A confusion about the Stern integral inequality

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Everything below is pretty much directly adopted from my Bachelor's Thesis (which more or less just followed [Bra+19]), though I have tried to keep track of additional terms (marked in red) to strengthen the encountered inequalities into equalities.

The following is adopted from [Bra+19, Proposition 4.2] but (surely due to some mistake I made somewhere) I have seemingly proven an equality without even requiring any additional terms.

Proposition 1. *Let (Ω, g) be a compact 3-dimensional oriented Riemannian manifold with piecewise smooth boundary $\partial\Omega = P_1 \sqcup P_2$, having outward unit normal ν. Let $u\colon \Omega \to \mathbb{R}$ be a harmonic function (i.e. $\Delta u = 0$) such that $\partial_\nu u = 0$ on P_1 and $|\nabla u| > 0$ on P_2. If $\overline{u}$ and $\underline{u}$ denote the maximum and minimum of u and S_t are t-level sets of u, then*

$$\int_{\underline{u}}^{\overline{u}} \int_{S_t} \frac{1}{2}\left(\frac{|\nabla^2 u|}{|\nabla u|^2} + R\right) dA + \int_{\partial S_t \cap P_1} H_{P_1}\, dl\, dt = \int_{\underline{u}}^{\overline{u}} \Big(2\pi\chi(S_t) - \int_{\partial S_t \cap P_2} \kappa_{\partial S_t}\, dl\Big)\, dt + \int_{\tilde{P}_2} \partial_\nu |\nabla u|\, dA\,,$$

where $\chi(S_t)$ denotes the Euler characteristic of the level sets, $\kappa_{\partial S_t}$ denotes the geodesic curvature of ∂S_t in S_t and H_{P_1} denotes the mean curvature of P_1.

Proof of Proposition 1. During the following proof, we will be considering

$$\phi_\varepsilon := \sqrt{|\nabla u|^2 + \varepsilon}$$

for $\varepsilon > 0$ instead of $|\nabla u|$, since we cannot control the behavior of integrands like $\Delta|\nabla u|$ and $\partial_\nu |\nabla u|$ at critical points of u (where $|\nabla u| = 0$).

We find

$$\begin{aligned}
\Delta\phi_\varepsilon &= \nabla_i \nabla^i \sqrt{|\nabla u|^2 + \varepsilon} \\
&= \nabla_i \frac{\nabla^i |\nabla u|^2}{2\phi_\varepsilon} \\
&= \frac{\Delta|\nabla u|^2}{2\phi_\varepsilon} - \frac{|\nabla|\nabla u|^2|^2}{4\phi_\varepsilon^3} \\
&\underset{\substack{\uparrow \\ \text{Bochner's identity}}}{=} {\phi_\varepsilon}^{-1}(|\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) - \phi_\varepsilon^{-2}|\nabla u|^2 |\nabla|\nabla u||^2.
\end{aligned} \tag{1}$$

Thus on a regular level set S [Bra+19, Lemma 4.1] yields

$$\Delta\phi_\varepsilon = \frac{1}{2\phi_\varepsilon}(|\nabla^2 u|^2 + |\nabla u|^2 (R_\Omega - R_S) + 2 \cdot (1 - \phi_\varepsilon^{-2}|\nabla u|^2)|\nabla|\nabla u||^2)\,. \tag{2}$$

Let now $\mathscr{A} \subset [\underline{u}, \overline{u}]$ be an open set containing all the critical values of u (images of points where $\nabla u = 0$), and let $\mathscr{B} = [\underline{u}, \overline{u}] \setminus \mathscr{A}$ be the complementary set.

