Mini Test 2

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1. Consider a particle of mass m and energy E moving in a one-dimensional region with potential energy where $U_0 > 0$ has the dimensions of energy, and $a > 0, b > 0$.

$$U(x) = U_0\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{b}\right)^4\right]$$

(a) Identify the locations of maxima and minima of $U(x)$ (1 MARK).

(b) Graphically sketch $U(x)$ versus x in the range $x\epsilon(-\infty,\infty)$(1 MARK).

(c) Find the range of energy E corresponding to bounded motion in the form $E_1 < E < E_2$ (i.e., find E_1 *and* E_2) (1 MARK).

(d) Sketch the complete phase portrait for the system, including bounded and unbounded trajectories as well as separatrix curves (2 MARKS).

(e) Write down the equation for the separatrix curve (1 MARK).

(f) Sketch the force $F(x)$ on the particle as a function of x in the range $x\epsilon(-\infty,\infty)$ (1 MARK).

a) $U(x) = U_0\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{b}\right)^4\right]$; $U_0, a, b > 0$

$\Rightarrow U'(x) = \frac{d}{dx}(U(x)) = \frac{2U_0}{a^2}x - \frac{4U_0}{b^4}x^3$

$\Rightarrow U''(x) = \frac{d^2}{dx^2}(U(x)) = \frac{d}{dx}(U'(x)) = \frac{2U_0}{a^2} - \frac{12U_0}{b^4}x^2$

For the location of maxima,
$U'(x) = 0$ and $U''(x) < 0$

$\Rightarrow \frac{2U_0}{a^2}x - \frac{4U_0}{b^4}x^3 = 0 \Rightarrow x\left(\frac{2U_0}{a^2} - \frac{4U_0}{b^4}x^2\right) = 0$

$x = 0$ or $x = \frac{-b^2}{\sqrt{2}a}$ or $x = \frac{b^2}{\sqrt{2}a}$

We get $U''\left(\frac{-b^2}{\sqrt{2}a}\right) = \frac{2U_0}{a^2} - \frac{12U_0}{b^4} \times \frac{b^4}{2a^2} = -\frac{4U_0}{a^2} < 0$

and $U''\left(\frac{b^2}{\sqrt{2}a}\right) = \frac{2U_0}{a^2} - \frac{12U_0}{b^4} \times \frac{b^4}{2a^2} = \frac{-4U_0}{a^2} < 0$

Hence for $x = \frac{-b^2}{\sqrt{2}a}$ and $x = \frac{b^2}{\sqrt{2}a}$, $U(x)$ has local maximas.

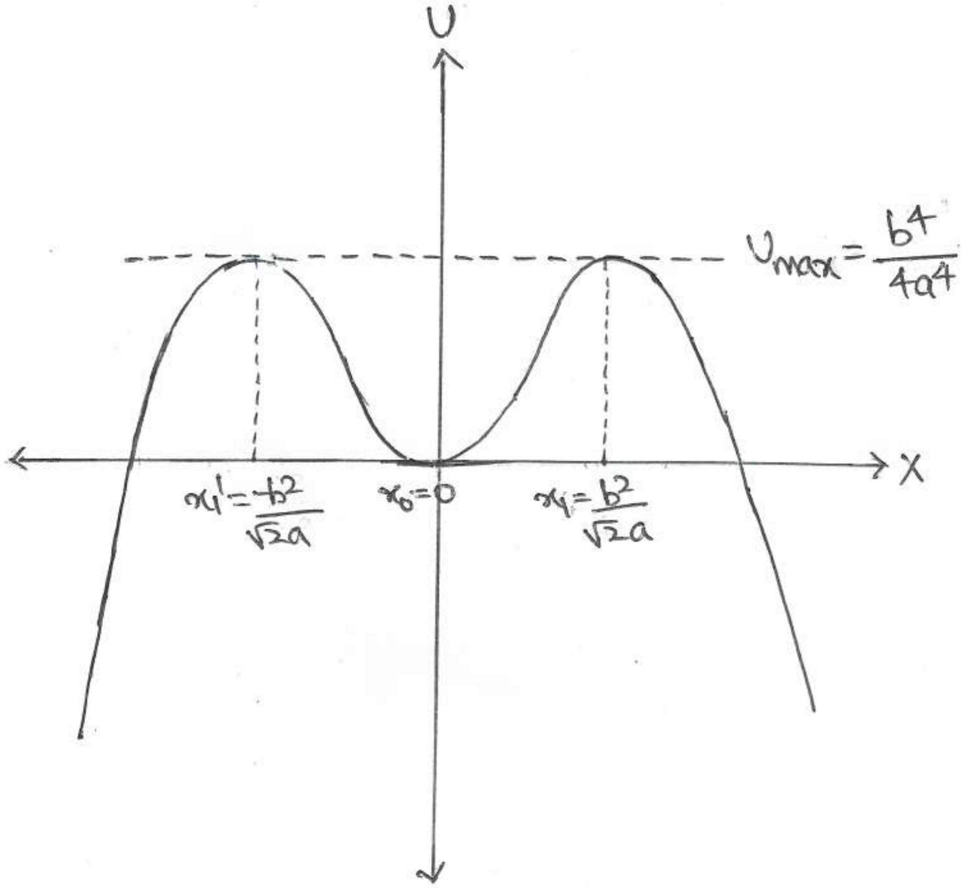
similarly, for the location of minima,

$U'(x)=0$ and $U''(x)>0$

We get $U''(0) = \frac{2U_0}{a^2} > 0$

Hence for $x=0$, $U(x)$ has local minima.

