$\frac{25}{25}$

Department of Physics, IIT Madras

PH1010 Physics I **Mid-Term Examination** **17.12.2022**
Time: *Answer all questions* **Max. marks: 25 Marks**

Name	Roll No.	Old Roll No.(if any)
Jonnalagadda Dattatreya Sastry	NA22B041	
Instructions: You must write the final answers only in the allotted box. Vectors must be indicated clearly using arrows. You may use the reverse side of all pages for rough work. All symbols have their usual meaning unless stated otherwise. You must use only black or blue ink for writing the answers. Calculators, cell phones or any internet connectable device must not be in your possession during the examination.		**Exam Hall No.** MSB 301 **Total Marks**

Part A

1. In the boxes provided clearly indicate your answers to each of the questions below. The answers to each of these questions **must be a number or an expression. Please note that the answers that are not filled in the appropriate boxes will not be evaluated.** Each box is worth 1 mark. [$7 \times 1 = 7$ marks]

(a) A particle of mass m executes oscillations of amplitude A about the origin along the x-axis. If the potential energy of the particle is kx^6, with $k > 0$, then the time period $T \propto A^\alpha$, where

$\alpha = 6\left(\frac{1}{6} - \frac{1}{2}\right) = -6/3 = -2$

1

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(b) Find the potential that corresponds to the force $\vec{F}(x,y,z) = x\hat{e_x} + 2y\hat{e_y} + 3z\hat{e_z}$.

$U(x,y,z) = \frac{x^2}{2} + y^2 + \frac{3}{2}z^2 + c$

1

(c) A particle of mass m is constrained to move on a circle of radius R. In terms of the plane polar co-ordinate system (ρ, ϕ), the kinetic energy is given by

$\frac{1}{2}mR^2\dot{\phi}^2 =$ $T =$

1

(d) The solution of an underdamped harmonic oscillator of mass m is described by $x(t) = 7e^{-8t}\cos(6t + \pi/4)$. The natural frequency ω_0 of the oscillator is

$\omega_0 = 10$

1

(e) The unit-vector $\hat{n}$ normal to the surface $x^2y + y^2z + z^2x + 1 = 0$ at the point (1,2,-1) is:

$\hat{n} = \frac{\pm 1}{\sqrt{38}}(5\hat{i} - 3\hat{j} + 2\hat{k})$

(f) A particle moves along a helical path under the influence of a force given by $\vec{F}(x, y, z) = -y\hat{e_x} + x\hat{e_y} + z\hat{e_z}$. If the path is paramaterized by $x = \cos t$, $y = \sin t$, and $z = t$, find the work done W in moving the particle from $(1, 0, 0)$ to $(-1, 0, \pi)$.

$W = \pi + \pi^2/2$

(g) An undamped harmonic oscillator of natural frequency ω_0 is subject to a driving force given by $F = F_0 \exp(-\alpha t)$. If the initial position and velocity, $x(0) = \dot{x}(0) = 0$, then the position at any later time t is given by

$x(t) = -\frac{F_0}{m(\alpha^2+\omega_0^2)}\cos\omega_0 t + \frac{F_0\alpha}{m\omega_0(\alpha^2+\omega_0^2)}\sin\omega_0 t + \frac{F_0}{m(\alpha^2+\omega_0^2)}e^{-\alpha t}$

Part B

2. In the box provided, indicate whether the following statements are True or False. **Please record your answer only in the box provided, and nowhere else).** [3 × 1 = 3 marks]

(a) The overdamped harmonic oscillator cannot pass the origin more than once. On the other hand, the critically damped harmonic oscillator can pass the origin more than once. False

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(b) The time derivative of the position vector in the plane polar co-ordinate system gives a velocity v which has no component in the $\hat{e_\phi}$ direction. False

(c) For conservative forces, the phase trajectories cannot intersect each other. True

