Ryan Jung
Winter 2016

Math 118 Final Review

Infinite Series

A) Convergence Testing
B) Finding the Sum of a Series
C) Power Series
 a. Taylor and Maclaurin Series
D) Parametric Equations/Curves

A) Convergence Testing

	Sequence of Partial Sums	For a series $\sum_{n=1}^{\infty} a_n$ n^{th} term of the series represents the sum of the first n terms of the series: $S_n = \sum_{k=1}^{n} a_k$	If the series converges to an exact sum S: • $\lim_{n\to\infty} S_n = S$ • $\sum_{n=1}^{\infty} a_n = S$ Otherwise, the series diverges/does not have a sum.
1)	*Geometric Series*	For a series: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ Where the series converges to/has the sum: $\frac{a}{1-r}$	The geometric series converges to $\frac{a}{1-r}$ if and only if: • $\lvert r\rvert < 1$ or $-1 < r < 1$
2)	*Divergence Test (nth-term test)*	For any series $\sum_{n=1}^{\infty} a_n$	If $\lim_{n\to\infty} a_n \neq 0$ (ie. some non-zero value / $\pm\infty$ / DNE etc.) • Then, the series diverges • If lim = 0, then test fails so use a different test
3)	*Integral Test*	For an integrable series $\sum_{n=1}^{\infty} a_n$	If a_n is: • Always positive (no alternating components) • Decreasing (prove using first derivative/induction) • Continuous function for all n (there is no 1/0 or discontinuity)
		Then, if $\int_{n=1}^{\infty} f(x)dx$ converges	Then, the series $\sum_{n=1}^{\infty} a_n$ also converges. Otherwise, diverges
4)	*P-Series*	For any series of the type $\sum_{n=1}^{\infty} \frac{1}{n^p}$	• If $p > 1$, series is convergent • If $p \leq 1$, series is divergent • If p is less than 0 (negative), use the divergence test
5)	*Comparison Test*	For $\sum_{n=1}^{\infty} a_n$ We can compare it to a similar series: $\sum_{n=1}^{\infty} b_n$ (where b_n is usually p-series/geometric)	If $0 \leq a_n \leq b_n$ (ie. a_n and b_n are both positive): • If b_n converges, a_n must also converge by comparison If $0 \leq b_n \leq a_n$ • If b_n diverges, a_n must also diverge by comparison
6)	*Limit Comparison Test (LCT)*	For $\sum_{n=1}^{\infty} a_n$ pick a similar series: $\sum_{n=1}^{\infty} b_n$ Where: $\lim_{n\to\infty} {a_n}/{b_n} = L$	If $0 < L < \infty$ then: • b_n and a_n do the same thing. ie. if b_n converges, a_n also converges. If b_n diverges a_n also diverges • Otherwise, test fails (use another test)
7)	*Alternating Series Test (AST)*	For an alternating series: $\sum_{n=1}^{\infty}(-1)^n a_n$	If a_n (the non-alternating component) of the series: - Is always positive, - Always decreasing, Then, if the $\lim_{n\to\infty} a_n = 0$ • The entire alternating series converges • Otherwise, test fails (use another test)
8)	*Ratio Test*	For $\sum_{n=1}^{\infty} a_n$ (where a_n usually has factorials, or nth powers) Take the limit: $\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert = L$	For the limit L: • If $L < 1$, series is absolutely convergent • If $L > 1$, series is divergent • If $L = 1$ test fails, use another test
9)	*Root Test*	For $\sum_{n=1}^{\infty} a_n$ (where series a_n is to some common power of n) Take the limit: $\lim_{n\to\infty} \lvert a_n\rvert^{1/n} = L$	For the limit L: • If $L < 1$, series is absolutely convergent • If $L > 1$, series is divergent • If $L = 1$ test fails, use another test

Ryan Jung
Winter 2016

1. Geometric Series

- Where ***a*** is the first term in the geometric series and ***r*** is the common ratio
- Use the geometric series when given a series that has a common ratio r (some constant value) to the power of n.

