Harmonic Functions and the Positive Mass Theorem for Asymptotically Flat Half-Spaces

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1 Introduction

The (spacetime) positive mass theorem is a central result in the study of general relativity and differential geometry, originally proved by Richard Schoen and Shing-Tung Yau in 1979 [**schoenProofPositiveMass1979**] employing stable minimal hypersurfaces and independently by Edward Witten in 1981 [**wittenNewProofPositive1981**] using spinor techniques.

In the following, we will explore a relatively new proof of the Positive Mass theorem using (spacetime) harmonic functions, and in particular consider the case of rigidity. We will then look at how these harmonic functions look in some simple example cases. Finally, we will will apply this method to the Positive Mass theorem on asymptotically flat half spaces with connected (non-compact) boundary, acquiring an explicit lower bound for the mass in the process, i.e. we will prove the following theorem (notation and concepts will be introduced later):

Theorem 1.1. *Let (M, g) be an asymptotically flat half-space (M, g) of dimension $n = 3$ with horizon boundary $\Sigma \subset M$, associated exterior region $M(\Sigma)$ and connected non-compact boundary ∂M. Let (x_1, x_2, x_3) be asymptotically flat coordinates such that outside of a compact set, M is diffeomorphic to $\{x \in \mathbb{R}^3_+ \mid |x| > r_0\}$ for some $r_0 > 0$. Assume that the following three conditions hold*

- $R \geqslant 0$ *in* $M(\Sigma)$.
- $H \geqslant 0$ *on* $M(\Sigma) \cap \partial M$.

Then there exists a unique harmonic function u on $M(\Sigma)$ asymptotic to x_3 fulfilling zero Dirichlet boundary conditions on $\partial M \cap M(\Sigma)$ and zero Neumann boundary conditions on Σ, and we have

$$m \geqslant \frac{1}{16\pi} \int_{M(\Sigma)} \left(\frac{|\nabla^2 u|^2}{|\nabla u|} + R|\nabla u| \right) dV + \frac{1}{8\pi} \int_{\partial M \cap M(\Sigma)} H|\nabla u| \, dA,$$

where m is the ADM mass of M. Equality $m = 0$ occurs if and only if (M, g) is isometric to $(\mathbb{R}^3_+, \delta)$.

On 15^{th} June, 2023, a preprint [**batistaHarmonicLevelSet2023**] was published on the arXiv, titled „A harmonic level set proof of a positive mass theorem“ by Rondinelle Batista and Levi Lopes de Lima. Batista and de Lima prove the same result while using primarily the same methods as this thesis (the proof of rigidity in this thesis is different and more elementary). I only noticed the arXiv preprint on 13^{th} September, 2023, after my own proof of Theorem 1.1 was already completed and written up.

1.1 Physical Motivation

The positive mass theorem was originally motivated by the study of general relativity, but is also (particularly in the so-called time-symmetric or Riemannian case) of independent importance to differential geometry. We will do a quick exposition of both of these perspectives. Physically, a less general version of the positive mass theorem can informally be expressed as the following:

Consider a static (i.e. time-independent) mass distribution ρ in $\mathbb{R}^3$ that is compactly supported in some finite volume V (it would suffice if the mass distribution fell off sufficiently quickly towards infinity, but this case is easier to reason about).

Then in the Newtonian theory of gravity, this mass distribution would at large distances look like a point mass of some total mass M. Due to the linear nature of Newtonian gravity, we can calculate that M is just $\int \rho \, dV$.

But when we consider Einstein's theory of gravity (via General Relativity), though we can still assign a total mass, now called the *ADM mass* M (in practice this takes the form of an integral expression over large coordinate spheres, where we take the limit as the radius goes to infinity), we lose the linearity of Newtonian Gravity and we cannot anymore identify M with the integral of the individual masses. Here our mass distribution bends spacetime in some (possibly very complicated) way, but the ADM mass tells us that his spacetime geometry asymptotically looks like the geometry around a Schwarzschild black hole of mass M.

The positive mass theorem now states that even though we lose the relation to the integral of the mass distribution, we retain at least some good behaviour of the mass: If the mass distribution is non-negative everywhere, then we also have $M \geqslant 0$, i.e. there exists no configuration of positive masses (however complicated) that acts like a black hole of negative mass (a white hole) at large distances. Compare [**leeGeometricRelativity2019**] for more details.

When expressing this theorem mathematically, we leave behind a lot of the physical details. In particular, we directly consider the scalar curvature R instead of the mass distribution (as the scalar curvature is proportional to mass density in static spacetimes). Since we define the ADM mass in terms of the asymptotic geometric behaviour as well, we reduce the physical statement to a purely geometric one. This leads us to another approach to motivate the theorem, at least for the time-symmetric case (the following formulation is from [**braySpacetimeHarmonicFunctions2021**]):

Every compactly supported perturbation of the Euclidean metric on $\mathbb{R}^n$ must somewhere decrease its scalar curvature. This is a kind of extremality property of the Euclidean metric. It follows directly from the Geroch conjecture – the fact that the torus $\mathbb{T}^n$ does not admit a metric of positive scalar curvature – by identifying the ends of a large coordinate cube (containing the compact set on which the perturbation takes place). The Riemannian positive mass theorem then is an extension of this extremality property to the nonnegativity of the ADM-mass on manifolds that are *asymptotically euclidean* instead of straight up equal to the euclidean geometry outside a compact set. We'll call these manifolds *asymptotically flat*.

These ideas extend naturally to *asymptotically flat half spaces* (which asymptotically look like $\mathbb{R}^n_+$ instead of $\mathbb{R}_+$). One application of the positive mass theorem for these asymptotically flat half-spaces is during the proof [**almarazConvergenceScalarflatMetrics2015**] of the convergence of a certain Yamabe-type flow on compact manifolds N with boundary ∂N. The asymptotically flat half spaces appear during a step where it is necessary to look at $N \setminus \{x\}$ for $x \in \partial N$.

2 Prerequisites

To read this thesis, a basic understanding of Riemannian manifolds as well as in particular some facts about Riemannian submanifolds are required. For anyone with basic knowledge about differential geometry (definitions of manifolds, tangent bundles and differential forms), an introduction of the relevant concepts from Riemannian geometry can be found in the appendix (Appendix B and Appendix C). For a more complete look at especially Riemannian Geometry, see [**petersenRiemannianGeometry2006**]. For an introduction to differential geometry and manifolds, see [**leeIntroductionSmoothManifolds201**

Further miscellaneous definitions and results that are generally common knowledge and in particular not specific to asymptotically flat half-spaces can be found in Appendix D.

Notation 2.1. We will in this thesis always let M denote a 3-dimensional Riemannian manifold, equipped with the unique Levi-Civita-Connection. Note that by ∇ we will always denote this covariant derivative / the Levi-Civita connection. In particular we will have $\nabla f = df$, and $\operatorname{grad} f = (df)^\sharp$, such that in coordinates df will be given by $\nabla_i f = \partial_i f$ and $\operatorname{grad} f$ will be given by $\nabla^i f$.

3 The mass of an asymptotically flat half-space

We will now establish the necessary definitions (mostly adapted from [**almarazPositiveMassTheorem2**
[**eichmairDoublingAsymptoticallyFlat2023**] and [**brayHarmonicFunctionsMass2019**]) to state the main result of this thesis.

Definition 3.1 (Asymptotically flat half-space)**.** Let (M, g) be a connected, complete Riemannian manifold of dimension 3, with scalar curvature R and a smooth non-compact boundary ∂M with mean curvature H (computed as the divergence along ∂M of an outward pointing unit normal ν, see Appendix C and Remark C.7 for more details on our definition and in particular the choice of sign).

We call (M, g) an *asymptotically flat half-space* with decay rate $\tau > 0$ if there exists a compact subset K such that $M \setminus K$ consists of a finite number of connected components M^i_{end} called *ends*, such that for each of these ends there exists a diffeomorphism $\Phi\colon M^i_{\text{end}} \to \{x \in \mathbb{R}^3_+ \mid |x| > r_0\}$ and such that in the coordinate system given by this diffeomorphism we have the following asymptotic as $r \to \infty$:

$$|\partial^l(g_{ij} - \delta_{ij})| = O(r^{-\tau - l}) \tag{1}$$

for $l = 0, 1, 2$. Here $r = |x|$ and δ is the Euclidean metric on $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 \mid x_3 \geqslant 0\}$.

Remark 3.2. We follow [**eichmairDoublingAsymptoticallyFlat2023**] in calling these manifolds „asymptotically flat half-spaces“. Note that they are often (in particular in [**almarazPositiveMassTheorem2016**]) also referred to as „asymptotically flat manifolds with noncompact boundary“.

Notation 3.3. In the following, we will often use the Einstein summation convention with the index ranges $i, j, \ldots = 1, \ldots, 3$ and $\alpha, \beta, \ldots = 1, 2$: Repeated Latin indices will be summed over from 1 up to 3 (in general this would be up to n, the dimension of our manifold) and repeated Greek indices will be summed over from 1 up to 2 (in general this would be up to $n - 1$).

Note that, along ∂M, $\{\partial_\alpha\}_\alpha$ spans $T_{\partial M}$, while ∂_n points inwards.

Definition 3.4. If R and H are integrable over M and ∂M respectively and $\tau > 1/2$, then the *mass* of each end of M is well defined and (introducing the notation G_i for the coordinate dependent quantity $\sum_j (g_{ij,j} - g_{jj,i})$, compare [**almarazPositiveMassTheorem2016**] where instead C_i is used) given by

$$\mathfrak{m}_{(M^i_{\text{end}}, g)} = \frac{1}{16\pi} \lim_{r \to \infty} \left(\int_{\mathbb{S}^2_{r,+}} G_i \mu^i \, dA + \int_{\mathbb{S}^1_r} g_{\alpha 3} \theta^\alpha \, dl \right),$$

where the integrals are computed in the asymptotically flat chart, $\mathbb{S}^2_{r,+}(0) = \{\mathbb{R}\}^3_+ \cap \mathbb{S}^2_r(0)$ is a large upper coordinate hemisphere with outward unit normal μ, and θ is the outward pointing unit co-normal to $\mathbb{S}^1_r = (\{\mathbb{R}\}^2 \times \{0\}) \cap \mathbb{S}^2_r(0) = \partial \mathbb{S}^2_{r,+}$, oriented as the boundary of $(\partial M)_r \subset \partial M$.

Remark 3.5. A vector $v \in T_pM$ is *co-normal* to a submanifold $\Sigma \subset M$ with boundary $\partial\Sigma$ if $p \in \partial\Sigma$ and $v \in T_p\Sigma \cap N_p\partial\Sigma \subset T_pM$, i.e. v is tangent to the submanifold but normal to its boundary. See also Fig. 1 for a picture.

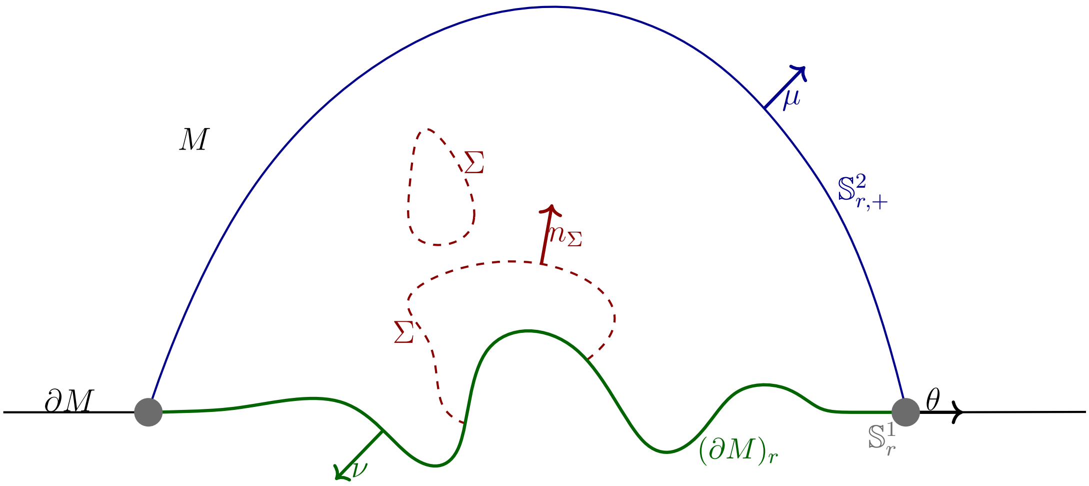

Figure 1: A cross section of a 3-dimensional asymptotically flat half space with horizon boundary and a large coordinate sphere (from Definition 3.4). Note that the gray points on the boundary of $(\partial M)_r$ are just the part of the circle $\mathbb{S}^1_r$ visible in this cross section.

Remark 3.6. In the definition above, the factor $1/(16\pi)$ is a normalization factor used also for the ADM mass of asymptotically flat manifolds with the full $\mathbb{R}^n$ as a model space, where it ensures that we recover the mass of the Schwarzschild solution.

Note that thus in our case of asymptotically flat half-space, the mass of a *Schwarzschild half-space* $M_{m,+} = \{x \in \mathbb{R}^n_+ \mid |x| \geqslant (m/2)^{1/(n-2)}\}$ with the conformal metric

$$g_m = \omega^4 \cdot \delta, \qquad \text{where } \omega = 1 + \frac{m}{2|x|}, \quad m > 0$$

will be

$$\mathfrak{m}_{(M_{m,+}, g_m)} = \frac{m}{2},$$

which is half the ADM mass of the standard Schwarzschild space.

In [**almarazPositiveMassTheorem2016**], Almaraz, Barbosa, and de Lima showed that this mass is well defined and a geometric invariant, and, in fact, non-negative under suitable energy conditions:

Theorem 3.7. *For (M, g) as above in Definition 3.4, if $R \geqslant 0$ and $H \geqslant 0$ on M and ∂M respectively, then*

$$\mathfrak{m}_{(M,g)} \geqslant 0,$$

with equality occurring if and only if (M, g) is isometric to $(\mathbb{R}^3_+, \delta)$.

For the positive mass theorem on 3-dimensional asymptotically flat manifolds modeled on the full $\mathbb{R}^3$, recently [**brayHarmonicFunctionsMass2019**] a new method using harmonic functions has been used to achieve a relatively elementary proof of the above theorem and in particular an explicit lower bound for the mass. The goal of this thesis is to establish an equivalent result for the case of asymptotically flat half spaces. We will need two further definitions adopted from [**eichmairDoublingAsymptoticallyFlat2023**] to unterstand the statement of our main result.

Definition 3.8. Let $\Sigma \subset M$ be a compact separating hypersurface satisfying $\partial\Sigma = \Sigma \cap \partial M$ with normal n_Σ pointing towards the closure $M(\Sigma)$ of the non-compact component of $M \setminus \Sigma$. We call a connected component Σ_0 of M *closed* if $\partial\Sigma_0 = \varnothing$ or a *free boundary hypersurface* if $\partial\Sigma_0 \neq 0$ and $n_{\Sigma_0}(x) \in T_x\partial M$ for every $x \in \Sigma_0 \cap \partial M = \partial\Sigma_0$ (i.e. if Σ_0 meets ∂M orthogonally along its boundary).

