Harmonic Functions and the Positive Mass Theorem for Asymptotically Flat Half-Spaces

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1 Introduction

TODO Add historical context and applications of the Positive Mass theorem. Todo

The (spacetime) positive mass theorem is a central result in the study of general relativity and differential geometry, originally proved by Richard Schoen and Shing-Tung Yau in 1979 [SY79] employing stable minimal hypersurfaces and independently by Edward Witten in 1981 [Wit81] using spinor techniques.

In the following, we will explore a relatively new proof of the Positive Mass theorem using (spacetime) harmonic functions, and in particular consider the case of rigidity. We will then look at how these harmonic functions look in some simple example cases. Finally, we will will apply this method to the Positive Mass theorem on asymptotically flat half spaces with connected (non-compact) boundary, acquiring an explicit lower bound for the mass in the process, i.e. we will prove the following theorem (notation and concepts will be introduced later):

Theorem 1.1. *Let (M, g) be an asymptotically flat half-space (M, g) of dimension $n = 3$ with horizon boundary $\Sigma \subset M$, associated exterior region $M(\Sigma)$ and connected non-compact boundary ∂M. Let (x_1, x_2, x_3) be asymptotically flat coordinates such that outside of a compact set, M is diffeomorphic to $\{x \in \mathbb{R}^3_+ \mid |x| > r_0\}$ for some $r_0 > 0$. Assume that the following three conditions hold*

- $R \geqslant 0$ *in* $M(\Sigma)$.
- $H \geqslant 0$ *on* $M(\Sigma) \cap \partial M$.

Then there exists a unique harmonic function u on $M(\Sigma)$ asymptotic to x_3 fulfilling zero Dirichlet boundary conditions on $\partial M \cap M(\Sigma)$ and zero Neumann boundary conditions on Σ, and we have

$$m \geqslant \int_{M(\Sigma)} \left(\frac{|\nabla^2 u|^2}{|\nabla u|} + R|\nabla u| \right) dV + \int_{\partial M \cap M(\Sigma)} H|\nabla u| \, dA \geqslant 0,$$

where m is the ADM mass of M. Equality $m = 0$ occurs if and only if (M, g) is isometric to $(\mathbb{R}^3_+, \delta)$.

1.1 Physical Motivation

The positive mass theorem was originally motivated by the study of general relativity, but is also (particularly in the so-called time-symmetric or Riemannian case) of independent importance to differential geometry. We will do a quick exposition of both of these perspectives.

Physically, a less general statement can informally be expressed as the following:

Consider a static (i.e. time-independent) mass distribution ρ in $\mathbb{R}^3$ that is compactly supported in some finite volume V (it would suffice if the mass distribution fell off sufficiently quickly towards infinity, but this case is easier to reason about).

Then in the Newtonian Theory of Gravity, this mass distribution would at large distances look like a point mass of some total mass M. Due to the linear nature of Newtonian gravity, we can calculate that M is just $\int \rho \, dV$.

But when we consider Einstein's Theory of Gravity (via General Relativity), though we can still assign a total mass, now called the *ADM mass* M (in practice this takes the form of an integral expression over large coordinate spheres, where we take the limit as the radius goes to infinity), we lose the linearity of Newtonian Gravity and we cannot anymore identify M with the integral of the individual masses anymore. Here our mass distribution bends spacetime in some (possibly very complicated) way, but the ADM mass tells us that his spacetime geometry asymptotically looks like the geometry around a Schwarzschild black hole of mass M.

The positive mass theorem now says that even though we lose the relation to the integral of the mass distribution, we retain at least some good behaviour of the mass: If the mass distribution is non-negative everywhere, then we also have $M \geqslant 0$, i.e. there exists no configuration of positive masses (however complicated) that acts like a black hole of negative mass (a white hole) at large distances. Compare [Lee19, Chapter 7] for more details.

When expressing this theorem mathematically, we leave behind a lot of the physical details. In particular, we directly consider the scalar curvature R instead of the mass distribution (since the scalar curvature is proportional to mass in General Relativity). Since we define the ADM mass in terms of the asymptotic geometric behaviour as well, we reduce the physical statement to a purely geometric one. This leads us to another approach to motivate the theorem, at least for the time-symmetric case (this formulation is from [Bra+21, page 1]):

Every compactly supported perturbation of the Euclidean metric on $\mathbb{R}^n$ must somewhere decrease its scalar curvature. This is a kind of extremality property of the Euclidean metric. It follows directly from the Geroch conjecture – the fact that the torus $\mathbb{T}^n$ does not admit a metric of positive scalar curvature – by identifying the ends of a large coordinate cube (containing the compact set on which the perturbation takes place). The Riemannian positive mass theorem then is an extension of this extremality property to the nonnegativity of the ADM-mass on manifolds that are *asymptotically euclidean* instead of straight up equal to the euclidean geometry outside a compact set. We'll call these manifolds *asymptotically flat.*

2 Prerequisites

In the following let M always be a 3-dimensional Riemannian manifold, equipped with the unique Levi-Civita-Connection.

Note that by ∇ we will always denote the covariant derivative / the Levi-Civita connection. In particular we will have $\nabla f = df$, and $\operatorname{grad} f = (df)^\sharp$, such that in coordinates df will be given by $\nabla_i f = \partial_i f$ and $\operatorname{grad} f$ will be given by $\nabla^i f$.

We will often be considering M as a three dimensional spacelike surface embedded in a larger $3+1$-dimensional Lorentzian manifold $\tilde{M}$ (for the signature of the metric of $\tilde{M}$ we choose the convention $(-,+,+,+)$), such that the induced metric on M is positive definite. M itself will also often have a boundary ∂M. Thus some basic facts about submanfolds will be helpful. Most of the following is from [Lee19, Chapter 2.1].

