Exercise Workbook to Accompany A Gentle Introduction to the Art of Mathematics

Version 3.2

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`http://www.southernct.edu/~fields/`.

The pdf and the source code repository can also be found on GitHub at

`http://http://osj1961.github.io/giam/`.

Chapter 1

Introduction and notation

1.1 Basic sets

Exercises — 1.1

1. Each of the quantities indexing the rows of the following table is in one or more of the sets which index the columns. Place a check mark in a table entry if the quantity is in the set.

	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$
17					
π					
$22/7$					
-6					
e^0					
$1+i$					
$\sqrt{3}$					
i^2					

2. Write the set $\mathbb{Z}$ of integers using a singly infinite listing.

3. Identify each as rational or irrational.

   (a) $5021.2121212121\ldots$

   (b) $0.2340000000\ldots$

   (c) $12.31331133311133331111\ldots$

   (d) π

   (e) $2.98765432198765432198765432 1\ldots$

4. The "see and say" sequence is produced by first writing a 1, then iterating the following procedure: look at the previous entry and say how many entries there are of each integer and write down what you just said. The first several terms of the "see and say" sequence are $1, 11, 21, 1112, 3112, 211213, 312213, 212223, \ldots$. Comment on the rationality (or irrationality) of the number whose decimal digits are obtained by concatenating the "see and say" sequence.

$$0.1112111123112211213...$$

5. Give a description of the set of rational numbers whose decimal expansions terminate. (Alternatively, you may think of their decimal expansions ending in an infinitely-long string of zeros.)

6. Find the first 20 decimal places of π, $3/7$, $\sqrt{2}$, $2/5$, $16/17$, $\sqrt{3}$, $1/2$ and $42/100$. Classify each of these quantity's decimal expansion as: terminating, having a repeating pattern, or showing no discernible pattern.

7. Consider the process of long division. Does this algorithm give any insight as to why rational numbers have terminating or repeating decimal expansions? Explain.

8. Give an argument as to why the product of two rational numbers is again a rational.

9. Perform the following computations with complex numbers

   (a) $(4 + 3i) - (3 + 2i)$

   (b) $(1 + i) + (1 - i)$

   (c) $(1 + i) \cdot (1 - i)$

   (d) $(2 - 3i) \cdot (3 - 2i)$

10. The *conjugate* of a complex number is denoted with a superscript star, and is formed by negating the imaginary part. Thus if $z = 3 + 4i$ then the conjugate of z is $z^* = 3 - 4i$. Give an argument as to why the product of a complex number and its conjugate is a real quantity. (I.e. the imaginary part of $z \cdot z^*$ is necessarily 0, no matter what complex number is used for z.)

1.2 Definitions: Prime numbers

Exercises — 1.2

1. Find the prime factorizations of the following integers.

   (a) 105

   (b) 414

   (c) 168

   (d) 1612

   (e) 9177

2. Use the sieve of Eratosthenes to find all prime numbers up to 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

3. What would be the largest prime one would sieve with in order to find all primes up to 400?

4. Characterize the prime factorizations of numbers that are perfect squares.

5. Complete the following table which is related to the conjecture that whenever p is a prime number, $2^p - 1$ is also a prime.

p	$2^p - 1$	prime?	factors
2	3	yes	1 and 3
3	7	yes	1 and 7
5	31	yes	
7	127		
11			

6. Find a counterexample for the conjecture that $x^2 - 31x + 257$ evaluates to a prime number whenever x is a natural number.

7. Use the second definition of "prime" to see that 6 is not a prime. In other words, find two numbers (the a and b that appear in the definition) such that 6 is not a factor of either, but *is* a factor of their product.

8. Use the second definition of "prime" to show that 35 is not a prime.

9. A famous conjecture that is thought to be true (but for which no proof is known) is the Twin Prime conjecture. A pair of primes is said to be twin if they differ by 2. For example, 11 and 13 are twin primes, as are 431 and 433. The Twin Prime conjecture states that there are an infinite number of such twins. Try to come up with an argument as to why 3, 5 and 7 are the only prime triplets.

10. Another famous conjecture, also thought to be true – but as yet unproved, is Goldbach's conjecture. Goldbach's conjecture states that every even number greater than 4 is the sum of two odd primes. There is a function $g(n)$, known as the Goldbach function, defined on the positive integers, that gives the number of different ways to write a given number as the sum of two odd primes. For example $g(10) = 2$ since $10 = 5 + 5 = 7 + 3$. Thus another version of Goldbach's conjecture is that $g(n)$ is positive whenever n is an even number greater than 4.

    Graph $g(n)$ for $6 \leq n \leq 20$.

1.3 More scary notation

Exercises — 1.3

1. How many quantifiers (and what sorts) are in the following sentence?

   "Everybody has *some* friend that thinks they know everything about a sport."

2. The sentence "Every metallic element is a solid at room temperature." is false. Why?

3. The sentence "For every pair of (distinct) real numbers there is another real number between them." is true. Why?

4. Write your own sentences containing four quantifiers. One sentence in which the quantifiers appear ($\forall\exists\forall\exists$) and another in which they appear ($\exists\forall\exists\forall$).

1.4 Definitions of elementary number theory

Exercises — 1.4

1. An integer n is *doubly-even* if it is even, and the integer m guaranteed to exist because n is even is itself even. Is 0 doubly-even? What are the first 3 positive, doubly-even integers?

2. Dividing an integer by two has an interesting interpretation when using binary notation: simply shift the digits to the right. Thus, $22 = 10110_2$ when divided by two gives 1011_2 which is $8 + 2 + 1 = 11$. How can you recognize a doubly-even integer from its binary representation?

3. The *octal* representation of an integer uses powers of 8 in place notation. The digits of an octal number run from 0 to 7, one never sees 8's or 9's. How would you represent 8 and 9 as octal numbers? What octal number comes immediately after 777_8? What (decimal) number is 777_8?

4. One method of converting from decimal to some other base is called *repeated division*. One divides the number by the base and records the remainder – one then divides the quotient obtained by the base and records the remainder. Continue dividing the successive quotients by the base until the quotient is smaller than the base. Convert 3267 to base-7 using repeated division. Check your answer by using the meaning of base-7 place notation. (For example 54321_7 means $5 \cdot 7^4 + 4 \cdot 7^3 + 3 \cdot 7^2 + 2 \cdot 7^1 + 1 \cdot 7^0$.)

5. State a theorem about the octal representation of even numbers.

6. In hexadecimal (base-16) notation one needs 16 "digits," the ordinary digits are used for 0 through 9, and the letters A through F are used to give single symbols for 10 through 15. The first 32 natural number in hexadecimal are: 1,2,3,4,5,6,7,8,9,A,B,C,D,E,F,10,11,12,13,14,15,16, 17,18,19,1A, 1B,1C,1D,1E,1F,20.

   Write the next 10 hexadecimal numbers after AB.

   Write the next 10 hexadecimal numbers after FA.

7. For conversion between the three bases used most often in Computer Science we can take binary as the "standard" base and convert using a table look-up. Each octal digit will correspond to a binary triple, and each hexadecimal digit will correspond to a 4-tuple of binary numbers. Complete the following tables. (As a check, the 4-tuple next to A in the table for hexadecimal should be 1010 – which is nice since A is really 10 so if you read that as "ten-ten" it is a good aid to memory.)

octal	binary
0	000
1	001
2	
3	
4	
5	
6	
7	

hexadecimal	binary
0	0000
1	0001
2	0010
3	
4	
5	
6	
7	
8	
9	
A	
B	
C	
D	
E	
F	

8. Use the tables from the previous problem to make the following conversions.

   (a) Convert 757_8 to binary.

   (b) Convert 1007_8 to hexadecimal.

   (c) Convert 100101010110_2 to octal.

   (d) Convert 1111101000110101_2 to hexadecimal.

   (e) Convert $FEED_{16}$ to binary.

   (f) Convert $FFFFFF_{16}$ to octal.

9. Try the following conversions between various number systems:

   (a) Convert 30 (base 10) to binary.

   (b) Convert 69 (base 10) to base 5.

   (c) Convert 1222_3 to binary.

   (d) Convert 1234_7 to base 10.

   (e) Convert $EEED_{15}$ to base 12. (Use $\{1, 2, 3 \ldots 9, d, e\}$ as the digits in base 12.)

   (f) Convert 678_9 to hexadecimal.

10. It is a well known fact that if a number is divisible by 3, then 3 divides the sum of the (decimal) digits of that number. Is this result true in base 7? Do you think this result is true in *any* base?

11. Suppose that 340 pounds of sand must be placed into bags having a 50 pound capacity. Write an expression using either floor or ceiling notation for the number of bags required.

12. True or false?

$$\left\lfloor \frac{n}{d} \right\rfloor < \left\lceil \frac{n}{d} \right\rceil$$

for all integers n and $d > 0$. Support your claim.

13. What is the value of $\lceil \pi \rceil^2 - \lceil \pi^2 \rceil$?

14. Assuming the symbols n,d,q and r have meanings as in the quotient-remainder theorem (see page 29 of GIAM). Write expressions for q and r, in terms of n and d using floor and/or ceiling notation.

15. Calculate the following quantities:

    (a) $3 \mod 5$

    (b) $37 \mod 7$

    (c) $1000001 \mod 100000$

    (d) 6 div 6

    (e) 7 div 6

    (f) 1000001 div 2

16. Calculate the following binomial coefficients:

    (a) $\binom{3}{0}$

    (b) $\binom{7}{7}$

    (c) $\binom{13}{5}$

    (d) $\binom{13}{8}$

    (e) $\binom{52}{7}$

17. An ice cream shop sells the following flavors: chocolate, vanilla, strawberry, coffee, butter pecan, mint chocolate chip and raspberry. How many different bowls of ice cream – with three scoops – can they make?

1.5 Some algorithms of elementary number theory

Exercises — 1.5

1. Trace through the division algorithm with inputs $n = 27$ and $d = 5$, each time an assignment statement is encountered write it out. How many assignments are involved in this particular computation?

2. Find the gcd's and lcm's of the following pairs of numbers.

a	b	$\gcd(a, b)$	$\mathrm{lcm}(a, b)$
110	273		
105	42		
168	189		

3. Formulate a description of the gcd of two numbers in terms of their prime factorizations in the general case (when the factorizations may include powers of the primes involved).

4. Trace through the Euclidean algorithm with inputs $a = 3731$ and $b = 2730$, each time the assignment statement that calls the division algorithm is encountered write out the expression $a = qb + r$. (With the actual values involved !)

1.6 Rational and irrational numbers

Exercises — 1.6

1. Rational Approximation is a field of mathematics that has received much study. The main idea is to find rational numbers that are very good approximations to given irrationals. For example, 22/7 is a well-known rational approximation to π. Find good rational approximations to $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and e.

2. The theory of base-n notation that we looked at in the sub-section on base-n can be extended to deal with real and rational numbers by introducing a decimal point (which should probably be re-named in accordance with the base) and adding digits to the right of it. For instance 1.1011 is binary notation for $1 \cdot 2^0 + 1 \cdot 2^{-1} + 0 \cdot 2^{-2} + 1 \cdot 2^{-3} + 1 \cdot 2^{-4}$ or $1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{16} = 1\frac{11}{16}$.
   Consider the binary number .1010010001000010000010000001 . . ., is this number rational or irrational? Why?

3. If a number x is even, it's easy to show that its square x^2 is even. The lemma that went unproved in this section asks us to start with a square (x^2) that is even and deduce that the UN-squared number (x) is even. Perform some numerical experimentation to check whether this assertion is reasonable. Can you give an argument that would prove it?

4. The proof that $\sqrt{2}$ is irrational can be generalized to show that $\sqrt{p}$ is irrational for every prime number p. What statement would be equivalent to the lemma about the parity of x and x^2 in such a generalization?

5. Write a proof that $\sqrt{3}$ is irrational.

1.7 Relations

Exercises — 1.7

1. Consider the numbers from 1 to 10. Give the set of pairs of these numbers that corresponds to the divisibility relation.