Part C

Each of the questions below are worth 2 marks. Pick the correct choice(s) for each of the questions below. Further, all constants have appropriate dimensions. Marks will be given only if no wrong choices are picked. [5 × 2 = 10 marks]

(a) The equation of motion for a particle of mass m and charge q placed in a region with a magnetic field $\vec{B}$ and an electric field $\vec{E}$ is given by

$$m\ddot{\vec{r}} = q(\vec{E} + \dot{\vec{r}} \times \vec{B}).$$

Here, $\vec{r}$ is the position vector of the particle at time t. Given $\vec{E} = E\hat{e_x}$, $\vec{B} = B\hat{e_y}$ (where E and B are uniform), and the particle starts from the origin at time $t = 0$ with initial velocity $\dot{\vec{r}}(0) = v_0\hat{e_z}$, then the following statements hold true:

☑ (A) The motion of the particle is confined to the xz plane.

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☐ (B) The motion of the particle is confined to the xy plane.
☑ (C) If $v_0 = 0$, the equation of motion for $x(t)$ resembles that of the undamped harmonic oscillator with a constant driving force.

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☐ (D) If $v_0 = E/B$, the equation of motion for $x(t)$ resembles that of the undamped harmonic oscillator with a constant driving force.

(b) Consider a particle of unit mass moving under the influence of a one dimensional potential given by:

$$U(x) = \frac{U_0 x^2}{x^2 + d^2},$$

where U_0 and d are positive constants. Then, :

☐ (A) There are two points of stable equilibrium for this potential.
☑ (B) The time period of small oscillation is $2\pi\sqrt{\frac{d^2}{2U_0}}$.
☐ (C) The turning point are at $\pm\frac{d}{\sqrt{3}}$, when the total energy is at $E = \frac{U_0}{3}$.
☑ (D) The turning point are at $\pm\frac{d}{\sqrt{2}}$, when the total energy is at $E = \frac{U_0}{3}$.

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(c) Consider a vector field

$$\vec{F}(x, y, z) = (xy^2 + z)\hat{e_x} + (x^2y + 2)\hat{e_y} + x\hat{e_z}.$$

The following holds true for the force field $\vec{F}(x, y, z)$

☑ (A) The integral of $\vec{F}$ around any closed loop is zero.
☑ (B) The integral $I = \int_A^B \vec{F} \cdot d\vec{r}$ depends only on the end points and not on the path taken to traverse from point A to point B.
☐ (C) The integral $I = \int_A^B \vec{F} \cdot d\vec{r}$ is dependent on the path taken to traverse from point A to point B.
☐ (D) The vector function $\vec{F}$ cannot be written as the gradient of a scalar function.

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(d) For a simple harmonic oscillator, the following facts hold true

☑ (A) The time average over one cycle of potential and kinetic energies are equal.
☐ (B) The time average over one cycle of the potential energy is twice the time average of the kinetic energy over one cycle.
☑ (C) The space average of the kinetic energy is twice the space average of the potential energy.
☐ (D) The space average of the potential energy is the same as the space average of the kinetic energy.

2

2

(e) Following facts hold true for the gradient of a scalar function:

☑ (A) A vector field can be written as the gradient of a scalar function if and only if the work done in moving a particle in such a vector field is path independent.
☑ (B) Gradient of a scalar function at any point indicates the direction in which

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the function has the greatest rate of change.

~~☐~~ (C) Gradient of a function is tangential to the surface where the scalar function has a constant value.

~~☐~~ (D) Gradient of a scalar function at any point indicates the direction in which the function has the smallest rate of change.

Part D

The question below is worth 5 marks. All constants have appropriate dimensions.

(a) A particle of unit mass is moving in a 1-dimensional potential given by:

$$U(x) = \frac{x^2}{(x^2 + a^2)^2}.$$

Here, a is a constant of appropriate dimensions.

[1 × 5 = 5 marks]

i. Sketch the potential, and clearly mark the stable and unstable equilibrium points

[2 marks]

ii. The phase trajectories are dependent on the energy E of the system. Sketch the phase trajectories for all possible representative values of E.
Important: Please indicate the direction in which the trajectory is traversed as time increases.