Then the divergence theorem yields

$$\begin{aligned}\int_{P_1 \cap u^{-1}(\mathscr{A})} \partial_\nu \phi_\varepsilon \, dA + \int_{P_1 \cap u^{-1}(\mathscr{B})} \partial_\nu \phi_\varepsilon \, dA + \int_{P_2} \partial_\nu \phi_\varepsilon \, dA = \int_{\partial\Omega} \partial_\nu \phi_\varepsilon \, dA \\ = \int_\Omega \Delta \phi_\varepsilon \, dV = \int_{u^{-1}(\mathscr{A})} \Delta \phi_\varepsilon \, dV + \int_{u^{-1}(\mathscr{B})} \Delta \phi_\varepsilon \, dV \, . \end{aligned} \quad (3)$$

We first deal with the integrals over $P_1 \cap u^{-1}(\mathscr{A})$ and $u^{-1}(\mathscr{A})$. Since

$$\frac{|\nabla u|}{\phi_\varepsilon} |\nabla |\nabla u|| = \frac{1}{2\phi_\varepsilon} \nabla(g(\nabla u, \nabla u)) = \frac{g(\nabla^2 u, \nabla u)}{\phi_\varepsilon} \underset{\substack{\uparrow \\ \text{Cauchy–Schwarz}}}{\leqslant} \frac{|\nabla^2 u| |\nabla u|}{\phi_\varepsilon} \underset{\substack{\uparrow \\ |\nabla u| \leqslant \phi_\varepsilon}}{\leqslant} |\nabla^2 u| \, ,$$

Eq. (1) and another application of Cauchy–Schwarz give on $u^{-1}(\mathscr{A})$

$$\Delta \phi_\varepsilon \geqslant {\phi_\varepsilon}^{-1} \mathrm{Ric}(\nabla u, \nabla u) \geqslant -|\mathrm{Ric}||\nabla u| \, .$$

Thus we can decompose into level sets of u using the coarea formula to get

$$\begin{aligned} -\int_{u^{-1}(\mathscr{A})} \Delta \phi_\varepsilon \, dV &\leqslant \int_{u^{-1}(\mathscr{A})} |\mathrm{Ric}||\nabla u| \, dV \, , \\ &= \int_{t \in \mathscr{A}} \int_{S_t} |\mathrm{Ric}| \, dA \, dt \\ &\leqslant C \int_{t \in \mathscr{A}} \mathscr{H}^2(S_t) \, dt \end{aligned} \quad (4)$$

where $\mathscr{H}^2(S_t)$ is the Hausdorff measure of the level sets and C is some constant bounding the Ricci curvature.

Similarly, on $P_1 \cap u^{-1}(\mathscr{A})$ we have

$$\begin{aligned} \partial_\nu \phi_\varepsilon &= \frac{\nu^i \nabla_i \nabla_j u \nabla^j u}{\phi_\varepsilon} \, , \\ &= \frac{\nabla^i u \nabla_i \nabla_j u \nu^j}{\phi_\varepsilon} \\ &= \frac{g(\nabla_{\nabla u} \nabla u, \nu)}{\phi_\varepsilon} \\ &= -\frac{g(\nabla_{\nabla u} \nu, \nabla u)}{\phi_\varepsilon} \\ &\leqslant |\nabla u||A_{P_1}| \leqslant |\nabla u| C \end{aligned}$$

where we have used $g(\nabla u, \nu) = 0$ by the Neumann boundary condition of u on P_1. We thus get by the coarea formula

$$\int_{P_1 \cap u^{-1}(\mathscr{A})} \partial_\nu \phi_\varepsilon \, dA \leqslant \int_{t \in \mathscr{A}} \int_{P_1 \cap S_t} |A_{P_1}| \, dl \, dt \leqslant C \int_{t \in \mathscr{A}} \mathscr{H}^1(\partial S_t \cap P_1) \, dt \, . \quad (5)$$

Let us now deal with the integrals over $P_1 \cap u^{-1}(\mathcal{B})$ and $u^{-1}(\mathcal{B})$. On P_1 we have as before

$$\partial_\nu \phi_\varepsilon = -\frac{g(\nabla_{\nabla u}\nu, \nabla u)}{\phi_\varepsilon}$$

where we have used the Neumann boundary condition in the last line. Let n now denote the normal vector $n^i = \frac{\nabla^i u}{|\nabla u|}$ to S_t. This yields

$$\partial_\nu \phi_\varepsilon = -{\phi_\varepsilon}^{-1}|\nabla u|^2 A_{P_1}(n,n) = -{\phi_\varepsilon}^{-1}|\nabla u|^2 (H_{P_1} - \operatorname{tr}_{S_t} A_{P_1})\,.$$