Let Σ^m be a submanifold of (pseudo-)Riemannian manifolds (M^n, g) (equipped with Levi-Civita connection ∇). We have an induced metric $\gamma = g|_{T\Sigma}$ (also called the *first fundamental form*).

Fact 2.1. *Denoting the Levi-Civita connection of* (Σ, γ) *by* $\hat{\nabla}$*, we have for any* $p \in \Sigma$*, tangent vector* $v \in T_p\Sigma$ *and* $Y \in \Gamma(T\Sigma)$*,*

$$\hat{\nabla}_X Y = (\nabla_X \tilde{Y})^\top,$$

where $\tilde{Y}$ *is any extension of* Y *to a vector field on* M*. Here* $(-)^\top$ *denotes the orthogonal projection from* T_pM *to* $T_p\Sigma$*.*

TODO *This might be too similar to [Lee19, p. 2.1]* Todo

Thus Σ intrinsically contains information about tangential parts of tangential derivatives. But this information does not determine the orthogonal part! This motivates the following definition:

Definition 2.2 (Second fundamental form)**.** The *second fundamental form* of Σ is a tensor $\mathbf{A} \in \Gamma(T^*\Sigma \otimes T^*\Sigma \otimes N\Sigma)$ such that

$$\mathbf{A}(X, Y) \coloneqq (\nabla_X \tilde{Y})^\perp,$$

where $(-)^\perp$ denotes the orthogonal projection from T_pM to $T_p\Sigma$.

Fact 2.3. $\mathbf{A}(X,Y) = \mathbf{A}(Y,X)$*, i.e.* $\mathbf{A}$ *is symmetric (since* $\nabla_X Y - \nabla_Y X = [X,Y] \in T_\Sigma$*).*

Definition 2.4. The *mean curvature vector* $\mathbf{H}$ is the trace of $\mathbf{A}$ over $T_p\Sigma$, i.e. for an orthonormal basis $e_1, \dots, e_n$ of $T_p\Sigma$ we define

$$\mathbf{H} \coloneqq \sum_{i=1}^m \mathbf{A}(e_i, e_i).$$

If Σ is an orientable hypersurface of M, we can choose a normal direction ν (if Σ has an interior and exterior, we typically implicitly choose ν to be the outward normal). Then we define

$$A(X,Y) \coloneqq g(\mathbf{A}(\mathbf{X},\mathbf{Y}),-\nu) \qquad H \coloneqq g(\mathbf{H},-\nu) = \operatorname{tr}_\gamma(A).$$

We also call A the second fundamental form and H the mean curvature. Note that we have

$$A(X,Y) = g(\nabla_X Y, -\nu) = \underbrace{\nabla_X(g(Y,-\nu))}_{=0} - g(Y, \nabla_X(-\nu)) = g(Y, \nabla_X \nu).$$

The following (which is [Exercise 2.3 in Lee19]) is one of the main facts we will require:

Fact 2.5. *Given a hypersurface Σ in (M,g) and a smooth function f on M,*

$$\Delta_M f = \Delta_\Sigma + \nabla_\nu \nabla_\nu f + H \nabla_\nu f.$$

Proof. Choose an orthonormal frame $e_1, \ldots, e_n$ of T_pM and $e_1, \ldots, e_{n-1} \in T_p\Sigma$ and $e_n = \nu$, then

$$\begin{aligned}
\Delta_M f &= \sum_{i=1}^n (\nabla\nabla f)(e_i, e_i) \\
&= \nabla_M \nabla_\nu f + \sum_{i=1}^{n-1} g(\nabla_{e_i}(\operatorname{grad}_M f), e_i) \\
&= \nabla_\nu \nabla_\nu f + \sum_{i=1}^{n-1} g(\nabla_{e_i}(\operatorname{grad}_\Sigma f + \nu \cdot \nabla_\nu f), e_i) \\
&= \nabla_\nu \nabla_\nu f + \sum_{i=1}^{n-1} (\gamma(\hat{\nabla}_{e_i}(\operatorname{grad}_\Sigma f), e_i) + \underbrace{g(\mathbf{A}(e_i, \operatorname{grad}\Sigma), e_i)}_{=0} \\
&\quad + \nabla_{e_i} \nabla_\nu f \cdot \underbrace{g(\nu, e_i)}_{=0} + \nabla_\nu f \cdot g(\nabla_{e_i} \nu, e_i)) \\
&= \nabla_\nu \nabla_\nu f + \sum_{i=1}^{n-1} (\hat{\nabla}\hat{\nabla} f)(e_i, e_i) + \nabla_\nu f \cdot \sum_{i=1}^{n-1} A(e_i, e_i) \\
&= \nabla_\nu \nabla_\nu f + \Delta_\Sigma f + \nabla_\nu f \cdot H.
\end{aligned}$$

■

We will have to also discuss some basic topological properties of the level sets of the functions we will be looking at.

Definition 2.6 (Euler Characteristic)**.** For a compact, connected, oriented surface Σ (two-dimensional manifold with boundary), the *Euler characteristic* is given by

$$\chi(\Sigma) = 2 - 2g - b, \tag{1}$$

where $g > 0$ is the genus and b is the number of connected boundary components.