$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots = \frac{a}{1-r}$ if and only if $-1 < r < 1$, otherwise diverges

eg)	$\sum_{n=1}^{\infty} \frac{1}{9^n} = \sum_{n=1}^{\infty} \left(\frac{1}{9}\right)^n$	First term in the series, $a = \left(\frac{1}{9}\right)^1 = \frac{1}{9}$ Common ratio, $r = \frac{1}{9}$	Since the common ratio, $\lvert r = \frac{1}{9} \rvert = \frac{1}{9} < 1$ The series converges to $\frac{a}{1-r} = \frac{1/9}{1-1/9} = \frac{1}{8}$
eg)	$\sum_{n=3}^{\infty} 4\left(\frac{1}{3}\right)^{n-1} = 4\left(\frac{1}{3}\right)^{-1} \sum_{n=3}^{\infty} \left(\frac{1}{3}\right)^n$	First term in the series, $a = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$ Common ratio, $r = \frac{1}{3}$	Since the common ratio, $\lvert r = \frac{1}{3} \rvert = \frac{1}{3} < 1$ The series converges to $4\left(\frac{1}{3}\right)^{-1}\left(\frac{a}{1-r}\right) = 12\left(\frac{\frac{1}{27}}{1-1/3}\right) = \frac{2}{3}$
or:	*Re-index series to start at 0:* $\sum_{n=3}^{\infty} 4\left(\frac{1}{3}\right)^{n-1} = \sum_{n=0}^{\infty} 4\left(\frac{1}{3}\right)^{n+2}$	$\sum_{n=0}^{\infty} 4\left(\frac{1}{3}\right)^{n+2} = \sum_{n=0}^{\infty} \frac{4}{9}\left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} ar^n$ $a = \frac{4}{9}$ & $r = \frac{1}{3}$	Since the common ratio, $\lvert r = \frac{1}{3} \rvert = \frac{1}{3} < 1$ The series converges to $\left(\frac{a}{1-r}\right) = \left(\frac{4/9}{1-1/3}\right) = \frac{2}{3}$

2. Divergence Test

- When given a series, should usually use divergence test first since it is the easiest test to use.
- This test works because we know that:
  - A convergent series <u>always</u> has a limit that approaches 0.
  - A divergent series might have a limit that can approach any value including 0.
- Therefore, if the limit is non-zero, we immediately know it is divergent, whereas, if the limit = 0, it can be either convergent/divergent.

eg)	$\sum_{n=1}^{\infty} \frac{n^2+2}{4n^2-n}$	$\lim_{n\to\infty} \frac{n^2+2}{4n^2-n} = \frac{1}{4}$	Since the limit $L = \frac{1}{4} \neq 0$, the series ***diverges***.
eg)	$\sum_{n=1}^{\infty} \frac{n^2+2}{4n^3-n}$	$\lim_{n\to\infty} \frac{n^2+2}{4n^3-n} = \lim_{n\to\infty} \frac{1+\frac{2}{n^2}}{4n-\frac{1}{n}} = 0$	Since the limit $L = 0$, the test fails, use another test.
eg)	$\sum_{n=1}^{\infty} (-1)^n n^2$	$\lim_{n\to\infty} (-1)^n n^2 = DNE$	Since the limit $L = DNE \neq 0$, the series ***diverges***.

3. Integral Test

- Use the integral test when the series is easy to integrate (simple integral/integration by substitution/by parts etc.)
- First show that the function is continuous, positive and decreasing.
- To show that function is always decreasing: take the first derivative or use induction.

eg)	$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ *Observe that we can use u-substitution*	• Since $n \geq 2$, function is always continuous (no 1/0 possibility). • Function is always positive (no negative terms) • Decreasing (first derivative): $f(x) = \frac{1}{x(\ln(x))^2} = x^{-1}(\ln(x))^{-2}$ $f'(x) = -\frac{1}{x^2(lnx)^2} - \frac{2}{x^2(lnx)^3}$ Since, $x \geq 2$ to ∞, the first derivative is always negative (decreasing)	Improper Integral: $\lim_{t\to\infty} \int_2^t \frac{1}{x(\ln(x))^2} dx$ $\quad u = lnx \quad du = \frac{1}{x} dx \quad$ $\lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u^2} du$ $\lim_{t\to\infty} \left[-\frac{1}{u}\right]_{ln2}^{lnt} = \lim_{t\to\infty} \left[-\frac{1}{\ln t} - \left(-\frac{1}{\ln 2}\right)\right] = \frac{1}{\ln 2}$ Therefore, series ***converges*** (not to $\frac{1}{\ln 2}$ however)
eg)	$\sum_{n=1}^{\infty} ne^{-2n}$	• Since $n \geq 1$, function is always continuous (no 1/0 possibility). • Function is always positive (no negative terms) • Decreasing (induction) if: $a_{n+1} < a_n$ $\frac{(n+1)}{e^{2(n+1)}} < \frac{n}{e^{2n}}$ $\quad \frac{e^{2n}}{e^{2n+2}} < \frac{n}{n+1}$ $\quad \frac{1}{e^2} < \frac{n}{n+1}$ $\frac{1}{e^2}(n+1) < n$ $\quad 0 < n - \frac{1}{e^2}n - \frac{1}{e^2}$ This inequality is always true for $n \geq 1$ to ∞, therefore, decreasing	Improper Integral: integration by parts: $\lim_{t\to\infty} \int_1^t xe^{-2x} dx$ $\quad u = x \quad dv = e^{-2x}dx$ $du = dx \quad v = -\frac{1}{2}e^{-2x}$ $= \lim_{t\to\infty} \int_1^t xe^{-2x} dx = \lim_{t\to\infty} \left[-\frac{x}{2}e^{-2x}\right]_1^t + \frac{1}{2}\lim_{t\to\infty} \int_1^t e^{-2x} dx$ $= \lim_{t\to\infty} \left[-\frac{t}{2}e^{-2t} - \left(-\frac{1}{2}e^{-2}\right)\right] + \frac{1}{2}\lim_{t\to\infty} \left[-\frac{1}{2}e^{-2x}\right]_1^t$ $= \frac{1}{2}e^{-2} + \frac{1}{2}\lim_{t\to\infty} \left[-\frac{1}{2}e^{-2t} - \left(-\frac{1}{2}e^{-2}\right)\right] = \frac{1}{2}e^{-2} + \frac{1}{2}\left(\frac{1}{2}e^{-2}\right)$ Therefore, series ***converges***

