A (NON-HARMONIC) EQUALITY VERSION OF THE STERN INTEGRAL INEQUALITY AND CONSEQUENCES FOR MASS

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1. Introduction

In [2], Bray, Kazaras, Khuri and Stern used an approach based on level sets of harmonic functions to prove the Riemannian positive mass theorem in 3 dimensions. The approach centrally employs a certain integral inequality ([2, Proposition 4.2]), a version of which was first established by Stern in [7]. It is well known (cf. [3, Section 4.1]) that in the case where the harmonic function of interest has no critical points, this inequality becomes an equality. We will prove that equality also holds in the general case, i.e., even in the presence of critical points. By keeping track of an additional term involving the Euler characteristic of level sets, one can then strengthen the lower bounds for the mass obtained in [2] and [3] into equalities. We will do this explicitly for [2].

TODO The approach of [2] can be generalized by replacing harmonic functions with other geometrically motivated elliptic equations, allowing to, e.g., prove the positive energy theorem for asymptotically flat initial data sets. For the sake of generality we thus follow [3] and do not assume harmonicity of our function for now.

Todo: This is very similar to the third paragraph of [3]. In general this whole intro is pretty rough right now.

2. The main identity

TODO Everything below is pretty much directly adopted from my Bachelor's Thesis (which more or less just followed [3]), though I have

(1) changed some stuff (marked in blue) to allow for non harmonic functions u, and
(2) kept track of some additional terms (marked in red) to strengthen the encountered inequalities into equalities.

If something is related to both, it'll be in purple.

Todo: Probably shouldn't keep this if I want to upload this to the ArXiv. Though maybe it's useful to readers just wanting to see what's different to before?

Proposition 1. *Let (Ω, g) be a compact oriented Riemannian manifold with boundary having outward unit normal ν. Let $u\colon \Omega \to \mathbb{R}$ be any smooth function such that there exists $C_0 > 0$ with*

$$|\Delta u| \leq C_0 \cdot |\nabla u|. \tag{1}$$

Denote by $\partial_{\neq 0}\Omega$ the open set of points in $\partial\Omega$ with $|\nabla u| \neq 0$. If $\overline{u}$ and $\underline{u}$ denote the maximum and minimum of u and S_t are t-level sets of u, then

$$\int_{\underline{u}}^{\overline{u}} \int_{S_t} \frac{1}{2}\left(\frac{|\nabla^2 u|^2 - (\Delta u)^2}{|\nabla u|^2} + R_\Omega - R_{S_t}\right) \mathrm{d}A \,\mathrm{d}t = \int_{\partial_{\neq 0}\Omega} \left(\partial_\nu |\nabla u| - \Delta u \frac{\langle \nu, \nabla u\rangle}{|\nabla u|}\right) \mathrm{d}A,$$

where R_Ω and R_{S_T} denote the scalar curvature of Ω and the level sets S_t respectively.

For the proof we follow the [Proof of Proposition 4.2 in 2] and use Bochner's identity as well as the following Lemma:

Lemma 2. *For $u\colon \Omega \to \mathbb{R}$ as above with regular level set S we have*

$$2\mathrm{Ric}(\nabla u, \nabla u) = |\nabla u|^2(R_\Omega - R_S) + 2|\nabla|\nabla u||^2 - |\nabla^2 u|^2 - 2\Delta u \nabla^2_{\nu\nu} u + (\Delta u)^2.$$

Proof. Let S be a regular level set of u with induced metric γ, second fundamental form A_{ij} and mean curvature H. The normal to S is then $\nu^i = \nabla^i u/|\nabla u|$ and we have

$$\begin{aligned} A_{ij} &= \gamma_i^k \gamma_j^l \nabla_k(\nabla_l u/|\nabla u|) \\ &= \gamma_i^k \gamma_j^l \frac{\nabla^2_{kl} u}{|\nabla u|} + (\dots)\gamma_j^l \nabla_l u \\ &= \gamma_i^k \gamma_j^l \frac{\nabla^2_{kl} u}{|\nabla u|} \\ &= (g_i^k - \nu_i \nu^k)(g_j^l - \nu^l \nu_j)\frac{\nabla^2_{kl} u}{|\nabla u|} \\ &= \frac{\overbrace{\nabla^2_{ij} u}^{\text{Term } T^1} - \overbrace{\nu_i \nu^k \nabla^2_{kj} u}^{T^2} - \overbrace{\nu_j \nu^l \nabla^2_{il} u}^{T^3} + \overbrace{\nu_i \nu_j \nu^k \nu^l \nabla_{kl} u}^{T^4}}{|\nabla u|} \end{aligned}$$

and thus (below we use c.w. to denote which term "contracted with" which other term)

$$\begin{aligned} |A|^2 &= \frac{\overbrace{|\nabla^2 u|^2}^{T^1 \text{ c.w. } T^1} + \overbrace{(\nabla^2_{\nu\nu} u)^2}^{T^4 \text{ c.w. } T^4}}{|\nabla u|^2} \\ &\quad + \frac{\overbrace{2\nu^k \nu_l \nabla^2_{kj} u (\nabla^2)^{lj} u}^{T^{2/3} \text{ c.w. } T^{2/3}} - \overbrace{4\nabla^2_{ij} u \nu^i \nu_k (\nabla^2)^{kj} u}^{T^1 \text{ c.w. } T^{2/3}}}{|\nabla u|^2} \\ &\quad + \frac{\overbrace{2(\nabla^2_{\nu\nu} u)^2}^{T^{2/3} \text{ c.w. } T^{3/2}} - \overbrace{4(\nabla^2_{\nu\nu} u)^2}^{T^{2/3} \text{ c.w. } T^4} + \overbrace{2(\nabla^2_{\nu\nu} u)^2}^{T^1 \text{ c.w. } T^4}}{|\nabla u|^2} \\ &= \frac{|\nabla^2 u|^2 - 2g^{jk} \nabla^2_{kl} u \nabla^l u \nabla^2_{ij} u \nabla^i u/(|\nabla u|^2) + (\nabla^2_{\nu\nu} u)^2}{|\nabla u|^2} \\ &= \frac{|\nabla^2 u|^2 - (\nabla_k(\nabla_l u \nabla^l u)\nabla^k(\nabla_l \nabla^l u))/(2|\nabla u|^2) + (\nabla^2_{\nu\nu} u)^2}{|\nabla u|^2} \\ &= \frac{|\nabla^2 u|^2 - |\nabla|\nabla u|^2|^2/(2|\nabla u|^2) + (\nabla^2_{\nu\nu} u)^2}{|\nabla u|^2} \\ &= \frac{1}{|\nabla u|^2}(|\nabla^2 u|^2 - 2|\nabla|\nabla u||^2 + (\nabla^2_{\nu\nu} u)^2). \end{aligned}$$

On the other hand contracting A_{ij} gives

$$H = \frac{1}{|\nabla u|}(\Delta u - \nabla^2_{\nu\nu} u).$$

and thus

$$|A|^2 - H^2 = |\nabla u|^{-2}(|\nabla^2 u|^2 - 2|\nabla|\nabla u||^2 + 2\Delta u \nabla^2_{\nu\nu} u - (\Delta u)^2).$$

Combining with the contracted Gauss-Codazzi equation

$$2\mathrm{Ric}((\nabla u)/|\nabla u|, (\nabla u)/|\nabla u|) = R_\Omega - R_S + H^2 - |A|^2,$$

then yields the result. $\square$

Proof of Proposition 1. During the following proof, we will be considering

$$\varphi_\varepsilon \coloneqq \sqrt{|\nabla u|^2 + \varepsilon}$$

for $\varepsilon > 0$ instead of $|\nabla u|$, since we cannot control the behavior of integrands like $\Delta|\nabla u|$ and $\partial_\nu |\nabla u|$ at critical points of u (where $|\nabla u| = 0$).

