1. (a) Angular momentum $\ell = mv_0R = \alpha m\sqrt{2GMR}$ **(0.5)**

Energy $E = \dfrac{mv_0^2}{2} - \dfrac{GMm}{R} = (\alpha^2 - 1)\dfrac{GMm}{R}$ **(0.5)**

(b) Radius of circular orbit $r_0 = \dfrac{\ell^2}{GMm^2} = 2\alpha^2 R$ **(0.5)**

Energy of circular orbit $E_0 = \dfrac{1}{2}U(r_0) = -\dfrac{GMm}{2r_0} = -\dfrac{GMm}{4\alpha^2 R}$ **(0.5)**

(c) Eccentricity $\epsilon = \sqrt{1 - E/E_0} = \sqrt{1 + 4\alpha^2(\alpha^2 - 1)} = \pm(1 - 2\alpha^2)$ **(1)**

OR $|1 - 2\alpha^2|$

(d) In order that the rocket becomes a satellite, we require

$E < 0$ **(condition for bounded orbit)** **(0.5)**

$\implies \alpha < 1$ **(0.5)**

Also,

$r_{max} > R$ **(OR EQUIVALENT)** **(0.5)**

$$\implies \frac{r_0}{1-\epsilon} > R \implies \epsilon > 1 - 2\alpha^2$$

Else, the rocket will crash into earth.

Choose - sign in the expression for ϵ, then $2\alpha^2 - 1 > 1 - 2\alpha^2$.

$\implies \alpha > \dfrac{1}{\sqrt{2}}$ **(0.5)**

Hence, the complete condition is $\dfrac{1}{\sqrt{2}} < \alpha < 1$.

(e) Given $r_{max} = \dfrac{r_0}{1-\epsilon} = \dfrac{\alpha^2}{1-\alpha^2}R$, for $\alpha = 3/4, r_{\max} = \dfrac{9}{7}R$. The corresponding height is $h_{\max} = r_{\max} - R = \dfrac{2}{7}R$. **(1)**

(f) For $\alpha = 1$, $E = 0$ OR $\epsilon = 1$.
Hence, the corresponding orbit is a parabola. **(1)**

2. (a) (i) $d\mathbf{S_1} = \rho d\phi dz \hat{\boldsymbol{\rho}}$ **(0.5)**

(ii) $d\mathbf{S_2} = \rho d\phi d\rho \hat{\mathbf{z}}$ **(0.5)**

(iii) $d\mathbf{S_3} = \rho d\phi dz(\hat{\boldsymbol{\rho}} + \hat{\mathbf{z}})$ **(0.5)**

OR (iii) $d\mathbf{S_3} = \rho d\phi dl \dfrac{(\hat{\boldsymbol{\rho}} + \hat{\mathbf{z}})}{\sqrt{2}}$, $dl =$ line element along the curved surface,

$dl = \sqrt{2}dz$, ($\rho = 2a - z$, when one goes from the bottom to the top of the cone).

(iv) $dV = \rho d\rho d\phi dz$ **(0.5)**

(b) (i) $\displaystyle\int_{S_1} \mathbf{F} \, . \, d\mathbf{S_1} = \int_{S_1} \mathbf{F} \, . \, (\rho d\phi dz \hat{\boldsymbol{\rho}}) = \int_{S_1} \rho^2 \sin^2\phi d\phi dz = \pi a^3$ **(1)**

(ii) $\displaystyle\int_{S_2} \mathbf{F} \, . \, d\mathbf{S_2} = \mathbf{F} \, . \, (\rho d\phi d\rho \hat{\mathbf{z}}) = \int_{S_2} (\rho d\phi d\rho)(4z) = 0$ **(1)**

(since z=0 for the bottom surface)

(iii) $\displaystyle\int_{S_3} \mathbf{F} \, . \, d\mathbf{S_3} = \int_{S_3} \mathbf{F} \, . \, (\rho d\phi dz(\hat{\boldsymbol{\rho}} + \hat{\mathbf{z}})) = \int_{S_3} \rho^2 \sin^2\phi d\phi dz + \int_{S_3} 4z\rho d\phi dz$

Substituting $\rho = (2a - z)$ and complete the integration (**Details in Appendix 1**)

$\displaystyle\int_{S_3} \mathbf{F} \, . \, d\mathbf{S_3} = \frac{17}{3}\pi a^3$ **(2)**

Total flux = $\displaystyle\int_{S_1} \mathbf{F} \, . \, d\mathbf{S_1} + \int_{S_2} \mathbf{F} \, . \, d\mathbf{S_2} + \int_{S_3} \mathbf{F} \, . \, d\mathbf{S_3} = \pi a^3 + 0 + \frac{17}{3}\pi a^3 = \frac{20}{3}\pi a^3$

