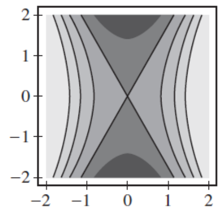
In Exercises 29–36, draw a contour map of $f(x, y)$ with an appropriate contour interval, showing at least six level curves.

36. $f(x, y) = 3x^2 - y^2$

SOLUTION The level curves are the hyperbolas $3x^2 - y^2 = c$, $c \neq 0$, and for $c = 0$ it is the two lines $y = \pm\sqrt{3}x$. We plot a contour map with contour interval $m = 2$ using $c = -4, -2, 0, 2, 4, 6$:



Partial Derivatives:

$$\boxed{f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}, \qquad f_y(a, b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k}}$$

Tangent Plane:

$$\boxed{z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)}$$

Taylor Polynomial:

Directional Derivative:

$$\boxed{D_{\mathbf{u}} f(P) = \nabla f_P \cdot \mathbf{u}}$$

$$D_{\mathbf{u}} f(P) = \nabla f_P \cdot \mathbf{u} = \|\nabla f_P\| \cos\theta \qquad \boxed{4}$$

60. Find the minimum and maximum values of $f(x, y, z) = x - z$ on the intersection of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$ (Figure 5).

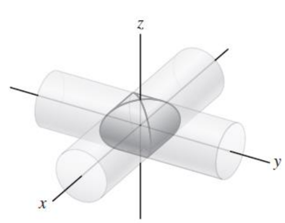


FIGURE 5

SOLUTION Let us use the Lagrange Multipliers method with two constraints for $f(x, y, z) = x - z$ subject to $g(x, y, z) = x^2 + y^2 - 1 = 0$ and $h(x, y, z) = x^2 + z^2 - 1 = 0$. The Lagrange condition would be $\nabla f = \lambda \nabla g + \mu \nabla h$. Noting here that we have $\nabla f = \langle 1, 0, -1 \rangle$, $\nabla g = \langle 2x, 2y, 0 \rangle$, and $\nabla h = \langle 2x, 0, 2z \rangle$. Therefore we have

$$\langle 1, 0, -1 \rangle = \lambda \langle 2x, 2y, 0 \rangle + \mu \langle 2x, 0, 2z \rangle$$

yielding the equations:

$$1 = 2\lambda x + 2\mu x, \quad 0 = 2\lambda y, \quad -1 = 2\mu z$$

Next, using the second equation, we find either $\lambda = 0$ or $y = 0$.

If $y = 0$, then using the first constraint equation, $x = \pm 1$ and using the second constraint equation we find $z = 0$. The derived critical points are then:

$$(1, 0, 0), \quad (-1, 0, 0)$$

If $\lambda = 0$, then using the first equation above we see $1 = 2\mu x$ which implies

$$\mu = \frac{1}{2x}$$

Using the last equation above we have:

$$-1 = 2 \cdot \frac{1}{2x} z \quad \Rightarrow \quad -x = z$$

Then using the second constraint equation, we have

$$2x^2 = 1 \quad \Rightarrow \quad x = \pm \frac{1}{\sqrt{2}}, z = \mp \frac{1}{\sqrt{2}}$$

Using the first constraint equation, we have

$$x^2 + y^2 = 1 \quad \Rightarrow \quad y^2 = \frac{1}{2} \quad \Rightarrow \quad y = \pm \frac{1}{\sqrt{2}}$$

We have four derived critical points here:

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Now to analyze $f(x, y, z) = x - z$ for maximum and minimum values:

$$f(1, 0, 0) = 1, \quad f(-1, 0, 0) = -1$$

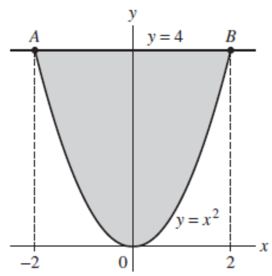
$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \sqrt{2}, \quad f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \sqrt{2}$$

$$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\sqrt{2}, \quad f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\sqrt{2}$$

Hence the maximum value of $f(x, y, z) = x - z$ subject to the two constraints is $\sqrt{2}$, while the minimum value is $-\sqrt{2}$.