b)



The graph $U(x)$ vs x has

Global Maximas at $x_1 = \frac{b^2}{\sqrt{2}a}$ and $x_1' = \frac{-b^2}{\sqrt{2}a}$

Local Minima at $x_0 = 0$

Global Minimas at $x \to -\infty$ and $x \to +\infty$

c) From the graph of $U(x)$ vs x we can see that the particle executes

Case 1: $E<0$

Unbounded Motion. Particle reaches $-\infty$ if $x<0$ and $+\infty$ if $x>0$

Case 2: $0<E<U_{max}$

Bounded Motion. If $\frac{-b^2}{\sqrt{2}a}<x<\frac{b^2}{\sqrt{2}a}$ particle oscillates about stable equilibrium.

Since, $U_{max} = \frac{U_0 b^4}{4a^4}$

Hence $E_1=0$ and $E_2 = \frac{U_0 b^4}{4a^4}$

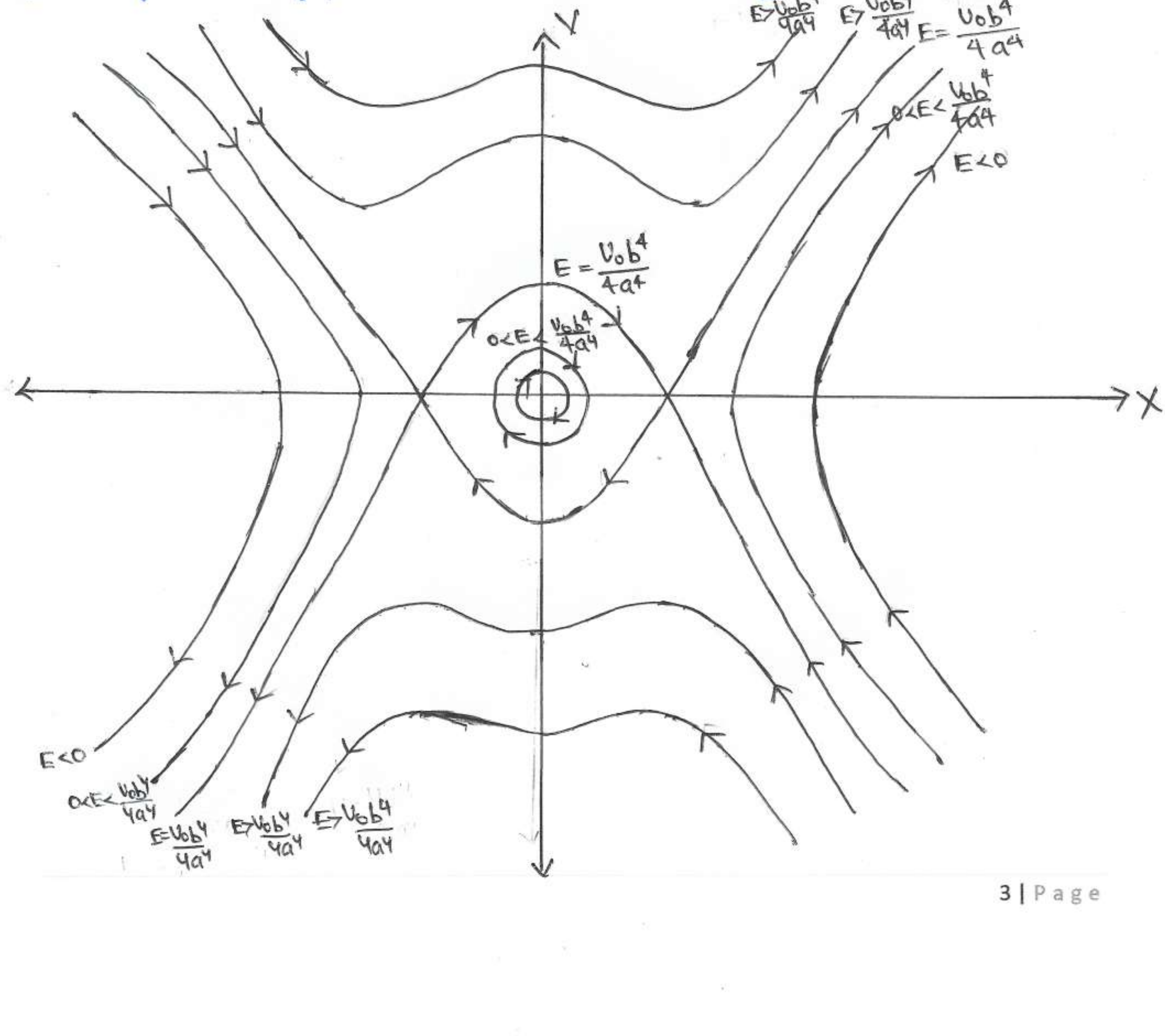
Case 3: $E>0$

Unbounded Motion

d) Let E denote Energy of the particle

$\Rightarrow E = \frac{1}{2}mv^2 + U(x) = \frac{1}{2}mv^2 + U_0\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{b}\right)^4\right]$

Therefore equation of phase potrait is $\frac{1}{2}mv^2 + U_0\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{b}\right)^4\right] = E$