4. P-Series

eg)	$\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$	Since $P = 1.5 > 1,$ the series ***converges***.
eg)	$\sum_{n=1}^{\infty} \frac{n^{1/2}}{n^{3/2}} = \sum_{n=1}^{\infty} n^{\frac{1}{2}-\frac{3}{2}} = \sum_{n=1}^{\infty} n^{-1}$	Since $P = 1 \leq 1,$ the series ***diverges.*** *(harmonic series coincidentally)*
eg)	$\sum_{n=1}^{\infty} \frac{1}{n^{-2}}$	Since $P = -2$ is negative, better to use divergence test: $\sum_{n=1}^{\infty} n^2$ $\lim_{n\to\infty} n^2 = \infty \neq 0$ therefore, ***diverges***.

5. Comparison Test

- Use when the above 4 tests are not applicable for the given series.
- Pick a comparable series that we know information about (usually p-series/geometric series).
- If the comparison test does not yield satisfactory results from the inequality, try moving on to the limit comparison test or pick a different p-series/geometric series to compare with.

eg)	$\sum_{n=1}^{\infty} \frac{2n-1}{4n^3+n}$	For $a_n = \frac{2n-1}{4n^3+n}$, notice that we can remove negligible terms on both the denominator and numerator: $b_n = \frac{2n}{4n^3}$	$a_n = \frac{2n-1}{4n^3+n} < \frac{2n}{4n^3} = b_n$ Since $b_n = \sum_{n=1}^{\infty} \frac{2n}{4n^3} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, $a_n = \sum_{n=1}^{\infty} \frac{2n-1}{4n^3+n}$ must also converge by comparison.
eg)	$\sum_{n=1}^{\infty} \frac{n^2+n+2}{4n^3-n}$	For $a_n = \frac{n^2+n+2}{4n^3-n}$, notice that we can remove negligible terms on both the denominator and numerator: $b_n = \frac{n^2}{4n^3}$	$a_n = \frac{n^2+n+2}{4n^3-n} > \frac{n^2}{4n^3} = b_n$ Since $b_n = \sum_{n=1}^{\infty} \frac{n^2}{4n^3}$ is a divergent p-series, $a_n = \sum_{n=1}^{\infty} \frac{n^2+2}{4n^3-n}$ must also diverge by comparison.
or	$\sum_{n=1}^{\infty} \frac{n^2+2}{4n^3-n}$	For $a_n = \frac{n^2+2}{4n^3-n}$, change the negligible terms on either the denominator and numerator into the same degree as the dominant term: $b_n = \frac{n^2+2}{4n^3-n^3}$ or $b_n = \frac{n^2+2n^2}{4n^3-n}$	$a_n = \frac{n^2+2}{4n^3-n} > \frac{n^2+2}{4n^3-n^3} > \frac{n^2}{3n^3} = b_n$ Since $b_n = \sum_{n=1}^{\infty} \frac{n^2}{3n^3}$ is a divergent p-series, $a_n = \sum_{n=1}^{\infty} \frac{n^2+2}{4n^3-n}$ must also diverge by comparison.

