Department of Physics
Indian Institute of Technology Madras

End-of-semester exam

From Newton's laws to circulation of vector fields

Date: November 17, 2017 Time: 09:00 AM – 12:00 NOON

Name:		**Roll No.:**	
Instructor:		**Batch:**	

Instructions

1. Begin by completing the information requested above. Please write your complete name, your roll number, the name of your instructor, and your batch number (out of I–XII). **The answer sheet will not be evaluated unless both your name and roll number are written.**
2. This question paper cum answer sheet booklet contains **fourteen** single-sided pages. Please check right away that all the pages are present.
3. As we had announced earlier, this exam consists of 4 true/false questions (for 1 mark each), 4 multiple choice questions with more than one correct option (for 2 marks each), 8 fill in the blanks (for 1 mark each), 5 questions involving detailed calculations (for 3 marks each) and 3 questions involving some plotting (for 5 marks each), adding to a total of 50 marks.
4. You are expected to answer **all** the questions. There are no negative marks.
5. **The answers have to be written in the boxes provided. Answers written elsewhere in the booklet will not be evaluated.**
6. **Kindly write the answers, including sketches, with a blue or black pen. Note that answers written with pencils or pens of other colors will not be evaluated.**
7. You can use the empty reverse sides for rough work. No extra sheets will be provided.
8. You are not allowed to use a calculator or any other electronic device during the exam. Please note that you will not be permitted to continue with the exam if you are found with any such device.
9. Make sure that you return question paper cum answer sheet booklet before you leave the examination hall.

For use by examiners
(Do not write in this space)

Q1–Q8	Q9-16	Q17-21	Q22-24	Total

✦ **True or false** (1 mark each, write True (T)/False (F) in the box provided)

1. The position vector of a particle in motion is given as $\vec{\boldsymbol{r}}(t) = r_0\,\hat{\boldsymbol{r}}$, where r_0 is a constant. This implies that the components of the velocity along the $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ directions are always zero.

2. Kepler's second law, viz. that a planet sweeps equal areas in equal intervals of time, is actually applicable to any trajectory in any central potential.

3. The orbital solution $r = r_0/\cos\phi$ describes the motion of a free particle, where r_0 is the radius of closest approach to the origin and ϕ is the angle measured with respect to the radius vector at the point of closest approach.

4. The force $\vec{\boldsymbol{F}} = k\,(y\,\hat{\boldsymbol{x}} + x\,\hat{\boldsymbol{y}} - 3\,z\,\hat{\boldsymbol{z}})$, where k is a constant, is conservative.

✦ **Multiple choice questions** (2 marks each, write the correct option(s) in the box provided. Note that there can be more than one correct option and zero marks will be given if you choose an incorrect option.)

5. A charged particle is moving in the field of a constant and uniform magnetic field. In general, the particle's

   [**A**] Speed is constant [**B**] Energy is conserved
   [**C**] Trajectory can be a helix [**D**] Trajectory can be a parabola

6. A particle moves under the influence of the gravitational force due to an infinite sheet of mass which lies confined to the x-y-plane. Which of the following kinematic quantities of the particle are conserved? (Note that E and $\vec{\boldsymbol{p}} = (p_x, p_y, p_z)$ denote the energy and momentum of the particle, respectively.)

   [**A**] E [**B**] p_x [**C**] p_y [**D**] p_z

7. A particle moves in the central potential $U(r) = k\,r^4$, where $k > 0$. If the particle has a reduced mass μ, angular momentum L and follows a circular orbit, the radius r_0 and energy E_0 of the particle are

   [**A**] $r_0 = \ [L^2/(4\,k\,\mu)]^{1/6}$ [**B**] $E_0 = (3/2)\,(L^4\,k/2\,\mu^2)^{1/3}$
   [**C**] $r_0 = 2\,[L^2/(4\,k\,\mu)]^{1/6}$ [**D**] $E_0 = (5/2)\,(L^4\,k/2\,\mu^2)^{1/3}$

8. A region has a uniform magnetic field $\vec{\boldsymbol{B}} = B_0\,\hat{\boldsymbol{z}}$ throughout. Consider a cone of height h, base radius R and axis of symmetry along the z-axis. Let the vertex of the cone be at the origin and the outward normal to the flat face be along the positive z-direction. If Φ_{F} and Φ_{C} denote the magnetic flux through the flat and the curved surfaces of the cone, then

   [**A**] $\Phi_{\mathrm{F}} = B_0\,\pi\,R^2$ [**B**] $\Phi_{\mathrm{F}} = -B_0\,\pi\,R^2$
   [**C**] $\Phi_{\mathrm{C}} = B_0\,\pi\,R^2$ [**D**] $\Phi_{\mathrm{C}} = -B_0\,\pi\,R^2$

✦ **Fill in the blanks** (1 mark each, write the answer in the box provided)

9. Consider the elastic collision of two different masses in one dimension. The relation between the relative velocity of the two masses before (say, $v_1 - v_2$) and after (say, $v_1' - v_2'$) the collision is

10. A particle exhibits bounded motion in the one-dimensional potential $U(x) = \alpha\, x^6$, where $\alpha > 0$. What is the relation between kinetic and potential energies of the particle, when averaged over one period?

11. A particle is moving on a plane with the velocity $\vec{\boldsymbol{v}}(t) = c\, t\, \hat{\boldsymbol{\phi}}$, where c is a constant. The acceleration $\vec{\boldsymbol{a}}$ of the particle is

12. A central potential allows a particle to move in a spiral orbit as follows: $r(\phi) = k\, \phi^2$, where k is a constant. Express the force $\vec{\boldsymbol{F}}(r)$ that gives rise to such a trajectory in terms of the angular momentum L, k and the reduced mass μ.

13. Recently, two neutron stars, one having mass $1.2\, M_{\mathrm{Sun}}$ and another $1.6\, M_{\mathrm{Sun}}$ were observed to have moved in an elliptical orbit about their common center of mass. If their maximum separation is 400 km, estimate the time period of their orbit. (In the central potential $U(r) = -\alpha/r$, according to Kepler's third law, $T^2 = 4\,\pi^2\,\mu\, a^3/\alpha$. Note that, $G = 6.7 \times 10^{-11}\,\mathrm{m^2\,kg^{-1}\,s^{-2}}$ and $M_{\mathrm{Sun}} = 2 \times 10^{30}\,\mathrm{kg}$.)

14. The infinite plane $z = a\,x + b\,y + c$ has a constant surface charge density σ. The electric field above and below the plane are given by

15. A sphere of radius 0.5 m has its centre at the origin and a current density $\vec{\boldsymbol{J}} = 2\,\hat{\boldsymbol{r}}\,\mathrm{A/m^2}$ over its surface. By how much does the amount of charge within the sphere change in one minute? (Note: The unit A denotes Ampère, i.e. Coulomb per unit time.)

