Unit- I.

Probability theory.

⇒ Probability deals with uncertainty.

⇒ It tells you how likely something can happen.

Defn:

1. Experiment:

A set of conditions under which some variable is observed. (observing something).

(1) Deterministic experiment

It is an experiment in which output is known.

Ex:

⇒ Sun rises in east

⇒ water boils at 100°c.

(2) Random experiment

(1) It has more than one possible out come.

(2) Not possible to predict out come in advance.

Ex: Tossing coint, rolling die, etc.

⇒ Outcome :- The result of the observation

Sample Space :-

The set of all possible outcomes of a random experiment is called sample space.

⇒ denote :- S, Ω ⇒ set of all possible outcomes.
S ↓ subset of Ω

Ex:

⇒ colors of cards = Ω = {R, B}

⇒ Suit of cards = Ω = {Hearts (R), Diamonds (R), Spades (B) and clubs (B)}

⇒ Face cards = Ω = {Jack, Queen, and King}.

⇒ gender of child = Ω = {B, G}

Event:

Any subset of sample space is called event.

Ex: ⇒ A = getting a head

B = getting a tail.

⇒ Rolling a die:

$S = \{1, 2, 3, 4, 5, 6\}$.

A = Getting a odd number $= \{1, 3, 5\}$.

B = getting a even number $= \{2, 4, 6\}$.

C = getting prime number $\{2, 3, 5\}$.

Finite Sample Space:

If the set Ω contains only a finite number of points, we say that (Ω, S) is a finite sample space.

Discrete Sample Space:

If Ω contains atmost a countable no. of points, then (Ω, S) is called discrete sample space.

Uncountable Sample Space:

Ω contains uncountably many points, then we say that $\{\Omega, S\}$ is called uncountable sample space.

A and B event:

$\Rightarrow$ $A \cup B$ = (i) occurence of atleast one event.

(ii) A occurs or B occurs or (both).

$\Rightarrow$ $A \cap B$ = ① A occurs and B occurs

② occurence of A and B simultaneously.

$\Rightarrow$ A^c = ① not happening of the event.

② the completement of A, is the event that occurs iff A does not occur.

$\Rightarrow$ $A - B$ = occurence of A but not B.

(or)

$= A \cap B^c$.

Equally likely outcomes:

All outcomes have same chance of occurence.

Ex: Tossing coin: H T

$\frac{1}{2}$ $\frac{1}{2}$

⇒ fair

Rolling die

$p(1) = \frac{1}{2}$ $p(2) = \frac{1}{2}$ ......

$p(6) = \frac{1}{2}$.

Exhaustive events:

$A_1, A_2, \ldots A_n$ are called exhaustive events if $\bigcup_{i=1}^{n} A_i = \Omega$

⇒ Events of a random experiment are said to be exhaustive if atleast one of them necessarily occur.

Ex: Rolling die $S = \{1, 2, 3, 4, 5, 6\}$.

A: Getting an even number $= \{2, 4, 6\}$

B: odd number $= \{1, 3, 5\}$.

C: Prime number $= \{2, 3, 5\}$.

$A \not\subset B \Rightarrow A \cup B = \Omega \rightarrow$ exhaustive events.

$B \not\subset C \Rightarrow B \cup C \neq \Omega \Rightarrow$ not exhaustive.

* In drawing two cards from a pack of 52 cards, the exhaustive number of cases in $^{52}C_2$.

Mutually Exclusive:

A and B are mutually exclusive, if $A \cap B = \phi$.

$\Rightarrow$ A and B cannot occur at same time.

A even B - odd.

A & B are mutually exclusive event.

$A \cap B = \phi$.

$\rightarrow$ Rolling die:

A odd = $\{1, 3, 5\}$.

B = getting number less than 4

= $\{1, 2, 3\}$.

A & B are not mutually exclusive.

$A \cap B \neq \phi$.

$\Rightarrow$ Pairwise disjoint :-

$A_1, A_2, A_3, \ldots A_n$. $A_i \cap A_j = \phi$
$\forall i = j$

~~Independent~~ ~~Event~~ :-

Definition of Probability

(1) (*) Classical defn or Mathematical defn.

(2) Empirical or Statistics defn.

(3) Axiomatic definition.

I Classical defn :-

It is given by Laplace and Bernoulli in 18th century.

Let a random experiment has N possible outcomes which are mutually exclusive, Exhaustive, equally likely. Let M of these

~~Cases~~ outcomes be favourable to the (?) happening of event A then the Prob of A defined by.

$$P(A) = \frac{M}{N} = \frac{\text{Number of favourable outcomes}}{\text{Total outcomes}}$$

Ex:-

1. Tossing of fair coin:-

$$\Omega = \{H, T\}$$

$$P(H) = \frac{1}{2} \quad P(T) = \frac{1}{2}.$$

2. Roll of fair die:-

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$A = \{2, 4, 6\}. \quad P(2) = \frac{1}{6}$$

$$P(4) = \frac{1}{6}$$

$$P(\text{even number}) = \frac{1}{2}. \quad P(6) = 1/6.$$

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}.$$

II. Emperical definition:-

Suppose a random experiment is conducted large number of times independently under certain conditions. Let A_n denote the number of times event A occurs in n trials of experiment.

$$P(A) = \lim_{n \to \infty} \frac{A_n}{n}$$

n trials

1. $A = \sqrt{n}$.

$$P(A) = \lim_{n \to \infty} \frac{\sqrt{n}}{n} = \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n}\sqrt{n}}$$

$$= \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0.$$

2. $A = n - \sqrt{n}$.

$$P(A) = \lim_{n \to \infty} \frac{n - \sqrt{n}}{n} = 1 \Rightarrow \text{Sure event.}$$

I Classical defn

Ex:. Rolling of a die

$S = \{1, 2, 3, 4, 5, 6\}$.