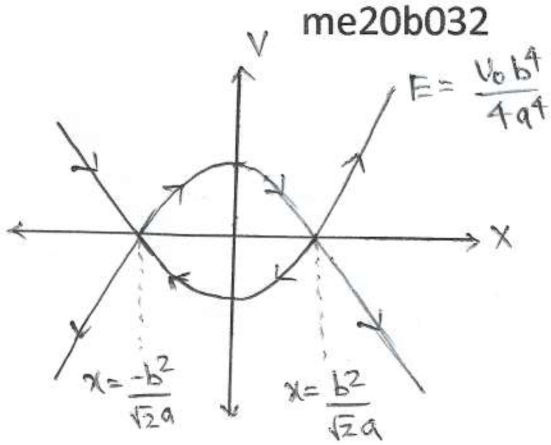


e) For separatrix, $E = U_{max}$

Therefore $E = \frac{U_0 b^4}{4a^4}$

Hence equation of separatrix is given by

$$\frac{U_0 b^4}{4a^4} = \frac{1}{2}mv^2 + U_0\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{b}\right)^4\right]$$

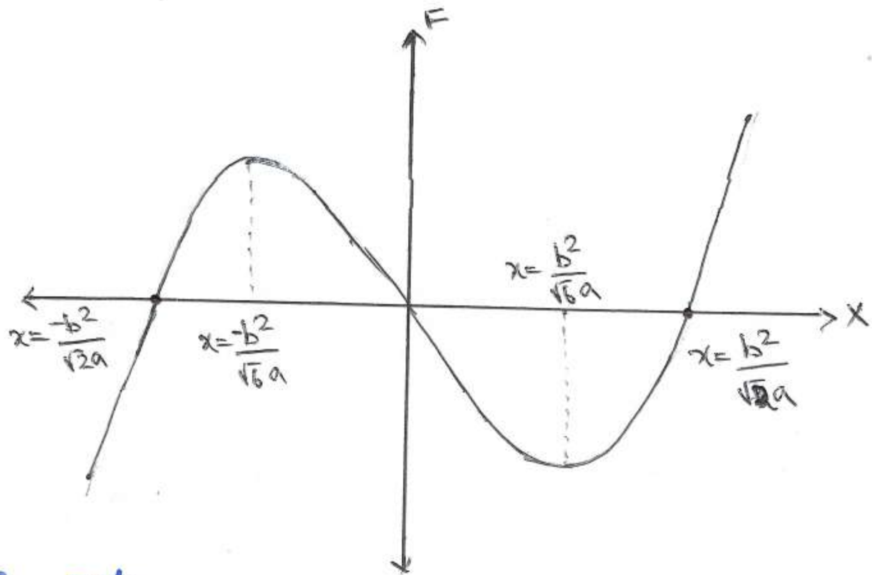


f) Since particle is moving in one dimension

$$F(x) = -\frac{dU(x)}{dx}$$

$$F(x) = \frac{4U_0}{b^4}x^3 - \frac{2U_0}{a^2}x$$

Therefore



$$F'(x) = \frac{12U_0}{b^4}x^2 - \frac{2U_0}{a^2}$$

For extrema's

$$F'(x) = 0 \Rightarrow x = \frac{-b^2}{\sqrt{6}a} \text{ and } x = \frac{b^2}{\sqrt{6}a}$$

For $x = \frac{-b^2}{\sqrt{6}a}$, F has a local maxima. $F_{max} = \frac{2\sqrt{6}}{9}\frac{U_0 b^2}{a^3}$ N

For $x = \frac{b^2}{\sqrt{6}a}$, F has local minima. $F_{min} = \frac{-2\sqrt{6}}{9}\frac{U_0 b^2}{a^3}$ N

2. A toy train consists of an engine and wagon of equal mass m each, connected by a spring with spring constant k. The relaxed length of the spring may be considered to be zero. The train is initially placed at the centre of a horizontal, circular turntable (see Fig. 1), and is free to move on a radial frictionless track on the turntable. The engine (alone) is now given an initial (radial) velocity v_0, and the turntable is independently set-in motion to rotate counter clockwise with an angular speed ω. Neglect the physical dimensions of the train.