We say that an (M, g) has horizon boundary Σ if Σ is a non-empty compact minimal (i.e. having zero mean curvature) hypersurface, whose connected components are all either closed or free boundary hypersurfaces such that $M(\Sigma) \setminus \Sigma$ contains no minimal closed or free boundary hypersurfaces. Σ is also called an *outer most minimal surface* and the region $M(\Sigma)$ outside Σ is called an *exterior region*.

Remark 3.9. By [**koerberRiemannianPenroseInequality2020**] if $H \geqslant 0$ on ∂M, then there either exists a unique horizon boundary $\Sigma \subset M$ or M contains no compact hypersurfaces.

The main result of this thesis is then the following, which will prove Theorem 3.7 as a corollary:

Theorem 3.10. *For (M, g) as above in Definition 3.4, if an exterior region $M(\Sigma)$ exists, then there exists a unique harmonic function u asymptotic to the linear function x_3, satisfying zero Dirichlet boundary conditions on ∂M and zero Neumann boundary conditions on the horizon boundary Σ, and we have*

$$\mathfrak{m}_{(M,g)} \geqslant \frac{1}{16\pi} \int_{M(\Sigma)} \left(\frac{|\nabla^2 u|^2}{|\nabla u|} + R|\nabla u| \right) dV + \frac{1}{8\pi} \int_{\partial M \cap M(\Sigma)} H|\nabla u| \, dA. \tag{2}$$

In particular, if R and H are nonnegative, then the existence of $M(\Sigma)$ is guaranteed and the above inequality also gives nonnegativity of the mass.

4 Main basic integral inequality

The main tool which also motivates our use of harmonic functions is the following integral inequality relating scalar curvature to derivatives of harmonic. The following is adopted from [**brayHarmonicFunctionsMass2019**], which we change only very slightly (we allow $|\nabla u| = 0$ on P_2 as in [**hirschSpacetimeHarmonicFunctions2021**]).

Proposition 4.1. *Let (Ω, g) be a compact 3-dimensional oriented Riemannian manifold with piecewise smooth boundary $\partial\Omega = P_1 \sqcup P_2$, having outward unit normal ν. Let $u\colon \Omega \to \mathbb{R}$ be a harmonic function (i.e. $\Delta u = 0$) such that $\partial_{n_{P_1}} u = 0$ on P_1. Let $\tilde{P}_2 = \{x \in P_1 \mid \nabla u \neq 0\}$ be the set of regular points of u in P_2. If $\bar{u}$ and $\underline{u}$ denote the maximum and minimum of u and S_t are t-level sets of u, then*

$$\int_{\underline{u}}^{\bar{u}} \int_{S_t} \frac{1}{2}\left(\frac{|\nabla^2 u|}{|\nabla u|^2} + R\right) dA + \int_{\partial S_t \cap P_1} H_{P_1}\, dl\, dt$$
$$\leqslant \int_{\underline{u}}^{\bar{u}} \Big(2\pi\chi(S_t) - \int_{\partial S_t \cap P_2} \kappa_{\partial S_t}\, dl\Big)\, dt + \int_{\tilde{P}_2} \partial_\nu |\nabla u|\, dA,$$

where $\chi(S_t)$ denotes the Euler characteristic of the level sets, $\kappa_{\partial S_t}$ denotes the geodesic curvature of ∂S_t in S_t and H_{P_1} denotes the mean curvature of P_1.

For the proof we follow the [**brayHarmonicFunctionsMass2019**]

Lemma 4.2. *For u harmonic with level set S we have*

$$\operatorname{Ric}(\operatorname{grad} u, \operatorname{grad} u) = \frac{1}{2}|\nabla u|^2 (R_\Omega - R_S) + |\nabla|\nabla u||^2 - \frac{1}{2}|\nabla^2 u|.$$

Proof. Let S be a level set of u with induced metric γ, second fundamental form A_{ij}

and mean curvature H. The normal to S is then $\nu^i = \nabla^i u/|\nabla u|$ and Eq. (11) yields

$$\begin{aligned}
A_{ij} &= \gamma_i^k \gamma_j^l \nabla_k(\nabla_l u/|\nabla u|) \\
&= \gamma_i^k \gamma_j^l \frac{\nabla^2_{kl} u}{|\nabla u|} + (\dots)\gamma_j^l \nabla_l u \\
&= \gamma_i^k \gamma_j^l \frac{\nabla^2_{kl} u}{|\nabla u|} \\
&= (g_i^k - \nu_i \nu^k)(g_j^l - \nu^l \nu_j)\frac{\nabla^2_{kl} u}{|\nabla u|} \\
&= \frac{\overbrace{\nabla^2_{ij} u}^{\text{Term } T^1} - \overbrace{\nu_i \nu^k \nabla^2_{kj} u}^{T^2} - \overbrace{\nu_j \nu^l \nabla^2_{il} u}^{T^3} + \overbrace{\nu_i \nu_j \nu^k \nu^l \nabla_{kl} u}^{T^4}}{|\nabla u|}
\end{aligned}$$

and thus (below we use c.w. to denote which term „contracted with“ which other term)

$$\begin{aligned}
|A|^2 = {} & \frac{\overbrace{|\nabla^2 u|^2}^{T^1 \text{ c.w. } T^1} + \overbrace{(\nabla^2_{\nu\nu} u)^2}^{T^4 \text{ c.w. } T^4}}{|\nabla u|^2} \\
& + \frac{\overbrace{2\nu^k \nu_l \nabla^2_{kj} u (\nabla^2)^{lj} u}^{T^{2/3} \text{ c.w. } T^{2/3}} - \overbrace{4\nabla^2_{ij} u \nu^i \nu_k (\nabla^2)^{kj} u}^{T^1 \text{ c.w. } T^{2/3}} + \overbrace{2(\nabla^2_{\nu\nu} u)^2}^{T^{2/3} \text{ c.w. } T^{3/2}} - \overbrace{4(\nabla^2_{\nu\nu} u)^2}^{T^{2/3} \text{ c.w. } T^4} + \overbrace{2(\nabla^2_{\nu\nu} u)^2}^{T^1 \text{ c.w. } T^4}}{|\nabla u|^2} \\
= {} & \frac{|\nabla^2 u|^2 - 2g^{jk} \nabla^2_{kl} u \nabla^l u \nabla^2_{ij} u \nabla^i u/(|\nabla u|^2) + (\nabla^2_{\nu\nu} u)^2}{|\nabla u|^2} \\
= {} & \frac{|\nabla^2 u|^2 - (\nabla_k(\nabla_l u \nabla^l u)\nabla^k(\nabla_l \nabla^l u))/(2|\nabla u|^2) + (\nabla^2_{\nu\nu} u)^2}{|\nabla u|^2} \\
= {} & \frac{|\nabla^2 u|^2 - |\nabla|\nabla u|^2|^2/(2|\nabla u|^2) + (\nabla^2_{\nu\nu} u)^2}{|\nabla u|^2} \\
= {} & \frac{1}{|\nabla u|^2}(|\nabla^2 u|^2 - 2|\nabla|\nabla u||^2 + (\nabla^2_{\nu\nu} u)^2)
\end{aligned}$$

On the other hand contracting A_{ij} gives

$$H = \frac{1}{|\nabla u|}(\underbrace{\Delta u}_{=0} - \nabla^2_{\nu\nu} u).$$

and thus

$$|A|^2 - H^2 = |\nabla u|^{-2}(|\nabla^2 u|^2 - 2|\nabla|\nabla u||^2).$$

Combining with Lemma C.8,

$$\operatorname{Ric}(\operatorname{grad} u/|\nabla u|, \operatorname{grad} u/|\nabla u|) = \frac{1}{2}(R_\Omega - R_S + H^2 - |A|^2),$$

then yields the result. ■

Proof of Proposition 4.1. During the following proof, we will be using $\phi = \sqrt{|\nabla u|^2 + \varepsilon}$ for $\varepsilon > 0$ instead of $|\nabla u|$, since otherwise we cannot control for the behaviour of integrands like $\Delta_\Omega |\nabla u|$ and $\partial_\nu |\nabla u|$ near critical points of u (where $|\nabla u| = 0$).

We find

$$\begin{aligned}
\Delta_\Omega \phi &= \nabla_i \nabla^i \sqrt{|\nabla u|^2 + \varepsilon} \\
&= \nabla_i \frac{\nabla^i |\nabla u|^2}{2\phi} \\
&= \frac{\Delta |\nabla u|^2}{2\phi} - \frac{|\nabla |\nabla u|^2|^2}{4\phi^3} \\
&\underset{\substack{\uparrow \\ \textit{Lemma D.4}}}{=} \phi^{-1}(|\nabla^2 u|^2 + \operatorname{Ric}(\nabla u, \nabla u) - \phi^{-2}|\nabla u|^2 |\nabla |\nabla u||^2),
\end{aligned} \tag{3}$$

where we have used Bochner's identity in the last line.

Thus on a regular level set S we can apply Lemma 4.2 to get

$$\Delta_\Omega \phi \geqslant \frac{1}{2}\phi^{-1}(|\nabla u|^2 + |\nabla u|^2 (R_\Omega - R_S)).$$

Let now $\mathcal{A} \subset [\underline{u}, \bar{u}]$ be an open set containing all the critical values of u (images of points where $\nabla u = 0$), and let $\mathcal{B} = [\underline{u}, \bar{u}] \setminus \mathcal{A}$ be the complementary set.

Then the divergence theorem yields (since $\Delta = \operatorname{div} \operatorname{grad}$)

$$\int_{P_1 \cap u^{-1}(\mathcal{A})} \partial_\nu \phi \, dA + \int_{P_1 \cap u^{-1}(\mathcal{B})} \partial_\nu \phi \, dA + \int_{P_2} \partial_\nu \phi \, dA = \int_{\partial\Omega} \partial_\nu \phi \, dA = \int_\Omega \Delta \phi \, dV = \int_{u^{-1}(\mathcal{A})} \Delta \phi \, dV + \int_{u^{-1}(\mathcal{B})} \Delta$$

We first deal with the integrals over $P_1 \cap u^{-1}(\mathcal{A})$ and $u^{-1}(\mathcal{A})$. On $u^{-1}(\mathcal{A})$ Eq. (3) and Cauchy-Schwarz give

$$\Delta_\Omega \phi \geqslant \Phi^{-1} \operatorname{Ric}(\nabla u, \nabla u) \geqslant -|\operatorname{Ric}||\nabla u|.$$

Then we can decompose into level sets of u using the coarea formula (note that we need to divide by $|\nabla u|$) to get

$$\begin{aligned} -\int_{u^{-1}(\mathscr{A})} \phi\, dV \leqslant \int_{u^{-1}(\mathscr{A})} |\mathrm{Ric}||\nabla u|\, dV & \qquad\qquad\qquad\qquad , \\ = int_{t\in\mathscr{A}} \int_{S_t} |\mathrm{Ric}| & \qquad \leqslant C \int_{t\in\mathscr{A}} \mathscr{H}^2(S_t)\, dt \end{aligned}$$

where $\mathscr{H}^2(S_t)$ is the Hausdorff measure of the level sets and C is some constant bounding the Ricci curvature.

Similarly, on $P_1 \cap u^{-1}(\mathscr{A})$ we have

$$\partial_\nu \phi = \frac{\nu^i \nabla^i \nabla_j u \nabla^j u}{\phi} = \frac{\nabla_i u \nabla^i \nabla_j u \nabla^j u}{\phi} = \frac{g(\nabla_{\nabla u} \nabla u)}{\nu} \leqslant |A_{P_1}| \leqslant C,$$

where we have used $g(\nabla u, \nu) = 0$ by the Neumann boundary condition of u on P_1. We thus get by the coarea formula

$$\int_{P_1 \cap u^{-1}(\mathscr{A})} \partial_\nu \phi\, dA \leqslant C \int_{t\in\mathscr{A}} \mathscr{H}^1(\partial S_t \cap P_1).$$

Let us now deal with the integrals over $P_1 \cap u^{-1}(\mathscr{B})$ and $u^{-1}(\mathscr{B})$. On P_1 we have away from critical points of u

$$\begin{aligned} \partial_\nu \phi &= \frac{\nu^i \nabla^i \nabla_j u \nabla^j u}{\phi} \\ &= \frac{\nabla^i u \nabla^i \nabla_j u \nu^j}{\phi} \\ &\underset{\substack{\uparrow \\ \nabla^i(\nu^j \nabla_j u) = \nabla^i(0) = 0}}{=} -\frac{\nabla^i u \nabla^j u \nabla_i \nu_j}{\phi}, \end{aligned}$$

where we have used the Neumann boundary condition in the last line. Let n now denote the normal vector $n^i = \frac{\nabla^i u}{|\nabla u|}$ to S_t. This yields

$$\partial_\nu |\phi| = -\phi^{-1} |\nabla u|^2 A_{P_1}(n, n) = -\phi^{-1} |\nabla u|^2 (H_{P_1} - \mathrm{tr}_{S_t} A_{P_1}).$$

Let $v \in T_p P_1 \cap T_p S_t$ be a normed vector (there are only two choices, since the vector space is one-dimensional). Then (as S_t is orthogonal to P_1 by the Neumann boundary condition of u on P_1)

$$\mathrm{tr}_{S_t} A_{P_1} = A_{\partial\Omega}(v, v) = |(\nabla_v v)^\top| = |\mathrm{proj}_{S_t}(\nabla_v v)| = \kappa_{S_t \cap P_1} = \kappa_{\partial S_t}.$$

Thus decomposing P_1 into level sets of u using the coarea formula yields

$$\int_{P_1 \cap u^{-1}(\mathcal{B})} \partial_\nu |\nabla u| \, dA = -\int_{t \in \mathcal{B}} \left(\int_{\partial S_t \cap P_1} \phi^{-1} |\nabla u| (H_{P_1} - \kappa_{\partial S_t}) \right) dt.$$

Meanwhile on $\mathcal{B}^{-1}$ applying the coarea formula and Section 4 produces

$$\int_{u^{-1}(\mathcal{B})} \Delta_\Omega \phi \, dV \leqslant \frac{1}{2} \int_{t \in \mathcal{B}} \int_{S_t} \phi^{-1} |\nabla u| \left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + (R_\Omega - R_{S_t}) \right) dA \, dt.$$

Note that

$$\partial_\nu \phi = \frac{n^i \nabla_{ij} u \nabla^j u}{\phi} = 0$$

at critical points of u (where $\nabla u = 0$) and thus we may replace P_2 with $\tilde{P}_2$ in the integral in Section 4 and below in Section 4. We can now combine all these pieces and find

$$\begin{aligned} \frac{1}{2} \int_{t \in \mathcal{B}} \int_{S_t} \frac{|\nabla u|}{\phi} \left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + R_\Omega \right) dA \, dt \leqslant & \int_{t \in \mathcal{B}} \left(\frac{1}{2} \int_{S_t} \frac{|\nabla u|}{\phi} R_{S_t} \, dA + \int_{\partial S_t \cap P_1} \frac{|\nabla u|}{\phi} (\kappa_{\partial S_t} - H_{P_1}) \right) dt \\ & + \int_{\tilde{P}_2} \partial_\nu \phi \, dA + C \int_{t \in \mathcal{A}} (\mathcal{H}^1(\partial S_t \cap P_1) + \mathcal{H}^2(S_t)) \, dt. \end{aligned}$$

Since Ω is compact and $\mathcal{B}$ closed, $|\nabla u|$ is uniformly bounded from below on $u^{-1}(\mathcal{B})$. Furthermore on $\tilde{P}_2$ (where $|\nabla u| \neq 0$) we have

$$\partial_\nu \phi = \frac{|\nabla u|}{\phi} \partial_\nu |\nabla u| \to \partial_\nu |\nabla u| \quad \text{as } \varepsilon \to 0$$

We can thus now take the limit $\varepsilon \to 0$ in Section 4 and get

$$\begin{aligned} \frac{1}{2} \int_{t \in \mathcal{B}} \int_{S_t} \left(\frac{|\nabla^2 u|^2}{|\nabla u|} + |\nabla u| R_\Omega \right) dA \, dt \leqslant & \int_{t \in \mathcal{B}} \left(\frac{1}{2} \int_{S_t} R_{S_t} \, dA + \int_{\partial S_t \cap P_1} (\kappa_{\partial S_t} - H_{P_1}) \right) dt \\ & + \int_{\tilde{P}_2} \partial_\nu |\nabla u| \, dA + C \int_{t \in \mathcal{A}} (\mathcal{H}^1(\partial S_t \cap P_1) + \mathcal{H}^2(S_t)) \, dt. \\ = & \int_{t \in \mathcal{B}} \Big(2 \\ & + \int_{\tilde{P}_2} \partial \end{aligned}$$

, where in the second step we have applied the Gauss-Bonnet theorem (Theorem D.1) to S_t.