[3 marks]

$$E = T + V = \frac{1}{2}m\dot{x}^2 + \frac{x^2}{(x^2+a^2)^2}$$

$$\dot{x} = \sqrt{\frac{2}{m}\left(E - \frac{x^2}{(x^2+a^2)^2}\right)} \quad — ①$$

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$$U(x) = \frac{x^2}{(x^2+a^2)^2}$$

$$U'(x) = \frac{2(x^2+a^2)(2x)x^2 - 2x(x^2+a^2)^2}{(x^2+a^2)^4}$$

$U'(x) = 0$

for stable & Unstable eqm. points

$$\Rightarrow \cancel{2}\cancel{x}^{2}(x^2+a^2)\,x^2 = \cancel{2x}(x^2+a^2)^2$$

$x = 0$

$$2x^2 = x^2 + a^2$$

$$x = \pm a$$

The 3 points are $0, +a, -a = x$.

$$U'(x) = \frac{4x^3(x^2+a^2) - 2x(x^2+a^2)^2}{(x^2+a^2)^4} = \frac{4x^5 + \cancel{4a^2x^3} - 2x^5 - 2a^4x - \cancel{4a^2x^3}}{(x^2+a^2)^4}$$

$$= \frac{2x^5 - 2a^4x}{(x^2+a^2)^4} = \frac{2x\,\cancel{(x^2+a^2)}(x^2-a^2)}{(x^2+a^2)^{\cancel{4}\,3}}$$

$$U'(x) = \frac{2x^3 - 2a^2x}{(x^2+a^2)^3}$$

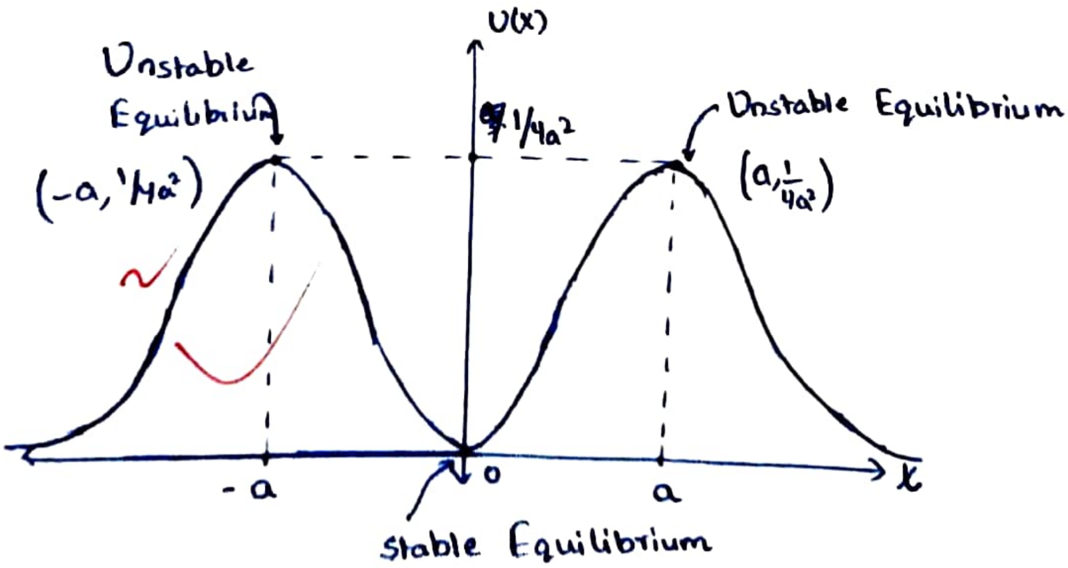
$$U''(x) = \frac{2x(x^2-a^2)(3)\cancel{(x^2+a^2)^2}(2x) - (x^2+a^2)^3(6x^2-2a^2)}{(x^2+a^2)^{\cancel{6}\,4}}$$

