The Laplace transform

Definition:

Suppose that $f(t)$ is a piecewise continuous function. The Laplace transform of $f(t)$ is denoted $\mathcal{L}\{f(t)\}$ and defined as

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

There is an alternate notation for Laplace transforms. For the sake of convenience, we will often denote Laplace transforms as,

$$\mathcal{L}\{f(t)\} = F(s)$$

Ex1: compute $\mathcal{L}\{1\}$.

Sol: let $f(t) = 1$ then

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt$$

$$= \lim_{A\to\infty} \int_0^A e^{-st}\, dt$$

$$= \left[\frac{-1}{s} e^{-st}\right]_0^A = \lim_{A\to\infty} \left[\frac{-1}{s} e^{-st}\right]_0^A$$

$$= \frac{-1}{s}\left[\lim_{A\to\infty} e^{-sA} - 1\right]$$

$$= \frac{-1}{s}\left[e^{-s\infty} - 1\right]$$

$$= \frac{-1}{s}[0 - 1]$$

$$= \frac{1}{s}$$

$$\therefore \mathcal{L}\{1\} = \frac{1}{s}$$

Ex2: compute $\mathcal{L}\{e^{at}\}$.

Sol:

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}\, e^{at}\, dt$$

$$= \int_0^\infty e^{(a-s)t}\, dt$$

$$= \frac{1}{(a-s)} \int_0^\infty (a-s) e^{(a-s)t}\, dt$$

$$= \frac{1}{(a-s)} \left[e^{(a-s)t}\right]_0^\infty$$

If $a - s > 0$, then $a > s$ it does not have limit (diverge)

If $a - s < 0$, then $s > a$ it has limit (converge)

$$= \frac{1}{(a-s)} (-1)$$

$$= \frac{1}{s-a}, s > a$$

$$\therefore \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

Ex3: compute $\mathcal{L}\{\sin at\}$.

Sol:

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st}\ \sin at\ \ dt = y$$

$$= \lim_{A\to\infty} \int_0^A e^{-st}\ \sin at\ \ dt$$

Integral by part

$$u = \sin at\,, \qquad dv = \ e^{-st}$$

$$du = a\cos at\,, \qquad v = \frac{-1}{s} e^{-st}$$

$$y = \frac{-1}{s} e^{-st} \sin at - \int_0^\infty (\frac{-1}{s} e^{-st})\ (\ a\cos at)\ \ dt$$

$$= \frac{-e^{-st}}{s} \sin at + \frac{a}{s}\int_0^\infty e^{-st}\ \cos at\ \ dt$$

We solve $\frac{a}{s}\int_0^\infty e^{-st}\ \cos at\ \ dt$

$$u = \cos at\,, \qquad dv = \ e^{-st}$$

$$du = -a\sin at\,, \qquad v = \frac{-1}{s} e^{-st}$$

$$\frac{a}{s}\int_0^\infty e^{-st}\ \cos at\ \ dt = \frac{a}{s}[\frac{-1}{s} e^{-st}\ \cos at - \int_0^\infty \frac{-1}{s} e^{-st}\ (-a\sin at)dt]$$

$$= \frac{-a}{s^2} e^{-st}\ \cos at - \frac{a^2}{s^2}\int_0^\infty e^{-st}\ \sin at\ \ dt$$

$since\ y = \int_0^\infty e^{-st}\ \sin at\ \ dt$

$$y = \frac{-1}{s} e^{-st} \sin at - \frac{a}{s^2} e^{-st} \cos at - \frac{a^2}{s^2} y$$

$$y + \frac{a^2}{s^2} y = \frac{-1}{s} e^{-st} \sin at - \frac{a}{s^2} e^{-st} \cos at$$

$$\left(\frac{s^2 + a^2}{s^2}\right) y = \frac{-1}{s} e^{-st} \sin at - \frac{a}{s^2} e^{-st} \cos at$$

$$= \left[-e^{-st} (\frac{1}{s} \sin at + \frac{a}{s^2} \cos at \right]_0^\infty$$

$$= 0 - (-1) \left(0 + \frac{a}{s^2}\right) = \frac{a}{s^2}$$

$$y = \left(\frac{s^2}{s^2 + a^2}\right) \left(\frac{a}{s^2}\right) = \frac{a}{s^2 + a^2}$$

$$\therefore \mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$$