Limits and continuity

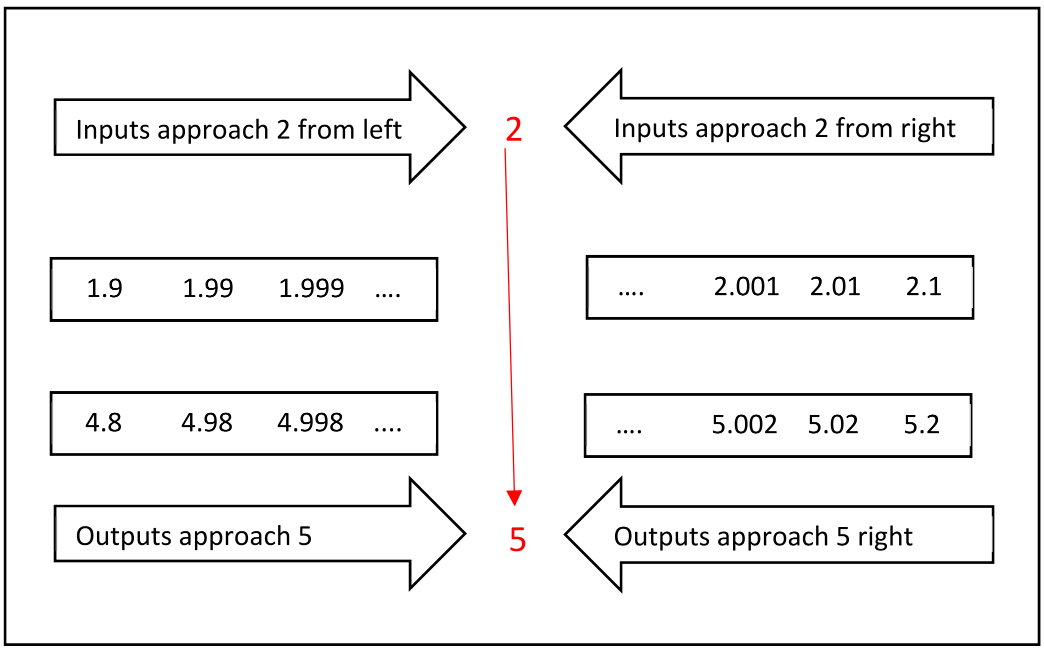
1- Limits

Def: If the values of a function f of x approach the value L as x approaches c, we say f has limit L as x approaches c and we write

$$\lim_{x \to c} f(x) = L$$

Ex: As fig. below suggest

$$\lim_{x \to 2} (2x + 1) = 5$$



Properties of limits

If $\lim_{x \to c} f(x) = L1$ and $\lim_{x \to c} g(x) = L2$, then

1- Sum	$\lim_{x \to c}[f(x) + g(x)] = L1 + L2$
2- Difference	$\lim_{x \to c}[f(x) - g(x)] = L1 - L2$
3- Product	$\lim_{x \to c}[f(x).g(x)] = L1.L2$
4- Constant multiply	$\lim_{x \to c}[K.f(x)] = K.L1 , \forall K \in R$

5- Quotient	$\lim_{x \to c} \left[\frac{f(x)}{g(x)}\right] = \frac{L1}{L2}, L2 \neq 0$

Ex1: if $f(x) = k$ then $\lim_{x \to c} f(x) = \lim_{x \to c} k = k$

Ex2: if $f(x) = x$ then $\lim_{x \to c} f(x) = \lim_{x \to c} x = c$

Ex3: if $f(x) = ax^2 + bx + K$ then

$\lim_{x \to c} f(x) = \lim_{x \to c} ax^2 + bx + K = ac^2 + bc + K$

Ex4: find $\lim_{x \to 2} \frac{x^2+2x+4}{x+2}$

Sol: $\lim_{x \to 2} \frac{x^2+2x+4}{x+2} = \frac{2^2+2(2)+4}{2+2} = \frac{4+4+4}{4} = 3$

Def:

- Right hand limit: $\lim_{x \to c^+} f(x) = L$, mean that $f(x) \to L$ as $x \to c$ from right
- Left hand limit: $\lim_{x \to c^-} f(x) = L$, mean that $f(x) \to L$ as $x \to c$ from left
- Limit exist: $\lim_{x \to c} f(x)$,exist and equal to L, iff $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$ from right