2. The *domain* of a function (or binary relation) is the set of numbers appearing in the first coordinate. The *range* of a function (or binary relation) is the set of numbers appearing in the second coordinate.

   Consider the set $\{0, 1, 2, 3, 4, 5, 6\}$ and the function $f(x) = x^2 \pmod 7$. Express this function as a relation by explicitly writing out the set of ordered pairs it contains. What is the range of this function?

3. What relation on the numbers from 1 to 10 does the following set of ordered pairs represent?

$$\begin{gathered}
\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(1,9),(1,10),\\
(2,2),(2,3),(2,4),(2,5),(2,6),(2,7),(2,8),(2,9),(2,10),\\
(3,3),(3,4),(3,5),(3,6),(3,7),(3,8),(3,9),(3,10),\\
(4,4),(4,5),(4,6),(4,7),(4,8),(4,9),(4,10),\\
(5,5),(5,6),(5,7),(5,8),(5,9),(5,10),\\
(6,6),(6,7),(6,8),(6,9),(6,10),\\
(7,7),(7,8),(7,9),(7,10),\\
(8,8),(8,9),(8,10),\\
(9,9),(9,10),\\
(10,10)\}
\end{gathered}$$

4. Draw a five-pointed star, label all 10 points. There are 40 triples of these labels that satisfy the betweenness relation. List them.

5. Sketch a graph of the relation

$$\{(x, y) \,|\, x, y \in \mathbb{R} \text{ and } y > x^2\}.$$

6. A function $f(x)$ is said to be *invertible* if there is another function $g(x)$ such that $g(f(x)) = x$ for all values of x. (Usually, the inverse function, $g(x)$ would be denoted $f^{-1}(x)$.) Suppose a function is presented to you as a relation – that is, you are just given a set of pairs. How can you distinguish whether the function represented by this list of input/output pairs is invertible? How can you produce the inverse (as a set of ordered pairs)?

7. There is a relation known as "has color" which goes from the set

$$F = \{orange, cherry, pumpkin, banana\}$$

to the set

$$C = \{orange, red, green, yellow\}.$$

What pairs are in "has color"?

Chapter 2

Logic and quantifiers

2.1 Predicates and Logical Connectives

Exercises — 2.1

1. Design a digital logic circuit (using and, or & not gates) that implements an exclusive or.

2. Consider the sentence “This is a sentence which does not refer to itself.” which was given in the beginning of this chapter as an example. Is this sentence a statement? If so, what is its truth value?

3. Consider the sentence “This sentence is false.” Is this sentence a statement?

4. Complete truth tables for each of the sentences $(A \wedge B) \vee C$ and $A \wedge (B \vee C)$. Does it seem that these sentences have the same logical content?

5. There are two other logical connectives that are used somewhat less commonly than $\vee$ and $\wedge$. These are the Scheffer stroke and the Peirce arrow – written $|$ and $\downarrow$, respectively — they are also known as NAND and NOR.

   The truth tables for these connectives are:

A	B	$A \mid B$
T	T	ϕ
T	ϕ	T
ϕ	T	T
ϕ	ϕ	T

   and

A	B	$A \downarrow B$
T	T	ϕ
T	ϕ	ϕ
ϕ	T	ϕ
ϕ	ϕ	T

   Find an expression for $(A \wedge \neg B) \vee C$ using only these new connectives (as well as negation and the variable symbols themselves).

6. The famous logician Raymond Smullyan devised a family of logical puzzles around a fictitious place he called "the Island of Knights and Knaves." The inhabitants of the island are either knaves, who always make false statements, or knights, who always make truthful statements.

   In the most famous knight/knave puzzle, you are in a room which has only two exits. One leads to certain death and the other to freedom. There are two individuals in the room, and you know that one of them is a knight and the other is a knave, but you don't know which. Your challenge is to determine the door which leads to freedom by asking a single question.

2.2 Implication

Exercises — 2.2

1. The transitive property of equality says that if $a = b$ and $b = c$ then $a = c$. Does the implication arrow satisfy a transitive property? If so, state it.

2. Complete truth tables for the compound sentences $A \implies B$ and $\neg A \vee B$.

3. Complete a truth table for the compound sentence $A \implies (B \implies C)$ and for the sentence $(A \implies B) \implies C$. What can you conclude about conditionals and the associative property?

4. Determine a sentence using the *and* connector ($\wedge$) that gives the negation of $A \implies B$.

5. Rewrite the sentence "Fix the toilet or I won't pay the rent!" as a conditional.

6. Why is it that the sentence "If pigs can fly, I am the king of Mesopotamia." true?

7. Express the statement $A \implies B$ using the Peirce arrow and/or the Scheffer stroke. (See Exercise 5 in the previous section.)

8. Find the contrapositives of the following sentences.

   (a) If you can't do the time, don't do the crime.

   (b) If you do well in school, you'll get a good job.

   (c) If you wish others to treat you in a certain way, you must treat others in that fashion.

   (d) If it's raining, there must be clouds.

   (e) If $a_n \leq b_n$, for all n and $\sum_{n=0}^{\infty} b_n$ is a convergent series, then $\sum_{n=0}^{\infty} a_n$ is a convergent series.

9. What are the converse and inverse of "If you watch my back, I'll watch your back."?

10. The integral test in Calculus is used to determine whether an infinite series converges or diverges: Suppose that $f(x)$ is a positive, decreasing, real-valued function with $\lim_{x\longrightarrow\infty} f(x) = 0$, if the improper integral $\int_0^\infty f(x)$ has a finite value, then the infinite series $\sum_{n=1}^\infty f(n)$ converges. The integral test should be envisioned by letting the series correspond to a right-hand Riemann sum for the integral, since the function is decreasing, a right-hand Riemann sum is an underestimate for the value of the integral, thus

$$\sum_{n=1}^{\infty} f(n) < \int_0^\infty f(x).$$

Discuss the meanings of and (where possible) provide justifications for the inverse, converse and contrapositive of the conditional statement in the integral test.

11. On the Island of Knights and Knaves (see page 31) you encounter two individuals named Locke and Demosthenes.

    Locke says, "Demosthenes is a knave."
    Demosthenes says "Locke and I are knights."

    Who is a knight and who a knave?

2.3 Logical equivalences

Exercises — 2.3

1. There are 3 operations used in basic algebra (addition, multiplication and exponentiation) and thus there are potentially 6 different distributive laws. State all 6 “laws” and determine which 2 are actually valid. (As an example, the distributive law of addition over multiplication would look like $x + (y \cdot z) = (x + y) \cdot (x + z)$, this isn’t one of the true ones.)

2. Use truth tables to verify or disprove the following logical equivalences.

   (a) $(A \wedge B) \vee B \;\cong\; (A \vee B) \wedge B$

   (b) $A \wedge (B \vee \neg A) \;\cong\; A \wedge B$

   (c) $(A \wedge \neg B) \vee (\neg A \wedge \neg B) \cong (A \vee \neg B) \wedge (\neg A \vee \neg B)$

   (d) The absorption laws.

3. Draw pairs of related digital logic circuits that illustrate DeMorgan's laws.

4. Find the negation of each of the following and simplify as much as possible.

   (a) $(A \vee B) \iff C$

   (b) $(A \vee B) \implies (A \wedge B)$

5. Because a conditional sentence is equivalent to a certain disjunction, and because DeMorgan's law tells us that the negation of a disjunction is a conjunction, it follows that the negation of a conditional is a conjunction. Find denials (the negation of a sentence is often called its "denial") for each of the following conditionals.

   (a) "If you smoke, you'll get lung cancer."

   (b) "If a substance glitters, it is not necessarily gold."

   (c) "If there is smoke, there must also be fire."

   (d) "If a number is squared, the result is positive."

   (e) "If a matrix is square, it is invertible."

6. The so-called “ethic of reciprocity” is an idea that has come up in many of the world’s religions and philosophies. Below are statements of the ethic from several sources. Discuss their logical meanings and determine which (if any) are logically equivalent.

   (a) “One should not behave towards others in a way which is disagreeable to oneself.” Mencius Vii.A.4 (Hinduism)

   (b) “None of you [truly] believes until he wishes for his brother what he wishes for himself.” Number 13 of Imam “Al-Nawawi’s Forty Hadiths.” (Islam)

   (c) “And as ye would that men should do to you, do ye also to them likewise.” Luke 6:31, King James Version. (Christianity)

   (d) “What is hateful to you, do not to your fellow man. This is the law: all the rest is commentary.” Talmud, Shabbat 31a. (Judaism)

   (e) “An it harm no one, do what thou wilt” (Wicca)

   (f) “What you would avoid suffering yourself, seek not to impose on others.” (the Greek philosopher Epictetus – first century A.D.)

   (g) “Do not do unto others as you expect they should do unto you. Their tastes may not be the same.” (the Irish playwright George Bernard Shaw – 20th century A.D.)

7. You encounter two natives of the land of knights and knaves. Fill in an explanation for each line of the proofs of their identities.

   (a) Natasha says, "Boris is a knave."
   Boris says, "Natasha and I are knights."

   Claim: Natasha is a knight, and Boris is a knave.

   Proof: If Natasha is a knave, then Boris is a knight.
   If Boris is a knight, then Natasha is a knight.
   Therefore, if Natasha is a knave, then Natasha is a knight.
   Hence Natasha is a knight.
   Therefore, Boris is a knave.

   Q.E.D.

   (b) Bonaparte says "I am a knight and Wellington is a knave."
   Wellington says "I would tell you that B is a knight."

   Claim: Bonaparte is a knight and Wellington is a knave.

   Proof: Either Wellington is a knave or Wellington is a knight.
   If Wellington is a knight it follows that Bonaparte is a knight.
   If Bonaparte is a knight then Wellington is a knave.
   So, if Wellington is a knight then Wellington is a knave (which is impossible!)
   Thus, Wellington is a knave.
   Since Wellington is a knave, his statement "I would tell you that Bonaparte is a knight" is false.
   So Wellington would in fact tell us that Bonaparte is a knave.
   Since Wellington is a knave we conclude that Bonaparte

is a knight.
Thus Bonaparte is a knight and Wellington is a knave (as claimed).

Q.E.D.

2.4 Two-column proofs

Exercises — 2.4

Write two-column proofs that verify each of the following logical equivalences.

1. $A \vee (A \wedge B) \;\cong\; A \wedge (A \vee B)$

2. $(A \wedge \neg B) \vee A \;\cong\; A$

3. $A \vee B \;\cong\; A \vee (\neg A \wedge B)$

4. $\neg(A \vee \neg B) \vee (\neg A \wedge \neg B) \;\cong\; \neg A$

5. $A \;\cong\; A \wedge ((A \vee \neg B) \vee (A \vee B))$

6. $(A \wedge \neg B) \wedge (\neg A \vee B) \;\cong\; c$

7. $A \;\cong\; A \wedge (A \vee (A \wedge (B \vee C)))$

8. $\neg(A \wedge B) \wedge \neg(A \wedge C) \;\cong\; \neg A \vee (\neg B \wedge \neg C)$

2.5 Quantified statements

Exercises — 2.5

1. There is a common variant of the existential quantifier, $\exists!$, if you write $\exists!\, x,\ P(x)$ you are asserting that there is a *unique* element in the universe that makes $P(x)$ true. Determine how to negate the sentence $\exists!\, x,\ P(x)$.

2. The order in which quantifiers appear is important. Let $L(x, y)$ be the open sentence "x is in love with y." Discuss the meanings of the following quantified statements and find their negations.

   (a) $\forall x\, \exists y\ L(x, y)$.

   (b) $\exists x\, \forall y\ L(x, y)$.

   (c) $\forall x\, \forall y\ L(x, y)$.

   (d) $\exists x\, \exists y\ L(x, y)$.

