Solutions to Quiz I

From Newton's laws to oscillations

✦ True or false

1. The total energy of a closed system is always conserved.

   Solution: True. Since the system is closed, there will be no transfer of energy (to or from the system) and hence the energy of the system will remain a constant.

2. The acceleration of an underdamped oscillator vanishes when its position is zero.

   Solution: False. For an underdamped oscillator, we have

   $$x(t) = A\,\mathrm{e}^{-\beta\, t} \cos\,(\omega_1\, t - \delta)\,,$$

   where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$. Therefore, we have

   $$v(t) = \dot{x}(t) = -A\,\mathrm{e}^{-\beta\, t}\,[\beta \cos\,(\omega_1\, t - \delta) + \omega_1 \sin\,(\omega_1\, t - \delta)]$$

   and

   $$\begin{aligned} a(t) = \dot{v}(t) &= \beta\, A\,\mathrm{e}^{-\beta\, t}\,[\beta \cos\,(\omega_1\, t - \delta) + \omega_1 \sin\,(\omega_1\, t - \delta)] \\ &\quad - A\,\mathrm{e}^{-\beta\, t}\,\left[-\beta\,\omega_1 \sin\,(\omega_1\, t - \delta) + \omega_1^2 \cos\,(\omega_1\, t - \delta)\right] \\ &= -2\,\beta\,\dot{x} - \omega_0^2\, x, \end{aligned}$$

   where we have set $\omega_1^2 = \omega_0^2 - \beta^2$. Note that $x(t)$ vanishes when the cosine term vanishes, i.e. when $\omega_1\, t - \delta = n\,\pi/2$, where $n = 1, 2, 3, \ldots$. However, since the velocity also contains a sine term, it will *not* vanish at these times. Hence, the acceleration will not vanish when the position is zero.

3. A particle moving in the potential $U(x) = -\alpha\, x - \beta\, x^2$, where $\alpha > 0$ and $\beta > 0$, never exhibits bounded motion.

   Solution: True. The potential

   $$U(x) = -\alpha\, x - \beta\, x^2,$$

   where $\alpha > 0$ and $\beta > 0$, has a maximum at $x = -\alpha/(2\,\beta)$. The potential vanishes at $x = 0$ and goes to negative infinity at large positive as well as negative values of x. So, the allowed range of energy is $-\infty < E < \infty$. Since there arises no minima, the system does not exhibit bounded motion for any value of the energy.

✦ Multiple choice questions

4. The unit vector perpendicular to the vectors $\vec{\boldsymbol{a}} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}$ and $\vec{\boldsymbol{b}} = 2\,\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} - 3\,\hat{\boldsymbol{z}}$ is

   [**A**] $(-2\,\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} - \hat{\boldsymbol{z}})\,/\sqrt{6}$ [**B**] $(2\,\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}})\,/\sqrt{6}$

   [**C**] $(-2\,\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}})\,/\sqrt{6}$ [**D**] $(2\,\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} - \hat{\boldsymbol{z}})\,/\sqrt{6}$

   Solution: A. We have, say,

   $$\vec{\boldsymbol{c}} = \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = (-2\,\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} - \hat{\boldsymbol{z}})\,,$$

   so that

   $$\hat{\boldsymbol{c}} = \frac{\vec{\boldsymbol{c}}}{|\boldsymbol{c}|} = (-2\,\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} - \hat{\boldsymbol{z}})\,/\sqrt{6}.$$

5. For a weakly damped oscillator, $x(t+T)$ is found to be 94% of $x(t)$, where T is the time period of the system. The quantity β/ω_1 (where β is the damping constant and ω_1 is the actual frequency) for the system is approximately

   [**A**] 10^{-5} [**B**] 10^{-4} [**C**] 10^{-3} [**D**] 10^{-2}

Solution: D. Recall that, for a underdamped oscillator,

$$\frac{x(t+T)}{x(t)} = \mathrm{e}^{-2\,\pi\,\beta/\omega_1}$$

so that, in our case, we have

$$0.94 = \mathrm{e}^{-2\,\pi\,\beta/\omega_1} \simeq 1 - \frac{2\,\pi\,\beta}{\omega_1},$$

or

$$\frac{\beta}{\omega_1} = \frac{\sqrt{\omega_0^2 - \omega_1^2}}{\omega_1} = \frac{0.06}{2\,\pi} \simeq 0.01 = 10^{-2}.$$

6. Consider a particle of mass m which is falling under the influence of gravity and subject to the drag force $\vec{\boldsymbol{F}} = -\alpha\,\vec{\boldsymbol{v}}$. The terminal velocity of the particle is given by $v_{\mathrm{term}} = g\,\tau$, where $\tau = m/\alpha$ denotes the characteristic time scale. If the particle falls from rest, it will reach 95% of v_{term} in time