For non connected surfaces $\Sigma = \bigsqcup_{i \in I} \Sigma_i$, where the Σ_i are the connected components of Σ, we have

$$\chi(\Sigma) = \sum_i \chi(\Sigma_i).$$

The following theorem (a version of the maximum principle) will then help control the Euler characteristic of the level sets of our harmonic functions:

Theorem 2.7. *Let Ω be a compact connected Riemannian manifold with boundary $\partial\Omega = P_1 \sqcup P_2$. Let $u \colon \Omega \to \mathbb{R}$ be harmonic (i.e. $\Delta u = 0$) with Dirichlet boundary condition $u = 0$ on P_1 and Neumann boundary condition $\partial_n u = 0$ on P_2, where n is normal to P_2.*

Then $u = 0$ on all of Ω.

Proof. We start from

$$0 = \int_\Omega u \cdot \Delta u \, d\Omega.$$

Integrating by parts then yields

$$\begin{aligned} 0 &= \int_\Omega u \cdot g^{ij} \nabla_i \nabla_j u \, dx \\ &= \int_\Omega \nabla_i u g^{ij} \nabla_j u \, dx - \int_{\partial\Omega} u \cdot g^{ij} \nabla_i u n_j \, dS \\ &= \int_\Omega |\nabla u|^2 \, dx - \int_{\partial\Omega} u \cdot \partial_n u \, dS. \end{aligned}$$

But we always have either $u = 0$ or $\partial_n u = 0$ on $\partial\Omega$, and we conclude that $|\nabla u| = 0$ everywhere, i.e. that u is constant on Ω (since Ω only has one connected component). ■

3 The mass of an asymptotically flat half-space

We will now establish the necessary definitions (mostly adapted from [ABL16], [EK23] and [Bra+19]) to state the main result of this thesis.

Definition 3.1 (Asymptotically flat half-space)**.** Let (M, g) be a connected, complete Riemannian manifold of dimension 3, with scalar curvature R and a non-compact boundary ∂M with mean curvature H (computed as the divergence along ∂M of an outward pointing unit normal ν).

We call (M, g) an *asymptotically flat half-space* with decay rate $\tau > 0$ if there exists a compact subset K such that $M \setminus K$ consists of a finite number of connected components M^i_{end} called *ends*, such that for each of these ends there exists a diffeomorphism $\Phi \colon M^i_{\text{end}} \to \{x \in \mathbb{R}^3_+ \mid |x| > r_0\}$ and such that in the coordinate system given by this diffeomorphism we have the following asymptotic as $r \to \infty$:

$$|\partial^l (g_{ij} - \delta_{ij})| = O(r^{-\tau - l}) \tag{2}$$

for $l = 0, 1, 2$. Here $r = |x|$ and δ is the Euclidean metric on $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 \mid x_3 \geqslant 0\}$.

In the following, we will often use the Einstein summation convention with the index ranges $i, j, \ldots = 1, \ldots, 3$ and $\alpha, \beta, \ldots = 1, 2$. We write $T_{(\text{indices}),i}$ and $T_{(\text{indices});i}$ for the partial and covariant derivative of T in the direction x_i. Note that, along ∂M, $\{\partial_\alpha\}_\alpha$ spans $T_{\partial M}$, while ∂_n points inwards.

Definition 3.2. If R and H are integrable over M and ∂M respectively and $\tau > 1/2$, then the *mass* of each end of M is well defined and (introducing the notation C_i for the coordinate dependent quantity $\sum_j (g_{ij,j} - g_{jj,i})$) given by

$$\mathfrak{m}_{(M^i_{\text{end}}, g)} = \frac{1}{16\pi} \lim_{r \to \infty} \left(\int_{\mathbb{S}^2_{r,+}} C_i \mu^i \, dA + \int_{\mathbb{S}^1_r} g_{\alpha 3} \theta^\alpha \, dl \right),$$

where the integrals are computed in the asymptotically flat chart, $\mathbb{S}^2_{r,+}(0) = \{\mathbb{R}\}^3_+ \cap \mathbb{S}^2_r(0)$ is a large upper coordinate hemisphere with outward unit normal μ, and θ is the outward pointing unit co-normal to $\mathbb{S}^1_r = (\{\mathbb{R}\}^2 \times \{0\}) \cap \mathbb{S}^2_r(0) = \partial \mathbb{S}^2_{r,+}$, oriented as the boundary of $(\partial M)_r \subset \partial M$.

TODO Explain what co-normal means. Todo

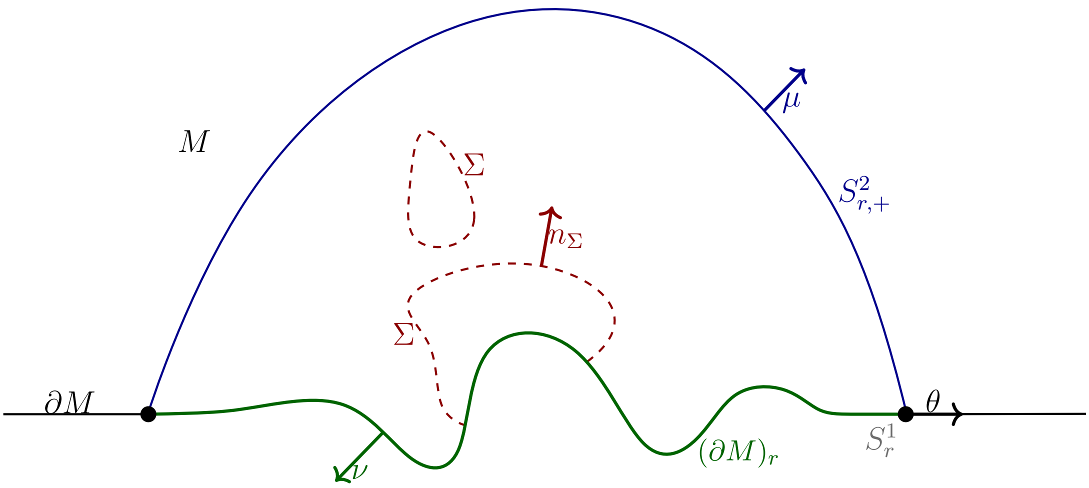

Figure 1: An asymptotically flat half space with horizon boundary and a large coordinate sphere (from Definition 3.2)

Remark 3.3. In the definition above, the factor $1/(16\pi)$ is a normalization factor used also for the ADM mass of asymptotically flat manifolds with the full $\mathbb{R}^n$ as a model space, where it ensures that we recover the mass of the Schwarzschild solution.