By Sard's theorem [**sardMeasureCriticalValues1942**], the set of critical values has measure 0 and we thus may take the measure of $\mathscr{A}$ to be arbitrarily small. Since

$$t \mapsto \mathscr{H}^1(\partial S_t \cap P_1) + \mathscr{H}^2(S_t)$$

is integrable by the coarea formula, taking $|\mathscr{A}| \to 0$ in Section 4 leads to

$$\int_{\underline{u}}^{\bar{u}} \int_{S_t} \frac{1}{2}\left(\frac{|\nabla^2 u|}{|\nabla u|^2} + R\right) dA + \int_{\partial S_t \cap P_1} H_{P_1}\, dl\, dt \\ \leqslant \int_{\underline{u}}^{\bar{u}} \Big(2\pi\chi(S_t) - \int_{\partial S_t \cap P_2} \kappa_{\partial S_t}\, dl\Big)\, dt + \int_{\tilde{P}_2} \partial_\nu |\nabla u|\, dA,$$

■

Remark 4.3. Note that if u had no critical points (i.e. if $|\nabla u| > 0$ everywhere), then we could have always worked with $|\nabla u|$ instead of ϕ and it is easy to check that under these conditions, Proposition 4.1 in fact becomes an equality!

5 Existence and uniqueness of asymptotically linear harmonic functions

To use Proposition 4.1, we will require harmonic functions with properties as in Theorem 3.10. More specifically we will require asymptotically linear harmonic coordinates on $M(\Sigma)$ (for Σ the horizon boundary of M) with certain boundary conditions.

That is we want functions $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ that

- are harmonic, i.e. $\Delta \tilde{x}_i = 0$ in M, for $i = 1, 2, 3$
- are asymptotic to our standard asymptotically half-euclidean coordinates on M_{end}, i.e. $\tilde{x}_i - x_i \in C^{2,\alpha}_{1-\tau+\varepsilon}(M)$ for some $\varepsilon > 0$ and $0 < \alpha < 1$. For a precise definition of the weighted Hölder space $C^{k,\alpha}_{\gamma}(M)$ see [**almarazPositiveMassTheorem2016**]. Important for us is mostly that this condition ensures that our $\tilde{x}_i$ themselves form an asymptotically flat coordinate system.
- fulfill boundary conditions on ∂M mimicking the behaviour of the standard coordinates on Euclidean half space, i.e.

$$\begin{cases} \partial_\nu \tilde{x}_\alpha = 0 & \text{on } \partial M \cap M(\Sigma), \text{ for } \alpha = 1, 2 \\ \tilde{x}_3 = 0 & \text{on } \partial M \cap M(\Sigma). \end{cases}$$

Later, when we compute the mass $\mathfrak{m}_{(M,g)}$ in the asymptotically flat coordinate system consisting of the $\tilde{x}_i$, these boundary conditions will significantly simplify our expression for the mass. To see this, note first that since ∂M is a level set of $\tilde{x}_3$, we have $\nabla \tilde{x}_3 = |\nabla \tilde{x}_3| \cdot \nu$. Then we can compute

$$g_{\alpha 3} = g(\partial_\alpha, \partial_3) = dx_\alpha(\partial_3) = \partial_3(x_\alpha) = |\nabla u| \cdot \partial_\nu(x_\alpha) = 0, \tag{4}$$

and thus the part of the mass given by the integral over $\mathbb{S}^1_L$ (see Definition 3.4) vanishes completely. This also shows that even though $\mathfrak{m}_{(M,g)}$ is coordinate independent, the two terms that define it (the integrals over $\mathbb{S}^2_{R,+}$ and $\mathbb{S}^1_r$) individually are coordinate dependent.

- fulfill a Neumann boundary condition on Σ, i.e. $\partial_{n_\Sigma} \tilde{x}_i = 0$. This will make boundary terms on Σ disappear completely in our calculation.

Even though we normally only work with a single end, our proof of the existence of these functions will use a reflection argument along Σ, which will require uniqueness and existence of our functions for the case of multiple ends and no horizon boundary, i.e. $\Sigma = \varnothing$. But it is not much harder to use this to prove existence and also uniqueness for the case with multiple ends and possibly non-empty boundary. It thus streamlines our proof to just state the following proposition for multiple ends, prove the case $\Sigma = \varnothing$, and then use the reflection argument to prove the general case.

Proposition 5.1. *Suppose (M, g) is an asymptotically flat half-space with decay-rate $\tau > 1/2$, asymptotically flat coordinates $\{x_i\}^j$ in each end M^j_{end} and horizon boundary Σ. Assume (e.g. by shrinking the ends a bit) that the closures of the ends M^j_{end} are disjoint. Then there exist (up to addition of constants) unique smooth functions $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \colon M(\Sigma) \to \mathbb{R}$ satisfying*

$$\begin{cases} \Delta \tilde{x}_\beta = 0 & \text{in } M(\Sigma), \\ \partial_\nu \tilde{x}_\beta = 0 & \text{on } \partial M \cap M(\Sigma), \\ \partial_{n_\Sigma} \tilde{x}_\beta = 0 & \text{on } \Sigma, \end{cases}$$

for $\beta = 1, 2$,

$$\begin{cases} \Delta \tilde{x}_3 = 0 & \text{in } M(\Sigma), \\ \tilde{x}_3 = 0 & \text{on } \partial M \cap M(\Sigma), \\ \partial_{n_\Sigma} \tilde{x}_3 = 0 & \text{on } \Sigma, \end{cases}$$

and

$$x^j_i - \tilde{x}_i \in C^{2,\alpha}_{1-\tau+\varepsilon} \quad \text{in } M^j_{\mathrm{end}}.$$

Moreover, for each end M^j_{end}, the functions $\{\tilde{x}_i\}$ form an asymptotically flat coordinate system in a neighborhood of infinity.

Proof. We first show existence and uniqueness for the case $\Sigma = \varnothing$, then extend to $\Sigma \neq \varnothing$ via a reflection argument along Σ.

Step 1. [**almarazPositiveMassTheorem2016**] proves existence for $\Sigma = \varnothing$ and one end. But by replacing the x_i in the proof with arbitrary smooth extensions of the x_i^j (which are defined on open sets with disjoint closures) with $x_i|_{M^j_{\text{end}}} = x_i^j$, $x_3|_{\partial M} = 0$, we can easily generalise the statement to multiple ends.

We want to show uniqueness for the case $\Sigma = \varnothing$. Let $\{\tilde{x}_i\}$ and $\tilde{x}'_i$ be two harmonic coordinates fulfilling all the properties. By [**almarazPositiveMassTheorem2016**] (the proof of which extends without changes to the case with multiple ends), there exist an orthogonal matrix $(Q_i^j)_{i,j=1}^3$ and constants $\{a_i\}_{i=1}^3$, such that

$$\tilde{x}'_i = Q_i^j \tilde{x}_i + a_i.$$

We have

$$\begin{aligned}(\delta_i^j - Q_i^j)\tilde{x}_j - a_i = \tilde{x}_i - \tilde{x}'_i = o(r^{1/2}) \quad \text{as } r \to \infty,\\ (\delta_i^j - Q_i^j)(x_j - \tilde{x}_j) + a_i = o(r^{1/2})\end{aligned}$$

which implies

$$(\delta_i^j - Q_i^j)x_j = o(r^1)$$

and thus we must have $Q_i^j = \delta_i^j$ (since otherwise $(\delta_i^j - Q_i^j)x_j$ would be linear and nonzero, which is not $o(r^1)$). Hence

$$\tilde{x}_i = \tilde{x}'_i + a_i.$$

Note that $a_3 = 0$, since $\tilde{x}_3 = 0 = \tilde{x}'_3$ on ∂M.

Step 2. Consider now the case $\Sigma \neq \varnothing$. We adapt the proof of [**eichmairDoublingAsymptoticallyF**

Consider the differentiable manifold $(\hat{M} = M \times \{-1, +1\})/_\sim$, where $(x, \pm 1) \sim (x, \mp 1)$ if and only if $x_1, x_2 \in \Sigma$ and $x_1 = x_2$ (i.e. $\hat{M}$ is constructed by gluing two copies of M along Σ). We equip $\hat{M}$ with the Riemannian metric $\hat{g}(\hat{x}) = \gamma(\pi(\hat{x}))$, where $\pi([(x, \pm 1)]) = x$.

Then by [**eichmairDoublingAsymptoticallyFlat2023**], $\hat{g}$ is of class C^2 away from $\pi^{-1}(\Sigma)$ and on $\pi^{-1}(\Sigma)$ the coefficients of $\Delta_{\hat{g}}$ are Lipschitz since Σ is minimal.

Note that $\hat{M}$ has twice as many ends as M, where we set $\hat{x}_i^{j,\pm}(\hat{x}) = x_i^j(\pi(x))$ to be the asymptotic coordinates in these ends.

We can thus apply the result from Step 1 to $\hat{M}$ (for which we don't consider any boundary conditions on horizon boundaries) to obtain asymptotically linear harmonic coordinates $\tilde{\hat{x}}_i$ on $\hat{M}$ with Dirichlet boundary condition on $\partial\hat{M}$. But note that $\tilde{\hat{x}}_i \circ \tau$ is another solution, where we let $\tau\colon \hat{M} \to \hat{M}$ be given by $\tau([(x, \pm 1)]) = [(x, \mp 1)]$. Then by the already established uniqueness for the case without horizon boundary, $\tilde{\hat{x}}_i \circ \tau = \tilde{\hat{x}}_i + a_i$ for some constants a_i. But since these must agree on $\pi^{-1}(\Sigma)$ (τ is the identity there), we have $a_i = 0$.

In particular we get on $\pi^{-1}(\Sigma)$

$$\partial_{n_\Sigma}\tilde{\hat{x}}_i = -\partial_{n_\Sigma}(\tilde{\hat{x}}_i \circ \tau) = -\partial_{n_\Sigma}(\tilde{\hat{x}}_i) = -\partial_{n_\Sigma}(\tilde{\hat{x}})$$

and thus $\tilde{\hat{x}}$ satisfies Neumann boundary conditions on $\pi^{-1}(\Sigma)$ (here we need to fix n_Σ, e.g. choose it to point towards $M \times \{+1\}$).

In particular we get a solution to our original problem on M by setting $\tilde{x}_i(x) = \tilde{\hat{x}}_i([x, +1])$.

The argument from [**eichmairDoublingAsymptoticallyFlat2023**] extends straightforwardly to also show uniqueness (up to adding constants) for the $\tilde{x}_i$ on $M(\Sigma)$.

■

6 Proof of the Mass Lower Bound

We proceed by constructing a proof parallel to [**brayHarmonicFunctionsMass2019**].

To this end, let (M, g) be an asymptotically flat half-space and horizon boundary Σ with asymptotically flat harmonic coordinates x_1, x_2, x_3 as in Proposition 5.1. Note that from now on we will again consider M to only have a single end M_{end} and that although x_1, x_2, x_3 are defined on all of $M(\Sigma)$ and we call them harmonic coordinates, they are only guaranteed to form a coordinate system in M_{end}, i.e. for $|x| > r_0$ for some $r_0 > 0$.

By [**almarazPositiveMassTheorem2016**], we can compute the mass in these harmonic coordinates. For $L > r_0$ define coordinate half-cylinders $C_L = D_L \cup T_L$ given by

$$D_L = \{x \in M_{\text{end}} \mid (x_1)^2 + (x_2)^2 \leqslant L^2,\ \ x_3 = L\}$$
$$T_L = \{x \in M_{\text{end}} \mid (x_1)^2 + (x_2)^2 = L^2,\ \ 0 \leqslant x_3 \leqslant L\}.$$

Further define

$$\mathbb{S}^1_L = \{x \in M_{\text{end}} \mid (x_1)^2 + (x_2)^2 = L^2,\ \ 0 = x_3\} = \partial C_L = C_L \cap \partial M$$
$$(\partial M)_L = \{x \in \partial M \cap M(\Sigma) \mid (x_1)^2 + (x_2)^2 \leqslant L\}$$

and let Ω_L be the closure of the bounded component of $M(\Sigma) \setminus C_L$. Since we chose $L > r_0$, and we can thus be sure that C_L looks as expected and that $\Sigma \subset \Omega_L$.

By Proposition A.1, which can be proven by just slightly modifying the proof of [**almarazPositiveMassTheorem2016**], we can compute the mass as

$$\mathfrak{m}_{(M,g)} = \lim_{L\to\infty}\left(\int_{C_L} G_i\mu^i\, dA + \int_{\mathbb{S}^1_L} g_{\alpha 3}\theta^\alpha\, dl\right)$$

where now μ is the outward unit normal to C_L and θ is as in Definition 3.4. We delegate the details here to the Appendix, see Proposition A.1.

To prove our main result, the inequality Theorem 3.10, we will recover the mass as the boundary term at infinity of Proposition 4.1 applied to $u = x_3$ and $\Omega = \Omega_L$.

Write $S^L_t := \{u = t\} \cap \Omega_L$. Setting $P_1 = \Sigma$ and $P_2 = C_L \cup (\partial M)_L$ (and $\tilde{P}_2 = P_2 \cap \{\nabla u \neq 0\}$) yields (since Σ is a minimal surface, i.e. $H_{P_1} = H_\Sigma = 0$)

$$\int_{\Omega_L} \frac{1}{2}\left(\frac{|\nabla^2 u|}{|\nabla u|} + R|\nabla u|\right) dV \leqslant \int_0^L \Big(2\pi\chi(S_t) - \int_{\partial S^L_t \cap T_L} \kappa_{t,L}\, dl\Big)\, dt + \int_{\tilde{P}_2} \partial_n |\nabla u|\, dA, \quad (5)$$

where $\kappa_{t,L}$ is the geodesic curvature of $S^L_t \cap T_L$ viewed as the boundary of S^L_t. We have used the coarea formula to express the integral on the left as being over Ω_L.