   [**A**] $t \simeq \tau$ [**B**] $t \simeq 3\,\tau$ [**C**] $t \simeq 5\,\tau$ [**D**] $t \simeq 10\,\tau$

   Solution: B. Recall that, in the case of a particle falling from rest, we have

$$v_y(t) = v_{\mathrm{term}}\,\left(1 - \mathrm{e}^{-t/\tau}\right),$$

   so that, when $t = 3\,\tau$,

$$\frac{v_y}{v_{\mathrm{term}}} = 1 - \frac{1}{\mathrm{e}^3} \simeq 1 - \frac{1}{2.72^3} \simeq 1 - \frac{1}{25} \simeq 1 - 0.04 \simeq 0.96.$$

✦ Fill in the blanks

7. A point mass has the trajectory $\vec{\boldsymbol{r}}(t) = A\cos\,(\omega\,t)\;\hat{\boldsymbol{x}} + \sin\,(\omega\,t)\;\hat{\boldsymbol{y}}$. If the angle between the velocity $\vec{\boldsymbol{v}}$ and the acceleration $\vec{\boldsymbol{a}}$ at time $t = \pi/(4\,\omega)$ is $60°$, what is the value of A?

   Solution: Since

$$\vec{\boldsymbol{r}}(t) = A\cos\,(\omega\,t)\;\hat{\boldsymbol{x}} + \sin\,(\omega\,t)\;\hat{\boldsymbol{y}},$$

   we have

$$\vec{\boldsymbol{v}}(t) = -A\,\omega\sin\,(\omega\,t)\;\hat{\boldsymbol{x}} + \omega\cos\,(\omega\,t)\;\hat{\boldsymbol{y}}, \quad \vec{\boldsymbol{a}}(t) = -A\,\omega^2\cos\,(\omega\,t)\;\hat{\boldsymbol{x}} - \omega^2\sin\,(\omega\,t)\;\hat{\boldsymbol{y}},$$

   so that

$$\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}} = \omega^3\,\left(A^2-1\right)\,\sin\,(\omega\,t)\,\cos\,(\omega\,t) = (\omega^3/2)\,\left(A^2-1\right)\,\sin\,(2\,\omega\,t)\,.$$

   At $\omega\,t = \pi/4$, we have

$$\vec{\boldsymbol{v}} = -\frac{A\,\omega}{\sqrt{2}}\,\hat{\boldsymbol{x}} + \frac{\omega}{\sqrt{2}}\,\hat{\boldsymbol{y}}, \quad \vec{\boldsymbol{a}}(t) = -\frac{A\,\omega^2}{\sqrt{2}}\,\hat{\boldsymbol{x}} - \frac{\omega^2}{\sqrt{2}}\,\hat{\boldsymbol{y}}$$

   and, at this time,

$$\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}} = |\vec{\boldsymbol{v}}|\,|\vec{\boldsymbol{a}}|\cos 60° = \frac{1}{2}\,\frac{\omega}{\sqrt{2}}\,\sqrt{A^2+1}\,\frac{\omega^2}{\sqrt{2}}\,\sqrt{A^2+1} = \frac{\omega^3}{2}\,\left(A^2-1\right),$$

   which leads to

$$\frac{A^2-1}{A^2+1} = \frac{1}{2},$$

   or $A = \pm\sqrt{3}$.

8. Consider a particle moving in the one-dimensional potential $U(x) = -U_0\,\exp\left[-(x-\mu)^2/(2\,\sigma^2)\right]$, where U_0, μ and σ are positive quantities of suitable dimensions. Determine the frequency of small oscillations about the minimum of the potential.

   <u>*Solution*</u>: Since
$$\frac{\mathrm{d}U}{\mathrm{d}x} = \frac{U_0\,(x-\mu)}{\sigma^2}\,\mathrm{e}^{-(x-\mu)^2/(2\,\sigma^2)}$$
   and
$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} = \frac{U_0}{\sigma^2}\,\mathrm{e}^{-(x-\mu)^2/(2\,\sigma^2)} - \frac{U_0\,(x-\mu)^2}{\sigma^4}\,\mathrm{e}^{-(x-\mu)^2/(2\,\sigma^2)},$$
   evidently, the minimum is located at $x = \mu$. Therefore, we have
$$k = \left(\frac{\mathrm{d}^2U}{\mathrm{d}x^2}\right)_{x=\mu} = \frac{U_0}{\sigma^2}$$
   and, hence, the frequency of small oscillations about the minimum is given by
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{U_0}{m\,\sigma^2}}.$$

9. Let x_1 and x_2 be two different positions of an undamped oscillator at times t_1 and t_2. Let v_1 and v_2 be the corresponding velocities. Express the angular frequency of the oscillations in terms of x_1, x_2, v_1 and v_2.