Note that thus in our case of asymptotically flat half-space, the mass of a *half Schwarzschild space* $M_m = \{x \in \mathbb{R}^3_+ \mid |x| > (m/2)^{1/(n-2)}\}$ with conformal metric

$$g_m = \left(1 + \frac{m}{2|x|}\right)^2 \delta, \quad m > 0$$

will be

$$\mathfrak{m}_{(M_m, g_m)} = \frac{m}{2},$$

which is half the ADM mass of the standard Schwarzschild space.

In [ABL16], Almaraz, Barbosa, and de Lima showed that this mass is well defined and a geometric invariant, and, in fact, non-negative under suitable energy conditions:

Theorem 3.4. *For (M, g) as above in Definition 3.2, if $R \geqslant 0$ and $H \geqslant 0$ on M and ∂M respectively, then*

$$\mathfrak{m}_{(M,g)} \geqslant 0,$$

with equality occurring if and only if (M, g) is isometric to $(\mathbb{R}^3_+, \delta)$.

For the positive mass theorem on 3-dimensional asymptotically flat manifolds modeled on the full $\mathbb{R}^3$, recently [Bra+19] a new method using harmonic functions has been used to achieve a relatively elementary proof of the above theorem and in particular an explicit lower bound for the mass. This thesis will attempt to establish an equivalent result for the case of asymptotically flat half spaces. We will need two further definitions adopted from [EK23] to unterstand the statement of our main result.

Definition 3.5. Let $\Sigma \subset M$ be a compact separating hypersurface satisfying $\partial\Sigma = \Sigma \cap \partial M$ with normal n_Σ pointing towards the closure $M(\Sigma)$ of the noncompact component of $M \setminus \Sigma$. We call a connected component Σ_0 of M *closed* if $\partial\Sigma_0 = \varnothing$ or a *free boundary hypersurface* if $\partial\Sigma_0 \neq 0$ and $n_{\Sigma_0}(x) \in T_x\partial M$ for every $x \in \Sigma_0 \cap \partial M = \partial\Sigma_0$ (i.e. if Σ_0 meets ∂M orthogonally along its boundary).

We say that an (M, g) has horizon boundary Σ if Σ is a non-empty compact minimal (i.e. having zero mean curvature) hypersurface, whose connected components are all either closed or free boundary hypersurfaces such that $M(\Sigma) \setminus \Sigma$ contains no minimal closed or free boundary hypersurfaces. Σ is also called an *outer most minimal surface* and the region $M(\Sigma)$ outside Σ is called an *exterior region*.

Remark 3.6. By [Koe20, Lemma 2.3] if $H \geqslant 0$ on ∂M, then there either exists a unique horizon boundary $\Sigma \subset M$ or M contains no compact hypersurfaces.

The main result of this thesis is then the following, which will prove Theorem 3.4 as a corollary:

Theorem 3.7. *For (M, g) as above in Definition 3.2, if an exterior region $M(\Sigma)$ exists, then there exists a unique harmonic function u asymptotic to the linear function x_3 and satisfying zero Dirichlet boundary conditions on ∂M and zero Neumann boundary conditions on the horizon boundary Σ, and we have*

$$\mathfrak{m}_{(M,g)} \geqslant \frac{1}{16\pi}\int_{M(\Sigma)} \left(\frac{|\nabla^2 u|^2}{|\nabla u|} + R|\nabla u|\right) dV + \frac{1}{16\pi}\int_{\partial M \cap M(\Sigma)} H(\partial M, g)|\nabla u|\, dA \geqslant 0.$$

In particular, if R and H are nonnegative, then the existence of $M(\Sigma)$ is guaranteed and the above inequality also gives nonnegativity of the mass.

4 Main basic identity

The basic identity underlying the method of harmonic functions is the [Bra+19, Proposition 4.2]:

Proposition 4.1. *Let (Ω, g) be a compact 3-dimensional manifold with boundary $\partial\Omega = P_1 \sqcup P_2$ (smooth almost everywhere), having outward unit normal n. Let $u\colon \Omega \to \mathbb{R}$ be a harmonic function (i.e. $\Delta u = 0$) such that $\partial_{n_{P_1}} u = 0$ on P_1. If $\bar{u}$ and $\underline{u}$ denote the maximum and minimum of u and Σ_t are t-level sets of u, then*

$$\int_{\underline{u}}^{\bar{u}} \Big(\int_{\Sigma_t} \frac{1}{2} \Big(\frac{|\nabla^2 u|}{|\nabla u|^2} + R \Big) \, dA + \int_{\partial\Sigma_t \cap P_1} H_{P_1} \, dl \Big) \, dt \leqslant \int_{\underline{u}}^{\bar{u}} \Big(2\pi\chi(\Sigma_t) - \int_{\partial\Sigma_t \cap P_2} k_{\partial\Sigma_t} \, dl \Big) \, dt + \int_{\tilde{P}_2} \partial_n |\nabla u| \, dA,$$

where $\chi(\Sigma_t)$ denotes the Euler characteristic of the level sets, $\kappa_{\partial\Sigma_t}$ denotes the geodesic curvature of $\partial\Sigma_t$ and H_{P_1} denotes the mean curvature of P_1.

