MODEL SOLUTIONS

Department of Physics
Indian Institute of Technology Madras

Quiz I

Date: September 10, 2018 Time: 08:00 – 08:50 AM

Name [ ] Roll No: [ ] Batch [ ]

Instructions

1. Please write your name, roll number and batch number.This booklet should contain 6 single-sided pages (please check).
2. The answers have to be written in the boxes provided. Answers written elsewhere will not be evaluated.
3. Write the answers and sketch your plots with a blue or black pen only.
4. For questions 11 to 13, answers without detailed steps will NOT be awarded full marks.
5. You can use the empty reverse sides for rough work. No extra sheets will be provided.
6. You are not allowed to use a calculator or any other electronic device during the quiz.

For use by examiners(Do not write in this space)

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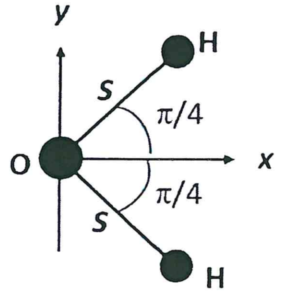
✦ State whether the statement is true or false(write **True** OR **False** in the box.) [1 x 5 = 5]

1. The position vector of a particle is given by $\mathbf{r}(t) = a\sin\omega t\ \hat{\mathbf{i}} + b\cos\omega t\ \hat{\mathbf{j}}$ where a, b, ω are positive constants and $b \neq a$. Then, the product $\mathbf{r}\cdot\mathbf{v}$ becomes zero twice in one complete period, where $\mathbf{v} = d\mathbf{r}/dt$ is the velocity.
2. The potential energy function $U(x) = x/(1+x^2)$, where $-\infty < x < \infty$, has no point of stable equilibrium.
3. The time period of small oscillations of a particle near a minimum of the potential energy function $U(x) = kx^2 + \lambda x^{-2}$ $(k, \lambda > 0)$ is independent of λ.
4. A pendulum consisting of an inextensible string of length ℓ and mass m, is released from rest, at an angle $\theta = \pi/2$ with respect to the vertical. The phase space trajectory representing its oscillatory motion is an ellipse.
5. All pseudo/fictitious forces acting on a particle are proportional to its mass.

1. $\boxed{\text{F}}$ 2. $\boxed{\text{F}}$ 3. $\boxed{\text{T}}$ 4. $\boxed{\text{F}}$ 5. $\boxed{\text{T}}$

✦ **Give the final answer in the box provided, no derivation need to be shown.** [1 x 5 =

6. Consider the model of the water molecule H_2O shown below, with the oxygen atom at the origin. Find the coordinates of the center of mass, (X_{CM}, Y_{CM}). Let the length of each O-H bond be 's.' Assume, for simplicity, that the angle between the two O-H bonds is $\pi/2$. Use, $M_O = 16M_H$.



$X_{CM} = \frac{s}{9\sqrt{2}}$ OR $\frac{\sqrt{2}\,s}{18}$ .... $Y_{CM} =$ ......0......

7. The displacement of an undamped simple harmonic oscillator is given by the equation $x(t) = A\cos\omega t + B\sin\omega t$, where A, B are positive constants such that $A > B$. The amplitude of the motion is given by

$\sqrt{A^2+B^2}$ OR $(A^2+B^2)^{\frac{1}{2}}$

8. For a particle at a point (x, y, z), the potential energy is given by the equation, $U(x, y, z) = x^2 - y^2 + yz$. At the point $(1, 0, 0)$, the unit vector in the direction in which U increases most rapidly is

$\hat{i}$ OR $2\hat{i}$ $(-\hat{i}$ also ok$)$

9. The interaction between two nucleons at a separation r may be approximately represented by the Yukawa potential

$$U(r) = -3\frac{r_0}{r}U_0 e^{-r/r_0}$$

with $U_0 = 50$ MeV and $r_0 = 1.6 \times 10^{-15}$ m. Use $e = 3$ and the conversion 1 eV $= 1.6 \times 10^{-19}$ Nm. The magnitude of the force, in Newtons, between two nucleons at $r = r_0$, is

$\approx 10^4$ N (OR $\frac{3}{e} \times 10^4$ N)

10. The quality factor (Q-factor) of a series LCR circuit with $L = 10\ mH$, $C = 1\ \mu F$ and $R = 2\ \Omega$ is given by

≈ 50

✦ **Answer in detail** (write the calculations and answers within the boxes provided)

11. A particle moves in a conservative force-field given by $\mathbf{F}(\mathbf{r}) = A\,\mathbf{b}\exp(-\mathbf{b}\cdot\mathbf{r})$, where $\mathbf{b} = b(\hat{\mathbf{i}}+\hat{\mathbf{j}})/\sqrt{2}$, $\mathbf{r} = x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$ and A is a constant of appropriate dimensions (also, $\exp(x) = e^x$).

