Continuity

We noticed in previous lecture that the limit of a function as x approaches a can often be found simply by computing the value of the function at a. Functions having this property are called continuous at a.

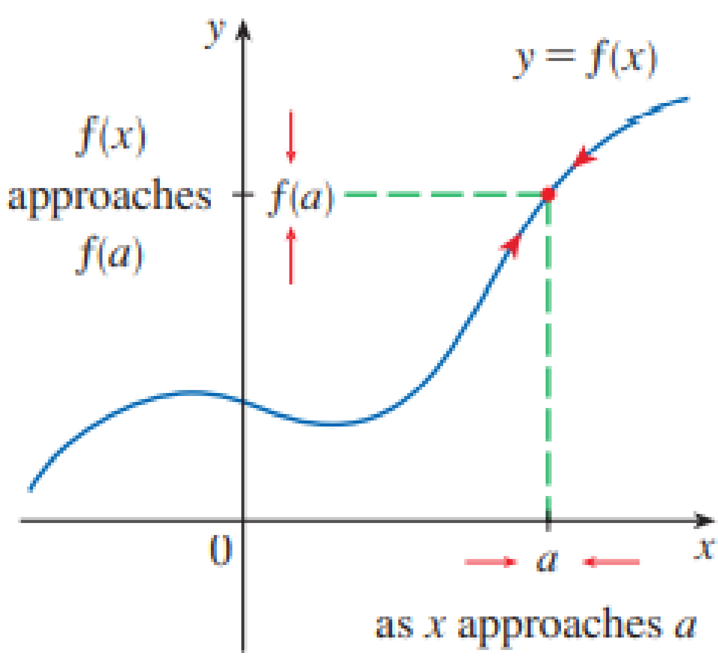
Def: A function f is continuous at a number (a) if

$$\lim_{x \to a} f(x) = f(a)$$

Notice that Definition above implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

The definition says that f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a. see fig. below



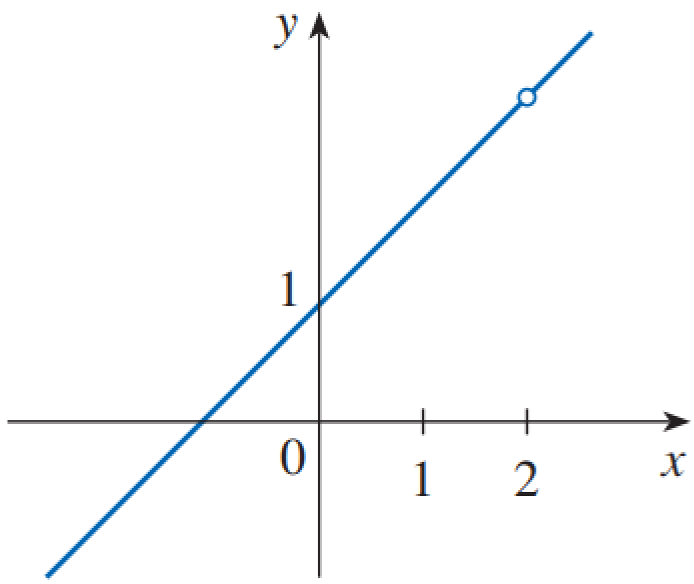
If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that f is discontinuous at a (or f has a discontinuity at a) if f is not continuous at a.

EX: Where are each of the following functions discontinuous?

a) $f(x) = \frac{x^2-x-2}{x-2}$

sol: take $x - 2 = 0 \rightarrow x = 2, D_f = \frac{R}{\{2\}}$

Notice that fs2d is not defined, so f is discontinuous at 2. See in the fig below:



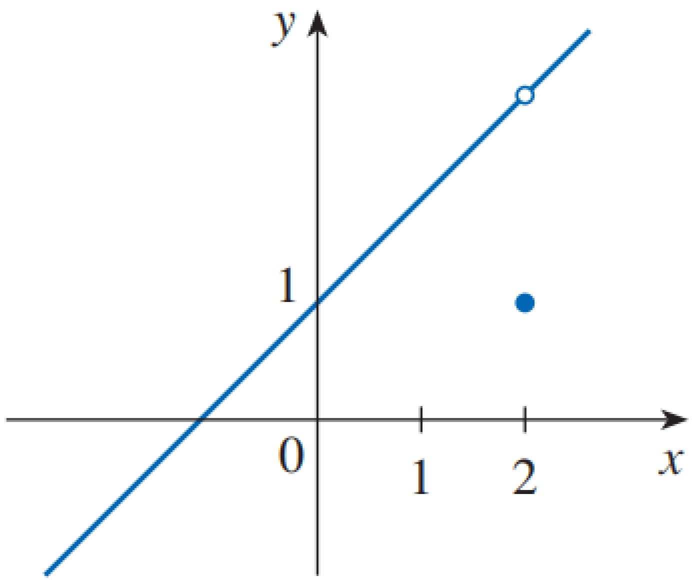
b) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & if\ x \neq 2 \\ 1 \quad if\ x = 2 \end{cases}$

sol: when $f(2) = 1$ is defined

$\lim_{x\to2} f(x) = \lim_{x\to2} \frac{x^2-x-2}{x-2} = \lim_{x\to2} \frac{(x+1)(x-2)}{x-2} = \lim_{x\to2}(x+1) = 3$

