Mini-Test 1

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1. (i) $\ddot{x}+2w_0\dot{x}+w_0^2x = f_0\cos wt$ —(1)

$\Rightarrow \ddot{y}+2w_0\dot{y}+w_0^2y = f_0\sin wt$ —(2)

(2) $\times i \Rightarrow i\ddot{y}+2iw_0\dot{y}+iw_0^2y = f_0(i\sin wt)$ —(3)

Adding (1) and (3), we get

$\Rightarrow \ddot{z}+2w_0\dot{z}+w_0^2z = f_0e^{iwt}$ (where $z=x+iy$)

$\Rightarrow$ Let $z=z_0e^{iwt}$

$\Rightarrow -z_0w^2+2iw_0z_0+w_0^2z_0 = f_0$

$\Rightarrow z_0 = \dfrac{f_0}{(w_0^2-w^2)+2iww_0} \Rightarrow z_0 = \dfrac{f_0\left[(w_0^2-w^2)-2iww_0\right]}{(w_0^2-w^2)^2+4w^2w_0^2}$

$\Rightarrow z_0 = \dfrac{f_0\left[(w_0^2-w^2)-2iww_0\right]}{(w_0^2+w^2)^2}$

Let $z_0 = A_0e^{i\phi}$

$\Rightarrow A_0 = \dfrac{f_0}{w_0^2+w^2}$, $\phi = \tan^{-1}\left(\dfrac{2ww_0}{w^2-w_0^2}\right)$

$\Rightarrow z = (A_0e^{i\phi})e^{iwt} \Rightarrow \boxed{z = A_0e^{i(wt+\phi)}}$

$\Rightarrow x_p(t) = Re(z)$, where $Re(z)$ represents the real part of z.

$\Rightarrow x_p(t) = A_0\cos(wt+\phi)$, where

$\boxed{A_0 = \dfrac{f_0}{(w_0^2+w^2)}}$, $\boxed{\phi = \tan^{-1}\left(\dfrac{2ww_0}{w^2-w_0^2}\right)}$

(ii) Given equation is $\ddot{x}+2w_0\dot{x}+w_0^2x=0$ —(1)

Let $x = te^{-\beta t} \Rightarrow \dot{x} = e^{-\beta t}-\beta te^{-\beta t}$

$\ddot{x} = -\beta e^{-\beta t}+\beta^2te^{-\beta t}-\beta e^{-\beta t} = (\beta^2t-2\beta)e^{-\beta t}$

$\Rightarrow \ddot{x}+2w_0\dot{x}+w_0^2x = (\beta^2t-2\beta)e^{-\beta t}+2w_0e^{-\beta t}-2w_0\beta te^{-\beta t}+w_0^2te^{-\beta t}$

$= e^{-\beta t}\left[t(\beta-w_0)^2-2(\beta-w_0)\right]$

From equation (1),

$e^{-\beta t}\left[t(\beta-w_0)^2-2(\beta-w_0)\right] = 0$

$\Rightarrow (\beta-w_0)\left[e^{-\beta t}\left[t(\beta-w_0)-2\right]\right] = 0 \Rightarrow \boxed{\beta = w_0}$

$\therefore$ By substitution, $x = te^{-w_0t}$ can be the second solution.

(iii) $x(t) = A_0\cos(\omega t+\phi) + (C_1 + C_2 t)e^{-\omega_0 t}$

$x(0) = 0 \Rightarrow C_1 + A\cos\phi = 0 \Rightarrow C_1 = -A\cos\phi$

$$= \frac{-f_0}{\omega_0^2+\omega^2}\left(\frac{\omega_0^2-\omega^2}{\omega_0^2+\omega^2}\right)$$

$$= \underline{\underline{\frac{f_0(\omega^2-\omega_0^2)}{(\omega_0^2+\omega^2)^2}}}$$