Recall first *Bochner's identity*,

$$\frac{1}{2}\Delta(|\nabla u|^2) = |\nabla^2 u|^2 + \langle \nabla \Delta u, \nabla u\rangle + \mathrm{Ric}(\nabla u, \nabla u).$$

We find

$$\begin{aligned}
\Delta\varphi_\varepsilon &= \nabla_i \nabla^i \sqrt{|\nabla u|^2 + \varepsilon} \\
&= \nabla_i \frac{\nabla^i |\nabla u|^2}{2\varphi_\varepsilon} \\
&= \frac{\Delta|\nabla u|^2}{2\varphi_\varepsilon} - \frac{|\nabla|\nabla u|^2|^2}{4\varphi_\varepsilon^3} \\
&= \varphi_\varepsilon^{-1}(|\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) - |\nabla|\nabla u|^2|^2/(4\varphi_\varepsilon^2) + \langle \nabla \Delta u, \nabla u\rangle), \qquad (2)
\end{aligned}$$

where we have used Bochner's identity on the last line.

On a regular level set S, Lemma 2 thus yields

$$\begin{aligned}
\Delta\varphi_\varepsilon = \frac{1}{2\varphi_\varepsilon}\Big(&|\nabla^2 u|^2 + |\nabla u|^2(R_\Omega - R_S) \\
&+ 2\cdot(1 - \varphi_\varepsilon^{-2}|\nabla u|^2)|\nabla|\nabla u||^2 \\
&+ 2\langle \nabla(\Delta u), \nabla u\rangle + (\Delta u)^2 - 2(\Delta u)\nabla^2_{\nu\nu} u\Big) \qquad (3)
\end{aligned}$$

.

Note that

$$\begin{aligned}
&\frac{(\Delta u)^2}{\varphi_\varepsilon} + \frac{\langle \nabla u, \nabla \Delta u\rangle}{\varphi_\varepsilon} - \mathrm{div}\Big(\Delta u \frac{\nabla u}{\varphi_\varepsilon}\Big) \\
&\quad = \frac{\Delta u}{\varphi_\varepsilon^2}\cdot \langle \nabla u, \nabla\varphi_\varepsilon\rangle = \frac{\Delta u}{2\varphi_\varepsilon^3}\cdot\langle \nabla u, \nabla|\nabla u|^2\rangle. \qquad (4)
\end{aligned}$$

and thus one obtains in general

$$\begin{aligned}
\mathrm{div}\Big(\nabla\varphi_\varepsilon - \Delta u\frac{\nabla u}{\varphi_\varepsilon}\Big) = \frac{1}{\varphi_\varepsilon}\Big(&|\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) - |\nabla|\nabla u|^2|/(4\varphi_\varepsilon^2) \\
&- (\Delta u)^2 + \Delta u\langle \nabla u, \nabla|\nabla u|^2\rangle/(2\varphi_\varepsilon^2)\Big). \qquad (5)
\end{aligned}$$

On a regular level set, note that

$$\langle \nabla u, \nabla|\nabla u|^2\rangle = 2\nabla^i u \nabla^j u \nabla^2_{ij} u = 2|\nabla u|^2 \nabla^2_{\nu\nu} u$$

and combine Eq. (4) with Eq. (3) to get

$$\operatorname{div}\left(\nabla\varphi_\varepsilon - \Delta u\frac{\nabla u}{\varphi_\varepsilon}\right) = \frac{1}{2\varphi_\varepsilon}\Big(|\nabla^2 u|^2 + |\nabla u|^2(R_\Omega - R_S) - (\Delta u)^2 + 2(1-|\nabla u|^2/(\varphi_\varepsilon^2))(|\nabla|\nabla u||^2 - (\Delta u)\nabla^2_{\nu\nu}u)\Big). \tag{6}$$

Let now $\mathcal{A} \subset [\underline{u}, \overline{u}]$ be an open set containing all the critical values of u, and let $\mathcal{B} = [\underline{u}, \overline{u}] \setminus \mathcal{A}$ be the complementary set. Then the divergence theorem yields

$$\int_{\partial\Omega}\left(\partial_\nu\varphi - \Delta u\frac{\langle\nu, \nabla u\rangle}{\varphi_\varepsilon}\right)\mathrm{d}A = \int_\Omega \operatorname{div}\left(\nabla\varphi_\varepsilon - \Delta u\frac{\nabla u}{\varphi_\varepsilon}\right)\mathrm{d}V. \tag{7}$$

We will deal with the right side of the equation by treating integrals over the preimages of $\mathcal{A}$ and $\mathcal{B}$ seperately.

Let us first consider the integral $u^{-1}(\mathcal{A})$. For convenience, in the following equations C will always refer to some nonnegative constant (independent of ε), but each appearance of C may denote a different constant. Note that

$$|\Delta u\langle\nabla u, \nabla|\nabla u|^2\rangle| \underset{\substack{\uparrow \\ \text{Cauchy-Schwarz and Eq. (1)}}}{\le} C\cdot|\nabla u|\cdot|\nabla u|\cdot|\nabla|\nabla u|^2| \underset{\substack{\uparrow \\ \text{Young's inequality with } \varepsilon = 1/2}}{\le} C\cdot|\nabla u|^4 + \frac{|\nabla|\nabla u|^2|^2}{8},$$

which then yields

$$\begin{aligned}|\nabla^2 u|^2 - \frac{|\nabla|\nabla u|^2|^2}{4\varphi_\varepsilon^2} &+ \frac{\Delta u\langle\nabla u, \nabla|\nabla u|^2\rangle}{2\varphi_\varepsilon^2} \\ &\ge -C\cdot|\nabla u|^2 + |\nabla^2 u|^2 - \frac{5}{4}\cdot\frac{|\nabla|\nabla|^2|^2}{4} \ge -C\cdot|\nabla u|^2,\end{aligned}$$

where the last step is possible due to the refined Kato's inequality [1, Eq. (C.2)]. Together with Eq. (5), one gets

$$\operatorname{div}\left(\nabla\varphi_\varepsilon - \Delta u\frac{\nabla u}{\varphi_\varepsilon}\right) \ge \frac{1}{\varphi_\varepsilon}(\overbrace{\operatorname{Ric}(\nabla u, \nabla u)}^{\ge -|\operatorname{Ric}||\nabla u|^2} - \underbrace{(\Delta u)^2}_{\le C|\nabla u|^2} - C\cdot|\nabla u|^2) \ge -C\cdot|\nabla u|^2,$$

where we have used Eq. (1) and that $|\operatorname{Ric}|$ is bounded since Ω is compact. In particular we can apply the coarea formula and obtain

$$-\int_{u^{-1}(\mathcal{A})}\operatorname{div}\left(\nabla\varphi_\varepsilon - \Delta u\frac{\nabla u}{\varphi_\varepsilon}\right)\mathrm{d}V \le \int_{u^{-1}(\mathcal{A})} C|\nabla u|\,\mathrm{d}V = C\int_{t\in\mathcal{A}}\mathcal{H}^2(S_t)\,\mathrm{d}t, \tag{8}$$

where $\mathcal{H}^2(S_t)$ is the Hausdorff measure of the level sets.

