$.

There is an exact sequences induced by the horizontal differential:

$$0 \to I_B^{*,*} \to I_Z^{*,*} \to I_H^{*,*} \to 0.$$

Since these objects are all injective, the short exact sequence splits and the sequence

$$0 \to F(I_B^{*,*}) \to F(I_Z^{*,*}) \to F(I_H^{*,*}) \to 0$$

is exact as well.

Therefore, we conclude $H^p_{hor}(F(I^{*,q})) = F(H^p_{hor}(I^{*,q}))$. Recall $H^p_{hor}(I^{*,q})$ is an injective resolution of $H^p(G(I^*)) = R^pG(A)$.

Taking the vertical differential then tells us that $E_2^{p,q} = (R^qF)(R^pG)(A)$. By our computation for the convergence of the column-filtered spectral sequence, we know that this converges to $R^{p+q}(FG)(A)$ as desired.

3 An Application: Constructing Dualizing Sheaves

You can find some immediate consequences of the Grothendieck spectral sequence. such as the Leray spectral sequence or the local-to-global Ext spectral sequence.

Recall the definition of sheaf cohomology. We let $\mathrm{Coh}(X)$ denote the abelian category of coherent sheaves over a scheme X.

Definition 3.1. *The functor* $\Gamma : \mathrm{Coh}(X) \to O_X(X) - \mathrm{Mod}$ *given by taking global sections is left-exact. For any* $F \in \mathrm{Coh}(X)$*, the* sheaf cohomology $H^i(X, F)$ *is the right-derived functor* $R^i\Gamma(F)$*.*

Theorem 3.2. *(Leray Spectral Sequence) Let* $f : X \to Y$ *be a morphism of schemes. The pushforward* f_* *of coherent sheaves on* X *to coherent sheaves on* Y *is left-exact. For any coherent sheaf* F *on* X*, there is a spectral sequence*

$$E_2^{p,q} = H^p(Y, R^q f_* F) \Rightarrow H^{p+q}(X, F).$$

This is a direct consequence (modulo technical conditions) of the Grothendieck spectral sequence applied to the composition of f_* and the global sections functor Γ.

Theorem 3.3. *(Local-to-global Ext Spectral Sequence) Let* F, G *be coherent sheaves over* X*. Then* $\mathrm{Ext}^i(F, G)$ *and* $\mathbf{Ext}^i(F, G)$ *are the right-derived functors of the Hom and sheaf-Hom functors* $\mathrm{Hom}(F, -)$ *and* $\mathbf{Hom}(F, -)$*. Then there is a spectral sequence*

$$H^p(X, \mathbf{Ext}^q(F, G)) \Rightarrow \mathrm{Ext}^{p+q}(F, G).$$

This one uses the composition of the functors $\mathbf{Hom}(F, -)$ and Γ and uses the fact that $\mathrm{Hom}(F, -) = \Gamma \circ \mathbf{Hom}(F, -)$.

In addition to these, I wanted to highlight a particular application of the Grothendieck spectral sequence to extending Serre duality.

Definition 3.4. *The functor* $\Gamma : \mathrm{Coh}(X) \to O_X(X) - \mathrm{Mod}$ *given by taking global sections is left-exact. For any* $F \in \mathrm{Coh}(X)$*, the* sheaf cohomology $H^i(X, F)$ *is the right-derived functor* $R^i\Gamma(F)$*.*

Serre duality is in some sense the sheaf cohomology analogue to Poincare duality of manifolds.

We present here without proof the statement for projective space.

Theorem 3.5. *Let $P = \mathbb{P}^n_k$ be projective space over an algebraically closed field k. Set ω_P to be the canonical sheaf $O(-n-1)$.*

Then there is a perfect pairing

$$H^i(P, F) \times \mathrm{Ext}^{n-i}(F, \omega_P) \to H^n(P, \omega_P) \simeq k.$$

In other words, $\mathrm{Ext}^{n-i}(F, \omega_P) \simeq H^i(P, F)^*$ *naturally.*

If F is just the twisted sheaf $O(k)$ its cohomology vanishes in all degrees but 0 and n. We can calculate $\mathrm{Ext}^n(O(k), \omega_P) = H^n(O(-n-1-k))$ by hand. This statement then reduces to a duality

$$H^n(P, O(-n-1-k)) \simeq H^0(P, O(k))^*.$$

The latter is just the space of homogeneous polynomials of degree k on projective space.

The sheaf ω_P is a more general case of a *dualizing sheaf.*

Definition 3.6. *Let X be a proper scheme of dimension r over a field k. A* dualizing sheaf *is a coherent sheaf ω_X along with a* trace $H^r(X, \omega_X) \to k$ *such that the natural pairing for any sheaf F given by*

$$\mathrm{Hom}(F, \omega_X) \times H^r(X, F) \to H^r(X, \omega_X) \to k$$

is perfect*, inducing an isomorphism* $\mathrm{Hom}(F, \omega_X) \to H^r(X, F)^*$.

It shouldn't be too bad to work out that:

1. Dualizing sheaves are unique up to isomorphism.

2. There is an analogue of Serre duality for projective space using the dualizing sheaf.

We will construct a dualizing sheaf for a projective scheme as an application of the Grothendieck spectral sequence. Let $\iota : X \hookrightarrow P = \mathbb{P}^n_k$ be a closed subscheme of dimension r.

For any coherent sheaf F on X, we know already that $H^r(X, F)^* \simeq H^r(P, \iota_* F)^*$ since $F, \iota_* F$ have the same global sections. By Serre duality, the latter is isomorphic to $\mathrm{Ext}^{n-r}_P(F, \omega_P)$.

We want to find a sheaf ω_X such that there is an isomorphism $\mathrm{Hom}_X(F, \omega_X) \simeq \mathrm{Ext}^{n-r}_P(F, \omega_P)$.

If we set $F = O_X$ and then sheafify the left-hand side, we can recover ω_X. The effect on the right-hand side is to turn it into the sheaf $\mathbf{Ext}^{n-r}_P(O_X, \omega_P)$.

We will stop short of verifying the perfect pairing, but what we will show is that there is in fact a natural isomorphism $\mathrm{Hom}_X(F, \omega_X) \simeq \mathrm{Ext}^{n-r}_P(F, \omega_P)$.

This will fall out of the local-to-global Ext spectral sequence (after some hand-waving)! We have a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_X(F, \mathbf{Ext}^q_P(O_X, \omega_P)) \Rightarrow \mathrm{Ext}^{p+q}_P(F, \omega_P).$$

By direct calculation, $\mathbf{Ext}^q_P(O_X, \omega_P) = 0$ for $q < n - r$.

We can calculate $E_2^{0,n-r} = \mathrm{Hom}_X(F, \mathbf{Ext}^{n-r}_P(O_X, \omega_P)) \simeq \mathrm{Hom}_X(F, \omega_X)$ and $E_\infty^{0,n-r} = \mathrm{Ext}^{n-r}_P(F, \omega_P)$.

The differential d_s sends $E_s^{0,n-r}$ to $E_s^{0,n-r-s+1} = 0$ for every $s \geq 2$. Therefore, $E_2^{0,n-r} \simeq E_\infty^{0,n-r}$ as desired.