A = number less than 5 = $\{1, 2, 3, 4\}$.

$$P(A) = \frac{4}{6} = \frac{2}{3}.$$

II: Emperical definition:-

Ex:. Tossing of coin 1000 times.

Head : 455 tail : 545

$$P(\text{Head}) = \frac{455}{1000} = 0.455.$$

III Axiomatic defn:.

Measure theory:-

It defines the prob as a function.

Let S, Ω sample space of a random experiment.

$\mathcal{B}$ = σ algebra of subsets of Ω

(or) Ω field.

Ex:.

Tossing of a coin:.

$\Omega = \{H, T\}$

$\mathcal{B} = \{\phi, \{H\}, \{T\}, \{H, T\}\}$

1. $\phi \in \mathcal{B}$
2. $A \in \mathcal{B} \Rightarrow A^c \in \mathcal{B}$ closed under complement.
3. Closed under countable union:-
   $\{A_i\} \in \mathcal{B}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$

★ $(\Omega, \mathcal{B})$ is measurable space.

Probability Space:-

Let $(\Omega, \mathcal{B})$ be a measurable space

A set fn $P : \mathcal{B} \to R$ is said to be prob fn if

1. P_1 (Axiom of positivity or axiom of non negativity)

$0 \leq P(A) \leq 1.$

(2) P_2 (Axiom of certainity or completeness)

$$P(\Omega)=1$$

(3) P_3 (Axiom of countable additivity)

For a seq of pairwise disjoint subsets $E_i \in \beta$.

mutually exclusive

$E_i \cap E_j = \phi$ $(i \neq j)$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

$$P(E_1 \cup E_2 \cup E_3 \ldots) = P(E_1) + P(E_2) + P(E_3) + \ldots$$

$\therefore (\Omega, \beta, P)$ is called probability space

Consequence of the axiomatic defn:

Theorem:1

Prob of Impossible event is zero,

$$P(\phi)=0.$$

Proof: Let A be any set; then A and ϕ are disjoint and

$A \cup \phi = A$ (by P_3)

$$P(A) = P(A \cup \phi)$$
$$= P(A) + P(\phi)$$
$$P(\phi) = 0$$

Theorem: 2

If A^c is the complement of an event A, then $P(A^c) = 1 - P(A)$.

Proof:

If A is any event in S

$\Rightarrow A$ and A^c are disjoint

from P_3, $A \cup A^c = \Omega$.

Using axiom P_3,

$$P(A \cup A^c) = P(\Omega)$$

$$P(A) + P(A^c) = 1 \quad \text{(by using axiom } P_2 \text{ and } P_3\text{)}$$

$$P(A^c) = 1 - P(A).$$

Theorem 3:

For any finite sequence $A_1, A_2, \ldots, A_n$ of pairwise disjoint subset in $\mathcal{F}$..

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P(A_1) + P(A_2) + \cdots P(A_n)$$
$$= \sum_{i=1}^{n} P(A_i)$$

Proof:

let $A_{n+1}, A_{n+2}, \ldots = \varphi$.

from P_3, $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$.

$P(A_1 \cup A_2 \ldots A_n \cup A_{n+1} \cup A_{n+2} \ldots)$.

$= P(A_1) + P(A_2) + P(A_3) + \ldots$

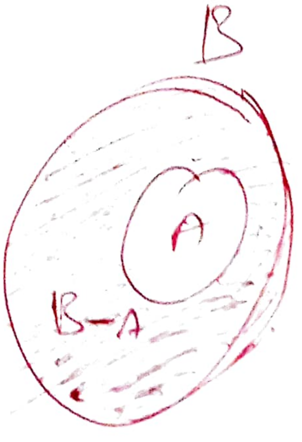
$P(A_n) + P(A_{n+1}) + P(A_{n+2}) + \ldots$

$= P(A_1) + P(A_2) + \ldots P(A_n) + \varphi + \varphi +$

$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$

Theorem: If $A \subseteq B$ then $P(A) \leq P(B)$.

Proof: $B = A \cup (B - A)$



where A, $B-A$ are disjoint sets.

$P(B) = P(A \cup (B - A))$

using P_3 (axiom),

$P(B) = P(A) + P(B - A)$

$P(B \cap A^c)$ using axiom P_1

$P(B - A) \geq 0$.

$P(B) \geq P(A)$ //.

$P(B \cap A^c)$
$P(\phi) = 0$

Ex: Rolling of die,

$\Omega = \{1, 2, 3, 4, 5, 6\}$.

A = odd numbers $= \{1, 3, 5\}$

B = less than 6 $= \{1, 2, 3, 4, 5\}$.

$A \subseteq B$.

$P(A) = \frac{1}{2} \left(\frac{3}{6}\right)$ $\quad P(B) = \frac{5}{6}$.

$P(A) \leq P(B)$.

Theorem: 5

For an event $A \in B$,

$0 \leq P(A) \leq 1$

Proof:

from Theo. 4, $A \subseteq \Omega$

$P(A) \leq P(\Omega)$

$P(A) \leq 1$.

From P_1 (Axiom)

$P(A) \geq 0$.

$\therefore 0 \leq P(A) \leq 1$.