Exponential Functions

The function $f(x) = 2^x$ is called an *exponential function* because the variable, x, is the exponent. It should not be confused with the power function $g(x) = x^2$, in which the variable is the base.

An exponential function is a function of the form

$$f(x) = b^x$$

Where b is a positive constant.

- If $x = n$, a positive integer, then $b^n = b.b \ldots \ldots .b$
- If $x = 0$, then $b^0 = 1$, and if $x = -n$, where n is a positive integer, then

$$b^{-n} = \frac{1}{b^n}$$

- If $x$is a rational number, $x = {}^p/_q$, where p and q are integers and $q > 0$, then

$$b^x = b^{p/q} = \sqrt[q]{b^p} = (\sqrt[q]{b})^p$$

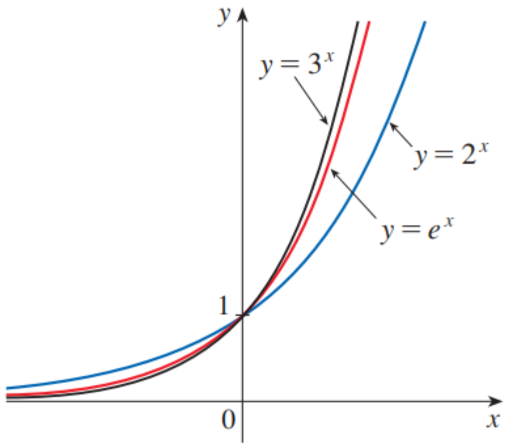
Laws of Exponents If a and b are positive numbers and x and y are any real numbers, then

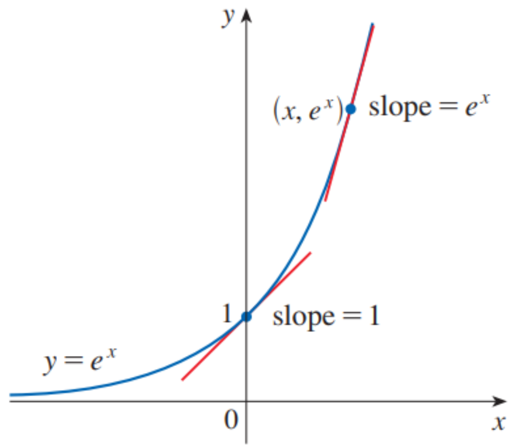
1. $b^{x+y} = b^x b^y$ **2.** $b^{x-y} = \dfrac{b^x}{b^y}$ **3.** $(b^x)^y = b^{xy}$ **4.** $(ab)^x = a^x b^x$

The number *e*

Def: e is the number such that $\lim_{h\to 0}\frac{e^h-1}{h}=1$.

Geometrically, this means that of all possible exponential functions $y=b^x$, the function $f(x)=e^x$ is the one whose tangent line at (0, 1) has a slope $f'(0)=1$. See fig. below





Derivative of natural exponential number function

$$\frac{d}{dx}(e^x)=e^x$$

If $f(x)=e^x-x$, find f' and f''. Compare the graphs of f and f'.

Sol:

Using the Difference Rule, we have

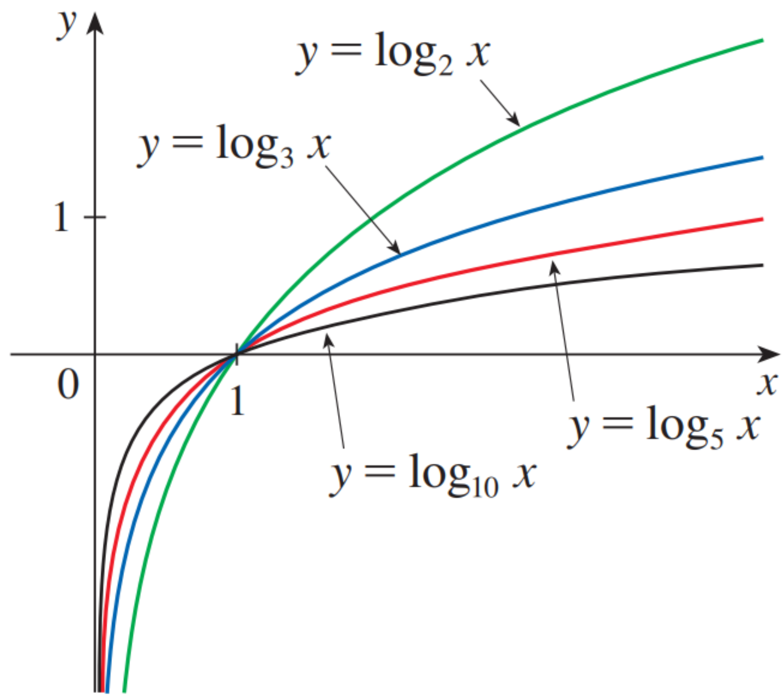
$$f'(x)=\frac{d}{dx}(e^x-x)=\frac{d}{dx}(e^x)-\frac{d}{dx}(x)=e^x-1$$

the second derivative as the derivative of f', so

$$f''(x)=\frac{d}{dx}(e^x-1)=\frac{d}{dx}(e^x)-\frac{d}{dx}(1)=e^x$$

Logarithmic function

The logarithmic function $f(x) = \log_b x$, where the base b is a positive constant, are the inverse functions of the exponential functions. Fig. below shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.



If we use the formulation of an inverse function

$$\boxed{\log_b x = y \iff b^y = x}$$

For example, $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$

Laws of Logarithms If x and y are positive numbers, then

1. $\log_b(xy) = \log_b x + \log_b y$
2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b(x^r) = r \log_b x$ (where r is any real number)

Ex: use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$

Sol: $\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right)$

$= \log_2 16$

$= 4$

Natural logarithms

If the base is the number e of the logarithm, is called the natural algorithm

$$\log_e x = \ln x$$

Then, the defining properties of the natural algorithm function become

$$\boxed{\ln x = y \iff e^y = x}$$

$$\boxed{\begin{array}{ll} \ln(e^x) = x & x \in \mathbb{R} \\ e^{\ln x} = x & x > 0 \end{array}}$$

In particular, if we set x=1, we get

$$\boxed{\ln e = 1}$$

From the property $e^{\ln x} \; = x$, we will get

$$x^r = \; e^{\mathrm{r}\ln x}$$

Ex: solve the equation $e^{5-3x} \; = 10$

Sol: we take ln of both sides of the equation

$$\ln(e^{5-3x}) \; = \ln(10)$$

$$5 - 3x = \ln 10$$

$$3x = 5 - \ln 10$$

$$x = \frac{1}{3}(5 - \ln 10)$$

Using a calculator, we get

$$x \approx 0.8991$$

Finally, we can take $\log_b x = \frac{\ln x}{\ln b}$

Ex: evaluate $\log_8 5$ correct to 6 decimal places.

Sol: $\log_8 5 = \frac{\ln 5}{\ln 8} \approx 0.773976$

Derivatives of logarithmic functions

The derivative of a logarithmic fun.

$$\frac{\partial}{\partial x}(\log_b x) = \frac{1}{x \ln b}$$

If we put $b = e$, we get

$$\frac{\partial}{\partial x}(\ln x) = \frac{1}{x}$$

In general, by using chain rule, we get

$$\frac{\partial}{\partial x}(\ln u) = \frac{1}{u} \cdot \frac{\partial u}{\partial x}$$

Ex: differentiate $f(x) = \sqrt{\ln x}$

Sol:

$$f'(x) = \tfrac{1}{2}(\ln x)^{-1/2}\frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$