II. Emperical definition:

Suppose a random experiment is conducted large number of times independently under certain conditions. Let A_n denote the number of times occurs in n trials of event A experiment.

$$P(A) = \lim_{n \to \infty} \frac{A_n}{n}$$

n trials

1. $A = \sqrt{n}$.

$$P(A) = \lim_{n \to \infty} \frac{\sqrt{n}}{n} = \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n}\sqrt{n}}$$

$$= \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0.$$

2. $A = n - \sqrt{n}$.

$$P(A) = \lim_{n \to \infty} \frac{n - \sqrt{n}}{n} = 1 \Rightarrow \text{Sure event.}$$

I Classical defn

Ex:- Rolling of a die

$S = \{1, 2, 3, 4, 5, 6\}$.

A = number less than 5 = $\{1, 2, 3, 4\}$.

$$P(A) = \frac{4}{6} = \frac{2}{3}.$$

II. Emperical definition:-

Ex:- Tossing of coin 1000 times.

Head : 455 tail : 545

$$P(\text{Head}) = \frac{455}{1000} = 0.455.$$

III Axiomatic defn:-

Measure theory:-

It defines the prob as a function.

Let S, Ω sample space of a random experiment.

$\mathcal{B}$ = σ algebra of subsets of Ω (or) Ω field.

Ex:.

Tossing of a coin:.

$\Omega = \{H, T\}$

$\mathcal{B} = \{\phi, \{H\}, \{T\}, \{HT\}\}$

(1) $\phi \in \mathcal{B}$

(2) $A \in \mathcal{B} \Rightarrow A^C \in \mathcal{B}$ closed under complement.

(3) closed under countable union:.

$\{A_i\} \in \mathcal{B}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$

$(\Omega, \mathcal{B})$ is measurable space.

Probability space:-

Let $(\Omega, \mathcal{B})$ be a measurable space

A set fn $P: \mathcal{B} \to R$ is said to be prob fn if

(1) P_1 (Axiom of positivity or axiom of non negativity)

$0 \leq P(A) \leq 1$.

(3) P_2 (Axiom of certainity or completeness)

$$p(\Omega)=1 \quad , \; P(\phi)=0$$

(4) P_3 (Axiom of countable additivity)

For a seq of pairwise disjoint subsets $E_i \in \beta$

mutually exclusive

$E_i \cap E_j = \phi$ $(i \neq j)$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} p(E_i)$$

$$P(E_1 \cup E_2 \cup E_3 \ldots) = P(E_1)+P(E_2)+P(E_3)+ \ldots$$

$\therefore (\Omega, \mathcal{B}, P)$ is called probability space

Consequence of the axiomatic defn:.

Theorem:1

Prob of Impossible event is zero,

$$P(\phi)=0.$$

Proof: Let A be any set; then A and ϕ are disjoint and

$$A \cup \phi = \cancel{A} \quad (\text{by } P_3)$$

$$P(A) = P(A \cup \phi)$$
$$= P(A) + P(\phi)$$
$$P(\phi) = 0$$

Theorem: 2

If A^c is the complement of an event A, then $P(A^c) = 1 - P(A)$.

Proof:

If A is any event in S

$\Rightarrow A$ and A^c are disjoint.

From P_3, $A \cup A^c = \Omega$.

Using axiom P_3,

$$P(A \cup A^c) = P(\Omega)$$

$$P(A) + P(A^c) = 1 \quad \text{(by using axiom } P_2 \text{ and } P_3\text{)}$$

$$P(A^c) = 1 - P(A).$$

$0 \le P(\Omega) \le 1$
$P(\Omega) = 1$
$P(\phi) = 0$

Theorem 3:

For any finite sequence $A_1, A_2, \ldots A_n$ of pairwise disjoint subset in $\mathcal{B}$.

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P(A_1) + P(A_2) + \cdots P(A_n)$$
$$= \sum_{i=1}^{n} P(A_i)$$

(8)

<u>Proof:-</u>

let $A_{n+1}, A_{n+2}, \ldots = \varphi$.

from P_3, $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$.

$P(A_1 \cup A_2 \ldots A_n \cup A_{n+1} \cup A_{n+2} \ldots)$.