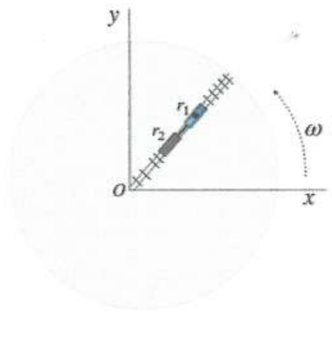


Figure 1

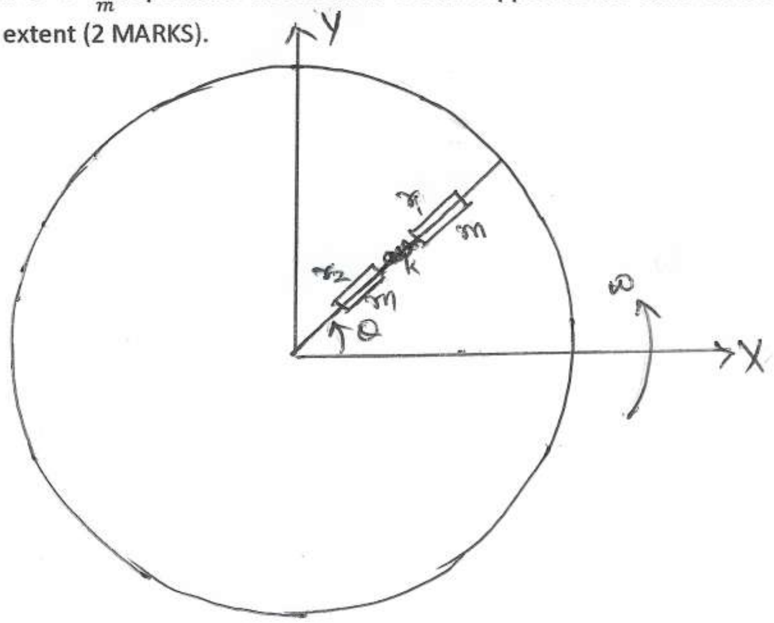
(a) Write down the equations of motion for the radial coordinates of the engine and the wagon, denoted by r_1 and r_2 (1 MARK).

(b) Using (a), write down the equation of motion for the radial coordinate $R(t)$ of the centre of mass (COM) of the train. Solve this equation subject to the given initial conditions and determine $R(t)$ (2 MARKS).

(c) Using (a), write down the equation of motion for the separation $r = r_1 - r_2$ between the engine and the wagon. Solve the equation and find $r(t)$ subject to the given initial conditions (assume that $\omega^2 < \frac{2k}{m}$) (2 MARKS).

(d) Find $r(t)$ if $\omega^2 > \frac{2k}{m}$. Speculate about what would happen to the train[1] in this case if the table is infinite in extent (2 MARKS).

a)



[1] i.e., to a "real" train, with a real spring!

Let $\vec{r_1}$ denote position vector of the engine $\Rightarrow \vec{r_1} = r_1\hat{r}$

$\vec{r_2}$ denote position vector of the wagon $\Rightarrow \vec{r_2} = r_2\hat{r}$

$\Rightarrow \vec{r} = \vec{r_1} - \vec{r_2}$ & $r = r_1 - r_2$

Therefore the equations of motion are

$\vec{r_1} = r_1\hat{r}$

$\Rightarrow \dot{\vec{r_1}} = \dot{r_1}\hat{r} + r_1\dot{\theta}\hat{\theta}$

$\ddot{\vec{r_1}} = (\ddot{r_1} - r_1\dot{\theta}^2)\hat{r} + (2\dot{r_1}\dot{\theta} + r_1\ddot{\theta})\hat{\theta}$

Therefore equating $\hat{r}$ and $\hat{\theta}$ components

$(F_{\hat{r}})_1 = m(\ddot{r_1} - r_1\dot{\theta}^2) = -k(r_1-r_2) = m(\ddot{r_1} - r_1 w^2)$ → 1A

$(F_{\hat{\theta}})_1 = m(2\dot{r_1}\dot{\theta} + r_1\ddot{\theta}) = N_1$ → 2A

$\vec{r_2} = r_2\hat{r}$

$\Rightarrow \dot{\vec{r_2}} = \dot{r_2}\hat{r} + r_2\dot{\theta}\hat{\theta}$

$\ddot{\vec{r}} = (\ddot{r_2} - r_2\dot{\theta}^2)\hat{r} + (2\dot{r_2}\dot{\theta} + r_2\ddot{\theta})\hat{\theta}$

Therefore equating $\hat{r}$ and $\hat{\theta}$ components

$(F_{\hat{r}})_2 = m(\ddot{r_2} - r_2\dot{\theta}^2) = k(r_1-r_2) = m(\ddot{r_2} - r_2 w^2)$ → 1B

$(F_{\hat{\theta}})_2 = m(2\dot{r_2}\dot{\theta} + r_2\ddot{\theta}) = N_2$ → 2B

Equations 1A and 1B are the required equations.

b) Let $\vec{R}$ denote position of centre of mass of the engine and wagon

$\Rightarrow \vec{R} = \frac{m\vec{r_1} + m\vec{r_2}}{m+m} = \frac{\vec{r_1}+\vec{r_2}}{2}$

Equation 1A+1B $\Rightarrow (\ddot{r_1+r_2}) - (r_1+r_2)\dot{\theta}^2 = 0$

$\Rightarrow 2\ddot{\vec{R}} - 2\vec{R}\dot{\theta}^2 = 0 \Rightarrow \ddot{\vec{R}} = w^2\vec{R}$, where $\dot{\theta} = w$ [constant]

This is a differential of order 2.

Solution is of the form

$R(t) = Ae^{wt} + Be^{-wt}$

The given initial conditions are $r_1(0)=0$, $r_2(0)=0$, $\dot{r_1}(0) = V_0$, $\dot{r_2}(0) = 0$

$\Rightarrow R(0) = 0$, $\dot{R}(0) = \frac{V_0}{2}$

$R(0) = A+B = 0$

$\dot{R}(0) = A\omega - B\omega = \frac{V_0}{2}$

Solving $A = \frac{V_0}{4\omega}$, $B = \frac{-V_0}{4\omega}$

$\Rightarrow R(t) = \frac{V_0}{4\omega} e^{\omega t} - \frac{V_0}{4\omega} e^{-\omega t} = \frac{V_0}{2\omega} \sinh(\omega t)$

c) Let $\vec{r} = \vec{r}_1 - \vec{r}_2$

since r_1 and r_2 are along $\hat{r}$, $r = r_1 - r_2$.

