Matrices: An *m* x *n matrix A* is a rectangular array of numbers, real or complex, with m rows and *n* columns. We shall write *dij* for the number that appears in the ith row and the *jth* column of *A;* this is called the *(i,j) entry* of *A.* We can either write *A* in the extended form

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Ex:

1- $\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$ 2- $[[3]]$ 3- $\begin{bmatrix} \sqrt{2} & 0.5 & 3 \\ 7 & 6 & 1+i \\ 18 & 20 & -3 \end{bmatrix}$

4- $\begin{bmatrix} 12 & 4 \\ 34 & 23 \\ 2 & -4 \end{bmatrix}$ 5- $\begin{bmatrix} 3 & 9 & 27 \end{bmatrix}$ 6- $\begin{bmatrix} -8 \\ 4 \\ -2 \end{bmatrix}$

Some special matrices

(i) A 1 x *n* matrix, or *n* — *row vector, A* has a single row *A* = (a_{11} a_{12} ... a_{1n}*)*

(ii) An *m* x 1 matrix, or *m-column vector, B* has just one column

$$B = \begin{bmatrix} b_{11} \\ \vdots \\ b_{m1} \end{bmatrix}$$

(iii) A matrix with the same number of rows and columns is said to be *square.*

(iv) A *zero matrix* is a matrix all of whose entries are zero. The zero m x *n* matrix is denoted by 0_{nm}

Ex: $0_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(v) *The identity nxn matrix* has l's on *the principal diagonal,* that is, from top left to bottom right, and zeros elsewhere; thus it has the form

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The identity matrix plays the role of the number 1 in matrix multiplication.

(vi) A square matrix is called *upper triangular* if it has only zero entries below the principal diagonal. Similarly, a matrix is *lower triangular* if all entries above the principal diagonal are zero. For example, the matrices

$$\begin{pmatrix} 1 & 3 \\ 0 & 5 \end{pmatrix} \; and \; \begin{pmatrix} a & 0 & 0 \\ e & b & 0 \\ f & g & d \end{pmatrix}$$

are respectively diagonal and scalar

(vii) A square matrix in which all the non-zero elements lie on the principal diagonal is called a *diagonal matrix.*

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

A scalar matrix is a diagonal matrix in which the elements on the principal diagonal are all equal.

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$$

Operations with Matrices

1- Addition and subtraction

Let *A* and *B* be two *mxn* matrices; as usual write *aij* and *bij* for their respective *(i,j)* entries. Define the *sum A + B* to be the *mxn* matrix whose *(i,j)* entry is aij + *bij;* thus to form the matrix *A + B we* simply add corresponding entries of *A* and *B.* Similarly, the *difference A — B* is the *mxn* matrix whose *(i,j)* entry is aij — *bij*. However, *A + B* and *A — B* are not defined if *A* and *B* do not have the same numbers of rows and columns.

Ex:

if $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \; and \; B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \end{pmatrix}$ find A+B, and B-A

Sol:

$$A + B = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ -1 & -3 & 2 \end{pmatrix}$$

$$B - A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -3 & 0 \end{pmatrix}$$

B+A, A-B H.W

2- Scalar multiplication

By a *scalar* we shall mean a number, as opposed to a matrix or array of numbers. Let c be a scalar and *A an mxn* matrix. The *scalar multiple cA* is the *mxn* matrix whose *(i, j)* entry is *caij*. Thus to form *cA* we multiply every entry of *A* by the scalar c. The matrix (- l)A is usually written *-A;* it is called the *negative* of *A* since it has the property that

A + *(-A)* = 0

Ex: find 2A, and 3B

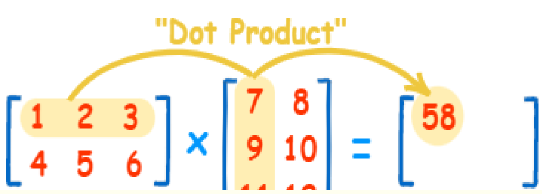
Sol: $2A = \begin{pmatrix} 2 & 4 & 0 \\ -2 & 0 & 2 \end{pmatrix}, 3B = A = \begin{pmatrix} 3 & 3 & 3 \\ 0 & -9 & 3 \end{pmatrix}$

Ex: find 2A+3B, and 2A-3B H.W.

3- *Matrix multiplication*

When, we want to multiple between two matrices, should follow these steps

1- The number of rows for the first matrix = The number of columns for the second matrix.
2- We need to do the dot products of rows and columns. For example



$(1, 2, 3) \bullet (7, 9, 11) = 1\times7 + 2\times9 + 3\times11 = 58$ — The 1^{st} row and 1^{st} column

$(1, 2, 3) \bullet (8, 10, 12) = 1\times8 + 2\times10 + 3\times12 = 64$ — The 1^{st} row and 2^{nd} column

$(4, 5, 6) \bullet (7, 9, 11) = 4\times7 + 5\times9 + 6\times11 = 139$ — The 2^{nd} row and 1^{st} column