On $u^{-1}(\mathcal{B})$ on the other hand we can apply the coarea formula directly to Eq. (6), which produces

$$\begin{aligned}&\int_{u^{-1}(\mathcal{B})}\operatorname{div}\left(\nabla\varphi_\varepsilon - \Delta u\frac{\nabla u}{\varphi_\varepsilon}\right)\mathrm{d}V \\ &\quad = \frac{1}{2}\int_{t\in\mathcal{B}}\int_{S_t}\frac{|\nabla u|}{2\varphi_\varepsilon}\Big(\frac{|\nabla^2 u|^2 - (\Delta u)^2}{|\nabla u|^2} + R_\Omega - R_{S_t} \\ &\qquad + 2\cdot(|\nabla u|^{-2} - \varphi_\varepsilon^{-2})(|\nabla|\nabla u||^2 - \Delta u\nabla^2_{\nu\nu})\Big)\,\mathrm{d}A\,\mathrm{d}t.\end{aligned} \tag{9}$$

Note that

$$\partial_\nu\varphi_\varepsilon = \frac{\nu^i\nabla_{ij}u\nabla^j u}{\varphi_\varepsilon} = 0$$

at critical points of u and thus we may replace $\partial\Omega$ by $\partial_{\neq 0}\Omega$ in the integral in Eq. (7) and below in Eq. (10).

We combine Eq. (9) with Eq. (7) and obtain

$$\begin{aligned}\frac{1}{2}\int_{t\in\mathcal{B}}\int_{S_t}\frac{|\nabla u|}{\varphi_\varepsilon}\left(\frac{|\nabla^2 u|^2-(\Delta u)^2}{|\nabla u|^2}+R_\Omega-R_{S_t}\right)\mathrm{d}A\,\mathrm{d}t&\\ -\int_{\partial_{\neq 0}\Omega}\left(\partial_\nu\varphi_\varepsilon-\Delta u\frac{\langle\nu,\nabla u\rangle}{\varphi_\varepsilon}\right)\mathrm{d}A&\\ =-\int_{u^{-1}(\dashv)}\operatorname{div}\left(\Delta\varphi_\varepsilon-\Delta\frac{\nabla u}{\varphi_\varepsilon}\right)\mathrm{d}V&\\ -\int_{t\in\mathcal{B}}\int_{S_t}(|\nabla u|^{-2}-\varphi_\varepsilon^{-2})\cdot(|\nabla|\nabla u||^2-\Delta u\nabla^2_{\nu\nu})\,\mathrm{d}A\,\mathrm{d}t.&\end{aligned}$$

Taking absolute values and using Eq. (8) yields

(10)
$$\begin{aligned}&\left|\frac{1}{2}\int_{t\in\mathcal{B}}\int_{S_t}\frac{|\nabla u|}{\varphi_\varepsilon}\left(\frac{|\nabla^2 u|^2-(\Delta u)^2}{|\nabla u|^2}+R_\Omega-R_{S_t}\right)\mathrm{d}A\,\mathrm{d}t\right.\\ &\left.-\int_{\partial_{\neq 0}\Omega}\left(\partial_\nu\varphi_\varepsilon-\Delta u\frac{\langle\nu,\nabla u\rangle}{\varphi_\varepsilon}\right)\mathrm{d}A\right|\\ &\qquad\le C\int_{t\in\mathcal{A}}\mathcal{H}^2(S_t)\,\mathrm{d}t-\int_{t\in\mathcal{B}}\int_{S_t}(|\nabla u|^{-2}-\varphi_\varepsilon^{-2})\cdot(|\nabla|\nabla u||^2-\Delta u\nabla^2_{\nu\nu})\,\mathrm{d}A\,\mathrm{d}t.\end{aligned}$$

Since Ω is compact and $\mathcal{B}$ closed, $|\nabla u|$ and $|\nabla|\nabla u||^2$ are uniformly bounded from below on $u^{-1}(\mathcal{B})$. Furthermore on $\partial_{\neq 0}\Omega$ (where $|\nabla u|\neq 0$) we have

$$\partial_\nu\varphi_\varepsilon=\frac{|\nabla u|}{\varphi_\varepsilon}\partial_\nu|\nabla u|\to\partial_\nu|\nabla u|\quad\text{as }\varepsilon\to 0$$

We can thus now take the limit $\varepsilon\to 0$ in Eq. (10) and get

$$\begin{aligned}(11)\qquad&\left|\frac{1}{2}\int_{\mathcal{B}}\int_{S_t}\left(\frac{|\nabla^2 u|^2-(\Delta u)^2}{|\nabla u|^2}+R_\Omega-R_{S_t}\right)\mathrm{d}A\,\mathrm{d}t\right.\\ &\left.-\int_{\partial_{\neq 0}\Omega}\left(\partial_\nu|\nabla u|-\Delta u\frac{\langle\nu,\nabla u\rangle}{|\nabla u|}\right)\mathrm{d}A\right|\le C\int_{t\in\mathcal{A}}\mathcal{H}^2(S_t)\,\mathrm{d}t.\end{aligned}$$

By Sard's theorem ([6]), the set of critical values has measure 0 and we thus may take the measure of $\mathcal{A}$ to be arbitrarily small. Since $t\mapsto\mathcal{H}^2(S_t)$ is integrable by the coarea formula, taking $|\mathcal{A}|\to 0$ in Eq. (11) leads to

$$\int_{\underline{u}}^{\overline{u}}\int_{S_t}\frac{1}{2}\left(\frac{|\nabla^2 u|^2-(\Delta u)^2}{|\nabla u|^2}+R_\Omega-R_{S_t}\right)\mathrm{d}A\,\mathrm{d}t=\int_{\partial_{\neq 0}\Omega}\left(\partial_\nu|\nabla u|-\Delta u\frac{\langle\nu,\nabla u\rangle}{|\nabla u|}\right)\mathrm{d}A,$$

and we have thus proven our claim. □

Remark 3. *Note that the requirement on u (namely that $|\Delta u|\le C\cdot|\nabla u|$) can be weakened a bit: Really one just needs that there exists some C such that $-C|\nabla u|$ bounds the RHS of Eq. (5) from below.*

3. Application to (spacetime) harmonic functions

Let us now apply Proposition 1 to the case of spacetime harmonic functions on initial data sets for the Einstein equations to obtain an equality version of [3, Proposition 4.2].