$\Rightarrow$ solving with R and r

$r_1 = R + \frac{r}{2}$, $r_2 = R - \frac{r}{2}$

Using equation 1A (Equation 1B can also be used, we will get the same expression)

$$\left(\ddot{R} + \frac{\ddot{r}}{2}\right) - \left(R + \frac{r}{2}\right)\omega^2 = \left(\ddot{R} - R\omega^2\right) + \left(\frac{\ddot{r}}{2} - \frac{r}{2}\omega^2\right) = \frac{-kr}{m}$$

But $\ddot{R} - R\omega^2 = 0$ [Derived in (b)]

$\Rightarrow \ddot{r} = \left(\omega^2 - \frac{2k}{m}\right) r$

If $\omega^2 < \frac{2k}{m}$, Let $\omega_1 = \left(\left|\omega^2 - \frac{2k}{m}\right|\right)^{1/2} = \sqrt{\frac{2k}{m} - \omega^2}$

We have $\ddot{r} = -\omega_1^2 r$

which is similar to Harmonic Oscillator and has solution of the form

$r(t) = A\sin(\omega_1 t + \phi)$

The given initial conditions are $r_1(0) = 0$, $r_2(0) = 0$, $\dot{r}_1(0) = V_0$, $\dot{r}_2(0) = 0$

$\Rightarrow r(0) = 0$, $\dot{r}(0) = V_0$

$r(0) = A\sin\phi = 0 \Rightarrow \sin\phi = 0$ (or $\cos\phi = 1$)

$\dot{r}(0) = A\omega_1\cos\phi = V_0$

$$\Rightarrow r(t) = \frac{V_0}{\omega_1}\sin(\omega_1 t) = \frac{V_0}{\sqrt{\frac{2k}{m} - \omega^2}} \sin\left(\sqrt{\frac{2k}{m} - \omega^2}\, t\right)$$

Therefore, the seperation between engine and wagon is a harmonic oscillator for $\omega^2 < \frac{2k}{m}$.

d) If $w^2 > \frac{2K}{M}$

We have $\ddot{r} = \left(w^2 - \frac{2K}{M}\right) r \Rightarrow \ddot{r} = w_1^2 r$ where $w_1 = \sqrt{w^2 - \frac{2K}{M}}$

The solution to this second order differential equation is of the form

$r(t) = A e^{w_1 t} + B e^{-w_1 t}$

The given initial conditions are $r(0) = 0$, $\dot{r}(0) = V_0$

$\Rightarrow r(0) = A + B = 0$

$\dot{r}(0) = (A - B) w_1 = V_0$

$A = \frac{V_0}{2w_1}$, $B = \frac{-V_0}{2w_1}$

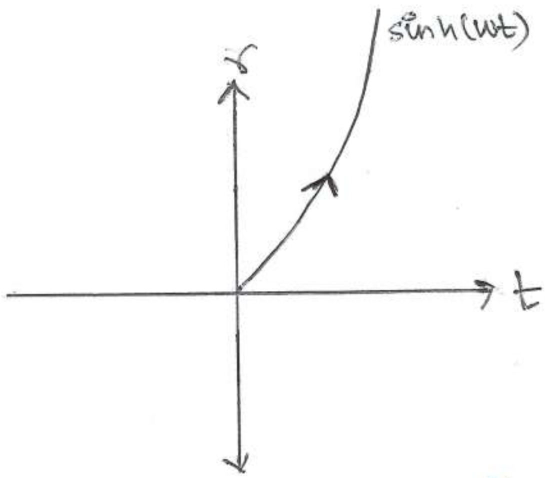
$\Rightarrow r(t) = \frac{V_0}{2w_1} e^{w_1 t} - \frac{V_0}{2w_1} e^{-w_1 t} = \frac{V_0}{2\sqrt{w^2 - \frac{2K}{M}}} \left(e^{\sqrt{w^2 - \frac{2K}{M}}\, t} - e^{-\sqrt{w^2 - \frac{2K}{M}}\, t} \right)$

If the spring is real,

For $t \to \infty$, $r \to \infty$

Hence the motion is unbounded.

This translates to the seperation between the train and the wagon as follows,



The separation between the train and wagon keeps on increasing.

Therefore the spring loses its character [after it reaches threshold length] eventually the spring will break thus disconnecting the engine from wagon breaking the train.

But the centre of mass the system will be moving with same $R(t)$

3. Consider a two-dimensional region (say, the $x-y$ plane), in which a particle (mass m) experiences a force-field, characterised by potential energy

$$U(\boldsymbol{r}) = -U_0\,, |\boldsymbol{r}| \leq a$$
$$U(\boldsymbol{r}) = 0,\ |\boldsymbol{r}| > a$$

where $U_0 > 0$ and $\boldsymbol{r} = x\hat{\imath} + y\hat{\jmath}$ (see Fig. 2)[2]. The particle approaches the well from $x = \infty$ with velocity $-v_0\hat{\imath}$ $(v_0 > 0)$, energy and angular momentum $\boldsymbol{L} = \ell\hat{k}$ $(\ell > 0)$.[3]

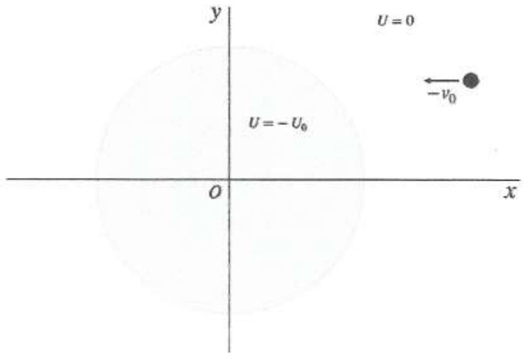


Figure 2

(a) Identify all conserved dynamical quantities associated with the motion of this particle (1 MARK).