(c) $\nabla \, . \, \mathbf{F} = 5$ **(0.5)**

The volume integral $\displaystyle\int_V (\nabla \, . \, \mathbf{F}) \, dV = \int_{V_{cylinder}} (\nabla \, . \, \mathbf{F}) \, dV + \int_{V_{cone}} (\nabla \, . \, \mathbf{F}) \, dV$

$$= 5\int_{\rho=0}^{a} \rho d\rho \int_{\phi=0}^{2\pi} d\phi \int_{z=0}^{a} dz + \int_{\phi=0}^{2\pi} d\phi \int_{z=a}^{2a} \left[\int_{\rho=0}^{2a-z} \rho d\rho\right] dz$$

$$= 5\pi a^3 + \frac{5}{3}\pi a^3 = \frac{20}{3}\pi a^3$$ **(0.5)**

3. (**Details added in Appendix 2**) **(1 MARK EACH)**

(a) $m_{\text{eff}} = m\left(1 + \frac{\rho^2}{4a^2}\right)$, $U_{\text{eff}} = \frac{mg}{4a}\rho^2 + \frac{\ell_z^2}{2m\rho^2}$ (**0.5+0.5=1)**

(b) $\epsilon = 3$ **(1)**

(c) $a = vT/\pi$ **(1)**

(d) $\ell \leq \sqrt{\alpha m/\lambda e}$ OR $\ell < \sqrt{\alpha m/\lambda e}$ **(1)**

(e) FALSE **(1)**

(f) $\mathbf{v} = \frac{\dot{M}}{4\pi\rho_m r^2}\hat{\mathbf{r}}$ **OR** $\mathbf{v} = \frac{M}{4\pi\rho_m r^2}\hat{\mathbf{r}}$ **(1)**

(g) TRUE **(1)**

APPENDIX1: DETAILS OF SURFACE INTEGRALS IN Q.2

(b) (i) $\int_{S_1} \mathbf{F} \, . \, d\mathbf{S_1} = \int_{S_1} \mathbf{F} \, . \, (\rho d\phi dz \hat{\boldsymbol{\rho}}) = \int_{S_1} \rho^2 \sin^2 \phi d\phi dz$

$$= a^2 \int_{\phi=0}^{2\pi} sin^2\phi d\phi \int_{z=0}^{a} dz = \pi a^3$$

(ii) $\int_{S_2} \mathbf{F} \, . \, d\mathbf{S_2} = \mathbf{F} \, . \, (\rho d\phi d\rho \hat{\mathbf{z}}) = \int_{S_2} (\rho d\phi d\rho)(4z) = 0$

(since z=0 for the bottom surface)

(iii) $\int_{S_3} \mathbf{F} \, . \, d\mathbf{S_3} = \int_{S_3} \mathbf{F} \, . \, (\rho d\phi dz(\hat{\boldsymbol{\rho}} + \hat{\mathbf{z}})) = \int_{S_2} \rho^2 \sin^2 \phi d\phi dz + \int_{S_2} 4z\rho d\phi dz$

substituting $\rho = (2a - z)$,

$$\int_{S_3} \mathbf{F} \, . \, d\mathbf{S_3} = 4a^2 \int_{\phi=0}^{2\pi} \sin^2 \phi d\phi \int_{z=a}^{2a} dz + \int_{\phi=0}^{2\pi} \sin^2 \phi d\phi \int_{z=a}^{2a} z^2 dz$$

$$-4a \int_{\phi=0}^{2\pi} \sin^2 \phi d\phi \int_{z=a}^{2a} z dz + 8a \int_{z=a}^{2a} z dz \int_{\phi=0}^{2\pi} d\phi - 4 \int_{z=a}^{2a} z^2 dz \int_{\phi=0}^{2\pi} d\phi$$

$$= 4\pi a^3 + \frac{7}{3}\pi a^3 - 6\pi a^3 + 24\pi a^3 - \frac{56}{3}\pi a^3 = \frac{17}{3}\pi a^3$$

APPENDIX 2: DETAILED ANSWERS FOR Q.3

(a) In cylindrical coordinate system,

$$E = \frac{m}{2}(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) + mgz$$

Substitute $z = \rho^2/4a \implies \dot{z} = (\rho/2a)\dot{\rho}$.

From angular momentum conservation, $m\rho^2\dot{\phi} = \ell_z$ is conserved, hence $\dot{\phi} = \ell_z/m\rho^2$.

Substitute both relations in the expression for energy to get the required result.