6. Limit Comparison Test (LCT)

- Pick b_n so that when we take the limit, we can divide both top and bottom by the highest degree of n.
- Or follow same logic as comparison test (remove negligible terms from top and bottom to get b_n).

eg)	$\sum_{n=1}^{\infty} \frac{n-5}{2n^2+3n-2}$	For $a_n = \frac{n-5}{2n^2+3n-2}$, notice that we can make both top and bottom the same highest power of n (n^2) by picking: $b_n = \frac{1}{n}$ $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n-5}{2n^2+3n-2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^2-5n}{2n^2+3n-2} = \frac{1}{2}$	Since the limit $L = \frac{1}{2}$, is $0 < L < \infty$, a_n and b_n share the same convergence properties. Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series, $\sum_{n=1}^{\infty} \frac{n-5}{2n^2+3n-2}$ is also a divergent series by LCT
eg)	$\sum_{n=1}^{\infty} \frac{\sqrt{n^2+2n}}{n^{5/2}+15n}$	For $a_n = \frac{\sqrt{n^2+2n}}{n^{5/2}+15n}$, notice that we can make both top and bottom the same highest power of n ($n^{5/2}$) by picking: $b_n = \frac{1}{n^{3/2}}$ $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n^2+2n}}{n^{5/2}+15n} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n^2+2n}}{n^{3/2}(n+15n^{2/3})} = 1$	Since the limit $L = 1$, is $0 < L < \infty$, a_n and b_n share the same convergence properties. Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series, $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+2n}}{n^{5/2}+15n}$ is also a convergent series by LCT
eg)	$\sum_{n=1}^{\infty} \frac{2^n-1}{3^n+2}$	For $a_n = \frac{2^n-1}{3^n+2}$, notice that we can make both top and bottom the same highest power n by picking: $b_n = \frac{2^n}{3^n}$ $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n-1}{3^n+2} \cdot \frac{3^n}{2^n} = \lim_{n\to\infty} \frac{3^n}{2^n} \cdot \frac{2^n}{3^n} \frac{\left(1-\frac{1}{2^n}\right)}{\left(1+\frac{2}{3^n}\right)} = 1$	Since the limit $L = 1$, is $0 < L < \infty$, a_n and b_n share the same convergence properties. Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2^n}{3^n}$ is convergent geometric series, $\sum_{n=1}^{\infty} \frac{2^n-1}{3^n+2}$ is also a convergent series by LCT

Ryan Jung
Winter 2016

7. Alternating Series Test (AST)

- First test that the non-alternating part of the series is continuous, positive, and decreasing first before taking limit.
- If the limit does not go to zero, use another test (like divergence test/ratio test etc.)

eg)	$\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{n^2+1}\right)$	For the non-alternating part of the series, $a_n = \frac{n}{n^2+1}$ • Since $n \geq 1$, function is always continuous (no 1/0 possibility). • Function is always positive (no negative terms) • Decreasing (can use induction): $check\ the\ ratio: \frac{a_{n+1}}{a_n} < 1$ $\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{(n+1)^2+1}}{\frac{n}{n^2+1}} = \frac{n^3+n^2+n+1}{n^3+2n^2+2n}$ Since, $n \geq 1\ to\ \infty$, the ratio is always < 1 (decreasing)	Check the limit: $\lim_{n\to\infty}\frac{n}{n^2+1} = 0$ Since the limit = 0, the alternating series $\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{n^2+1}\right)$ Is ***at least conditionally convergent*** (have to check absolute)
eg)	$\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n^2+3}}{n^2+5}$	For the non-alternating part of the series, $a_n = \frac{\sqrt{n^2+3}}{n^2+5}$ • Since $n \geq 1$, function is always continuous (no 1/0 possibility). • Function is always positive (no negative terms) • Decreasing (can use induction): check $a_{n+1} < a_n \quad \frac{a_{n+1}}{a_n} < 1$ $\frac{a_{n+1}}{a_n} = \frac{\sqrt{(n+1)^2+3}}{(n+1)^2+5}\cdot\frac{n^2+5}{\sqrt{n^2+3}}$ $= \frac{\sqrt{n^2+2n+4}}{\sqrt{n^2+3}}\cdot\frac{n^2+5}{n^2+2n+6}$ Since, $n \geq 1\ to\ \infty$, the ratio is always < 1 (decreasing)	Check the limit: $\lim_{n\to\infty}\frac{\sqrt{n^2+3}}{n^2+5} = 0$ Since the limit = 0, the alternating series $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n^2+3}}{n^2+5}$ Is ***at least conditionally convergent*** (have to check absolute)