16. A conservative force field is given by $\vec{\boldsymbol{F}} = f_0\,[-\rho\cos(2\,\phi)\,\hat{\boldsymbol{\rho}} + \rho\sin(2\,\phi)\,\hat{\boldsymbol{\phi}} + z\,\hat{\boldsymbol{z}}]$, where f_0 is a constant. Calculate the work done in moving a particle from $(\rho, \phi, z) = (4, \pi/2, 2)$ to $(4, \pi, 1)$.

✦ **Questions with detailed answers** (write the calculations and answers within the boxes provided)

17. *Integrating vectors:* Evaluate the following integrals: 1.5+1.5 marks

$$\text{(a)} \int \mathrm{d}t \left(\vec{\boldsymbol{A}} \times \frac{\mathrm{d}^2\vec{\boldsymbol{A}}}{\mathrm{d}t^2}\right), \quad \text{(b)} \int \mathrm{d}t \left(\frac{1}{r}\frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} - \frac{\vec{\boldsymbol{r}}}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\right)$$

18. *Frequency of small oscillations:* Consider the one-dimensional potential

$$U(x) = U_0 \left(\frac{x}{d}\right)^{2\,n} \mathrm{e}^{-x/d},$$

where U_0 and d are positive constants and $n \geq 1$ is an integer. (a) Plot the potential as a function of (x/d) for $n = 1$. (b) Determine the frequency of small oscillations about the minima for $n \geq 1$ (and integer n). 1.5+1.5 marks

19. *Parabolic trajectory in the Kepler problem:* Consider a particle moving in the central Kepler potential $U(r) = -\alpha/r$, where $\alpha > 0$. (a) Write down the equation of motion governing the radial trajectory of the particle. (b) Obtain the solution to the above equation of motion when the energy of particle is zero. (Note that it would be easier to express t in terms of r.) $\boxed{\text{1+2 marks}}$

20. An electric dipole of moment $\vec{\boldsymbol{p}} = p\,\hat{\boldsymbol{z}}$ is located at the origin. The dipole creates the electrostatic potential

$$\phi(\boldsymbol{r}) = \frac{\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{r}}}{4\,\pi\,\epsilon_0\,r^3}.$$

(a) Determine the corresponding electric field $\vec{\boldsymbol{E}}$ in terms of $\vec{\boldsymbol{p}}$, $\hat{\boldsymbol{r}}$ and r. (b) Evaluate the work done by the electric field to move a charge q from $z = 1$ to $z = \infty$. 2+1 marks

21. *Vortex in a fluid and Stokes' theorem:* A fluid flowing in the x-y-plane has the following velocity field: $\vec{\boldsymbol{v}} = a\,(-y\,\hat{\boldsymbol{x}} + x\,\hat{\boldsymbol{y}})/(x^2 + y^2)^n$, where a is a constant. (a) Evaluate the vorticity field $\vec{\boldsymbol{\Omega}}$ corresponding to the above velocity field (recall that $\vec{\boldsymbol{\Omega}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}$). (b) Assuming $n < 1$, verify the Stokes' theorem for the velocity field $\vec{\boldsymbol{v}}$ by considering a circular path in the x-y-plane, with its centre at the origin. $\boxed{\text{1+2 marks}}$

22. *Behavior of the damped, driven oscillator:* Recall that, a damped, driven oscillator satisfies the following equation of motion:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\,\beta\,\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2\,x = f_0\cos(\omega\,t).$$

Assume that the oscillator is underdamped, say, $\beta/\omega_0 \simeq 10^{-2}$. (a) Obtain the solution to the above equation for times such that $\beta\,t \gg 1$, expressing the amplitude (say, A) and phase (say, δ) of the solution in terms of the parameters ω_0, β, f_0 and ω. (b) Plot the square of the amplitude A^2 and phase δ as a function of ω/ω_0. (c) Assuming $\omega = \omega_0$, plot the position $x(t)$, the velocity $v(t) = (\mathrm{d}x/\mathrm{d}t)$ and the forcing term $f(t) = f_0\cos(\omega\,t)$ as a function of $(\omega\,t)$, in particular highlighting the difference in phases between these quantities. $\boxed{\text{1+2+2 marks}}$

23. *Trajectories in phase space:* A particle moves in the one-dimensional potential $U(x) = \alpha\, x^2\, \mathrm{e}^{-x^2}$, where $\alpha > 0$. (a) Plot the potential, specifically indicating the locations of the maxima and minima. (b) What is the allowed range of energy, and what is the range of energy (and position) for which the particle exhibits bounded motion? (c) Illustrate each type of bounded and unbounded trajectories that are possible in the phase space, indicating the direction of motion with arrows. $\boxed{\text{2+1+2 marks}}$

24. *Electric field in and around a charged rod:* A very long rod of radius R carries a charge per unit length λ that is uniformly distributed throughout the rod. It is surrounded by a coaxial cylindrical shell of radius $3\,R$ which is oppositely charged and carries the charge per unit length $-\lambda$. (a) Calculate the electric field at the following locations: (i) inside the charged rod, (ii) between the charged rod and the coaxial shell and (iii) outside the coaxial shell. (b) Plot the magnitude of the electric field as a function of the radial distance from the centre of the rod (i.e. from 0) to $5\,R$. $\boxed{\text{3+2 marks}}$

Solutions to the end-of-semester exam

From Newton's laws to circulation of vector fields

✦ True or false

1. The position vector of a particle in motion is given as $\vec{\boldsymbol{r}}(r,\theta,\phi) = r_0\,\hat{\boldsymbol{r}}$, where r_0 is a constant. This implies that the components of the velocity along the $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ directions are always zero.

   <u>*Solution:*</u> **False**. The velocity is

   $$\frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} = r_0\,\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t} = r_0\,\left(\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{\phi}\,\sin\theta\,\hat{\boldsymbol{\phi}}\right).$$

   In general $\dot{\theta}$ and $\dot{\phi}$ are not equal to zero and hence the statement is false. Note that the motion is that of a particle constrained to move on the surface of a sphere of radius r_0.

2. Kepler's second law, viz. that a planet sweeps equal areas in equal intervals of time, is actually applicable to any trajectory in any central potential.

   <u>*Solution:*</u> **True.** The derivation of the second law only assumes the conservation of angular momentum, which is true for any central potential.

3. The orbital solution $r = r_0/\cos\phi$ describes the motion of a free particle, where r_0 is the radius of closest approach to the origin and ϕ is the angle measured with respect to the radius vector at the point of closest approach.