53. Find the global extrema of $f(x, y) = 2xy - x - y$ on the domain $\{y \leq 4, y \geq x^2\}$.

SOLUTION The region is shown in the figure.



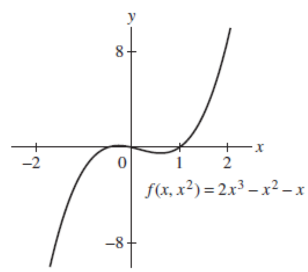
Step 1. Finding the critical points. We find the critical points in the interior of the domain by setting the partial derivatives equal to zero and solving. We get

$$f_x = 2y - 1 = 0$$
$$f_y = 2x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{2}, \quad y = \frac{1}{2}$$

The critical point is $\left(\frac{1}{2}, \frac{1}{2}\right)$. (It lies in the interior of the domain since $\frac{1}{2} < 4$ and $\frac{1}{2} > \left(\frac{1}{2}\right)^2$).

Step 2. Finding the global extrema on the boundary. We consider the two parts of the boundary separately.

The parabola $y = x^2$, $-2 \leq x \leq 2$:



On this curve, $f(x, x^2) = 2 \cdot x \cdot x^2 - x - x^2 = 2x^3 - x^2 - x$. Using calculus in one variable or the graph of the function, we see that the minimum of $f(x, x^2)$ on the interval occurs at $x = -2$ and the maximum at $x = 2$. The corresponding points are $(-2, 4)$ and $(2, 4)$.

The segment $\overline{AB}$: On this segment $y = 4$, $-2 \leq x \leq 2$, hence $f(x, 4) = 2 \cdot x \cdot 4 - x - 4 = 7x - 4$. The maximum value occurs at $x = 2$ and the minimum value at $x = -2$. The corresponding points on the segment $\overline{AB}$ are $(-2, 4)$ and $(2, 4)$

Step 3. Conclusions. Since the global extrema occur either at critical points in the interior of the domain or on the boundary of the domain, the candidates for global extrema are the following points:

$$\left(\frac{1}{2}, \frac{1}{2}\right), \quad (-2, 4), \quad (2, 4)$$

We compute the values of $f = 2xy - x - y$ at these points:

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -\frac{1}{2}$$

$$f(-2, 4) = 2 \cdot (-2) \cdot 4 + 2 - 4 = -18$$
$$f(2, 4) = 2 \cdot 2 \cdot 4 - 2 - 4 = 10$$

We conclude that the global maximum is $f(2, 4) = 10$ and the global minimum is $f(-2, 4) = -18$.

■ **EXAMPLE 6** Let $f(x, y) = xe^y$, $P = (2, -1)$, and $\mathbf{v} = \langle 2, 3 \rangle$. Calculate the directional derivative in the direction of $\mathbf{v}$.

Solution First note that $\mathbf{v}$ is NOT a unit vector. So, we first replace it with the vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 2, 3 \rangle}{\sqrt{13}} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$$

Then compute the gradient at $P = (2, -1)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle e^y, xe^y \rangle \quad \Rightarrow \quad \nabla f_P = \nabla f_{(2,-1)} = \left\langle e^{-1}, 2e^{-1} \right\rangle$$

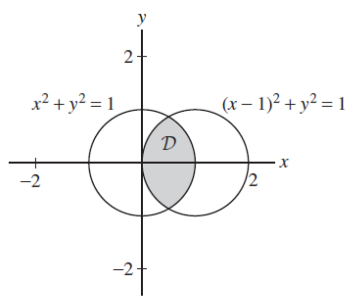
Next use Theorem 3:

$$D_{\mathbf{u}} f(P) = \nabla f_P \cdot \mathbf{u} = \left\langle e^{-1}, 2e^{-1} \right\rangle \cdot \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle = \frac{8e^{-1}}{\sqrt{13}} \approx 0.82. \qquad ■$$

This means that if we think of this function as representing a mountain, then at coordinate $(x, y) = (2, -1)$, we should expect that if we head 1 unit in the direction of vector $\mathbf{v}$, we would have to step up in the vertical direction by approximately 0.82 units.

