1st Class

2016-2017

Discrete Mathematics

الهياكل المتقطعة

أستاذ المادة : م.د.إقباس عز الدين

References

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SETS AND ELEMENTS

A set is a collection of objects called the elements or members of the set. The ordering of the elements is not important and repetition of elements is ignored,
for example

$$\{1, 3, 1, 2, 2, 1\} = \{1, 2, 3\}.$$

One usually uses capital letters, A,B,X, Y, . . . , to denote sets, and lowercase letters, a, b, x, y, . . ., to denote elements of sets. Below you'll see a sampling of items that could be considered as sets:

- The items in a store
- The English alphabet
- Even numbers

A set could have as many entries as you would like. It could have one entry, 10 entries, 15 entries, infinite number of entries, or even have no entries at all! For example, in the above list the English alphabet would have 26 entries, while the set of even numbers would have an infinite number of entries.

Each entry in a set is known as an **element or member**

Sets are written using curly brackets "{" and "}", with their elements listed in between.
For example:

1- the English alphabet could be written as
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z}

2- even numbers could be {0,2,4,6,8,10,...}

Principles:

$\in$ belong to

$\notin$ not belong to

$\subseteq$ subset

$\subset$ proper subset (is a non-equal subset)
For example, {a, b} is a proper subset of {a, b, c}, but {a, b, c} is not a proper subset of {a, b, c}.

$\not\subset$ not subset

So we could replace the statement: "a is belong to the alphabet" with:

$a \in \{\text{alphabet}\}$

and replace the statement "3 is not belong to the set of even numbers" with:

$3 \notin \{\text{Even numbers}\}$

Now if we named our sets we could go even further.
Give the set consisting of the **alphabet** the name A,
and give the set consisting of **even numbers** the name E.

We could now write
$a \in A$
and
$3 \notin E$.

Problem

Let A = {2, 3, 4, 5} and C = {1, 2, 3, . . ., 8, 9}, Show that A is a proper subset of C.

Answer

Each element of A belongs to C, so $A \subseteq C$. On the other hand,

$1 \in C$ but $1 \notin A$. Hence $A \neq C$. Therefore A is a proper subset of C.

There are three ways to specify a particular set:

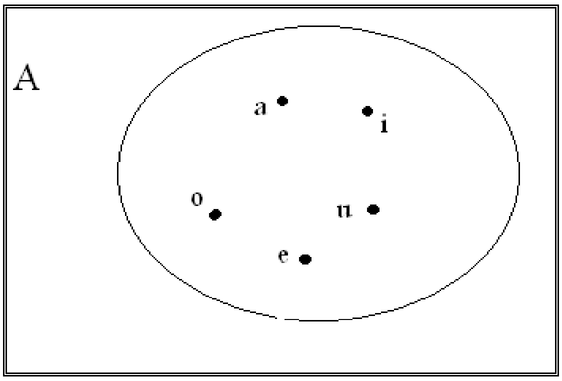
1) By list its members separated by commas and contained in braces{ }, (if it is possible), for example,

   A= {a,e,i,o,u}

2) By state those properties which characterize the elements in the set, for example,

   A={x:x is a letter in the English alphabet, x is a vowel}

3) Venn diagram: (A graphical representation of sets).



Example (1)

A={x:x is a letter in the English alphabet, x is a vowel}
$e \in A$ (e is belong to A)
$f \notin A$ (f is not belong to A)

Example (2)
X is the set {1,3,5,7,9}
$3 \in X$ and
$4 \notin X$

Example (3)
Let $E = \{x \mid x^2 - 3x + 2 = 0\} \rightarrow (x-2)(x-1)=0 \rightarrow x=2$ & $x=1$
$E = \{2, 1\}$, and
$2 \in E$

Empty Set
A set with no elements is called an *empty set*.
An empty is denoted by { } or ∅.
For example,
- $\emptyset = \{x$: x is an integer and $x^2 + 5 = 0\}$
- ∅ = {*x*: *x* are living beings who never die}
- ∅ = {*x*: *x* is the UOT student of age below 15}
- ∅ = { *x*: *x* is the set of persons of age over 200}

Universal set:
In any application of the theory of sets, the members of all sets under investigation usually belong to some fixed large set called the universal set.
For example:
In human population studies the universal set consists of all the people in the world.
We will let the symbol U denotes the universal set.

Subsets:
Every element in a set A is also an element of a set B, then A is called a subset of B.
We also say that B contains A.
This relationship is written:

$$A \subset B \qquad \text{or} \qquad B \supset A$$

If A is not a subset of B, i.e. if at least one element of A dose not belong to B, we write $A \not\subset B$.

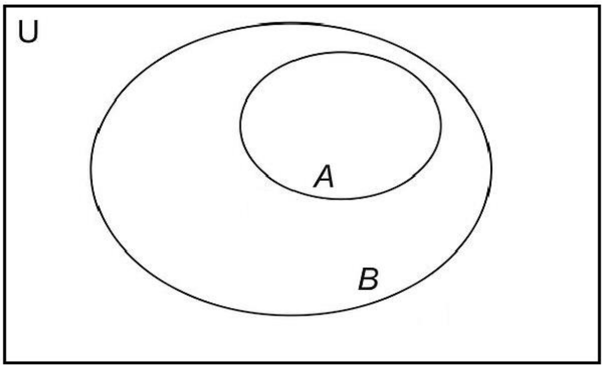
Example 4:
Consider the sets:
$A = \{1,3,4,5,8,9\}$, $B = \{1,2,3,5,7\}$ and $C =\{1,5\}$
Then $C \subset A$ and $C \subset B$
since 1 and 5, the element of C, are also members of A and B.

But $B \not\subset A$ since some of its elements, e.g. 2 and 7, do not belong to A.
Furthermore, since the elements of A, B and C must also belong to the universal set U,
we have that U must at least the set $\{1,2,3,4,5,7,8,9\}$.

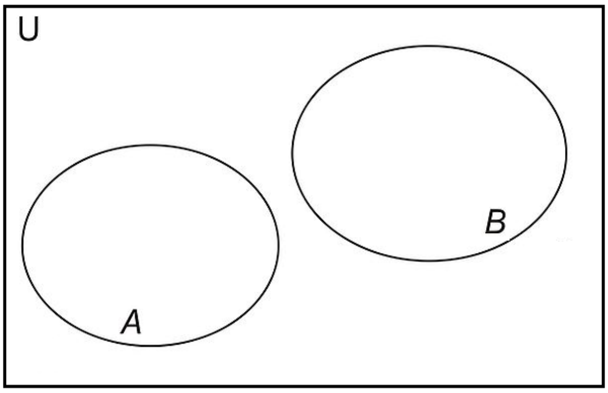
$A \subset B$: $\{\forall\ x \in A \quad \Rightarrow \quad x \in B$
$A \not\subset B$: $\{\exists\ x \in A$ but $x \notin B$

$\forall$: For all لكل
$\exists$: There exists يوجد على الاقل

The notion of subsets is graphically illustrated below:



A is entirely within B so $A \subset B$.



A and *B* are disjoint or ($A \cap B = \varnothing$) so we could write $A \not\subset B$ and $B \not\subset A$.

Set of numbers:

Several sets are used so often, they are given special symbols.

N = the set of natural numbers or positive integers

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

Z = the set of all integers: . . . ,−2,−1, 0, 1, 2, . . .

$$\mathbb{Z} = \mathbb{N} \cup \{\ldots, -2, -1\}$$

Q = the set of rational numbers

$$\mathbb{Q} = \mathbb{Z} \cup \{\ldots, -1/3, -1/2, 1/2, 1/3, \ldots, 2/3, 2/5, \ldots\}$$

Where $Q = \{ a/b : a , b \in Z, b \neq 0\}$

R = the set of real numbers

$$\mathbb{R} = \mathbb{Q} \cup \{\ldots, -\pi, -\sqrt{2}, \sqrt{2}, \pi, \ldots\}$$

C = the set of complex numbers

$$\mathbb{C} = \mathbb{R} \cup \{i, 1+i, 1-i, \sqrt{2}+\pi i, \ldots\}$$

Where $C = \{ x + iy \,;\, x , y \in R;\, i = \sqrt{-1}\}$

Observe that $N \subset Z \subset Q \subset R \subset C$.

Theorem 1:

For any set A, B, C:

1. $\varnothing \subset A \subset U$.
2. $A \subset A$.
3. If $A \subset B$ and $B \subset C$, then $A \subset C$.
4. $A = B$ if and only if $A \subset B$ and $B \subset A$.

Set operations:

1) UNION:

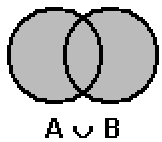
The *union* of two sets *A* and *B*, denoted by $A \cup B$, is the set of all elements which belong to *A* or to *B*;

$$A \cup B = \{ x : x \in A \text{ or } x \in B\}$$

Example

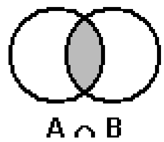
A={1,2,3,4,5}   B={5,7,9,11,13}

$A \cup B = \{1,2,3,4,5,7,9,11,13\}$



2) **INTERSECTION**

The intersection of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B;



$$A \cap B = \{ x : x \in A \text{ and } x \in B\}.$$

Example 1

A={1,3,5,7,9}   B={2,3,4,5,6}

The elements they have in common are 3 and 5

$A \cap B = \{3,5\}$

Example 2

A={The English alphabet}   B={vowels}

So $A \cap B$ = {vowels}

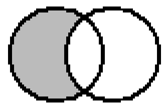
Example 3

A={1,2,3,4,5}   B={6,7,8,9,10}

In this case A and B have nothing in common. $A \cap B = \varnothing$

3) THE DIFFERENCE:

The difference of two sets A\B or A-B is those elements which belong to A but which do not belong to B.

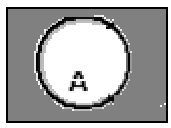


A – B

$A \backslash B = \{x : x \in A,\ x \notin B\}$

4) COMPLEMENT OF SET:

Complement of set A^c or A' , is the set of elements which belong to U but which do not belong to A .



U

$A^c = \{x : x \in U,\ x \notin A\}$

Example 1:

let $A=\{1,2,3\}$
$B = \{3,4\}$
$U=\{1,2,3,4,5,6\}$

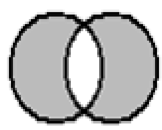
Find:

$A \cup B = \{1, 2, 3, 4\}$
$A \cap B = \{3\}$
$A - B = \{1, 2\}$
$A^c = \{4, 5, 6\}$

5) Symmetric difference of sets

The symmetric difference of sets A and B, denoted by $A \oplus B$, consists of those elements which belong to A or B but not to both. That is,

$A \oplus B = (A \cup B) \backslash (A \cap B)$ or
$A \oplus B = (A \backslash B) \cup (B \backslash A)$



A ⊕ B

Example:

Suppose $U = N = \{1, 2, 3, \ldots\}$ is the universal set.

Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$,

$C = \{2, 3, 8, 9\}$, $E = \{2, 4, 6, 8, \ldots\}$

Then:

$A^c = \{5, 6, 7, \ldots\}$,

$B^c = \{1, 2, 8, 9, 10, \ldots\}$,

$C^c = \{1,4,5,6,7,10,\ldots\}$

$E^c = \{1, 3, 5, 7, \ldots\}$

$A \backslash B = \{1, 2\}$,

$A \backslash C = \{1, 4\}$,

$B \backslash C = \{4, 5, 6, 7\}$,

$A \backslash E = \{1, 3\}$,

$B \backslash A = \{5, 6, 7\}$,

$C \backslash A = \{8, 9\}$,

$C \backslash B = \{2, 8, 9\}$,

$E \backslash A = \{6, 8, 10, 12, \ldots\}$.

Furthermore:

$A \oplus B = (A \backslash B) \cup (B \backslash A) = \{1, 2, 5, 6, 7\}$,

$B \oplus C = \{2, 4, 5, 6, 7, 8, 9\}$,

$A \oplus C = (A \backslash C) \cup (B \backslash C) = \{1, 4, 8, 9\}$,

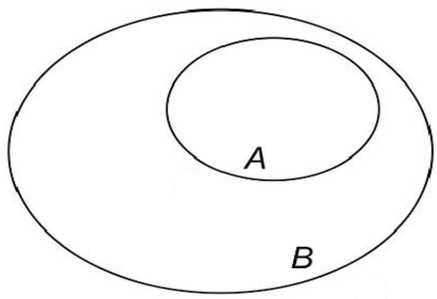
$A \oplus E = \{1, 3, 6, 8, 10, \ldots\}$.

Theorem 2 :

$A \subset B$,

$A \cap B = A$,

$A \cup B = B$ are equivalent



Theorem 3: (Algebra of sets)
Sets under the above operations satisfy various laws or identities which are listed below:

1- $A \cup A = A$
$A \cap A = A$

2- $(A \cup B) \cup C = A \cup (B \cup C)$ Associative laws
$(A \cap B) \cap C = A \cap (B \cap C)$

3- $A \cup B = B \cup A$ Commutativity
$A \cap B = B \cap A$

4- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ Distributive laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

5- $A \cup \varnothing = A$ Identity laws
$A \cap U = A$

6- $A \cup U = U$ Identity laws
$A \cap \varnothing = \varnothing$

7- $(A^c)^c = A$ Double complements

8- $A \cup A^c = U$ Complement intersections and unions

$A \cap A^c = \varnothing$

9- $U^c = \varnothing$
$\varnothing^c = U$

10- $(A \cup B)^c = A^c \cap B^c$ De Morgan's laws
$(A \cap B)^c = A^c \cup B^c$

We discuss two methods of proving equations involving set operations. The first is to break down what it means for an object x to be an element of each side, and the second is to use Venn diagrams.
For example:
Consider the first of De Morgan's laws:

$$(A \cup B)^c = A^c \cap B^c$$

We must prove: 1) $(A \cup B)^c \subset A^c \cap B^c$

2) $A^c \cap B^c \subset (A \cup B)^c$

We first show that $(A \cup B)^c \subset A^c \cap B^c$

Let's pick an element at random $x \in (A \cup B)^c$. We don't know anything about x, it could be a number, a function. All we do know about x, is that:

$x \in (A \cup B)^c$, so
$x \notin A \cup B$

because that's what complement means. Therefore

$x \notin A$ and $x \notin B$,

by pulling apart the union. Applying complements again we get

$x \in A^c$ and $x \in B^c$

Finally, if something is in 2 sets, it must be in their intersection, so

$x \in A^c \cap B^c$

So, any element we pick at random from: $(A \cup B)^c$ is definitely in, $A^c \cap B^c$

so by definition

$(A \cup B)^c \subset A^c \cap B^c$

Next we show that $(A^c \cap B^c) \subset (A \cup B)^c$.