(b) The change in energy due to the thrust at apogee is

$$\Delta E = \frac{m}{2} v_a^2(\lambda^2 - 1),$$

where $v_a = \dfrac{\ell}{mr_a} = \dfrac{\ell}{mr_0}(1-\epsilon)$ is the orbital speed at apogee, where ϵ is the eccentricity of the original orbit.

Hence, we find $\Delta E = -(\lambda^2-1)(1-\epsilon)^2 E_0$ after substituting for ℓ and r_0. Here, E_0 is the energy of circular orbit.

Compare with Eq.41, Q7, PS-8 Solution: the only difference is $1+\epsilon \to 1-\epsilon$. Hence, from Eq. 42 in the same problem, we find, for the eccentricity of the new orbit,

$$\epsilon' = \pm\left[\lambda^2(1-\epsilon)-1\right]$$

For $\lambda = 2$ and $\epsilon = 0$ as given, we find $\epsilon' = 3$.

(c) From Kepler's 3rd law,

$$T^2 = \frac{4\pi^2}{GM_e}a^3$$

Here, $a = r_0/(1-\epsilon^2)$, hence

$$a^2 = \frac{GM_e}{4\pi^2}\frac{1-\epsilon^2}{r_0}T^2 \tag{1}$$

Write $1-\epsilon^2 = (1-\epsilon)(1+\epsilon)$.

Now, $\epsilon = \dfrac{r_a - r_p}{r_a + r_p} = \dfrac{v_p - v_a}{v_p + v_a}$, where $p = \text{perigee}, a = \text{apogee}$.

The above relation follows from $v_p = \dfrac{\ell}{mr_p}$ and $v_a = \dfrac{\ell}{mr_a}$.

Therefore, $1-\epsilon^2 = \dfrac{v_p v_a}{(v_p+v_a)^2}$. (2)

Now, $v_p + v_a = \dfrac{2\ell}{mr_0}$ (3)

Substitute (3) in (2) and use $r_0 = \ell^2/GM_e m^2$ to find

$$a^2 = \frac{T^2}{4\pi^2}v_p v_a \tag{4}$$

Taking the square root and substituting $v_p = 4v_a = 4v$ gives the required result.

(d) Write the effective potential $U_{\text{eff}}(r) = \dfrac{\ell^2}{2mr^2} + U(r)$:

For circular orbit, $U'(r_0) = -\dfrac{\ell^2}{mr_0^3} - F(r_0) = 0.$

Use the given expression for $F(r)$ to find the condition

$$r_0 e^{-\lambda r_0} = \frac{\ell^2}{\alpha m}$$

The LHS, as a function of r_0, as a maximum at $r_0 = \lambda^{-1}$, at which, its value is $(\lambda e)^{-1}$. Hence, intersection of LHS and RHS is possible only if

$\ell^2 \leq \dfrac{\alpha m}{\lambda e}$, which is the required condition.

(e) Divergence of a purely radial vector field $\mathbf{A} = A_r \hat{\mathbf{r}}$ is

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{d}{dr}(r^2 A_r).$$

For $A_r = 1,\ \nabla \cdot \mathbf{A} = \dfrac{2}{r}$, which is non-zero everywhere.

(f) Since the fluid is incompressible, outside the source, the velocity field should satisfy

$$\nabla \cdot \mathbf{v} = 0.$$

By symmetry, $\mathbf{v} = v_r(r)\hat{\mathbf{r}}$. From the expression in (e), therefore,

$$\nabla \cdot \mathbf{v} = 0 \implies v_r = \frac{f(t)}{r^2}, \tag{1}$$

where $f(t)$ is an undetermined function of t.

The net out-flux of fluid leaving a spherical region of radius R is

$I_{\text{out}} = \oint \mathbf{J} \cdot d\mathbf{S}$ integrated over the surface of the sphere. (2)

Here, $\mathbf{J} = \rho_m \mathbf{v}$ is the current density vector. Use (1) in (2) to find

$$I_{\text{out}} = 4\pi\rho_m f(t) \tag{3}$$

Since the fluid is incompressible, we need $I_{\text{out}} = I_{\text{in}} = \dot{M}$, which is the mass of fluid being ejected by the source per unit time. Using this relation in (3) gives $f(t) = \dot{M}/4\pi\rho_m$, which when substituted in (1) gives the final answer.

(g) For a loop lying in $x - y$ plane, we can write $\hat{\mathbf{n}} = \hat{\mathbf{t}} \times \hat{\mathbf{k}}$ where $\hat{\mathbf{t}}$ is the unit tangent vector, oriented CCW along the loop.

Hence, $\oint \hat{\mathbf{n}} dl = \oint \hat{\mathbf{t}} dl \times \hat{\mathbf{k}}$.

But $\oint \hat{\mathbf{t}} dl = 0$ for any closed loop, thus the result follows.