   <u>*Solution:*</u> **True**. Define the point of closest approach to be along the x axis so that

   $$x = r\cos\phi = r_0\ ,\quad y = r\sin\phi = r_0\tan\phi,$$

   which parametrically describes the straight line $x = r_0$ with $\phi = (-\frac{\pi}{2}, \frac{\pi}{2})$.

4. The force $\vec{\boldsymbol{F}} = k\,(y\,\hat{\boldsymbol{x}} + x\,\hat{\boldsymbol{y}} - 3\,z\,\hat{\boldsymbol{z}})$, where k is a constant, is conservative.

   <u>*Solution:*</u> **True**. We take the curl to check if the force is conservative:

   $$\boldsymbol{\nabla}\times\boldsymbol{F} = k\begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & -3z \end{vmatrix} = \left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}y\right)\hat{\boldsymbol{z}} = 0.$$

   Therefore, the force is conservative.

✦ Multiple choice questions (2 marks each, write the correct option(s) in the box provided. Note that there can be more than one correct option and zero marks will be given if you choose an incorrect option.)

5. A charged particle is moving in the field of a constant and uniform magnetic field. In general, the particle's

   [**A**] Speed is constant [**B**] Energy is conserved

   [**C**] Trajectory can be a helix [**D**] Trajectory can be a parabola

   <u>*Solution:*</u> **A**, **B** and **C**. The magnetic force is conservative but does no work. Therefore, the energy and kinetic energy are constant and hence **A** and **B** are true. It is well known that a charged particle moves in a circle in the plane perpendicular to the magnetic field. If there is any velocity component in the direction of the field, the trajectory is a helix of constant pitch, so **C** is correct.

6. A particle moves under the influence of the gravitational force due to an infinite sheet of mass which lies confined to the x-y-plane. Which of the following kinematic quantities of the particle

are conserved? (Note that E and $\vec{\boldsymbol{p}} = (p_x, p_y, p_z)$ denote the energy and momentum of the particle, respectively.)

[A] E **[B]** p_x **[C]** p_y **[D]** p_z

<u>*Solution:*</u> **A**, **B** and **C**. As the potential is time-independent, the energy is conserved. The gravitational field is uniform and in the $\pm z$ direction. Therefore, the potential only depends on z, so it is invariant under translation in the x and y directions. Hence, the momentum components in those directions are conserved.

7. A particle moves in the central potential $U(r) = k\, r^4$, where $k > 0$. If the particle has a reduced mass μ, angular momentum L and follows a circular orbit, the radius r_0 and energy E_0 of the particle are

[A] $r_0 = \ [L^2/(4\, k\, \mu)]^{1/6}$ **[B]** $E_0 = (3/2)\, (L^4\, k/2\, \mu^2)^{1/3}$

[C] $r_0 = 2\, [L^2/(4\, k\, \mu)]^{1/6}$ **[D]** $E_0 = (5/2)\, (L^4\, k/2\, \mu^2)^{1/3}$

<u>*Solution:*</u> **A** and **B**. The effective potential for this motion is

$$U_{\mathrm{eff}} = k\, r^4 + \frac{L^2}{2\, \mu\, r^2}$$

and hence

$$\frac{\mathrm{d}U_{\mathrm{eff}}}{\mathrm{d}r} = 4\, k\, r^3 - \frac{L^2}{\mu\, r^3}.$$

The circular orbit will occur at the radius r_0 when $(\mathrm{d}U_{\mathrm{eff}}/\mathrm{d}r)_{r_0} = 0$. Therefore, we have

$$r_0 = \left(\frac{L^2}{4\, \mu\, k}\right)^{1/6},$$

so **A** is correct. In a circular orbit $\dot{r} = 0$, so that

$$\begin{aligned} E_0 &= U_{\mathrm{eff}}(r_0) = k\, r_0^4 + \frac{L^2}{2\, \mu\, r_0^2} \\ &= k\, \left(\frac{L^2}{4\, \mu\, k}\right)^{2/3} + \frac{L^2}{2\, \mu}\, \left(\frac{4\, \mu\, k}{L^2}\right)^{1/3} \\ &= \left(\frac{L^4 k}{\mu^2}\right)^{1/3} \left(\frac{1}{4^{2/3}} + \frac{4^{1/3}}{2}\right) = \frac{3}{2}\, \left(\frac{L^4\, k}{2\, \mu^2}\right)^{1/3}, \end{aligned}$$

and hence the correct option is **B**.

8. A region has a uniform magnetic field $\vec{\boldsymbol{B}} = B_0\, \hat{\boldsymbol{z}}$ throughout. Consider a cone of height h, base radius R and axis of symmetry along the z-axis. Let the vertex of the cone be at the origin and the outward normal to the flat face be along the positive z-direction. If Φ_{F} and Φ_{C} denote the magnetic flux through the flat and the curved surfaces of the cone, then

[A] $\Phi_{\mathrm{F}} = B_0\, \pi\, R^2$ **[B]** $\Phi_{\mathrm{F}} = -B_0\, \pi\, R^2$

[C] $\Phi_{\mathrm{C}} = B_0\, \pi\, R^2$ **[D]** $\Phi_{\mathrm{C}} = -B_0\, \pi\, R^2$

<u>*Solution:*</u> **A** and **D**. According to Gauss' divergence theorem, for magnetic fields, we have

$$\oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{A}} = 0$$

over any closed surface, since $\vec{\nabla} \cdot \vec{\boldsymbol{B}} = 0$. Therefore,

$$\Phi_{\mathrm{C}} + \Phi_{\mathrm{F}} = \int_{\mathrm{C}} \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{A}} + \int_{\mathrm{F}} \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{A}} = 0$$

and hence

$$\Phi_{\rm C} = -\Phi_{\rm F} = -\int_0^R \mathrm{d}\rho\,\rho \int_0^{2\pi} \mathrm{d}\phi\,(B_0\,\hat{\boldsymbol{z}})\cdot\hat{\boldsymbol{z}} = -B_0\,\pi\,R^2,$$

so that

$$\Phi_{\rm F} = B_0\,\pi\,R^2.$$

✦ **Fill in the blanks** (1 mark each, write the answer in the box provided)

9. Consider the elastic collision of two different masses in one dimension. The relation between the relative velocity of the two masses before (say, $v_1 - v_2$) and after (say, $v_1' - v_2'$) the collision is

   Solution: From conservation of momentum and energy, we have

   $$m_1\,v_1 + m_2\,v_2 = m_1\,v_1' + m_2\,v_2',$$

   and

   $$\frac{m_1\,v_1^2}{2} + \frac{m_2\,v_2^2}{2} = \frac{m_1\,v_1'^2}{2} + \frac{m_2\,v_2'^2}{2}.$$

   These can be written as

   $$m_1\,\left(v_1 - v_1'\right) = m_2\,\left(v_2' - v_2\right)$$

   and

   $$\frac{m_1}{2}\,\left(v_1^2 - v_1'^2\right) = \frac{m_1}{2}\,\left(v_1 + v_1'\right)\,\left(v_1 - v_1'\right) = \frac{m_2}{2}\,\left(v_2'^2 - v_2^2\right) = \frac{m_2}{2}\,\left(v_2 + v_2'\right)\,\left(v_2' - v_2\right),$$

   so that, upon using these two expressions, we arrive at

   $$v_1 + v_1' = v_2 + v_2'$$

   or

   $$v_1 - v_2 = v_2' - v_1',$$

   which is the required result.