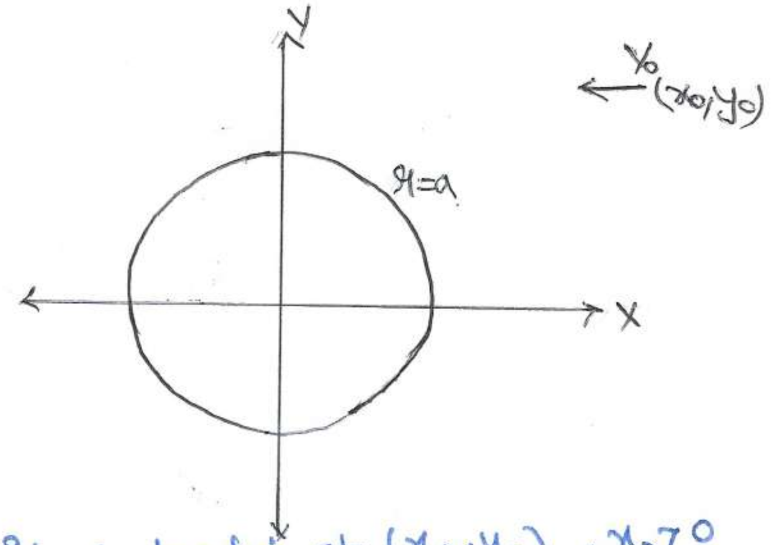
(b) Determine the condition on ℓ such that the particle will eventually enter the potential well (1 MARK).

a) It is given that U, Potential energy of the particle depends only on $\vec{r}$.
i.e $\frac{\partial U}{\partial t} = 0$
Hence the system has time translational symmetry.
⇒ Energy of the particle is conserved.

Also the particle undergoes motion in XY plane and U depends on $\vec{r}$ only.
i.e $\frac{\partial U}{\partial \theta} = 0$
Particle has rotational symmetry about z axis
⇒ Angular Momentum of the particle is conserved.
Since it undergoes motion in XY plane, $L_x = 0$ & $L_y = 0$ ⇒ $\vec{L} = L_z\hat{k}$ is conserved.

[2] Imagine a circular "potential well" of depth U_0 and radius a, see Fig.2.
[3] measured with respect to the centre of the well (the origin here).

b)



Let the particle be located at (x_0, y_0), $x_0 > 0$

Since $\vec{v} = v_0(-\hat{i})$, Particle moves parallel to X axis

Therefore for the particle to enter the potential well whose boundary is given by $r=a$,

$y_0 \leq a$

We have $\vec{r} = x\hat{i} + y_0\hat{j}$, $\vec{v} = -v_0\hat{i}$

$\Rightarrow \vec{L} = m\vec{r}\times\vec{v} = +mv_0y_0\,\hat{k} = l\,\hat{k}$

$\Rightarrow l = mv_0|y_0| \leq mv_0 a \quad [|y_0| \leq a]$

$\Rightarrow$ If $|l| \leq mv_0 a$, the particle will enter the potential well

Therefore energy of the particle will always remain as

$E = \frac{1}{2}mv_0^2$ J

and Angular momentum of the particle will always be

$l = mv_0y_0$ kgm^2s^{-1}, where y_0 is Y coordinate of particle at start.

NOTE:
If $|y_0| > a$, particle never enters the potential well, hence linear momentum of the particle is also conserved :

$\vec{P} = -mv_0\hat{i}$ $kgms^{-1}$

4. In problem 3 above, consider specific values $\ell = \frac{mv_0 a}{\sqrt{2}}$ and $U_0 = \frac{E}{2}$

(a) Determine the velocity of the particle, immediately AFTER it enters the well, **in plane polar coordinates** (1 MARK).

(b) Use the relevant conservation laws (refer to 3 (a) above) to express the radial speed $\dot{r}$ in terms of the radial coordinate r (2 MARKS).

(c) By solving the equation in (b) or otherwise (show details), determine the time T it takes for the particle to escape from the well (2 MARKS).

Since the system has time translational symmetry, energy of the particle is conserved

$\Rightarrow E_{\text{Before entering well}} = E_{\text{After entering well}}$

Similarly, Angular momentum of the particle in z direction is conserved

Let $\vec{r} = r\hat{r}$ be the position vector of the particle

where $\hat{r} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$

$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

$\Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$

$\Rightarrow \vec{L} = m\vec{r}\times\vec{v} = mr^2\dot{\phi}\,\hat{k} = \ell\hat{k}$ [Given]

$\Rightarrow$ We have $\frac{d\phi}{dt} = \frac{\ell}{mr^2} = \frac{\frac{mav_0}{\sqrt{2}}}{mr^2} = \frac{av_0}{\sqrt{2}r^2}$ $\left[\text{Given } \ell = \frac{mav_0}{\sqrt{2}}\right]$

Also

$$E = \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 - U_0$$
$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}\times\left(\frac{1}{2}mv_0^2\right) \quad \left[\text{Given } U_0 = \frac{E}{2}\right]$$

a) Velocity of the particle just after entering the well will be directed towards the origin