3. Determine a useful denial of:

   $\forall \epsilon > 0\, \exists \delta > 0\, \forall x\, (|x - c| < \delta) \implies (|f(x) - l| < \epsilon).$

   The denial above gives a criterion for saying $\lim_{x \to c} f(x) \neq l$.

4. A *Sophie Germain prime* is a prime number p such that the corresponding odd number $2p + 1$ is also a prime. For example 11 is a Sophie Germain prime since $23 = 2 \cdot 11 + 1$ is also prime. Almost all Sophie Germain primes are congruent to $5 \pmod 6$, nevertheless, there are exceptions – so the statement "There are Sophie Germain primes that are not 5 mod 6." is true. Verify this.

5. Alvin, Betty, and Charlie enter a cafeteria which offers three different entrees, turkey sandwich, veggie burger, and pizza; four different beverages, soda, water, coffee, and milk; and two types of desserts, pie and pudding. Alvin takes a turkey sandwich, a soda, and a pie. Betty takes a veggie burger, a soda, and a pie. Charlie takes a pizza and a soda. Based on this information, determine whether the following statements are true or false.

   (a) $\forall$ people p, $\exists$ dessert d such that p took d.

   (b) $\exists$ person p such that $\forall$ desserts d, p did not take d.

   (c) $\forall$ entrees e, $\exists$ person p such that p took e.

   (d) $\exists$ entree e such that $\forall$ people p, p took e.

   (e) $\forall$ people p, p took a dessert $\iff$ p did not take a pizza.

   (f) Change one word of statement 5d so that it becomes true.

(g) Write down the negation of 5a and compare it to statement 5b.

Hopefully you will see that they are the same! Does this make you want to modify one or both of your answers to 5a and 5b?

2.6 Deductive reasoning and Argument forms

Exercises — 2.6

1. In the movie "Monty Python and the Holy Grail" we encounter a medieval villager who (with a bit of prompting) makes the following argument.

   If she weighs the same as a duck, then she's made of wood.
   If she's made of wood then she's a witch.
   Therefore, if she weighs the same as a duck, she's a witch.

   Which rule of inference is he using?

2. In constructive dilemma, the antecedent of the conditional sentences are usually chosen to represent opposite alternatives. This allows us to introduce their disjunction as a tautology. Consider the following proof that there is never any reason to worry (found on the walls of an Irish pub).

   Either you are sick or you are well.
   If you are well there's nothing to worry about.
   If you are sick there are just two possibilities:
   Either you will get better or you will die.
   If you are going to get better there's nothing to worry about.
   If you are going to die there are just two possibilities:
   Either you will go to Heaven or to Hell.
   If you go to Heaven there is nothing to worry about. If you go to Hell, you'll be so busy shaking hands with all your friends there won't be time to worry . . .

   Identify the three tautologies that are introduced in this "proof."

3. For each of the following arguments, write it in symbolic form and determine which rules of inference are used.

   (a) You are either with us, or you're against us. And you don't appear to be with us. So, that means you're against us!

   (b) All those who had cars escaped the flooding. Sandra had a car – therefore, Sandra escaped the flooding.

   (c) When Johnny goes to the casino, he always gambles 'til he goes broke. Today, Johnny has money, so Johnny hasn't been to the casino recently.

   (d) (A non-constructive proof that there are irrational numbers a and b such that a^b is rational.) Either $\sqrt{2}^{\sqrt{2}}$ is rational or it is irrational. If $\sqrt{2}^{\sqrt{2}}$ is rational, we let $a = b = \sqrt{2}$. Otherwise, we let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. (Since $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}} = 2$, which is rational.) It follows that in either case, there are irrational numbers a and b such that a^b is rational.

2.7 Validity of arguments and common errors

Exercises — 2.7

1. Determine the logical form of the following arguments. Use symbols to express that form and determine whether the form is valid or invalid. If the form is invalid, determine the type of error made. Comment on the soundness of the argument as well, in particular, determine whether any of the premises are questionable.

   (a) All who are guilty are in prison.
   George is not in prison.
   Therefore, George is not guilty.

   (b) If one eats oranges one will have high levels of vitamin C.
   You do have high levels of vitamin C.
   Therefore, you must eat oranges.

(c) All fish live in water.
The mackerel is a fish.
Therefore, the mackerel lives in water.

(d) If you're lazy, don't take math courses.
Everyone is lazy.
Therefore, no one should take math courses.

(e) All fish live in water.
The octopus lives in water.
Therefore, the octopus is a fish.

(f) If a person goes into politics, they are a scoundrel.
Harold has gone into politics.
Therefore, Harold is a scoundrel.

2. Below is a rule of inference that we call extended elimination.

$$\begin{array}{ll} & (A \vee B) \vee C \\ & \neg A \\ & \neg B \\ \hline \therefore & C \end{array}$$

Use a truth table to verify that this rule is valid.

3. If we allow quantifiers and open sentences in an argument form we get a couple of new argument forms. Arguments involving existentially quantified premises are rare – the new forms we are speaking of are called "universal modus ponens" and "universal modus tollens." The minor premises may also be quantified or they may involve particular elements of the universe of discourse – this leads us to distinguish argument subtypes that are termed "universal" and "particular."

   For example $\begin{array}{ll} & \forall x, A(x) \implies B(x) \\ & A(p) \\ \hline \therefore & B(p) \end{array}$ is the particular form of universal modus ponens (here, p is not a variable – it stands for some particular element of the universe of discourse) and $\begin{array}{ll} & \forall x, A(x) \implies B(x) \\ & \forall x, \neg B(x) \\ \hline \therefore & \forall x, \neg A(x) \end{array}$ is the universal form of (universal) modus tollens.

   Reexamine the arguments from problem (1), determine their forms (including quantifiers) and whether they are universal or particular.

4. Identify the rule of inference being used.

   (a) The Buley Library is very tall.
   Therefore, either the Buley Library is very tall or it has many levels underground.

   (b) The grass is green.
   The sky is blue.
   Therefore, the grass is green and the sky is blue.

   (c) g has order 3 or it has order 4.
   If g has order 3, then g has an inverse.
   If g has order 4, then g has an inverse.
   Therefore, g has an inverse.

   (d) x is greater than 5 and x is less than 53.
   Therefore, x is less than 53.

   (e) If $a|b$, then a is a perfect square.
   If $a|b$, then b is a perfect square.
   Therefore, if $a|b$, then a is a perfect square and b is a perfect square.

5. Read the following proof that the sum of two odd numbers is even. Discuss the rules of inference used.

   Proof: Let x and y be odd numbers. Then $x = 2k + 1$ and $y = 2j + 1$ for some integers j and k. By algebra,

   $$x + y = 2k + 1 + 2j + 1 = 2(k + j + 1).$$

   Note that $k+j+1$ is an integer because k and j are integers. Hence $x + y$ is even.

   Q.E.D.

6. Sometimes in constructing a proof we find it necessary to "weaken" an inequality. For example, we might have already deduced that $x < y$ but what we need in our argument is that $x \leq y$. It is okay to deduce $x \leq y$ from $x < y$ because the former is just shorthand for $x < y \vee x = y$. What rule of inference are we using in order to deduce that $x \leq y$ is true in this situation?

Chapter 3

Proof techniques I — Standard methods

As a convenience, the table containing the definitions of elementary number theory is reproduced on the following page.

Even

$$\forall n \in \mathbb{Z}, \quad n \text{ is even} \iff \exists k \in \mathbb{Z},\ n = 2k$$

Odd

$$\forall n \in \mathbb{Z}, \quad n \text{ is odd} \iff \exists k \in \mathbb{Z},\ n = 2k + 1$$

Divisibility

$$\forall n \in \mathbb{Z}, \forall \quad d > 0 \in \mathbb{Z}, \quad d \,|\, n \iff \exists k \in \mathbb{Z},\ n = kd$$

Floor

$$\forall x \in \mathbb{R}, \quad y = \lfloor x \rfloor \iff y \in \mathbb{Z} \ \wedge \ y \leq x < y + 1$$

Ceiling

$$\forall x \in \mathbb{R}, \quad y = \lceil x \rceil \iff y \in \mathbb{Z} \ \wedge \ y - 1 < x \leq y$$

Quotient-remainder theorem, Div and Mod

$$\forall n, d > 0 \in \mathbb{Z},$$
$$\exists! q, r \in \mathbb{Z},\ n = qd + r \ \wedge \ 0 \leq r < d$$
$$n \text{ div } d = q$$
$$n \bmod d = r$$

Prime

$$\forall p \in \mathbb{Z}$$
$$p \text{ is prime} \iff$$
$$(p > 1) \quad \wedge \quad (\forall x, y \in \mathbb{Z}^+,\ p = xy \implies x = 1 \vee y = 1)$$

Table 3.1: The definitions of elementary number theory restated.

3.1 Direct proofs of universal statements

Exercises — 3.1

1. Every prime number greater than 3 is of one of the two forms $6k+1$ or $6k+5$. What statement(s) could be used as hypotheses in proving this theorem?

2. Prove that 129 is odd.

3. Prove that the sum of two rational numbers is a rational number.

4. Prove that the sum of an odd number and an even number is odd.

5. Prove that if the sum of two integers is even, then so is their difference.

6. Prove that for every real number x, $\frac{2}{3} < x < \frac{3}{4} \implies \lfloor 12x \rfloor = 8$.

7. Prove that if x is an odd integer, then x^2 is of the form $4k+1$ for some integer k.

8. Prove that for all integers a and b, if a is odd and $6 \mid (a+b)$, then b is odd.

9. Prove that $\forall x \in \mathbb{R},\, x \notin \mathbb{Z} \implies \lfloor x \rfloor + \lfloor -x \rfloor = -1$.

10. Define the *evenness* of an integer n by:

$$\text{evenness}(n) = k \iff 2^k \,|\, n \,\wedge\, 2^{k+1} \nmid n$$

State and prove a theorem concerning the evenness of products.

11. Suppose that a, b and c are integers such that $a \mid b$ and $b \mid c$. Prove that $a \mid c$.

12. Suppose that a, b, c and d are integers with $a \neq c$. Further, suppose that x is a real number satisfying the equation

$$\frac{ax + b}{cx + d} = 1.$$

Show that x is rational. Where is the hypothesis $a \neq c$ used?

13. Show that if two positive integers a and b satisfy $a \mid b$ *and* $b \mid a$ then they are equal.

3.2 More direct proofs

Exercises — 3.2

1. Suppose you have a savings account which bears interest compounded monthly. The July statement shows a balance of \$ 2104.87 and the September statement shows a balance \$ 2125.97. What would be the balance on the (missing) August statement?

2. Recall that a quadratic equation $ax^2 + bx + c = 0$ has two real solutions if and only if the discriminant $b^2 - 4ac$ is positive. Prove that if a and c have different signs then the quadratic equation has two real solutions.

3. Prove that if $x^3 - x^2$ is negative then $3x + 4 < 7$.

4. Prove that for all integers a, b, and c, if $a|b$ and $a|(b + c)$, then $a|c$.

5. Show that if x is a positive real number, then $x + \frac{1}{x} \geq 2$.

6. Prove that for all real numbers a, b, and c, if $ac < 0$, then the quadratic equation $ax^2 + bx + c = 0$ has two real solutions.
   Hint: The quadratic equation $ax^2 + bx + c = 0$ has two real solutions if and only if $b^2 - 4ac > 0$ and $a \neq 0$.

7. Show that $\binom{n}{k} \cdot \binom{k}{r} = \binom{n}{r} \cdot \binom{n-r}{k-r}$ (for all integers r, k and n with $r \leq k \leq n$).

8. In proving the *product rule* in Calculus using the definition of the derivative, we might start our proof with:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\cdot g(x)\right)$$

$$= \lim_{h\longrightarrow 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

The last two lines of our proof should be:

$$= \lim_{h\longrightarrow 0} \frac{f(x+h) - f(x)}{h}\cdot g(x) \;+\; f(x)\cdot \lim_{h\longrightarrow 0} \frac{g(x+h) - g(x)}{h}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right)\cdot g(x) \;+\; f(x)\cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)\right)$$

Fill in the rest of the proof.