10. A particle exhibits bounded motion in the one-dimensional potential $U(x) = \alpha\,x^6$, where $\alpha > 0$. What is the relation between kinetic and potential energies of the particle, when averaged over one period?

    Solution: The virial theorem for a $U(x) = \alpha\,x^n$ is

    $$\langle T\rangle = (n/2)\,\langle U\rangle.$$

    Therefore, for $n = 6$, we have

    $$\langle T\rangle = 3\,\langle U\rangle.$$

11. A particle is moving on a plane with the velocity $\vec{\boldsymbol{v}}(t) = c\,t\,\hat{\boldsymbol{\phi}}$, where c is a constant. The acceleration $\vec{\boldsymbol{a}}$ of the particle is

    Solution: First we find the velocity, i.e.

    $$\vec{\boldsymbol{a}} = \dot{\vec{\boldsymbol{v}}} = c\,\left(\hat{\boldsymbol{\phi}} - \dot{\phi}\,t\,\hat{\boldsymbol{\rho}}\right).$$

12. A central potential allows a particle to move in a spiral orbit as follows: $r = k\,\phi^2$, where k is a constant. Express the force $\vec{\boldsymbol{F}}(r)$ that gives rise to such a trajectory in terms of the angular momentum L, k and the reduced mass μ.

    Solution: We write the energy

    $$E = \frac{\mu}{2}\,\left(\dot{r}^2 + r^2\,\dot{\phi}^2\right) + U(r)$$

which implies that

$$U(r) = -\frac{\mu}{2}\,(\dot{r}^2 + r^2\,\dot{\phi}^2) + E.$$

Now,

$$\dot{r} = 2\,k\,\phi\,\dot{\phi} = 2\,\sqrt{k\,r}\,\dot{\phi}$$

and $\dot{\phi} = L/(\mu\,r^2)$ so that

$$U(r) = -\frac{L^2}{2\,\mu\,r^4}\,\left(4\,k\,r + r^2\right) + E = -\frac{L^2}{2\,\mu}\,\left(\frac{4\,k}{r^3} + \frac{1}{r^2}\right) + E,$$

and, hence, the force is

$$F_r = -\frac{\mathrm{d}U}{\mathrm{d}r} = -\frac{L^2}{\mu}\,\left(\frac{1}{r^3} + \frac{6\,k}{r^4}\right).$$

13. Recently, two neutron stars, one having mass $1.2\,M_{\mathrm{Sun}}$ and another $1.6\,M_{\mathrm{Sun}}$ were observed to have moved in an elliptical orbit about their common center of mass. If their maximum separation is 400 km, estimate the time period of their orbit. (In the central potential $U(r) = -\alpha/r$, according to Kepler's third law, $T^2 = 4\,\pi^2\,\mu\,a^3/\alpha$. Note that, $G = 6.7 \times 10^{-11}\,\mathrm{m^2\,kg^{-1}\,s^{-2}}$ and $M_{\mathrm{Sun}} = 2 \times 10^{30}\,\mathrm{kg}$.)

    Solution: First we note $\alpha = G\,M_1\,M_2$ and $\mu = M_1\,M_2/(M_1 + M_2)$, so that

$$T^2 = \frac{4\,\pi^2\,M_1\,M_2\,a^3}{(M_1 + M_2)\,G\,M_1\,M_2} = \frac{4\,\pi^2\,a^3}{2.8\,M_{\mathrm{Sun}}\,G} = \frac{4^4\,\pi^2 \times 10^{15}}{5.6 \times 10^{30} \times 6.7 \times 10^{-11}} = \frac{4^4\pi^2}{37.5 \times 10^4}\ \mathrm{s},$$

    which leads to

$$T \approx \sqrt{\frac{4^3}{10^4}} \approx \frac{8}{10^2} = 0.08\,\mathrm{s}.$$

14. The infinite plane $z = a\,x + b\,y + c$ has a constant surface charge density σ. The electric field above and below the plane are given by

    Solution: The electric field will be given by $\vec{\boldsymbol{E}} = \pm(\sigma/2\,\epsilon_0)$, $\hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is the normal to the plane. Given that the plane can be described as $f(x, y, z) = c = z - a\,x - b\,y$, we have the normal to be

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{\nabla} f}{|\boldsymbol{\nabla} f|} = \frac{-a\,\hat{\boldsymbol{x}} - b\,\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}}{\sqrt{a^2 + b^2 + 1}},$$

    so that

$$\vec{\boldsymbol{E}} = \pm\frac{\sigma}{2\,\epsilon_0}\,\left(\frac{-a\,\hat{\boldsymbol{x}} - b\,\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}}{\sqrt{a^2 + b^2 + 1}}\right).$$

15. A sphere of radius 0.5 m has its centre at the origin and a current density $\vec{\boldsymbol{J}} = 2\,\hat{\boldsymbol{r}}\,\mathrm{A/m^2}$ over its surface. By how much does the amount of charge within the sphere change in one minute? (Note: The unit A denotes Ampère, i.e. Coulomb per unit time.)

    Solution: We will use the integrated continuity equation, viz.

$$\oint \vec{\boldsymbol{J}} \cdot \mathrm{d}\vec{\boldsymbol{A}} = -\frac{\mathrm{d}Q_{\mathrm{enc}}}{\mathrm{d}t}$$

    which implies that

$$\frac{\mathrm{d}Q_{\mathrm{enc}}}{\mathrm{d}t} = -\int_0^{\pi} \mathrm{d}\theta \int_0^{2\,\pi} \mathrm{d}\phi\ (2\,\hat{\boldsymbol{r}}) \cdot \left(R^2\,\sin\,\theta\,\hat{\boldsymbol{r}}\right) = -8\,\pi\,R^2 = -2\,\pi\,\mathrm{C/s}$$

    because $R = 0.5\,\mathrm{m}$. Therefore, the change in charge in one minute is $60 \times (\mathrm{d}Q_{\mathrm{enc}}/\mathrm{d}t) = -120\,\pi\,\mathrm{C} = -372\,\mathrm{C}$.