This follows a very similar way.

Firstly, we pick an element at random from the first set,

$x \in (A^c \cap B^c)$

Using what we know about intersections, that means
$x \in A^c$ and $x \in B^c$

Now, using what we know about complements,

$x \notin A$ and $x \notin B$.

If something is in neither A nor B, it can't be in their union, so

$x \notin A \cup B$,

And finally

$\therefore \quad x \in (A \cup B)^c$

We have prove that every element of $(A \cup B)^c$ belongs to $A^c \cap B^c$ and that every element of $A^c \cap B^c$ belongs to $A \cup B)^c$. Together, these inclusions prove that the sets have the same elements, i.e. that $(A \cup B)^c = A^c \cap B^c$

Power set

The power set of some set S, denoted P(S), is the set of all subsets of S (including S itself and the empty set)

$P(S) = \{e : e \subseteq S\}$

Example 1:

Let A = { 1,2 3}

Power set of set A = P(A)

={1},{2},{3},{1,2},{1,3},{2,3},{},A]

Example 2:

P({0,1})={{},{0},{1},{0,1}}

Classes of sets:

Collection of subset of a set with some properties

Example:

Suppose A = { 1,2 3} ,

let X2 be the class of subsets of A which contain exactly two elements of A. Then

class X0 = [{}]
class X1 = [{1},{2},{3}]
class X2 = [{1,2},{1,3},{2,3}]
class X3 = [{1,2,3}]

Cardinality

The cardinality of a set S, denoted **|S|**, is simply the number of elements a set has, so

|{a,b,c,d**}| =** 4,

The cardinality of the power set

Theorem:
If |A| = n then |P(A)| = 2^n
(Every set with n elements has 2^n subsets)

Problem set

Write the answers to the following questions.

1. |{1,2,3,4,5,6,7,8,9,0}|
2. |P({1,2,3})|
3. P({0,1,2})
4. P({1})

Answers

1. 10
2. $2^3=8$
3. {{},{0},{1},{2},{0,1},0,2},{1,2},{0,1,2}}
4. {{},{1}}

The Cartesian product

The Cartesian Product of two sets is the set of all tuples made from elements of two sets.

We write the Cartesian Product of two sets A and B as A × B. It is defined as:

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

It may be clearer to understand from examples;

$$\{0,1\} \times \{2,3\} = \{(0,2),(0,3),(1,2),(1,3)\}$$
$$\{a,b\} \times \{c,d\} = \{(a,c),(a,d),(b,c),(b,d)\}$$
$$\{0,1,2\} \times \{4,6\} = \{(0,4),(0,6),(1,4),(1,6),(2,4),(2,6)\}$$

Example:
If A = {1, 2, 3} and B = {x, y} then

A . B = {(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)}
B . A = {(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)}

It is clear that, the cardinality of the Cartesian product of two sets A and B is:

$$|A \times B| = |A||B|$$

A Cartesian Product of two sets A and B can be produced by making tuples of each element of A with each element of B; this can be visualized as a grid (which *Cartesian* implies) or table: if, *e.g.*,

A = { 0, 1 } and B = { 2, 3 }, the grid is

×		A	
		0	**1**
B	**2**	(0,2)	(1,2)
	3	(0,3)	(1,3)

Problem set

Answer the following questions:

1. {2,3,4}×{1,3,4}
2. {0,1}×{0,1}
3. |{1,2,3}×{0}|
4. |{1,1}×{2,3,4}|

Answers

1. {(2,1),(2,3),(2,4),(3,1),(3,3),(3,4),(4,1),(4,3),(4,4)}
2. {(0,0),(0,1),(1,0),(1,1)}
3. 3
4. 6

EXAMPLE
What is the Cartesian product $A \times B \times C$, where
$A = \{0, 1\}$,
$B = \{1, 2\}$, and
$C = \{0, 1, 2\}$?
Solution:
The Cartesian product $A \times B \times C$ consists of all ordered triples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$. Hence,

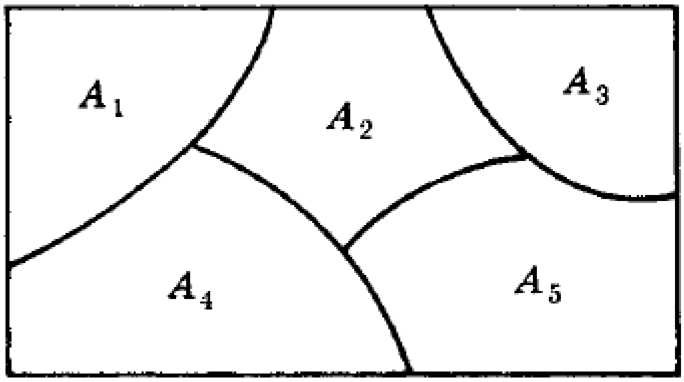
$$A\times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}.$$

EXAMPLE
Suppose that $A = \{1, 2\}$. It follows that
$A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ and
$A^3 = \{(1,1,1), (1,1,2), (1,2,1), (1,2, 2), (2,1,1), (2,1,2), (2, 2,1), (2, 2, 2)\}$.

Partitions of set:
Sets are disjoint if they share no elements. Often when modeling, we will take some set S and divide its members into disjoint subsets called partitions where each member of S belongs to exactly one partition.
Let S be any nonempty set. A partition ($\prod$) of S is a subdivision of S into nonoverlapping, nonempty subsets.
A partition of S is a collection {Ai} of non-empty subsets of S such that:

1) $Ai \neq \varnothing$, where i=1,2,3,……
2) The sets of {Ai } are mutually disjoint
   or $Ai \cap Aj = \varnothing$ where $i \neq j$.
3) $UA_i = S$, where $A_1 \cup A_2 \cup .... \cup A_i = S$

The partition of a set into five cells, A1, A2,A3,A4,A5, can be represented by Venn diagram



Example 1:
let A = {1,2,3,n}
and A1 = {1}, A2 = {3,n}, A3 = {2}

∏ = {A1, A2, A3} is a partition on A because it satisfy the three above conditions.

Example 2 :
Consider the following collections of subsets of

S = {1,2,3,4,5,6,7,8,9}

(i) [{1,3,5},{2,6},{4,8,9}]
(ii) [{1,3,5},{2,4,6,8},{5,7,9}]
(iii) [{1,3,5},{2,4,6,8},{7,9}]

Then

(i) is not a partition of S since 7 in S does not belong to any of the subsets.
(ii) is not a partition of S since {1,3,5} and {5,7,9} are not disjoint.
(iii) is a partition of S.

Computer Representation of Sets

There are various ways to represent sets using a computer. One method is to store the elements of the set in an unordered fashion. However, if this is done, the operations of computing the union, intersection, or difference of two sets would be time-consuming, because each of these operations would require a large amount of searching for elements. We will present a method for storing elements using an arbitrary ordering of the elements of the universal set. This method of representing sets makes computing combinations of sets easy.

Assume that the universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used). First, specify an arbitrary ordering of the elements of U, for instance:

$a1, a2, \ldots, an$. Represent a subset A of U with the bit string of length n, where the ith bit in this string is 1 if ai belongs to A and is 0 if ai does not belong to A.

Example

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is,

$ai = i$. What bit strings represent

1- the subset of all odd integers in U,
2- The subset of all even integers in U, and
3- the subset of integers not exceeding 5 in U?

Solution:

1- The bit string that represents the set of odd integers in U, namely, $\{1, 3, 5, 7, 9\}$, has a one bit in the first, third, fifth, seventh, and ninth positions, and a zero elsewhere. It is
10 1010 1010.

2- we represent the subset of all even integers in U, namely, $\{2, 4, 6, 8, 10\}$, by the string 01 0101 0101.

3- The set of all integers in U that do not exceed 5, namely, $\{1, 2, 3, 4, 5\}$, is represented by the String 11 1110 0000.

Using bit strings to represent sets, it is easy to find complements of sets and unions, intersections, and differences of sets. To find the bit string for the complement of a set from the bit string for that set, we simply change each 1 to a 0 and each 0 to 1, because $x \in A$ if and only if $x \notin \bar{A}$. Note that this operation corresponds to taking the negation of each bit when we associate a bit with a truth value—with 1 representing true and 0 representing false.

Example

We have seen that the bit string for the set {1, 3, 5, 7, 9} (with universal set {1, 2, 3, 4,5, 6, 7, 8, 9, 10}) is 10 1010 1010.What is the bit string for the complement of this set?

Solution:

The bit string for the complement of this set is obtained by replacing 0s with 1s and vice versa. This yields the string 01 0101 0101, which corresponds to the set {2, 4, 6, 8, 10}.

To obtain the bit string for the union and intersection of two sets we perform bitwise Boolean operations on the bit strings representing the two sets.

The bit in the *i*th position of the bit string of the **union** is 1 if either of the bits in the *i*th position in the two strings is 1 (or both are 1), and is 0 when both bits are 0. Hence, the bit string for the union is the bitwise *OR* of the bit strings
for the two sets. The bit in the *i*th position of the bit string of the **intersection** is 1 when the bits in the corresponding position in the two strings are both 1, and is 0 when either of the two bits
is 0 (or both are). Hence, the bit string for the intersection is the bitwise *AND* of the bit strings for the two sets.

EXAMPLE

The bit strings for the sets {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

Solution:

The bit string for the **union** of these sets is:

11 1110 0000 ∨ 10 1010 1010 = 11 1110 1010, which corresponds to the set {1, 2, 3, 4, 5, 7, 9}.

The bit string for the **intersection** of these sets is
11 1110 0000 ∧ 10 1010 1010 = 10 1010 0000, which corresponds to the set {1, 3, 5}.

Finite Sets and Counting Principle:

A set is said to be finite if it contains exactly m distinct elements, where m denotes some nonnegative integer. Otherwise, a set is said to be infinite.

For example:
- The empty set ∅ and the set of letters of English alphabet are finite sets,
- The set of even positive integers, {2,4,6,…...}, is infinite.

If a set A is finite, we let n(A) or #(A) denote the number of elements of A.

Example:
If A ={1,2,a,w} then
$n(A) = \#(A) = |A| = 4$

Lemma: If A and B are finite sets and disjoint Then $A \cup B$ is finite set and:

$$n(A \cup B) = n(A) + n(B)$$

Theorem (Inclusion–Exclusion Principle): Suppose A and B are finite sets. Then
$A \cup B$ and $A \cap B$ are finite and

$$|A \cup B| = |A| + |B| - |A \cap B|$$

That is, we find the number of elements in A or B (or both) by first adding n(A) and n(B) (inclusion) and then subtracting $n(A \cap B)$ (exclusion) since its elements were counted twice.
We can apply this result to obtain a similar formula for three sets:

Corollary:

If A, B, C are finite sets then

$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B \cap C|$

Example (1) :

A= {1,2,3}

B= {3,4}

C= {5,6}

$A \cup B \cup C = \{1,2,3,4,5,6\}$

$|A \cup B \cup C| = 6$

$|A| = 3$, $|B| = 2$, $|C| = 2$

$A \cap B = \{3\}$, $|A \cap B| = 1$

$A \cap C = \{\ \}$, $|A \cap C| = 0$

$B \cap C = \{\ \}$, $|B \cap C| = 0$

$A \cap B \cap C = \{\ \}$, $|A \cap B \cap C| = 0$

$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B \cap C|$

$|A \cup B \cup C| = 3 + 2 + 2 - 1 - 0 - 0 + 0 = 6$

Example (2):

Suppose a list A contains the 30 students in a mathematics class, and a list B contains the 35 students in an English class, and suppose there are 20 names on both lists. Find the number of students:

(a) only on list A

(b) only on list B

(c) on list $A \cup B$

Solution:

(a) List A has 30 names and
20 are on list B;
hence $30 - 20 = 10$ names are only on list A.

(b) Similarly, $35 - 20 = 15$ are only on list B.

(c) We seek $n(A \cup B)$. By inclusion–exclusion,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30 + 35 - 20 = 45.$$

Example (3):

Suppose that 100 of 120 computer science students at a college take at least one of languages: French, German, and Russian:

65 study French (F).

45 study German (G).

42 study Russian (R).

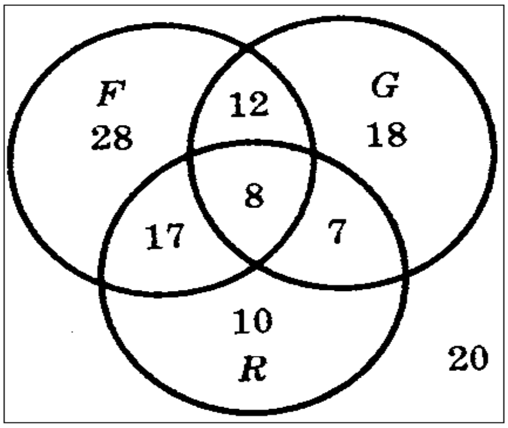
20 study French & German $F \cap G$.

25 study French & Russian $F \cap R$.

15 study German & Russian $G \cap R$.