$\left.\frac{d\phi}{dt}\right|_{r=a} = \frac{v_0}{\sqrt{2}a} \Rightarrow a\dot{\phi} = \frac{v_0}{\sqrt{2}}$

Also $E = \frac{1}{2}mv_0^2 = \frac{1}{2}m\left(\dot{r}(a)^2 + a^2\times\frac{v_0^2}{2a^2}\right) - \frac{1}{4}mv_0^2$

$\Rightarrow \dot{r}(a) = -v_0$

$\Rightarrow \vec{v} = -v_0\,\hat{r} + \frac{v_0}{\sqrt{2}}\hat{\phi}$ m/s.

b) From Conservation of Angular momentum

We have $l = mr^2 \frac{d\phi}{dt} \Rightarrow \frac{d\phi}{dt} = \frac{l}{mr^2} = \frac{V_0 a}{\sqrt{2} r^2}$ $\left[\text{Given } l = \frac{mV_0 a}{\sqrt{2}}\right]$

From conservation of Energy

We have $E = \frac{1}{2} mV_0^2 = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{4} mV_0^2$ $\left[\text{Given } V_0 = \frac{E}{2}\right]$

substituting $\frac{d\phi}{dt}$ as $\frac{V_0 a}{\sqrt{2} r^2}$

$$\frac{1}{2} mV_0^2 = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} mr^2 \times \frac{V_0^2 a^2}{2r^4} - \frac{1}{4} mV_0^2$$

$$\Rightarrow \frac{1}{2} m\dot{r}^2 = \frac{3}{4} mV_0^2 - \frac{1}{4} m \frac{V_0^2 a^2}{r^2}$$

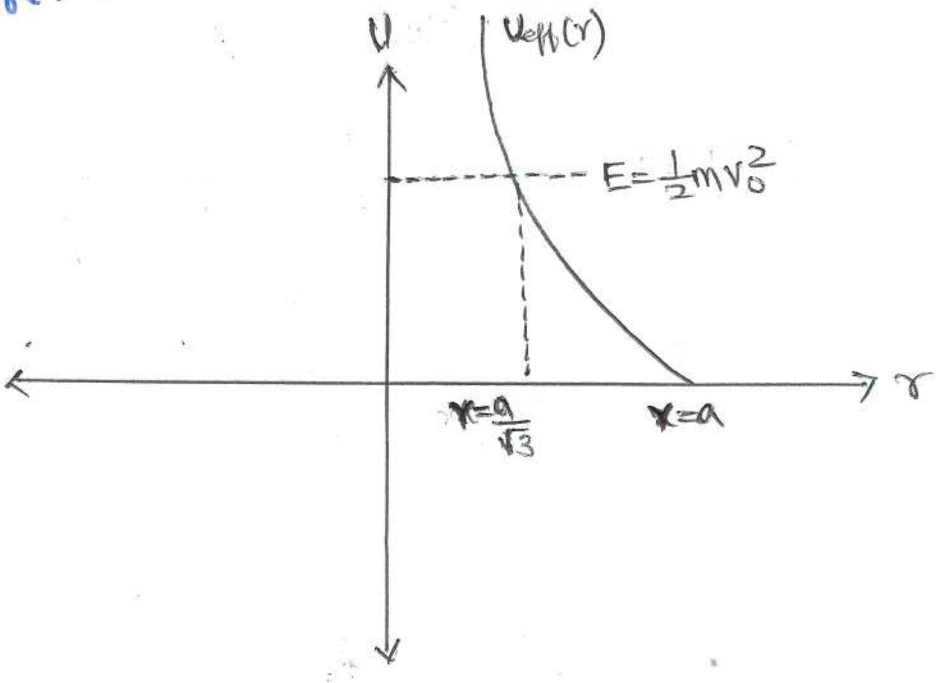
$$\Rightarrow \dot{r}^2 = \frac{V_0^2}{2}\left(3 - \frac{a^2}{r^2}\right)$$

To determine the direction of $\vec{V}$ in $\hat{r}$ direction

we can consider a similar 1D problem whose potential energy is $U_{eff}(r)$

$$E = \frac{1}{2} m\dot{r}^2 + \underbrace{\frac{1}{4} m \frac{V_0^2 a^2}{r^2} - \frac{1}{4} mV_0^2}_{U_{eff}(r)}$$