We claim that if $t \in (0, L)$ is a regular value of u (i.e. $|\nabla u| \neq 0$ on S_t), then S^L_t consists of a single component, which intersects T_L along a circle. Assume otherwise, i.e. that there is a regular value $t \in (0, L)$ such that $S' \subset S^L_t$ is a connected component disjoint from T_L. Then, since $M(\Sigma)$ is diffeomorphic to the compliment of a finite number of balls in $\mathbb{R}^3_+$, there exists a compact domain $E \subset \Omega_L$ with $\overline{\partial E \setminus \Sigma} = S'$ (since surely $\Sigma \not\subset S^L_t$, since otherwise $|\nabla u|$ would vanish on Σ and t would not be a regular value).

Note now $u - t$ is still harmonic, has Dirichlet boundary condition $u - t = 0$ on S', and Neumann boundary condition $\partial_n(u - t) = 0$ on Σ. Hence we can apply Theorem D.3 and get that u must be constant on E, which contradicts the assumption that t is a regular value.

Thus S^L_t consists of a single connected component and meets T_L along a circle. In particular, we can apply Eq. (13) with $b \geqslant 1$ and get $\chi(S^L_t) \leqslant 1$. Then Eq. (5) becomes

$$\int_{\Omega_L} \frac{1}{2}\left(\frac{|\nabla^2 u|}{|\nabla u|} + R|\nabla u|\right) dV \leqslant 2\pi L - \int_0^L \int_{\partial S^L_t \cap T_L} \kappa_{t,L}\, dl\, dt + \int_{\tilde{P}_2} \partial_n |\nabla u|\, dA. \quad (6)$$

It now only remains to compute the boundary terms in Eq. (6). Note first that for large enough L, we can be sure that $\nabla u \neq 0$ on C_L due to the asymptotic behaviour of u. We thus define $\widetilde{\partial M_L} = \partial M_L \cap \{\nabla u \neq 0\}$ so that we have $\tilde{P}_2 = C_L \cup \widetilde{\partial M_L}$ for large enough L (which we assume from now on, since we are taking the limit $L \to \infty$ later on anyway).

The following is just the equivalent of [**brayHarmonicFunctionsMass2019**] for our half (instead of full) cylinder (note also that we choose the cylinder with symmetry axis in direction x_3 and $u = x_3$, while Bray et al. choose the symmetry axis in direction x_1 and $u = x_1$). The proof proceeds just as in [**brayHarmonicFunctionsMass2019**]

Lemma 6.1. *In the notation fixed above, we have*

$$\begin{aligned}\int_{C_L} \partial_\nu |\nabla u| \, dA = \frac{1}{2} \int_{D_L} \sum_j (g_{3j,j} - g_{jj,3}) \, dA \\ + \frac{1}{2L} \int_{T_L} [x_2(g_{23,3} - g_{11,2}) + x_1(g_{13,3} - g_{33,1})] \, dA + O(L^{1-2\tau})\end{aligned}$$

and

$$\begin{aligned}\int_0^L \int_{\partial S_t^L \cap T_L} \kappa_{t,L} \, dl \, dt = 2\pi L + \frac{1}{2L} \int_{T_L} [x_1(g_{11,2} - g_{21,1}) + x_2(g_{22,1} - g_{12,2})] \, dA \\ + O(L^{1-2\tau} + L^{-\tau}).\end{aligned}$$

It remains to consider the boundary term on $\widetilde{(\partial M)}_L$:

Lemma 6.2. *In the notation established above, we have*

$$\int_{\widetilde{(\partial M)}_L} \partial_\nu |\nabla u| = - \int_{\widetilde{(\partial M)}_L} H |\nabla u| \, dA.$$

Proof. Note that, since ∂M is a level set of u, we know that ∇u is orthogonal to ∂M and hence $\nu = -\nabla u / |\nabla u|$ (recall that ν points outside of M, i.e. towards $x_3 < 0$).

Thus we can compute

$$\begin{aligned}
\partial_\nu |\nabla u| &= \partial_\nu \sqrt{|\nabla u|^2} \\
&= \frac{\partial_\nu |\nabla u|^2}{2|\nabla u|} \\
&= -\frac{\nabla^i u \nabla_i (\nabla_j u \nabla^j u)}{2|\nabla u|^2} \\
&= -\frac{\nabla^i u \nabla^j u \nabla_i \nabla_j u}{|\nabla u|^2} \\
&= -n^i n^j \nabla_i \nabla_j u \\
&= -\nabla_\nu \nabla_\nu u \\
&\underset{\substack{\uparrow \\ \text{Fact C.9}}}{=} -\Delta_M u + \Delta_{\partial M} u + H \cdot \nabla_\nu u.
\end{aligned}$$

But u is harmonic, so $\Delta_M u = 0$. Also, u is constant on ∂M, and thus $\Delta_{\partial M} u = 0$. Then recognizing that $\nabla_\nu u = -|\nabla u|$ yields the statement of the Lemma, where we can integrate over $(\partial M)_L$ instead of $\widetilde{(\partial M)}_L$ since on the critical points the integrand $H|\nabla u|$ is zero anyway. ■

Proof of Theorem 3.10. We can now combine Lemma 6.1 and Lemma 6.2 with Eq. (6) to get

$$\begin{aligned}
&\int_{\Omega_L} \frac{1}{2}\left(\frac{|\nabla^2 u|}{|\nabla u|} + R|\nabla u|\right) dV + \int_{\partial M_L} H|\nabla u|\, dA \\
&\leqslant 2\pi L - 2\pi L \\
&\quad + \frac{1}{2}\int_{T_L} [\frac{x_1}{L}(g_{21,1} - g_{11,2}) + \frac{x_2}{L}(g_{22,1} - g_{12,2})]\, dA \\
&\quad + \frac{1}{2}\int_{T_L} [\frac{x_1}{L}(g_{13,3} - g_{33,1}) + \frac{x_2}{L}(g_{23,3} - g_{11,2})]\, dA \\
&\quad + \frac{1}{2}\int_{D_L} \sum_j (g_{3j,j} - g_{jj,3})\, dA \\
&\quad + O(L^{1-2\tau} + L^{-\tau}) \\
&= \frac{1}{2}\int_{C_L} \sum_j (g_{ij,j} - g_{jj,i})\tilde{\mu}^i\, dA + O(L^{1-2\tau} + L^{-\tau}) \\
&= \frac{1}{2}\int_{C_L} G_i \tilde{\mu}^i\, dA + O(L^{1-2\tau} + L^{-\tau}),
\end{aligned}$$

where $\tilde{\mu}$ is the outward unit normal to C_L. Recalling that (see Eq. (4))

$$\int_{\mathbb{S}^1_L} g_{\alpha 3}\theta^\alpha \, dl = 0,$$

we get (after also dividing by 8π)

$$\begin{aligned}\int_{\Omega_L} \frac{1}{16\pi}\left(\frac{|\nabla^2 u|}{|\nabla u|} + R|\nabla u|\right) dV + \frac{1}{8\pi}\int_{\partial M_L} H|\nabla u| \, dA \\ \leqslant \frac{1}{16\pi}\int_{C_L} G_i \tilde{\mu}^i \, dA + \frac{1}{16\pi}\int_{\mathbb{S}^1_L} g_{\alpha 3}\theta^\alpha \, dl + O(L^{1-2\tau} + L^{-\tau}).\end{aligned}$$

Since $\tau > \frac{1}{2}$, we can take the limit $L \to \infty$ and arrive at the statement of Theorem 3.10:

$$\frac{1}{16\pi}\int_{M(\Sigma)}\left(\frac{|\nabla^2 u|^2}{|\nabla u|} + R|\nabla u|\right) dV + \frac{1}{16\pi}\int_{\partial M \cap M(\Sigma)} H|\nabla u| \, dA \leqslant \mathfrak{m}_{(M,g)}.$$

■

Now the positive mass theorem follows as corollary:

Proof of Theorem 3.7. Since H is nonnegative, the horizon boundary required by Theorem 3.10 is guaranteed to exist by Remark 3.9. Then Eq. (2) together with $R \geqslant 0$ and $H \geqslant 0$ directly implies

$$\mathfrak{m}_{(M,g)} \geqslant 0.$$

It remains to show rigidity (i.e. that $\mathfrak{m}_{(M,g)} = 0$ if and only if (M, g) is isometric to $(\mathbb{R}^3_+, \delta)$):

$(M, g) \cong (\mathbb{R}^3_+, \delta) \implies \mathfrak{m}_{(M,g)} = 0$: In standard coordinates, g is constant and thus G_i is surely 0. Since also $g_{13} = g_{23} = 0$ everywhere, we have $\mathfrak{m}_{(M,g)} = 0$ directly from Definition 3.4.

$\mathfrak{m}_{(M,g)} = 0 \implies (M, g) \cong (\mathbb{R}^3_+, \delta)$:

Notice that Eq. (2) (together with $H \geqslant 0$) directly implies that $H = 0$. This then allows us to apply Proposition 4.1 to $u = x_\alpha$ for $\alpha = 1, 2$. We do this by using horizontal half-cylinders (let $\beta = 3 - \alpha$ be the other direction with Neumann boundary condition on ∂M), defining

$$\begin{aligned}
D_L^{\pm} &\coloneqq \{x \in M_{\text{end}} \mid (x_\beta)^2 + (x_3)^2 \leqslant L^2,\ \ x_3 \geqslant 0,\ \ x_\alpha = \pm L\},\\
T_L &\coloneqq \{x \in M_{\text{end}} \mid (x_\beta)^2 + (x_3)^2 = L^2, \tau 0 \leqslant x_3,\ \ -L \leqslant x_\alpha \leqslant L\},\\
C_L &\coloneqq D_L^+ \cup T_L \cup D_L^-,\\
\Omega_L &\coloneqq \text{ closure of bounded component of } M(\Sigma) \setminus C_L,\\
(\partial M)_L &\coloneqq \partial M \cap \Omega_L,\\
S_t^L &= \{x_\alpha = t\} \cap \Omega_L,
\end{aligned}$$

and setting $P_1 = \Sigma \cup (\partial M)_L$ and $P_2 = C_L$ (instead of $P_1 = \Sigma$ and $P_2 = C_L \cup (\partial M)_L$, as we did for $u = x_3$). Since all of P_1 is a minimal surface (has zero mean curvature), we get the following equivalent of Eq. (5)

$$\frac{1}{2}\int_{\Omega_L} \frac{1}{2}\left(\frac{|\nabla^2 x_\alpha|}{|\nabla x_\alpha|} + R|\nabla x_\alpha|\right) dV \leqslant \int_{-L}^{L} (2\pi\chi(S_t^L) - \int_{\partial S_t^L \cap T_L} \kappa_{t,L}) + \int_{C_L} \partial_n |\nabla u|.$$

A similar argument as was used for $u = x_3$ now shows that if $t \in (-L, L)$ is a regular value of x_α, then S_t^L consists of a single component meeting T_L along a half-circle: Otherwise there would exist a regular value $t \in (-L, L)$ with S' a connected component of S_t^L disjoint from T_L. As before then there would exist a compact domain $E \subset \Omega_L$ with $\overline{\partial E \setminus \Sigma} = S'$, to which we could apply *Theorem D*.3 to give $E \subset S_t^L$, contradicting our assumption that t is regular.

Thus we can again conclude that $\chi(S_t^L) \leqslant 1$. Repeating the following calculations with these slightly changed cylinders then leads to the equivalent of Eq. (2) for x_α:

$$\frac{1}{16\pi}\int_{M(\Sigma)} \left(\frac{|\nabla^2 u|^2}{|\nabla u|} + R|\nabla u|\right) dV\, dA \leqslant \mathfrak{m}_{(M,g)}. \tag{7}$$

While deriving Eq. (6) we also used the inequality

$$\int_0^L 2\pi\chi(S_t)\, dt \leqslant 2\pi,$$

and then the fact that we have equality in Eq. (2) also implies that we get equality here, i.e. the Euler characteristic of the level sets of $u = x_3$ is constant $\chi(S_t^L) = 1$. Thus there is always exactly one boundary component $S_t^L \cap T_L$ and a horizon boundary Σ cannot exist. Hence $M(\Sigma) = M$ and M is at least diffeomorphic to $\mathbb{R}^3_+$ (since $M(\Sigma)$ is always diffeomorphic to $\mathbb{R}^3_+$ with a finite number of balls removed).

Equation (2) and Eq. (7) immediately imply $\nabla^2 x_i = 0$ for all $i = 1, 2, 3$. Then

$$\begin{aligned}\nabla(g(\partial_j, \partial_k)) &= \nabla(g(\nabla x_j, \nabla x_k)) \\ &= g(\nabla^2 x_j, \nabla x_k) + g(\nabla x_j, \nabla^2 x_k) \\ &= 0,\end{aligned}$$

i.e. the metric is constant in these coordinates. We can then easily transform our coordinates linearly (while only rescaling x_3, such that on ∂M we still have $x_3 = 0$) to get $g = \delta$ everywhere. Thus we have found an isometry $(M, g) \cong (\mathbb{R}^3_+, \delta)$.

■

7 An example calculation

In the following, we consider harmonic functions on some concrete example manifolds and also (numerically) compute the actual lower bound for the mass given by our result.

7.1 Introducing some different Schwarzschild half-spaces

In Remark 3.6 we introduced the half Schwarzschild space $M_{m,+}$ (which has mass $m/2$). There are many other, less symmetric half-spaces resulting from the full Schwarzschild space. One possibility is

$$M_{m,\geqslant a} = \{x \in \mathbb{R}^3 \mid x_3 \geqslant a\}$$

for some $a > 0$, equipped with the same metric $g = \omega^4 \cdot \delta$ as the normal Schwarzschild space. We also identify $M_{m,+}$ with $M_{m,\geqslant 0}$.

Note that $g_m = \omega^4 \delta$ is just the Euclidean metric multiplied by a strictly positive function. Metrics that are related in such a way are called *conformal* (since there is a *conformal transformation*, i.e. an angle preserving map, between the manifolds $(M_{m,+}, g_m)$ and $(M_{m,+}, \delta)$).

It is then easy to see that for both $M_{m,+}$ and $M_{m,\geqslant a}$ the mass Definition 3.4 gets no contribution from the integral over $g_{\alpha 3} = 0$. The other part of the mass then is then the same for $M_{m,+}$ and all the $M_{m,\geqslant a}$ ($g_{ij,k}$ is of order $O(r^{-\tau-1})$ and thus the difference between the integrals, given by an integral over a piece approaching $\mathbb{S}^1_r \times [0, a]$, is 0 in the limit $r \to \infty$). In particular, the mass of $M_{m,\geqslant a}$ is the same as the mass of $M_{m,+}$, i.e. $m/2$.