(a) Determine the corresponding potential energy $U(x, y, z)$.

(b) Using the result in (a) or otherwise, evaluate the work done by the force in moving a particle from the origin to the point $(\sqrt{2}/b, 0, 0)$.

2+1=3 marks

(a)

either +1

$$-\frac{\partial U}{\partial x} = A\cdot\frac{b}{\sqrt{2}}\, e^{-\frac{b}{\sqrt{2}}(x+y)} = F_x \quad -(1)$$

$$-\frac{\partial U}{\partial y} = A\,\frac{b}{\sqrt{2}}\, e^{-\frac{b}{\sqrt{2}}(x+y)} = F_y \quad -(2)$$

$$-\frac{\partial U}{\partial z} = 0 = F_z \quad -(3)$$

From (1), $U = -\int F_x\,dx + g(y,z)$

$$= e^{-\frac{b}{\sqrt{2}}(x+y)} + g(y,z) \quad -(4)$$

Use (4) in (2): $\frac{b}{\sqrt{2}}\cancel{e^{-\frac{b}{\sqrt{2}}(x+y)}} = \frac{b}{\sqrt{2}}\cancel{e^{-\frac{b}{\sqrt{2}}(x+y)}} + \frac{\partial g}{\partial y}$

$$\Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = g(z) \quad (5)$$

From (3), using (5), $\frac{\partial g}{\partial z} = 0 \Rightarrow g(z) = \text{constant}$.

+1

$$\therefore\ U(x,y,z) = e^{-\frac{b}{\sqrt{2}}(x+y)} + \text{const} = e^{-\vec{b}\cdot\vec{r}} + \text{const}.$$

(b) Using U

either +1

$$W = U(0,0,0) - U\left(\frac{\sqrt{2}}{b}, 0, 0\right) = 1 - 1/e.$$

By direct integration

$$W = \int_0^{\sqrt{2}/b} F_x(x,0,0)\,dx = \frac{b}{\sqrt{2}}\int_0^{\sqrt{2}/b} e^{-\frac{b}{\sqrt{2}}x}\,dx = 1 - 1/e.$$

12. A **critically damped** oscillator of unit mass with natural angular frequency ω_0 is subjected to an external force $f(t) = f_0 e^{-\alpha t}$, where α is a positive constant and $\alpha \neq \omega_0$. The equation of motion has the form

$$\ddot{x} + 2\omega_0 \dot{x} + \omega_0^2 x = f(t). \tag{1}$$

(a) Determine the particular (steady state) solution $x_p(t)$ to Eq.(1). (Hint: try $x_p(t) = Pe^{Qt}$, find P and Q by substitution).

(b) Write down (no derivation required) the general solution $x_h(t)$ to the corresponding homogeneous differential equation.

(c) Find the complete solution $x(t)$ to Eq. (1) using the initial conditions, $x(0) = 0$, $\dot{x}(0) = 0$. [Hint: Use your results in (a) and (b)].

(d) Find the limiting form of the solution in (c), when $\alpha \to \omega_0$. Your answer should involve only f_0, ω_0 and t.

1+1+1+1=4 marks

(a) $\ddot{x}_p + 2\omega_0 \dot{x}_p + \omega_0^2 x_p = f_0 e^{-\alpha t}$

Substitute $x_p = Pe^{Qt}$

$\Rightarrow (Q^2 + 2\omega_0 Q + \omega_0^2) Pe^{Qt} = f_0 e^{-\alpha t}$

$\frac{1}{2} + \frac{1}{2}$ $\Rightarrow Q = -\alpha, \quad P = \dfrac{f_0}{(\alpha - \omega_0)^2}.$

+1 (b) $x_h(t) = (A + Bt) e^{-\omega_0 t}$; A and B arbitrary.

($\frac{1}{2}$ for each independent solution).

(c) $x(t) = x_h(t) + x_p(t)$

$= (A + Bt) e^{-\omega_0 t} + \dfrac{f_0}{(\alpha - \omega_0)^2} e^{-\alpha t}$

Using initial conditions: $x(0) = A + \dfrac{f_0}{(\alpha - \omega_0)^2} = 0$

$\frac{1}{2}$ $\Rightarrow A = -\dfrac{f_0}{(\alpha - \omega_0)^2}$ —(1).