   <u>*Solution*</u>: We can write
$$x(t) = A\cos\left(\omega\,t - \delta\right),$$
   where ω is the frequency of the oscillator. Then, we have
$$v(t) = -\omega\,A\sin\left(\omega\,t - \delta\right),$$
   so that
$$\begin{aligned} x_1 = x(t_1) &= A\cos\left(\omega\,t_1 - \delta\right), \quad v_1 = v(t_1) = -\omega\,A\sin\left(\omega\,t_1 - \delta\right),\\ x_2 = x(t_2) &= A\cos\left(\omega\,t_2 - \delta\right), \quad v_2 = v(t_2) = -\omega\,A\sin\left(\omega\,t_2 - \delta\right), \end{aligned}$$
   and, hence,
$$x_1^2 + \frac{v_1^2}{\omega^2} = x_2^2 + \frac{v_2^2}{\omega^2} = A^2$$
   or
$$\omega = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}},$$
   which is the required result.

10. Given the function $f(r) = 1/r$, where $r = \sqrt{x^2+y^2+z^2}$, determine $\vec{\boldsymbol{\nabla}} f$.

    <u>*Solution*</u>: We have
$$\vec{\boldsymbol{\nabla}} f = \frac{\partial f}{\partial x}\,\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\,\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\,\hat{\boldsymbol{z}},$$
    where
$$\frac{\partial f}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}r}\,\frac{\partial r}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\mathrm{d}f}{\mathrm{d}r}\,\frac{\partial r}{\partial y}, \quad \frac{\partial f}{\partial z} = \frac{\mathrm{d}f}{\mathrm{d}r}\,\frac{\partial r}{\partial z}.$$
    Since $r = \sqrt{x^2+y^2+z^2}$, we have
$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}, \quad \frac{\partial r}{\partial x} = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{r}, \quad \frac{\partial r}{\partial x} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{r},$$

so that

$$\vec{\boldsymbol{\nabla}} f = \frac{\mathrm{d}f}{\mathrm{d}r}\,\frac{x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}}}{r} = \frac{\mathrm{d}f}{\mathrm{d}r}\,\frac{\vec{\boldsymbol{r}}}{r} = \frac{\mathrm{d}f}{\mathrm{d}r}\,\hat{\boldsymbol{r}}$$

and, in the case wherein $f(r) = 1/r$, we obtain that

$$\vec{\boldsymbol{\nabla}} f = -\frac{1}{r^2}\,\hat{\boldsymbol{r}}.$$

✦ Questions with detailed answers

11. *Dropping from a plane:* A plane flying horizontally at a constant speed v_0 and at a height h above the sea, drops a bundle on a small static raft. (a) Neglecting air resistance, write down Newton's second law for the bundle as it falls from the plane, and solve the equations to obtain the bundle's position as a function of time t. (b) If $v_0 = 50\,\mathrm{m/s}$, $h = 125\,\mathrm{m}$ and $g \simeq 10\,\mathrm{m/s^2}$, what is the horizontal distance before the raft that the pilot must drop the bundle to hit the raft? (c) Determine the interval of time ($\pm\Delta t$) within which the pilot must drop the bundle if it is to land within $\pm 10\,\mathrm{m}$ of the raft.

    Solution: The equations of motion along the horizontal (x, pointing rightward) and vertical (y, pointing upward) directions are given by

    $$\ddot{x} = 0, \qquad \ddot{y} = -g,$$

    which can be integrated to yield

    $$x = v_0\,t, \quad y = h - \frac{g}{2}\,t^2.$$

    The time taken for the bundle to drop on the raft is (i.e. when $y = 0$)

    $$t_{\mathrm{fall}} = \sqrt{2\,h/g}$$

    and, over this time, the horizontal distance traveled by the bundle is

    $$x = v_0\,t_{\mathrm{fall}} = v_0\,\sqrt{2\,h/g}.$$

    For $v_0 = 50\,\mathrm{m/s}$, $h = 100\,\mathrm{m}$, and $g \simeq 10\,\mathrm{m/s^2}$,

    $$x = 50\,\sqrt{2 \times 125/10}\,\mathrm{m} = 250\,\mathrm{m}.$$

    Note that, if the drop is delayed by a time Δt, then the bundle will overshoot the raft by $\Delta x = v_0\,\Delta t$. Therefore, we have

    $$\Delta t = \frac{\Delta x}{v_0} = \frac{10\,\mathrm{m}}{50\,\mathrm{m/s}} = 0.2\,\mathrm{s},$$

    so that $\Delta t = \pm 0.2\,\mathrm{s}$.