We will see in the Lemma below that the horizon boundary of $M_{m,+}$ is given in our coordinates by $\Sigma = \{x \in \mathbb{R}^3_+ \mid |x| = m/2\}$. In fact, this space is just the exterior region of some bigger manifold, which includes some of the interior of the horizon and will then also have smooth boundary as required in Definition 3.1 (which is not the case for $M_{m,+}$). Consequently, we should actually consider Σ to not be part of the non-compact boundary $\partial M_{m,+}$. To avoid confusion, we will denote this by $\widetilde{\partial M_{m,+}} \coloneqq \overline{\partial M_{m,+} \setminus \Sigma}$.

Lemma 7.1. *The boundary ∂ has positive mean curvature for $a > 0$. Similarly, $\widetilde{\partial M_{m,+}}$ has zero mean curvature (in particular it is non-negative).*

Σ is a free horizon boundary components of $M_{m,+}$ (i.e. has zero mean curvature and is orthogonal to $\widetilde{\partial M_{m,+}}$).

We already know the mean curvature of these different surfaces when considered as surfaces under the background metric δ. It will thus be helpful to find a formula for how the mean curvature changes under a conformal change of metric. We delegate the details of this to Appendix D.3.

Proof. Let $f = \ln(\omega^2)$. Lemma D.6 then yields

$$H_g = \frac{1}{\omega^2}(H_\delta + 2g(\operatorname{grad} f, n)),$$

' where H_g and H_δ are the mean curvature of some surface under the metrics g and δ respectively, and where n is the normal vector to the surface used to compute the mean curvature.

We can compute

$$\begin{aligned}
\nabla_i f &= \partial_i f \\
&= \frac{1}{\omega^2} \cdot 2\omega \cdot \partial_i \omega \\
&= -\frac{2}{\omega} \cdot \frac{m x_i}{2|x|^3}. \\
\implies (\operatorname{grad} f)^i &= \nabla^i f \\
&= g^{ij} \nabla_j f \\
&= -\frac{1}{\omega^4} \cdot \frac{2}{\omega} \cdot \frac{m x_i}{2|x|^3} \\
&= -\frac{m x_i}{|x|^3 \omega^5}.
\end{aligned}$$

Note that the outward pointing normal vector ν to $\widetilde{\partial M_{m,+}}$ is $-\partial_3/\omega^2$. We can then calculate

$$g(\operatorname{grad} f, \nu) = \omega^4 \cdot \frac{m}{|x|^3 \cdot \omega^7} x_3 = \frac{m}{|x|^3 \cdot \omega^3},$$

and thus (since the mean curvature under δ of both $\widetilde{\partial M_{m,+}}$ and $\partial M_{m,\geqslant a}$ is clearly zero)

$$H_g = \frac{2m}{|x|^3 \cdot \omega^3} x_3.$$

In particular, on $\widetilde{\partial M_{m,+}}$ we have $H_g = 0$ (since there $x_3 = 0$) and on $\partial M_{m,\geqslant a}$ (for $a > 0$) we have $H_g > 0$ (since there $x_3 > 0$).

Similarly, the normal vector n_Σ to Σ pointing out of our manifold is

$$n_\Sigma = -\frac{1}{\omega^2 \cdot |x|} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

and thus

$$g(\operatorname{grad} f, n_\Sigma) = \omega^4 \cdot \frac{|x|^2 \cdot m}{\omega^5 |x|^4} = \frac{m}{|x|^2 \cdot \omega} \underset{|x|=m/2}{\overset{\uparrow}{=}} 2m.$$

On the other hand, the mean curvature of a sphere of radius r in $\mathbb{R}^3$ is (under our definition and with respect to our normal vector) $-\frac{2}{r}$. Thus for Σ we have

$$H_g = \frac{1}{\omega^2} \cdot \left(-\frac{2}{m/2} + 4m \right) = 0.$$

Lastly, since angles under g and δ are the same, $\Sigma$and $M_{m,+}$ are surely orthogonal. ■

Remark 7.2. We could also define $M_{m,\geqslant a}$ for $a < 0$, but then the mean curvature of the relevant portion of $\partial M_{m,\geqslant a}$ is negative and the positive mass theorem does not apply. Thus we are only interested in $a \geqslant 0$.

7.2 Finding the asymptotically linear harmonic coordinates

We can reduce the Laplacian Δ_g for a metric conformally euclidean metric g to Δ_δ:

Lemma 7.3. *In the above notation, we have on* $\mathbb{R}^3 \setminus \{0\}$

$$\Delta_g(u) = \omega^{-5} \cdot \Delta_\delta(\omega \cdot u).$$

Remark 7.4. This formula is specific to our case, where in particular $M_{m,\geqslant a}$ and $M_{m,+}$ have zero scalar curvature. In general the above conformal invariance is fulfilled not by Δ_g but by the conformal Laplacian $L_g = \Delta_g - \frac{n-2}{4(n-1)}$ (see also [**curryIntroductionConformalGeometr**

Proof of Lemma 7.3. Note first that $\Delta_\delta \omega = 0$ on $\mathbb{R}^3 \setminus \{0\}$. Using Lemma B.15, we then obtain

$$\begin{aligned}
\Delta_g(u) &= \frac{1}{\sqrt{\det g}} \partial_a (\sqrt{\det g} g^{ab} \partial_b u) \\
&= \frac{1}{\omega^6} \partial_a (\omega^2 \cdot \delta^{ab} \partial_b u) \\
&= \frac{1}{\omega^6} \delta^{ab} \partial_a (\omega^2 \partial_b (\omega \cdot u)) \\
&= \frac{1}{\omega^5} \delta^{ab} (\partial_a (\omega \partial_b u) + \partial_a \omega \cdot \partial_b u) \\
&\underset{\substack{\uparrow \\ \Delta_\delta \omega = 0 \text{ and symmetry of } \delta}}{=} \frac{1}{\omega^5} \delta^{ab} (\partial_a (\omega \cdot \partial_b u) + \partial_a u \cdot \partial_b \omega + u \cdot \partial_a \partial_b \omega) \quad = \frac{1}{\omega^5} \delta^{ab} \partial_a (\omega \cdot \partial_b u + u \cdot \partial_b \omega) \\
&= \frac{1}{\omega^5} \delta^{ab} \partial_a \partial_b (\omega \cdot u) \\
&= \frac{1}{\omega^5} \Delta_\delta (\omega \cdot u).
\end{aligned}$$

■

Thus, finding harmonic functions u_i under the metric g reduces to finding harmonic functions $\tilde{u}_i = \omega \cdot u_i$ under the euclidean metric. For this, we use spherical coordinates (r, θ, φ) and expand $\tilde{u}$ in terms of solid harmonics,

$$\tilde{u}_i(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} \tilde{u}_{i,\ell}^k \left(r^\ell Y_\ell^k(\theta, \varphi) + \sum_{\ell=-\infty}^{\ell=-1} + \tilde{u}_{i,-\ell-1}^k \cdot r^{-\ell-1} Y_\ell^k(\theta, \varphi) \right),$$

where $\tilde{u}_{i,\ell}^k$ are real coefficients and Y_ℓ^k the spherical harmonics.

We want our u_i to be asymptotic to the asymptotically flat coordinates x_i. A necessary condition is in particular that (see beginning of Section 5 and the definition of the weighted Hölder space in [**almarazPositiveMassTheorem2016**])

$$\sup_M |\nabla_j u_i - x_i| |x|^{1-\tau+\varepsilon} < \infty$$

for all i, j and any $\varepsilon > 0$. Note that for M_g the strongest possible choice for the asymptotic falloff parameter τ is $\tau = 1$ (compare Eq. (1)).Thus for $j = 0$ in Section 7.2 we have $\sup_M |u_i - x_i| \cdot |x|^\varepsilon < \infty$, i.e. $u_i - x_i = O(|x|^0)$. Then for $\tilde{u}_i$ this implies

$$\begin{aligned}(\tilde{u}_i - x_i) &= u_i \cdot (1 + m/2|x|) - x_i \\ &= (u_i - x_i) - m\frac{m \cdot x_i}{2|x|} \\ &= O(|x|^0).\end{aligned}$$

Hence we can conclude for the expansion in terms of solid harmonics, that $\tilde{u}^k_{i,\ell} = 0$ for all $\ell > 1$ and that

$$\sum_{k=-1}^{1} \tilde{u}^k_{i,1} \cdot r \cdot Y^k_l(\theta, \varphi) = x_i. \tag{8}$$

It remains to find the remaining $\tilde{u}^k_{i,\ell}$ such that the boundary conditions on the u_i are satisfied. We will treat the cases $a = 0$ and $a > 0$ separately. Note that while we are interested in determining all the u_i for $a = 0$ (so that we can look at the resulting coordinate system), for $a > 0$ we will not go this extra effort and instead only determine u_3.

$a = 0$: Recall from the proof of Lemma 7.1 that ∂_r is orthogonal to Σ. The Neumann boundary condition on Σ then becomes

$$\begin{aligned}0 &= \partial_r(\tilde{u}_i(r, \theta, \varphi)/\omega)|_{r=m/2} \\ &= \left.\frac{2(r(m + 2r)\partial_r \tilde{u}_i(r, \theta, \varphi) + m \cdot \tilde{u}_i(r, \theta, \varphi))}{(m + 2r)^2}\right|_{r=m/2} \\ &= \frac{m\partial_r \tilde{u}_i(m/2, \theta, \varphi) + \tilde{u}_i(m/2, \theta, \varphi)}{2m} \\ \implies 0 &= m\partial_r \tilde{u}_i(m/2, \theta, \varphi) + \tilde{u}_i(m/2, \theta, \varphi) \\ &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^k_\ell(\theta, \varphi) \cdot [\tilde{u}^k_{i,\ell} \cdot ((m/2)^\ell + m \cdot \ell \cdot (m/2)^{\ell-1}) \\ &\qquad + \tilde{u}^k_{i,-\ell-1}((m/2)^{-\ell-1} + m \cdot (-\ell - 1) \cdot (m/2)^{-\ell-2})] \\ \underset{\substack{\uparrow \\ Y^k_\ell \text{ are linearly independent}}}{\implies} 0 &= Y^m_\ell \quad \text{or} \quad \tilde{u}^k_{i,\ell} \cdot (m/2)^\ell \cdot (1 + 2\ell) - \tilde{u}^k_{i,-\ell-1} \cdot (m/2)^{-\ell-1}(1 + 2\ell).\end{aligned}$$

Since $\tilde{u}^k_{i,\ell} = 0$ for $\ell > 1$ we see also that $\tilde{u}^k_{i,\ell} = 0$ for $\ell < -2$. We further have

$$
\begin{aligned}
\tilde{u}^0_{i,-1} &= \frac{m}{2}\tilde{u}^0_{i,0}, \\
\sum_{k=-1}^{1} \tilde{u}^k_{i,-2} Y^k_1(\theta,\varphi) &= \frac{3m/2}{3 \cdot 4/m^2} \sum_{k=-1}^{1} \tilde{u}^k_{i,1} Y^k_1(\theta,\varphi) \\
&\underset{\substack{\uparrow \\ Eq.\ (8)}}{=} \frac{m^3}{8} \cdot \frac{x_i}{r}.
\end{aligned}
$$

In particular our solution so far looks like

$$
\tilde{u}_i = x_i - \frac{m^3}{8} \cdot \frac{x_i}{r^3} + C_i + \frac{C_i \cdot m}{2 \cdot r} = x_i - \frac{m^3}{8} \cdot \frac{x_i}{r^3} + C_i \cdot \omega.
$$

For $i = 3$ the Dirichlet boundary condition $u_3 = 0$ (and thus also $\tilde{u}_3 = \omega \cdot u_3 = 0$) on $\widetilde{\partial M_{m,+}}$ (where $x_3 = 0$) then gives $C_3 = 0$. For $i = 1, 2$ the choice of C_i is free and we choose $C_1 = C_2 = 0$. We can check that the Neumann boundary condition on $\widetilde{\partial M_{m,+}}$ for these coordinates (we use $\alpha = 1, 2$ instead of i to make clear that we do not consider $i = 3$) is fulfilled:

$$
\begin{aligned}
\partial_\nu(\tilde{u}_\alpha/\omega) &= x_\alpha \cdot \partial_\nu(1/\omega) - \frac{m^3}{8} \cdot x_\alpha \cdot \partial_\nu(1/(r^3 \cdot \omega)) \\
&= 0
\end{aligned}
$$

Here we have for the first equality used that $\partial_\nu x_\alpha = 0$. For the second equality note that both expressions in the derivatives with respect to ν only depended on r, but

$$
\partial_\nu r = \partial_{x_3} \sqrt{x_1^2 + x_2^2 + x_3^2}\Big|_{x_3 = 0} = 0.
$$

We thus have for all $i = 1, 2, 3$

$$
\tilde{u}_i = x_i \cdot \Big(1 - \frac{m^3}{8r^3}\Big).
$$

$a > 0$: Here we only have to consider the boundary conditions on $\partial M_{m,\geqslant a}$. To make this easier to work with we shift u_3 such that $u_3 = a$ on $\partial M_{m,\geqslant a}$ (no arguments in this thesis require the boundary to be anything more than a level set, we just normally use $u_3 = 0$ on the boundary for convenience).

For $\tilde{u}_3$ this then yields (where C is some constant)

$$\begin{aligned} a \cdot \omega &= \tilde{u}_3(a/\cos(\theta), \theta, \varphi) \\ &= x_3|_{x_3=a} + C + \sum_{\ell=0}^{\infty} \tilde{u}_{3,\ell}^m a^{-\ell-1} \cdot (\cos(\theta))^{\ell+1} \cdot Y_\ell^m(\theta, \varphi) \\ \implies \frac{m}{2} \cdot \cos(\theta) &= C + \sum_{\ell=0}^{\infty} \tilde{u}_{3,\ell}^m a^{-\ell-1} \cdot (\cos(\theta))^{\ell+1} \cdot Y_\ell^m(\theta, \varphi). \end{aligned}$$

It is then easy to guess the solution $C = 0$ and $\tilde{u}_{3,-1}^0 \cdot Y_0^0(\theta, \varphi) = \frac{m}{2} \cdot a$, $\tilde{u}_{3,\ell}^m = 0$ for $\ell < -1$, which gives

$$\tilde{u}_3 = x_3 + \frac{m \cdot a}{2r}.$$

For u_1 and u_2, finding a solution for the Neumann boundary condition on $\partial M_{m,\geqslant a}$ gets a lot more involved.

u_3 suffices for calculating the lower bound in Theorem 3.10, and thus we do not deal with u_1 and u_2 here.

7.3 Computing the lower bound

We can now compute the lower bound for $\mathfrak{m}_{(M_{m,\geqslant a}, g_m)}$ given in Theorem 3.10. For this we use the Python library GRAVIPY. All the calculations are contained in a Jupyter notebook [**fischerhenryrubenHarmonicFunctionMethod2023**]. The code can be found at , in which there is also a simple Python script that will run the same computations and output the same values and graphs.