Let $v \in T_pP_1 \cap T_pS_t$ be a normed vector (there are only two choices, since the vector space is one-dimensional). Then (as S_t is orthogonal to P_1 by the Neumann boundary condition of u on P_1)

$$\operatorname{tr}_{S_t} A_{P_1} = A_{\partial\Omega}(v,v) = \langle \nabla_v v, -n\rangle = \kappa_{S_t\cap P_1} = \kappa_{\partial S_t}\,.$$

Thus decomposing P_1 into level sets of u using the coarea formula yields

$$\int_{P_1\cap u^{-1}(\mathcal{B})} \partial_\nu \phi_\varepsilon \, dA = -\int_{t\in\mathcal{B}} \left(\int_{\partial S_t \cap P_1} {\phi_\varepsilon}^{-1}|\nabla u|(H_{P_1} - \kappa_{\partial S_t})\right) dt\,. \tag{6}$$

Meanwhile on $\mathcal{B}^{-1}$ applying the coarea formula and Eq. (2) produces

$$\int_{u^{-1}(\mathcal{B})} \Delta\phi_\varepsilon \, dV = \\ \frac{1}{2}\int_{t\in\mathcal{B}}\int_{S_t} \frac{|\nabla u|}{\phi_\varepsilon}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + (R_\Omega - R_{S_t}) + 2\cdot(|\nabla u|^{-2} - \phi_\varepsilon^{-2})\cdot|\nabla|\nabla u||^2\right) dA\, dt\,. \tag{7}$$

We combine Eq. (7) and Eq. (6) with Eq. (3) and obtain

$$\frac{1}{2}\int_{t\in\mathcal{B}}\int_{S_t} \frac{|\nabla u|}{\phi_\varepsilon}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + R_\Omega\right) dA\, dt \\ - \int_{t\in\mathcal{B}}\left(\frac{1}{2}\int_{S_t}\frac{|\nabla u|}{\phi_\varepsilon} R_{S_t}\, dA + \int_{\partial S_t\cap P_1} \frac{|\nabla u|}{\phi_\varepsilon}(\kappa_{\partial S_t} - H_{P_1})\right) dt - \int_{P_2} \partial_\nu\phi_\varepsilon\, dA = \\ \int_{P_1\cap u^{-1}(\mathcal{A})} \partial_\nu \phi_\varepsilon\, dA - \int_{u^{-1}(\mathcal{A})} \Delta\phi_\varepsilon\, dV + \int_{t\in\mathcal{B}}\int_{S_t} (\phi_\varepsilon^{-2} - |\nabla u|^{-2})\cdot|\nabla|\nabla u||^2.$$

Taking absolute values and using Eq. (4) and Eq. (5) yields

$$\left|\frac{1}{2}\int_{t\in\mathcal{B}}\int_{S_t} \frac{|\nabla u|}{\phi_\varepsilon}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + R_\Omega\right) dA\, dt \right. \\ \left. - \int_{t\in\mathcal{B}}\left(\frac{1}{2}\int_{S_t}\frac{|\nabla u|}{\phi_\varepsilon} R_{S_t}\, dA + \int_{\partial S_t\cap P_1} \frac{|\nabla u|}{\phi_\varepsilon}(\kappa_{\partial S_t} - H_{P_1})\right) dt - \int_{P_2} \partial_\nu\phi_\varepsilon\, dA\right| \leqslant \\ C\int_{t\in\mathcal{A}} (\mathcal{H}^1(\partial S_t\cap P_1) + \mathcal{H}^2(S_t))\, dt + \int_{t\in\mathcal{B}}\int_{S_t} (\phi_\varepsilon^{-2} - |\nabla u|^{-2})\cdot|\nabla|\nabla u||^2. \tag{8}$$

Since Ω is compact and $\mathscr{B}$ closed, $|\nabla u|$ and $|\nabla|\nabla u||$ are uniformly bounded from below on $u^{-1}(\mathscr{B})$. In particular note that the additional term is bounded! Also, on P_2 (where $|\nabla u| \neq 0$) we have

$$\partial_\nu \phi_\varepsilon = \frac{|\nabla u|}{\phi_\varepsilon} \partial_\nu |\nabla u| \to \partial_\nu |\nabla u| \quad \text{as } \varepsilon \to 0$$