We first recall some definitions (see also [5, Definition 7.16]):

Definition 4. *An* initial data set *(IDS) (M, g, k) is a Riemannian manifold (M, g) equipped with a symmetric 2-tensor k, representing an embedded spacelike hypersurface in $(3+1)$-dimensional spacetime, where g is the induced metric on the hypersurface (also called first fundamental form) and k is the extrinsic curvature tensor (or second fundamental form). These objects satisfy the* constraint equations

$$J = \operatorname{div}(k - (\operatorname{tr} k)g), \qquad \mu = \frac{1}{2}(R + (\operatorname{tr} k)^2 - |k|^2). \tag{12}$$

Here R is the scalar curvature and μ and J represent energy and momentum density respectively and $\bar{\nabla}^2$ is called the spacetime Hessian*.*

We say the IDS fulfills the dominant energy condition *if $\mu \geq |J|$.*

Definition 5. *Let $u\colon M \to \mathbb{R}$ be a smooth function. We define the* spacetime hessian*,*

$$\bar{\nabla}^2 u \coloneqq \nabla^2 u + k|\nabla u|,$$

and call u a spacetime harmonic function *if it satisfies the equation*

$$\operatorname{tr} \bar{\nabla}^2 u = 0,$$

i.e. if

$$\Delta u = -\operatorname{tr}(k)|\nabla u|.$$

Corollary 6 (Equality version of [3, Proposition 4.2])**.** *Let (Ω, g, k) be an oriented compact initial data set (IDS) with smooth boundary $\partial\Omega$, having outward unit normal ν. Let $u\colon \Omega \to \mathbb{R}$ be a spacetime harmonic function, and let $\partial_{\neq 0}\Omega$, $\underline{u}$, $\overline{u}$, S_t and K be as in Proposition 1. Then*

$$\int_{\partial_{\neq 0}\Omega} (\partial_\nu |\nabla u| + k(\nabla u, \nu))\, \mathrm{d}A = \int_{\underline{u}}^{\overline{u}} \int_{S_t} \left(\frac{1}{2} \frac{|\bar{\nabla}^2 u|}{|\nabla u|^2} + \mu + J(\nabla u / |\nabla u|) - K \right) \mathrm{d}A\, \mathrm{d}t,$$

where K is the level set Gauss curvature.

Note that the original [3, Proposition 4.2] uses ν differently than we do. In the above, ν is still the normal vector of the level sets.

Proof. Let C_0 be any constant bounding $|\operatorname{tr} k|$. Then since u is spacetime harmonic, we have $|\Delta u| \leq C_0 \cdot |\nabla u|$ and can thus apply Proposition 1 to get

$$\int_{\underline{u}}^{\overline{u}} \int_{S_t} \frac{1}{2} \left(\frac{|\nabla^2 u|}{|\nabla u|^2} - (\operatorname{tr} k)^2 + R - 2K \right) \mathrm{d}A\, \mathrm{d}t = \int_{\partial_{\neq 0}\Omega} (\partial_\nu |\nabla u| + (\operatorname{tr} k)\langle \nu, \nabla u \rangle)\, \mathrm{d}A.$$

Combining

$$\begin{aligned} \langle k, \nabla^2 u \rangle &= \nabla^i (k_{ij} \nabla^j u) - \nabla^j u \nabla^i k_{ij} \\ &= \operatorname{div}(k(\cdot, \nabla u)) - (\operatorname{div} k)(\nabla u) \end{aligned}$$

and

$$\begin{aligned}\operatorname{div}((\operatorname{tr} k)\nabla u) &= \nabla_i(\operatorname{tr} k)\nabla^i u + (\operatorname{tr} k)\Delta u\\ &= \nabla_i((\operatorname{tr} k)g_{ij})\nabla_j u - (\operatorname{tr} k)^2|\nabla u|\\ &= -J(\nabla u) + (\operatorname{div} k)(\nabla u) - (\operatorname{tr} k)^2|\nabla u|\end{aligned}$$

gives

$$\int_\Omega (J(\nabla u) + (\operatorname{tr} k)^2|\nabla u| + \langle k, \nabla^2 u\rangle)\,\mathrm{d}V = \int_{\partial\Omega} (k(\nu, \nabla u) - (\operatorname{tr} k)\langle \nu, \nabla u\rangle)\,\mathrm{d}A.$$

Then using

$$|\nabla^2 u| = |\bar\nabla^2 u| - 2\langle k, \nabla^2 u\rangle|\nabla u| - |k|^2|\nabla u|^2$$

and the coarea formula yields

$$\begin{aligned}&\int_{\underline{u}}^{\overline{u}}\int_{S_t}\frac{1}{2}\left(\frac{|\bar\nabla^2 u|}{|\nabla u|^2} + R - |k|^2 + 2J(\nabla u/|\nabla u|) + (\operatorname{tr} k)^2 - 2K\right)\mathrm{d}A\,\mathrm{d}t\\ &\quad= \int_{\partial_{\neq 0}\Omega}(\partial_\nu|\nabla u| + k(\nu, \nabla u))\,\mathrm{d}A.\end{aligned}$$

Applying the definition of μ completes the proof.

□

The Riemannian case can be obtained by setting $k = 0$:

Corollary 7 (Equality version of [2, Proposition 3.2]). *Let (Ω, g, k) be an oriented compact initial data set (IDS) with smooth boundary $\partial\Omega$, having outward unit normal ν. Let $u\colon \Omega \to \mathbb{R}$ be a spacetime harmonic function, and let $\partial_{\neq 0}\Omega$, $\underline{u}$, $\overline{u}$, S_t and K be as in Proposition 1. Then*

$$\int_{\partial_{\neq 0}\Omega}(\partial_\nu|\nabla u| + k(\nabla u, \nu))\,\mathrm{d}A = \int_{\underline{u}}^{\overline{u}}\int_{S_t}\left(\frac{1}{2}\frac{|\bar\nabla^2 u|}{|\nabla u|^2} + \mu + J(\nabla u/|\nabla u|) - K\right)\mathrm{d}A\,\mathrm{d}t,$$

where K is the level set Gauss curvature.

4. Consequences for mass

We can now carry this improvement of [2, Propositon 3.2] through the rest of [2]. We will notice that by just keeping track of one additional term involving the Euler characteristic $\chi(S_t)$, all the inequalities can be promoted to equalities. In particular we will obtain a new description of the mass of an asymptotically flat manifold instead of the usual lower bound.

In an effort to keep this note mostly self-contained we will repeat pretty much all the computations of [2]. Note that from now on, we will only consider 3-dimensional manifolds.

Definition 8. *We say that a 3-dimensional Riemannian manifold (M, g) is asymptotically flat if there exists a compact set $C \subset M$ such that we can write $M \setminus C$ as a disjoint union of finitely many* ends *M^ℓ_{end}, where each end is diffeomorphic to the complement of a ball $\mathbb{R}^n \setminus B_1$ (equipped with the standard metric δ) and we have the following asymptotic behavior in the* asymptotically flat coordinates *x_1, x_2, x_3 defined by this diffeomorphism:*

$$|\partial^l(g_{ij} - \delta_{ij})| = O(|u|^{-q-l}), \quad l = 0, 1, 2. \tag{13}$$

for some $q > \frac{n-2}{2}$. We assume that the scalar curvature is integrable, $R \in L^1(M)$, such that the ADM mass of each end is well-defined and given by

$$m = \lim_{\rho\to\infty} \frac{1}{16\pi} \int_{S_\rho} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j})\nu^j \,\mathrm{d}A,$$

where ν is the outer unit normal to S_ρ.

By [4, Lemma 4.1], there exists for each end M_{end} an *exterior region* ($M_{\mathrm{ext}} \supset M_{\mathrm{end}}$), which is diffeomorphic to the complement of a finite number of open balls (with disjoint closure) in $\mathbb{R}^3$ and has minimal boundary.

