DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH1010 Physics I | **Endsemester Examination** | **22/11/2013**
Time: 01:00-4:00 PM. | **Max. marks: 50**

Instructions:

1. This question paper contains **4 numbered pages**. Check this **now**.
2. **ANSWER ALL QUESTIONS**
3. Use ONLY the answer sheet provided. Use the **last TWO pages for ROUGH work.**
4. **Write your NAME, ROLL No. and BATCH.**
   Note that the BATCH (A to J) MUST be clearly indicated next to the Course No. box.
5. Write your answers in blue or black ink. **This includes the plots.** Do **not** use a pencil or red ink.
6. There are **three parts** to this question paper. Answers to PART-A, PART-B and PART-C questions must NOT be mixed up. **In particular PART-A must be written in a single page.** Clearly mark on your answer booklet where part-A and part-B and part-C are attempted.
7. **NO cellphones, calculators** etc.
8. Please note that if the instructions above are not fully complied with, your answer book will **not** be graded.
9. All symbols have their usual meaning, unless otherwise specified. All constants are assumed to be of appropriate dimensions.

Good Luck!

PART-A

1. Answer **TRUE or FALSE**. No reasoning required. [**1 mark each**]

   (a) The total angular momentum of N particles, any two of them interacting via a noncentral force, and in the absence of any external forces, is constant. F

   (b) Phase space trajectories of a particle trapped in the potential $-x^2/2$ actually cross at the equilibrium point. F

   (c) The power delivered to a forced underdamped harmonic oscillator is maximum exactly when the driving frequency and natural frequency match. T

(d) The rate of decay of the amplitude is faster for the overdamped harmonic oscillator than for the critically damped. T

(e) The time period of a particle trapped in the one-dimensional potential $x^4/4$ is independent of energy. T

(f) The radial velocity of a planet in an elliptic Kepler orbit vanishes twice per period. F

(g) The number of generalized coordinates (or number of degrees of freedom) of a bead on a helical wire is 3. F

(h) If the spherical polar coordinates of a point are $(r, \theta, \phi) = (1, \pi/4, \pi/4)$, its Cartesian coordinates are $(1/2, 1/2, 1/2)$. F

(i) The value of $\epsilon_{ijk}\epsilon_{ijk}$ is equal to 1 (summation convention is implied, i, j, k take values 1, 2, 3). T

(j) The value of $\nabla \cdot \boldsymbol{r}$ is equal to 3. ($\boldsymbol{r}$ is position vector in three dimensions). T

PART-B

[2 marks each]

$F(\rho)\hat{e}_\rho$

2. A particle of mass m is in a central force field $F(\rho)\hat{e}_\phi$. On one of its orbits given by $\rho\phi = C$ it has an angular momentum L, where C is an appropriate constant and (ρ, ϕ) are polar coordinates in the plane of motion. Find the form of $F(\rho)$.

$L = m\rho^2\dot{\phi} = m\rho^2 \cdot C/\rho^2 = -mC$

3. Find the work done by the force $\boldsymbol{F} = -\dfrac{3a^2}{\rho^2}\hat{e}_\rho + 5b\hat{e}_\phi$, where a, b are constant, for a particle to move around a closed circular path of radius R and centered on the origin in the $x - y$ plane. Determine the condition under which the force is conservative.

4. A current density is given by

$$\boldsymbol{J} = \frac{e^{-t}}{1 + x^2 + y^2 + z^2}\hat{e}_z.$$

Use the continuity equation to find the density $\rho(x, y, z, t)$ given that $\rho(x, y, z, 0) = 0$.

5. Show that $\nabla \times (\boldsymbol{a} \times \boldsymbol{r}) = 2\boldsymbol{a}$ where $\boldsymbol{a}$ is a constant vector.

6. Given $\boldsymbol{V} = y\hat{e}_x + x\hat{e}_y + 2z\hat{e}_z$ find $\int(\nabla \times \boldsymbol{V}) \cdot \boldsymbol{n}dS$ over the upper hemisphere $x^2 + y^2 + z^2 = a^2$.