The integrand for the integral over P_2 is defined when $|\nabla u| \neq 0$. By Sard's Theorem [Sar42] the set where $|\nabla u| = 0$ has measure 0 and we'll ignore it by abuse of notation (we should really write the integral over as being over $\tilde{P}_2 = P_2 \cap \{|\nabla u| \neq 0\}$, but that gets cumbersome).

5 Existence and uniqueness of asymptotically linear harmonic functions

To use Proposition 4.1, we will require harmonic functions with properties as in Theorem 3.7. More specifically we will require asymptotically linear harmonic coordinates on $M(\Sigma)$ (for Σ the horizon boundary of M) with certain boundary conditions.

That is we want functions $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ that

- are harmonic, i.e. $\Delta\tilde{x}_i = 0$ in M, for $i = 1, 2, 3$
- are asymptotic to our standard asymptotically half-euclidean coordinates on M_{end}, i.e. $\tilde{x}_i - x_i \in C^{2,\alpha}_{1-\tau+\varepsilon}(M)$ for some $\varepsilon > 0$ and $0 < \alpha < 1$. For a definition of the weighted Hölder space $C^{k,\alpha}_{\gamma}(M)$ see [ABL16, p. 682]. Important for us is mostly that this condition ensures that our $\tilde{x}_i$ themselves form an asymptotically flat coordinate system.
- fulfill boundary conditions on ∂M mimicking the behaviour of the standard coordinates on Euclidean half space, i.e.

$$\begin{cases} \partial_\nu \tilde{x}_\alpha = 0 & \text{on } \partial M \cap M(\Sigma), \text{ for } \alpha = 1, 2 \\ \tilde{x}_3 = 0 & \text{on } \partial M \cap M(\Sigma). \end{cases}$$

Later, when we will relate the mass $\mathfrak{m}_{(M,g)}$ to the behaviour of $\tilde{x}_3$, this will significantly simplify the boundary terms occurring on ∂M.

- fulfill a Neumann boundary condition on Σ, i.e. $\partial_{n_\Sigma}\tilde{x}_i = 0$. This will make boundary terms on Σ disappear completely in our calculation.

Even though we normally only work with a single end, our proof of the existence of these functions will use a reflection argument along Σ, which will require uniqueness and existence of our functions for the case of multiple ends and no horizon boundary, i.e. $\Sigma = \varnothing$. But it is not much harder to use this to prove existence and also uniqueness for the case with multiple ends and possibly non-empty boundary. I thus streamlines our proof to just state the following proposition for multiple ends, prove the case $\Sigma = \varnothing$, and then use the reflection argument to prove the general case.

Proposition 5.1. *Suppose (M, g) is an asymptotically flat half-space with decay-rate $\tau > 1/2$, asymptotically flat coordinates $\{x_i\}^j$ in each end M^j_{end} and horizon boundary Σ. Assume (e.g. by shrinking the ends a bit) that the closures of the ends M^j_{end} are disjoint. Then there exist (up to constant shift) unique smooth functions $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \colon M(\Sigma) \to \mathbb{R}$ satisfying*

$$\begin{cases} \Delta \tilde{x}_\beta = 0 & \text{in } M(\Sigma), \\ \partial_\nu \tilde{x}_\beta = 0 & \text{on } \partial M \cap M(\Sigma), \\ \partial_{n_\Sigma} \tilde{x}_\beta = 0 & \text{on } \Sigma, \end{cases}$$

for $\beta = 1, 2$,

$$\begin{cases} \Delta \tilde{x}_3 = 0 & \text{in } M(\Sigma), \\ \tilde{x}_3 = 0 & \text{on } \partial M \cap M(\Sigma), \\ \partial_{n_\Sigma} \tilde{x}_3 = 0 & \text{on } \Sigma, \end{cases}$$

and

$$x^j_i - \tilde{x}_i \in C^{2,\alpha}_{1-\tau+\varepsilon} \quad \text{in } M^j_{\text{end}}.$$

Moreover, for each end M^j_{end}, the functions $\{\tilde{x}_i\}$ form an asymptotically flat coordinate system in a neighborhood of infinity.

TODO *Maybe change the wording here to align with our wording elsewhere?* Todo

Proof. We first show existence and uniqueness for the case $\Sigma = \varnothing$, then extend to $\Sigma \neq \varnothing$ via a reflection argument along Σ.

Step 1. [ABL16, Proposition 3.8] proves existence for $\Sigma = \varnothing$ and one end. But by replacing the x_i in the proof with arbitrary smooth extensions of the x_i^j (which are defined on open sets with disjoint closures) with $x_i|_{M^j_{\mathrm{end}}} = x_i^j$, $x_3|_{\partial M} = 0$, we can easily generalise the statement to multiple ends.