16. A conservative force field is given by $\vec{\boldsymbol{F}} = f_0\,[-\rho\cos(2\,\phi)\,\hat{\boldsymbol{\rho}} + \rho\sin(2\,\phi)\,\hat{\boldsymbol{\phi}} + z\,\hat{\boldsymbol{z}}]$, where f_0 is a constant. Calculate the work done in moving a particle from $(\rho, \phi, z) = (4, \pi/2, 2)$ to $(4, \pi, 1)$.

<u>*Solution:*</u> Note that

$$\mathrm{d}\vec{\boldsymbol{l}} = \mathrm{d}\rho\,\hat{\boldsymbol{\rho}} + \rho\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}} + \mathrm{d}z\,\hat{\boldsymbol{z}},$$

so that

$$\int \vec{\boldsymbol{F}}\cdot\mathrm{d}\vec{\boldsymbol{l}} = f_0 \int_{(4,\frac{\pi}{2},2)}^{(4,\pi,1)} \left[-\rho\cos(2\,\phi)\,\mathrm{d}\rho + \rho^2\sin(2\,\phi)\,\mathrm{d}\phi + z\,\mathrm{d}z\right]$$

and, since the path is along a constant ρ, viz. $\rho = 4$, we have

$$\begin{aligned}\int \vec{\boldsymbol{F}}\cdot\mathrm{d}\vec{\boldsymbol{l}} &= f_0\left(16\int_{\pi/2}^{\pi}\mathrm{d}\phi\sin(2\,\phi) + \int_2^1 \mathrm{d}z\,z\right)\\ &= f_0\left\{8\,\left[-\cos(2\,\phi)\right]_{\pi/2}^{\pi} + \left(\frac{z^2}{2}\right)_2^1\right\} = f_0\left(-16 - \frac{3}{2}\right) = -\frac{35}{2}\,f_0.\end{aligned}$$

✦ **Questions with detailed answers** (write the calculations and answers within the boxes provided)

17. <u>*Integrating vectors:*</u> Evaluate the following integrals:

$$\text{(a)}\ \int \mathrm{d}t\,\left(\vec{\boldsymbol{A}}\times\frac{\mathrm{d}^2\vec{\boldsymbol{A}}}{\mathrm{d}t^2}\right),\quad \text{(b)}\ \int \mathrm{d}t\,\left(\frac{1}{r}\frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} - \frac{\vec{\boldsymbol{r}}}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\right)$$

<u>Solution:</u> (a) Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}\times\dot{\boldsymbol{A}}\right) = \dot{\boldsymbol{A}}\times\dot{\boldsymbol{A}} + \boldsymbol{A}\times\ddot{\boldsymbol{A}} = \boldsymbol{A}\times\ddot{\boldsymbol{A}},$$

because the vector product of a vector with itself is zero. Therefore,

$$\int \mathrm{d}t\left(\boldsymbol{A}\times\ddot{\boldsymbol{A}}\right) = \int \mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}\times\dot{\boldsymbol{A}}\right) = \boldsymbol{A}\times\dot{\boldsymbol{A}} + \boldsymbol{C},$$

where $\boldsymbol{C}$ is a constant vector.

(b) Note that, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\vec{\boldsymbol{r}}}{r}\right) = \frac{1}{r}\frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} - \frac{\vec{\boldsymbol{r}}}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t},$$

using the chain rule. Therefore, we obtain that

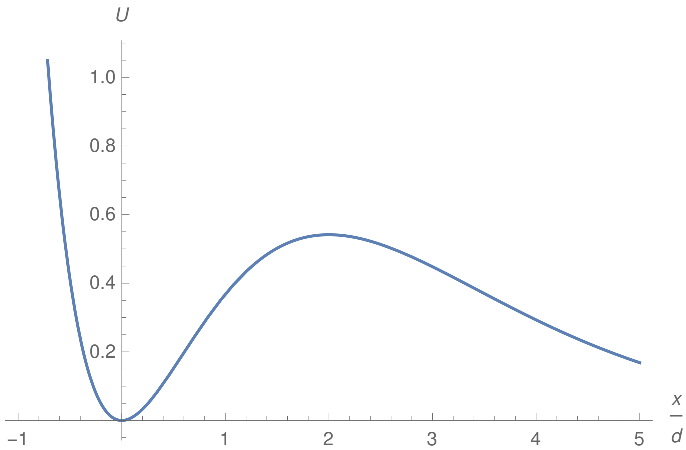
$$\int \mathrm{d}t\,\left(\frac{1}{r}\frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} - \frac{\vec{\boldsymbol{r}}}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\right) = \frac{\vec{\boldsymbol{r}}}{r} + \boldsymbol{C}.$$

18. <u>*Frequency of small oscillations:*</u> Consider the one-dimensional potential

$$U(x) = U_0\,\left(\frac{x}{d}\right)^{2\,n}\,\mathrm{e}^{-x/d},$$

where U_0 and d are positive constants and $n \geq 1$. (a) Plot the potential as a function of (x/d) for $n = 1$. (b) Determine the frequency of small oscillations about the minima for $n \geq 1$.

<u>*Solution:*</u> The potential for $n = 1$ behaves as follows:



We find that

$$\frac{\mathrm{d}U}{\mathrm{d}x} = \frac{U_0}{d}\,\left(\frac{x}{d}\right)^{2\,n}\,\mathrm{e}^{-x/d}\,\left(2\,n - \frac{x}{d}\right),$$

and

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} = \frac{U_0}{d^2}\,\left(\frac{x}{d}\right)^{2\,n-2}\,\mathrm{e}^{-x/d}\,\left[-\frac{x}{d}\,\left(2\,n - \frac{x}{d}\right) - \frac{x}{d} + (2\,n-1)\,\left(2\,n - \frac{x}{d}\right)\right].$$

But, all these are not needed! It is clear that the minimum occurs at $x = 0$ for all $n \geq 1$. If we Taylor expand the potential about $x = 0$, we obtain that

$$U(x) \simeq U_0\,\left(\frac{x}{d}\right)^{2\,n}.$$

The potential has a point of inflection for $n > 1$. For $n = 1$, the frequency ω of small oscillations is determined by the relation

$$\frac{m\,\omega^2}{2} = \frac{U_0}{d^2},$$

or

$$\omega = \sqrt{\frac{2\,U_0}{m\,d^2}}.$$

19. *Parabolic trajectory in the Kepler problem:* Consider a particle moving in the central Kepler potential $U(r) = -\alpha/r$, where $\alpha > 0$. (a) Write down the equation of motion governing the radial trajectory of the particle. (b) Obtain the solution $r(t)$ to the equation of motion when the energy of particle is zero.