Our goal will be to prove the following:

Theorem 9 (Equality version of [2, Theorem 1.2])**.** *Let (M_{ext}, g) be an exterior region of a complete asymptotically flat Riemannian 3-manifold (M, g) with mass m. Then there exists a harmonic function u on (M_{ext}, g) asymptotic to one of the coordinate functions of the associated end and satisfying Neumann boundary conditions on $\partial M_{\mathrm{ext}}$, and we have*

$$m \geq \frac{1}{16\pi} \int_{M_{\mathrm{ext}}} \left(\frac{|\nabla^2 u|^2}{|\nabla u|} + R|\nabla u| \right) \mathrm{d}V.$$

In particular if R is nonnegative everywhere, then the mass is nonnegative.

Todo: Finish this later.

Proof of Theorem 9. Consider harmonic coordinates x_1, x_2, x_3, see [2, Section 3.2].
TODO □

5. An attempt at generalizing this method to higher dimensions

The following is a consequence of Proposition 1. We are careless below in always using $\mathrm{d}A$ instead of whatever measure would be appropriate for the current integration (though we only use it for integration over submanifolds of Ω, but they need not necessarily be hypersurfaces).

Corollary 10. *Let $u_1, \dots, u_n$ be coordinates (in particular, none of the u_i should have any critical points) on (Ω, g) as above, with maxima $\overline{u}_i$, minima $\underline{u}_i$, and level sets S^i_t. Let further $I^k_j = \prod_{i=j}^k [\underline{u}_i, \overline{u}_i]$. For $\mathbf{t} = (t_1, \dots, t_k)$ let (for $j \leq k$) $\Sigma^j_{\mathbf{t}} := \bigcap_{i=1}^{j-1} S^i_{t_i}$. Note that $\Sigma^1_{\mathbf{t}} = \Omega$. Define also $\Sigma_{\mathbf{t}} := \Sigma^{n+1}_{\mathbf{t}} = \bigcap_{i=1}^n S^i_{t_i}$ (this will, since we consider coordinates, only consist of one point).*

Then

$$\int_{I_1^n} \int_{\Sigma_{\mathbf{t}}} \frac{1}{2} \left(\sum_{i=1}^n \frac{|\nabla^2_{\Sigma^i_{\mathbf{t}}} u_i| - (\Delta_{\Sigma^i_{\mathbf{t}}} u_i)^2}{|\nabla_{\Sigma^i_{\mathbf{t}}} u_i|^2 \cdot N_{\mathbf{t}}} + \frac{R_\Omega}{N_{\mathbf{t}}} \right) \mathrm{d}A \,\mathrm{d}\mathbf{t}$$
$$= \sum_{i=1}^n \int_{I_1^{i-1}} \int_{\partial\Omega \cap \Sigma^i_{\mathbf{t}}} \frac{\partial_{\nu_{\Sigma^i_{\mathbf{t}}}} |\nabla_{\Sigma^i_{\mathbf{t}}} u_i| - \Delta_{\Sigma^i_{\mathbf{t}}} u_i \cdot \frac{\langle \nu_{\Sigma^i_{\mathbf{t}}}, \nabla_{\Sigma^i_{\mathbf{t}}} u_i \rangle}{|\nabla_{\Sigma^i_{\mathbf{t}}} u_i|}}{\prod_{j=1}^{i-1} |\nabla_{\Sigma^j_{\mathbf{t}}} u_j|} \,\mathrm{d}A \,\mathrm{d}\mathbf{t},$$

where $N_{\mathbf{t}} = \prod_{i=2}^n |\nabla_{\Sigma^i_{\mathbf{t}}} u_i|$ and $\nu_{\Sigma^i_{\mathbf{t}}}$ is the normal vector to $\partial\Omega \cap \Sigma^i_{\mathbf{t}}$ inside $\Sigma^i_{\mathbf{t}}$.

Proof. We prove this via induction on the dimension of the space. Proposition 1 proves the case $n = 1$ (after an application of the coarea formula, and noting that the scalar curvature of the level sets is 0 since they are 0-dimensional). Thus assume now that we have proven the claim for $n - 1$, and let Ω be n-dimensional.

We start by considering Proposition 1 for $u = u_1$,

$$\int_{\underline{u}_1}^{\overline{u}_1} \int_{S^1_{t_1}} \frac{1}{2}\left(\frac{|\nabla^2 u_1|^2 - (\Delta u_1)^2}{|\nabla u_1|^2} + R_\Omega - R_{S^1_{t_1}}\right) \mathrm{d}A \, \mathrm{d}t_1$$
$$= \int_{\partial\Omega} \left(\partial_\nu |\nabla u_1| - \Delta u_1 \cdot \frac{\langle \nu, \nabla u_1\rangle}{|\nabla u_1|}\right) \mathrm{d}A.$$

Since none of the coordinates have any critical points, we can apply the coarea formula $n-1$ times, developing in order along $u_2, \ldots, u_n$. Using that $\nabla = \nabla_\Omega = \nabla_{\Sigma^1_{\mathbf{t}}}$, we obtain

$$\begin{aligned}(14)\quad &\int_{I^n_1} \int_{\Sigma_{\mathbf{t}}} \frac{1}{2}\left(\frac{|\nabla^2_{\Sigma^1_{\mathbf{t}}} u_1|^2 - (\Delta_{\Sigma^1_{\mathbf{t}}} u_1)^2}{|\nabla_{\Sigma^1_{\mathbf{t}}} u_1|^2 \cdot N_{\mathbf{t}}} + \frac{R_\Omega}{N_{\mathbf{t}}} - \frac{R_{S^1_{t_1}}}{N_{\mathbf{t}}}\right) \mathrm{d}A \, \mathrm{d}\mathbf{t} \\ &= \int_{\partial\Omega\cap\Sigma^1_{\mathbf{t}}} \left(\partial_{\nu_{\Sigma^1_{\mathbf{t}}}} |\nabla_{\Sigma^1_{\mathbf{t}}} u_1| - \Delta_{\Sigma^1_{\mathbf{t}}} u_1 \cdot \frac{\langle \nu_{\Sigma^1_{\mathbf{t}}}, \nabla_{\Sigma^1_{\mathbf{t}}} u_1\rangle}{|\nabla_{\Sigma^1_{\mathbf{t}}} u_1|}\right) \mathrm{d}A.\end{aligned}$$

Note that $S^1_{t_1}$ (for any fixed t_1) is $(n-1)$-dimensional and has $u_2, \ldots, u_n$ as coordinates. Thus we can apply the induction hypothesis to $S^1_{t_1}$ and divide by $|\nabla_{\Sigma^2_{\mathbf{t}}} u_2|$ to get

$$\int_{\{t_1\}\times I^n_2} \int_{\Sigma_{\mathbf{t}}} \frac{1}{2}\left(\sum_{i=2}^n \frac{|\nabla^2_{\Sigma^i_{\mathbf{t}}} u_i| - (\Delta_{\Sigma^i_{\mathbf{t}}} u_i)^2}{|\nabla_{\Sigma^i_{\mathbf{t}}} u_i|^2 \cdot N_{\mathbf{t}}} + \frac{R_{S^1_{t_1}}}{N_{\mathbf{t}}}\right) \mathrm{d}A \, \mathrm{d}\mathbf{t}$$
$$= \sum_{i=2}^n \int_{I^{i-1}_2} \int_{\partial\Omega\cap\Sigma^i_{\mathbf{t}}} \frac{\partial_{\nu_{\Sigma^i_{\mathbf{t}}}} |\nabla_{\Sigma^i_{\mathbf{t}}} u_i| - \Delta_{\Sigma^i_{\mathbf{t}}} u_i \cdot \frac{\langle \nu_{\Sigma^i_{\mathbf{t}}}, \nabla_{\Sigma^i_{\mathbf{t}}} u_i\rangle}{|\nabla_{\Sigma^i_{\mathbf{t}}} u_i|}}{\prod_{j=2}^i |\nabla_{\Sigma^j_{\mathbf{t}}} u_j|} \mathrm{d}A \, \mathrm{d}\mathbf{t},$$

Integrating this over $t_1 \in [\underline{u}_1, \overline{u}_1]$ and adding it to Eq. (14) concludes the induction step. □

Note in particular that in the above corollary, we get rid (via a telescoping sum) of any level set scalar curvature term (which was problematic in the generalization of the harmonic level set method to higher dimensions). Using the above corollary, we might thus hope to be able to apply a variant of the method in arbitrary dimension in the case where we have global *nested harmonic coordinates* (defined below) on all of M.

