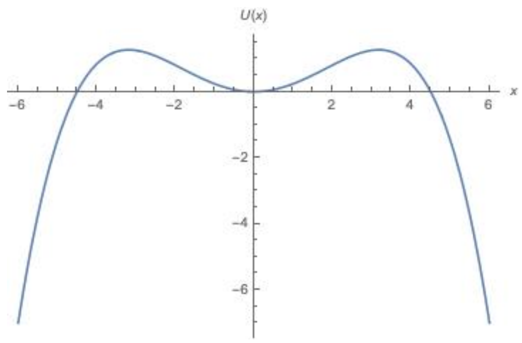
1. (a) Given $U(x) = U_0\{(x/a)^2 - (x/b)^4\}$,

$U'(x) = U_0 x\{2/a^2 - 4x^2/b^4\}$

(a) $U'(x) = 0 \implies x = 0 \text{ and} x = \pm \dfrac{b^2}{a\sqrt{2}}$ **(0.5 + 0.5= 1 MARK)**

(b)

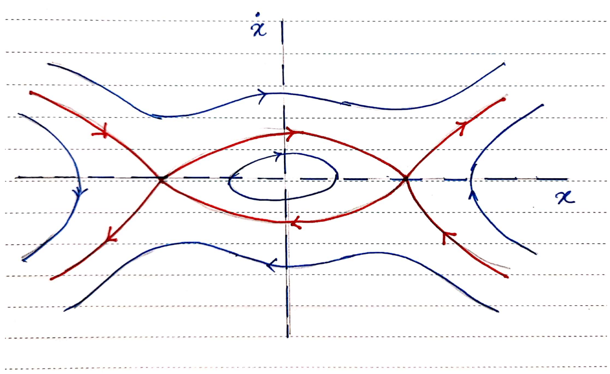


(1 MARK)

(c) For bounded motion, the particle needs to be "trapped" inside the potential well, for which the necessary condition is $0 \leq E < U_{\max}$,

where $U_{\max} = U(\pm b^2/a\sqrt{2}) = \dfrac{U_0}{4}\left(\dfrac{b}{a}\right)^4$ **(0.5+0.5=1 MARK)**

(d) Phase portrait



(e) Equation for separatrix curves is

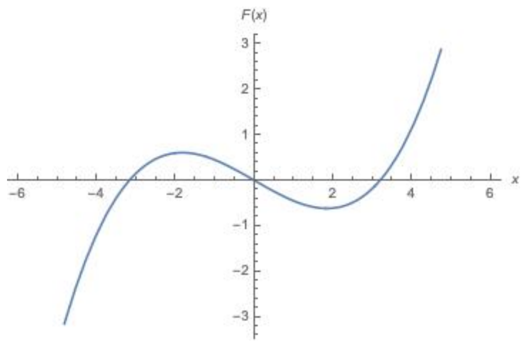
$$\frac{m}{2}\dot{x}^2 + U(x) = U_{\max} \quad \textbf{OR} \quad \frac{p_x^2}{2m} + U(x) = U_{\max} \quad \textbf{OR} \quad \frac{p^2}{2m} + U(x) = U_{\max}$$

$$\frac{m}{2}\dot{x}^2 + U_0[(x/a)^2 - (x/b)^4] = \frac{U_0}{4}\left(\frac{b}{a}\right)^4$$ **(1 MARK)**

OR

$$\frac{p_x^2}{2m} + U_0[(x/a)^2 - (x/b)^4] = \frac{U_0}{4}\left(\frac{b}{a}\right)^4$$

(f) $F(x) = -U'(x) = -U_0 x\left[\frac{2}{a^2} - \frac{4x^2}{b^4}\right]$



(1 MARK)

2. (a) Equations of motion for r_1 and r_2:

$$m(\ddot{r}_1 - \omega^2 r_1) = -k(r_1 - r_2) \tag{2.1}$$

$$m(\ddot{r}_2 - \omega^2 r_2) = k(r_1 - r_2) \tag{2.2}$$

(0.5+0.5=1 MARK)

(b) COM radial coordinate $R = (r_1 + r_2)/2.$ (2.3)