8. Ratio Test

- Use the ratio test whenever there are factorials, powers of n, alternating components in the series.
- Ratio test can be used to tell if a series is **absolutely convergent** right off the bat for any alternating series.

eg)	$\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$	$\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert = \lim_{n\to\infty}\left\lvert\frac{(2(n+1))!}{((n+1)!)^2}\cdot\frac{(n!)^2}{(2n)!}\right\rvert = \lim_{n\to\infty}\left\lvert\frac{(2n+2)!}{((n+1)!)^2}\cdot\frac{(n!)^2}{(2n)!}\right\rvert$ $= \lim_{n\to\infty}\left\lvert\frac{(2n+2)(2n+1)(2n)!}{(n+1)^2(n!)^2}\cdot\frac{(n!)^2}{(2n)!}\right\rvert = \lim_{n\to\infty}\left\lvert\frac{4n^2+6n+2}{n^2+2n+1}\right\rvert = 4$	Since the limit $L = 4 > 1$, The series is ***divergent*** by the ratio test.
eg)	$\sum_{n=1}^{\infty}(-1)^n\left(\frac{3^n}{n^2}\right)$	$\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert = \lim_{n\to\infty}\left\lvert(-1)^{n+1}\left(\frac{3^{n+1}}{(n+1)^2}\right)\cdot\frac{n^2}{(-1)^n\,3^n}\right\rvert$ $= \lim_{n\to\infty}\left\lvert\frac{3^n\cdot 3\cdot n^2}{(n^2+2n+1)3^n}\right\rvert = 3\lim_{n\to\infty}\left\lvert\frac{n^2}{(n^2+2n+1)}\right\rvert = 3$	Since the limit $L = 3 > 1$, The series is ***divergent*** by the ratio test.
eg)	$\sum_{n=1}^{\infty}n\left(\frac{3}{4}\right)^n$	$\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert = \lim_{n\to\infty}\left\lvert(n+1)\left(\frac{3}{4}\right)^{n+1}\cdot\frac{1}{n\left(\frac{3}{4}\right)^n}\right\rvert$ $= \lim_{n\to\infty}\left\lvert(n+1)\left(\frac{3}{4}\right)\cdot\frac{1}{n}\right\rvert = \frac{3}{4}$	Since the limit $L = \frac{3}{4} < 1$, The series is ***absolutely convergent*** by the ratio test.

9. Root Test

- Use the root test whenever the entire series/all the terms are to the same power of n.
- Root test can be used to tell if a series is **absolutely convergent** right off the bat for any series.
- Root test uses same conditions as the ratio test.

eg)	$\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\tan^{-1}n\right)^{3n}$	$\lim_{n\to\infty}\lvert a_n\rvert^{\frac{1}{n}} = \lim_{n\to\infty}\left\lvert\left(\frac{n}{n+1}\tan^{-1}n\right)^{3n}\right\rvert^{1/n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\tan^{-1}n\right)^{3} = \left(\frac{\pi}{2}\right)^{3}$	Since the limit $L = \left(\frac{\pi}{2}\right)^3 > 1$, The series is ***divergent*** by the root test.
eg)	$\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{2n+1}\right)^{n}$	$\lim_{n\to\infty}\lvert a_n\rvert^{\frac{1}{n}} = \lim_{n\to\infty}\left\lvert(-1)^n\left(\frac{n}{2n+1}\right)^{n}\right\rvert^{1/n} = \lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2}$	Since the limit $L = \frac{1}{2} < 1$, The series is ***absolutely convergent*** by the root test.

- **Notes:**
- Remember: $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ or $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$
- For ANY alternating series:
  - Absolute convergence also implies conditional convergence.
  - Conditional convergences does not imply absolute convergence.
  - Could be good idea to first take absolute value of an alternating series, then test for absolute convergence first. If the absolute value of the series diverges, then test for conditional convergence using something like the AST/ratio/divergence etc.

B) Finding the Sum of a Series (Using Approximations)

- Only way to find exact sum of a series is to use a geometric series ($\frac{a}{1-r}$)/partial sums with $\lim_{n\to\infty} S_n = S$ (infinite terms)
- For any other series, we have to use an approximation with an error bound.