Plotting $U_{eff}(r)$ vs r



As the particle enters the potential well,

i.e $|r_{initial}| = a$ & $\dot{r}_{initial} = -V_0$

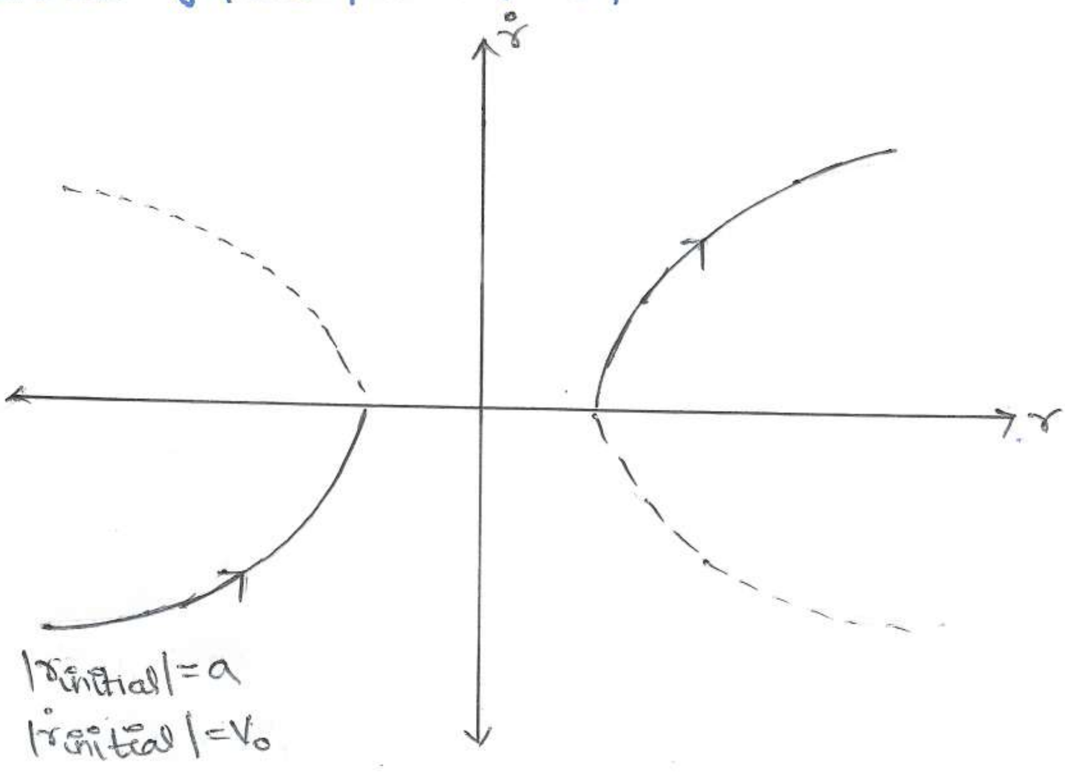
(Therefore particle approaches origin)

Potential energy of the particle increases. But since the energy of the particle is constant, $E = \frac{1}{2}mV_0^2$, Particle approaches till $|r| = \frac{a}{\sqrt{3}}$ and then reverses its direction.

Thus,

$$\dot{r} = \begin{cases} -\frac{V_0}{\sqrt{2}}\sqrt{3-\frac{a^2}{r^2}}, & \vec{r} = -a\hat{r} \text{ to } \vec{r} = \frac{-a}{\sqrt{3}}\hat{r} \\ \frac{V_0}{\sqrt{2}}\sqrt{3-\frac{a^2}{r^2}}, & \vec{r} = \frac{a}{\sqrt{3}}\hat{r} \text{ to } \vec{r} = a\hat{r} \end{cases}$$

A rough estimate of phase potrait of the particle resembles



Therefore radial speed $\dot{r}(r) = \frac{V_0}{\sqrt{2}}\sqrt{3-\frac{a^2}{r^2}}$ m/s.

c) When the particle enters the potential well, its velocity is directed towards the origin.
Later it changes direction and eveventually exits the potential well.

⇒ Timetaken to leave the well = Time taken for the particle to reach $r = \frac{a}{\sqrt{3}}$ $\left[r=-a \text{ to } r=-\frac{a}{\sqrt{3}}\right]$ (T1)
+ Time taken for the particle to reach $r=a$ $\left[r=\frac{a}{\sqrt{3}} \text{ to } r=a\right]$ (T_2)

We have,

$$\frac{dr}{dt} = \dot{r} = \begin{cases} -\frac{V_0}{\sqrt{2}}\sqrt{3-\frac{a^2}{r^2}}, & r=-a \text{ to } r=-\frac{a}{\sqrt{3}} \\ \frac{V_0}{\sqrt{2}}\sqrt{3-\frac{a^2}{r^2}}, & r=\frac{a}{\sqrt{3}} \text{ to } r=a \end{cases}$$

$$\Rightarrow \int_{-a}^{-\frac{a}{\sqrt{3}}} \frac{r\,dr}{\sqrt{3r^2-a^2}} = \int_0^{T_1} \frac{-V_0}{\sqrt{2}}dt \quad \text{and} \quad \int_{\frac{a}{\sqrt{3}}}^{a} \frac{r\,dr}{\sqrt{3r^2-a^2}} = \int_{T_1}^{T_1+T_2} \frac{V_0}{\sqrt{2}}dt$$

$$\Rightarrow \left[\frac{\sqrt{3r^2-a^2}}{3}\right]_{-a}^{-\frac{a}{\sqrt{3}}} = \frac{-V_0}{\sqrt{2}}T1 \quad \text{and} \quad \left[\frac{\sqrt{3r^2-a^2}}{3}\right]_{\frac{a}{\sqrt{3}}}^{a} = \frac{V_0}{\sqrt{2}}T_2$$

$$\Rightarrow -\frac{\sqrt{2}a}{3} = \frac{-V_0}{\sqrt{2}}T_1 \quad \text{and} \quad \frac{\sqrt{2}a}{3} = \frac{V_0}{\sqrt{2}}T_2$$

Clearly $T_1 = T_2$

And $T = T_1 + T_2 = \frac{2a}{3V_0} + \frac{2a}{3V_0} = \frac{4a}{3V_0}$ s

Hence the particle will leave the potential well after $T = \frac{4a}{3V_0}$ s once it enters the well.