$\dot{x}(0) = 0 \Rightarrow -A_0\omega\sin\phi - C_1\omega_0 + C_2 = 0$

$\Rightarrow C_2 = A_0\omega\sin\phi + C_1\omega_0$

$$= \frac{-f_0\omega}{(\omega^2+\omega_0^2)} \times \frac{2\omega\omega_0}{(\omega^2+\omega_0^2)} + \left[+\frac{f_0(\omega^2-\omega_0^2)}{(\omega_0^2+\omega^2)^2}\omega_0\right]$$

$$= -\frac{f_0\omega_0}{\omega^2+\omega_0^2}$$

$\therefore$ $$\boxed{x(t) = \frac{f_0}{(\omega^2+\omega_0^2)}\cos\left(\omega t + \tan^{-1}\left(\frac{2\omega\omega_0}{\omega^2-\omega_0^2}\right)\right) + \left[\frac{f_0(\omega^2-\omega_0^2)}{(\omega^2+\omega_0^2)^2} - \frac{f_0\omega_0 t}{(\omega^2+\omega_0^2)}\right]e^{-\omega_0 t}}$$

$\boxed{2.}$ $U(r) = \dfrac{U_0 e^{-\vec{P}\cdot\vec{r}}}{r}$

(i) $\vec{F} = -\dfrac{dU}{dr} = -\left[-\dfrac{U_0 e^{-\vec{P}\cdot\vec{r}}}{r^2}(\hat{r}) + \dfrac{U_0 e^{-\vec{P}\cdot\vec{r}}}{r}(-\vec{P})\right]$ [$\because$ using Product Rule]

$$= \boxed{\frac{U_0 e^{-P\cdot r}}{r}\left[\frac{\hat{r}}{r} + \vec{P}\right]}$$

(ii) Initially, the body is at $(0, \frac{1}{P}, 0)$, $\hat{r}$ is along y axis and $\vec{P}$ is also along y-axis. Since the force is along y-axis, the body continues to move along the y-axis.

$\therefore$ $U_\infty = 0$, At $(0, \frac{1}{P}, 0) \Rightarrow U = \dfrac{U_0 e^{-1}}{1/P} = \dfrac{PU_0}{e}$

$\therefore$ Mechanical energy is conserved $\Rightarrow \Delta K + \Delta U = 0$

$\Rightarrow \Delta K = -\Delta U \Rightarrow \frac{1}{2}(1)(V_\infty^2 - 0) = \frac{PU_0}{e}$

$$\Rightarrow \boxed{V_\infty = \sqrt{\frac{2PU_0}{e}}}$$

$\therefore$ The final velocity is $V_\infty = \sqrt{\dfrac{2PU_0}{e}}\,(\hat{j})$.

3. (i) Given,

$$r(t) = R\{(\omega t - \sin\omega t)\hat{i} + (1-\cos\omega t)\hat{j}\}$$

$$v(t) = \frac{d}{dt}[r(t)] = R\omega\{(1-\cos\omega t)\hat{i} + \sin\omega t\,\hat{j}\}$$

$$V_{max}(t) = |v(t)|\Big|_{max}$$

$$= \left|\sqrt{R^2\omega^2[(1-\cos\omega t)^2 + \sin^2\omega t]}\right|_{max}$$

$$= \left|\sqrt{2R^2\omega^2(1-\cos\omega t)}\right|_{max}$$

$$= \left|\sqrt{4R^2\omega^2\sin^2\left(\frac{\omega t}{2}\right)}\right|_{max} = \left|2R\omega\sin\left(\frac{\omega t}{2}\right)\right|_{max}$$

since, the maximum value of $\sin\left(\frac{\omega t}{2}\right) = 1$, then

$$\Rightarrow \boxed{V_{max}(t) = 2R\omega}$$

$\therefore$ Expression for speed at time t is

$$v(t) = 2R\omega\sin\left(\frac{\omega t}{2}\right)$$

The maximum value V_{max} is $2R\omega$.

(ii) $$\overline{V_x} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} V_x\,dt$$

$$= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} R\omega(1-\cos\omega t)\,dt$$

$$= \frac{R\omega^2}{2\pi}\left(\frac{1}{\omega}(\omega t - \sin\omega t)\right)\Big|_0^{2\pi/\omega}$$

$$= \frac{R\omega^2}{2\pi}\left(\frac{2\pi}{\omega}\right) = R\omega \Rightarrow \therefore \boxed{\overline{V_x} = R\omega}$$

$$\overline{V_y} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} V_y\,dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} R\omega\sin\omega t\,dt = 0$$

$$\therefore \boxed{\overline{V_y} = 0}$$