20. $f(x, y) = y$; $\quad x^2 + y^2 \le 1, \quad (x-1)^2 + y^2 \le 1$

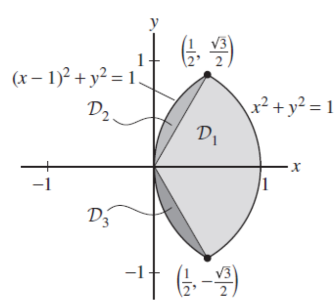
SOLUTION $\mathcal{D}$ is the common region of the two circles shown in the figure.



To evaluate the integral we decompose $\mathcal{D}$ into three regions $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ shown in the figure. Thus,

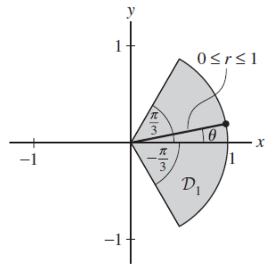
$$\iint_{\mathcal{D}} y\,dA = \iint_{\mathcal{D}_1} y\,dA + \iint_{\mathcal{D}_2} y\,dA + \iint_{\mathcal{D}_3} y\,dA$$

We compute each integral separately.



$\mathcal{D}_1$: The domain $\mathcal{D}_1$ lies in the angular sector $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$ (where $\frac{\pi}{3} = \tan^{-1}\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \tan^{-1}\sqrt{3}$). The circle $x^2 + y^2 = 1$ has polar equation $r = 1$, therefore $\mathcal{D}_1$ has the following definition:

$$\mathcal{D}_1 : -\frac{\pi}{3} \le \theta \le \frac{\pi}{3},\ 0 \le r \le 1$$



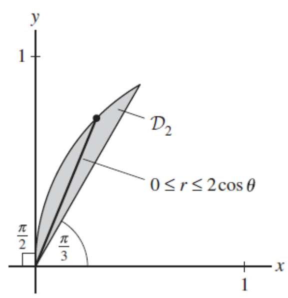
We use change of variables in the integral to write

$$\iint_{\mathcal{D}_1} y\,dA = \int_{-\pi/3}^{\pi/3}\int_0^1 (r\sin\theta)r\,dr\,d\theta = \int_{-\pi/3}^{\pi/3}\int_0^1 r^2\sin\theta\,dr\,d\theta = \int_{-\pi/3}^{\pi/3} \frac{r^3\sin\theta}{3}\bigg|_{r=0}^{1} d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \frac{\sin\theta}{3}\,d\theta = -\frac{\cos\theta}{3}\bigg|_{-\pi/3}^{\pi/3} = 0 \tag{2}$$

$\mathcal{D}_2$: The angle θ is changing in $\mathcal{D}_2$ from $\frac{\pi}{3}$ to $\frac{\pi}{2}$. The circle $(x-1)^2 + y^2 = 1$ has polar equation $r = 2\cos\theta$. Therefore, $\mathcal{D}_2$ has the following description:

$$\mathcal{D}_2 : \frac{\pi}{3} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta$$



Using the Change of Variables Formula gives

$$\iint_{\mathcal{D}_2} y\,dA = \int_{\pi/3}^{\pi/2}\int_0^{2\cos\theta} (r\sin\theta)r\,dr\,d\theta = \int_{\pi/3}^{\pi/2}\int_0^{2\cos\theta} r^2\sin\theta\,dr\,d\theta$$

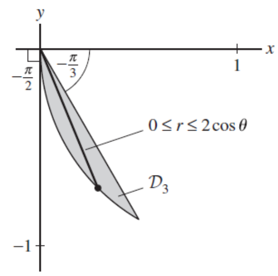
$$= \int_{\pi/3}^{\pi/2} \frac{r^3\sin\theta}{3}\bigg|_{r=0}^{2\cos\theta} d\theta = \int_{\pi/3}^{\pi/2} \frac{8}{3}\cos^3\theta\sin\theta\,d\theta$$