Definition 11. *We say $u_1, \ldots, u_m$ are* nested harmonic*, if we have for $\Sigma^i_{\mathbf{t}}$ as above (in Corollary 10)*

$$\Delta_{\Sigma^i_{\mathbf{t}}} u_i = 0,$$

i.e. u_1 is harmonic on Ω, u_2 is harmonic on the level sets of u_1, and u_i in general is harmonic on the intersection of the level sets of the previous nested harmonic functions.

If $u_1, \ldots, u_n$ are nested harmonic and also provide a coordinate chart of Ω, we say they are nested harmonic coordinates.

Corollary 12. *Let $u_1, \dots, u_n$ be nested harmonic coordinates. Then with the notation fixed above we have*

$$\int_{I_1^n} \int_{\Sigma_{\mathbf{t}}} \frac{1}{2} \left(\sum_{i=1}^{n} \frac{|\nabla^2_{\Sigma^i_{\mathbf{t}}} u_i|}{|\nabla_{\Sigma^i_{\mathbf{t}}} u_i|^2 \cdot N_{\mathbf{t}}} + \frac{R_\Omega}{N_{\mathbf{t}}} \right) \mathrm{d}A \, \mathrm{d}\mathbf{t}$$
$$= \sum_{i=1}^{n} \int_{I_1^{i-1}} \int_{\partial\Omega \cap \Sigma^i_{\mathbf{t}}} \frac{\partial_{\nu_{\Sigma^i_{\mathbf{t}}}} |\nabla_{\Sigma^i_{\mathbf{t}}} u_i|}{\prod_{j=1}^{i-1} |\nabla_{\Sigma^j_{\mathbf{t}}} u_j|} \mathrm{d}A \, \mathrm{d}\mathbf{t}.$$

Remark 13. *If it is possible to find nested harmonic functions exhibiting some bounding behavior like*

$$\frac{|\nabla^2_{\Sigma^i_{\mathbf{t}}} u_1|}{|\nabla_{\Sigma^i_{\mathbf{t}}} u_1|^2} - R_{\Sigma^i_{\mathbf{t}}} \geq -C \cdot |\nabla u_j|$$

for any $j > i$ even in the presence of critical points, then I believe one could show a version of Corollary 12 for any such set of harmonic functions. One might hope that one can find (on any given asymptotically flat manifold) such a set of functions that also forms an asymptotically flat coordinate system. This of course again needs some bounding of the level set scalar curvature, so we might have just shifted that difficulty of controlling the level set scalar curvature to another place.

I am relatively certain however that, if one can prove a version of Corollary 12 in the presence of critical points (for some set of function for which we can guarantee existence on arbitrary asymptotically flat manifolds), one should be able to prove the positive mass theorem in general dimension. Below is a proof under the restrictive assumption that a set of nested harmonic global coordinates exists (that is also itself an asymptotically flat coordinate system). In particular one has $M_{\text{ext}} = M$.

In general I would expect that the nested harmonic functions $u_1, \dots, u_n$ just form an asymptotically flat coordinate system on some $M_{\text{end}} \subset M_{\text{ext}}$ and otherwise fulfill Dirichlet boundary conditions on ∂M_{ext}. Then an argument like that in [3, Section 6.2] should show that we get no additional contribution from the boundary integral over ∂M_{ext}, assuming that the boundary of the exterior has zero mean curvature (see Appendix B for more details). All this of course depends on the equality Corollary 12 working without modifications for more general nested harmonic functions.

5.1. **An attempt at a positive mass theorem in general dimension (under very restrictive assumptions).** The nested harmonic functions below definitely do not exist on any manifold that is not diffeomorphic to $\mathbb{R}^n$, so keep in mind that all the calculations here are not particularly interesting without some way to generalize this.

Let (M, g) be diffeomorphic to $\mathbb{R}^n$ and asymptotically flat with integrable scalar curvature R. Assume we have nested harmonic asymptotically flat coordinates $u_1, \dots, u_n$ (none of which have any critical points) and let C_L be the coordinate cube of side length $2L$, centered at the origin of our coordinates. Let Ω_L be the closure of the compact component of $M \setminus C_L$. Let also $F_{\pm L,i}$ denote the face of C_L with constant $u_i = \pm L$, such that $C_L = \bigcup_{i=1}^n F_{+L,i} \cup F_{-L,i}$. Below the notation $\int_{F_{\pm L,i}} \pm f$ always represents $\int_{F_{+L,i}} f - \int_{F_{-L,i}} f$.

Recall that the mass is then given by

$$m = \lim_{L \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{C_L} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j}) \, \mathrm{d}A.$$

On the other hand we obtain from Corollary 12

$$\begin{aligned}\int_{[-L,L]^n}\int_{\Sigma_{L,\mathbf{t}}}\frac{1}{2}\left(\sum_{i=1}^{n}\frac{|\nabla^2_{\Sigma^i_{\mathbf{t}}}u_i|}{|\nabla_{\Sigma^i_{\mathbf{t}}}u_i|^2\cdot N_{\mathbf{t}}}+\frac{R}{N_{\mathbf{t}}}\right)\mathrm{d}A\,\mathrm{d}\mathbf{t}\\ =\sum_{i=1}^{n}\int_{[-L,L]^{i-1}}\int_{C_L\cap\Sigma^i_{\mathbf{t}}}\frac{\partial_{\nu_{\Sigma^i_{\mathbf{t}}}}|\nabla_{\Sigma^i_{\mathbf{t}}}u_i|}{\prod_{j=1}^{i-1}|\nabla_{\Sigma^j_{\mathbf{t}}}u_j|}\,\mathrm{d}A\,\mathrm{d}\mathbf{t},\end{aligned}\tag{15}$$

where $\Sigma_{L,\mathbf{t}} \coloneqq \Omega_L \cap \Sigma_{\mathbf{t}} = \Omega_L \cap \{u_i = t_i \quad \forall i\}$.

Let us now compute the boundary terms. In the following, let $g_{\Sigma^i_{\mathbf{t}}}$ be the induced metric on $\Sigma^i_{\mathbf{t}}$. It is shown in the appendix (Eq. (17)) that $(\Sigma^i_{\mathbf{t}}, g_{\Sigma^i_{\mathbf{t}}})$ is also asymptotically flat with asymptotically flat coordinates $u_i, \dots, u_n$ and the same decay rate ($q > (n-2)/2$) as (M, g). Then the proof of [2, Lemma 6.1] carries over with only two modifications, namely

(1) we use coordinate hypercubes instead of cylinders as domains and thus get terms integrating over the different faces of instead of the mantle and disks of the cylinder,
(2) and we need to exploit (in our equivalent of [2, Equation 6.8]) that

$$\frac{1}{\prod_{j=1}^{i-1}|\nabla_{\Sigma^j_{\mathbf{t}}}u_j|}=\frac{1}{\prod_{j=1}^{i-1}(\delta^{jj}+O_2(|u|^{-q}))}=1+O_2(|u|^{-q})\tag{16}$$

by some computations in the appendix.