Find the number of students who study:

1) All three languages ($F \cap G \cap R$)
2) The number of students in each of the eight regions of the Venn diagram



Solution:

$|F \cup G \cup R| = |F| + |G| + |R| - |F \cap G| - |F \cap R| - |G \cap R| + |F \cap G \cap R|$

$100 = 65 + 45 + 42 - 20 - 25 - 15 + |F \cap G \cap R|$

$100 = 92 + |F \cap G \cap R|$

$\therefore |F \cap G \cap R| = 8$ students study the 3 languages

$20 – 8 = 12$ $(F \cap G) - R$

$25 – 8 = 17$ $(F \cap R) - G$

$15 – 8 = 7$ $(G \cap R) - F$

65 – 12 – 8 – 17 = 28 students study French only

45 – 12 – 8 7 = 18 students study German only

42 – 17 – 8 7 = 10 students study Russian only

120 – 100 = 20 students do not study any language

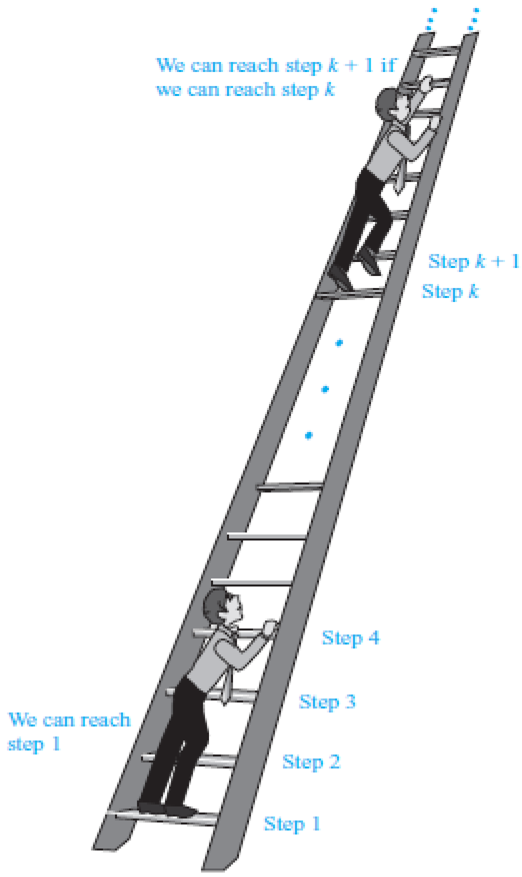
Mathematic induction:

Suppose that we have an infinite ladder and we want to know whether we can reach every step on this ladder. We know two things:

l. We can reach the first rung of the ladder.

2. If we can reach a particular rung of the ladder, then we can reach the next rung.

Can we conclude that we can reach every rung? By (1), we know that we can reach the first rung of the ladder. Moreover, because we can reach the first rung, by (2), we can also reach the second rung; it is the next rung after the first rung. Applying (2) again, because we can reach the second rung, we can also reach the third rung. Continuing in this way, we can show that we can reach the fourth rung, the fifth rung, and so on. For example, after 100 uses of (2), we know that we can reach the 101 st rung.



We can verify using an important proof technique called mathematical induction. That is, we can show that *P(n)* is true for every positive integer *n,* where *P(n)* is the statement that we can reach the *nth* rung of the ladder.
Mathematical induction is an important proof technique that can be used to prove assertions of this type. Mathematical induction is used to prove results about a large variety of discrete objects. For example, it is used to prove results about the complexity of algorithms, the correctness of certain types of computer programs, theorems about graphs and trees, as well as a wide range of identities and inequalities.

In general, mathematical induction can be used to prove statements that assert that *P(n)* is true for all positive integers *n,* where *P(n)* is a propositional function.

PRINCIPLE OF *MATHEMATICAL INDUCTION*
To prove that *P(n)* is true for all positive integers *n,* where *P(n)* is a propositional function, we complete two steps:

(i)BASIS *STEP:* We verify that *P(1)* is true.
(ii)INDUCTIVE STEP*:* We show that the conditional statement *P(k)→P(k* + I) is true for all positive integers *k.*

EXAMPLE1:
Show that if *n* is a positive integer, then

$$1+2+ \dots +n = \frac{n\,(n+1)}{2}.$$

Prove P (for n ≥ 1)

Solution:

Let *P(n)* be the proposition that the sum of the first *n* positive integers is *n(n* + 1)/2
We must do two things to prove that *P(n)* is true for
n = 1, 2, 3, ...
Namely, we must show that *P(1)* is true and that the conditional statement *P(k)* implies *P(k* + 1) is true for *k* = 1,2.3, ...

(i)BASIS STEP: P(1) IS true, because $1 = \frac{1(1+1)}{2}$

left side =1 & Right side =2/2 = 1

left side = Right side

(ii)INDUCTIVE STEP: For the inductive hypothesis we assume that *P(k)* holds for an arbitrary positive integer *k*. That is, we assume that P(k) is true

$$1+2+\dots+k = \frac{k(k+1)}{2}$$

Under this assumption, it must be shown that $P(k+1)$ is true, namely, that

to prove that P(k+1) is true

$$1 + 2 + 3 + 4 + \dots + k + (k+1) = 1/2 * k * (k+1) + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= 1/2\,(k+1)(k+2)$$

So P is true for all $n \geq k$

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Example 2:
Conjecture a formula for the sum of the first *n* positive odd integers. Then prove your conjecture using mathematical induction.
Solution*:*
The sums of the first *n* positive odd integers for
$n = 1,2,3,4,5$ are:

$1 = 1$, $1 + 3 = 4$, $1 + 3 + 5 = 9$,
$1+3+5+7=16$, $1 + 3 + 57 + 9 = 25$.

From these values it is reasonable to conjecture that the sum of the first *n* positive odd integers is n^2, that is,

$$1 + 3 + 5 + ... + (2n - 1) = n^2$$

We need a method to *prove* that this *conjecture* is correct, if in fact it is.
Let *P(n)* denote the proposition that the sum of the first *n* odd positive integers is n^2

(i)BASIS STEP: P(1) states that the sum of the first one odd positive integer is 1^2. This is true because the sum of the first odd positive integer is 1.

(ii)INDUCTIVE STEP:
we first assume the inductive hypothesis.
The inductive hypothesis is the statement that *P(k)* is true, that is,

$$1 + 3 + 5 + ... + (2k - 1) = k^2$$

(ii) n=k; Assuming P(k) is true,
We add (2k-1)+2 = 2K + 1 to both sides of P(k), obtaining:

$$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$$
$$= (k + 1)^2$$

Which is P(k + 1).
That is, P(k + 1) is true whenever P(k) is true.
By the principle of mathematical induction:
P is true for all n ≥.k.

Example 3:
Prove the following proposition (for $n \geq 0$):

$$P(n) : 1 + 2 + 2^2 + 2^3 + \ldots +2^n = 2^{n+1} - 1$$

solution :

(i) P(0) : left side =1
Right side $=2^1-1=1$

(ii) Assuming P(k) is true ; n=k

$$P(k) : 1 + 2 + 2^2 + 2^3 + \ldots +2^k = 2^{k+1} - 1$$

We add 2^{k+1} to both sides of P(k), obtaining

$$1 + 2 + 2^2 + 2^3 + \ldots +2^k + 2^{k+1} = 2^{k+1} - 1+ 2^{k+1}$$
$$= 2(2^{k+1}) - 1 = 2^{k+2} - 1$$

which is P(k +1). That is, P(k +1) is true whenever P(k) is true.
By the principle of induction:
P(n) is true for all n.

Example 4:.
Use mathematical induction to prove that *$(n^3 - n)$* is divisible by 3 whenever *n* is a positive integer.

Solution:
Let *P(n)* denote the proposition: "$n^3 - n$ is divisible by 3."

(i)BASIS STEP: The statement *P(*1*)* is true because
$1^3 - 1 = 0$ is divisible by 3.

(ii)INDUCTIVE STEP:
For the inductive hypothesis we assume that *P(k)* is true; that is, we assume that $k^3 - k$ is divisible by 3 for an arbitrary positive integer *k*.

To complete the inductive step, we must show that when we assume the inductive hypothesis, it follows that $P(k + 1)$, the statement that $(k + 1)^3 - (k + 1)$ is divisible by 3, is also true.

That is, we must show that $(k + 1)^3 - (k + 1)$ is divisible by 3.
Note that

$$(k + 1)^3 - (k + 1) = (k^3 + 3k^2 + 3k + 1) - (k + 1)$$

$$= (k^3 - k) + 3(k^2 + k).$$

Using the inductive hypothesis, we conclude that the first term $k^3 - k$ is divisible by 3.
The second term is divisible by 3 because it is 3 times an integer. So,
Because we have completed both the basis step and the inductive step, by the principle of mathematical induction we know that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

Homework:
Prove by induction:

1) $2 + 4 + 6 + \ldots\ldots + 2n = n(n + 1)$
2) $1 + 4 + 7 + \ldots\ldots + (3n - 2) = 1/2\, n(3n - 1)$

Relations

The important aspect of the any set is the relationship between its elements. The association of relationship established by sharing of some common feature proceeds comparing of related objects. For example, assume a set of students, where students are related with each other if their sir names are same. Conversely, if set is formed a class of students then we say that students are related if they belong to same class etc.

Relation is a predefined alliance of objects. The examples of relations are viz. brother and sister, and mathematical relation such as less than, greater than, and equal etc.

The relations can be classifying on the basis of its association among the objects. For example, relations said above are all association among two objects so these relations are called binary relation. Similarly, relations of parent to their children, boss and subordinates, brothers and sisters etc. are the examples of relations among three/more objects known as tertiary relation, quadratic relations and so on. In general an *n*-ary relation is the relation framed among *n* objects.

Product sets:

Consider two arbitrary sets A and B. The set of all ordered pairs (a,b) where $a \in A$ and $b \in B$ is called the product, or cartesian product, of A and B.

$$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$$

One frequently writes A^2 instead of $A \times A$.

Example

R denotes the set of real numbers and so

$\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ is the set of ordered pairs of real numbers.

The geometrical representation of $\mathbf{R}^2$ as points in the plane as in Fig.-1. Here each point *P* represents an ordered pair *(a, b)* of real numbers and vice versa; the vertical line through P meets the x-axis at a, and the horizontal line through P meets the y-axis at b. $\mathbf{R}^2$ is called the *Cartesian plane*.

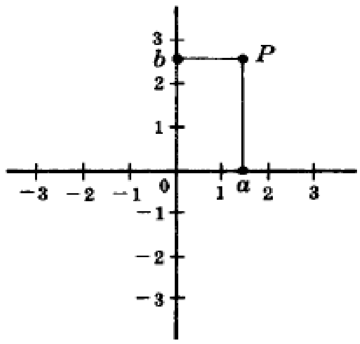


Fig. -1

Example:
a)Let A = {1,2} and B = {a ,b ,c} then

$A \times B = \{(1,a), (1,b),(1,c),(2,a),(2,b),(2,c)\}$, Also,

$A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

The order in which the sets are considered is important, so

$A\times B \neq B \times A$.

$n(A\times B) = n(A)\times n(B) = 2 \times 3 = 6$

Binary relation:
A relation between two objects is a binary relation and it is given by a set of ordered couples.
Let A and B be sets. A *binary relation* from A to B is a subset of $A \times B$.
Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair $a \in A$ and $b \in B$, exactly one of the following is true:

(i) $(a, b) \in R$; we then say "a is R-related to b", written aRb.
(ii) $(a, b) \notin R$; we then say "a is not R-related to b", written $a \not R b$.

If R is a relation from a set A to itself, that is, if R is a subset of $A^2 = A \times A$, then we say that R is a relation *on* A.

Domain Set and Range Set of Binary Relation

Let R be a binary relation, then *domain set* (domain) of relation R denoted by D(R), contains all first components of the ordered couples i.e. $D(R) = \{x: (x, y) \in R\}$

The *range set* (range) of relation R denoted by R(R), contains all second components of the ordered couples i.e.
$R(R) = \{y: (x, y) \in R\}$
Further, if $R = X \times Y$ then $R(R) \subseteq Y$.

Example
(a) A = (1, 2, 3) and B = {x, y, z}, and let
R = {(1, y), (1, z), (3, y)}. Then R is a relation from A to B since R is a subset of A × B.
With respect to this relation

$$1Ry,\ 1Rz,\ 3Ry \quad \text{but} \quad (1,x) \notin R \ \ \& \ \ (2,x) \notin R$$

The domain of R is {1, 3} and the range is {y, z}.

(b) Set inclusion $\subseteq$ is a relation on any collection of sets. For, given any pair of set A and B, either $A \subseteq B$ *or* or $A \not\subseteq B$.

(c) A familiar relation on the set **Z** of integers is "*m* divides *n*." A common notation for this relation is to write
$m|n$ when m divides n. Thus $6 \mid 30$ but $7 \nmid 25$.

(d) Consider the set L of lines in the plane. Perpendicularity, written "$\perp$," is a relation on L. That is, given any pair of lines a and b, either $a \perp b$ or a not$\perp$ b. Similarly, "is parallel to," written "$\|$" is a relation on L since either $a \| b$ or $a \nparallel b$.

(e) Let A be any set. Then $A \times A$ and $\emptyset$ are subsets of $A \times A$ and hence are relations on A called the *universal relation* and *empty relation*, respectively.

Example,

_ Let a relation R = $\{(x1, y1), (x1, y2), (x2, y2), (x3, y1)\}$ then its domain set and the range set will be

D(R) = $\{x1, x2, x3\}$ and

R(R) = $\{y1, y2\}$ correspondingly.

Example

Let R = $\{(x, x): x \in I+\}$ where I+ = {1, 2, 3, ……} then

D(R) = {1, 2, 3 , …..} and

R(R) = {1, 2, 3, ……}.

Example

Let R = $\{(x, \log_7 x): x \in N_0\}$ where N_0 = {0, 1, 2, ……} then

D(R) = N0 = {0, 1, 2, …..} and

R(R) = $\{ \log_7 0, \log_7 1, \log_7 2, \ldots\}$

Example :

Let A = {1, 2, 3, 4}. Define a relation R on A by writing

(x, y) $\in$ R if x < y. Then

R = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}.