$(4, 5, 6) \bullet (8, 10, 12) = 4\times8 + 5\times10 + 6\times12 = 154$ — The 2^{nd} row and 2^{nd} column

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

Ex: if $A = \begin{bmatrix} 2 & 4 & 6 \\ -1 & 32 & 2 \\ 1 & 0 & 5 \end{bmatrix}, and\ B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

Find a) 3A x 2B H.W

b) can we do B x A, why? H.W

4- Power of a matrix

Once matrix products have been defined, it is clear how to define a non-negative power of a square matrix. Let *A* be an *n* x *n* matrix; then the mth *power* of *A*, where *m* is a nonnegative integer, is defined by the equations

$$A^0 = I, and\ A^{m+1} = A^m A.$$

Ex: if $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Then

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, A^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, and\ A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

5- The transpose of matrix

If *A* is an *m* x n matrix, the *transpose* of *A*,

$$A^T$$

is the *n* x *m* matrix whose *(i,j)* entry equals the *(j,i)* entry of *A*. Thus the columns of become the rows of A^T.

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\ then\ \ A^T = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$$

Note: A matrix which equals its transpose is called *symmetric*.

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}\ then\ \ A^T = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

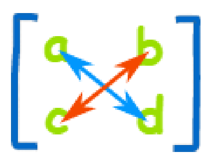
Theorem:
(a) A + B = B + A, {commutative law of addition)]
(b) (A + B) + C = A + (B + C), (associative law of addition);
(c) A + 0 = A;
(d) (AB)C = A(BC), (associative law of multiplication)]
(e) AI = A = I A;
(f) A(B + C) = AB + AC, {distributive law);
(g) (A + B)C = AC + BC, (distributive law);
(h) A-B = A + (-1)B;
(i) (cd)A = c(dA);
(i)c(AB) = (cA)B = A(cB);
(k) c(A + B) = cA + cB;
(1) (c + d)A = cA + dA;
(m) $(A+B)^T = A^T + B^T$;
(n) $(AB)^T = B^T A^T$.

Determinant of matrix
The determinant is a **special number** that can be calculated from a matrix. The matrix has to be square (same number of rows and columns)

1- If matrix has dimension 2x2 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
then the determinant $|A| = ad - cb$

It is easy to remember when you think of a cross:

- Blue is positive (+ad),
- Red is negative (−bc)



Ex: find the determinant of $C = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$
Sol: $|C| = (4*8) - (3*6)$
$= 32 - 18$
$= 14$

2- If matrix has 3x3 or more $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$
The determinant $|A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$
$$|A| = a(ei - hf) - b(di - gf) + c(dh - ge)$$

Example:

$$D = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

$$\begin{aligned} |D| &= 6\times(-2\times7 - 5\times8) - 1\times(4\times7 - 5\times2) + 1\times(4\times8 - (-2\times2)) \\ &= 6\times(-54) - 1\times(18) + 1\times(36) \\ &= \mathbf{-306} \end{aligned}$$

Note: It follows that the coefficient of a_{ik} in $\det(A)$ is $(-1)^{i+k}M_{ik}$, which is just the definition of A_{ik}.

Cofactors:
The (i,j)*cofactor* A_{ik} of A is simply the minor with an appropriate sign:

$$A_{ij} = (-1)^{i+j}M_{ij}$$

For example, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

Then $M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

and $A_{23} = (-1)^{2+3}M_{23} = -(a_{11}a_{32} - a_{31}a_{12}) = a_{31}a_{12} - a_{11}a_{32}$

The adjoint matrix
Let $A = a_{ij}$ be an *n* x *n* matrix. Then the *adjoint matrix*

$$adj(A)$$

of *A* is defined to be the *nxn* matrix whose *(i,j)* element is the *(j,i)* cofactor *Aji* of *A*. Thus $adj(A)$ *is the transposed matrix of cofactors of A.*

Ex: if $A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 3 \\ 2 & -3 & 4 \end{bmatrix}$ find the $adj(A)$

Sol: first we need to find cofactor for each element of A,

$$A_{11} = (-1)^{1+1}M_{11} = \begin{vmatrix} -1 & 3 \\ -3 & 4 \end{vmatrix} = 5$$

$$A_{12} = (-1)^{1+2}M_{12} = (-1)\begin{vmatrix} 6 & 3 \\ 2 & 4 \end{vmatrix} = -18$$

$$A_{13} = (-1)^{1+3}M_{13} = \begin{vmatrix} 6 & -1 \\ 2 & -3 \end{vmatrix} = -16$$

the same way we find the other elements.

$A_{21} = -11, A_{22} = 2, A_{23} = 7, A_{31} = 7, A_{32} = 3, and A_{33} = -13$

And then we find the transpose for the new matrix

$$\therefore adj(A) = \begin{bmatrix} 5 & -11 & 7 \\ -18 & 2 & 3 \\ -16 & 7 & -13 \end{bmatrix}$$

The inverse of matrix

If A is an invertible matrix, then $A^{-1} = (1/\det(A))adj(A)$.

Ex: if $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ find A^{-1}

Sol: the adjoint of A is

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Then $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$