We compute the integral using the substitution $u = \cos\theta$, $du = -\sin\theta\,d\theta$:

$$\iint_{\mathcal{D}_2} y\,dA = \int_{1/2}^{0} \frac{8}{3}u^3(-du) = \int_0^{1/2} \frac{8}{3}u^3\,du = \frac{2}{3}u^4\bigg|_0^{1/2} = \frac{1}{24} \tag{3}$$

$\mathcal{D}_3$: The domain $\mathcal{D}_3$ has the following description:

$$\mathcal{D}_3 : -\frac{\pi}{2} \le \theta \le -\frac{\pi}{3},\ 0 \le r \le 2\cos\theta$$



We obtain the integral (the inner integral was computed previously)

$$\iint_{\mathcal{D}_3} y\,dA = \int_{-\pi/2}^{-\pi/3} \int_0^{2\cos\theta} (r\sin\theta) r\,dr\,d\theta = \int_{-\pi/2}^{-\pi/3} \left(\int_0^{2\cos\theta} r^2 \sin\theta\,dr \right) d\theta = \int_{-\pi/2}^{-\pi/3} \frac{8}{3}\cos^3\theta \sin\theta\,d\theta$$

We use the substitution $\omega = -\theta$ and the integral computed previously to obtain

$$\iint_{\mathcal{D}_3} y\,dA = \int_{\pi/2}^{\pi/3} \frac{8}{3}\cos^3\omega(\sin -\omega)(-d\omega) = -\int_{\pi/3}^{\pi/2} \frac{8}{3}\cos^3\omega \sin\omega\,d\omega = -\frac{1}{24} \qquad (4)$$

Finally, we combine (1), (2), (3), and (4) to obtain the following solution:

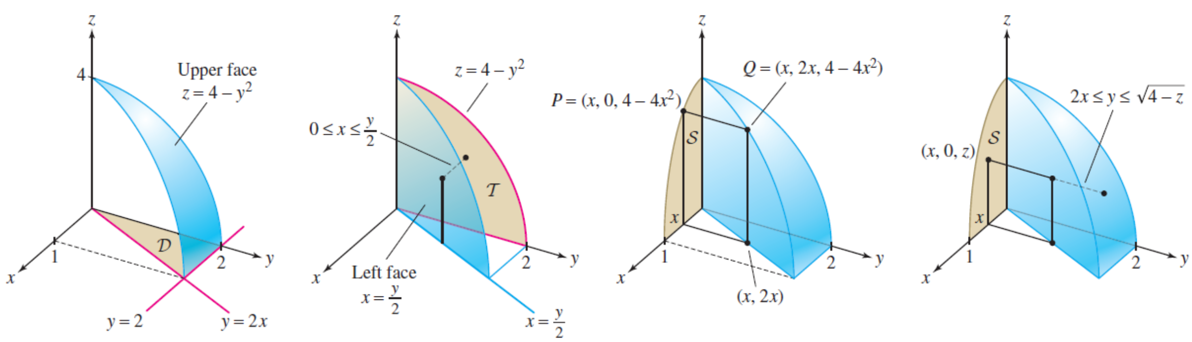
$$\iint_{\mathcal{D}} y\,dA = 0 + \frac{1}{24} - \frac{1}{24} = 0$$

Remark: The integral is zero since the average value of the y-coordinates of the points in $\mathcal{D}$ is zero ($\mathcal{D}$ is symmetric with respect to the x-axis).

EXAMPLE 5 **Writing a Triple Integral in Three Ways** The region $\mathcal{W}$ in Figure 7 is bounded by

$$z = 4 - y^2, \qquad y = 2x, \qquad z = 0, \qquad x = 0$$

Express $\iiint_{\mathcal{W}} xyz\,dV$ as an iterated integral in three ways, by projecting onto each of the three coordinate planes (but do not evaluate).



(A) Projection to xy-plane (B) Projection to yz-plane (C) Projection to xz-plane (D) The y-coordinates of points in the solid satisfy $2x \le y \le \sqrt{4 - z}$.