1. Alternating Series Estimation Theorem (ASET)

- Can only be used for any alternating series
- The truncation error is less than the first term omitted (the n+1th term).

$$\boldsymbol{|S - S_n| \le |a_{n+1}|}$$

eg)	Use 5 terms to approximate the sum of: $\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{n^2+1}\right)$	$S \approx S_5 = -\frac{1}{2}+\frac{2}{5}-\frac{3}{10}+\frac{4}{17}-\frac{5}{26} \approx -0.3570136$	Error bound: $\lvert a_{n+1}\rvert$ $\lvert error\rvert \le \lvert a_6\rvert$ $\lvert error\rvert \le \left\lvert(-1)^6\left(\frac{6}{6^2+1}\right)\right\rvert$ $\lvert error\rvert \le 0.162\ldots < 0.2$	$\lvert S - S_5\rvert \le 0.2$ $\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{n^2+1}\right) = -0.357 \pm 0.2$
eg)	Find the number of terms to obtain error < 10^{-3} for $\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{n^2+1}\right)$	We need to find the error bound at the n+1th term: $\left\lvert(-1)^{N+1}\left(\frac{N+1}{(N+1)^2+1}\right)\right\rvert < 10^{-3}$	$\left(\frac{N+1}{N^2+2N+2}\right) < 10^{-3}$ $1000N + 1000 < N^2 + 2N + 2$ $N^2 - 998N - 998 > 0$	$N = \frac{-(-998)\pm\sqrt{(-998)^2-4(1)(-998)}}{2(1)}$ $N = \frac{998\pm1000}{2(1)}$ reject (-) $N = \frac{998\pm1000}{2(1)} = 999$ terms required

2. Error Bounds for Estimation Using the Integral Test of a Converging Series

- If a series converges using the integral test, we can approximate it and provide an upper/lower bound on the error:

$$\boldsymbol{\int_{n+1}^{\infty} f(x)dx \le |S - S_n| \le \int_{n}^{\infty} f(x)dx}$$

ie. $lower\ bound \le |error| \le upper\ bound$

C) Power Series

- Of the form (where the x-variable is to some power of n):

$\sum_{n=0}^{\infty} a_n(x-c)^n$	For x centered at c.
$\sum_{n=0}^{\infty} a_n x^n$	For x centered at 0.

- Concerned with finding the following things:
  1) Radius and Interval of convergence of the power series

2) Generating a power series given a function
3) The sum/function the series converges to

1. Finding Radius and Interval of Convergence

- Two methods to find radius of convergence (of what value x can be for a series to still converge):

1)	Ratio Test	$\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert = \lvert L\rvert$ where $\lvert L\rvert < 1$
2)	Find R directly	$R = \lim_{n\to\infty}\left\lvert\frac{a_n}{a_{n+1}}\right\rvert$

- After finding the radius, we have to check the endpoints of the open interval since we don't know whether the endpoints converge or not (since the conditions of the ratio test when L=1 are unknown).