$= P(A_1) + P(A_2) + P(A_3) + \ldots$

$P(A_n) + P(A_{n+1}) + P(A_{n+2}) + \ldots$

$= P(A_1) + P(A_2) + \ldots P(A_n) + \varphi + \varphi + \ldots$

$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$

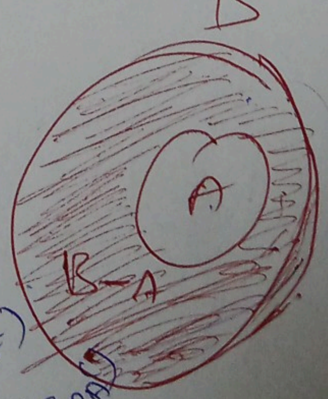
<u>Theorem:4</u>

If $A \subseteq B$ then $P(A) \leq P(B)$.

A and $B \setminus A$ are mutually exclusive events

<u>Proof:</u> $B = A \cup (B - A)$

where A, $B - A$ are disjoint sets.



$P(B) = P(A \cup (B - A))$

using P_3 (axiom), $= P(A \cup (B \cap A^c))$

$P(A) + P(B \cap A^c)$

≥ 0.

$A \subseteq B$
$B \cap C$

$$P(B) = P(A) + P(B - A).$$

$P(B \cap A^c)$ using axiom P_1,

$P(B - A) \geq 0$.

$$P(B) \geq P(A).$$

$P(B \cap A^c)$
$P(\phi) = 0$

<u>Ex:</u> Rolling of die,

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

A = odd numbers = $\{1, 3, 5\}$

B = less than 6 = $\{1, 2, 3, 4, 5\}$.

$$A \subseteq B.$$

$$P(A) = \frac{1}{2} = \left(\frac{3}{6}\right) \qquad P(B) = \frac{5}{6}.$$

$$P(A) \leq P(B).$$

Theorem: 5

For an event $A \in B$,

$$0 \leq P(A) \leq 1$$

<u>Proof:</u>

from Theo. 4, $A \subseteq \Omega$.

$$P(A) \leq P(\Omega)$$

$$P(A) \leq 1.$$

from P_1 (Axiom)

$$P(A) \geq 0.$$

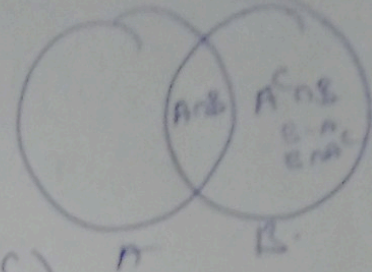
$$\therefore 0 \leq P(A) \leq 1.$$

T.Prove

theorem: $P(A^c \cap B) = P(B) - P(A \cap B)$.

$P(B-A)$

$B = (A \cap B) \cup (B \cap A^c)$



$P(B) = P((A \cap B) \cup (B \cap A^c))$

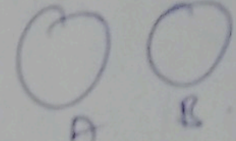
by P_3 axiom,

$P(B) = P(A \cap B) + P(B \cap A^c)$

$P(A^c \cap B) = P(B) - P(A \cap B)$. //

(2) TPT If $A \cap B = \phi$ then $P(A) \leq P(B^c)$.

$A \subseteq B^c$.



from the theorem $A \subseteq B$

$P(A) \leq P(B)$,

$A \subseteq B$

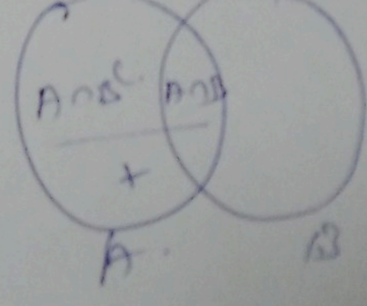
$P(A) \leq P(B)$.

$\Rightarrow P(A) \leq P(B^c)$.

(3) T.P.T $P(A \setminus B) = P(A) - P(A \cap B)$.

$\downarrow$

$A \cap B^c$.