We obtain the following:

Lemma 14. *In the notation fixed above, we have for all* $\mathbf{t} = (t_1, \dots, t_{i-1}) \in (-L, L)^{i-1}$ *(and thus for almost all* $\mathbf{t} \in [-L, L]^{i-1}$*)*

$$\begin{aligned}\int_{C_L\cap\Sigma^i_{\mathbf{t}}}\frac{\partial_{\nu_{\Sigma^i_{\mathbf{t}}}}|\nabla_{\Sigma^i_{\mathbf{t}}}u_i|}{\prod_{j=1}^{i-1}|\nabla_{\Sigma^j_{\mathbf{t}}}u_j|}\,\mathrm{d}A &= \frac{1}{2}\sum_{j=i}^{n}\int_{F_{\pm L,i}\cap\Sigma^i_{\mathbf{t}}}\pm((g_{\Sigma^i_{\mathbf{t}}})_{ij,j}-(g_{\Sigma^i_{\mathbf{t}}})_{jj,i})\,\mathrm{d}A\\ &\quad+\frac{1}{2}\sum_{j=i+1}^{n}\int_{F_{\pm L,j}\cap\Sigma^i_{\mathbf{t}}}\pm((g_{\Sigma^i_{\mathbf{t}}})_{ji,i}-(g_{\Sigma^i_{\mathbf{t}}})_{ii,j})\,\mathrm{d}A\\ &\quad+O(L^{(n-i)-1-2q})\end{aligned}$$

Let $\pi_{F_{\pm L,i}}$ now be the projection onto $TF_{\pm L,i}$. $\pi_{F_{\pm L,i}}$ is clearly a bounded operator and we have $\pi_{F_{\pm L,i}}(\partial_j) = \partial_j$ for all $j \neq i$. We again (as a consequence of Lemma 15) get $|\pi_{F_{\pm L,i}}(\nabla_{\Sigma^j_t})| = \delta_{jj} + O_2(|u|^{-q})$ and thus also

$$\prod_{j=1}^{i-1}|\pi_{F_{\pm L,i}}(\nabla_{\Sigma^j_{\mathbf{t}}}u_j)| = 1 + O_2(|u|^{-q}).$$

By combining this with the fact that $(g_{\Sigma^i_{\mathbf{t}}})_{ik,l} = O_1(|u|^{-1-q})$ we obtain

$$
\begin{aligned}
&\int_{[-L,L]^{i-1}} \int_{C_L \cap \Sigma^i_{\mathbf{t}}} \frac{\partial_{\nu_{\Sigma^i_{\mathbf{t}}}} |\nabla_{\Sigma^i_{\mathbf{t}}} u_i|}{\prod_{j=1}^{i-1} |\nabla_{\Sigma^j_{\mathbf{t}}} u_j|} \,\mathrm{d}A\,\mathrm{d}\mathbf{t} \\
&\quad = \frac{1}{2} \int_{[-L,L]^{i-1}} \sum_{j=i}^{n} \int_{F_{\pm L,i} \cap \Sigma^i_{\mathbf{t}}} \pm \frac{((g_{\Sigma^i_{\mathbf{t}}})_{ij,j} - (g_{\Sigma^i_{\mathbf{t}}})_{jj,i})}{\prod_{j=1}^{i-1} |\pi_{F_{\pm L,i}}(\nabla_{\Sigma^j_{\mathbf{t}}} u_j)|} \,\mathrm{d}A\,\mathrm{d}\mathbf{t} \\
&\qquad + \frac{1}{2} \int_{[-L,L]^{i-1}} \sum_{j=i+1}^{n} \int_{F_{\pm L,j} \cap \Sigma^i_{\mathbf{t}}} \pm \frac{((g_{\Sigma^i_{\mathbf{t}}})_{ji,i} - (g_{\Sigma^i_{\mathbf{t}}})_{ii,j})}{\prod_{j=1}^{i-1} |\pi_{F_{\pm L,i}}(\nabla_{\Sigma^j_{\mathbf{t}}} u_j)|} \,\mathrm{d}A\,\mathrm{d}\mathbf{t} \\
&\qquad + O(L^{n-2-2q}).
\end{aligned}
$$

But now (since $\pi_{F_{\pm L,i}}(\nabla_{\Sigma^j_{\mathbf{t}}}) = \nabla_{F_{\pm L,j} \cap \Sigma^j_{\mathbf{t}}}$) the right hand side has exactly the right form so that we can repeatedly apply the coarea formula on $F_{\pm L,i}$. This process yields

$$
\begin{aligned}
&\int_{[-L,L]^{i-1}} \int_{C_L \cap \Sigma^i_{\mathbf{t}}} \frac{\partial_{\nu_{\Sigma^i_{\mathbf{t}}}} |\nabla_{\Sigma^i_{\mathbf{t}}} u_i|}{\prod_{j=1}^{i-1} |\nabla_{\Sigma^j_{\mathbf{t}}} u_j|} \,\mathrm{d}A\,\mathrm{d}\mathbf{t} \\
&\quad = \frac{1}{2} \sum_{j=i}^{n} \int_{F_{\pm L,i}} \pm((g_{\Sigma^i_{\mathbf{t}}})_{ij,j} - (g_{\Sigma^i_{\mathbf{t}}})_{jj,i}) \,\mathrm{d}A \\
&\qquad + \frac{1}{2} \sum_{j=i+1}^{n} \int_{F_{\pm L,j}} \pm((g_{\Sigma^i_{\mathbf{t}}})_{ji,i} - (g_{\Sigma^i_{\mathbf{t}}})_{ii,j}) \,\mathrm{d}A \\
&\qquad + O(L^{n-2-2q}).
\end{aligned}
$$

We can now combine this with Eq. (15) to obtain

$$
\begin{aligned}
&\int_{[-L,L]^n} \int_{\Sigma_{L,\mathbf{t}}} \frac{1}{2} \left(\sum_{i=1}^{n} \frac{|\nabla^2_{\Sigma^i_{\mathbf{t}}} u_i|}{|\nabla_{\Sigma^i_{\mathbf{t}}} u_i|^2 \cdot N_{\mathbf{t}}} + \frac{R}{N_{\mathbf{t}}} \right) \mathrm{d}A\,\mathrm{d}\mathbf{t} + O(L^{n-2-2q}) \\
&\quad = \frac{1}{2} \sum_{i=1}^{n} \Bigg(\sum_{j=i+1}^{n} \int_{F_{\pm L,j}} \pm((g_{\Sigma^i_{\mathbf{t}}})_{ji,i} - (g_{\Sigma^i_{\mathbf{t}}})_{ii,j}) \,\mathrm{d}A \\
&\qquad\qquad + \sum_{j=i}^{n} \int_{F_{\pm L,i}} \pm((g_{\Sigma^i_{\mathbf{t}}})_{ij,j} - (g_{\Sigma^i_{\mathbf{t}}})_{jj,i}) \,\mathrm{d}A \Bigg) \\
&\quad = \frac{1}{2} \sum_{i,j=1}^{n} \int_{F_{\pm L,i}} \pm((g_{\Sigma^i_{\mathbf{t}}})_{ij,j} - (g_{\Sigma^i_{\mathbf{t}}})_{jj,i}) \,\mathrm{d}A.
\end{aligned}
$$