Solution We consider each coordinate plane separately.

Step 1. **The xy-plane.**

This is a z-simple region. The upper face $z = 4 - y^2$ intersects the first quadrant of the xy-plane ($z = 0$) in the line $y = 2$ [Figure 7(A)]. Therefore, the projection of $\mathcal{W}$ onto the xy-plane is a triangle $\mathcal{D}$ defined by $0 \le x \le 1$, $2x \le y \le 2$, and

$$\mathcal{W} : 0 \le x \le 1, \quad 2x \le y \le 2, \quad 0 \le z \le 4 - y^2$$

$$\iiint_{\mathcal{W}} xyz\, dV = \int_{x=0}^{1} \int_{y=2x}^{2} \int_{z=0}^{4-y^2} xyz\, dz\, dy\, dx \qquad \boxed{4}$$

Step 2. **The yz-plane.**

This is also an x-simple region. The projection of $\mathcal{W}$ onto the yz-plane is the domain $\mathcal{T}$ [Figure 7(B)]:

$$\mathcal{T} : 0 \le y \le 2, \quad 0 \le z \le 4 - y^2$$

The region $\mathcal{W}$ consists of all points lying between $\mathcal{T}$ and the "left face" $x = \frac{1}{2}y$. In other words, the x-coordinate must satisfy $0 \le x \le \frac{1}{2}y$. Thus,

$$\mathcal{W} : 0 \le y \le 2, \quad 0 \le z \le 4 - y^2, \qquad 0 \le x \le \frac{1}{2}y$$

$$\iiint_{\mathcal{W}} xyz\, dV = \int_{y=0}^{2} \int_{z=0}^{4-y^2} \int_{x=0}^{y/2} xyz\, dx\, dz\, dy$$

Step 3. **The xz-plane.**

This is a y-simple region. The challenge in this case is to determine the projection of $\mathcal{W}$ onto the xz-plane, that is, the region $\mathcal{S}$ in Figure 7(C). We need to find the equation of the boundary curve of $\mathcal{S}$. A point P on this curve is the projection of a point $Q = (x, y, z)$ on the boundary of the left face. Since Q lies on both the plane $y = 2x$ and the surface $z = 4 - y^2$, $Q = (x, 2x, 4 - 4x^2)$. The projection of Q is $P = (x, 0, 4 - 4x^2)$. We see that the projection of $\mathcal{W}$ onto the xz-plane is the domain

$$\mathcal{S} : 0 \le x \le 1, \quad 0 \le z \le 4 - 4x^2$$

This gives us limits for x and z variables, so the triple integral can be written

$$\iiint_{\mathcal{W}} xyz\, dV = \int_{x=0}^{1} \int_{z=0}^{4-4x^2} \int_{y=??}^{??} xyz\, dy\, dz\, dx$$

What are the limits for y? The equation of the upper face $z = 4 - y^2$ can be written $y = \sqrt{4-z}$. Referring to Figure 7(D), we see that $\mathcal{W}$ is bounded by the left face $y = 2x$ and the upper face $y = \sqrt{4-z}$. In other words, the y-coordinate of a point in $\mathcal{W}$ satisfies

$$2x \le y \le \sqrt{4-z}$$

Now we can write the triple integral as the following iterated integral:

$$\iiint_{\mathcal{W}} xyz\, dV = \int_{x=0}^{1} \int_{z=0}^{4-4x^2} \int_{y=2x}^{\sqrt{4-z}} xyz\, dy\, dz\, dx \qquad \blacksquare$$

The **average value** of a function of three variables is defined as in the case of two variables:

$$\overline{f} = \frac{1}{\text{Volume}(\mathcal{W})} \iiint_{\mathcal{W}} f(x, y, z)\, dV \qquad \boxed{5}$$

where $\text{Volume}(\mathcal{W}) = \iiint_{\mathcal{W}} 1\, dV$. And, as in the case of two variables, $\overline{f}$ lies between the minimum and maximum values of f on $\mathcal{D}$, and the Mean Value Theorem holds: If $\mathcal{W}$ is connected and f is continuous on $\mathcal{W}$, then there exists a point $P \in \mathcal{W}$ such that $f(P) = \overline{f}$.