Example :

let A = {1,2,3} and

R = {(1,2),(1,3),(3,2)}. Then R is a relation on A since it is a subset of A×A with respect to this relation:

1R2, 1R3, 3R2 but $(1,1) \notin R$ & $(2,1) \notin R$

The domain of R is {1,3} and

The range of R is {2,3}

Example :

Let X = {1, 2, 3, 4, 5} and Y = {7, 11, 13} are two sets.

(i) Consider a relation R i.e

R = {(x, y): x ∈ X and y ∈ Y and (y – x) is a perfect square}
then relation R contains the following ordered couples,
R = {(3, 7), (2, 11), (4, 13)}

(ii) Consider another relation R′ i.e.

R′ = {(x, y): x ∈ X and y ∈ Y and (y – x) is divisible by 6} then relation R' will be,
R′ = {(1, 7), (5, 11), (1, 13)}

(iii) R ∪ R′ = {(1, 7), (1, 13), (2, 11), (3, 7), (4, 13), (5, 11)}.

(iv) R ∩ R′ = { } or ∅.

Example :

Let A = {1, 2, 3}. Define a relation R on A by writing
(x, y) ∈ R , such that a≥b, list the element of R

$$aRb \leftrightarrow a \geq b \ , \ a,b \in A$$

∴ R = {(1,1),(2,1), (2,2), (3,1), (3,2), (3,3)}.

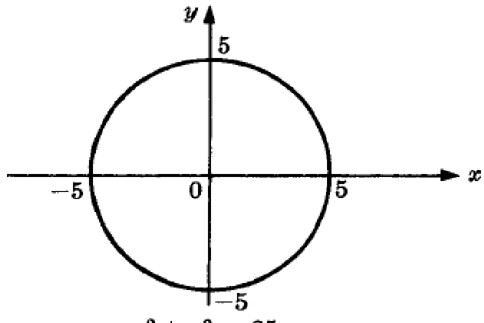
Pictorial representation of relations

There are various ways of picturing relations.

i- By coordinate plane

Let *S* be a relation on the set R of real numbers; that is, *S* is a subset of $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$. Frequently, *S* consists of all ordered pairs of real numbers which satisfy some given equation
$E(x, y) = 0$ (such as $x^2 + y^2 = 25$).
Since $\mathbf{R}^2$ can be represented by the set of points in the plane, we can picture *S* by emphasizing those points in the plane which belong to *S*. The pictorial representation of the relation is called the *graph* of the relation.

For example,
the graph of the relation $x^2+y^2 = 25$ is a circle having its center at the origin and radius 5.



$x^2 + y^2 = 25$

Fig. 2

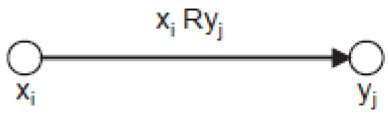
ii-Directed Graphs of Relations on Sets

Relation can be represented pictorially by drawing its *graph* (directed graph). Consider a relation R be defined between two sets

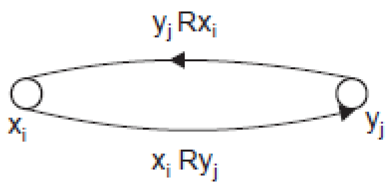
X = {x1, x2, …….., xl} and

Y = { y1, y2, …….., ym}

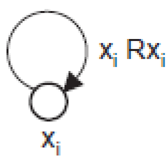
i.e., xi R yj , that is ordered couple (xi, yj) ∈ R where 1 ≤ i ≤ 1 and 1 ≤ j ≤ m. The elements of sets X and Y are represented by small circle called nodes. The existence of the ordered couple such as (xi, yj) is represented by means of an edge marked with an arrow in the direction from xi to yj



While all nodes related to the ordered couples in R are connected by proper arrows, we get a directed graph of the relation R. For the ordered couples xi Ryj and yj Rxi we draw two arcs between nodes xi and yj,



If ordered couple is like xiRxi or (xi, xi) ∈ R then we get self-loop over the node xi.



example,
relation R on the set A = {1, 2, 3, 4}:
R = {(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)}
Fig. 3 shows the directed graph of *R*

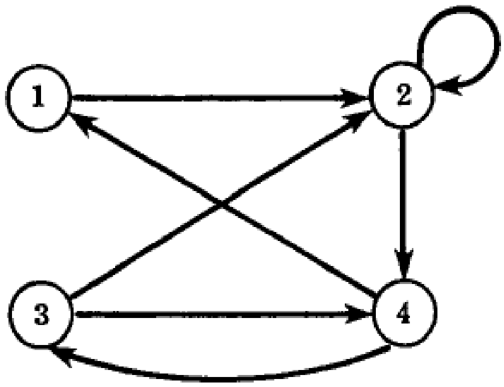


Fig. -3

iii- matrix

Form a rectangular array (matrix) whose rows are labeled by the elements of *A* and whose columns are labeled by the elements of *B*. Put a 1 or 0 in each position of the array according as $a \in A$ is or is not related to $b \in B$. This array is called the *matrix of the relation*.
example,
let A = {1, 2, 3} and B = {x, y, z}.
R = {(1,y),(1,z),(3,y)}
Fig. 4 shows the matrix of *R*

	x	*y*	*z*
1	0	1	1
2	0	0	0
3	0	1	0

Fig. 4

iv- arrow from

Write down the elements of *A* and the elements of *B* in two disjoint disks, and then draw an arrow from
$a \in A$ to $b \in B$ whenever *a* is related to *b*. This picture will be called the *arrow diagram* of the relation.
Fig. 5 pictures the relation *R* in the previous example by the arrow form.

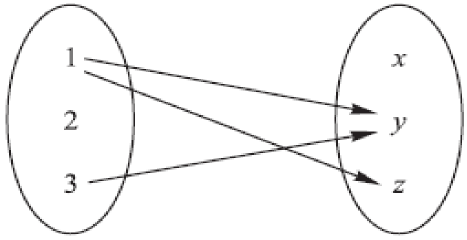


Fig. 5

Properties of binary relations (Types of relations)

Let R be a relation on the set A

1)Reflexive :

R is said to be *reflexive* if ordered couple $(x, x) \in$ R for $\forall x \in$ X.

$\forall$ a $\in$A $\rightarrow$ aRa or (a,a) $\in$ R ; $\forall$ a, b $\in$A. .

Thus R is not reflexive if there exists a $\in$ A such that (a, a) $\notin$ R.

Example i:

Consider the following five relations on the set $A = \{1, 2, 3, 4\}$:

$R1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$

$R2 = \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

$R3 = \{(1, 3), (2, 1)\}$

$R4 = \emptyset$, the empty relation

$R5 = A \times A$, the universal relation

Determine which of the relations are reflexive.

Since A contains the four elements 1, 2, 3, and 4, a relation R on A is reflexive if it contains the four pairs $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$.

Thus only $R2$ and the universal relation $R5 = A \times A$ are reflexive.

Note that $R1$,$R3$, and $R4$ are not reflexive since, for example, $(2, 2)$ does not belong to any of them.

Example ii

Consider the following five relations:

(1) Relation $\leq$ (less than or equal) on the set **Z** of integers.

(2) Set inclusion $\subseteq$ on a collection C of sets.

(3) Relation $\perp$ (perpendicular) on the set L of lines in the plane.
(4) Relation ∥ (parallel) on the set L of lines in the plane.
(5) Relation | of divisibility on the set **N** of positive integers. (Recall $x \mid y$ if there exists z such that $xz = y$.)

Determine which of the relations are reflexive.

The relation (3) is not reflexive since no line is perpendicular to itself.
Also (4) is not reflexive since no line is parallel to itself.
The other relations are reflexive; that is,
$x \leq x$ for every $x \in \mathbf{Z}$,
$A \subseteq A$ for any set $A \in C$, and
$n \mid n$ for every positive integer $n \in \mathbf{N}$.

2) **Symmetric** :
R is said to be *symmetric* if, ordered couple $(x, y) \in R$ and also ordered couple $(y, x) \in R$ for $\forall x, \forall y \in X$.

aRb $\rightarrow$ bRa $\forall$ a,b $\in$A. [if whenever (a, b) $\in$ R then (b, a) $\in$ R.]

Thus R is not symmetric if there exists a, b $\in$ A such that (a, b) $\in$ R but (b, a) $\notin$R.

Example
(a) Determine which of the relations in Example i are symmetric
$R1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$
$R2 = \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
$R3 = \{(1, 3), (2, 1)\}$
$R4 = \emptyset$, *the empty relation*
$R5 = A \times A$, *the universal relation*

R1 is not symmetric since (1, 2) $\in$ R1 but (2, 1) $\notin$ R1.
R3 is not symmetric since (1, 3) $\in$ R3 but (3, 1) $\notin$ R3.
The other relations are symmetric.

(b) Determine which of the relations in Example ii are symmetric.

(1) Relation $\leq$ (less than or equal) on the set **Z** of integers.
(2) Set inclusion $\subseteq$ on a collection *C* of sets.
(3) Relation $\perp$ (perpendicular) on the set *L* of lines in the plane.
(4) Relation $\parallel$ (parallel) on the set *L* of lines in the plane.
(5) Relation | of divisibility on the set **N** of positive integers.

The relation $\perp$ is symmetric since if line *a* is perpendicular to line *b* then *b* is perpendicular to *a*.
Also, $\parallel$ is symmetric since if line *a* is parallel to line *b* then *b* is parallel to line *a*.
The other relations are not symmetric. For example:
$3 \leq 4$ but 4 not$\leq$ 3; $\{1, 2\} \subseteq \{1, 2, 3\}$ but $\{1, 2, 3\}$ not$\subseteq$ $\{1, 2\}$; and 2 | 6 but 6 not | 2.

3) **Transitive** :
R is said to be *transitive* if ordered couple $(x, z) \in R$ whenever both ordered couples $(x, y) \in R$ and $(y, z) \in R$.
aRb $\wedge$ bRc $\rightarrow$ aRc. that is, if whenever $(a, b), (b, c) \in R$ then $(a, c) \in R$.
Thus R is not transitive if there exist $a, b, c \in R$ such that $(a, b), (b, c) \in R$ but $(a, c) \notin R$.

Example
(a) Determine which of the relations in example i are transitive.
$R1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$
$R2 = \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
$R3 = \{(1, 3), (2, 1)\}$
$R4 = \emptyset$, *the empty relation*
$R5 = A \times A$, *the universal relation*

The relation R3 is not transitive since $(2, 1), (1, 3) \in R3$ but $(2, 3) \notin R3$. All the other relations are transitive.

(b) Determine which of the relations in example ii are transitive.
(1) Relation $\leq$ (less than or equal) on the set **Z** of integers.
(2) Set inclusion $\subseteq$ on a collection *C* of sets.

(3) Relation $\perp$ (perpendicular) on the set *L* of lines in the plane.
(4) Relation $\parallel$ (parallel) on the set *L* of lines in the plane.
(5) Relation | of divisibility on the set **N** of positive integers.

The relations $\leq$, $\subseteq$, and | are transitive, but certainly not $\perp$.
Also, since no line is parallel to itself, we can have
a $\parallel$ b and b $\parallel$ a, but a $\parallel$ a. Thus $\parallel$ is not transitive.

4)Equivalence relation :
A binary relation on any set is said an *equivalence* relation if it is reflexive, symmetric, and transitive

R is an equivalence relation on *S* if it has the following three properties:

a - For every $a \in S$, aRa. (reflexive)
b- If aRb, then bRa. (symmetric)
c- If aRb and bRc, then aRc. (transitive)

5) **Irreflexive** :
$\forall\ a \in A\ (a,a) \notin R$

6) **AntiSymmetric** :
if $(x, y) \in R$ but $(y,x) \notin R$ unless $x = y$.
or
if aRb and bRa then a=b,
that is, if $a \neq b$ and aRb then $(b,a) \notin R$.
Thus *R* is not antisymmetric if there exist distinct elements *a* and *b* in *A* such that *aRb* and *bRa*.

the relations $\geq, \leq$ and $\subseteq$ are antisymmetric

Example
(a) Determine which of the relations in Example i are antisymmetric.
$R1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$
$R2 = \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
$R3 = \{(1, 3), (2, 1)\}$

R4 = ∅, the empty relation
R5 = A × A, the universal relation

$R2$ is not antisymmetric since $(1, 2)$ and $(2, 1)$ belong to $R2$, but $1 \neq 2$. Similarly,
the universal relation $R3$ is not antisymmetric.
All the other relations are antisymmetric.

(b) Determine which of the relations in Example ii are antisymmetric.
(1) Relation $\leq$ (less than or equal) on the set **Z** of integers.
(2) Set inclusion $\subseteq$ on a collection C of sets.
(3) Relation $\perp$ (perpendicular) on the set L of lines in the plane.
(4) Relation $\parallel$ (parallel) on the set L of lines in the plane.
(5) Relation | of divisibility on the set **N** of positive integers.

The relation$\leq$ is antisymmetric since whenever $a \leq b$ and $b \leq a$ then $a = b$.
Set inclusion $\subseteq$ is antisymmetric since whenever
$A \subseteq B$ and $B \subseteq A$ then $A = B$. Also,
divisibility on **N** is antisymmetric since whenever
$m|n$ and $n\,|m$ then $m = n$.
(Note that divisibility on **Z** is not antisymmetric since
$3|-3$ and $-3 \mid 3$ but $3 \neq -3$.)
The relations $\perp$ and $\parallel$ are not antisymmetric

7) **compatibility** :
if a relation is only reflexive and symmetric then it is called a *compatibility* relation. So, we can say that: every equivalence relation is a compatibility relation, but not every compatibility relation is an equivalence relation.

Example :
Let N = {1, 2, 3, ...} then show that relation
R = {(x, y):(x – y) is divisible by 2 for every x and y ∈ N} is an equivalence relation.