<table>
<tr><td>eg)</td><td>Find the radius and interval of convergence of the power series:
$\sum_{n=1}^{\infty}\frac{1}{n^2}x^{2n}$</td><td>Use Ratio Test:
$\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert = \lim_{n\to\infty}\left\lvert\frac{x^{2(n+1)}}{(n+1)^2}\cdot\frac{n^2}{x^{2n}}\right\rvert$
$\lim_{n\to\infty}\left\lvert\frac{x^{2n+2}}{x^{2n}}\cdot\frac{n^2}{n^2+2n+1}\right\rvert = \lvert x^2\rvert$</td><td>Therefore, convergence using ratio test:
$\lvert x^2\rvert < 1 \quad \lvert x\rvert < 1^{\frac{1}{2}} \quad \lvert x\rvert < 1$
ie. $-1 < x < 1$
which implies the radius, $R = 1$</td><td>Now test endpoints of the interval:
$-1 < x < 1$
Since we don't know what happens at $x = 1$ and $x = -1$</td></tr>
<tr><td colspan="5">When $x = 1,\ \sum_{n=1}^{\infty}\frac{1}{n^2}(1)^{2n} = \sum_{n=1}^{\infty}\frac{1}{n^2}$ Convergent p-series (n = 2). Therefore, the series converges when $x = 1$</td></tr>
<tr><td colspan="5">When $x = -1,\ \sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^{2n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ Convergent by AST. Therefore, the series converges at $x = -1$</td></tr>
<tr><td colspan="5">Therefore, Interval of Convergence is: $[-1,\ 1]$ with square brackets</td></tr>
<tr><td>eg)</td><td>Find the radius and interval of convergence of the power series:
$\sum_{n=1}^{\infty}\frac{n}{(n+1)^3 2^n}(x-1)^{2n}$</td><td>Use Ratio Test:
$\lim_{n\to\infty}\left\lvert\frac{(n+1)(x-1)^{2(n+1)}}{(n+2)^3 2^{n+1}}\cdot\frac{(n+1)^3 2^n}{n(x-1)^{2n}}\right\rvert$
$\lim_{n\to\infty}\left\lvert\frac{(x-1)^{2n+2}}{(x-1)^{2n}}\cdot\frac{(n+1)^4 2^n}{n(n+2)^3 2^{n+1}}\right\rvert = \left\lvert\frac{(x-1)^2}{2}\right\rvert$</td><td>∴ convergence using ratio test:
$\left\lvert\frac{(x-1)^2}{2}\right\rvert < 1 \quad \lvert(x-1)^2\rvert < 2$
$\lvert x-1\rvert < \sqrt{2}$ x centered at x=1
ie. $-\sqrt{2} < (x-1) < \sqrt{2}$
which implies the radius, $R = \sqrt{2}$</td><td>Now test endpoints of the interval:
$-\sqrt{2}+1 < x < \sqrt{2}+1$
When x is centered at x = 1</td></tr>
<tr><td colspan="5">When $x = -\sqrt{2}+1,\ \sum_{n=1}^{\infty}\frac{n}{(n+1)^3 2^n}\left(-\sqrt{2}+1-1\right)^{2n} = \sum_{n=1}^{\infty}\frac{n}{(n+1)^3 2^n}\left(-\sqrt{2}\right)^{2n} = \sum_{n=1}^{\infty}\frac{n2^n}{(n+1)^3 2^n} = \sum_{n=1}^{\infty}\frac{n}{(n+1)^3}$ use LCT, $b_n = \frac{1}{n^2}$</td></tr>
<tr><td colspan="5">Converges by the limit comparison test. Therefore, series converges when $x = -\sqrt{2}+1$</td></tr>
<tr><td colspan="5">When $x = \sqrt{2}+1,\ \sum_{n=1}^{\infty}\frac{n}{(n+1)^3 2^n}\left(\sqrt{2}+1-1\right)^{2n} = \sum_{n=1}^{\infty}\frac{n}{(n+1)^3 2^n}\left(\sqrt{2}\right)^{2n} = \sum_{n=1}^{\infty}\frac{n2^n}{(n+1)^3 2^n} = \sum_{n=1}^{\infty}\frac{n}{(n+1)^3}$ use LCT, $b_n = \frac{1}{n^2}$</td></tr>
<tr><td colspan="5">Converges by the limit comparison test. Therefore, series converges when $x = \sqrt{2}+1$</td></tr>
<tr><td colspan="5">Therefore, Interval of Convergence is: $[-\sqrt{2}+1,\ \sqrt{2}+1]$ with square brackets</td></tr>
<tr><td>eg)</td><td>Find the radius and interval of convergence of the power series:
$\sum_{n=0}^{\infty}\frac{1}{4^n}(x-2)^n$</td><td>Notice that the power series is geometric, therefore, we don't need to use ratio test:
$\sum_{n=0}^{\infty}\frac{1}{4^n}(x-2)^n = \sum_{n=0}^{\infty}\left[\frac{(x-2)}{4}\right]^n$</td><td>∴ the geometric series converges when:
$\left\lvert\frac{(x-2)}{4}\right\rvert < 1 \quad \lvert x-2\rvert < 4$
x centered at x = 2
ie. $-4 < (x-2) < 4$
which implies the radius, $R = 4$</td><td>The endpoints of the interval are:
$-4+2 < x < 4+2$
$-2 < x < 6$
When x is centered at x = 2</td></tr>
<tr><td colspan="5">We do not need to test endpoints for a geometric series, since we know they diverge when the common ratio $R = 1$. Therefore, the interval of convergence is $(-2, 6)$</td></tr>
</table>

2. Generating a Power Series Given a Function

- Remember how to use the following 4 Maclaurin power series to make your life easier (where x is centered at 0):

1)	Geometric Power Series:	$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$	where $\lvert x\rvert < 1$
2)	e^x	$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$	converges for all x
3)	$\sin x$	$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$	converges for all x
4)	$\cos x$	$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$	converges for all x