In Exercises 9–14, evaluate $\iiint_{\mathcal{W}} f(x, y, z)\, dV$ *for the function* f *and region* $\mathcal{W}$ *specified.*

12. $f(x, y, z) = x; \quad \mathcal{W} : x^2 + y^2 \le z \le 4$

12. $f(x, y, z) = x; \quad \mathcal{W} : x^2 + y^2 \le z \le 4$

SOLUTION Here, $\mathcal{W}$ is the upper half of the solid cone underneath the plane $z = 4$. First we must determine the projection $\mathcal{D}$ of $\mathcal{W}$ onto the xy-plane. First considering the intersection of the cone and the plane $z = 4$ we have:

$$x^2 + y^2 = 4$$

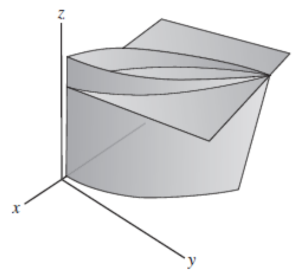
which is a circle centered at the origin with radius 2. The projection into the xy-plane is still $x^2 + y^2 = 4$.

The triple integral can be written as the following iterated integral:

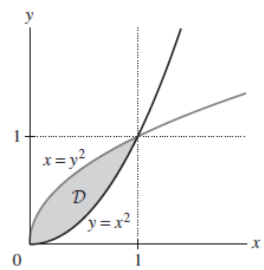
$$\begin{aligned}
\iiint_{\mathcal{W}} x\, dV &= \iint_{\mathcal{D}} \left(\int_{x^2+y^2}^{4} x\, dz \right) dA = \iint_{\mathcal{D}} xz\Big|_{z=x^2+y^2}^{4} dA \\
&= \iint_{\mathcal{D}} x(4 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^2 r\cos\theta(4 - r^2\cos^2\theta - r^2\sin^2\theta) r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 4r^2\cos\theta - r^4\cos\theta\, dr\, d\theta \\
&= \int_0^{2\pi} \frac{4}{3}r^3\cos\theta - \frac{1}{5}r^5\cos\theta\Big|_0^2 d\theta \\
&= \int_0^{2\pi} \frac{64}{15}\cos\theta\, d\theta = \frac{64}{15}\sin\theta\Big|_0^{2\pi} = 0
\end{aligned}$$

20. Find the volume of the solid in $\mathbf{R}^3$ bounded by $y = x^2$, $x = y^2$, $z = x + y + 5$, and $z = 0$.

SOLUTION The solid $\mathcal{W}$ is shown in the following figure:



The upper surface is the plane $z = x + y + 5$ and the lower surface is the plane $z = 0$. The projection of $\mathcal{W}$ onto the xy-plane is the region in the first quadrant enclosed by the curves $y = x^2$ and $x = y^2$.



We use the formula for the volume as a triple integral to write

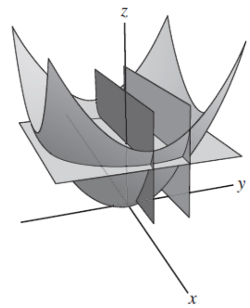
$$\text{Volume}(\mathcal{W}) = \iiint_{\mathcal{W}} 1\, dV$$

The triple integral is equal to the following iterated integral:

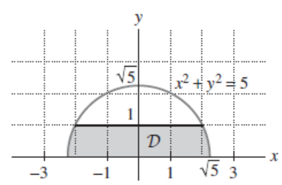
$$\begin{aligned}
\text{Volume}(\mathcal{W}) &= \iiint_{\mathcal{W}} 1\, dV = \iint_{\mathcal{D}} \left(\int_0^{x+y+5} 1\, dz \right) dA = \iint_{\mathcal{D}} z\Big|_{z=0}^{x+y+5} dA \\
&= \iint_{\mathcal{D}} (x + y + 5)\, dA = \int_0^1 \left(\int_{x^2}^{\sqrt{x}} (x + y + 5)\, dy \right) dx = \int_0^1 xy + \frac{y^2}{2} + 5y\Big|_{y=x^2}^{\sqrt{x}} dx \\
&= \int_0^1 \left(x\sqrt{x} + \frac{x}{2} + 5\sqrt{x} - \left(x^3 + \frac{x^4}{2} + 5x^2 \right) \right) dx \\
&= \int_0^1 \left(-\frac{x^4}{2} - x^3 - 5x^2 + x^{3/2} + \frac{x}{2} + 5x^{1/2} \right) dx \\
&= -\frac{x^5}{10} - \frac{x^4}{4} - \frac{5x^3}{3} + \frac{2}{5}x^{5/2} + \frac{x^2}{4} + \frac{10}{3}x^{3/2}\Big|_0^1 \\
&= -\frac{1}{10} - \frac{1}{4} - \frac{5}{3} + \frac{2}{5} + \frac{1}{4} + \frac{10}{3} = \frac{59}{30} = 1\frac{29}{30}
\end{aligned}$$

22. Calculate $\iiint_{\mathcal{W}} y\,dV$, where $\mathcal{W}$ is the region above $z = x^2 + y^2$ and below $z = 5$, and bounded by $y = 0$ and $y = 1$.

SOLUTION The region $\mathcal{W}$ is shown in the figure:



The upper surface is the plane $z = 5$ and the lower surface is the paraboloid $z = x^2 + y^2$. The projection of $\mathcal{W}$ onto the xy-plane is the part of the disk $x^2 + y^2 \le 5$ between the lines $y = 0$ and $y = 1$.



The triple integral of $f(x, y, z) = y$ over $\mathcal{W}$ is equal to the following iterated integral:

$$\begin{aligned}
\iiint_{\mathcal{W}} y\,dV &= \iint_{\mathcal{D}} \left(\int_{x^2+y^2}^{5} y\,dz \right) dA = \iint_{\mathcal{D}} yz \Big|_{z=x^2+y^2}^{5} dA = \iint_{\mathcal{D}} y\left(5 - x^2 - y^2\right) dA \\
&= \int_0^1 \left(\int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} y\left(5 - x^2 - y^2\right) dx \right) dy = 2\int_0^1 y\left(5x - \frac{x^3}{3} - y^2 x\right) \Big|_{x=0}^{\sqrt{5-y^2}} dy \\
&= 2\int_0^1 y\left(\left(5 - y^2\right)x - \frac{x^3}{3}\right) \Big|_{x=0}^{\sqrt{5-y^2}} dy = 2\int_0^1 y\left(\left(5 - y^2\right)^{3/2} - \frac{1}{3}\left(5 - y^2\right)^{3/2}\right) dy \\
&= \int_0^1 \frac{4}{3}\left(5 - y^2\right)^{3/2} y\,dy \qquad (1)
\end{aligned}$$

We compute the integral using the substitution $u = 5 - y^2$, $du = -2y\,dy$:

$$\begin{aligned}
\iiint_{\mathcal{W}} y\,dV &= \int_0^1 \frac{4}{3}\left(5 - y^2\right)^{3/2} y\,dy = \int_5^4 -\frac{2}{3}u^{3/2}\,du = \int_4^5 \frac{2}{3}u^{3/2}\,du = \frac{4}{15}u^{5/2}\Big|_4^5 \\
&= \frac{4}{15}\left(5^{5/2} - 4^{5/2}\right) \approx 6.37
\end{aligned}$$

29. Let

$$\mathcal{W} = \left\{(x, y, z) : \sqrt{x^2 + y^2} \le z \le 1\right\}$$

(see Figure 15). Express $\iiint_{\mathcal{W}} f(x, y, z)\, dV$ as an iterated integral in the order $dz\, dy\, dx$ (for an arbitrary function f).