Solution:

Since,

1- For any $x \in N$, $(x - x)$ is divisible by 2 therefore, relation R is reflexive.

2- For any $x, y \in N$, if $(x - y)$ is divisible by 2 then also $(y - x)$ is divisible by 2 therefore, relation R is symmetric.

3- For any x, y, and $z \in N$, if $(x - y)$ is divisible by 2 and $(y - z)$ is divisible by 2 then also $(x - z)$ is divisible by 2; Therefore, relation R is transitive.

Hence, relation R is an equivalence relation.

Example:

Determine the properties of the relation $\subset$ of set (inclusion on any collection of sets):

1) $A \subset A$ for any set, so $\subset$ is reflexive
2) $A \subset B$ dose not imply $B \subset A$, so $\subset$ is not symmetric
3) If $A \subset B$ and $B \subset C$ then $A \subset C$, so $\subset$ is transitive
4) $\subset$ is reflexive, not symmetric & transitive, so $\subset$ is not equivalence relations
5) $A \subset A$, so $\subset$ is not Irreflexive
6) If $A \subset B$ and $B \subset A$ then $A = B$, so $\subset$ is anti-symmetric
7) $\subset$ *is reflexive and not symmetric then it is not compatibility* relation.
8)

Example:

If $A = \{1,2,3\}$ and $R=\{(1,1),(1,2),(2,1),(2,3)\}$, is R equivalence relation ?

1) 2 is in A but $(2,2) \notin R$, so R is not reflexive
2) $(2,3) \in R$ but $(3,2) \notin R$, so R is not symmetric
3) $(1,2) \in R$ and $(2,3) \in R$ but $(1,3) \notin R$, so R is not transitive

So R is not Equivalence relation

Example:

What is the properties of the relation = ?

1) a=a for any element a $\in$ A, so = is reflexive
2) If a = b then b = a, so = is symmetric
3) If a = b and b = c then a = c, so = is transitive
4) = is (reflexive + symmetric + transitive), so = is equivalence
5) a = a, so = is not Irreflexive
6) If a = b and b = a then a = b, so = is anti-symmetric
7) = *is reflexive and symmetric then it is compatibility* relation.

Remark:

The properties of being symmetric and being antisymmetric are not negatives of each other.

For example,

the relation R = {(1, 3), (3, 1), (2, 3)} is neither symmetric nor antisymmetric.

On the other hand, the relation R = {(1, 1), (2, 2)} is both symmetric and antisymmetric.

From the directed graph of a relation we can easily examine some of its properties. For example if a relation is reflexive, then we must get a self-loop at each node. Conversely if a relation is irreflexive, then there is no self-loop at any node.

For symmetric relation if one node is connected to another, then there must be a return arc from second node to the first node. For antisymmetric relation there is no such direct return arc exist. Similarly we examine the transitivity of the relation in the directed graph.

Equivalence Relations, Partitions, and Equivalence Class

Recall first that a partition *P* of *S* is a collection $\{A_i\}$ of nonempty subsets of *S* with the following two properties:

(1) Each $a \in S$ belongs to some A_i .

(2) If $A_i \neq A_j$ then $A_i \cap A_j = \emptyset$.

In other words, a partition P of S is a subdivision of S into disjoint nonempty sets.

Suppose R is an equivalence relation on a set S. For each $a \in S$, let [a] denote the set of elements of S to which a is related under R; that is:

$$[a] = \{x \mid (a, x) \in R\}$$

We call $[a]$ the *equivalence class* of a in S; any $b \in [a]$ is called a *representative* of the equivalence class.

The collection of all equivalence classes of elements of S under an equivalence relation R is denoted by S/R, that is,

$S/R = \{[a] \mid a \in S\}$, it is called the *quotient set* of S by R.

EXAMPLE

Let R_5 be the relation of congruence modulo 5 on the set Z of integers denoted by

$$x \equiv y \pmod 5$$

This means that the difference $x-y$ is divisible by 5. Then R_5 is an equivalence relation on $\mathbf{Z}$. The quotient set $\mathbf{Z}/R_5$ contains the following five equivalence classes:

$$A0 = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$$
$$A1 = \{\ldots, -9, -4, 1, 6, 11, \ldots\}$$
$$A2 = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$$
$$A3 = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$$
$$A4 = \{\ldots, -6, -1, 4, 9, 14, \ldots\}$$

Any integer x, uniquely expressed in the form $x = 5q + r$ where $0 \leq r < 5$, is a member of the equivalence class A_r , where r is the remainder. As expected, $\mathbf{Z}$ is the disjoint union of equivalence classes $A1, A2, A3, A4$.

Usually one chooses $\{0, 1, 2, 3, 4\}$ or $\{-2, -1, 0, 1, 2\}$ as a set of representatives of the equivalence classes.

Example:

Consider a relation

R = {(x, y): x, y ∈ I+ and (x – y) is divisible by 3}

where I+ is the set of positive integers.

Find the set of equivalence classes generated by the elements of set I+.

Solution:

The equivalence classes are,

[0] = {0, 3, 6, 9, …….} (when $(x – y)$ % 3 = 0),

[1] = {1, 4, 7, 10, ……} (when $(x – y)$ % 3 = 1), and

[2] = {2, 5, 8, 11, ……} (when $(x – y)$ % 3 = 2)

See, unions of these equivalence classes return the set I+, i.e.

I+ = [0] ∪ [1] ∪ [2] = {0, 1, 2, …….

Let x and y are two elements from the set of integers I+, then the relation R is said to be *congruent relation* such that,

R = {(x, y): x, y ∈ I and (x – y) is divisible by m(∈ I+)}

Hence, this relation is a congruent relation of modulo 3.

Closure properties

Reflexive Closures

Let R be a relation on a set A. Then:

$R \cup \{(a, a) \mid a \in A\}$ is the reflexive closure of R.

In other words, **reflexive(*R*)** is obtained by simply adding to R those elements *(a, a)* in the diagonal which do not already belong to R.

Symmetric Closures

$R \cup R^{-1}$ is the symmetric closure of R. in other words, symmetric(R) is obtained by adding to R all pairs (b, a) whenever (a, b) belongs to R.

EXAMPLE :
Consider the relation
R = {(1, 1), (1, 3), (2, 4), (3, 1), (3, 3), (4, 3)} on the set
A = {1, 2, 3, 4}.Then
reflexive(R) = R $\cup$ {(2, 2), (4, 4)} and
symmetric(R) = R $\cup$ {(4, 2), (3, 4)}

Transitive Closure

R^* is the transitive closure of *R, where:*

$R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$ and

$R^2 = R \circ R$ and

$R^n = R^{n-1} \circ R$

Theorem :
Suppose *A* is a finite set with *n* elements and Let *R* be a relation on a set *A* with *n* elements. Then :
transitive$(R) = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$

EXAMPLE :
Consider the relation
$R = \{(1, 2), (2, 3), (3, 3)\}$ on $A = \{1, 2, 3\}$. Then:

$transitive(R) = R \cup R^2 \cup R^3$

$R^2 = R \circ R = \{(1, 3), (2, 3), (3, 3)\}$ and
$R^3 = R^2 \circ R = \{(1, 3), (2, 3), (3, 3)\}$ then

transitive $(R) = \{(1, 2), (2, 3), (3, 3), (1, 3)\}$

Inverse relations:

$R^{-1} = \{(b,a) : (a,b) \in R\}$

Example :
Let R be the following relation on A ={1,2,3}
R = {(1,2),(1,3),(2,3)}

$\therefore$ $R^{-1} = \{(2,1),(3,1),(3,2)\}$

The matrix for R :

$$MR = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$MR^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

MR^{-1} is the transpose of matrix R

Composition of relations:

When a relation is formed over stages such that let R be one relation defined from set X to Y, and S be another relation defined from set Y to Z,

then a relation W denoted by R ° S is a composite relation, i.e

$W = R \circ S = \{(x,z) : \exists\ x \in X \text{ for which } (x,y) \in R \text{ and } (y,z) \in S\}$

Composite relation W can also represented by a diagram

$$X \longrightarrow Y \longrightarrow Z \quad \Rightarrow \quad X \longrightarrow Z$$

Example :
let A ={1,2,3,4}
B = {a, b, c, d}
C = {x, y, z}

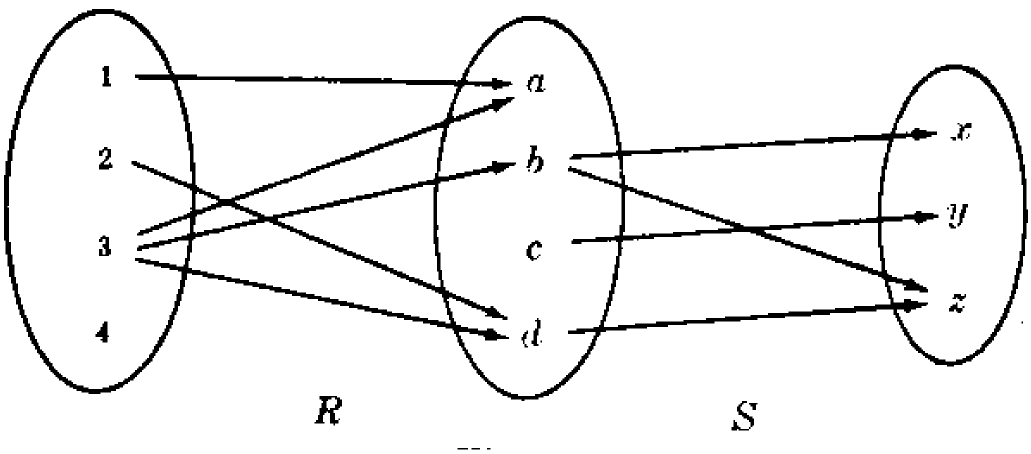
and R = {(1,a),(2,d),(3,a),(3,d),(3,b)}

$S = \{(b,x),(b,z),(c,y),(d,z)\}$

Find $R \circ S$?

Solution :

1) The first way by arrow form



There is an arrow (path) from 2 to d which is followed by an arrow from d to z

$2Rd$ and $dSz \Rightarrow 2(R \circ S)z$

and $3(R\circ S)x$ and $3(R\circ S)z$

so $R \circ S = \{(3,x),(3,z),(2,z)\}$

2) The second way by matrix:

$$MR = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad MS = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$R \circ S = M_R \cdot M_S =$

$$\begin{array}{c|ccc|} & x & y & z \\ \hline 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 2 \\ 4 & 0 & 0 & 0 \\ \hline \end{array}$$

$R \circ S = \{(2,z),(3,x),(3,z)\}$

Example,
let R1 = {(*p*, *q*), (*r*, *s*), (*t*, *u*), (*q*, *s*)} and
R2 = {(*q*, *r*), (*s*, *v*), (*u*, *w*)} are two relations then,

R1 ◦ R2 = {(*p*, *r*), (*r*, *v*), (*t*, *w*), (*q*, *v*)}, and
R2 ◦ R1 = {(*q*, *s*)}

Theorem:
Let A, B, C and D be sets. Suppose
R is a relation from A to B,
S is a relation from B to C, and
T is a relation from C to D. Then

$$(R \circ S) \circ T = R \circ (S \circ T)$$

Partial ordered relation

A binary relation R is said to be partial ordered relation if it is reflexive, antisymmetric, and transitive.

Example,
R={(w,w), (x, x), (y, y), (z, z),(w, x), (w, y), (w, z),(x, y), (x, z)}

In a partial ordered relation objects are related through superior/inferior criterion.

Example

In the arithmetic relation 'less than or equal to' or '$\leq$' (or 'greater than or equal to' or '$\geq$') are partial ordered relations.

Since,

(1) every number is equated to itself so it is reflexive.

(2) Also, if m and n are two numbers then ordered couple (m, n) $\in$ R if m = n $\Rightarrow$ n $\nleq$ m so (n, m) $\notin$ R hence, relation is antisymmetric.

(3) if (m, n) $\in$ R and (n, k) $\in$R $\Rightarrow$ m = n and n = k $\Rightarrow$ m = k so (m, k) $\in$ R hence, R is transitive.

Example

(a) The relation $\subseteq$ of set inclusion is a partial ordering on any collection of sets since set inclusion has the three desired properties. That is,

(1) A $\subseteq$ A for any set A (reflexive).

(2) If A $\subseteq$ B and B $\subseteq$ A, then A = B (antisymmetric).

(3) If A $\subseteq$ B and B $\subseteq$ C, then A $\subseteq$ C (transitive).

(b) The relation "a divides b," written a | b, is a partial ordering on the set N of positive integers.

However, "a divides b" is not a partial ordering on the set Z of integers since a | b and b | a need not imply a = b.

For example, $3|-3$ and $-3 \mid 3$ but $3 \neq -3$.

n-ARY RELATIONS

All the relations discussed above were binary relations. By an *n-ary relation*, we mean a set of ordered *n*-tuples. For any set *S*, a subset of the product set S^n is called an *n*-ary relation on *S*. In particular, a subset of S^3 is called a *ternary relation* on *S*.

EXAMPLE

(a) Let *L* be a line in the plane. Then "betweenness" is a ternary relation *R* on the points of *L*; that is, $(a, b, c) \in R$, if *b* lies between *a* and *c* on *L*.

(b) The equation $x2 + y2 + z2 = 1$
determines a ternary relation T on the set **R** of real numbers. That is, a triple (x, y, z) belongs to T if (x, y, z) satisfies the equation, which means (x, y, z) is the coordinates of a point in **R**3 on the sphere S with radius 1 and center at the origin $O = (0, 0, 0)$.

Home work:

1) Consider the following relations on the set $A = \{1, 2, 3\}$:

$R = \{(1, 1), (1, 2), (1, 3), (3, 3)\}$,
$S = \{(1, 1)(1, 2), (2, 1)(2, 2), (3, 3)\}$,
$T = \{(1, 1), (1, 2), (2, 2), (2, 3)\}$
$\varnothing$ = empty relation
$A \times A$ = universal relation

Determine whether or not each of the above relations on A is:
(*a*) reflexive; (*b*) symmetric; (*c*) transitive; (*d*) antisymmetric.