For $m = 1$ and our example $M_{m,+} = M_{m,\geqslant 0}$ we get a lower bound of 0.33 with an estimated absolute error of 0.14. The actual mass of $M_{m,+}$ is $m/2 = 1/2$. The error term here is large, since the integrand of the integral over M has a singularity near the intersection of the x_3-axis with the horizon. We thus cannot make strong statements here about how good the lower bound is.

For the modified example $M_{m,\geqslant a}$ for $a > 0$ and $m = 1$, the actual mass is also $m/2$ as noted at the beginning of this section. We can compute the lower bound for different values of a, for the results see Fig. 2. In particular note that the lower bound is consistently very near the actual mass of 1/2, where the contribution to the lower bound from the integral over the boundary increases as the contribution from the integral over $M_{m,\geqslant a}$ decreases for higher a.

Thus at least in this very special case (conformally euclidean, no horizon boundary), the lower bound from our main result Theorem 3.10 seems to be quite strong. We

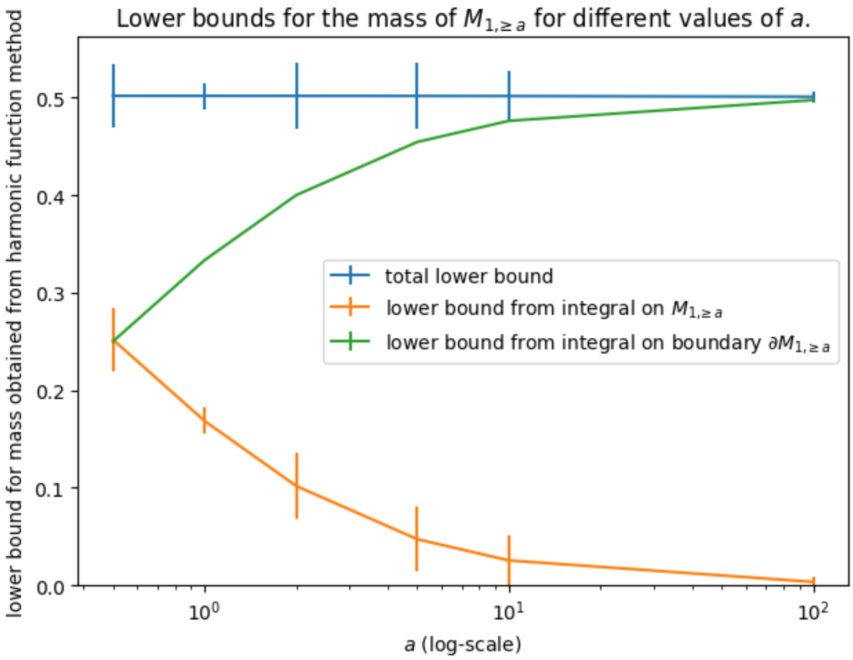
Lower bounds for the mass of $M_{1,\geq a}$ for different values of a.
lower bound for mass obtained from harmonic function method
0.0
0.1
0.2
0.3
0.4
0.5
10^0
10^1
10^2
a (log-scale)
total lower bound
lower bound from integral on $M_{1,\geq a}$
lower bound from integral on boundary $\partial M_{1,\geq a}$

Figure 2

might even have equality in Eq. (2) for this case, but our purely numerical calculations do not allow us to make a judgement on this one way or another.

7.4 Plotting the harmonic level sets

Using the same Jupyter notebook [**fischerhenryrubenHarmonicFunctionMethod2023**] as before, we also generate some plots of the level sets of u_2 (which look the same as those for u_1, just rotated) and of u_3 for $M_{m,+}$ (see Section 7.4) and of just x_3 for $M_{m,\geqslant a}$ (but here for different a) (see Section 7.4)

Note that the method of plotting implicit surfaces we use here is susceptible to rounding errors and thus sometimes the level set if u_3 on $M_{m,\geqslant a}$ corresponding to the boundary of the manifold is rendered with some slight imperfections (the level set should look flat in our coordinates). We can however symbolically verify even within the Jupyter notebook that in fact the boundary of the half-space is a level set.

Note also that even though in the level sets of u_2 on $M_{m,+}$ there seem to be disconnected components of semi-spherical shape (seemingly contradicting our argument for $\xi(S_t^L) \leqslant 1$ directly preceding Eq. (6)), this portion of the level set is entirely contained within the horizon boundary Σ, i.e. surely not contained in M_{ext}, and thus also not in $S_t^L \subset M_{\mathrm{ext}}$.

These plots do not seem to provide much more insight into the harmonic function method, though they do help in visualizing in particular how the boundary conditions (both asymptotic and on horizon and noncompact boundary) get fulfilled.

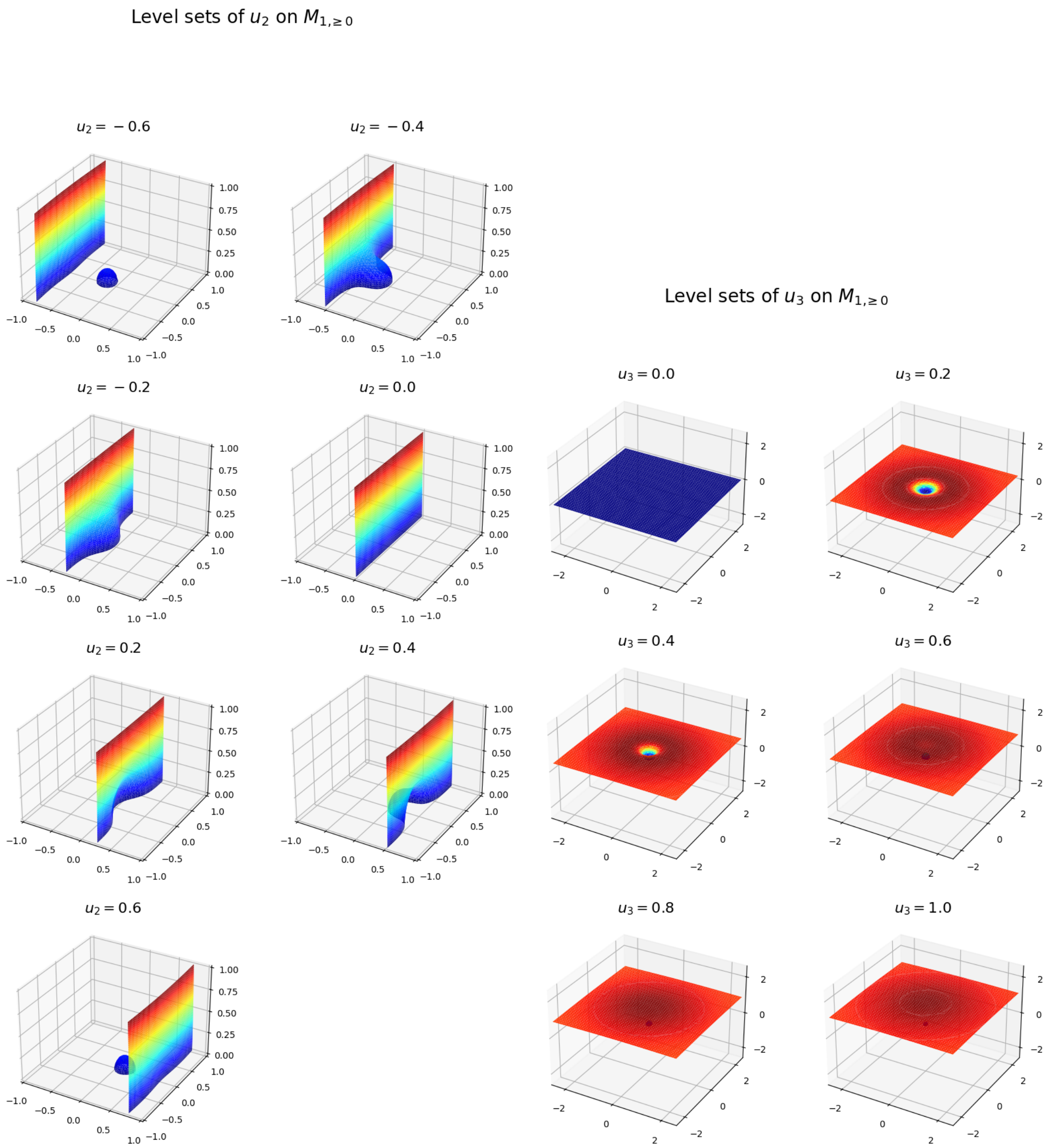

Figure 3: Level sets of harmonic coordinates u_i on $M_{m,+}$. Note that for better visibility of any curvature, the height of points on the surfaces is also visualized using color.

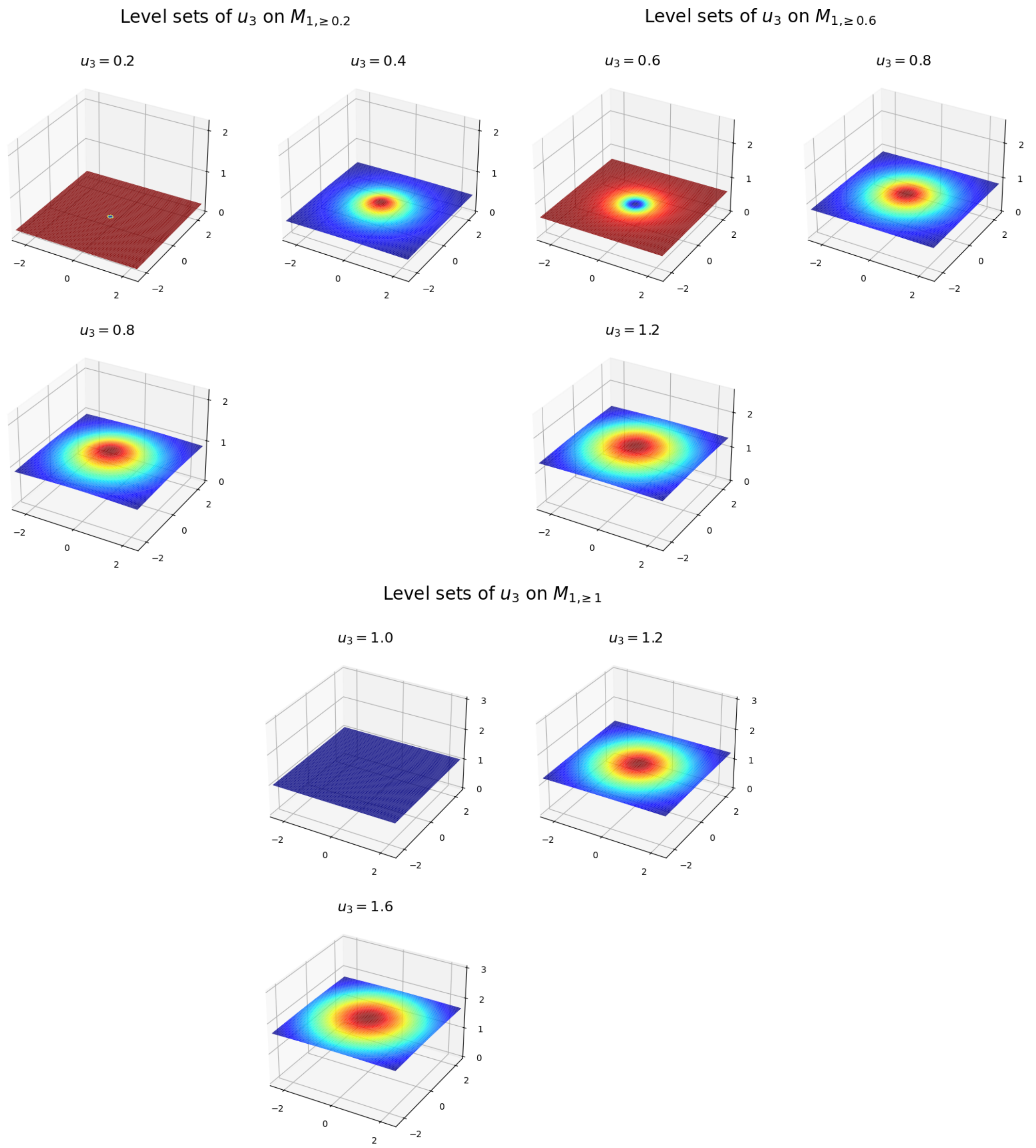

Figure 4: Level sets of u_3 on $M_{m,\geqslant a}$ for different values of a. Note that for better visibility of any curvature, the height of points on the surfaces is also visualized using color.

A Different Exhausting Sequences for Computation of the Mass

Proposition A.1. *Suppose that (M, g) is an asymptotically flat half space with asymptotically flat coordinates x_1, x_2, x_3 on M_{end} (fulfilling the conditions of Definition 3.4). Let $\{D_k^3\}_{k=1}^{\infty}$ be an exhaustion of M by closed sets with $\partial D_k = S_k \cup (D_k \cap \partial M)$, where S_k is a connected 2-dimensional piecewise smooth submanifold of the end M_{end} with $\partial S_k = \partial M \cap S_k$ such that*

$$R_k := \inf_{x \in S_k} |x| \to \infty \quad \text{as } k \to \infty,$$
$$R_k^2 \cdot |S_k| \text{ is bounded as } k \to \infty,$$

and $R_1 \geqslant R_0$, where $|S_k|$, the area of S_k, and $|x|$ are as usual calculated with respect to the euclidean background metric (possible since we are in M_{end}). Then

$$\mathfrak{m}_{(M,g)} = \lim_{k \to \infty} \int_{S_k} \int_{\mathbb{S}^2_{r,+}} G_i \tilde{\mu}^i \, dA + \int_{\partial S_k} g_{\alpha 3} \tilde{\theta}^\alpha \, dl$$

is independent of the sequence S_k, where as in Definition 3.4 $\tilde{\mu}^i$ is the outward normal to S_k and $\tilde{\theta}^\alpha$ the co-normal to ∂S_k oriented as the boundary of the compact component of $\partial M \setminus \partial S_k$.

Proof. Let $\tilde{D}_k := \{x \in D_k \mid |x| \geqslant R_k\}$ (this is the part of D_k extending beyond the biggest coordinate hemisphere that is possible to inscribe in D_k). Then $\partial \tilde{D}_k = S_k \cup \mathbb{S}^2_{R_k,+} \cup (\tilde{D}_k \cap \partial M)$ and $\partial(\tilde{D}_k \cap \partial M) = (S_k \cap \partial M) \cup (\mathbb{S}^1_{R_k})$.