We want to show uniqueness for the case $\Sigma = \varnothing$. Let $\{\tilde{x}_i\}$ and $\tilde{x}_i'$ be two harmonic coordinates fulfilling all the properties. By [ABL16, Proposition 3.9] (the proof of which extends without changes to the case with multiple ends), there exist an orthogonal matrix $(Q_i^j)_{i,j=1}^3$ and constants $\{a_i\}_{i=1}^3$, such that

$$\tilde{x}_i = Q_i^j \tilde{x}_i' + a_i.$$

We have

$$(\delta_i^j - Q_i^j)\tilde{x}_j - a_i = \tilde{x}_i - \tilde{x}_i' = o(r^{1/2}) \quad \text{as } r \to \infty,$$

and further

$$(\delta_i^j - Q_i^j)(x_j - \tilde{x}_j) - a_i = o(r^{1/2})$$

which implies

$$(\delta_i^j - Q_i^j)x_j = o(r^1)$$

and thus we must have $Q_i^j = \delta_i^j$ (since otherwise $(\delta_i^j - Q_i^j)x_j$ would be linear). Hence

$$\tilde{x}_i = \tilde{x}_i' + a_i.$$

Note that $a_3 = 0$, since $\tilde{x}_i = 0 = \tilde{x}_i'$ on ∂M.

Step 2. Consider now the case $\Sigma \neq \varnothing$. We adapt the proof of [EK23, Proposition 46].

Consider the differentiable manifold $(\hat{M} = M \times \{-1, +1\})/_\sim$, where $(x, \pm 1) \sim (x, \mp 1)$ if and only if $x_1, x_2 \in \Sigma$ and $x_1 = x_2$ (i.e. $\hat{M}$ is constructed by gluing two copies of M along Σ). We equip $\hat{M}$ with the Riemannian metric $\hat{g}(\hat{x}) = \gamma(\pi(\hat{x}))$, where $\pi([(x, \pm 1)]) = x$.

Then by [EK23, Lemma 19], $\hat{g}$ is of class C^2 away from $\pi^{-1}(\Sigma)$ and on $\pi^{-1}(\Sigma)$ the coefficients of $\Delta_{\tilde{g}}$ are Lipschitz since Σ is minimal.

Note that $\hat{M}$ has twice as many ends as M, where we set $\hat{x}_i^{j,\pm}(\hat{x}) = x_i^j(\pi(x))$ to be the asymptotic coordinates in these ends.

We can thus apply the result from Step 1 to $\hat{M}$ (for which we don't consider any boundary conditions on horizon boundaries) to obtain asymptotically linear harmonic coordinates $\tilde{\hat{x}}_i$ on $\hat{M}$ with Dirichlet boundary condition on $\partial\hat{M}$. But note that $\tilde{\hat{x}}_i \circ \tau$ is another solution, where we let $\tau\colon \hat{M} \to \hat{M}$ be given by $\tau([(x, \pm 1)]) = [(x, \mp 1)]$. Then by the already established uniqueness for the case without horizon boundary, $\tilde{\hat{x}}_i \circ \tau = \tilde{\hat{x}}_i + a_i$ for some constants a_i. But since these must agree on $\pi^{-1}(\Sigma)$ (τ is the identity there), we have $a_i = 0$.

In particular we get on $\pi^{-1}(\Sigma)$

$$\partial_{n_\Sigma} \tilde{\hat{x}}_i = -\partial_{n_\Sigma}(\tilde{\hat{x}}_i \circ \tau) = -\partial_{n_\Sigma}(\tilde{\hat{x}}_i) = -\partial_{n_\Sigma}(\tilde{\hat{x}})$$

and thus $\tilde{\hat{x}}$ satisfies Neumann boundary conditions on $\pi^{-1}(\Sigma)$ (here we need to fix n_Σ, e.g. choose it to point towards $M \times \{+1\}$).

In particular we get a solution to our original problem on M by setting $\tilde{x}_i(x) = \tilde{\hat{x}}_i([x, +1])$.

The argument from [EK23, Proposition 3.9] extends straightforwardly to also show uniqueness (up to adding constants) for the $\tilde{x}_i$ on $M(\Sigma)$.

■

6 Proof of the Mass Lower Bound

We proceed by constructing a proof parallel to [Bra+19, Section 6].

To this end, let (M, g) be an asymptotically flat half-space and horizon boundary Σ with asymptotically flat harmonic coordinates x_1, x_2, x_3 as in Proposition 5.1. Note that from now on we will again consider M to only have a single end M_{end} and that although x_1, x_2, x_3 are defined on all of $M(\Sigma)$ and we call them harmonic coordinates, they are only guaranteed to form a coordinate system in M_{end}, i.e. for $|x| > r_0$ for some $r_0 > 0$.

TODO The above wording is very similar to [HKK21, Section 6]. Todo

By [ABL16, Proposition 3.7], we can compute the mass in these harmonic coordinates. For $L > r_0$ define coordinate half-cylinders $C_L = D_L \cup T_L$ given by

$$\begin{aligned} D_L &= \{x \in M_{\text{end}} \mid (x_1)^2 + (x_2)^2 \leqslant L^2,\ \ x_3 = L\} \\ T_L &= \{x \in M_{\text{end}} \mid (x_1)^2 + (x_2)^2 = L^2,\ \ 0 \leqslant x_3 \leqslant L\}. \end{aligned}$$

Further define

$$\begin{aligned} \mathbb{S}^1_L &= \{x \in M_{\text{end}} \mid (x_1)^2 + (x_2)^2 = L^2,\ \ 0 = x_3\} = \partial C_L = C_L \cap \partial M \\ (\partial M)_L &= \{x \in \partial M \cap M(\Sigma) \mid (x_1)^2 + (x_2)^2 \leqslant L\} \end{aligned}$$

and let Ω_L be the compact component of $M(\Sigma) \setminus C_L$. Since we choe choose $L > r_0$, and we can thus be sure that C_L looks as expected and that $\Sigma \subset \Omega_L$.