$\dot{x}(0) = -\omega_0 A + B - \dfrac{\alpha f_0}{(\alpha - \omega_0)^2} = 0$

$\frac{1}{2}$

$$\Rightarrow \quad B = \frac{\alpha f_0}{(\alpha - w_0)^2} - \frac{w_0 f_0}{(\alpha - w_0)^2}$$

$$= \frac{f_0}{(\alpha - w_0)} \quad \text{— (2)}.$$

$$\therefore \quad x(t) = \frac{f_0}{(\alpha - w_0)} t e^{-w_0 t} + \frac{f_0}{(\alpha - w_0)^2} \left[e^{-\alpha t} - e^{-w_0 t} \right] \longrightarrow \bigstar$$

(d) Consider the limit $\alpha \to w_0$.

$$e^{-\alpha t} - e^{-w_0 t} = e^{-w_0 t} \left[e^{-(\alpha - w_0) t} - 1 \right]$$

$$\xrightarrow[\alpha \to w_0]{} e^{-w_0 t} \left[-(\alpha - w_0) t + \frac{(\alpha - w_0)^2}{2} t^2 + \dots \right]$$

+1

$$\Rightarrow \quad x(t) = \frac{f_0}{2} t^2 e^{-w_0 t} \quad \text{when } \alpha \to w_0. \quad (3)$$

ALTERNATE METHOD for (d).

Note that $\bigstar$ has a 0/0 form for $\alpha \to w_0$

Use L'Hospital's rule to get:

$$\lim_{\alpha \to w_0} x(t) = \frac{\cancel{f_0 (-1) t e^{-w_0 t}} + f_0 t (\alpha - w_0)(-t) e^{-w_0 t} - \cancel{f_0 t e^{-w_0 t}}}{2(\alpha - w_0)(-1)}$$

(upon differentiating with w_0)

$$= \frac{f_0 t^2 e^{-w_0 t}}{2} \quad \text{— same as (3)}$$

13. The potential energy for a particle of mass m moving in the one dimensional region $x \geq 0$ is given by

$$U(x) = (1 - \alpha x)e^{-\alpha x},$$

where $\alpha > 0$.

(a) Sketch $U(x)$ versus x for $x \geq 0$.

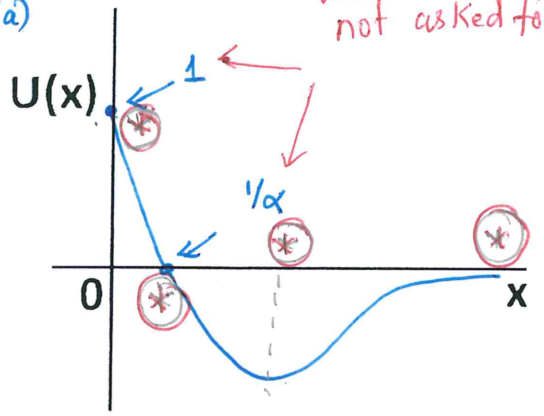
(b) Determine the equilibrium point(s) and the nature of equilibrium (stable/unstable/neutral) at each of these points.

(c) Find the angular frequency of small oscillations about the stable equilibrium point(s).

$\boxed{1+1+1=3 \text{ marks}}$

(a)

+1



Values not asked for!

Look for the (*) points
deduct marks for significant deviations

(b) $$\frac{dU}{dx} = \left(-\alpha e^{-\alpha x} - \alpha e^{-\alpha x} + \alpha^2 x e^{-\alpha x}\right)$$

$$= (\alpha^2 x - 2\alpha)e^{-\alpha x} = 0 \text{ at minima/maxima.}$$

$\frac{1}{2}$ $\Rightarrow$ $x = 2/\alpha$ is an equilibrium point.

$$\left.\frac{d^2U}{dx^2}\right|_{x=2/\alpha} = \left. e^{-\alpha x}(3\alpha^2 - \alpha^3 x)\right|_{x=2/\alpha}$$

$\frac{1}{2}$ $$= \alpha^2/e^2 > 0 \Rightarrow \text{the equilibrium point is } \underline{\text{stable}}.$$

(c) $$\omega_0 = \sqrt{\frac{U''(2/\alpha)}{m}} = \frac{\alpha}{e\sqrt{m}}.$$

$\frac{1}{2}$ $\frac{1}{2}$