A CONFUSION ABOUT THE STERN INTEGRAL INEQUALITY

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At the end of my Bachelor's thesis about the harmonic level set method for a specific positive mass theorem, I wanted to find out how close the lower bound from the method is to the actual value of the mass, and I did some numerical calculations for a half space version of the Schwarzschild black hole. The convergence of the numerical integral near the critical value of the harmonic function I used was quite terrible, and consequently my values were very rough estimates of the real lower bound, but still this numerical calculation got remarkably close to the actual values of the mass.

Consequently I wanted to understand better what actually prevents the main relevant inequality [1, Proposition 4.2], also referred to as *Stern's inequality*, from being an equality. Below are my calculations where I try to redo the proof of [1, Proposition 4.2] (in some additional generality) while accounting for "error terms" (marked in red), that were originally just left out (resulting in statements afterwards only being inequalities). On the way we can do a sanity check: The resulting error term should of course have the same sign everywhere so that we can recover the inequality again.

The following statement is similar to [2, Section 4.1], but we allow $|\nabla u| = 0$ at some points. I'm very sure that I must have gotten confused somewhere in the proof, the result seems too strong and would in particular imply that the lower bounds for the mass of [1] and [3] could be easily strengthened into equalities (just by not replacing the Euler characteristic of the level set $\chi(S_t) \leq 1$ with 1).

Theorem 1: Let (Ω, g) be a compact 3-dimensional oriented Riemannian manifold with boundary having outward unit normal ν. Let $u\colon \Omega \to \mathbb{R}$ be any smooth function such that there exists $C > 0$ with $|\Delta u| \leq C|\nabla u|$, and denote the open subset of $\partial\Omega$ on which $|\nabla u| \neq 0$ by $\partial_{\neq 0}\Omega$.

If $\overline{u}$ and $\underline{u}$ denote the maximum and minimum of u and S_t are t-level sets of u, then

$$\int_{\partial_{\neq 0}\Omega} \left(\partial_\nu |\nabla u| - \Delta u \cdot \frac{\langle \nu, \nabla u \rangle}{|\nabla u|} \right) \mathrm{d}A = \int_\Omega \frac{1}{2} \left(\frac{|\nabla^2 u|^2}{|\nabla u|} + R|\nabla u| - \frac{(\Delta u)^2}{|\nabla u|} \right) \mathrm{d}V - \int_{\underline{u}}^{\overline{u}} \left(2\pi\chi(S_t) - \int_{\partial S_t} \kappa_{\partial S_t} \right) \mathrm{d}t$$

where $\chi(S_t)$ denotes the Euler characteristic of the level sets and $\kappa_{\partial S_t}$ denotes the geodesic curvature of $\partial S_t \subset S_t$.

Proof: We follow [3, Section 3] and [1, Section 4]. We will consider $\varphi_\varepsilon := \sqrt{|\nabla u| + \varepsilon}$ for $\varepsilon > 0$ to avoid dealing with $\Delta|\nabla u|$ at critical values of u.

First, recall Bochner's identity

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta u \rangle,$$

which yields in particular

$$\begin{aligned} \Delta\varphi_\varepsilon &= \frac{\Delta|\nabla u|^2}{2\varphi_\varepsilon} - \frac{|\nabla|\nabla u|^2|^2}{4\varphi_\varepsilon^3} \\ &= \frac{1}{\varphi_\varepsilon}\Big(|\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) - \varphi_\varepsilon^{-2}|\nabla u|^2 |\nabla|\nabla u||^2 + \langle \nabla u, \nabla \Delta u \rangle \Big)) \end{aligned}$$

On regular level sets we have

$$\Delta\varphi_\varepsilon = \frac{1}{2\varphi_\varepsilon}\Big(\left|\nabla^2 u\right|^2 + |\nabla u|^2(R_\Omega - R_S) + 2\langle\nabla\Delta u, \nabla u\rangle + (\Delta u)^2 - 2\Delta u\nabla^2_{\nu\nu}u$$

$$+2\cdot\left(1-\varphi_\varepsilon^{-2}|\nabla u|^2\right)|\nabla|\nabla u||^2\Big)$$

Note that

$$\begin{aligned}\operatorname{div}\left(\Delta u\frac{\nabla u}{\varphi_\varepsilon}\right) &= \frac{(\Delta u)^2}{\varphi_\varepsilon} + \frac{\langle\nabla u, \nabla\Delta u\rangle}{\varphi_\varepsilon} - \frac{\Delta u}{\varphi_\varepsilon^2}\cdot\nabla_i u\nabla^i\varphi_\varepsilon \\ &= \frac{(\Delta u)^2}{\varphi_\varepsilon} + \frac{\langle\nabla u, \nabla\Delta u\rangle}{\varphi_\varepsilon} - \frac{\Delta u}{2\varphi_\varepsilon^3}\cdot\nabla_i u\nabla^i|\nabla u|^2 \\ &= \frac{(\Delta u)^2}{\varphi_\varepsilon} + \frac{\langle\nabla u, \nabla\Delta u\rangle}{\varphi_\varepsilon} - \frac{\Delta u}{2\varphi_\varepsilon^3}\cdot\nabla_i u\cdot 2\cdot\nabla^i\nabla_j u\nabla^j u \\ &= \frac{(\Delta u)^2}{\varphi_\varepsilon} + \frac{\langle\nabla u, \nabla\Delta u\rangle}{\varphi_\varepsilon} - \frac{\Delta u}{\varphi_\varepsilon^3}\cdot|\nabla u|^2\cdot\nabla^2_{\nu\nu}u\end{aligned}$$

And thus in general

$$\operatorname{div}\left(\nabla\varphi_\varepsilon - \Delta u\frac{\nabla u}{\varphi_\varepsilon}\right) = \frac{1}{\varphi_\varepsilon}\left(\left|\nabla^2 u\right|^2 + \operatorname{Ric}(\nabla u, \nabla u) - \varphi_\varepsilon^{-2}\left|\nabla|\nabla u|^2\right|^2/4 - (\Delta u)^2 + \frac{\Delta u}{2\varphi_\varepsilon^2}\nabla_i u\nabla^i|\nabla u|^2\right).$$

and on regular level sets

$$\operatorname{div}\left(\nabla\varphi_\varepsilon - \Delta u\frac{\nabla u}{\varphi_\varepsilon}\right) = \frac{1}{2\varphi_\varepsilon}\Big(\left|\nabla^2 u\right|^2 + |\nabla u|^2(R_\Omega - R_S) - (\Delta u)^2$$

$$+2\left(1-\varphi_\varepsilon^{-2}|\nabla u|^2\right)\left(|\nabla|\nabla u||^2 - \Delta u\nabla^2_{\nu\nu}u\right)\Big).$$

The additional $2\left(1-\varphi_\varepsilon^{-2}|\nabla u|^2\right)(\cdots)$ term vanishes again at the $\varepsilon\to 0$ step.

We thus have

$$\int_{\partial_{\neq 0}\Omega}\left(\partial_\nu|\nabla u| - \Delta u\cdot\frac{\nabla u}{|\nabla u|}\right)\mathrm{d}A = \int_\Omega\frac{1}{2}\left(\frac{\left|\nabla^2 u\right|^2}{|\nabla u|} + R|\nabla u| - \frac{(\Delta u)^2}{|\nabla u|}\,\mathrm{d}V\right) - \int_{\underline{u}}^{\overline{u}} 2\pi\chi(S_t)\,\mathrm{d}t$$

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Bibliography

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