3.3 Indirect proofs: contradiction and contraposition

Exercises — 3.3

1. Prove that if the cube of an integer is odd, then that integer is odd.

2. Prove that whenever a prime p does not divide the square of an integer, it also doesn't divide the original integer. ($p \nmid x^2 \implies p \nmid x$)

3. Prove (by contradiction) that there is no largest integer.

4. Prove (by contradiction) that there is no smallest positive real number.

5. Prove (by contradiction) that the sum of a rational and an irrational number is irrational.

6. Prove (by contraposition) that for all integers x and y, if $x + y$ is odd, then $x \neq y$.

7. Prove (by contraposition) that for all real numbers a and b, if ab is irrational, then a is irrational or b is irrational.

8. A *Pythagorean triple* is a set of three natural numbers, a, b and c, such that $a^2 + b^2 = c^2$. Prove that, in a Pythagorean triple, at least one of a and b is even. Use either a proof by contradiction or a proof by contraposition.

9. Suppose you have 2 pairs of positive real numbers whose products are 1. That is, you have (a, b) and (c, d) in $\mathbb{R}^2$ satisfying $ab = cd = 1$. Prove that $a < c$ implies that $b > d$.

3.4 Disproofs

Exercises — 3.4

1. Find a polynomial that assumes only prime values for a reasonably large range of inputs.

2. Find a counterexample to the conjecture that $\forall a, b, c \in \mathbb{Z}, a \mid bc \implies a \mid b \vee a \mid c$ using only powers of 2.

3. The alternating sum of factorials provides an interesting example of a sequence of integers.

$$1! = 1$$

$$2! - 1! = 1$$

$$3! - 2! + 1! = 5$$

$$4! - 3! + 2! - 1! = 19$$

et cetera

Are they all prime? (After the first two 1's.)

4. It has been conjectured that whenever p is prime, $2^p - 1$ is also prime. Find a minimal counterexample.

5. True or false: The sum of any two irrational numbers is irrational. Prove your answer.

6. True or false: There are two irrational numbers whose sum is rational. Prove your answer.

7. True or false: The product of any two irrational numbers is irrational. Prove your answer.

8. True or false: There are two irrational numbers whose product is rational. Prove your answer.

9. True or false: Whenever an integer n is a divisor of the square of an integer, m^2, it follows that n is a divisor of m as well. (In symbols, $\forall n \in \mathbb{Z}, \forall m \in \mathbb{Z}, n \mid m^2 \implies n \mid m$.) Prove your answer.

10. In an exercise in Section 3.2 we proved that the quadratic equation $ax^2 + bx + c = 0$ has two solutions if $ac < 0$. Find a counterexample which shows that this implication cannot be replaced with a biconditional.

3.5 Even more direct proofs: By cases and By exhaustion

Exercises — 3.5

1. Prove that if n is an odd number then $n^4 \pmod{16} = 1$.

2. Prove that every prime number other than 2 and 3 has the form $6q+1$ or $6q+5$ for some integer q. (Hint: this problem involves thinking about cases as well as contrapositives.)

3. Show that the sum of any three consecutive integers is divisible by 3.

4. There is a graph known as K_4 that has 4 nodes and there is an edge between every pair of nodes. The pebbling number of K_4 has to be at least 4 since it would be possible to put one pebble on each of 3 nodes and not be able to reach the remaining node using pebbling moves. Show that the pebbling number of K_4 is actually 4.

5. Find the pebbling number of a graph whose nodes are the corners and whose edges are the, uhmm, edges of a cube.

6. A *vampire number* is a $2n$ digit number v that factors as $v = xy$ where x and y are n digit numbers and the digits of v are the union of the digits in x and y in some order. The numbers x and y are known as the "fangs" of v. To eliminate trivial cases, pairs of trailing zeros are disallowed.

   Show that there are no 2-digit vampire numbers.

   Show that there are seven 4-digit vampire numbers.

7. Lagrange's theorem on representation of integers as sums of squares says that every positive integer can be expressed as the sum of at most 4 squares. For example, $79 = 7^2 + 5^2 + 2^2 + 1^2$. Show (exhaustively) that 15 can not be represented using fewer than 4 squares.

8. Show that there are exactly 15 numbers x in the range $1 \leq x \leq 100$ that can't be represented using fewer than 4 squares.

9. The *trichotomy property* of the real numbers simply states that every real number is either positive or negative or zero. Trichotomy can be used to prove many statements by looking at the three cases that it guarantees. Develop a proof (by cases) that the square of any real number is non-negative.

10. Consider the game called "binary determinant tic-tac-toe" which is played by two players who alternately fill in the entries of a 3×3 array. Player One goes first, placing 1's in the array and player Zero goes second, placing 0's. Player One's goal is that the final array have determinant 1, and player Zero's goal is that the determinant be 0. The determinant calculations are carried out mod 2.

    Show that player Zero can always win a game of binary determinant tic-tac-toe by the method of exhaustion.

3.6 Proofs and disproofs of existential statements

Exercises — 3.6

1. Show that there is a perfect square that is the sum of two perfect squares.

2. Show that there is a perfect cube that is the sum of three perfect cubes.

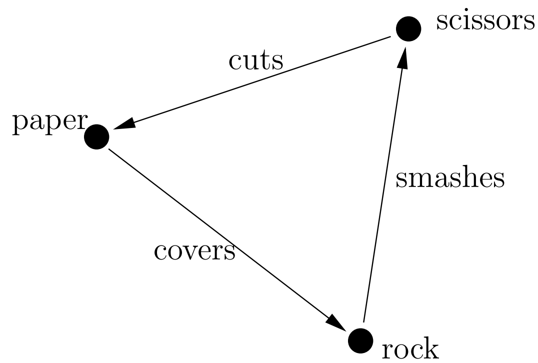
3. Show that the WOP doesn't hold in the integers. (This is an existence proof, you show that there is a subset of $\mathbb{Z}$ that doesn't have a smallest element.)

4. Show that the WOP doesn't hold in $\mathbb{Q}^+$.

5. In the proof of Theorem 3.6.4 we weaseled out of showing that $d \mid b$. Fill in that part of the proof.

6. Give a proof of the unique existence of q and r in the division algorithm.

7. A *digraph* is a drawing containing a collection of points that are connected by arrows. The game known as *scissors-paper-rock* can be represented by a digraph that is *balanced* (each point has the same number of arrows going out as going in). Show that there is a balanced digraph having 5 points.



Chapter 4

Sets

No more turkey, but I'd like some more of the bread it ate. –Hank Ketcham

4.1 Basic notions of set theory

Exercises — 4.1

1. What is the power set of $\emptyset$? Hint: if you got the last exercise in the chapter you'd know that this power set has $2^0 = 1$ element.

2. Try iterating the power set operator. What is $\mathcal{P}(\mathcal{P}(\emptyset))$? What is $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$?

3. Determine the following cardinalities.

   (a) $A = \{1, 2, \{3, 4, 5\}\}$ $\quad |A| =$______

   (b) $B = \{\{1, 2, 3, 4, 5\}\}$ $\quad |B| =$______

4. What, in Logic, corresponds the notion $\emptyset$ in Set theory?

5. What, in Set theory, corresponds to the notion t (a tautology) in Logic?

6. What is the truth set of the proposition $P(x) =$ "3 divides x and 2 divides x"?

7. Find a logical open sentence such that $\{0, 1, 4, 9, \ldots\}$ is its truth set.

8. How many singleton sets are there in the power set of $\{a, b, c, d, e\}$? "Doubleton" sets?

9. How many 8 element subsets are there in

$$\mathcal{P}(\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p\})?$$

10. How many singleton sets are there in the power set of $\{1, 2, 3, \ldots n\}$?

4.2 Containment

Exercises — 4.2

1. Insert either $\in$ or $\subseteq$ in the blanks in the following sentences (in order to produce true sentences).

   i) 1 ______ $\{3, 2, 1, \{a, b\}\}$

   ii) $\{a\}$ ______ $\{a, \{a, b\}\}$

   iii) $\{a, b\}$ ______ $\{3, 2, 1, \{a, b\}\}$

   iv) $\{\{a, b\}\}$ ______ $\{a, \{a, b\}\}$

2. Suppose that p is a prime, for each n in $\mathbb{Z}^+$, define the set $P_n = \{x \in \mathbb{Z}^+ \mid p^n \mid x\}$. Conjecture and prove a statement about the containments between these sets.

3. Provide a counterexample to dispel the notion that a subset must have fewer elements than its superset.

4. We have seen that $A \subseteq B$ corresponds to $M_A \implies M_B$. What corresponds to the contrapositive statement?

5. Determine two sets A and B such that both of the sentences $A \in B$ and $A \subseteq B$ are true.

6. Prove that the set of perfect fourth powers is contained in the set of perfect squares.

	Intersection version	Union version
Commutative laws	$A \cap B = B \cap A$	$A \cup B = B \cup A$
Associative laws	$A \cap (B \cap C) = (A \cap B) \cap C$	$A \cup (B \cup C) = (A \cup B) \cup C$
Distributive laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
DeMorgan's laws	$\overline{A \cap B} = \overline{A} \cup \overline{B}$	$\overline{A \cup B} = \overline{A} \cap \overline{B}$
Double complement	$\overline{\overline{A}} = A$	same
Complementarity	$A \cap \overline{A} = \emptyset$	$A \cup \overline{A} = U$
Identity laws	$A \cap U = A$	$A \cup \emptyset = A$
Domination	$A \cap \emptyset = \emptyset$	$A \cup U = U$
Idempotence	$A \cap A = A$	$A \cup A = A$
Absorption	$A \cap (A \cup B) = A$	$A \cup (A \cap B) = A$

Table 4.1: Basic set theoretic equalities.

4.3 Set operations

Exercises — 4.3

1. Let $A = \{1, 2, \{1, 2\}, b\}$ and let $B = \{a, b, \{1, 2\}\}$. Find the following:

   (a) $A \cap B$

   (b) $A \cup B$

   (c) $A \setminus B$

   (d) $B \setminus A$

   (e) $A \triangle B$

2. In a standard deck of playing cards one can distinguish sets based on face-value and/or suit. Let $A, 2, \ldots 9, 10, J, Q$ and K represent the sets of cards having the various face-values. Also, let $\heartsuit$, $\spadesuit$, $\clubsuit$ and $\diamondsuit$ be the sets of cards having the possible suits. Find the following

   (a) $A \cap \heartsuit$

   (b) $A \cup \heartsuit$

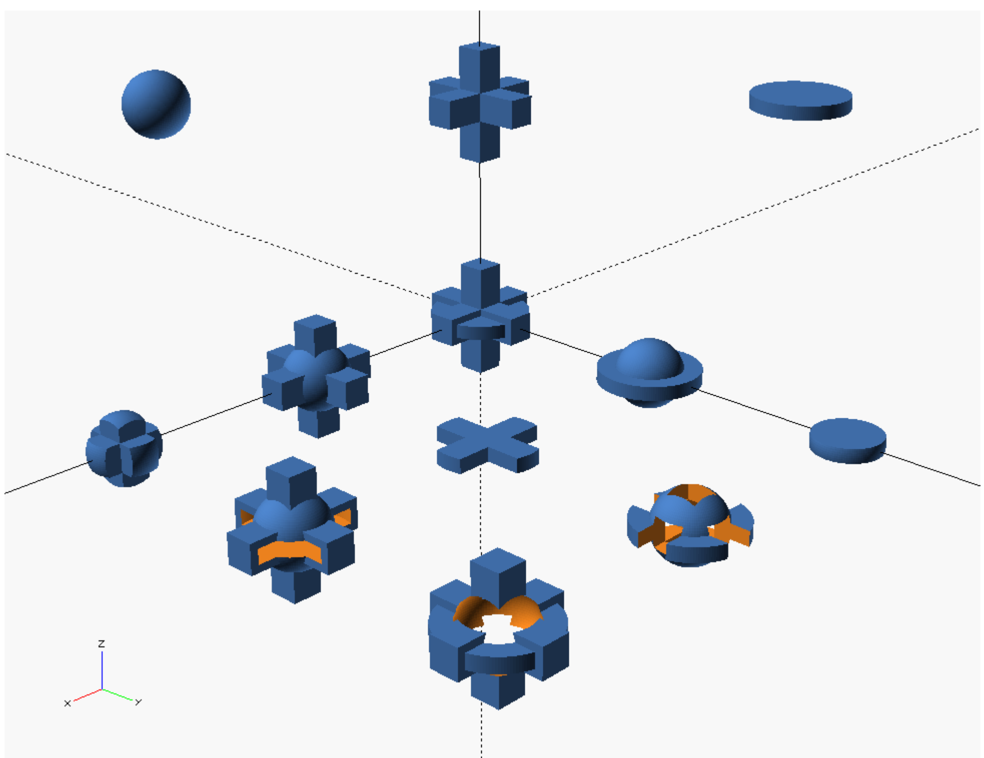
   (c) $J \cap (\spadesuit \cup \heartsuit)$

   (d) $K \cap \heartsuit$

   (e) $A \cap K$

   (f) $A \cup K$

3. The following is a screenshot from the computational geometry program OpenSCAD (very hand for making models for 3-d printing...) In computational geometry we use the basice set operations together with a few other types of transformations to create interesting models using simple components. Across the top of the image below we see 3 sets of points in $\mathbb{R}^3$, a ball, a sort of 3-dimensional plus sign, and a disk. Let's call the ball A, the plus sign B and the disk C. The nine shapes shown below them are made from A, B and C using union, intersection and set difference. Identify them!