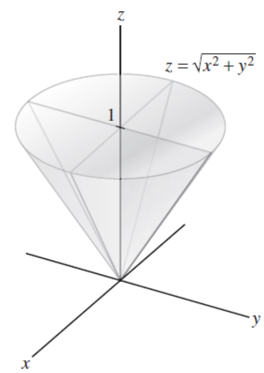
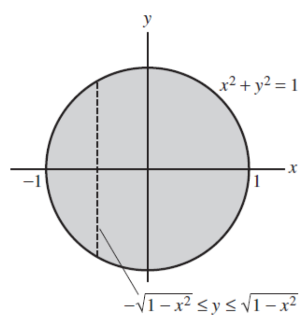


FIGURE 15

SOLUTION To express the triple integral as an iterated integral in order $dz\, dy\, dx$, we must find the projection of $\mathcal{W}$ onto the xy-plane. The upper circle is $\sqrt{x^2 + y^2} = 1$ or $x^2 + y^2 = 1$, hence the projection of $\mathcal{W}$ onto the xy plane is the disk

$$\mathcal{D} : x^2 + y^2 \le 1$$



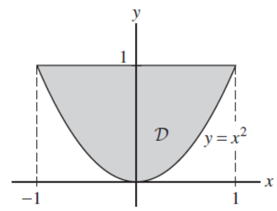
The upper surface is the plane $z = 1$ and the lower surface is $z = \sqrt{x^2 + y^2}$, therefore the triple integral over $\mathcal{W}$ is equal to the following iterated integral:

$$\iiint_{\mathcal{W}} f(x, y, z)\, dV = \iint_{\mathcal{D}} \left(\int_{\sqrt{x^2+y^2}}^{1} f(x, y, z)\, dz \right) dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} f(x, y, z)\, dz\, dy\, dx$$

31. Let $\mathcal{W}$ be the region bounded by $z = 1 - y^2$, $y = x^2$, and the planes $z = 0$, $y = 1$. Calculate the volume of $\mathcal{W}$ as a triple integral in the order $dz\,dy\,dx$.

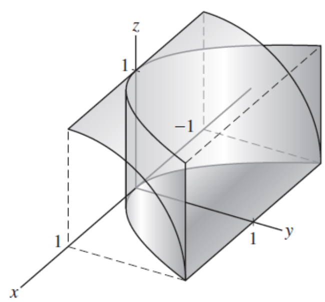
SOLUTION $dz\,dy\,dx$:

The projection of $\mathcal{W}$ onto the xy-plane is the region $\mathcal{D}$ bounded by the curve $y = x^2$ and the line $y = 1$. The region $\mathcal{W}$ consists of all points lying between $\mathcal{D}$ and the cylinder $z = 1 - y^2$.



Therefore, $\mathcal{W}$ can be described by the following inequalities:

$$-1 \le x \le 1, \quad x^2 \le y \le 1, \quad 0 \le z \le 1 - y^2$$



We use the formula for the volume as a triple integral, write the triple integral as an iterated integral, and compute it. We obtain

$$\begin{aligned}
\text{Volume}(\mathcal{W}) &= \iiint_{\mathcal{W}} 1\,dV = \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y^2} 1\,dz\,dy\,dx = \int_{-1}^{1}\int_{x^2}^{1} z\Big|_{z=0}^{1-y^2} dy\,dx \\
&= \int_{-1}^{1}\int_{x^2}^{1}\left(1 - y^2\right) dy\,dx = \int_{-1}^{1} y - \frac{y^3}{3}\Big|_{y=x^2}^{1} dx = \int_{-1}^{1}\left(1 - \frac{1}{3} - \left(x^2 - \frac{x^6}{3}\right)\right) dx \\
&= 2\int_{0}^{1}\left(\frac{x^6}{3} - x^2 + \frac{2}{3}\right) dx = 2\left(\frac{x^7}{21} - \frac{x^3}{3} + \frac{2x}{3}\right)\Big|_{0}^{1} = 2\left(\frac{1}{21} - \frac{1}{3} + \frac{2}{3}\right) = \frac{16}{21}
\end{aligned}$$