2) for the relation $R = \{(a, a), (a, b), (b, c), (c, c)\}$ on the set $A = \{a, b, c\}$.
Find: (*a*) reflexive(R); (*b*) symmetric(R); (*c*) transitive(R).

Function:

In many instances we assign to each element of a set a particular element of a second set. For example, suppose that each student in a discrete mathematics class is assigned a letter grade from the set {*A*,*B*,*C*,*D*, *F*}. And suppose that the grades are *A* for Adams, *C* for Chou, *B* for Goodfriend, *A* for Rodriguez, and *F* for Stevens. This assignment is an example of a function. This assignment is an example of a function. The concept of a function is extremely important in mathematics and computer science.

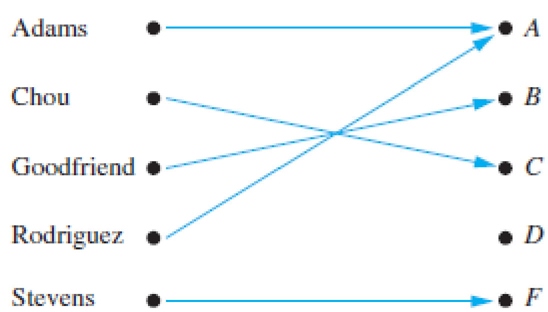


Fig. 1 Assignment of Grades in a Discrete Mathematics Class.

Function is a class of relation. it establishes the relationship between objects. For example, in computer system input is fed to the system in form of data or objects and the system generates the output that will be the function of input. So, function is the mapping or transformation of objects from one form to other.

Definition:

Let *A* and *B* be nonempty sets. A *function F: A→B is a rule which associates with each element of A a unique element in B.*

If f is a function from A to B, we say that A is the domain of f and B is the codomain of f. The range, of f is the set of all images of elements of A.

EXAMPLE 1 What are the domain, and range of the function that assigns grades to students described in the first paragraph of the introduction of this section?

Solution*:*

Let G be the function that assigns a grade to a student $G($Adams$) = A$, for instance.

The domain of G is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens}, and

The codomain is the set $\{A, B, C, D, F\}$.

The range of G is the set $\{A, B, C, F\}$, because each grade except D is assigned to some student.

EXAMPLE 2

Let R be the relation with ordered pairs (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24), and (Felicia, 22). Here each pair consists of a graduate student and this student's age. Specify a function determined by this relation.

Solution*:*

If f is a function specified by R, then

$f\,($Abdul$) = 22$,

$f\,($Brenda$) = 24$,

$f\,($Carla$) = 21$,

$f\,($Desire$) = 22$,

$f\,($Eddie$) = 24$, and

$f\,($Felicia$) = 22$. (Here, $f\,(x)$ is the age of x, where x is a student.)

The domain, = {Abdul, Brenda, Carla, Desire, Eddie, Felicia}.

The codomain, which needs to contain all possible ages of students. Because it is highly likely that all students are less than 100 years old, we can take the set of positive integers less than 100 as the codomain.

The range of the function is the set of different ages of these students, which is the set {21, 22, 24}.

EXAMPLE 3

Let f be the function that assigns the last two bits of a bit string of length 2 or greater to that string.
For example, $f(11010) = 10$.
The domain of f is the set of all bit strings of length 2 or greater, and both
the codomain and range are the set {00, 01, 10, 11}.

EXAMPLE 4

Let $f : \mathbf{Z} \to \mathbf{Z}$ assign the square of an integer to this integer. Then, $f(x) = x^2$, where
the domain of f is the set of all integers,
the codomain of f is the set of all integers, and the
range of f is the set of all integers that are perfect squares, namely, $\{0, 1, 4, 9, \ldots\}$.

EXAMPLE 5

The domain and codomain of functions are often specified in programming languages. For instance, the Java statement

int **floor**(float real){. . .}

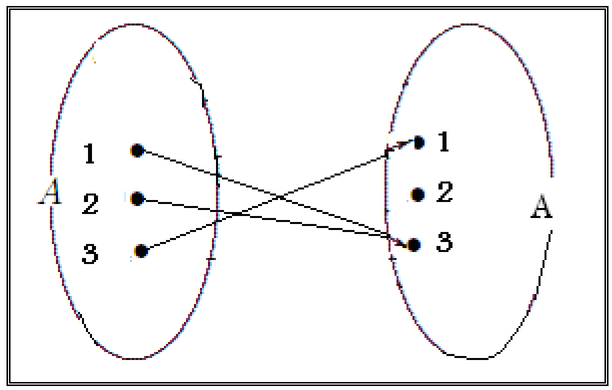
and the C++ function statement

int **function** (float *x*){. . .}

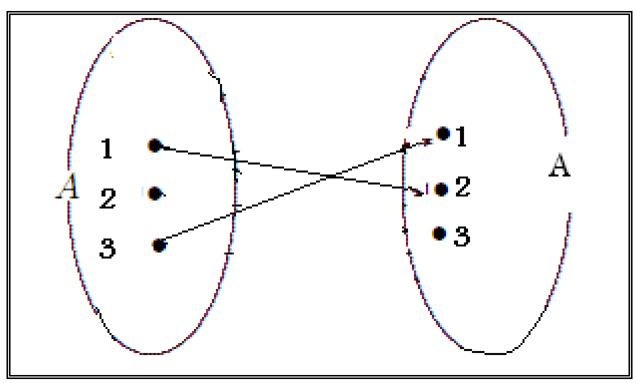
both tell us that the
domain of the floor function is the set of real numbers (represented by floating point numbers) and its
codomain is the set of integers.
A function is called **real-valued** if its codomain is the set of real numbers, and it is called **integer-valued** if its codomain is the set of integers.

Example 6:
Consider the function f (x) = x^3, i.e., f assigns to each real number its cube. Then the image of 2 is 8, and so we may write f (2) = 8.

Example 7 :
consider the following relation on the set A={1,2,3}
F = {(1,3),(2,3),(3,1)}
F is a function



G = {1,2},(3,1)}
G is not a function from A to A



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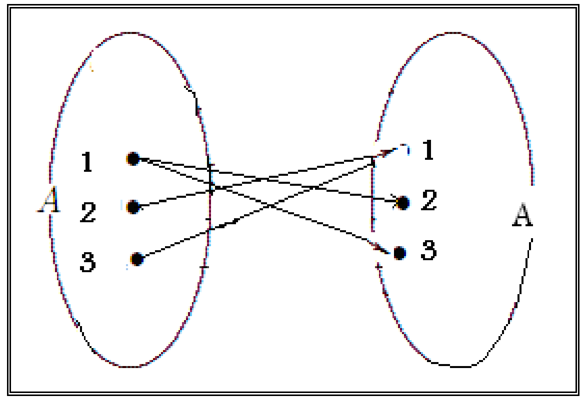
H = {(1,3),(2,1),(1,2),(3,1)}
H is not a function



Classification of functions:
(One-to-one ,onto and invertible functions) :

Some functions never assign the same value to two different domain elements. These functions are said to be one-to-one.

1) **One –to-one** :

a function F:A→B is said to be one-to-one if different elements in the domain (A) have distinct images.

Or If F(a) =F(a') $\Rightarrow$ a = a'

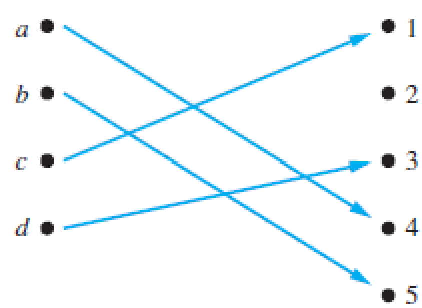


Fig 2: A One-to-One Function.

EXAMPLE 8

Determine whether the function f from {a, b, c, d} to {1, 2, 3, 4, 5} with

$f(a) = 4$,

$f(b) = 5$,

$f(c) = 1$, and

$f(d) = 3$ is one-to-one.

Solution:
The function f is one-to-one because f takes on different values at the four elements of its domain. This is illustrated in Figure 2.

EXAMPLE 9
Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Solution:
The function $f(x) = x^2$ is not one-to-one because, for instance,
$f(1) = f(-1) = 1$, but $1 \neq -1$.
Note that the function $f(x) = x^2$ with its domain restricted to $\mathbf{Z}^+$ is one-to-one. (Technically, when we restrict the domain of a function, we obtain a new function whose values agree with those of the original function for the elements of the restricted domain).

EXAMPLE 10
Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.

Solution:
The function $f(x) = x + 1$ is a one-to-one function. To demonstrate this, note that
$x + 1 \neq y + 1$ when $x \neq y$.

EXAMPLE 11
Suppose that each worker in a group of employees is assigned a job from a set of possible jobs, each to be done by a single worker. In this situation, the function f that assigns a job to each worker is one-to-one. To see this, note that if x and y are two different workers, then
$f(x) \neq f(y)$ because the two workers x and y must be assigned different jobs.

2) **Onto** :

F:A→B is said to be an onto function if each element of B is the image of some element of A.

$\forall\ b \in B \quad \exists \quad a \in A : F(a) = b$

EXAMPLE 12

Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by

$f(a) = 3$,

$f(b) = 2$,

$f(c) = 1$, and

$f(d) = 3$.

Is f an onto function?

Solution:

Because all three elements of the codomain are images of elements in the domain, we see that f is onto. This is illustrated in Figure 3.

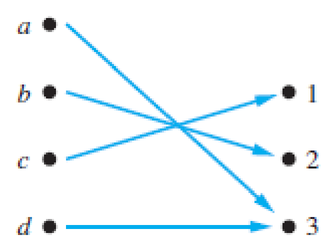


Fig. 3 An Onto Function

Note that if the codomain were $\{1, 2, 3, 4\}$, then f would not be onto.

EXAMPLE 13

Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution:

The function f is not onto because there is no integer x with $x^2 = -1$, for instance.

EXAMPLE 14

Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?

Solution:

This function is onto, because for every integer y there is an integer x such that $f(x) = y$. To see this, note that $f(x) = y$ if and only if $x + 1 = y$, which holds if and only if

$x = y - 1$.

EXAMPLE 15

Consider the function f in Example 11 that assigns jobs to workers.

The function f is onto if for every job there is a worker assigned this job. The function f is not onto when there is at least one job that has no worker assigned it.

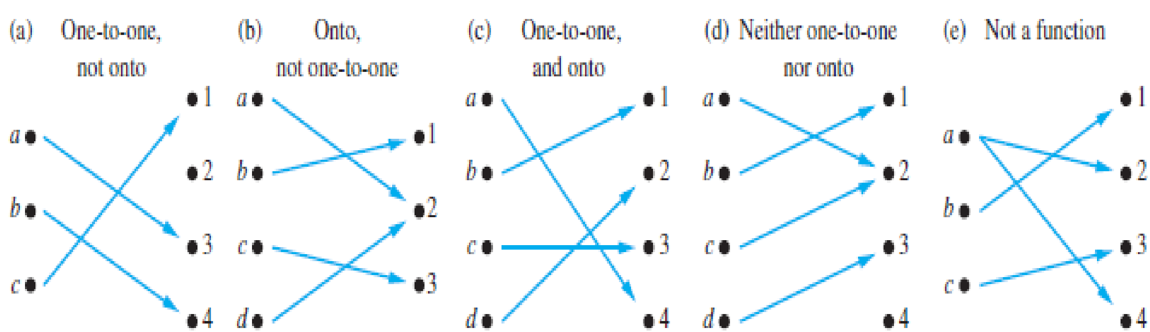


Fig 4 . Examples of Different Types of Correspondences.

3) **Invertible (One-to-one correspondence**)

F:A→ B is invertible if and only if F is **both** one-to-one and onto

$F:A\rightarrow B$ is invertible if its inverse relation f^{-1} is a function $F:B \rightarrow A$

$$F^{-1} : \{(b,a) \; \forall \; (a,b) \in F\}$$

EXAMPLE 16

Let *f* be the function from {*a, b, c, d*} to {1, 2, 3, 4} with

f (a) = 4,

f (b) = 2,

f (c) = 1, and

f (d) = 3. Is *f* an invertible?

Solution:

The function *f* is one-to-one and onto.

It is one-to-one because no two values in the domain are assigned the same function value.

It is onto because all four elements of the codomain are images of elements in the domain. Hence, *f* is a invertible.

Figure 4 displays four functions where the first is one-to-one but not onto,

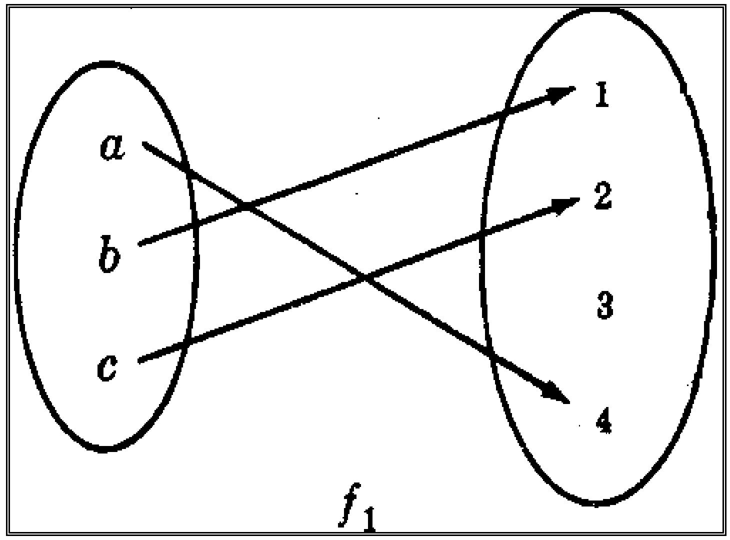
the second is onto but not one-to-one,t

the third is both one-to-one and onto, and

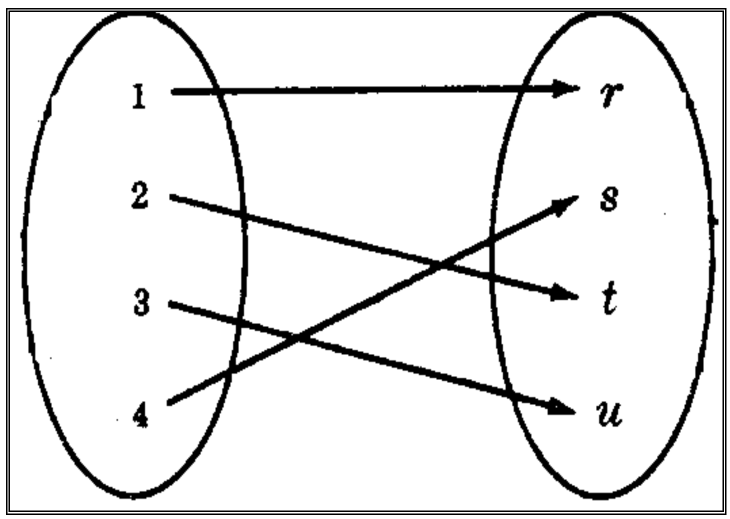
the fourth is neither one-to-one nor onto.