Is exists, but

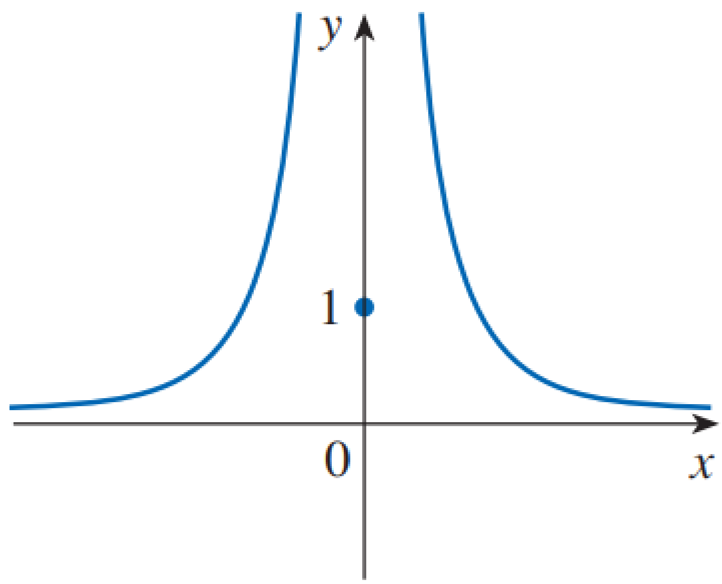
$\lim_{x\to2} f(x) \neq f(2)$



c) $f(x) = \begin{cases} \frac{1}{x^2} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}$

sol: when we take $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \frac{1}{x^2}$

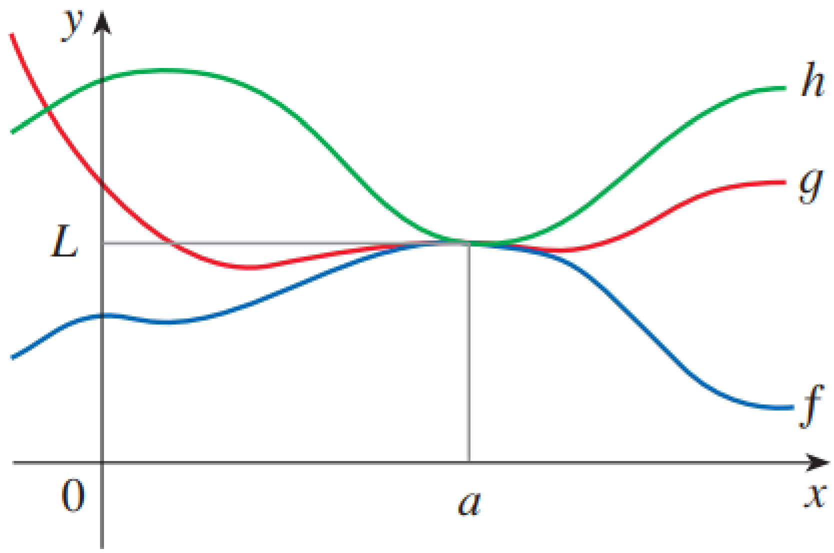
then the limit is not exist, so f discontinuous at 0.



Theorem: $If\ f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then $\lim\limits_{x\to a} f(x) \leq \lim\limits_{x\to a} g(x)$

The Squeeze Theorem: If $(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} h(x) = L \text{ then } \lim_{x\to a} g(x) \leq L$$



Ex) let $f(x) = \begin{cases} \frac{x^2-9}{x-3} & if\ x < 3 \\ kx - 3 & if\ x \geq 3 \end{cases}$ is continuous at x=3, find the value of K

Sol: since $f(x)$ is continuous at x=3 then

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} kx - 3$$

$$6 = 3x - 3$$

$$3x = 9$$

$$\therefore x = 3$$

Ex) Show that f is continuous at x = 1, $f(x) = \begin{cases} 1 - x^2 & if\ x \le 1 \\ \ln 1 & if\ x > 1 \end{cases}$

Sol:

1) $f(1) = 0$ is defined

2) $\lim_{x \to 1^+} 1 - x^2 = 0 \ldots \ldots . L1$

$\lim_{x \to 1^-} \ln x = 0 \ldots \ldots . L2$

3) $since\ L1 = L2$

Then f is continuous

The derivative of a function

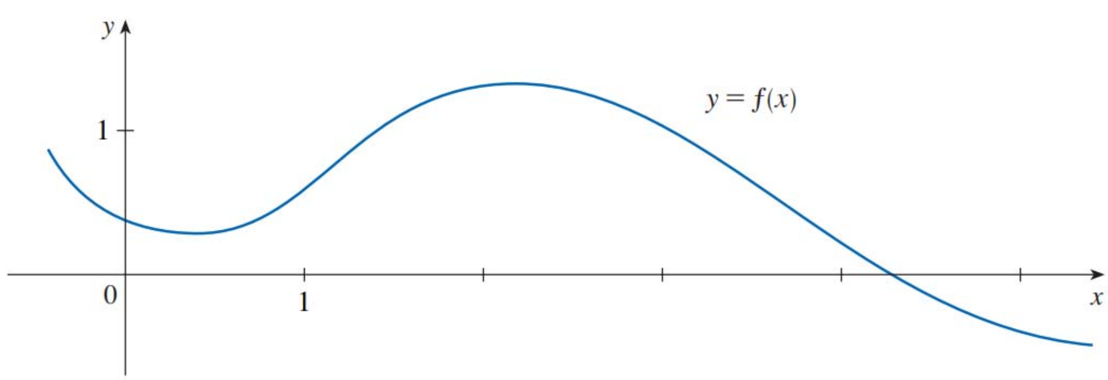
Def: the derivative of a function f at x, it is defined by the equation:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

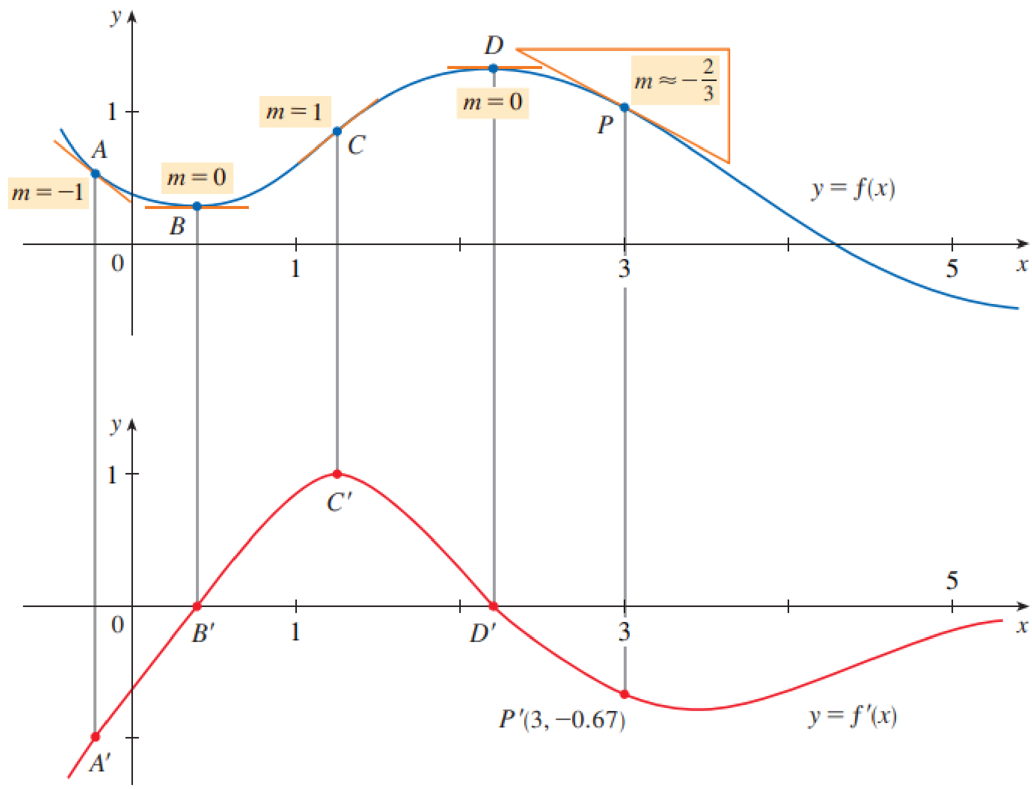
Given any number x for which this limit exists, we assign to x the number$f(x)$. So we can regard f' as a new function, called the derivative of f and defined by Equation above.

Note: $f'(x)$, can be explained geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

Ex1: The graph of a function f is given in fig below. Use it to sketch the graph of the derivative f'



Sol: We can estimate the value of the derivative at any value of x by drawing the tangent at the point $(x, f(x))$ and estimating its slope. For instance, for x = 3 we draw a tangent at P in Fig. below and estimate its slope to be about $\frac{-2}{3}$



Ex: if $f(x) = x^2$ find $f'(x)$

Sol: we use the equation

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2h(x+h)}{h}$$

$$f'(x) = 2\lim_{h\to 0}(x+h) = 2x$$