Using Eq. (18) we can replace $g_{\Sigma^i_{\mathbf{t}}}$ with g, incurring only another

$$
\int_{F_{\pm L,i}} O(|u|^{-1-2q}) \,\mathrm{d}A = O(|u|^{n-2-2q})
$$

term. We thus obtain (after dividing by $(n-1)\omega_{n-1}$)

$$\begin{aligned}\int_{[-L,L]^n}\int_{\Sigma_{L,\mathbf{t}}}\frac{1}{2(n-1)\omega_{n-1}}\left(\sum_{i=1}^{n}\frac{|\nabla^2_{\Sigma^i_{\mathbf{t}}}u_i|}{|\nabla_{\Sigma^i_{\mathbf{t}}}u_i|^2\cdot N_{\mathbf{t}}}+\frac{R}{N_{\mathbf{t}}}\right)\mathrm{d}A\,\mathrm{d}\mathbf{t}\\ =\frac{1}{2(n-1)\omega_{n-1}}\sum_{i,j=1}^{n}\int_{F_{\pm L,i}}\pm(g_{ij,j}-g_{jj,i})\,\mathrm{d}A+O(|u|^{n-2-2q}).\end{aligned}$$

Letting L go to ∞ and noticing that $n-2-2q<0$, we obtain our main result,

$$\int_{\mathbb{R}^n}\int_{\Sigma_{\mathbf{t}}}\frac{1}{2(n-1)\omega_{n-1}}\left(\sum_{i=1}^{n}\frac{|\nabla^2_{\Sigma^i_{\mathbf{t}}}u_i|}{|\nabla_{\Sigma^i_{\mathbf{t}}}u_i|^2\cdot N_{\mathbf{t}}}+\frac{R}{N_{\mathbf{t}}}\right)\mathrm{d}A\,\mathrm{d}\mathbf{t}=m.$$

In particular the mass must be positive if $R\geq 0$.

Appendix A. Some computations of asymptotic behavior

We compute the following to justify Eq. (16). Let from now on $n_{\Sigma^i_{\mathbf{t}}}$ be the normal vector to $\Sigma^i_{\mathbf{t}}$ inside $\Sigma^{i+1}_{\mathbf{t}}$.

Lemma 15. *For all $i=1,\dots,n$, and $|u|=L$ we have*

$$|\nabla_{\Sigma^i_{\mathbf{t}}}u_i|^2=g^{ii}+O_2(|u|^{-2q})$$

and

$$n_{\Sigma^i_{\mathbf{t}}}=\partial_i+O_2(|u|^{-q}).$$

Proof. Note first that for all i, the normal to the level set $S^i_{t_i}$ has coordinates

$$g^{jk}\delta_{ik}=\delta_{ik}+O_2(|u|^{-q})$$

(since it is equal to $(\mathrm{d}u_i)^\sharp$, and $\mathrm{d}u_i$ has coordinates δ_{ik}). In particular, the second claim is true for $n=1$.

Note also that $|\nabla u_i|^2=|\mathrm{d}u_i|^2=g^{ii}=\delta^{ii}+O(|u|^{-q})$ (by a Taylor expnasion of the matrix inverse).

We proceed further by induction. Assume the second claim is true for all $i\leq j$. Then we obtain on one hand

$$|\nabla_{\Sigma^j_{\mathbf{t}}}u_j|^2=|\nabla u_j|^2-\sum_{k=1}^{j}\langle n_{\Sigma^k_{\mathbf{t}}},\nabla u_j\rangle^2=g^{jj}+O_2(|u|^{-2q}),$$

since

$$\langle n_{\Sigma^k_{\mathbf{t}}},\nabla u_j\rangle=|\nabla u_j|\cdot(\langle\partial_k,\partial_j\rangle+O_2(|u|^{-q}))=O_2(|u|^{-q}).$$

On the other hand we get

$$\begin{aligned}n_{\Sigma^{j+1}_{\mathbf{t}}}&=\partial_{j+1}+O_2(|u|^{-q})-\sum_{k=1}^{j}n_{\Sigma^j_{\mathbf{t}}}\cdot\langle n_{\Sigma^k_{\mathbf{t}}},\partial_{j+1}+O_2(|u|^{-q})\rangle\\&=\partial_{j+1}+O_2(|u|^{-q}).\end{aligned}$$

□

Notice also that, for $j,k\geq i$,

$$g_{jk}=(g_{\Sigma^i_{\mathbf{t}}})_{jk}+\sum_{l=1}^{i-1}\langle n_{\Sigma^l_{\mathbf{t}}},\partial_j\rangle\langle\Sigma^l_{\mathbf{t}},\partial_k\rangle=(g_{\Sigma^i_{\mathbf{t}}})_{jk}+O_2(|u|^{-2q}),\tag{17}$$

where we have used the second part of Lemma 15. In particular we obtain

$$(g_{\Sigma^i_{\mathbf{t}}})_{jk,l} = g_{jk,l} + O_1(|u|^{-1-2q}). \tag{18}$$

Appendix B. What happens if we have a boundary?

If we want to generalize the computations of Section 5.1, we will have to compute another boundary term over ∂M_{ext}, namely

$$\sum_{i=1}^{n} \int_{\mathbb{R}^{i-1}} \int_{\partial M_{\text{ext}} \cap \Sigma^i_{\mathbf{t}}} \frac{\partial_{\nu_{\Sigma^i_{\mathbf{t}}}} |\nabla_{\Sigma^i_{\mathbf{t}}} u_i|}{\prod_{j=1}^{i-1} |\nabla_{\Sigma^j_{\mathbf{t}}} u_j|} \, \mathrm{d}A \, \mathrm{d}\mathbf{t}.$$

Here we assume that all the u_i admit ∂M_{ext} as a (connected component of a) regular level set. But then there exists some $\mathbf{t} \in \mathbb{R}^n$ such that $\partial M_{\text{ext}} \subset \Sigma^i_{\mathbf{t}}$ for all $i \leq n$. Note however that for $i \geq 2$, $\Sigma^i_{\mathbf{t}}$ is at most $(n-1)$ dimensional, and thus in particular the normal vector $\nu_{\Sigma^i_{\mathbf{t}}}$ to ∂M_{ext} inside $\Sigma^i_{\mathbf{t}}$ must be zero. Thus we are only left with the term with $i = 1$, i.e.

$$\int_{\partial M_{\text{ext}}} \partial_\nu |\nabla u_1| \, \mathrm{d}A.$$

We can now repeat the computation of [3, Section 6.2] with $k = 0$ (and thus also $\mathcal{K} = 0$) and obtain

$$\int_{\partial M_{\text{ext}}} \partial_\nu |\nabla u_1| \, \mathrm{d}A = \int_{\partial M_{\text{ext}}} H \cdot |\nu(u_1)| \, \mathrm{d}A,$$

where H and ν respectively are the mean curvature of and normal vector to ∂M_{ext} inside M_{ext}.

Noting now that the boundary of our exterior regions is minimal (i.e. $H = 0$), we see that this additional term vanishes entirely.

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