Hyperbolic functions

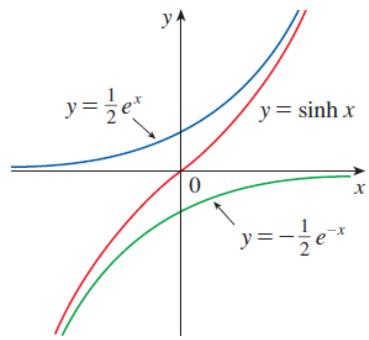
Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

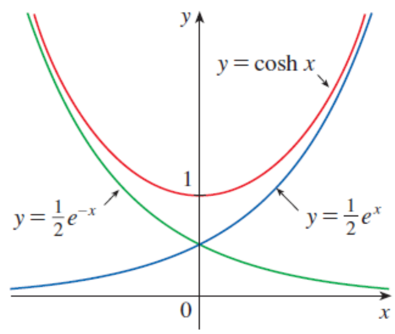
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

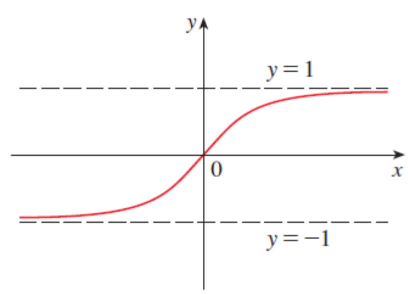
1- If $y = \sinh x$ this mean the $D_f = (-\infty, \infty)$ and $R_f = (-\infty, \infty)$. The graph below explains that:



2- If $y = \sinh x$ this mean the $D_f = (-\infty, \infty)$ and $R_f = (-\infty, \infty)$. The graph below explains that:



3- If $y = \tanh x$ this mean the $D_f = (-\infty, \infty)$ and $R_f = (-1, 1)$. The graph below explains that:



Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

Ex: if $\sinh x = \frac{3}{4}$, find $\tanh x$.

Sol: since $\tanh x = \frac{\sinh x}{\cosh x}$ and $\sinh x = \frac{3}{4}$ then we need to know value of $\cosh x$

Since $\cosh^2 x - \sinh^2 x = 1 \Longrightarrow \cosh^2 x - \frac{9}{16} = 1$

$\cosh^2 x = \frac{25}{16} \Rightarrow \cosh x = \frac{5}{4}$ since $(\cosh x \geq 1)$

$\therefore \tanh x = \frac{3}{5}$

Ex: if $\tanh x = \frac{-4}{5}$, show that $\sinh x + \cosh x = \frac{1}{3}$ H.W

1 Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

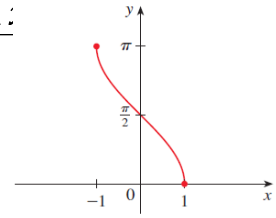
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Ex: if $y = \cosh\sqrt{x}$, find $\frac{dy}{dx}$.

Sol: $\frac{dy}{dx} = \sinh\sqrt{x} \cdot \frac{1}{2\sqrt{x}} \Longrightarrow \frac{dy}{dx} = \frac{\sinh\sqrt{x}}{2\sqrt{x}}$

Integral of hyperbolic function

1- $\int \sinh x \ dx = \cosh x + c$	2- $\int \cosh x \ dx = \sinh x + c$
3- $\int \operatorname{sech}^2 x \ dx = \tanh x + c$	4- $\int \operatorname{csch}^2 x \ dx = -\coth x + c$
5- $\int \operatorname{sech} x \tanh x \ dx = -\operatorname{sech} x + c$	6- $\int \operatorname{csch}$ [illegible] $\operatorname{sch} x + c$



Ex: $\int \coth 5x \ dx = \int \frac{\cosh 5x}{\sinh 5x} \ dx = \frac{1}{5} \ln|5x| + c$

Ex: $\int \sinh^2 x \ dx = \int \frac{1}{2}(\cosh 2x - 1) \ dx$

$\frac{1}{4} \int 2 \cosh 2x \ dx - \frac{1}{2} \int \ dx = \frac{1}{4} \sinh 2x - \frac{1}{2} x + c$

Ex: $\int \cosh^2 x \ dx$ H.W

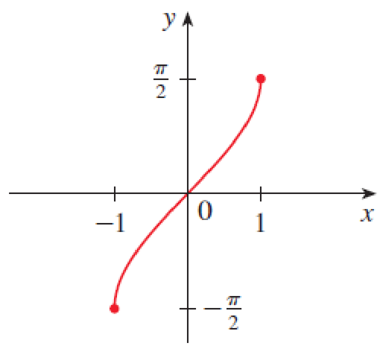
Ex: $\int \sinh^3 x \ dx$ H.W

Ex: find $\cosh 0$, $\cosh 1$, $\sinh 0$, and $\sinh 1$ H.W

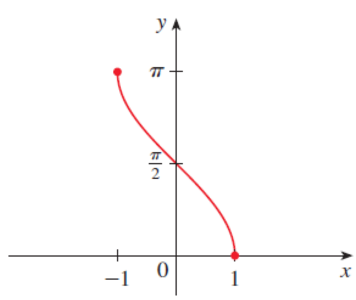
Ex: $\int \tanh x \operatorname{sech}^2 x \ dx$ H.W