Solution: The energy equation is given by

$$E = \frac{\mu}{2}\,\left(\dot{r}^2 + r^2\,\dot{\phi}^2\right) + U(r) = \frac{\mu}{2}\,\dot{r}^2 + \frac{L^2}{2\,\mu\,r^2} + U(r) = \frac{\mu}{2}\,\dot{r}^2 + U_{\mathrm{eff}}(r),$$

which upon differentiation leads to

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = -\frac{\alpha}{r^2} + \frac{L^2}{\mu\,r^3}.$$

For $E = 0$, we have

$$\frac{\mu}{2}\,\dot{r}^2 = \frac{\alpha}{r} - \frac{L^2}{2\,\mu\,r^2}$$

so that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\frac{2}{\mu}}\,\left(\frac{\alpha}{r} - \frac{L^2}{2\,\mu\,r^2}\right)^{1/2}$$

or

$$t - t_0 = \int \mathrm{d}t = \int \frac{\mathrm{d}r\,\sqrt{\mu/2}}{\sqrt{\frac{\alpha}{r} - \frac{L^2}{2\,\mu\,r^2}}} = \mu \int \frac{\mathrm{d}r\; r}{\sqrt{2\,\mu\,\alpha\,r - L^2}}$$

and, if we set

$$x = \sqrt{2\,\mu\,\alpha\,r - L^2}, \quad \mathrm{d}x = \frac{\mathrm{d}r\;(\mu\,\alpha)}{\sqrt{2\,\mu\,\alpha\,r - L^2}},$$

we obtain

$$t - t_0 = \frac{1}{2\,\mu\,\alpha^2} \int \mathrm{d}x\,\left(x^2 - L^2\right) = \frac{1}{2\,\mu\,\alpha^2}\left(\frac{x^3}{3} + L^2\,x\right) = \frac{x}{2\,\mu\,\alpha^2}\left(\frac{x^2}{3} + L^2\right)$$

which can be expressed as

$$t - t_0 = \frac{1}{2\,\mu\,\alpha^2}\sqrt{2\,\mu\,\alpha\,r - L^2}\left(\frac{2}{3}\,\mu\,\alpha\,r - \frac{L^2}{3} + L^2\right) = \frac{1}{3\,\mu\,\alpha^2}\sqrt{2\,\mu\,\alpha\,r - L^2}\,\left(\mu\,\alpha\,r + L^2\right).$$

20. An electric dipole of moment $\vec{\boldsymbol{p}} = p\,\hat{\boldsymbol{z}}$ is located at the origin. The dipole creates the electrostatic potential

$$\phi(\boldsymbol{r}) = \frac{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{r}}}{4\,\pi\,\epsilon_0\,r^3}.$$

(a) Determine the corresponding electric field $\vec{\boldsymbol{E}}$ in terms of $\vec{\boldsymbol{p}}$, $\hat{\boldsymbol{r}}$ and r. (b) Evaluate the work done by the electric field to move a charge q along the z-axis from $z = 1$ to $z = \infty$.

<u>*Solution:*</u> (a) As $\boldsymbol{p} = p\,\hat{\boldsymbol{z}}$, we have

$$\begin{aligned}
\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\phi &= -\frac{p\,z}{4\,\pi\,\epsilon_0}\frac{\partial}{\partial x}\left[\left(x^2+y^2+z^2\right)^{-3/2}\right]\hat{\boldsymbol{x}} - \frac{p\,z}{4\,\pi\,\epsilon_0}\frac{\partial}{\partial y}\left[\left(x^2+y^2+z^2\right)^{-3/2}\right]\hat{\boldsymbol{y}} \\
&\quad -\frac{p}{4\,\pi\,\epsilon_0}\frac{\partial}{\partial z}\left[z\,\left(x^2+y^2+z^2\right)^{-3/2}\right]\hat{\boldsymbol{z}} \\
&= \frac{3\,p\,x\,z}{4\,\pi\,\epsilon_0}\frac{\hat{\boldsymbol{x}}}{(x^2+y^2+z^2)^{5/2}} + \frac{3\,p\,y\,z}{4\,\pi\,\epsilon_0}\frac{\hat{\boldsymbol{y}}}{(x^2+y^2+z^2)^{5/2}} + \frac{3\,p\,z^2}{4\,\pi\,\epsilon_0}\frac{\hat{\boldsymbol{z}}}{(x^2+y^2+z^2)^{5/2}} \\
&\quad -\frac{p}{4\,\pi\,\epsilon_0}\,\frac{\hat{\boldsymbol{z}}}{(x^2+y^2+z^2)^{3/2}} \\
&= \frac{1}{4\,\pi\,\epsilon_0\,r^3}\left[3\,(\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}} - \boldsymbol{p}\right].
\end{aligned}$$

(b) The work done is

$$W = -\Delta\phi = -q\,\phi(0,0,\infty) + q\,\phi(0,0,1) = \left.\frac{q\,p\,z}{4\,\pi\,\epsilon_0\,z^3}\right|_{z=1} = \frac{q\,p}{4\,\pi\,\epsilon_0}.$$

21. <u>*Vortex in a fluid and Stokes' theorem:*</u> A fluid flowing in the x-y-plane has the following velocity field $\vec{\boldsymbol{v}} = a\,(-y\,\hat{\boldsymbol{x}} + x\,\hat{\boldsymbol{y}})/(x^2+y^2)^n$, where a is a constant. (a) Evaluate the vorticity field $\vec{\boldsymbol{\Omega}}$ corresponding to the above velocity field (recall that $\vec{\boldsymbol{\Omega}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}$). (b) Assuming $n < 1$, verify the Stokes' theorem for the velocity field $\vec{\boldsymbol{v}}$ by considering a circular path in the x-y-plane, with its centre at the origin. $\boxed{\text{1+2 marks}}$

<u>*Solution:*</u> The vorticity field is given by

$$\vec{\boldsymbol{\Omega}} = a\begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ \frac{x}{(x^2+y^2)^n} & \frac{-y}{(x^2+y^2)^n} & 0 \end{vmatrix} = \frac{2\,a\,(1-n)}{(x^2+y^2)^n}\,\hat{\boldsymbol{z}},$$

and, if we write in terms of cylindrical polar coordinates, we have

$$\vec{\boldsymbol{\Omega}} = \frac{2\,a\,(1-n)}{\rho^{2\,n}}\,\hat{\boldsymbol{z}}.$$

Note that, we can also write

$$\vec{\boldsymbol{v}} = \frac{a}{\rho^{2\,n-1}}\,\hat{\boldsymbol{\phi}}.$$

If we calculate the line integral along a circle of radius R, we have

$$\oint \vec{\boldsymbol{v}}\cdot \mathrm{d}\vec{\boldsymbol{l}} = \frac{2\,\pi\,a\,R}{R^{2\,n-1}} = \frac{2\,\pi\,a}{R^{2\,n-2}}.$$

Similarly, we have

$$\begin{aligned}\int_{\mathrm{s}} \vec{\boldsymbol{\Omega}}\cdot \mathrm{d}\vec{\boldsymbol{A}} &= \int_0^R \mathrm{d}\rho\,\rho\int_0^{2\,\pi}\mathrm{d}\phi\,\frac{2\,a\,(1-n)}{\rho^{2\,n}}\,\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{z}} = 4\,\pi\;a\,(1-n)\int_0^R \frac{\mathrm{d}\rho}{\rho^{2\,n-1}}\\ &= 4\,\pi\;a\,(1-n)\left(\frac{\rho^{2-2\,n}}{2-2\,n}\right)_0^R = \frac{4\,\pi\;a\,(1-n)}{2\,(1-n)}\,R^{2-2\,n} = \frac{2\,\pi\,a}{R^{2\,n-2}},\end{aligned}$$

for $n < 1$, which is the result we have obtained above from the line integral.