By Proposition A.1, which can be proven by just slightly modifying the proof of [ABL16, Proposition 3.7], we can compute the mass as

$$\mathfrak{m}_{(M,g)} = \lim_{L \to \infty} \left(\int_{C_L} C_i \mu^i \, dA + \int_{\mathbb{S}^1_L} g_{\alpha 3} \theta^\alpha \, dl \right)$$

where now μ is the outward unit normal to C_L and θ is as in $Definition$ 3.2. We delegate the details here to the Appendix, see Proposition A.1.

To prove our main result, the inequality Theorem 3.7, we will recover the mass as the boundary term at infinity of Proposition 4.1 applied to $u = x_3$ and $\Omega = \Omega_L$.

Write $\Sigma^L_t := \{u = t\} \cap \Omega_L$. Setting $P_1 = \Sigma$ and $P_2 = C_L \cup (\partial M)_L$ yields (since Σ is a minimal surface, i.e. $H_{P_1} = \Sigma$)

$$\int_{\Omega_L} \frac{1}{2}\Big(\frac{|\nabla^2 u|}{|\nabla u|^2} + R\Big) \, dA \, dt \leqslant \int_{-L}^{L} \Big(2\pi\chi(\Sigma_t) - \int_{\partial\Sigma_t \cap T_L} k_{\kappa_{t,L}} \, dl\Big) \, dt + \int_{C_L} \partial_n |\nabla u| \, dA, \quad (3)$$

where $\kappa_{t,L}$ is the geodesic curvature of $\Sigma^L_t \cap T_L$ viewed as the boundary of Σ^L_t.

We claim that if $t \in (0, L)$ is a regular value of u (i.e. $|\nabla u| \neq 0$ on Σ_t), then Σ^L_t consists of a single component, which intersects T_L along a circle. Assume otherwise, i.e. that there is a regular value $t \in (0, L)$ such that $\Sigma' \subset \Sigma^L_t$ is a connected component disjoint from T_L. Then, since $M(\Sigma)$ is diffeomorphic to the compliment of a finite number of balls in $\mathbb{R}^3_+$, there exists a compact domain $E \subset \Omega_L$ with $\partial E \setminus \Sigma = \Sigma'$.

Note now $u - t$ is still harmonic, has Dirichlet boundary condition $u - t = 0$ on Σ', and Neumann boundary condition $\partial_n(u - t) = 0$ on Σ. Hence we can apply **??** and get that u must be constant on E, which contradicts the assumption that t is a regular value.

Thus Σ^L_t consists of a single connected component and meets T_L along a circle. In particular, we can apply (1) with $b \geqslant 1$ and get $\chi(\Sigma^L_t) \leqslant 1$. Then (3) becomes

$$\int_{\Omega_L} \frac{1}{2}\Big(\frac{|\nabla^2 u|}{|\nabla u|^2} + R\Big) \, dA \, dt \leqslant 2\pi L - \int_0^L \int_{\partial\Sigma_t \cap T_L} k_{\kappa_{t,L}} \, dl \, dt + \int_{C_L \cup (\partial M)_L} \partial_n |\nabla u| \, dA. \quad (4)$$

It now only remains to compute the boundary terms in (**??**). We start with the following, which insert just the equivalent of [Bra+19, Lemma 6.1 and Lemma 6.2] for our half cylinder (note also that we choose the cylinder with symmetry axis in direction x_3 and $u = x_3$, while Bray et al. choose the symmetry axis in direction x_1 and $u = x_1$). The proof proceed just as in [Bra+19]

Lemma 6.1. *In the notation fixed above, we have*

$$\int_{C_L} \partial_\nu |\nabla u|\, dA = \frac{1}{2}\int_{D_L} C_3\, dA + \frac{1}{L}\int_{T_L} [x_2(g_{23,3}-g_{11,2})+x_1(g_{13,3}-g_{33,1})]\, dA + O(L^{1-2\tau})$$

and

$$\int_0^L \int_{\partial\Sigma_t\cap T_L} k_{\kappa_{t,L}}\, dl\, dt = 2\pi L + \frac{1}{L}\int_{T_L} [x_1(g_{11,2}-g_{21,1})+x_2(g_{22,1}-g_{12,2})]\, dA + O(L^{1-2\tau}+L^{-\tau}).$$

It remains to consider the boundary term on $(\partial M)_L$:

Lemma 6.2. *In the notation established above, we have*

$$\int_{(\partial M)_L} \partial_\nu |\nabla u| = \int_{\mathbb{S}^1_L} g_{\alpha 3}\theta^\alpha\, dl - \int_{(\partial M)_L} H|\nabla u|\, dA.$$

Proof. Note that, since ∂M is a level set of u, we know that ∇u is orthogonal to ∂M and hence $\nu = -\nabla u \big/ |\nabla u|$ (recall that ν points outside of M, i.e. towards $x_3 < 0$). Thus we can compute

$$\begin{aligned}
\partial_\nu |\nabla_u| &= \partial_\nu \sqrt{|\nabla u|^2} \\
&= \frac{\partial_\nu |\nabla u|^2}{2|\nabla u|} \\
&= -\frac{\nabla^i u \nabla_i (\nabla_j u \nabla^j u)}{2|\nabla u|^2} \\
&= -\frac{\nabla^i u \nabla^j u \nabla_i \nabla_j u}{|\nabla u|^2} \\
&= -n^i n^j \nabla_i \nabla_j u \\
&= -\nabla_\nu \nabla_\nu u \\
&\underset{\overset{\uparrow}{??\ 2.5}}{=} -\Delta_M u + \Delta_\Sigma u + H \cdot \nabla_\nu u.
\end{aligned}$$