4. Do element-chasing proofs (show that an element is in the left-hand side if and only if it is in the right-hand side) to prove each of the following set equalities.

   (a) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

   (b) $A \cup B = A \cup (\overline{A} \cap B)$

   (c) $A \triangle B = (A \cup B) \setminus (A \cap B)$

   (d) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$

5. For each positive integer n, we'll define an interval I_n by

$$I_n = [-n, 1/n).$$

Find the union and intersection of all the intervals in this infinite family.

$$\bigcup_{n\in\mathbb{N}} I_n \quad =$$

$$\bigcap_{n\in\mathbb{N}} I_n \quad =$$

6. There is a set X such that, for all sets A, we have $X \triangle A = A$. What is X?

7. There is a set Y such that, for all sets A, we have $Y \triangle A = \overline{A}$. What is Y?

8. In proving a set-theoretic identity, we are basically showing that two sets are equal. One reasonable way to proceed is to show that each is contained in the other. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by showing that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

9. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ by showing that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

10. Prove the set-theoretic versions of DeMorgan's laws using the technique discussed in the previous problems.

11. The previous technique (showing that $A = B$ by arguing that $A \subseteq B \wedge B \subseteq A$) will have an outline something like

    Proof: First we will show that $A \subseteq B$.
    Towards that end, suppose $x \in A$.

    $$\vdots$$

    Thus $x \in B$.

    Now, we will show that $B \subseteq A$.
    Suppose that $x \in B$.

    $$\vdots$$

    Thus $x \in A$.

    Therefore $A \subseteq B \wedge B \subseteq A$ so we conclude that $A = B$.

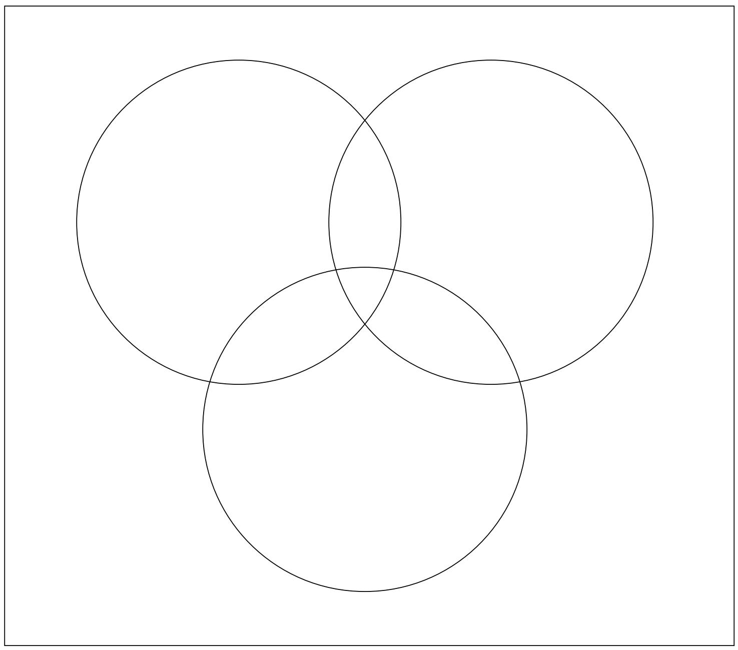
    Q.E.D.

    Formulate a proof that $A \triangle B = (A \cup B) \setminus (A \cap B)$ that follows this outline.

4.4 Venn diagrams

Exercises — 4.4

1. Let $A = \{1, 2, 4, 5\}$, $B = \{2, 3, 4, 6\}$, and $C = \{1, 2, 3, 4\}$. Place each of the elements $1, \ldots, 6$ in the appropriate regions of a three-set Venn diagram.

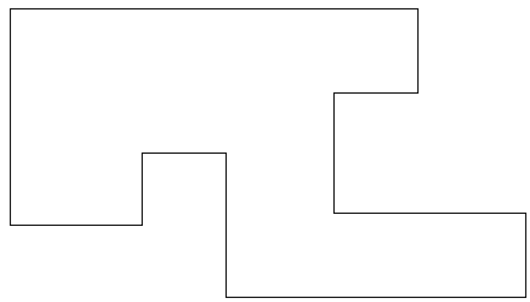


2. Prove or disprove:

$$(A \cap C \subseteq B \cap C) \quad \Longrightarrow \quad A \subseteq B$$

3. Venn diagrams are usually made using simple closed curves with no further restrictions. Try creating Venn diagrams for 3, 4 and 5 sets (in general position) using rectangular simple closed curves.

4. We call a curve *rectilinear* if it is made of line segments that meet at right angles. If you have ever played with an Etch-a-Sketch you'll know what we mean by the term "rectilinear." The following example of a rectilinear curve may also help to clarify this notion.



Use rectilinear simple closed curves to create a Venn diagram for 5 sets.

5. Argue as to why rectilinear curves will suffice to build any Venn diagram.

6. Find the disjunctive normal form of $A \cap (B \cup C)$.

7. Find the disjunctive normal form of $(A \triangle B) \triangle C$

8. The prototypes for the *modus ponens* and *modus tollens* argument forms are the following:

All men are mortal. Socrates is a man. Therefore Socrates is mortal.	and	All men are mortal. Zeus is not mortal. Therefore Zeus is not a man.

Illustrate these arguments using Venn diagrams.

9. Use Venn diagrams to convince yourself of the validity of the following containment statement

$$(A \cap B) \cup (C \cap D) \ \subseteq \ (A \cup C) \cap (B \cup D).$$

Now prove it!

10. Use Venn diagrams to show that the following set equivalence is false.

$$(A \cup B) \cap (C \cup D) \ = \ (A \cup C) \cap (B \cup D)$$

4.5 Russell's Paradox

Exercises — 4.5

1. Verify that $(A \implies \neg A) \wedge (\neg A \implies A)$ is a logical contradiction in two ways: by filling out a truth table and using the laws of logical equivalence.

2. One way out of Russell's paradox is to declare that the collection of sets that don't contain themselves as elements is not a set itself. Explain how this circumvents the paradox.

Chapter 5

Proof techniques II — Induction

5.1 The principle of mathematical induction

Exercises — 5.1

1. Consider the sequence of numbers that are 1 greater than a multiple of 4. (Such numbers are of the form $4j + 1$.)

$$1, 5, 9, 13, 17, 21, 25, 29, \ldots$$

The sum of the first several numbers in this sequence can be expressed as a polynomial.

$$\sum_{j=0}^{n} 4j + 1 = 2n^2 + 3n + 1$$

Complete the following table in order to provide evidence that the formula above is correct.

n	$\sum_{j=0}^{n} 4j+1$	$2n^2+3n+1$
0	1	1
1	$1+5=6$	$2\cdot 1^2+3\cdot 1+1=6$
2	$1+5+9=$	
3		
4		

2. What is wrong with the following inductive proof of "all horses are the same color."?

   Theorem Let H be a set of n horses, all horses in H are the same color.

   Proof: We proceed by induction on n.

   Basis: Suppose H is a set containing 1 horse. Clearly this horse is the same color as itself.

   Inductive step: Given a set of $k+1$ horses H we can construct two sets of k horses. Suppose $H = \{h_1, h_2, h_3, \ldots h_{k+1}\}$. Define $H_a = \{h_1, h_2, h_3, \ldots h_k\}$ (i.e. H_a contains just the first k horses) and $H_b = \{h_2, h_3, h_4, \ldots h_{k+1}\}$ (i.e. H_b contains the last k horses). By the inductive hypothesis both these sets contain horses that are "all the same color." Also, all the horses from h_2 to h_k are in both sets so both H_a and H_b contain only horses of this (same) color. Finally, we conclude that all the horses in H are the same color.

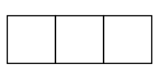
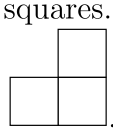
   Q.E.D.

3. For each of the following theorems, write the statement that must be proved for the basis – then prove it, if you can!

   (a) The sum of the first n positive integers is $(n^2 + n)/2$.

   (b) The sum of the first n (positive) odd numbers is n^2.

   (c) If n coins are flipped, the probability that all of them are "heads" is $1/2^n$.

   (d) Every $2^n \times 2^n$ chessboard – with one square removed – can be tiled perfectly[1] by L-shaped trominoes. (A trominoe is like a domino but made up of 3 little squares. There are two kinds, straight  and L-shaped . This problem is only concerned with the L-shaped trominoes.)

[1] Here, "perfectly tiled" means that every trominoe covers 3 squares of the chessboard (nothing hangs over the edge) and that every square of the chessboard is covered by some trominoe.

4. Suppose that the rules of the game for PMI were changed so that one did the following:

    - Basis. Prove that $P(0)$ is true.
    - Inductive step. Prove that for all k, P_k implies P_{k+2}

    Explain why this would not constitute a valid proof that P_n holds for all natural numbers n. How could we change the basis in this outline to obtain a valid proof?

5. If we wanted to prove statements that were indexed by the integers,

    $$\forall z \in \mathbb{Z},\ P_z,$$

    what changes should be made to PMI?

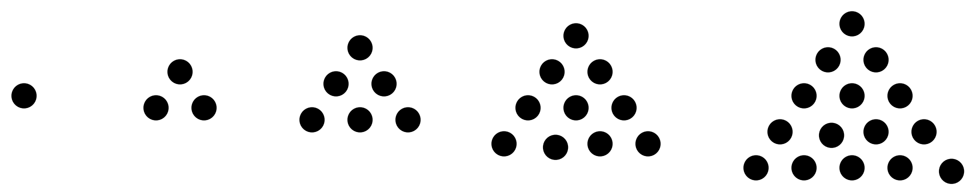
5.2 Formulas for sums and products

Exercises — 5.2

1. Write an inductive proof of the formula for the sum of the first n cubes.

2. Find a formula for the sum of the first n fourth powers.

3. The sum of the first n natural numbers is sometimes called the n-th triangular number T_n. Triangular numbers are so-named because one can represent them with triangular shaped arrangements of dots.



The first several triangular numbers are 1, 3, 6, 10, 15, et cetera.

Determine a formula for the sum of the first n triangular numbers $\left(\sum_{i=1}^{n} T_i\right)$ and prove it using PMI.