The fifth correspondence in Figure 4 is not a function,

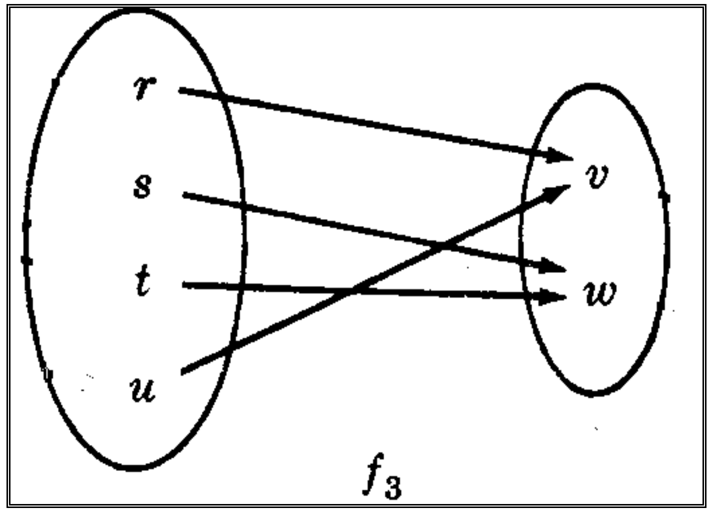
because it sends an element to two different elements.



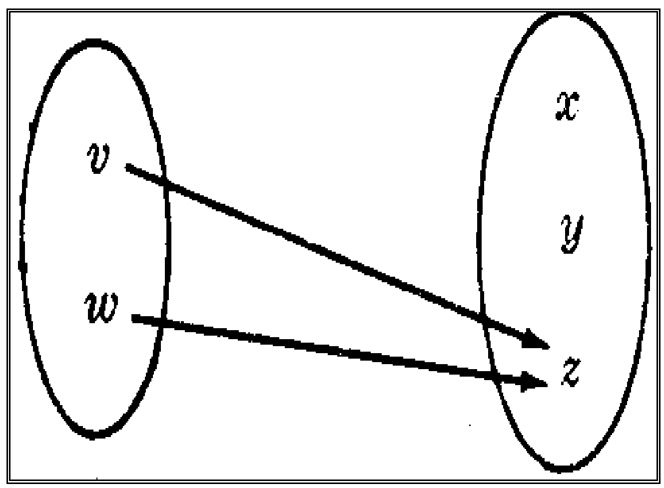
one to one but not onto (3∈B but it is not the image under f1)



both one to one & onto
(or one to one correspondence between A and B)



not one to one & onto



not one to one & not onto

Inverse Functions

Now consider a one-to-one correspondence *f* from the set *A* to the set *B*. Because *f* is an onto function, every element of *B* is the image of some element in *A*. Furthermore, because *f* is also a one-to-one function, every element of *B* is the image of a *unique* element of *A*. Consequently, we can define a new function from *B* to *A* that reverses the correspondence given by *f* .

DEFINITION: Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$.

If a function f is not a one-to-one correspondence, we cannot define an inverse function of f. When f is not a one-to-one correspondence, either it is not one-to-one or it is not onto.

If f is not one-to-one, some element b in the codomain is the image of more than one element in the domain.

If f is not onto, for some element b in the codomain, no element a in the domain exists for which $f(a) = b$. Consequently, if f is not a one-to-one correspondence, we cannot assign to each element b in the codomain a unique element a in the domain such that $f(a) = b$ (because for some b there is either more than one such a or no such a).
A one-to-one correspondence is called invertible because we can define an inverse of this function. A function is not invertible if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

EXAMPLE 17
Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that
$f(a) = 2$,
$f(b) = 3$, and
$f(c) = 1$.
Is f invertible, and if it is, what is its inverse?

***Solution*:**
The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so

$F^{-1}(1) = c$,
$F^{-1}(2) = a$, and
$F^{-1}(3) = b$.

EXAMPLE 18

Let $f : Z \rightarrow Z$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Solution:

The function f has an inverse because it is a one-to-one correspondence. To reverse the correspondence, suppose that y is the image of x, so that $y = x + 1$. Then $x = y - 1$.

Consequently, $f^{-1}(y) = y - 1$.

EXAMPLE 19

Let f be the function from R to R with $f(x) = x^2$. Is f invertible?

Solution:

Because $f(-2) = f(2) = 4$, f is not one-to-one.
If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible.
(Note we can also show that f is not invertible because it is not onto.)

Sometimes we can restrict the domain or the codomain of a function, or both, to obtain an invertible function, as Example 20 illustrates.

EXAMPLE 20

Show that if we restrict the function $f(x) = x^2$ in Example 19 to a function from the set of all nonnegative real numbers to the set of all nonnegative real numbers, then f is invertible.

Solution:
The function $f(x) = x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one.
To see this, note that

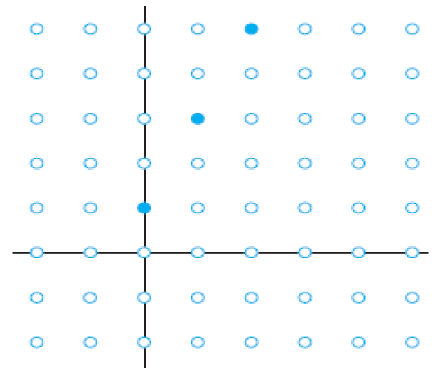
If $f(x) = f(y)$, then
$x^2 = y^2$, so
$x^2 - y^2 = (x + y)(x - y) = 0$.
This means that $x + y = 0$ or $x - y = 0$, so
$x = -y$ or $x = y$.
Because both x and y are nonnegative, we must have $x = y$. So, this function is one-to-one.

Furthermore, $f(x) = x^2$ is onto when the codomain is the set of all nonnegative real numbers, because each nonnegative real number has a square root. That is, if y is a nonnegative real number, there exists a nonnegative real number x such that
$x = \sqrt{y}$, which means that $x^2 = y$.
Because the function $f(x) = x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one and onto, it is invertible. Its inverse is given by the rule
$F^{-1}(y) = \sqrt{y}$.

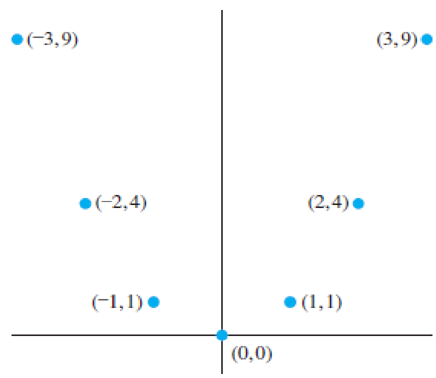
Graph of a function:
We can associate a set of pairs in $A \times B$ to each function from A to B. This set of pairs is called the **graph** of the function and is often displayed pictorially to aid in understanding the behavior of the function.

EXAMPLE 21
Display the graph of the function $f(n) = 2n + 1$ from the set of integers to the set of integers.

Solution:
The graph of f is the set of ordered pairs of the form $(n, 2n + 1)$, where n is an integer.



EXAMPLE 22
Display the graph of the function $f(x) = x^2$ from the set of integers to the set of integers.
Solution:
The graph of f is the set of ordered pairs of the form $(x, f(x)) = (x, x^2)$, where x is an integer.



By a *real polynomial function*, we mean a function f: $\mathbf{R} \rightarrow \mathbf{R}$ of the form

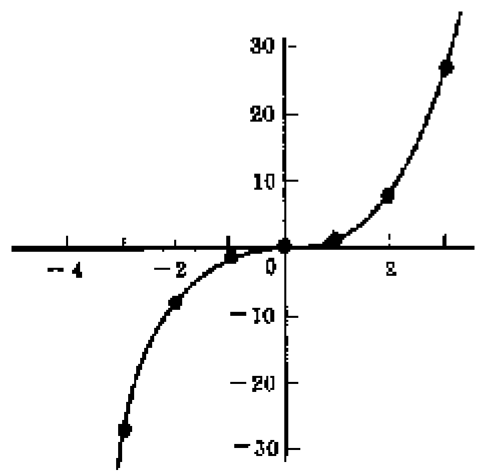
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the *ai* are real numbers. Since **R** is an infinite set, it would be impossible to plot each point of the graph. However, the graph of such a function can be approximated by first plotting some of its points and then drawing a smooth curve though these points. The table points are usually obtained from a table where various values are assigned to x and the corresponding value of f(x) computed.

Example 23 : let f: R→R and f(x)= x^3 , find f(x)

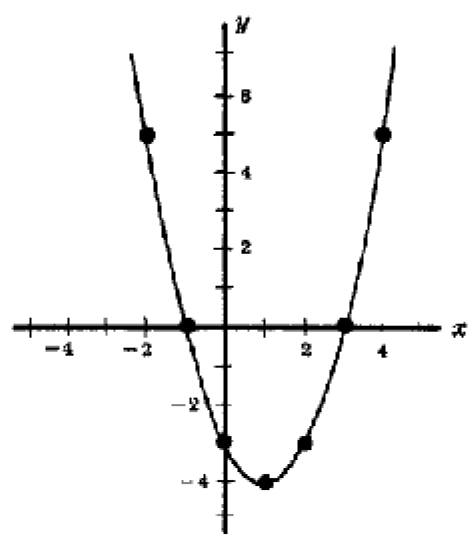
$f(3) = 3^3 = 27$
$f(-2) = (-2)^3 = -8$

x	$f(x)$
−3	−27
−2	8
1	−1
0	0
1	1
2	8
3	27



Example 24: let f: R→R and $f(x) = x^2 - 2x - 3$, , find f(x)

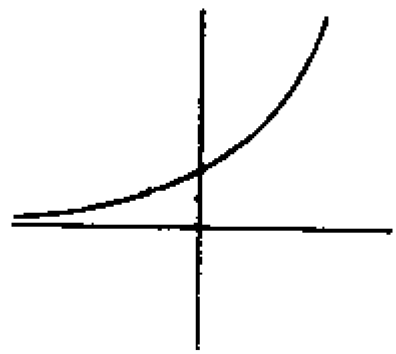
x	$f(x)$
−2	5
−1	0
0	−3
1	−4
2	−3
3	0
4	5



Geometrical Characterization of One-to-One and Onto Functions

For the functions of the form $f : R \to R$. the graphs of such functions may be plotted in the Cartesian plane and functions may be identified with their graphs, so the concepts of being one-to-one and onto have some geometrical meaning :

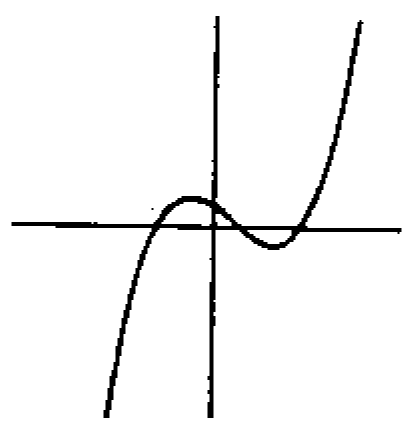
(1) $f : R \to R$ is said to be one-to-one if there are no 2 distinct pairs (a1,b) and (a2,b) in the graph one-to-one or if each horizontal line intersects the graph of f in at most one point.



$f_2(x) = 2^x$

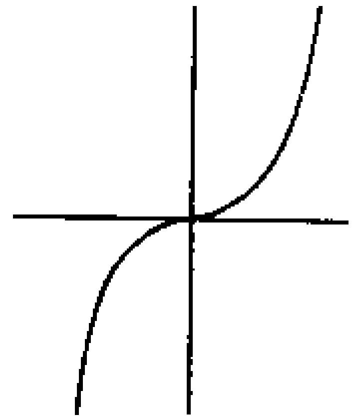
---

(2) $f : R \to R$ is an onto function if each horizontal line intersects the graph of f at one or more points (at least once)

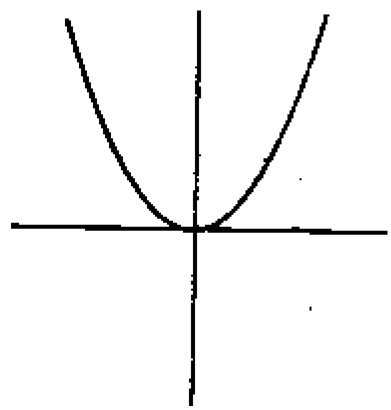


$f_3(x) = x^3 - 2x^2 - 5x + 6$

(3) if f is both one-to-one and onto, i.e. invertible, then each horizontal line will intersect the graph of f <u>at exactly one</u> point.