$A = (A \cap B^c) \cup (A \cap B)$

$P(A) \Rightarrow P[(A \cap B^c) \cup (A \cap B)]$

by axiom P_3,

$P(A) = P(A \cap B^c) + P(A \cap B)$

$P(A \cap B^c) = P(A) - P(A \cap B)$ //

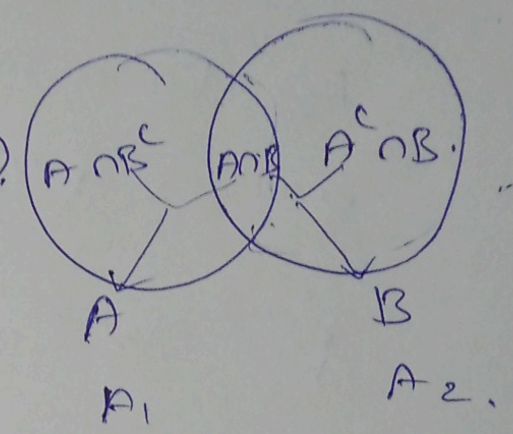
Addition theorem:

For any two events A_1 and A_2 are not disjoint events. $A, B \in \mathcal{B}$.

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

(or)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



$\cancel{P(A \cup B) = P}$

$\cancel{A \cup B = A - B -}$

$$A \cup B = A \cup (B - A \cap B)$$

$$P(A \cup B) = P(A \cup (B - (A \cap B)))$$

$$\Rightarrow P(A) + P(B - (A \cap B))$$

$$= P(A) + P(B) - P(A \cap B).$$

General <u>Addition Rule</u>: Principle of inclusion & exclusion.

For any events $A_1 A_2 . A_n \in \mathcal{B}$

then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i \neq j}^{n} \sum P(A_i \cap A_j)$$

$$+ \sum_{i<j<k} \sum P(A_i \cap A_j \cap A_k)$$

$$+ \ldots + \cancel{\Sigma} (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

<u>Theorem</u>: <u>Subadditivity of Prob function</u>.

For $A_1, A_2 \ldots A_n \in \mathcal{B}$ then

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

<u>Proof</u>: Basis step let $n=2$.

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$\leq P(A_1) + P(A_2) \quad (P(A_1 \cap A_2) \geq 0 \text{ by axiom } 1)$$

Inductive Step:

Assume $n=k$ given Statement true.

$\Rightarrow$ $n=k+1$ is also true.

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i) \text{ is true.}$$

To S.T

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) \text{ is true.}$$

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right).$$

$$\leq P\left(\bigcup_{i=1}^{k} A_i\right) + P\left(A_{k+1}\right).$$

$$\leq \sum_{i=1}^{k} P(A_i) + P\left(A_{k+1}\right)$$

$$= \sum_{i=1}^{k+1} P(A_i).$$

(**) For any countable seq $\{A_i\} \in \mathcal{B}$.

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i).$$

$$\cancel{P\left(\bigcup_{i=1}^{n} A_i\right.}$$

Theorem:- Bonferroni's Inequality

Given $n > 1$ events $A_1, A_2 \ldots A_n \in \mathcal{B}$.

$$\cancel{P\left(\bigcup_{i=1}^{n} A_i\right) \leq} \sum_{i=1}^{n} P(A_i) - \sum_{i<j}^{n} P(A_i \cap A_j)$$

$$\leq P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

2 events:

$$\Rightarrow P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq P(A_1 \cup A_2)$$
$$\leq P(A_1) + P(A_2).$$

$$\Rightarrow P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3)$$
$$- P(A_2 \cap A_3) \leq P(A_1 \cup A_2 \cup A_3)$$
$$\leq P(A_1) + P(A_2) + P(A_3)$$

Boole's Inequality:

Let $\{A_n\}$ be a seq of sets in $\mathcal{B}$

Then $P\left(\bigcap_{i=1}^{\infty} A_i\right) \geq 1 - \sum_{i=1}^{\infty} P(A_i^c)$

2 events:

$$P(A_1 \cap A_2) \geq 1 - P(A_1^c) - P(A_2^c)$$

Proof:

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1 - P\left[\bigcap_{i=1}^{\infty} A_i\right]^c$$

$(P(A) = 1 - P(A^c))$

by demorgan's law

$$= 1 - P\left[\bigcup_{i=1}^{\infty} A_i^c\right]$$

$$\geq 1 - \sum_{i=1}^{\infty} P(A_i^c)$$

$$\boxed{P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)}$$

Example Problem

① If $(\Omega, \mathcal{B})$ is measurable space.