4. Consider the alternating sum of squares:

$$1$$
$$1 - 4 = -3$$
$$1 - 4 + 9 = 6$$
$$1 - 4 + 9 - 16 = -10$$
$$\text{et cetera}$$

Guess a general formula for $\sum_{i=1}^{n}(-1)^{i-1}i^2$, and prove it using PMI.

5. Prove the following formula for a product.

$$\prod_{i=2}^{n}\left(1-\frac{1}{i}\right)=\frac{1}{n}$$

6. Prove $\displaystyle\sum_{j=0}^{n}(4j+1) \;=\; 2n^2+3n+1$ for all integers $n \geq 0$.

7. Prove $\displaystyle\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$ for all natural numbers n.

8. The *Fibonacci numbers* are a sequence of integers defined by the rule that a number in the sequence is the sum of the two that precede it.

$$F_{n+2} = F_n + F_{n+1}$$

The first two Fibonacci numbers (actually the zeroth and the first) are both 1.

Thus, the first several Fibonacci numbers are

$$F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_5 = 8, F_6 = 13, F_7 = 21, \text{ et cetera}$$

Use mathematical induction to prove the following formula involving Fibonacci numbers.

$$\sum_{i=0}^{n} (F_i)^2 \; = \; F_n \cdot F_{n+1}$$

5.3 Divisibility statements and other proofs using PMI

Exercises — 5.3

Give inductive proofs of the following

1. $\forall x \in \mathbb{N},\ 3 \,|\, x^3 - x$

2. $\forall x \in \mathbb{N},\ 3 \,|\, x^3 + 5x$

3. $\forall x \in \mathbb{N},\ 11 \,|\, x^{11} + 10x$

4. $\forall n \in \mathbb{N},\ 3 \,|\, 4^n - 1$

5. $\forall n \in \mathbb{N},\ 6 \,|\, (3n^2 + 3n - 12)$

6. $\forall n \in \mathbb{N},\ 5 \,|\, (n^5 - 5n^3 + 14n)$

7. $\forall n \in \mathbb{N},\ 4 \mid (13^n + 4n - 1)$

8. $\forall n \in \mathbb{N},\ 7 \mid 8^n + 6$

9. $\forall n \in \mathbb{N},\ 6 \,|\, 2n^3 - 2n - 12$

10. $\forall n \geq 3 \in \mathbb{N},\ 3n^2 + 3n + 1 < 2n^3$

11. $\forall n > 3 \in \mathbb{N},\ n^3 < 3^n$

12. $\forall n \geq 3 \in \mathbb{N},\ n^3 + 3 > n^2 + 3n + 1$

13. $\forall x \geq 4 \in \mathbb{N},\ x^2 2^x \leq 4^x$

5.4 The strong form of mathematical induction

Exercises — 5.4

Give inductive proofs of the following

1. A "postage stamp problem" is a problem that (typically) asks us to determine what total postage values can be produced using two sorts of stamps. Suppose that you have 3¢ stamps and 7¢ stamps, show (using strong induction) that any postage value 12¢ or higher can be achieved. That is,

$$\forall n \in \mathbb{N}, n \geq 12 \implies \exists x, y \in \mathbb{N}, n = 3x + 7y.$$

2. Show that any integer postage of 12¢ or more can be made using only 4¢ and 5¢ stamps.

3. The polynomial equation $x^2 = x + 1$ has two solutions, $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Show that the Fibonacci number F_n is less than or equal to α^n for all $n \geq 0$.

Chapter 6

Relations and functions

6.1 Relations

Exercises — 6.1

1. The *lexicographic order*, $<_{\text{lex}}$, is a relation on the set of all words, where $x <_{\text{lex}} y$ means that x would come before y in the dictionary. Consider just the three letter words like "iff", "fig", "the", et cetera. Come up with a usable definition for $x_1x_2x_3 <_{\text{lex}} y_1y_2y_3$.

2. What is the graph of "=" in $\mathbb{R} \times \mathbb{R}$?

3. The *inverse* of a relation R is denoted R^{-1}. It contains exactly the same ordered pairs as R but with the order switched. (So technically, they aren't *exactly* the same ordered pairs ...)

$$\mathsf{R}^{-1} = \{(b, a) \mid (a, b) \in \mathsf{R}\}$$

Define a relation S on $\mathbb{R} \times \mathbb{R}$ by $\mathsf{S} = \{(x, y) \mid y = \sin x\}$. What is S^{-1}? Draw a single graph containing S and S^{-1}.

4. The "socks and shoes" rule is a very silly little mnemonic for remembering how to invert a composition. If we think of undoing the process of putting on our socks and shoes (that's socks first, then shoes) we have to first remove our shoes, *then* take off our socks.

   The socks and shoes rule is valid for relations as well.

   Prove that $(\mathsf{S} \circ \mathsf{R})^{-1} = \mathsf{R}^{-1} \circ \mathsf{S}^{-1}$.

6.2 Properties of relations

Exercises — 6.2

1. Consider the relation S defined by $\mathsf{S} = \{(x, y) \mid x \text{ is smarter than } y\}$.
   Is S symmetric or anti-symmetric? Explain.

2. Consider the relation A defined by $\mathsf{A} = \{(x, y) \mid x \text{ has the same astrological sign as } y\}$.
   Is A symmetric or anti-symmetric? Explain.

3. Explain why both of the relations just described (in problems 1 and 2) have the transitive property.

4. For each of the five properties, name a relation that has it and a relation that doesn't.

5. Show by counterexample that "$\div$" (divisibility) is not symmetric as a relation on $\mathbb{Z}$.

6. Prove that "$\div$" is an o

6.3 Equivalence relations

Exercises — 6.3

1. Consider the relation A defined by

$$\mathsf{A} = \{(x, y) \mid x \text{ has the same astrological sign as } y\}.$$

   Show that A is an equivalence relation. What equivalence class under A do you belong to?

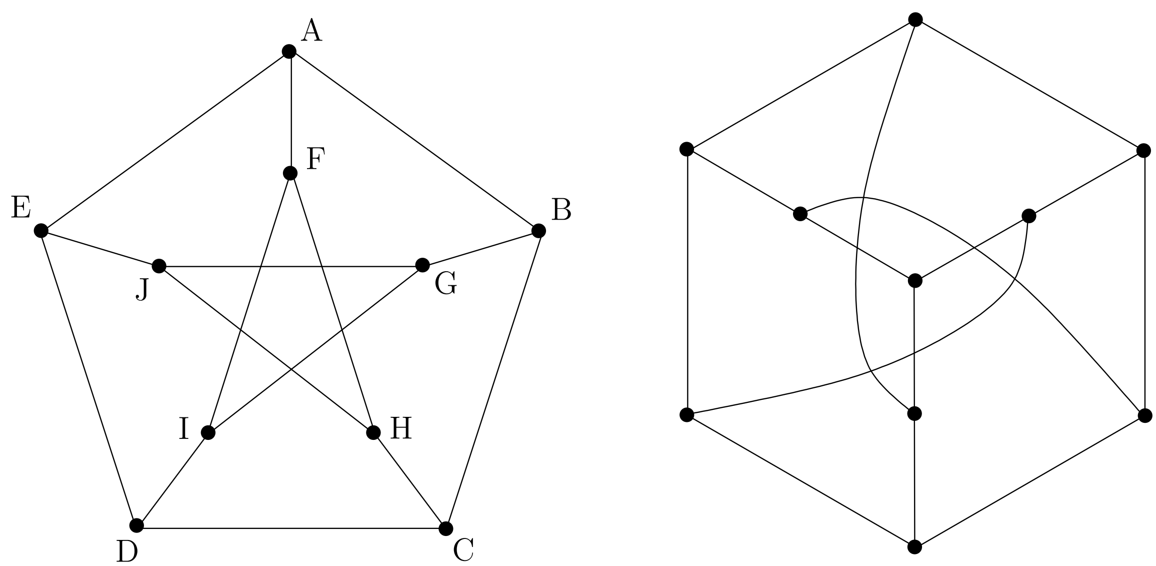
2. Define a relation $\Box$ on the integers by $x \Box y \iff x^2 = y^2$. Show that $\Box$ is an equivalence relation. List the equivalence classes $x/\Box$ for $0 \leq x \leq 5$.

3. Define a relation A on the set of all words by

$$w_1 \mathsf{A} w_2 \iff w_1 \text{ is an anagram of } w_2.$$

Show that A is an equivalence relation. (Words are anagrams if the letters of one can be re-arranged to form the other. For example, 'ART' and 'RAT' are anagrams.)

4. The two diagrams below both show a famous graph known as the Petersen graph. The picture on the left is the usual representation which emphasizes its five-fold symmetry. The picture on the right highlights the fact that the Petersen graph also has a three-fold symmetry. Label the right-hand diagram using the same letters (A through J) in order to show that these two representations are truly isomorphic.



5. We will use the symbol $\mathbb{Z}^*$ to refer to the set of all integers *except* 0. Define a relation Q on the set of all pairs in $\mathbb{Z} \times \mathbb{Z}^*$ (pairs of integers where the second coordinate is non-zero) by $(a, b)\mathsf{Q}(c, d) \iff ad = bc$. Show that Q is an equivalence relation.

6. The relation Q defined in the previous problem partitions the set of all pairs of integers into an interesting set of equivalence classes. Explain why

$$\mathbb{Q} \quad = \quad (\mathbb{Z} \times \mathbb{Z}^*)/\mathsf{Q}.$$

Ultimately, this is the "right" definition of the set of rational numbers!

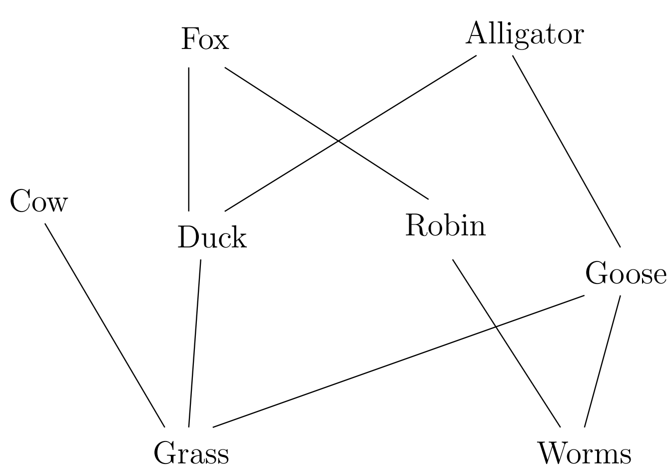
7. Reflect back on the proof in problem 5. Note that we were fairly careful in assuring that the second coordinate in the ordered pairs is non-zero. (This was the whole reason for introducing the $\mathbb{Z}^*$ notation.) At what point in the argument did you use this hypothesis?

6.4 Ordering relations

Exercises — 6.4

1. In population ecology there is a partial order "predates" which basically means that one organism feeds upon another. Strictly speaking this relation is not transitive; however, if we take the point of view that when a wolf eats a sheep, it is also eating some of the grass that the sheep has fed upon, we see that in a certain sense it is transitive. A chain in this partial order is called a "food chain" and so-called apex predators are said to "sit atop the food chain". Thus "apex predator" is a term for a maximal element in this poset. When poisons such as mercury and PCBs are introduced into an ecosystem, they tend to collect disproportionately in the apex predators – which is why pregnant women and young children should not eat shark or tuna but sardines are fine.

   Below is a small example of an ecology partially ordered by "predates"

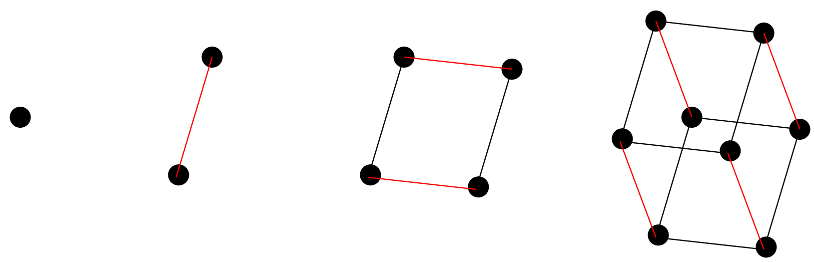


   Find the largest antichain in this poset.