22. *Behavior of the damped, driven oscillator:* Recall that, a damped, driven oscillator satisfies the following equation of motion:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\,\beta\,\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2\,x = f_0\cos(\omega\,t).$$

Assume that the oscillator is underdamped, i.e. $\beta \ll \omega_0$. (a) Obtain the solution to the above equation for times such that $\beta\,t \gg 1$, expressing the amplitude (say, A) and phase (say, δ) of the solution in terms of the parameters ω_0, β, f_0 and ω. (b) Plot the square of the amplitude A^2 and phase δ as a function of ω/ω_0. (c) Assuming $\omega = \omega_0$, plot the position $x(t)$, the velocity $v(t) = (\mathrm{d}x/\mathrm{d}t)$ and the forcing term $f(t) = f_0\cos(\omega\,t)$ as a function of $(\omega\,t)$, in particular highlighting the difference in phases between these quantities.

Solution: Recall that the solution to the damped oscillator at late times (i.e. when $\beta\,t \gg 1$) can be expressed as

$$x(t) = A\cos(\omega\,t - \delta),$$

where $f_0 =$, the amplitude A is given by

$$A^2 = \frac{f_0^2}{(\omega_0^2-\omega^2)^2 + 4\,\beta^2\,\omega_0^2}$$

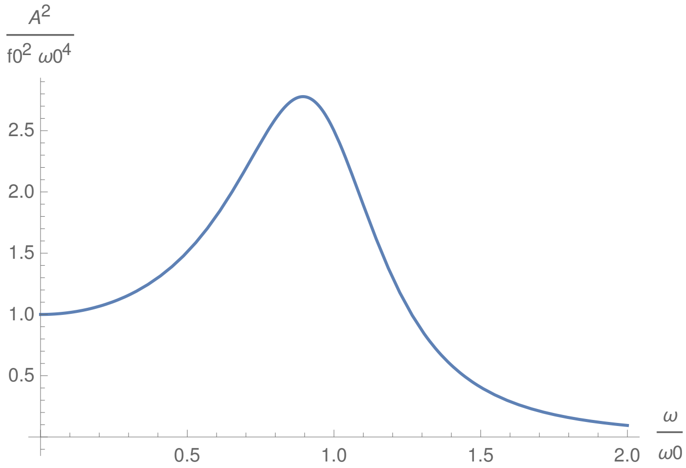
and the phase δ is defined through the relation

$$\tan\delta = \frac{2\,\beta\,\omega}{\omega_0^2-\omega^2}.$$

Note that, we can write

$$\frac{A^2}{f_0^2/\omega_0^4} = \frac{1}{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + 4\,\frac{\beta^2}{\omega_0^2}\,\frac{\omega^2}{\omega_0^2}}$$

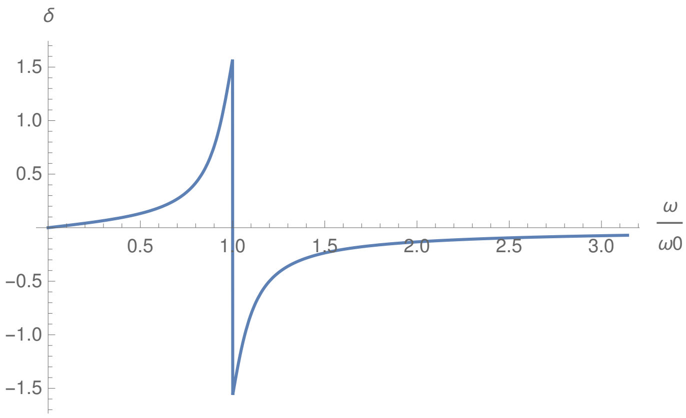
which is plotted for $\beta/\omega_0 = 0.1$ in the figure below.



Similarly, we have

$$\tan\delta = \frac{2\,(\beta/\omega_0)\,(\omega/\omega_0)}{1 - \frac{\omega^2}{\omega_0^2}},$$

which is plotted in the figure below for $\beta/\omega_0 = 0.1$



Note that when $\delta = \pi/2$, we have

$$x(t) = A\sin(\omega\, t).$$

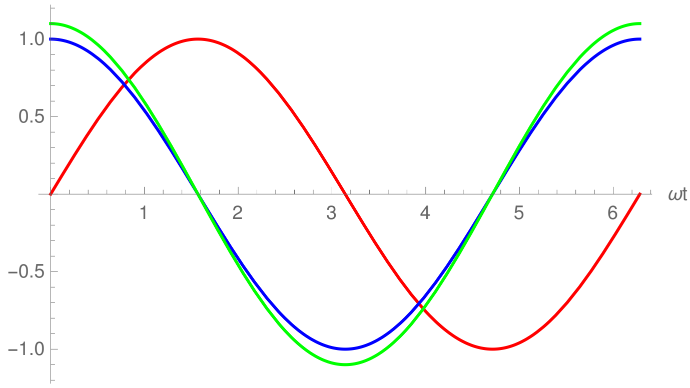
The velocity of the oscillator is given by

$$v(t) = \dot{x}(t) = -\omega\, A\sin(\omega\, t - \delta).$$

Now, when $\omega = \omega_0$, $\tan\delta$ is infinite or $\delta = \pi/2$ and hence we obtain that

$$v(t) = \omega\, A\cos(\omega\, t),$$

which is exactly in phase with the forcing term. The quantities $x(t)$ (in red), $v(t)$ (in blue) and $f(t)$ (in green) are plotted below.



23. *Trajectories in phase space:* A particle moves in the one-dimensional potential $U(x) = \alpha\, x^2\, \mathrm{e}^{-x^2}$, where $\alpha > 0$. (a) Plot the potential, specifically indicating the locations of the maxima and minima. (b) What is the allowed range of energy, and what is the range of energy (and position) for which the particle exhibits bounded motion? (c) Illustrate each type of bounded and unbounded trajectories that are possible in the phase space, indicating the direction of motion with arrows.