Inverse of trigonometric functions

1- $\sin^{-1} x = y \Leftrightarrow y = \sin x$, $D_f = [-1, 1], R_f = \left[\frac{\pi}{2}, -\frac{\pi}{2}\right]$.
   $\therefore \sin^{-1}(\sin x) = x$

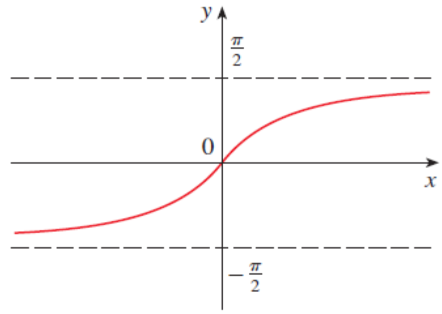


2- $\cos^{-1} x = y \Leftrightarrow y = \cos x$, $D_f = [-1, 1], R_f = [0, \pi]$.
   $\therefore \cos^{-1}(\cos x) = x$



3- $\tan^{-1} x = y \Leftrightarrow y = \tan x$, $D_f = [-\infty, \infty], R_f = \left[\frac{\pi}{2}, -\frac{\pi}{2}\right]$.

$\therefore \cos^{-1}(\cos x) = x$



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$$y = \csc^{-1}x \; (|x| \geqslant 1) \iff \csc y = x \text{ and } y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$

$$y = \sec^{-1}x \; (|x| \geqslant 1) \iff \sec y = x \text{ and } y \in [0, \pi/2) \cup [\pi, 3\pi/2)$$

$$y = \cot^{-1}x \; (x \in \mathbb{R}) \iff \cot y = x \text{ and } y \in (0, \pi)$$

Ex: solve for x in the equation $\tan^{-1}(2x - 3) = \frac{\pi}{4}$

Sol: taking tan for both side

$\tan(\tan^{-1}(2x - 3)) = \tan\frac{\pi}{4} \rightarrow 2x - 3 = \tan\frac{\pi}{4}$

$\rightarrow 2x - 3 = 1 \rightarrow 2x = 4 \rightarrow x = 2$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

Ex: if $y = \tan^{-1} 3x$, find $\frac{dy}{dx}$.

Sol: $\frac{dy}{dx} = \frac{1}{1+9x^2}.(3) = \frac{3}{1+9x^2}$

Ex: if $y = \sec^{-1}(2x+1)$, find $\frac{dy}{dx}$.

Sol: $\frac{dy}{dx} = \frac{1}{(2x+1)\sqrt{(2x+1)^2-1}}.(2) = \frac{2}{(2x+1)\sqrt{(2x+1)^2-1}}$

The integral of inverse trigonometric functions

1- $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$
2- $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$
3- $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$

Ex: $\int \frac{dx}{1+3x^2} = \int \frac{dx}{1+(\sqrt{3}x)^2} = \frac{1}{\sqrt{3}} \int \frac{\sqrt{3}}{1+(\sqrt{3}x)^2} dx = \frac{1}{\sqrt{3}} \tan^{-1} \sqrt{3}x + c$

Remark:

1- $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a} + c$
2- $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$
3- $\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a} \sec^{-1} \frac{x}{a} + c$

Ex: $\int \frac{dx}{\sqrt{2-x^2}} = \sin^{-1} \frac{x}{\sqrt{2}} + c$

Technique of integration

Integration by parts:

The formula for integration by parts comes from the product rule.

$\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$

In its differential form, the rule becomes

$d(u.v) = udv + vdu$

Which is then written as

$udv = \frac{d}{dx}(u.v) - vdu$

And the integrated to give the following formula

The integration by parts formula

$$\int u\, dv = uv - \int v\, du$$

Ex: $\int x \cos x \; dx$

Sol: let $u = x$, and $dv = \cos x \; dx$

$du = dx$, and $v = \sin x$

$\int x \cos x \; dx = x \sin x - \int \sin x \; dx$

$= x \sin x + \cos x + c$

Ex: $\int \ln x \; dx$

Sol: $u = \ln x$, $dv = dx$

$du = \frac{1}{x} dx$, $v = x$

$\therefore \int \ln x \; dx = \ln x \,.\, x - \int x \frac{1}{x} dx$

$= \ln x \,.\, x - x + c$

Ex: $\int x e^x \, dx$ H.W

Ex: $\int x^2 e^x \, dx$

Sol: $u = x^2$, $dv = e^x dx$

$du = 2x \, dx$, $v = e^x$

$$\therefore \int x^2 e^x \, dx = x^2 e^x - \int 2x \, e^x \, dx$$

$$\therefore \int x^2 e^x \, dx = x^2 e^x - 2 \int x \, e^x \, dx \ldots (1)$$

We take $\int x \, e^x \, dx$

$u = x$, $dv = e^x dx$

$du = 1 \, dx$, $v = e^x$

$\int x e^x \, dx = x e^x - \int \; e^x \, dx$

$= x e^x - e^x \ldots (2)$

Substitute (2) in (1)

$$\int x^2 e^x \, dx = x^2 e^x - 2(x e^x - e^x) = x^2 e^x - 2x e^x - 2 e^x + c$$

Ex: $\int e^x \cos x \; dx$ H.W

Ex: $\int x^3 \sin x \; dx$ H.W