Adding (2.1) and (2.2), we find

$$\ddot{R} - \omega^2 R(t) = 0$$ **(0.5 MARK)** (2.4)

Given initial conditions are

$$r_1(0) = 0 = r_2(0),\, \dot{r}_1(0) = v_0;\, \dot{r}_2(0) = 0 \qquad (2.5)$$

Using (2.5) in (2.3), we find $R(0) = 0 \quad ; \dot{R}(0) = \dfrac{v_0}{2}$ **(0.5 MARK)** (2.6)

General solution to (2.4) is

$$R(t) = Ae^{\omega t} + Be^{-\omega t} \qquad \textbf{(0.5 MARK)} \qquad (2.7)$$

Using IC in (2.6), we find $A = v_0/2\omega = -B$, and hence

$$R(t) = \frac{v_0}{4\omega}(e^{\omega t} - e^{-\omega t}) \ \textbf{OR}\ \frac{v_0}{2\omega}\sinh \omega t \qquad \textbf{(0.5 MARK)} \qquad (2.8)$$

(c) Subtracting (2.2) from (2.1), we find

$$m(\ddot{r} - \omega^2 r) = -2kr \implies \ddot{r} + (2k/m - \omega^2)r = 0 \quad \textbf{(0.5 MARK)} \qquad (2.9)$$

If $\omega^2 < 2k/m$, define $\omega_1^2 = 2k/m - \omega^2$, Eq. 2.9 becomes the SHM oscillator equation with solution

$$r(t) = C\cos\omega_1 t + D\sin\omega_1 t \qquad \textbf{(0.5 MARK)} \qquad (2.11)$$

From the IC in (2.5), it follows that $r(0) = 0, \dot{r}(0) = v_0$ **(0.5 MARK)** (2.12)

Using (2.10) in (2.9), we find $C = 0,\ D = \dfrac{v_0}{\omega_1}$: Use in 2.11 to find

$$r(t) = \frac{v_0}{\omega_1}\sin\omega_1 t \qquad \textbf{(0.5 MARK)} \qquad (2.13)$$

If elastic collision between engine and wagon (when $r(t) = 0$) is taken into account, the solution becomes (no derivation needed)

$$r(t) = \frac{v_0}{\omega_1}|\sin\omega_1 t| \qquad (2.13A)$$

(d) If $\omega^2 > 2k/m$, define $\alpha^2 = \omega^2 - 2k/m$, so 2.9 becomes

$\ddot{r} = \alpha^2 r$, with solution (follow the same reasoning as (a), with IC in 2.12):

$$r(t) = \frac{v_0}{2\alpha}(e^{\alpha t} - e^{-\alpha t}) \text{ OR } \frac{v_0}{\alpha}\sinh\alpha t, \qquad \textbf{(1 MARK)} \qquad (2.14)$$

$r(t)$ **increases continuously with time.** **(1 MARK)**

AND/OR

The spring will break/snap eventually, and the engine gets separated from the wagon.

OR

Any equivalent statement.

3. (a) Total energy E and angular momentum (z- component) L_z is conserved. **(0.5+0.5=1 MARK)**

From given conditions, $L_z = \ell$ and $E = mv_0^2/2$ (4.1)

(b) Let $b > 0$ be the y-coordinate of the particle before it reaches the well. Then, $\ell = mv_0 b$. To enter the potential well, we require $b < a$, hence $\ell < mv_0 a$ is the condition. **(1 MARK)**

4. (a) The particle enters the potential well at the point with coordinates $(a/\sqrt{2}, a/\sqrt{2})$. At this point, the 2-d polar angle $\theta_0 = \pi/4$.

Let v_r and v_θ be the radial and tangential components of velocity, IMMEDIATELY AFTER the particle has entered the well.

The tangential velocity component

$$v_\theta = v_0 \sin\theta_0 = v_0/\sqrt{2}$$ **(0.5 MARK)** (4.2)

is conserved as the particle enters the well, since there is no tangential force.

The radial component v_r changes from its pre-entry value, $-v_0\cos\theta_0$, since there is a radial force.