$$= \frac{12x^2(x^2-a^2) - 6x^2(x^2+a^2) + 2a^2(x^2+a^2)}{(x^2+a^2)^4}$$

$$U''(0) = \frac{2a^4}{a^8} = \frac{2}{a^4} > 0 \longrightarrow \text{Stable Equilibrium}$$

$$U''(a) = \frac{-6a^2(2a^2) + 2a^2(2a^2)}{(2a^2)^4} = \frac{-12a^4 + 4a^4}{16a^8}$$

$$U''(a) = \frac{-1}{2a^4} = U''(-a) < 0 \longrightarrow \text{Unstable Equilibrium}$$



Potential Energy Graph

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$$\dot{x} = \sqrt{\frac{2}{m}\left(E - \frac{x^2}{(x^2+a^2)^2}\right)} \qquad \text{from (1)}$$

$0 < E \ll \frac{1}{4a^2} \longrightarrow$ Simple Harmonic Motion

$\frac{1}{8a^2} < E < \frac{1}{4a^2} \longrightarrow$ Oscillations, but not harmonic

$E = \frac{1}{4a^2} \longrightarrow$ No Oscillations

$$\dot{x} = \sqrt{2\left(\frac{1}{4a^2} - \frac{x^2}{(x^2+a^2)^2}\right)}$$

$E > \frac{1}{4a^2} \longrightarrow$ No Oscillations

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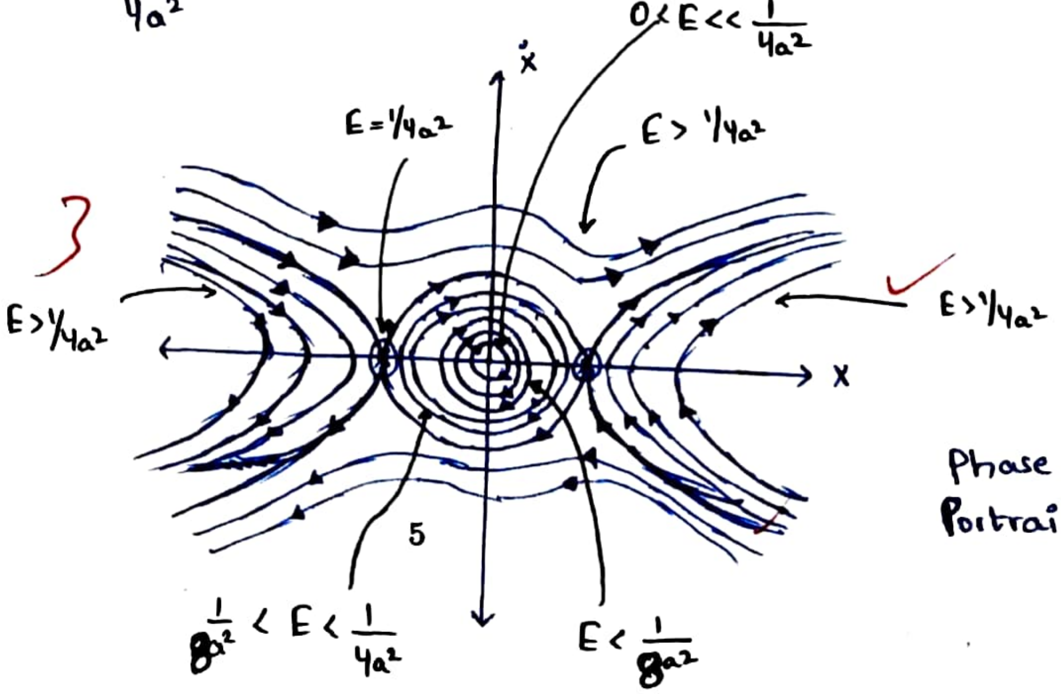

Phase Portrait