$f_4(x) = x^3$



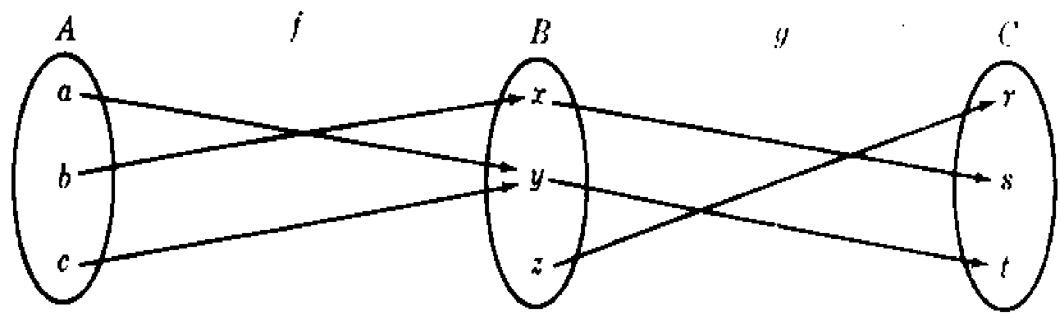
$f_1(x) = x^2$

f(x)NOT (ONE-TO-ONE) & NOT (ONTO)

Composition of function:

Let f:A→B and g:B→C, to find the composition function gof:A→C

$(g_\circ f)(a) = g(f(a)) = g(y) = t$
$(g_\circ f)(b) = g(f(b)) = g(x) = s$
$(g_\circ f)(c) = g(f(c)) = g(y) = t$



EXAMPLE 25
Let f and g be the functions from the set of integers to the set of integers defined by
$f(x) = 2x + 3$ and
$g(x) = 3x + 2$.
What is the composition of f and g?
What is the composition of g and f?

Solution:
Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$
and
$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$.

Some Important Functions
Next, we introduce two important functions in discrete mathematics, namely, the floor and ceiling functions.
Let x be a real number. The floor function rounds x down to the closest integer less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$.
The ceiling function rounds x up to the closest integer greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

These functions are often used when objects are counted. They play an important role in the analysis of the number of steps used by procedures to solve problems of a particular size.

EXAMPLE 26

These are some values of the floor and ceiling functions:

$\lfloor 1/2 \rfloor = 0$

$\lceil 1/2 \rceil = 1$

$\lfloor -1/2 \rfloor = -1$

$\lceil -1/2 \rceil = 0$

$\lfloor 3.1 \rfloor = 3$

$\lceil 3.1 \rceil = 4$

$\lceil 7 \rceil = 7$

EXAMPLE 27

Data stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8 bits. How many bytes are required to encode 100 bits of data?

Solution:

To determine the number of bytes needed, we determine the smallest integer that is at least as large as the quotient when 100 is divided by 8, the number of bits in a byte. Consequently,

$\lceil 100/8 \rceil = \lceil 12.5 \rceil = 13$ bytes are required.

EXAMPLE 28

In asynchronous transfer mode (ATM) (a communications protocol used on networks), data are organized into cells of 53 bytes. How many ATM cells can be transmitted in 1 minute over a connection that transmits data at the rate of 500 kilobits per second?

Solution:

In 1 minute, this connection can transmit:

500,000 x 60 = 30,000,000 bits.

Each ATM cell is 53 bytes long, which means that it is:

53 x 8 = 424 bits long.

To determine the number of cells that can be transmitted in 1 minute, we determine the largest integer not exceeding the quotient when 30,000,000 is divided by 424. Consequently,

30,000,000/424 = 70754.71698113208

$\lfloor 30{,}000{,}000/424 \rfloor$ = 70,754 ATM cells

Sequences of sets

A sequence is a discrete structure used to represent an ordered list.

For example,

1, 2, 3, 5, 8 is a sequence with five terms (called a *list*)

1, 3, 9, 27, 81 , . . . , 3*n*, . . . is an infinite sequence.

A *sequence* is a function from subset of the set of integers (usually either the set {0, 1, 2, . . .} or the set {1, 2, 3, . . .}) to a set *S*.. The notation a_n is used to denote the image of the integer n that called the term of the sequence and used to describe the sequence . Thus a sequence is usually denoted by

$a1, a2, a3, \ldots$

We describe sequences by listing the terms of the sequence in order of increasing subscripts.

EXAMPLE 1

Consider the sequence {*an*}, where

$$a_n = \frac{1}{n};$$

The list of the terms of this sequence, beginning with a_1, namely,

$a_1, a_2, a_3, a_4, \ldots,$

starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

EXAMPLE 2

a-The sequences {bn} with $b_n = (-1)^n$

if we start at n = 0, the list of terms begins with 1,−1, 1,−1, 1, . . .

b-The sequences {cn} with $c_n = 2 \times 5^n$

if we start at n = 0, the list of terms begins with

2, 10, 50, 250, 1250, . . .

c- The sequences {dn} with $d_n = 6 \times (1/3)^n$

if we start at n = 0, The list of terms begins with

$$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots.$$

d- The sequences {bn} with $b_n = 2^{-n}$

if we start at n = 0, The list of terms begins with

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

e- The sequences {bn} with $a_n = \frac{1}{n}$

if we start at n = 1, The list of terms begins with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

Summation Symbol, Sums

Here we introduce the summation symbol $\sum$ (the Greek letter sigma). Consider a sequence $a1$, $a2$, $a3$, ...Then we define the following:

$$\sum_{j=1}^{n} aj = a1 + a2 + \cdot \quad \cdot \quad \cdot + an$$

EXAMPLE 3:

$$\sum_{j=2}^{5} j^{2} = 2^2 + 3^2 + 4^2 + 5^2 = 4 + 9 + 16 + 25 = 54$$

$$\sum_{j=1}^{n} j = 1 + 2 + \cdot \quad \cdot + n = n(n + 1)/2,$$

for example, $1 + 2 + \cdot \quad \cdot + 50 = (50 \times 51)/2 = 1275$

Recurrence Relations

another way to specify a sequence is to provide one or more initial terms together with a rule for determining subsequent terms from those that precede them.

EXAMPLE 4

Let $\{an\}$ be a sequence that satisfies the recurrence relation

$a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots,$

and suppose that $a_0 = 2$. What are $a1$, $a2$, and $a3$?

Solution:
We see from the recurrence relation that
$a1 = a0 + 3 = 2 + 3 = 5$. It then follows that
$a_2 = 5 + 3 = 8$ and
$a_3 = 8 + 3 = 11$.

EXAMPLE 5
Let $\{an\}$ be a sequence that satisfies the recurrence relation
$a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots,$
and suppose that
$a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Solution:
We see from the recurrence relation that
$a_2 = a_1 - a_0 = 5 - 3 = 2$ and
$a_3 = a_2 - a_1 = 2 - 5 = -3$.
We can find $a4$, $a5$, and each successive term in a similar way.

EXAMPLE 6
What is the value of

$$\sum_{k=4}^{8} (-1)^k \ ?$$

Solution:
We have

$$\sum_{k=4}^{8}(-1)^k = (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8$$

$$= 1 + (-1) + 1 + (-1) + 1$$
$$= 1.$$

RECURSIVELY DEFINED FUNCTIONS

A function is said to be *recursively defined* if the function definition refers to itself. In order for the definition not to be circular, the function definition must have the following two properties:

(1) There must be certain arguments, called *base values*, for which the function does not refer to itself.

(2) Each time the function does refer to itself, the argument of the function must be closer to a base value.

A recursive function with these two properties is said to be *well-defined*.

Factorial Function

The product of the positive integers from 1 to n, inclusive, is called "n factorial" and is usually denoted by $n!$. That is,

$$n! = n(n-1)(n-2) \cdot \cdot \cdot 3 \cdot 2 \cdot 1$$

where

0! = 1, so that the function is defined for all nonnegative integers.

Thus:

We have: f(0) = 0! = 1

$$f(1) = 1! = 1,$$
$$f(2) = 2! = 1 \cdot 2 = 2,$$
$$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720,$$

and

f (20) = 1x 2 x3x 4 x 5 x 6 x 7 x 8 x 9 x 10 x 11x 12 x 13 x 14 x 15x16 x 17 x 18 x 19 x20 = 2,432,902,008,176,640,000.

the factorial function grows extremely rapidly as n grows.

This is true for every positive integer n; that is,

$$n! = n \cdot (n-1)!$$

Accordingly, the factorial function may also be defined as follows:

Definition of Factorial Function:

(a) If $n = 0$, then $n! = 1$.

(b) If $n > 0$, then $n! = n \cdot (n - 1)!$

The definition of $n!$ is recursive, since it refers to itself when it uses $(n - 1)!$. However:
(1) The value of $n!$ is explicitly given when $n = 0$ (thus 0 is a base value).
(2) The value of $n!$ for arbitrary n is defined in terms of a smaller value of n which is closer to the base value 0.

Accordingly, the definition is not circular, or, in other words, the function is well-defined.

EXAMPLE 7: the 4! Can be calculated in 9 steps using the recursive definition .

(1) $4! = 4 \cdot 3!$

(2) $3! = 3 \cdot 2!$

(3) $2! = 2 \cdot 1!$

(4) $1! = 1 \cdot 0!$

(5) $0! = 1$

(6) $1! = 1 \cdot 1 = 1$

(7) $2! = 2 \cdot 1 = 2$

(8) $3! = 3 \cdot 2 = 6$

(9) $4! = 4 \cdot 6 = 24$

Fibonacci Sequence

The Fibonacci sequence is a particularly useful sequence that is important for many applications, including modeling the population growth of rabbits. It is usually denoted by F0, F1, F2, . . and can be defined by:

$$0, \quad 1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad 21, \quad 34, \quad 55, \quad \ldots$$

That is, F0 = 0 and F1 = 1 and each succeeding term is the sum of the two preceding terms. For example, the next two terms of the sequence are

$$34 + 55 = 89 \text{ and}$$
$$55 + 89 = 144$$

Fibonacci Sequence can be defined:

(a) If n = 0, or n = 1, then Fn = n.

(b) If n > 1, then Fn = $F_{n-1} + F_{n-2}$.

Where : The base values are 0 and 1, and the value of Fn is defined in terms of smaller values of n which are closer to the base values. Accordingly, this function is well-defined.

EXAMPLE 8

Suppose that a person deposits $10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Solution:

To solve this problem, let *Pn* denote the amount in the account after *n* years. Because the amount in the account after *n* years equals the amount in the account after *n*−1 years plus interest for the *n*th year, we see that the sequence {*Pn*} satisfies the recurrence relation

$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}$.

The initial condition is *P*0 = 10,000.

We can use an iterative approach to find a formula for *Pn*. Note that

$P_1 = (1.11).P_0$

$P_2 = (1.11).P_1 = (1.11)^2.P_0$

$P_3 = (1.11)P_2 = (1.11)^3 P_0$
...
$P_n = (1.11)P_{n-1} = (1.11)^n P_0$.

When we insert the initial condition $P_0 = 10{,}000$, the formula $Pn = (1.11)^n\ 10{,}000$ is obtained.

Inserting $n = 30$ into the formula $Pn = (1.11)^n\ 10{,}000$ shows that after 30 years the account contains

$P_{30} = (1.11)^{30}\ 10{,}000 = \$228{,}922.97$.

Special Integer Sequences

A common problem in discrete mathematics is finding a closed formula, a recurrence relation, or some other type of general rule for constructing the terms of a sequence. Sometimes only a few terms of a sequence solving a problem are known; the goal is to identify the sequence.
When trying to deduce a possible formula, recurrence relation, or some other type of rule
for the terms of a sequence when given the initial terms, try to find a pattern in these terms.

EXAMPLE 9

Find formulae for the sequences with the following first five terms:
(a) 1, 1/2, 1/4, 1/8, 1/16
(b) 1, 3, 5, 7, 9
(c) 1, −1, 1, −1, 1.

Solution:
(a) We recognize that the denominators are powers of 2. The sequence with $an = 1/2^n$,
$n = 0, 1, 2, \ldots$ is a possible match.

(b) We note that each term is obtained by adding 2 to the previous term. The sequence

with $a_n = 2n + 1$, $n = 0, 1, 2, \ldots$ is a possible match.

(c) The terms alternate between 1 and −1. The sequence with $a_n = (-1)^n$, $n = 0, 1, 2 \ldots$ is a possible match.

EXAMPLE 10

How can we produce the terms of a sequence if the first 10 terms are 1, 2, 2, 3, 3, 3, 4, 4, 4, 4?

Solution:

In this sequence, the integer 1 appears once, the integer 2 appears twice, the integer 3 appears three times, and the integer 4 appears four times. A reasonable rule for generating this sequence is that the integer n appears exactly n times,
so the next five terms of the sequence would all be 5,
the following six terms would all be 6, and so on. The sequence generated this way is a possible match.

EXAMPLE 11

How can we produce the terms of a sequence if the first 10 terms are 5, 11, 17, 23, 29, 35, 41, 47, 53, 59?

Solution:

Note that each of the first 10 terms of this sequence after the first is obtained by adding 6 to the previous term. (We could see this by noticing that the difference between consecutive terms is 6.) Consequently, the nth term could be produced by starting with 5 and adding 6 a total of $n - 1$ times; that is, a reasonable guess is that the nth term is $5 + 6(n - 1) = 6n - 1$.

EXAMPLE 12

How can we produce the terms of a sequence if the first 10 terms are 1, 3, 4, 7, 11, 18, 29, 47, 76, 123?

Solution:

Observe that each successive term of this sequence, starting with the third term, is the sum of the two previous terms. That is,

$4 = 3 + 1$,
$7 = 4 + 3$,
$11 = 7 + 4$, and so on.

Consequently, if L_n is the nth term of this sequence, we guess that the sequence is determined by the recurrence relation

$L_n = L_{n-1} + L_{n-2}$

with initial conditions $L_1 = 1$ and $L_2 = 3$ (the same recurrence relation as the Fibonacci sequence, but with different initial conditions). This sequence is known as the Lucas sequence.