The new value can be found from conservation of energy:

$$\frac{m}{2}v_r^2 - U_0 = \frac{m}{2}v_0^2\cos^2\theta_0 \implies v_r = \pm\sqrt{v_0^2\cos^2\theta_0 + \frac{2}{m}U_0}.$$

Choose - sign, since for $U_0 = 0$, we require $v_r = -v_0\cos\theta_0$.

Hence, $v_r = -\sqrt{v_0^2\cos^2\theta_0 + \frac{2}{m}U_0} = -v_0$ (4.3)

After substituting $\cos\theta_0 = 1/\sqrt{2}$ and $U_0 = mv_0^2/4$ as given, we find

$v_r = -v_0$ **(0.5 MARK)** (4.3A)

Combine (3.2) and (3.3):

$$\mathbf{v} = -v_0 \hat{\boldsymbol{r}} + \frac{v_0}{\sqrt{2}}\,\hat{\boldsymbol{\theta}} \tag{4.4}$$

(and speed $v = v_0\sqrt{3/2}$: NOT REQUIRED FOR CREDIT). (4.4A)

(b) *Motion inside the well:*

In 2d polar coordinates, angular momentum

$$\ell = mr^2\dot{\phi} = mv_0 a/\sqrt{2} \quad \textbf{(0.5 MARK)} \tag{4.5}$$

Energy

$$E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - U_0 \quad \textbf{(0.5 MARK)} \tag{4.6}$$

Use (4.1) and (4.5) in (4.6):

$$\frac{m}{2}\dot{r}^2 + \frac{mv_0^2a^2}{4r^2} = E + U_0 = \frac{3}{4}mv_0^2. \quad \textbf{(0.5 MARK)} \tag{4.7}$$

After some simplifications, 4.7 can be rewritten in the form

$$\dot{r} = -\frac{v_0}{r}\sqrt{\frac{3}{2}\left(r^2 - \frac{a^2}{3}\right)}, \quad \textbf{(0.5 MARK)} \tag{4.8}$$

(c) *Integration of 4.8*:

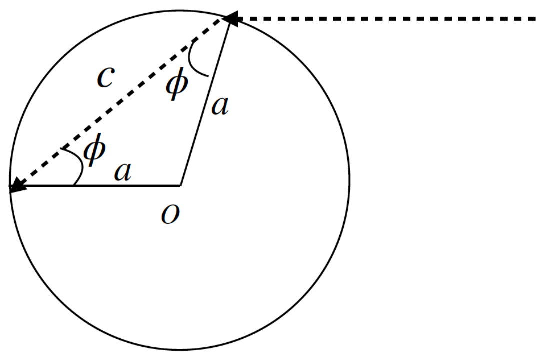
$$\int_a^r \frac{r'dr'}{\sqrt{r'^2 - \frac{a^2}{3}}} = -v_0 t\sqrt{\frac{3}{2}} \tag{4.9}$$

(Initial condition $r(0) = a$ has been used, where $t = 0$ is the instant where the particle enters the well).

After completing the elementary integration,

$$r^2 = a^2 + 2v_0t\left(\frac{3v_0t}{4} - a\right) \quad \textbf{(1 MARK)} \tag{4.10}$$

At the instant of exit, from well, $r = a$, which happens at time



$$T = \frac{4a}{3v_0} \quad \textbf{(1 MARK)} \tag{4.11}$$

ALTERNATIVE SOLUTION FOR 4(c):

Time spent by the particle inside the well is $T = \frac{c}{v} = \sqrt{\frac{2}{3}}\frac{c}{v_0}$,

where c is the "chord length" in the figure, and v is speed of particle inside well (see 4.4A).

From figure, $c^2 = 2a^2[1 - \cos(\pi - 2\phi)] = 2a^2(1 + \cos 2\phi) = 4a^2\cos^2\phi$.

Hence $c = 2a\cos\phi$. **(1 MARK)**

Here, ϕ is the angle made by the velocity vector with the radial unit vector at the point of entry.

From Eq. 4.4, we find $\cos\phi = \frac{|v_r|}{v} = \frac{v_0}{v} = \sqrt{\frac{2}{3}}$

Hence, $c = 2\sqrt{\frac{2}{3}}a$, and $T = \frac{4a}{3v_0}$. **(1 MARK)**