eg)	Find a Maclaurin Series for the function: $f(x)=\frac{1}{1+2x^2}$	Relate $f(x)$ to the geometric power series: $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ where $\lvert x\rvert<1$ $\frac{1}{1+2x^2}=\sum_{n=0}^{\infty}(-2x^2)^n=\sum_{n=0}^{\infty}(-1)^n2^nx^{2n}$	Therefore, the Maclaurin Series representing the function is: $\frac{1}{1+2x^2}=\sum_{n=0}^{\infty}(-1)^n2^nx^{2n}$ And converges when, $\lvert -2x^2\rvert<1$ $\quad \lvert x^2\rvert<\frac{1}{2}$ $\quad \lvert x\rvert<\frac{1}{\sqrt{2}}$ That is, the radius of convergence is $R=\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$
eg)	Find a Maclaurin Series for the function: $f(x)=e^{x^2}$	Relate $f(x)$ to the e^x Maclaurin series: $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ where it converges for all x $e^{x^2}=\sum_{n=0}^{\infty}\frac{(x^2)^n}{n!}=\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$	Therefore, the Maclaurin Series representing the function is: $e^{x^2}=\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$ And converges for all x.
eg)	Find a Maclaurin Series for the function: $f(x)=x\sin(x^2)$	Relate $f(x)$ to the $\sin x$ Maclaurin series: $\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$ converges for all x $\sin(x^2)=\sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{(2n+1)!}$ $x\sin(x^2)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+3}}{(2n+1)!}$	Therefore, the Maclaurin Series representing the function is: $x\sin(x^2)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+3}}{(2n+1)!}$ And converges for all x.
eg)	Find the Taylor Series centered at $c=3$ for the function: $f(x)=\frac{1}{5-x}$	Relate $f(x)$ to the geometric power series, and re-center function to $c=3$, so when we generate it's power series, we will get a $(x-3)^n$ $5-x=-(x-3)+2$ $\quad f(x)=\frac{1}{2-(x-3)}=\frac{1}{2}\left(\frac{1}{1-\frac{(x-3)}{2}}\right)$ $f(x)=\frac{1}{2}\left(\frac{1}{1-\frac{(x-3)}{2}}\right)=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{[x-3]}{2}\right)^n$	Therefore, the Taylor Series centered at $c=3$ of the function is: $f(x)=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}(x-3)^n$ And converges when: $\left\lvert\frac{(x-3)}{2}\right\rvert<1$ $\quad \lvert x-3\rvert<2$ That is, x is centered at 3 and $R=2$, Interval of convergence of: $(1,5)$

- If you can't relate the function to one of the 4 basic Maclaurin power series, observe if you can:

a) Factor/simplify function (substitution/partial fractions, etc.) and then add/subtract two or more series together.

b) Take the derivative/integral of one of the 4 basic Maclaurin power series to see if it resembles the given function.

Ryan Jung
Winter 2016

eg)	Find a Maclaurin Series for the Function: $f(x) = \ln(1+x^2)$	We know, the derivative of the function will closely resemble a geometric Maclaurin series: $$\frac{d}{dx}[f(x)] = \frac{2x}{1+x^2} = 2x\left(\frac{1}{1+x^2}\right)$$ $$= 2x\sum_{n=0}^{\infty}(-x^2)^n = 2x\sum_{n=0}^{\infty}(-1)^n x^{2n} = 2\sum_{n=0}^{\infty}(-1)^n x^{2n+1}$$	Therefore, $\frac{d}{dx}[f(x)] = 2\sum_{n=0}^{\infty}(-1)^n x^{2n+1}$ To obtain the series for the original function, $f(x)$, we have to integrate both sides of the equation: $$\int \frac{d}{dx}[f(x)]dx = f(x) = 2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{2n+2} + c$$ Therefore, $f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1} + c$
To get rid of c, set $x = 0$, $f(0) = \ln(1+0) = 0 = \sum_{n=0}^{\infty}\frac{(-1)^n (0)^{2n+2}}{n+1} + c$ Therefore, $c = 0$, and our final answer is: $\boldsymbol{ln(1+x^2) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1}}$			
eg)	Find a Maclaurin Series for the Function: $f(x) = \frac{1}{(x-2)^2}$	We know, the derivative of the geometric Maclaurin series function will closely resemble the original function: $\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1}$ Note: If we put in n=0 for the first term of our series, it will return 0, therefore, it is more apt to start the series at n=1.	The original function rearranged: $$f(x) = \frac{1}{(x-2)^2} = \frac{1}{(-2+x)^2} = \frac{1}{2^2}\left(\frac{1}{\left(1-\frac{x}{2}\right)^2}\right)$$ Now substitute into our series generated for $g(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$, $$f(x) = \frac{1}{2^2}\sum_{n=1}^{\infty} n\left(\frac{x}{2}\right)^{n-1} = \sum_{n=1}^{\infty}\frac{n}{2^{n+1}}x^{n-1}$$
Therefore our final answer is: $\boldsymbol{\frac{1}{(x-2)^2}} = \sum_{n=1}^{\infty}\frac{n}{2^{n+1}}x^{n-1}$			