12. *Decaying force:* An undamped oscillator with natural frequency ω_0 is subject to the external force $F(t) = F_0\,\mathrm{e}^{-\alpha\,t}$, where F_0 and α are positive quantities. (a) Determine the homogeneous and inhomogeneous (i.e. particular) solutions. (b) Which of the two solutions dominates at late times, i.e. when $\alpha\,t \gg 1$?

    Solution: We have

    $$\ddot{x} + \omega_0^2\,x = \frac{F_0}{m}\,\mathrm{e}^{-\alpha\,t},$$

    and the homogeneous solution is evidently given by

    $$x_{\mathrm{h}}(t) = A\cos\left(\omega_0\,t - \delta\right),$$

    where A and δ are constants determined by the initial conditions.

Let us propose the inhomogeneous solution to be $x_{\rm ih}(t) = C\,{\rm e}^{-\alpha\,t}$. Upon substituting this solution in the above differential equation, we obtain that

$$\left(\alpha^2+\omega_0^2\right)\,C\,{\rm e}^{-\alpha\,t} = \frac{F_0}{m}\,{\rm e}^{-\alpha\,t}$$

or, equivalently,

$$C = \frac{F_0/m}{\alpha^2+\omega_0^2},$$

so that we have

$$x_{\rm ih}(t) = \frac{F_0/m}{\alpha^2+\omega_0^2}\,{\rm e}^{-\alpha\,t}.$$

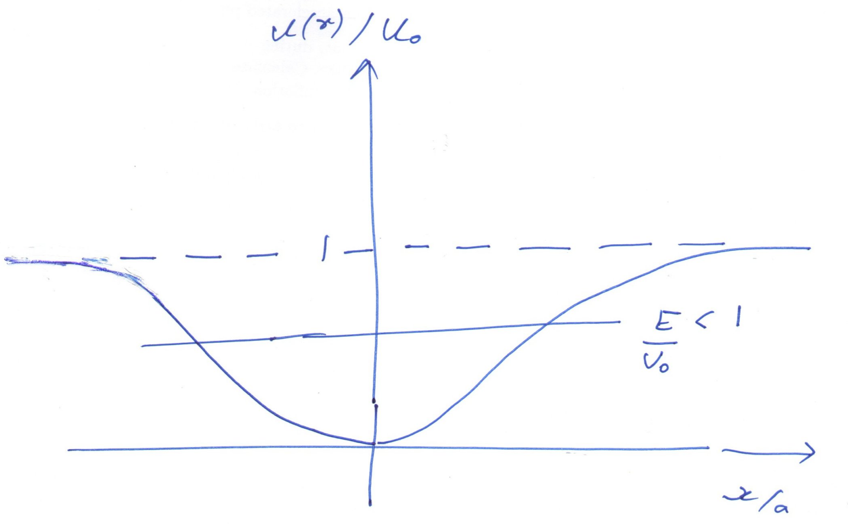
Clearly, at late times such that $\alpha\,t \gg 1$, $x_{\rm ih}(t)$ dies away. Therefore, at such times, it is the homogeneous solution $x_{\rm h}(t)$ that will dominate.

13. *Motion in one dimension:* Consider the one-dimensional potential

$$U(x) = \frac{U_0\,x^2}{x^2+a^2},$$

where U_0 and a are positive constants. (a) Plot the quantity $U(x)/U_0$ against x/a. (b) Identify the stable and/or unstable point(s) of equilibrium. (c) Find the turning points for $E = U_0/4$.

Solution: Note that the potential is symmetric in x. Also, $U(x) = 0$ at $x = 0$ and $U(x) \to U_0$ as $x \to \pm\infty$.



Moreover, as

$$\frac{{\rm d}U}{{\rm d}x} = \frac{2\,U_0\,x}{x^2+a^2} - \frac{2\,U_0\,x^3}{(x^2+a^2)^2} = \frac{2\,U_0\,a^2\,x}{(x^2+a^2)^2},$$

which vanishes only at $x = 0$. Further,

$$\frac{{\rm d}^2U}{{\rm d}x^2} = \frac{2\,U_0\,a^2}{(x^2+a^2)^2} - \frac{8\,U_0\,a^2\,x^2}{(x^2+a^2)^3},$$

so that $({\rm d}^2U/{\rm d}x^2) = 2\,U_0/a^2$ at $x = 0$. Clearly, the potential has one stable equilibrium point at $x = 0$.

For $E = U_0/4$, we have, when $E = U(x_*)$,

$$\frac{U_0}{4} = \frac{U_0\, x_*^2}{x_*^2 + a^2}$$

so that

$$x_*^2 + a^2 = 4\, x_*^2,$$

and, hence, the turning points are given by $x_* = \pm a/\sqrt{3}$.