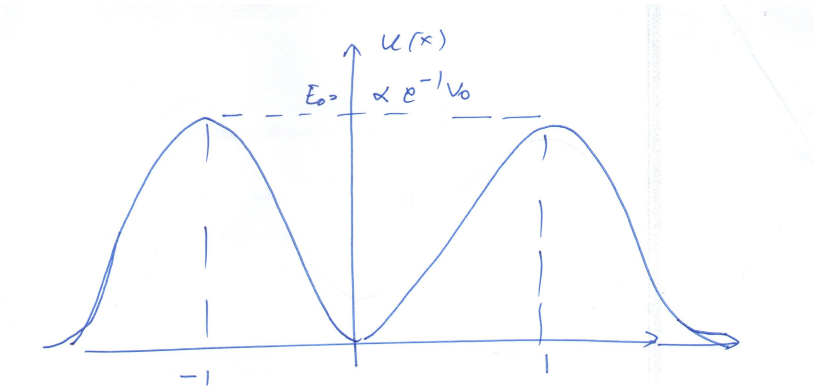
Solution: One finds that

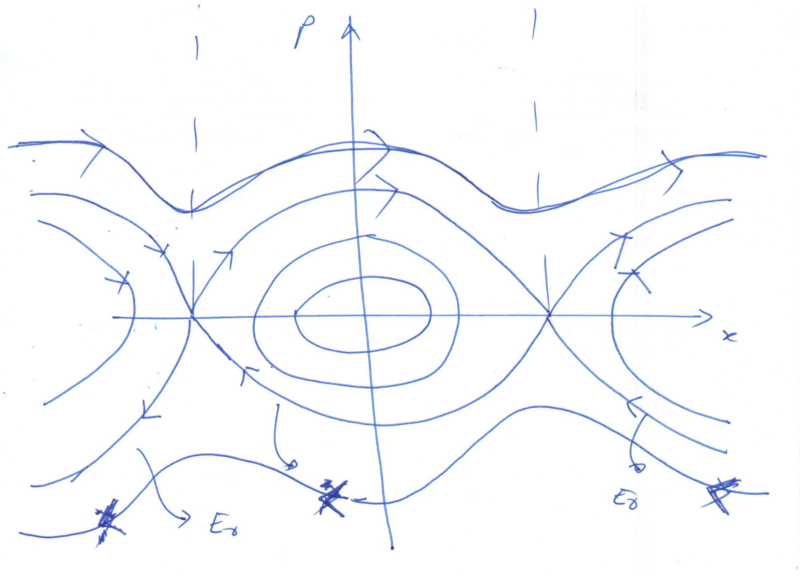
$$\frac{\mathrm{d}U}{\mathrm{d}x} = 2\, U_0\, \mathrm{e}^{-x^2}\, x\, (1 - x^2)$$

and

$$\frac{\mathrm{d}^2 U}{\mathrm{d}x^2} = 2\, U_0\, \mathrm{e}^{-x^2} (1 - 5\, x^2 + 2\, x^4)$$

so that, clearly, the extrema are located at $x = 0$ and $x = 1$. It is also evident that $x = 0$ is a minimum, while $x = 1$ is a maximum. The potential and the phase portrait are illustrated in the two figures below.





24. *Electric field in and around a charged rod:* A very long rod of radius R carries a charge per unit length λ that is uniformly distributed throughout the rod. It is surrounded by a coaxial tube of radius $3\,R$ which is oppositely charged and carries the charge per unit length $-\lambda$. (a) Calculate the electric field at the following locations: (i) inside the charged rod, (ii) between the charged rod and the coaxial tube and (iii) outside the coaxial tube. (b) Plot the magnitude of the electric field as a function of the radial distance from the centre of the tube (i.e. from 0) to $5\,R$. $\boxed{\text{3+2 marks}}$

Solution: (a) We use Gauss' law and exploit the cylindrical symmetry to compute the $\vec{\boldsymbol{E}} = E(\rho)\,\hat{\boldsymbol{\rho}}$. We will use a cylinder of length L about the axis of symmetry as the Gaussian volume. Also note that the charge density in the rod is $\varrho = \lambda/(\pi R^2)$. Therefore, since

$$\iint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{a}} = \frac{Q_{\mathrm{enc}}}{\epsilon_0},$$

(i) when $\rho < R$, we have

$$E(\rho)\,(2\,\pi\,\rho\,L) = \frac{\varrho\,\pi\,\rho^2\,L}{\epsilon_0},$$

which implies that

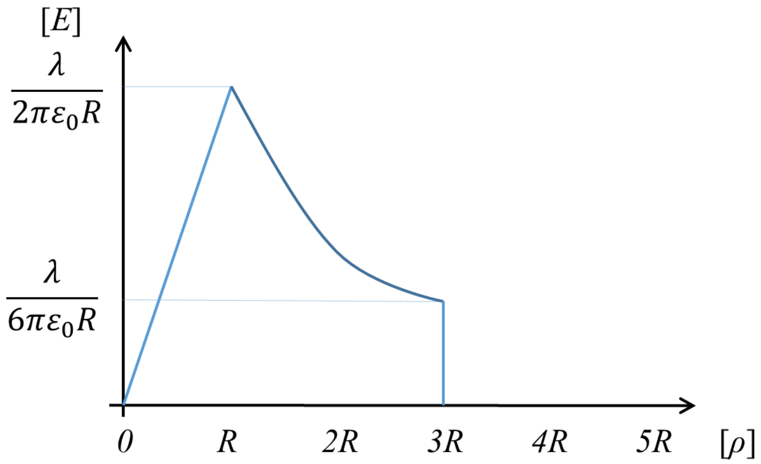
$$\vec{\boldsymbol{E}} = \frac{\lambda\,\rho}{2\,\pi\,\epsilon_0\,R^2}\,\hat{\boldsymbol{\rho}}.$$

(ii) And, when $R \le \rho < 3\,R$, we have

$$E(\rho)(2\,\pi\,\rho L) = \frac{\lambda L}{\epsilon_0} \Rightarrow \boldsymbol{E} = \frac{\lambda}{2\pi\rho\epsilon_0}\hat{\boldsymbol{\rho}}.$$

(iii) $\rho \ge 3R$

$$E(\rho)(2\pi\rho L) = \frac{\lambda L - \lambda L}{\epsilon_0} \Rightarrow \boldsymbol{E} = 0\,.$$


[E]
$\frac{\lambda}{2\pi\varepsilon_0 R}$
$\frac{\lambda}{6\pi\varepsilon_0 R}$
0
R
2R
3R
4R
5R
[ρ]