2. Referring to the poset given in exercise 1, match the following.

1. An (non-maximal) antichain	a. Grass
2. A maximal antichain	b. Goose
3. A maximal element	c. Fox
4. A (non-maximal) chain	d. $\{\text{Grass}, \text{Duck}\}$
5. A maximal chain	e. There isn't one!
6. A cover for "Worms"	f. $\{\text{Fox}, \text{Alligator}, \text{Cow}\}$
7. A least element	g. $\{\text{Cow}, \text{Duck}, \text{Goose}\}$
8. A minimal element	h. $\{\text{Worms}, \text{Robin}, \text{Fox}\}$

3. The graph of the edges of a cube is one in an infinite sequence of graphs. These graphs are defined recursively by "Make two copies of the previous graph then join corresponding nodes in the two copies with edges." The 0-dimensional 'cube' is just a single point. The 1-dimensional cube is a single edge with a node at either end. The 2-dimensional cube is actually a square and the 3-dimensional cube is what we usually mean when we say "cube."



Make a careful drawing of a *hypercube* – which is the name of the graph that follows the ordinary cube in this sequence.

4. Label the nodes of a hypercube with the divisors of 210 in order to produce a Hasse diagram of the poset determined by the divisibility relation.

5. Label the nodes of a hypercube with the subsets of $\{a, b, c, d\}$ in order to produce a Hasse diagram of the poset determined by the subset containment relation.

6. Complete a Hasse diagram for the poset of divisors of 11025 (partially ordered by divisibility).

7. Find a collection of sets so that, when they are partially ordered by $\subseteq$, we obtain the same Hasse diagram as in the previous problem.

6.5 Functions

Exercises — 6.5

1. For each of the following functions, give its domain, range and a possible codomain.

   (a) $f(x) = \sin(x)$

   (b) $g(x) = e^x$

   (c) $h(x) = x^2$

   (d) $m(x) = \frac{x^2+1}{x^2-1}$

   (e) $n(x) = \lfloor x \rfloor$

   (f) $p(x) = \langle \cos(x), \sin(x) \rangle$

2. Find a bijection from the set of odd squares, $\{1, 9, 25, 49, \ldots\}$, to the non-negative integers, $\mathbb{Z}^{\text{noneg}} = \{0, 1, 2, 3, \ldots\}$. Prove that the function you just determined is both injective and surjective. Find the inverse function of the bijection above.

3. The natural logarithm function $\ln(x)$ is defined by a definite integral with the variable x in the upper limit.

$$\ln(x) = \int_{t=1}^{x} \frac{1}{t}\,\mathrm{d}t.$$

From this definition we can deduce that $\ln(x)$ is strictly increasing on its entire domain, $(0, \infty)$. Why is this true?

We can use the above definition with $x = 2$ to find the value of $\ln(2) \approx .693$. We will also take as given the following rule (which is valid for all logarithmic functions).

$$\ln(a^b) = b\ln(a)$$

Use the above information to show that there is neither an upper bound nor a lower bound for the values of the natural logarithm. These facts together with the information that ln is strictly increasing show that $\mathrm{Rng}(\ln) = \mathbb{R}$.

4. Georg Cantor developed a systematic way of listing the rational numbers. By "listing" a set one is actually developing a bijection from $\mathbb{N}$ to that set. The method known as "Cantor's Snake" creates a bijection from the naturals to the non-negative rationals. First we create an infinite table whose rows are indexed by positive integers and whose columns are indexed by non-negative integers – the entries in this table are rational numbers of the form "column index" / "row index." We then follow a snake-like path that zig-zags across this table – whenever we encounter a rational number that we haven't seen before (in lower terms) we write it down. This is indicated in the diagram below by circling the entries.

	0	1	2	3	4	5	6	7	8
1	0/1	1/1	2/1	3/1	4/1	5/1	6/1	7/1	8/1
2	0/2	1/2	2/2	3/2	4/2	5/2	6/2	7/2	8/2
3	0/3	1/3	2/3	3/3	4/3	5/3	6/3	7/3	8/3
4	0/4	1/4	2/4	3/4	4/4	5/4	6/4	7/4	8/4
5	0/5	1/5	2/5	3/5	4/5	5/5	6/5	7/5	8/5
6	0/6	1/6	2/6	3/6	4/6	5/6	6/6	7/6	8/6
7	0/7	1/7	2/7	3/7	4/7	5/7	6/7	7/7	8/7
8	0/8	1/8	2/8	3/8	4/8	5/8	6/8	7/8	8/8

Effectively this gives us a function f which produces the rational number that would be found in a given position in this list. For example $f(1) = 0/1, f(2) = 1/1$ and $f(5) = 1/3$.

What is $f(26)$? What is $f(30)$? What is $f^{-1}(3/4)$? What is $f^{-1}(6/7)$?

6.6 Special functions

Exercises — 6.6

1. The n-th triangular number, denoted $T(n)$, is given by the formula $T(n) = (n^2 + n)/2$. If we regard this formula as a function from $\mathbb{R}$ to $\mathbb{R}$, it fails the horizontal line test and so it is not invertible. Find a suitable restriction so that T is invertible.

2. The usual algebraic procedure for inverting $T(x) = (x^2+x)/2$ fails. Use your knowledge of the geometry of functions and their inverses to find a formula for the inverse. (Hint: it may be instructive to first invert the simpler formula $S(x) = x^2/2$ — this will get you the right vertical scaling factor.)

3. What is $\pi_2(W(t))$?

4. Find a right inverse for $f(x) = |x|$.

5. In three-dimensional space we have projection functions that go onto the three coordinate axes (π_1, π_2 and π_3) and we also have projections onto coordinate planes. For example, $\pi_{12} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$, defined by

   $$\pi_{12}((x, y, z)) = (x, y)$$

   is the projection onto the x–y coordinate plane.

   The triple of functions $(\cos t, \sin t, t)$ is a parametric expression for a helix. Let $H = \{(\cos t, \sin t, t) \mid t \in \mathbb{R}\}$ be the set of all points on the helix. What is the set $\pi_{12}(H)$? What are the sets $\pi_{13}(H)$ and $\pi_{23}(H)$?

6. Consider the set $\{1, 2, 3, \ldots, 10\}$. Express the characteristic function of the subset $S = \{1, 2, 3\}$ as a set of ordered pairs.

7. If S and T are subsets of a set D, what is the product of their characteristic functions $1_S \cdot 1_T$?

8. Evaluate the sum

$$\sum_{i=1}^{10} \frac{1}{i} \cdot [i \text{ is prime}].$$

Chapter 7

Proof techniques III — Combinatorics

7.1 Counting

1. Determine the number of entries in the following sequences.

   (a) $(999, 1000, 1001, \ldots 2006)$

   (b) $(13, 15, 17, \ldots 199)$

   (c) $(13, 19, 25, \ldots 601)$

   (d) $(5, 10, 17, 26, 37, \ldots 122)$

(e) $(27, 64, 125, 216, \ldots 8000)$

(f) $(7, 11, 19, 35, 67, \ldots 131075)$

2. How many “full houses” are there in Yahtzee? (A full house is a pair together with a three-of-a-kind.)

3. In how many ways can you get “two pairs” in Yahtzee?

4. Prove that the binomial coefficients $\binom{n+k-1}{k}$ and $\binom{n+k-1}{n-1}$ are equal.

5. The “Cryptographer’s alphabet” is used to supply small examples in coding and cryptography. It consists of the first 6 letters, $\{a, b, c, d, e, f\}$. How many “words” of length up to 6 can be made with this alphabet? (A word need not actually be a word in English, for example both “fed” and “dfe” would be words in the sense we are using the term.)

6. How many “words” are there of length 4, with distinct letters from the Cryptographer’s alphabet, in which the letters appear in increasing order alphabetically? (“Acef” would be one such word, but “cafe” would not.)

7. How many “words” are there of length 4 from the Cryptographer’s alphabet, with repeated letters allowed, in which the letters appear in non-decreasing order alphabetically?

8. How many subsets does a finite set have?

9. How many handshakes will transpire when n people first meet?

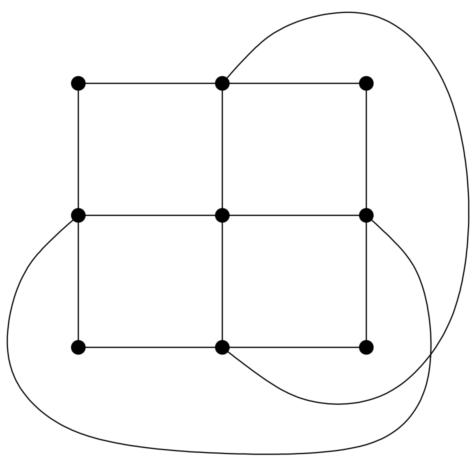
10. How many functions are there from a set of size n to a set of size m?

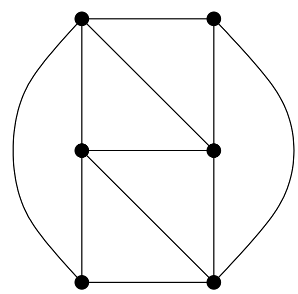
11. How many relations are there from a set of size n to a set of size m?

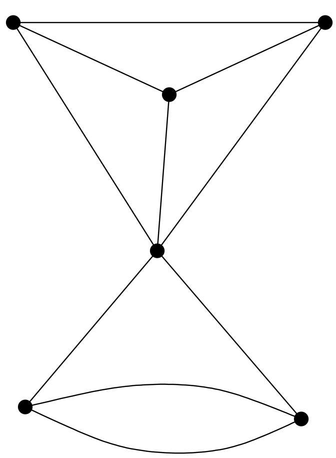
7.2 Parity and Counting arguments

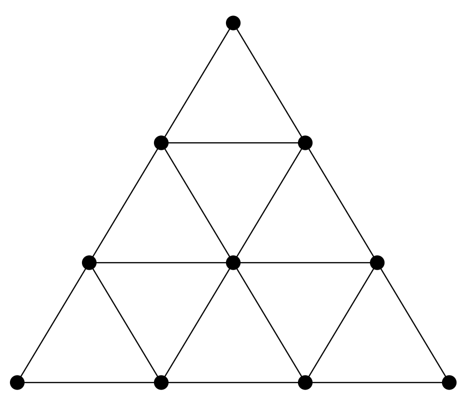
Exercises — 7.2

1. A walking tour of Königsberg such as is described in this section, or more generally, a circuit through an arbitrary graph that crosses each edge precisely once and begins and ends at the same node is known as an *Eulerian circuit*. An *Eulerian path* also crosses every edge of a graph exactly once but it begins and ends at distinct nodes. For each of the following graphs determine whether an Eulerian circuit or path is possible, and if so, draw it.







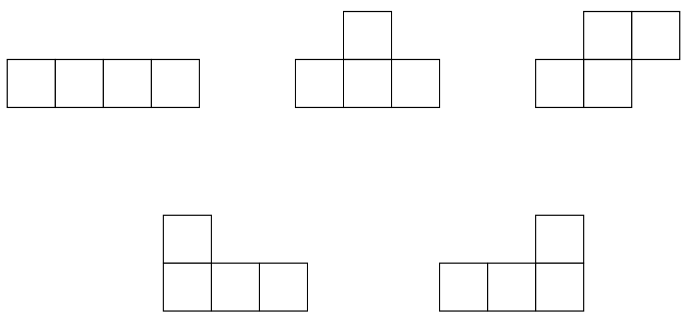


2. Complete the proof of the fact that "Every graph has an even number of odd nodes."

3. Provide an argument as to why an 8×8 chessboard with two squares pruned from diagonally opposite corners cannot be tiled with dominoes.

4. Prove that, if n is odd, any $n \times n$ chessboard with a square the same color as one of its corners pruned can be tiled by dominoes.