As in [**almarazPositiveMassTheorem2016**], we get (using [**almarazPositiveMassTheorem2016**]

$$\begin{aligned}
\int_{\tilde{D}_k} R \, dV = & \int_{S_k} G_i \tilde{\mu}^i \, dA - \int_{\mathbb{S}^2_{R_k,+}} G_i \mu^i \, dA \\
& + \int_{\tilde{D}_k \cap \partial M} G_i \nu^i \, dA + \int_{\tilde{D}_k} O(r^{-2\tau-2}), \\
\int_{\tilde{D}_k \cap \partial M} G_i \nu^i \, dA = & \int_{S_k \cap \partial M} g_{\alpha 3} \tilde{\theta}^\alpha \, dl - \int_{\mathbb{S}{R_k}^1} g_{\alpha,3} \theta^\alpha \, dl \\
& -2 \int_{\tilde{D}_k \cap \partial M} H + \int_{\tilde{D} \cap \partial M} O(r^{-2\tau-1}),
\end{aligned}$$

and thus

$$
\begin{aligned}
&\Big|\int_{S_k} G_i\tilde{\mu}^i\,dA + \int_{S_k\cap\partial M} g_{\alpha 3}\tilde{\theta}^\alpha\,dl - \Big(\int_{\mathbb{S}^2_{R_k,+}} G_i\mu^i\,dA + \int_{\mathbb{S}^1_{R_k}} g_{\alpha 3}\theta^\alpha\,dl\Big)\Big| \\
&\leqslant \int_{\tilde{D}_k} O(r^{-2\tau-2}) + |R|\,dV + \int_{\tilde{D}_k\cap\partial M} O(r^{-2\tau-1}) + |H|\,dA \\
&\leqslant \int_{M\setminus D_k} O(r^{-2\tau-2}) + |R|\,dV + \int_{(\partial M)\setminus D_k} O(r^{-2\tau-1}) + |H|\,dA
\end{aligned}
$$

Since $R \in L^1(M)$ and $H \in L^1(\partial M)$, the fact that the D_k exhaust M (together with $r > R_k$ in $M \setminus D_k$) implies that the integrals over R and H on the right hand side vanish in the limit $k \to \infty$. Similarly, since $\tau > 1/2$, the integrals over $O(r^{-2\tau-2})$ and $O(r^{-2\tau-1})$ also vanish in this limit.

We learn that using the S_k to compute the mass yields the same result as using coordinate spheres (as we used in our original Definition 3.4). ■

B Basic Riemannian Geometry

Remark B.1. The following section will require some basic knowledge about manifolds, as e.g. taught in [**leeIntroductionSmoothManifolds2012**]. But in many physics courses a basic understanding / intuition for this topic is developed as well, via talking about different coordinate systems (called *charts* in the language of differential geometry) and how objects transform between them. It should hopefully be possible to follow this introduction, and via that knowledge also the rest of this thesis, with only this „physicist's understanding of manifolds“ and by ignoring any unfamiliar notation.

The following are only some basic translation tools to understand the notation we will be using:

- A n-dimensional manifold M is some space which we can (at least locally) describe using n coordinates. Think e.g. of the sphere with spherical coordinates (φ, θ).
- T_pM is the collection of vectors tangent at p tangent to M.
- $X \in \Gamma(TM)$ is a *vector field* on M. In general a (k,l)-tensor (k-times covariant and l-times contravariant) $T^{a_1\cdots a_k}_{b_1\cdots b_l}$ will be written as

$$T \in \Gamma(\underbrace{TM \otimes TM}_{l\text{-times}} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{k\text{-times}}).$$

Definition B.2. A *Riemannian manifold* (M, g) is a smooth manifold M with a positive-definite, inner product g_p smoothly assigned to each point in M, i.e. a positive definite section $g \in \Gamma(T_M^* \odot T_M^*)$ (where $\odot$ is the symmetric product). We call g a *Riemannian metric*.

Remark B.3. A *Pseudo-Riemannian manifold* is equipped with a *Pseudo-Riemannian metric* instead: Here the inner product on each tangentspace is not necessarily positive-definite anymore. We will most often consider $3+1$-dimensional *Lorentzian manifolds*, where the metric has signature $(-,+,+,+)$. We call the first coordinate the *time coordinate* and the other three *spacial coordinates*. All of the following constructions extend without modification to Lorentzian manifolds.

Remark B.4. A metric allows us to convert between vectors and covectors, i.e. we have an isomorphism (called the *musical isomorphism*)

$$T_pM \to T_p^*M \qquad v \mapsto v^\flat,$$

(with inverse $\alpha \mapsto \alpha^\sharp$), where for $w \in T_pM$ we define

$$v^\flat(w) = g(v, w).$$

In coordinates this map is given by lowering the index of our vector i.e. $v^a \rightsquigarrow g_{ab}v^b = v_a$.

Remark B.5. Recall that on any manifold, we can always contract one covariant and one contravariant index of a tensor. This is equivalent to taking the trace of the endomorphism $T_pM \to T_pM$ given by this $\Gamma(TM \otimes T^*M)$ part of the tensor.

In the presence of a metric, the *musical isomorphism* allows us to raise and lower arbitrary indices and thus contract (originally) covariant with covariant indices and contravariant with contravariant tensors, i.e. if we have a tensor $T_{ij\dots}$ we can compute the contraction $\mathrm{tr}_g(T)_{\dots} = g^{ij}T_{ij\dots}$. This is equivalent to choosing an orthonormal basis X_i and dual basis X^i and evaluating

$$\mathrm{tr}_g(T)(\cdots) = \sum_i T(X_i, X_i, \cdots).$$

Riemannian manifolds enable us to not only take derivatives of functions as is possible on all smooth manifolds, but of all tensor fields:

Definition B.6. A *covariant derivative* (or sometimes *(affine) connection*) is an $\mathbb{R}$-linear map $\nabla\colon \Gamma(T_M) \to \Gamma(T_M^* \otimes T_M)$ such that the product rule

$$\nabla(f \cdot X) = df \otimes X + f \cdot \nabla X$$

is fulfilled for any function $f\colon M \to \mathbb{R}$. We often write $\nabla_Y X$ for $(\nabla X)(Y)$.

The covariant derivative also extends to arbitrary tensors, such that for T a (k,l)-tensor, ∇T is a $(k, l+1)$ tensor. Here we demand 2 further properties:

(i) $\nabla(T \otimes T') = \nabla T \otimes T' + T \otimes \nabla T'$, i.e. a Leibniz rule for the tensor product.

(ii) The covariant derivative commutes with taking traces (contracting a covariant and a contravariant part of a tensor):

$$\nabla_Y(\operatorname{tr} T) = \operatorname{tr}(\nabla_Y(T)).$$

Notation B.7. We write $T_{(\text{indices}),i}$ and $T_{(\text{indices});i}$ for the partial and covariant derivative of T in the direction x_i.

Remark B.8. Note that $\nabla_Y X$ is tensorial in Y (since ∇X is a tensor), i.e. if Y is a vector field then $(\nabla_Y X)(p)$ only depends on $Y(p)$ at a point $p \in M$. But $\nabla_Y X$ is *not* tensorial in X, only linear (i.e. $\nabla_Y(X + aX') = \nabla_Y X + a\nabla_Y X'$), instead $(\nabla_Y X)(p)$ depends on the behaviour of X around p (as is to be expected for a derivative)

Remark B.9. If we set

$$\nabla_Y f = df(Y) = Y(f),$$

then the first product rule involving functions and vector fields is just another incarnation of our product rule for tensors.

Remark B.10. We can define higher covariant derivatives, e.g.

$$\nabla^2_{X,Y}(Z)(\cdots) = (\nabla(\nabla Z))(X,Y).$$

In coordinates we can then compute that

$$\begin{aligned}(\nabla^2_{X,Y} Z)^a &= X^b Y^c \nabla_b \nabla_c Z^a \\ &= X^b \nabla_b (Y^c \nabla_c Z^a) - (X^b \nabla_b Y^c)(\nabla_c Z^a) \\ &= (\nabla_X \nabla_Y Z)^a - (\nabla_{\nabla_X Y} Z)^a.\end{aligned}$$

Note that, since coordinate vector fields commute, we have

$$\nabla^2_{a,b} = \nabla_a \nabla_b - \nabla_b \nabla_a.$$

Remark B.11. Writing ∇_i for ∇_{∂_i} in coordinates, we note that the covariant derivative differs from the (coordinate dependent) partial derivative by a linear correction term given by the *Christoffel symbol* (also sometimes called the connection coefficients) Γ^a_{bc}:

$$\nabla_a X^b = \partial_a X^b + \Gamma^b_{ac} X^c.$$

Recall B.12. The commutator of two vector fields X, Y is another vector field fulfilling

$$[X,Y](f) = X(Y(f)) - Y(X(f)).$$

Theorem B.13. *Let (M,g) be a Riemannian manifold. We call a connection ∇ the* Levi-Civita connection *if it is*

(i) Metric compatible: $\nabla g = 0$, and thus in particular

$$\nabla_Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y),$$

(ii) Torsion free: For vector fields X and Y we have

$$\nabla_Y X - \nabla_X Y = [X,Y].$$

This can also, in coordinates, be expressed as

$$\Gamma^a_{bc} = \Gamma^a_{cb}.$$

There always exists a unique Levi-Civita connection on (M,g). Its coefficients in coordinates are (this is also called the Koszul formula)

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}). \tag{9}$$

Definition B.14. We define the Laplacian (also called Laplace-Beltrami-Operator)

$$\Delta f = \mathrm{tr}_g(\nabla^2 f) = \nabla^a \nabla_a f.$$

Lemma B.15. *If M is Riemannian and equipped with the Levi-Civita-connection,*

$$\Delta f = \frac{1}{\sqrt{\det g}} \partial_a(\sqrt{\det g}\,\partial^a f).$$

Proof. Our proof proceeds entirely in coordinates. Note first that (for I the identity matrix)

$$\det'(I) = \mathrm{tr}\,.$$

Then consider the function

$$f(A) = \det A = \det g \cdot \det(g^{-1} \cdot A).$$

Taking the derivative and evaluating at $A = g$ then gives

$$\det'(g)(T) = \det g \cdot \operatorname{tr} g^{-1} \cdot T,$$

and thus

$$\begin{aligned}\partial_a \det g &= \det g \cdot \operatorname{tr} g^{-1} \cdot \partial_i g \\ &= \det g g^{bc} \partial_a g_{bc}.\end{aligned}$$

Thus we can compute

$$\begin{aligned}\Gamma^a_{ab} &= \frac{1}{2} g^{ac}(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}) \\ &= \frac{1}{2} g^{bc} \partial_b g_{ac} \qquad\qquad = \frac{1}{2\det g}\partial_b(\det g) \\ &= \frac{1}{\sqrt{\det g}}\partial_b(\sqrt{\det g}),\end{aligned}$$

where the second equality is due to the fact that (because of symmetry of g)

$$g^{ac}\partial_a g_{bc} = g^{ca}\partial_c g_{ba}.$$

Then

$$\begin{aligned}\nabla^a \nabla_a f &= \nabla^a \partial_a f \\ &= \partial^a \partial_a f + \Gamma^a_{ab}\partial^b f \\ &= \frac{1}{\sqrt{\det g}} \cdot \sqrt{\det g}\partial^a \partial_a f + \frac{1}{\sqrt{\det g}} \cdot (\partial_b \sqrt{\det g})(\partial^b f) \\ &= \frac{1}{\sqrt{\det g}}\partial_a(\sqrt{\det g}\partial^a f).\end{aligned}$$

■

One major difference between the covariant and our usual partial derivatives is that covariant derivatives in different directions do not necessarily commute. The failure of this commutativity is one way to understand and measure curvature:

Definition B.16 (Riemann curvature tensor)**.** We define the *Riemann curvature tensor*

$$R \in \Gamma(T^*M \otimes TM \otimes TM \otimes TM)$$

by writing it as a map taking at each point three tangent vectors X, Y, Z and returning another tangent vector, which we denote by $R(X, Y)Z$:

$$\operatorname{Rm}(X, Y)Z \coloneqq \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The $\nabla_{[X,Y]}Z$ term ensures that Rm is a tensor. We denote Rm in coordinates by

$$\mathrm{Rm}_{cab}^{d} Z^c = \nabla_a \nabla_b Z^d - \nabla_b \nabla_a Z^d.$$

Remark B.17. Note that we still have $\nabla_a \nabla_b f = \nabla_b \nabla_a f$ for any function f.

Oftentimes, the 4 indices of the Riemann curvature tensor are both unwieldy to work with and not necessary to describe many phenomena. By contracting / taking traces we can get two further measures of curvature:

Definition B.18. We define the *Ricci curvature* as

$$\mathrm{Ric}(X, Y) := \mathrm{tr}(X \mapsto R(X, Y)Z)$$

or, in coordinates,

$$\mathrm{Ric}_{ab} = \mathrm{Rm}_{acb}^{c}.$$

We will require the Ricci curvature only once during this thesis, as a stepping stone for our main integral inequality. A lot more important will be the scalar curvature, which in General Relativity is proportional to mass density for static spacetimes, and thus is central formulating the positive mass theorem in purely geometric terms:

Definition B.19. We define the *scalar curvature* as

$$R := \mathrm{tr}_g \mathrm{Ric} = g^{ij} \mathrm{Ric}_{ij}.$$

C Riemannian submanifolds

We will often be considering a three dimensional spacelike hypersurface (defined in Remark C.1) M embedded in a larger $3+1$-dimensional Lorentzian manifold $\tilde{M}$ (for the signature of the metric of $\tilde{M}$ we choose the convention $(-,+,+,+)$), such that the induced metric on M is positive definite. M itself will also often have a boundary ∂M. Thus some basic facts about Riemannian submanifolds will be helpful. Most of the following is from [**leeGeometricRelativity2019**].

Let Σ^m be a submanifold of (pseudo-)Riemannian manifolds (M^n, g) (equipped with Levi-Civita connection ∇).

Remark C.1. g induces a metric

$$\gamma = g|_{T\Sigma} \tag{10}$$

(also called the *first fundamental form*) on Σ^m.

If Σ^m is Riemannian inside a Lorentzian manifold M^n, i.e. if the induced metric γ is positive definite, then we call Σ^m a *spacelike submanifold*.

This metric γ in turn defines a Levi-Civita connection on Σ, which fulfills the following relation:

Fact C.2. *Denoting the Levi-Civita connection of* (Σ, γ) *by* $\hat{\nabla}$*, we have for any* $p \in \Sigma$*, tangent vector* $X \in T_p\Sigma$ *and* $Y \in \Gamma(T\Sigma)$*,*

$$\hat{\nabla}_X Y = (\nabla_X \tilde{Y})^\top,$$

where $\tilde{Y}$ *is any extension of* Y *to a vector field on* M*. Here* $(-)^\top$ *denotes the orthogonal projection from* T_pM *to* $T_p\Sigma$ *(which in coordinates can be done via* $X^i \mapsto \gamma^i_j X^j$*).*

Thus Σ intrinsically contains information about tangential parts of tangential derivatives. But this information does not determine the orthogonal part! This motivates the following definition:

Definition C.3 (Second fundamental form)**.** The *second fundamental form* of Σ is a tensor $\mathbf{A} \in \Gamma(T^*\Sigma \otimes T^*\Sigma \otimes N\Sigma)$ such that for $X, Y \in T_p\Sigma$

$$\mathbf{A}(X, Y) \coloneqq (\nabla_X \tilde{Y})^\perp,$$

where $(-)^\perp$ denotes the orthogonal projection from T_pM to the normal space $N_p\Sigma$. Here $\tilde{Y}$ is again any extension of Y to a vector field on M.

Note that although ∇ is not a tensor (i.e. it depends on the behaviour of the vector field $\tilde{Y}$ around a point p), the second fundamental form $\mathbf{A}$ *is* tensorial. In both of the above definitions, we did not have to extend X to a vector field, since ∇ only depends on the value of X at p.