But u is harmonic, so $\Delta_M u = 0$. Furthermore, writing $\Delta_\Sigma u$ as $\div_\Sigma(\operatorname{grad}_\Sigma u)$ enables us to use the divergence theorem to get

$$\int_{(\partial M_L)} \partial_\nu |\nabla_u|\, dA = \int_{(\partial M_L)} H \cdot \nabla_\nu u\, dA + \int_{\mathbb{S}^1_L} (\operatorname{grad}_\Sigma u)_\alpha \theta^\alpha. \tag{5}$$

Note that since $\mathbb{S}^1_L$ is entirely contained in M_{end}, we can be sure that the x_i form a coordinate system. Denote the vectors ∂_{x_i} as e_i, and again use Greek indices α, β when only summing over $1, 2$. Recall that the x_α fulfill a Neumann boundary condition on $\partial M \cap M(\Sigma)$, which gives

$$\nabla_{e_3}(x_\alpha) = 0 \implies g(e_3, e_\alpha) = 0.$$

In particular we can apply this to (5), where we see that

$$(\operatorname{grad}_\Sigma u)_\alpha \theta^\alpha = g_{\alpha\beta}\theta^\alpha \nabla^\beta(u) = 0$$

■

A Different Exhausting Sequences for Computation of the Mass

Proposition A.1. *Suppose that (M, g) is an asymptotically flat half space with asymptotically flat coordinates x_1, x_2, x_3 on M_{end} (fulfilling the conditions of Definition 3.2). Let $\{D_k^3\}_{k=1}^{\infty}$ be an exhaustion of M by closed sets with $\partial D_k = S_k \cup (D_k \cap \partial M)$, where S_k is a connected 2-dimensional submanifold (smooth almost everywhere) of the end M_{end} with $\partial S_k = \partial M \cap S_k$ such that*

$$R_k \coloneqq \inf_{x \in S_k} |x| \to \infty \quad \text{as } k \to \infty,$$
$$R_k^2 \cdot |S_k| \text{ is bounded as } k \to \infty,$$

and $R_1 \geqslant R_0$, where $|S_k|$, the area of S_k, and $|x|$ are as usual calculated with respect to the euclidean background metric (possible since we are in M_{end}). Then

$$\mathfrak{m}_{(M,g)} = \lim_{k\to\infty} \int_{S_k} \int_{\mathbb{S}^2_{r,+}} C_i \tilde{\mu}^i \, dA + \int_{\partial S_k} g_{\alpha 3} \tilde{\theta}^{\alpha} \, dl$$

is independent of the sequence S_k, where as in Definition 3.2 $\tilde{\mu}^i$ is the outward normal to S_k and $\tilde{\theta}^{\alpha}$ the co-normal to ∂S_k oriented as the boundary of the compact component of $\partial M \setminus \partial S_k$.

Proof. Let $\tilde{D}_k \coloneqq \{x \in D_k \mid |x| \geqslant R_k\}$ (this is the part of D_k extending beyond the biggest coordinate hemisphere that is possible to inscribe in D_k). Then $\partial \tilde{D}_k = S_k \cup \mathbb{S}^2_{R_k,+} \cup (\tilde{D}_k \cap \partial M)$ and $\partial(\tilde{D}_k \cap \partial M) = (S_k \cap \partial M) \cup (\mathbb{S}^1_{R_k})$.

As in [ABL16, Proposition 3.7], we get (using [ABL16, Equations 3.16 and 3.17])

$$\begin{aligned}
\int_{\tilde{D}_k} R \, dV = &\int_{S_k} C_i \tilde{\mu}^i \, dA - \int_{\mathbb{S}^2_{R_k,+}} C_i \mu^i \, dA \\
&+ \int_{\tilde{D}_k \cap \partial M} C_i \nu^i \, dA + \int_{\tilde{D}_k} O(r^{-2\tau-2}), \\
\int_{\tilde{D}_k \cap \partial M} C_i \nu^i \, dA = &\int_{S_k \cap \partial M} g_{\alpha 3} \tilde{\theta}^{\alpha} \, dl - \int_{\mathbb{S}{R_k}^1} g_{\alpha,3} \theta^{\alpha} \, dl \\
&-2 \int_{\tilde{D}_k \cap \partial M} H + \int_{\tilde{D} \cap \partial M} O(r^{-2\tau-1}),
\end{aligned}$$

and thus

$$\begin{aligned}
&\Big|\int_{S_k} C_i\tilde{\mu}^i\,dA + \int_{S_k\cap\partial M} g_{\alpha 3}\tilde{\theta}^\alpha\,dl - \Big(\int_{\mathbb{S}^2_{R_k,+}} C_i\mu^i\,dA + \int_{\mathbb{S}^1_{R_k}} g_{\alpha 3}\theta^\alpha\,dl\Big)\Big| \\
&\leqslant \int_{\tilde{D}_k} O(r^{-2\tau-2}) + |R|\,dV + \int_{\tilde{D}_k\cap\partial M} O(r^{-2\tau-1}) + |H|\,dA \\
&\leqslant \int_{M\setminus D_k} O(r^{-2\tau-2}) + |R|\,dV + \int_{(\partial M)\setminus D_k} O(r^{-2\tau-1}) + |H|\,dA
\end{aligned}$$

Since $R \in L^1(M)$ and $H \in L^1(\partial M)$, the fact that the D_k exhaust M (together with $r > R_k$ in $M \setminus D_k$) implies that the integrals over R and H on the right hand side vanish in the limit $k \to \infty$. Similarly, since $\tau > 1/2$, the integrals over $O(r^{-2\tau-2})$ and $O(r^{-2\tau-1})$ also vanish in this limit.

We learn that using the S_k to compute the mass yields the same result as using coordinate spheres (as we used in our original Definition 3.2). ∎

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