Taylor and Maclaurin Series

Definitions of Taylor Series and Maclaurin Series

We start by supposing that f is a function that can be represented by a power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots \quad |x-a| < R \ldots (1)$$

We can differentiate the series in equation 1 term by term

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \quad |x-a| < R \ldots (2)$$

And substation x = a in eq. (2) gives

$$f'(x) = c_1$$

Now we differentiate both sides of Equation 2 and obtain

$$f''(x) = 2c_2 + 6c_3(x-a)^1 + 12c_4(x-a)^3 + \cdots \quad |x-a| < R \ldots (3)$$

Again we put x − a in Equation 3. The result is

$$f''(x) = 2c_2$$

Let's apply the procedure one more time. Differentiation of the series in Equation 3 gives

$$f'''(x) = 6c_3 + 36c_4(x-a)^4 + \cdots \quad |x-a| < R \ldots (4)$$

And substation x = a in eq. (4) gives

$$f'''(x) = 6c_3 = 3!\, c_3$$

By now you can see the pattern. If we continue to differentiate and substitute $x = a$, we obtain

$$f^n(a) = 2 \cdot 3 \cdot 4 \cdot \cdots \cdot nc_n = n!\, c_n$$

Solving this equation for the nth coefficient c_n, we get

$$c_n = \frac{f^n(a)}{n!}$$

This formula remains valid even for $n = 0$ if we adopt the conventions that $0! = 1$ and $f^{(0)}(a) = f$. Thus we have proved the following theorem

Theorem: If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^n(a)}{n!} \ldots (5)$$

Substituting this formula for c_n back into the series, we see that if f(x) has a power series expansion at a, then it must be of the following form.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \quad (6)$$

The series in Equation 6 is called the **Taylor series of the function f at a.** For the special case $a = 0$ the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(o)}{n!}(x)^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots (7)$$

This special case is called **Maclaurin series**

Ex: if $f(x) = \frac{1}{x-1}$ find Maclaurin series.

Sol: we compute

$$f(x) = \frac{1}{1-x} \qquad f(0) = 1$$

$$f'(x) = \frac{1}{(1-x)^2} \qquad f'(0) = 1$$

$$f''(x) = \frac{1 \cdot 2}{(1-x)^3} \qquad f''(0) = 1 \cdot 2$$

$$f'''(x) = \frac{1 \cdot 2 \cdot 3}{(1-x)^4} \qquad f'''(0) = 1 \cdot 2 \cdot 3$$

$$\frac{1}{x-1} = 1 + \frac{1}{1!}x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \cdots = 1 + x + x^2 + x^3 + \cdots = \sum_{n-0}^{\infty} x^n$$

To find the radius of convergence we let $a_n = x^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = |x| < 1$$

The $R = 1$

Ex: For the function $f(x) = e^x$, find the Maclaurin series and its radius of convergence. H.W

Note: there are three cases for radius (R)

1- $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 0$ this mean $R = \infty$ converge for all x
2- $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \infty$ this mean $R = \infty$ converge at $x = c$
3- $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{R}|x - c| < 1$ this mean $|x - c| < R$

Ex: For the function $f(x) = \sin x$, find the Maclaurin series.
Sol: We arrange our computation in two columns:

$$
\begin{aligned}
f(x) &= \sin x & f(0) &= 0 \\
f'(x) &= \cos x & f'(0) &= 1 \\
f''(x) &= -\sin x & f''(0) &= 0 \\
f'''(x) &= -\cos x & f'''(0) &= -1 \\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0
\end{aligned}
$$

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Ex: For the function $f(x) = \cos x$, find the Maclaurin series. H.W.

The Binomial Series If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Where $\binom{k}{n} = \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}$

Ex: For the function $\frac{1}{\sqrt{4-x}}$, find the Maclaurin series and its radius of convergence.

Sol: We rewrite f(x) in a form where we can use the binomial series:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\dfrac{x}{4}\right)}} = \frac{1}{2\sqrt{1-\dfrac{x}{4}}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n$$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-\frac{x}{4}\right)^3\right.$$

$$\left. + \cdots + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}\left(-\frac{x}{4}\right)^n + \cdots\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{8}x + \frac{1\cdot 3}{2!8^2}x^2 + \frac{1\cdot 3\cdot 5}{3!8^3}x^3 + \cdots + \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!8^n}x^n + \cdots\right]$$

Using the binomial series with $k = \frac{-1}{2}$, and with x replaced by $\frac{-x}{4}$, we have

Note: the binomial series always converges when $|x| < 1$.

Then by this note, this series is convergence when $\left|\frac{-x}{4}\right| < 1$, that is $|x| < 4$, the radius of convergence is $R = 4$.