PART-C

[5 marks each]

7. A projectile of mass m is fired from the origin at speed v_0 and angle θ. It is attached to the origin by a spring with spring constant k and relaxed length zero.

   (a) Find $x(t)$ and $y(t)$.

   (b) Show that for small $\omega = \sqrt{k/m}$ the dynamics reduces to a pure projectile one, while for large ω it reduces to a pure harmonic oscillation. State more precisely what "small" and "large" here means.

   (c) What value should ω take so that the projectile hits the ground when it is moving straight downward?

8. The potential energy of particle of mass m is $V(x) = ax^2/2 - bx^3/3$, where a and b are positive constants with appropriate units.

   (a) Sketch the potential.

   (b) Find all the equilibrium points and if they are stable or not.

   (c) Find the frequency of small oscillations about any stable equilibrium point(s).

   (d) Sketch the phase space trajectories including all the equilibrum points.

9. A particle moves in a central force field given by

$$F = -\frac{k}{r^2}e^{-r/\lambda}\hat{e}_r,$$

   where k and λ are positive constants with appropriate units.

   (a) Show that the radius r_0 of the circular orbit satisfies the equation

$$ke^{-r_0/\lambda} = \frac{L^2}{mr_0}.$$

   (b) Determine the condition under which the circular orbit is stable.

   (c) Find the frequency of small oscillations about the circular orbit, and state conditions that lead to closed orbits in this approximation.

10. Verify Gauss' theorem using the vector field

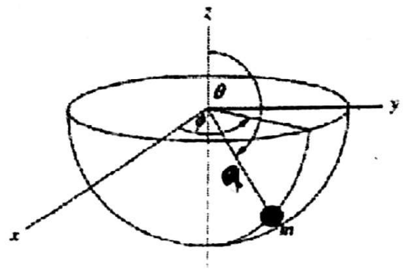
$$4xy\hat{e}_x - 3y^2\hat{e}_y + 2yz\hat{e}_z$$

$4y \; -6y \; + 2y$

   in the cubical region of dimensions $-a \le x \le a$, $-a \le y \le a$, $-a \le z \le a$.

$ke^{+3\lambda/x} = \frac{L^2}{mr_0} = \frac{L^2}{m(}$

11. A hemispherical bowl of radius a is held fixed with its rim upwards and horizontal. A particle of mass m moves without friction on its inner surface while being subject to the gravitational force $-mg\hat{\mathbf{e}}_z$.



(a) Find the Lagrangian of the system in terms of spherical polar coordinates.

(b) Derive the corresponding equations of motion using the Euler-Lagrange equation.

(c) Show that it is possible to have circular orbits with $\theta = \theta_0$ a constant. Find an equation of θ_0 in terms of m, g, a and a constant of motion.

12. Consider a particle of mass m in the gravitational central potential $-k/r$, having a total energy E and magnitude of angular momentum L. ($k > 0$ is an appropriate constant).

(a) Prove that

$$\boldsymbol{h} = \boldsymbol{v} - \frac{k}{L}\hat{\mathbf{e}}_\phi$$

is a constant vector. Here $\boldsymbol{v}$ is the velocity vector and $\hat{\mathbf{e}}_\phi$ the unit vector in the angular direction.

(b) Using the previous part show that the velocity vector (v_x, v_y), (with the $x - y$ plane being the orbit plane) sweeps out a circle, and find the radius of this circle. That is, in *velocity space* with axes v_x and v_y, all Kepler orbits are either entire circles or parts of circles.

(c) Show that

$$E = \frac{m}{2}\left(h^2 - \frac{k^2}{L^2}\right),$$

where $h^2 = \boldsymbol{h} \cdot \boldsymbol{h}$. Hence find how the velocity space circles (called *hodographs*) differ when the position space orbits are circular, elliptic, parabolic or hyperbolic, by roughly sketching the circles with the center and radius specified in each case.

The End

Message from coordinators of PH1020: See the Tutorials webpage of the current PH1010 course for the first Tutorial of PH1020, you are requested to solve this over the break!