Ex: If $f(x) = \begin{cases} x^2 - 2 & , x \geq 2, \\ x & x < 2. \end{cases}$

Then

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 - 2) = 2 \ldots\ldots\ldots L1$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x = 2 \ldots\ldots\ldots L2$

$\because L1 = L2$

$\therefore \lim_{x\to 2} f(x)$ is exist and $\lim_{x\to 2} f(x) = 2$

Ex: find $\lim_{x\to 0} \frac{|x|}{x}$

Note: To find limits, use substitution

- If you get real number, then the limit exist. Or $-\infty, \infty$, then the limit does not exist.
- If you get $\frac{0}{0}, \frac{\infty}{\infty}, 1^{\infty}, \infty^{0}, \infty, -\infty$ we have do some operations before decided the limit exist or not.

Examples: find

1- $\lim_{x\to 2} \frac{x^2-4}{x-2} = \frac{0}{0}$

Sol: $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4$

2- $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = \frac{0}{0}$

Sol: $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = \lim_{x\to 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1} = \lim_{x\to 1}\left(\sqrt{x}+1\right) = 2$

3- $\lim_{x\to 1} \frac{\sqrt{x+9}-3}{x} = \frac{0}{0}$

Sol: $\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x} = \lim_{x\to 0} \frac{\sqrt{x+9}-3}{x} \times \frac{\sqrt{x+9}+3}{\sqrt{x+9}+3} = \lim_{x\to 0} \frac{\left(\sqrt{x+9}-3\right)\left(\sqrt{x+9}+3\right)}{x\left(\sqrt{x+9}+3\right)} =$
$\lim_{x\to 0} \frac{x+9-9}{x\left(\sqrt{x+9}+3\right)} = \lim_{x\to 0} \frac{x+9-9}{x\left(\sqrt{x+9}+3\right)} = \lim_{x\to 0} \frac{1}{\sqrt{x+9}+3} = \frac{1}{6}$

4- - $\lim_{x\to 0} \frac{1-\cos x}{sinx}$

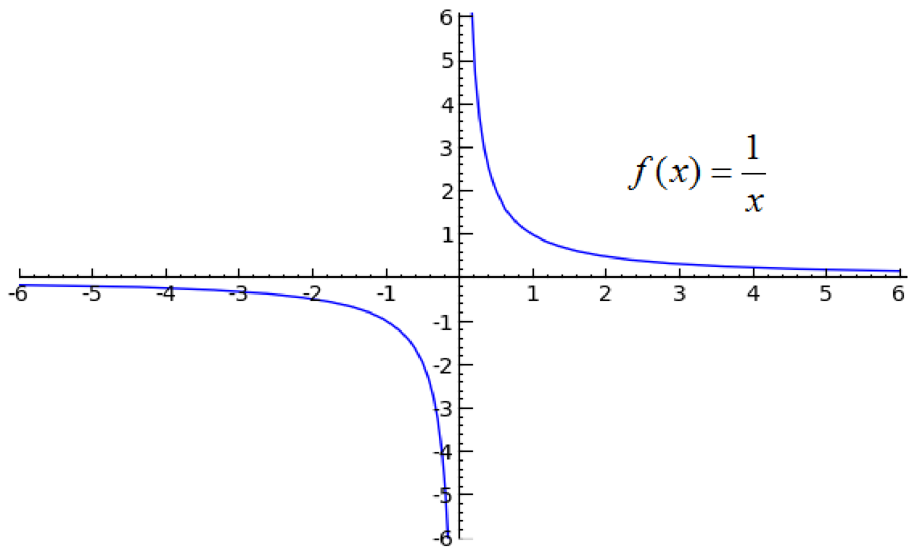
5- - $\lim_{x\to 0} \frac{\tan x - \sin x}{\sin^3 x}$

6- - $\lim_{x\to 2} \frac{1}{x^2-4}$

Limit involving infinity

Limits as $x \to \infty$, or $x \to -\infty$ for example

$$\lim_{x\to\infty} \frac{1}{x} = 0 \text{ or } \lim_{x\to-\infty} \frac{1}{x} = 0$$



Examples

1- $\lim_{x\to\infty} \left(5 + \frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{1}{x} = 5 + 0 = 0$

2- $\lim_{x\to-\infty} \frac{4}{x^2} = \lim_{x\to-\infty} 4. \lim_{x\to-\infty} \frac{1}{x^2} = 0$

3- $\lim_{x\to-\infty} \frac{\sin x}{x} = 0$

4- $\lim_{x\to-0} \frac{\sin x}{x} = 1$