Tossing coin $\Omega = \{H, T\}$

$P(H) = 1/3 \quad P(T) = \frac{2}{3}$

Probability space : $(\Omega, \mathcal{B}, P)$

where P is a valid probability fn.

① axiom of positivity $P(A) \geq 0$.

$\therefore$ since $P(H), P(T) \geq 0$

② axiom of certainty

$1 = P(\Omega) = P(H) + P(T) = \frac{1}{3} + \frac{2}{3} = 1.$

③ axiom of countable additivity.

$$P\left(\bigcup_{i=1} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

$P(H \cup T) = P(H) + P(T)$

~~② If $P(A) = 3/5$, $P(B) = 1/2$ $A, B \in B$.~~

~~$\leq P(AA$~~

(7) If $P(A) = \frac{3}{5}$, $P(B) = \frac{1}{2}$, $A, B \in \mathcal{B}$

$$\_\_\_\_ \leq P(A \cap B) \leq \_\_\_\_$$

Soln: If $A \subseteq B$ then $P(A) \leq P(B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1$$

$$\frac{3}{5} + \frac{1}{2} - P(A \cap B) \leq 1$$

$$\frac{3}{5} + \frac{1}{2} - 1 \leq P(A \cap B)$$

$$P(A \cap B) \geq \frac{1}{10}.$$

$\frac{1}{10} \leq \_\_ \leq \frac{1}{2}.$

$A \cap B \subset B.$

$P(A \cap B) \leq P(B)$

$P(A \cap B) \leq \frac{1}{2}$

$A \cap B \subset A.$

$P(A \cap B) \leq P(A)$

$P(A \cap B) \leq \frac{3}{5}.$

(8) If $P(A) = \frac{1}{3}$, $P(B) = \frac{3}{4}$, $P(A \cap B) = \frac{1}{6}$

Find $P(A \cap B')$

$$P(A \cap B') = P(A) - P(A \cap B).$$

$$= \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$$

(4) A die is rolled twice. Let $\Omega = \{(i,j),\ i,j = (1,2 \ldots 6\}$ are equally likely.

A = number $\leq 2 = \{1,2\}$.

B = atleast $5 = \{5,6\}$

$A \cap B = \{(1,5)\ (1,6),\ (2,5),(2,6)\}$.

Find $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

$$= \frac{2}{6} + \frac{2}{6} - \frac{4}{36} = \frac{5}{9}.$$

(5) A coin is tossed 3 times.

Let A be event that there is atleast one head shown up in 3 thrown?

$P(A) = 1 - P(A^c)$

$= 1 - P(\text{no head})$

$= 1 - \frac{1}{8}$

$= \frac{7}{8}.$

⑥ A Prob of horse A winning a race is $\frac{1}{5}$. The Prob of horse B winning race is $\frac{1}{4}$. Find the Prob

(i) Either of them win

(ii) none them will win.

Soln:

1. $P(A \cup B) = P(A) + P(B)$

$= \frac{1}{5} + \frac{1}{4} = \frac{9}{20}$.

2. $P[(A \cup B)'] = 1 - P(A \cup B)$

$= 1 - \frac{9}{20} = \frac{11}{20}$.

⑦ Let $A, B, C \in \mathcal{B}$, $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{5}$

$P(C) = \frac{1}{6}$, $P(A \cap B) = 0$, $P(B \cap C) = 0$

& $P(C \cap A) = \frac{1}{8}$

find

$\_\_\_\_ \leq P(A \cup B \cup C) \leq \_\_\_\_$

Soln:
By Bonferroni's inequality.

$$\sum_{i=1}^{n} P(A_i) - \sum P(A_i \cap A_j) \le P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$$

$$P(A)+P(B)+P(C)-P(A\cap B) - P(B\cap C)-P(C\cap A) \le P(A\cup B\cup C) \le P(A)+P(B)+P(C)$$

Soln: $\frac{59}{120} \le P(A\cup B\cup C) \le 37/60.$