We can thus now take the limit $\varepsilon \to 0$ in Eq. (3) and get

$$\begin{aligned} &\Bigg| \frac{1}{2} \int_{t \in \mathscr{B}} \int_{S_t} \left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + R_\Omega \right) dA\, dt \\ &\quad - \int_{t \in \mathscr{B}} \left(2\pi \chi(S_t) - \int_{\partial S_t \cap P_2} \kappa_{\partial S_t} - \int_{\partial S_t \cap P_1} H_{P_1} \right) dt - \int_{P_2} \partial_\nu |\nabla u|\, dA \Bigg| \leqslant \\ &\qquad C \int_{t \in \mathscr{A}} (\mathscr{H}^1(\partial S_t \cap P_1) + \mathscr{H}^2(S_t))\, dt + 0\,. \end{aligned} \tag{9}$$

where we have also applied the Gauss–Bonnet theorem to S_t.

By Sard's theorem ([Sar42]), the set of critical values has measure 0 and we thus may take the measure of $\mathscr{A}$ to be arbitrarily small. Since

$$t \mapsto \mathscr{H}^1(\partial S_t \cap P_1) + \mathscr{H}^2(S_t)$$

is integrable by the coarea formula, taking $|\mathscr{A}| \to 0$ in Eq. (9) leads to

$$\begin{aligned} &\int_{\underline{u}}^{\overline{u}} \int_{S_t} \frac{1}{2} \left(\frac{|\nabla^2 u|}{|\nabla u|^2} + R \right) dA + \int_{\partial S_t \cap P_1} H_{P_1}\, dl\, dt \\ &\qquad = \int_{\underline{u}}^{\overline{u}} \left(2\pi \chi(S_t) - \int_{\partial S_t \cap P_2} \kappa_{\partial S_t}\, dl \right) dt + \int_{P_2} \partial_\nu |\nabla u|\, dA\,. \end{aligned}$$

■

By carrying this result through the calculations in [Bra+19, Section 6], while only changing the step in Equation 6.4, where we do not do not use $\chi(S_t) \leqslant 1$ and just leave $\chi(S_t)$ as it is, one arrives at

Corollary 2 (Equality version of [Bra+19, Theorem 1.2])**.** *Let (M_{ext}, g) be an exterior region of an asymptotically flat Riemannian 3-manifold (M, g) with mass m. Let u be a harmonic function on (M_{ext}, g) satisfying Neumann boundary conditions at ∂M, and which is asymptotic to one of the asymptotically flat coordinate functions of the associated end. Then there exists a closed region Ω bounded by an infinite coordinate cylinder $\partial\Omega$ such that all the level sets S_t of u meet $\partial\Omega$ transversally and have Euler characteristic $\chi(S_t \cap \Omega) \leqslant 1$, and we have*

$$m = \frac{1}{16\pi} \int_{M_{\text{ext}}} \left(\frac{|\nabla^2 u|^2}{|\nabla u|} + R_g |\nabla u| \right) dV + \frac{1}{2} \int_{-\infty}^{\infty} (1 - \chi(S_t \cap \Omega))\, dt.$$

In particular, if the scalar curvature is nonnegative, then $m \geqslant 0$. Furthermore, if $m = 0$ then $(M, g) = (\mathbb{R}^3, \delta)$.

References

[Bra+19] Hubert L. Bray, Demetre P. Kazaras, Marcus A. Khuri, and Daniel L. Stern. “Harmonic Functions and The Mass of 3-Dimensional Asymptotically Flat Riemannian Manifolds”. Nov. 15, 2019. arXiv: `1911.06754 [gr-qc]`. URL: `http://arxiv.org/abs/1911.06754` (visited on 03/22/2022).

[Sar42] Arthur Sard. “The Measure of the Critical Values of Differentiable Maps”. In: *Bulletin of the American Mathematical Society* 48.12 (1942), pp. 883–890. ISSN: 0273-0979, 1088-9485. DOI: `10.1090/S0002-9904-1942-07811-6`. URL: `https://www.ams.org/bull/1942-48-12/S0002-9904-1942-07811-6/` (visited on 09/03/2023).