Fact C.4. $\mathbf{A}(X, Y) = \mathbf{A}(Y, X)$*, i.e.* $\mathbf{A}$ *is symmetric, since for any extensions* $\tilde{X}, \tilde{Y}$ *of* X, Y *we have* $\nabla_X Y - \nabla_Y X = [X, Y] \in T_\Sigma$*.*

Proof. Recall that when treating our vector fields as derivations we can also compute the Lie Bracket as $[X, Y](f) = X(Y(f)) - Y(X(f))$. This does obviously not depend on whether the ambient manifold of X and Y is Σ or M, and thus the resulting vector field $[X, Y]$ must be a vector field on Σ. ■

Definition C.5. The *mean curvature vector* $\mathbf{H}$ is the trace of $\mathbf{A}$ over $T_p\Sigma$, i.e. for an orthonormal basis $e_1, \ldots, e_n$ of $T_p\Sigma$ we define

$$\mathbf{H} \coloneqq \sum_{i=1}^{m} \mathbf{A}(e_i, e_i).$$

Definition C.6. If $\Sigma \subset \Sigma' \subset M$ is a one-dimensional submanifold with normed tangent vector v of another submanifold Σ' of some ambient manifold M, then we define the *geodesic curvature of* Σ as

$$|\operatorname{proj}_{\Sigma} \nabla_v v|.$$

If Σ is an orientable hypersurface of M, we can choose a normal direction ν (if Σ has an interior and exterior, we typically implicitly choose ν to be the outward normal). Then we define

$$A(X,Y) := g(\mathbf{A}(X,Y), -\nu) \qquad H := g(\mathbf{H}, -\nu) = \operatorname{tr}_{\gamma} + A.$$

We also call A the second fundamental form and H the mean curvature. Note that we have

$$A(X,Y) = g(\nabla_X Y, -\nu) = \underbrace{\nabla_X(g(Y,-\nu))}_{=0} - g(Y, \nabla_X(-\nu)) = g(Y, \nabla_X \nu).$$

Thus by using the projection γ^i_j we can write A in coordinates as

$$A_{ij} = \gamma^k_i \gamma^l_j \nabla_k \nu_l. \tag{11}$$

The above in particular also implies that

$$H = \operatorname{div}_g \nu,$$

if we extend ν to a normal vector field in a neighborhood of Σ. To see this, note that $0 = \nabla_\nu(1) = \nabla_\nu(g(\nu,\nu))$ implies $g(\nu, \nabla_\nu \nu) = 0$.

Remark C.7. With this definition, the mean curvature of a twodimensional surface is the sum of the principal curvatures, not the mean. Another common definition of the mean curvature is

$$H = \frac{1}{n-1} \operatorname{tr}_{\gamma}(A),$$

where n is the dimension of the manifold M, but most of our references use the same definition as we do.

Note also that we follow in particular [**almarazPositiveMassTheorem2016**] in choosing a sign convention for A and H opposite to that of their classical definition (given some normal ν). See also [**leeGeometricRelativity2019**] for why this choice is usually made.

We are now equipped to express an important formula relating the curvature of Σ to the curvature of M.

Lemma C.8. *Let* $\mathrm{Rm}, \tilde{\mathrm{Rm}}$ *denote the Riemann curvature of* M *and* Σ *respectively. For* $X, Y, Z, W \in T_\Sigma$*, the* Gauss-Codazzi equation *states*

$$\langle \mathrm{Rm}_(X, Y)Z, W\rangle = \langle \tilde{\mathrm{Rm}}(X, Y)Z, W\rangle + A(X, Z)A(Y, W) - A(X, W)A(Y, Z) \tag{12}$$

or in coordinates

$$\gamma^i_{i'}\gamma^j_{j'}\gamma^k_{k'}\gamma^l_{l'}\mathrm{Rm}_{ijkl} = \tilde{\mathrm{Rm}}_{i'j'k'l'} + A_{i'k'}A_{j'l'} - A_{i'l'}A_{j'k'}.$$

Contracting with $g^{i'k'}\gamma^{j'l'}$ *also yields*

$$\mathrm{Ric}(\nu, \nu) = \frac{1}{2}(R_M - R_\Sigma + H^2 - |A|^2).$$

The following (which is [**leeGeometricRelativity2019**]) is the main fact we will require to deal with calculations on the non-compact boundary of our objects of interest (asymptotically flat half-spaces):

Fact C.9. *Given a hypersurface* Σ *in* (M, g) *and a smooth function* f *on* M*,*

$$\Delta_M f = \Delta_\Sigma + \nabla_\nu\nabla_\nu f + H\nabla_\nu f.$$

Proof. Choose an orthonormal frame $e_1, \ldots, e_n$ of T_pM and $e_1, \ldots, e_{n-1} \in T_p\Sigma$ and $e_n = \nu$, then

$$\begin{aligned}
\Delta_M f &= \sum_{i=1}^{n}(\nabla\nabla f)(e_i, e_i) \\
&= \nabla_M\nabla_\nu f + \sum_{i=1}^{n-1} g(\nabla_{e_i}(\mathrm{grad}_M f), e_i) \\
&= \nabla_\nu\nabla_\nu f + \sum_{i=1}^{n-1} g(\nabla_{e_i}(\mathrm{grad}_\Sigma f + \nu \cdot \nabla_\nu f), e_i) \\
&= \nabla_\nu\nabla_\nu f + \sum_{i=1}^{n-1}(\gamma(\hat{\nabla}_{e_i}(\mathrm{grad}_\Sigma f), e_i) + \underbrace{g(\mathbf{A}(e_i, \mathrm{grad}\,\Sigma), e_i)}_{=0} \\
&\qquad + \nabla_{e_i}\nabla_\nu f \cdot \underbrace{g(\nu, e_i)}_{=0} + \nabla_\nu f \cdot g(\nabla_{e_i}\nu, e_i)) \\
&= \nabla_\nu\nabla_\nu f + \sum_{i=1}^{n-1}(\hat{\nabla}\hat{\nabla} f)(e_i, e_i) + \nabla_\nu f \cdot \sum_{i=1}^{n-1} A(e_i, e_i) \\
&= \nabla_\nu\nabla_\nu f + \Delta_\Sigma f + \nabla_\nu f \cdot H.
\end{aligned}$$

■

D Miscellaneous definitions and results

D.1 Gauss-Bonnet and the Euler characteristic

One of the main reasons for why the current technique does not readily seem to extend to higher dimensions is its reliance on applying the following theorem (which is specific to two dimensions) to level sets of harmonic functions on three-dimensional space:

Theorem D.1 (Gauss-Bonnet Theorem)**.** *Let Σ be a compact two-dimensional Riemannian manifold with boundary $\partial\Sigma$. Let R be the scalar curvature of Σ and let $\kappa_{\partial\Sigma}$ be the geodesic curvature of $\partial\Sigma$ in Σ. Then*

$$\int_\Sigma R/2\,dA + \int_{\partial\Sigma} \kappa_{\partial\Sigma}\,dt = 2\pi\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic (for a definition see below) of Σ.

For a proof see [**petersenRiemannianGeometry2006**].

Definition D.2 (Euler Characteristic)**.** For a compact, connected, oriented surface Σ (two-dimensional manifold with boundary), the *Euler characteristic* is given by

$$\chi(\Sigma) = 2 - 2g - b, \tag{13}$$

where $g \geqslant 0$ is the genus and b is the number of connected boundary components.

For non connected surfaces $\Sigma = \bigsqcup_{i\in I} \Sigma_i$, where the Σ_i are the connected components of Σ, we have

$$\chi(\Sigma) = \sum_i \chi(\Sigma_i).$$

If a non-compact space S results from a puncture of a compact space (is homotopy equivalent to $\sigma \setminus \{x_1, \ldots, x_p\}$, where p is the number of punctures), then we set

$$\chi(S) = \chi(\Sigma) - p.$$

The following theorem (a version of the maximum principle) will then help control the Euler characteristic of the level sets of our harmonic functions:

Theorem D.3. *Let Ω be a compact connected Riemannian manifold with boundary $\partial\Omega = P_1 \sqcup P_2$. Let $u\colon \Omega \to \mathbb{R}$ be harmonic (i.e. $\Delta u = 0$) with Dirichlet boundary condition $u = 0$ on P_1 and Neumann boundary condition $\partial_n u = 0$ on P_2, where n is normal to P_2.*

Then $u = 0$ on all of Ω.

Proof. We start from

$$0 = \int_\Omega u \cdot \Delta u \, d\Omega.$$

Integrating by parts then yields

$$\begin{aligned} 0 &= \int_\Omega u \cdot g^{ij} \nabla_i \nabla_j u \, dx \\ &= \int_\Omega \nabla_i u \cdot g^{ij} \nabla_j u \, dx - \int_{\partial\Omega} u \cdot g^{ij} \nabla_i u \cdot n_j \, dS \\ &= \int_\Omega |\nabla u|^2 \, dx - \int_{\partial\Omega} u \cdot \partial_n u \, dS. \end{aligned}$$

But we always have either $u = 0$ or $\partial_n u = 0$ on $\partial\Omega$, and we conclude that $|\nabla u| = 0$ everywhere, i.e. that u is constant on Ω (since Ω only has one connected component). ■

D.2 Bochner's identity

A major ingredient that allows us to connect the derivatives of our harmonic functions to the curvature of the surrounding space is Bochner's identity:

Lemma D.4. *For any smooth function u on a Riemannian manifold (m, g),*

$$\frac{1}{2}\Delta(|\nabla u|^2) = |\nabla^2 u|^2 + g(\operatorname{grad}(\Delta u), \operatorname{grad} u) + \operatorname{Ric}(\operatorname{grad} u, \operatorname{grad} u).$$

Proof. A straightforward calculation in coordinates yields

$$\begin{aligned} \frac{1}{2}\Delta|\nabla u|^2 &= \frac{1}{2}\nabla^j \nabla_j (\nabla_i u \nabla^i u) \\ &= \nabla^j((\nabla_j \nabla_i u)(\nabla^i u)) \\ &= (\nabla^j \nabla_j \nabla_i u)(\nabla^i u) + (\nabla_j \nabla_i u)(\nabla^j \nabla^i u) \\ &\underset{\substack{\uparrow \\ \text{Remark B.17}}}{=} \nabla^j((\nabla_i \nabla_j u)(\nabla^i u)) \\ &= (\nabla_j \nabla_i \nabla^j u)(\nabla^i u) + |\nabla^2 u|^2 \\ &\underset{\substack{\uparrow \\ \text{see Definition B.16}}}{=} (\nabla_i \nabla_j \nabla^j u + \operatorname{Rm}_{kji}^{j} \nabla^k u)(\nabla^i u) + |\nabla^2 u|^2 \\ &= g(\operatorname{grad} \Delta u, \operatorname{grad} u) + |\nabla^2 u|^2 + \operatorname{Ric}_{kj}(\nabla^k u)(\nabla^j u) \\ &= |\nabla^2 u|^2 + g(\operatorname{grad}(\Delta u), \operatorname{grad} u) + \operatorname{Ric}(\operatorname{grad} u, \operatorname{grad} u). \end{aligned}$$

■

D.3 The mean curvature under conformal changes of metric

In Section 7 we consider the Schwarzschild half-space as an example. In that context we also have to compute the mean curvature of some surfaces. But, since the Schwarzschild metric is conformal to the euclidean metric, we are able to simplify our calculations in Section 7 using a formula derived below. For a more in depth treatment of conformal transformations, see [**curryIntroductionConformalGeometry2015**].

In the following, let $(M, \bar{g})$ always be conformal to (M, g), i.e. let $\bar{g} = \Omega^2 g$ for some real function Ω. Denote all quantities with a bar (e.g. $\bar{\nabla}, \bar{R}, \bar{H}$) when they are computed with respect to $\bar{g}$ and without a bar when they are computed with respect to g. Let Σ be some hypersurface of M with normal vectors $\nu, \bar{\nu}$. Second fundamental forms $A, \bar{A}$ and mean curvatures $H, \bar{H}$ will follow the definitions from Appendix C.

Lemma D.5.

$$\bar{\nabla}_X Y = \nabla_X Y + X(f) \cdot Y + Y(f) \cdot X - g(X, Y) \operatorname{grad} f.$$

Proof. We will be using the Koszul formula Eq. (9),

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}).$$

In particular we note that (since $\bar{g}^{ad} = e^{-2f} g^{ad}$)

$$\begin{aligned}
\bar{\Gamma}^c_{ab} &= \frac{1}{2} e^{-2f} g^{cd} (\partial_a (e^{2f} g_{bd}) + \partial_b (e^{2f} g_{ad}) - \partial_d (e^{2f} g_{ab})) \\
&= \Gamma^c_{ab} + \delta^c_a \partial_b f + \delta^c_b \partial_a f - g_{ab} \nabla^c f.
\end{aligned}$$

Then

$$\begin{aligned}
(\bar{\nabla}_X Y)^c &= X^a \bar{\nabla}_a Y^c \\
&= X^a \partial_a Y^c + X^a \bar{\Gamma}^c_{ab} Y^b \\
&= X^a \partial_a Y^c + X^a \Gamma^c_{ab} Y^b + X^c Y^b \partial_b f + Y^c X^a \partial_a f - \nabla^c f g_{ab} X^a Y^b \\
&= (\nabla_X Y)^c + X^c \cdot Y(f) + Y^c \cdot X(f) - \nabla^c f \cdot g(X, Y),
\end{aligned}$$

as required. ∎

Lemma D.6.

$$\bar{H} = e^{-f} (H + (n - 1) g(\operatorname{grad} f, \nu)).$$

Proof. Let $e_1, \dots, e_n$ be an orthonormal basis at some tangent space under g such that $e_1, \dots, e_{n-1} \in T_\Sigma$, then $\bar{e}_a = e^{-f} e_a$ is such an orthonormal basis under $\bar{g}$. Similarly we have $\bar{\nu} = e^{-f}\nu$.

We can then compute

$$\begin{aligned}
\bar{A}(X,Y) &= \bar{A}(X,Y) \\
&= \bar{g}(\bar{\nabla}_X Y, -e^{-f}\nu) \\
&= e^f [A(X,Y) + X(f)g(Y,-\nu) + Y(f)g(X,-\nu) - g(\nabla f, -\nu) \cdot g(X,Y)]
\end{aligned}$$

This then leads directly to

$$\begin{aligned}
\bar{H} &= \sum_{\alpha=1}^{n-1} \bar{A}(\bar{e}_\alpha, \bar{e_\alpha}) \\
&= \sum_{\alpha=1}^{n-1} e^{-2f} \bar{A}(e_\alpha, e_\alpha) \\
&\underset{\substack{\uparrow \\ g(e_\alpha,\nu)=0}}{=} \sum_{\alpha=1}^{n-1} e^{-f} (A(e_i, e_i) + 0 + 0 - g(\operatorname{grad} f, -\nu) \cdot 1) \\
&= e^{-f}(H + (n-1) \cdot g(\operatorname{grad} f, \nu)).
\end{aligned}$$

■