5. The five tetrominoes (familiar to players of the video game Tetris) are relatives of dominoes made up of four small squares.



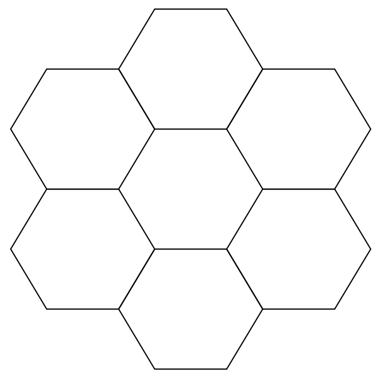
   All together these five tetrominoes contain 20 squares so it is conceivable that they could be used to tile a 4×5 chessboard. Prove that this is actually impossible.

6. State necessary and sufficient conditions for the existence of an Eulerian circuit in a graph.

7. State necessary and sufficient conditions for the existence of an Eulerian path in a graph.

8. Construct magic squares of order 4 and 5.

9. A magic hexagon of order 2 would consist of filling-in the numbers from 1 to 7 in the hexagonal array below. The magic condition means that each of the 9 “lines” of adjacent hexagons would have the same sum. Is this possible?



10. Is there a magic hexagon of order 3?

7.3 The pigeonhole principle

Exercises — 7.3

1. The statement that there are two non-bald New Yorkers with the same number of hairs on their heads requires some careful estimates to justify it. Please justify it.

2. A mathematician, who always rises earlier than her spouse, has developed a scheme – using the pigeonhole principle – to ensure that she always has a matching pair of socks. She keeps only blue socks, green socks and black socks in her sock drawer – 10 of each. So as not to wake her husband she must select some number of socks from her drawer in the early morning dark and take them with her to the adjacent bathroom where she dresses. What number of socks does she choose?

3. If we select 1001 numbers from the set $\{1, 2, 3, \ldots, 2000\}$ it is certain that there will be two numbers selected such that one divides the other. We can prove this fact by noting that every number in the given set can be expressed in the form $2^k \cdot m$ where m is an odd number and using the pigeonhole principle. Write-up this proof.

4. Given any set of 53 integers, show that there are two of them having the property that either their sum or their difference is evenly divisible by 103.

5. Prove that if 10 points are placed inside a square of side length 3, there will be 2 points within $\sqrt{2}$ of one another.

6. Prove that if 10 points are placed inside an equilateral triangle of side length 3, there will be 2 points within 1 of one another.

7. Prove that in a simple graph (an undirected graph with no loops or parallel edges) having n nodes, there must be two nodes having the same degree.

7.4 The algebra of combinations

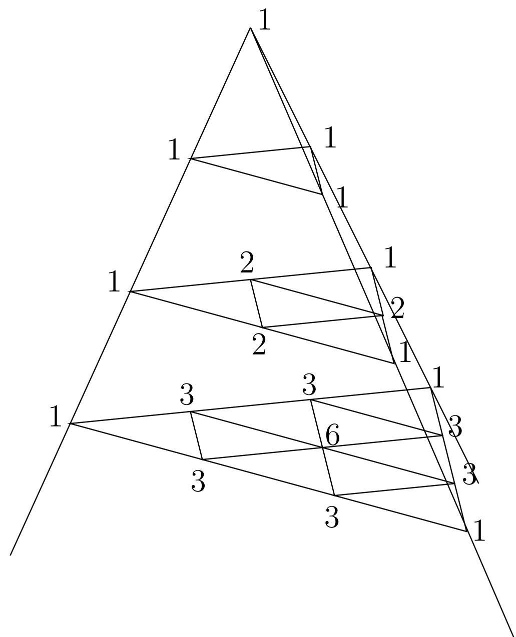
Exercises — 7.4

1. Use the binomial theorem (with $x = 1000$ and $y = 1$) to calculate 1001^6.

2. Find $(2x + 3)^5$.

3. Find $(x^2 + y^2)^6$.

4. The following diagram contains a 3-dimensional analog of Pascal's triangle that we might call "Pascal's tetrahedron." What would the next layer look like?



5. The student government at Lagrange High consists of 24 members chosen from amongst the general student body of 210. Additionally, there is a steering committee of 5 members chosen from amongst those in student government. Use the multiplication rule to determine two different formulas for the total number of possible governance structures.

6. Prove the identity

$$\binom{n}{k} \cdot \binom{k}{r} = \binom{n}{r} \cdot \binom{n-r}{k-r}$$

combinatorially.

7. Prove the binomial theorem.

$$\forall n \in \mathbb{N},\ \forall x, y \in \mathbb{R},\ (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Chapter 8

Cardinality

8.1 Equivalent sets

Exercises — 8.1

1. Name four sets in the equivalence class of $\{1, 2, 3\}$.

2. Prove that set equivalence is an equivalence relation.

3. Construct a Venn diagram showing the relationships between the sets of sets which are finite, infinite, countable, denumerable and uncountable.

4. Place the sets $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{C}$, $\mathbb{N}_{2007}$ and $\emptyset$; somewhere on the Venn diagram above. (Note to students (and graders): there are no wrong answers to this question, the point is to see what your intuition about these sets says at this point.)

8.2 Examples of set equivalence

Exercises — 8.2

1. Prove that positive numbers of the form $3k + 1$ are equinumerous with positive numbers of the form $4k + 2$.

2. Prove that $f(x) = c + \dfrac{(x-a)(d-c)}{(b-a)}$ provides a bijection from the interval $[a, b]$ to the interval $[c, d]$.

3. Prove that any two circles are equinumerous (as sets of points).

4. Determine a formula for the bijection from $(-1, 1)$ to the line $y = 1$ determined by vertical projection onto the upper half of the unit circle, followed by projection from the point $(0, 0)$.

5. It is possible to generalize the argument that shows a line segment is equivalent to a line to higher dimensions. In two dimensions we would show that the unit disk (the interior of the unit circle) is equinumerous with the entire plane $\mathbb{R} \times \mathbb{R}$. In three dimensions we would show that the unit ball (the interior of the unit sphere) is equinumerous with the entire space $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Here we would like you to prove the two-dimensional case.

   Gnomonic projection is a style of map rendering in which a portion of a sphere is projected onto a plane that is tangent to the sphere. The sphere's center is used as the point to project from. Combine vertical projection from the unit disk in the x–y plane to the upper half of the unit sphere $x^2 + y^2 + z^2 = 1$, with gnomonic projection from the unit sphere to the plane z = 1, to deduce a bijection between the unit disk and the (infinite) plane.

8.3 Cantor's theorem

Exercises — 8.3

1. Determine a substitution rule – a consistent way of replacing one digit with another along the diagonal so that a diagonalization proof showing that the interval (0, 1) is uncountable will work in decimal. Write up the proof.

2. Can a diagonalization proof showing that the interval (0, 1) is uncountable be made workable in base-3 (ternary) notation?

3. In the proof of Cantor's theorem we construct a set S that cannot be in the image of a presumed bijection from A to $\mathcal{P}(A)$. Suppose $A = \{1, 2, 3\}$ and f determines the following correspondences: $1 \longleftrightarrow \emptyset$, $2 \longleftrightarrow \{1, 3\}$ and $3 \longleftrightarrow \{1, 2, 3\}$. What is S?

4. An argument very similar to the one embodied in the proof of Cantor's theorem is found in the Barber's paradox. This paradox was originally introduced in the popular press in order to give laypeople an understanding of Cantor's theorem and Russell's paradox. It sounds somewhat sexist to modern ears. (For example, it is presumed without comment that the Barber is male.)

   > In a small town there is a Barber who shaves those men (and only those men) who do not shave themselves. Who shaves the Barber?

   Explain the similarity to the proof of Cantor's theorem.

5. Cantor's theorem, applied to the set of all sets leads to an interesting paradox. The power set of the set of all sets is a collection of sets, so it must be contained in the set of all sets. Discuss the paradox and determine a way of resolving it.

6. Verify that the final deduction in the proof of Cantor's theorem, "$(y \in S \implies y \notin S) \wedge (y \notin S \implies y \in S)$," is truly a contradiction.

8.4 Dominance

Exercises — 8.4

1. How could the clerk at the Hilbert Hotel accommodate a countable number of new guests?

2. Let F be the collection of all real-valued functions defined on the real line. Find an injection from $\mathbb{R}$ to F. Do you think it is possible to find an injection going the other way? In other words, do you think that F and $\mathbb{R}$ are equivalent? Explain.

3. Fill in the details of the proof that dominance is an ordering relation. (You may simply cite the C-B-S theorem in proving anti-symmetry.)

4. We can inject $\mathbb{Q}$ into $\mathbb{Z}$ by sending $\pm\frac{a}{b}$ to $\pm 2^a 3^b$. Use this and another obvious injection to (in light of the C-B-S theorem) reaffirm the equivalence of these sets.

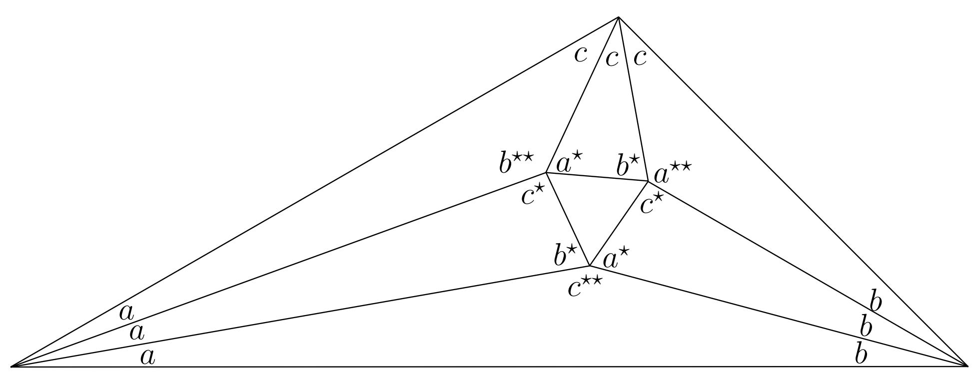
Chapter 9

Proof techniques IV — Magic

9.1 Morley's miracle

Exercises — 9.1

1. What value should we get if we sum all of the angles that appear around one of the interior vertices in the finished diagram? Verify that all three have the correct sum.



2. In this section we talked about similarity. Two figures in the plane are similar if it is possible to turn one into the other by a sequence of mappings: a translation, a rotation and a scaling.

   Geometric similarity is an equivalence relation. To fix our notation, let $T(x, y)$ represent a generic translation, $R(x, y)$ a rotation and $S(x, y)$ a scaling – thus a generic similarity is a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ that can be written in the form $S(R(T(x, y)))$.

   Discuss the three properties of an equivalence relation (reflexivity, symmetry and transitivity) in terms of geometric similarity.

9.2 Five steps into the void

Exercises — 9.2

1. Do the algebra (and show all your work!) to prove that invariant defined in this section actually has the value 1 for the set of all the men occupying the x-axis and the lower half-plane.

2. “Escape of the clones” is a nice puzzle, originally proposed by Maxim Kontsevich. The game is played on an infinite checkerboard restricted to the first quadrant – that is the squares may be identified with points having integer coordinates (x, y) with $x > 0$ and $y > 0$. The “clones” are markers (checkers, coins, small rocks, whatever…) that can move in only one fashion – if the squares immediately above and to the right of a clone are empty, then it can make a “clone move.” The clone moves one space up and a copy is placed in the cell one to the right. We begin with three clones occupying cells $(1, 1), (2, 1)$ and $(1, 2)$ – we'll refer to those three checkerboard squares as “the prison.” The question is this: can these three clones escape the prison?

   You must either demonstrate a sequence of moves that frees all three clones or provide an argument that the task is impossible.

9.3 Monge's circle theorem

Exercises — 9.3

1. There is a scenario where the proof we have sketched for Monge's circle theorem doesn't really work. Can you envision it? Hint: consider two relatively large spheres and one that is quite small.