Mini-Test 2

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1. (a) $U(x) = U_0\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{b}\right)^4\right]$

$\Rightarrow \frac{dU(x)}{dx} = 0$ (at minima and maxima)

$\Rightarrow U_0\left[\frac{2x}{a^2} - \frac{4x^3}{b^4}\right] = 0 \Rightarrow \boxed{x = 0, \pm\frac{b^2}{\sqrt{2}a}}$

$U''(0) = \frac{2U_0}{a^2} - \frac{12U_0(0)^2}{b^4} > 0$

$U''\left(\pm\frac{b^2}{\sqrt{2}a}\right) = \frac{2U_0}{a^2} - \frac{12U_0\left(\frac{b^4}{2a^2}\right)}{b^4} = -\frac{4U_0}{a^2} < 0$

$\therefore$ $x = 0$ is a point of minima.

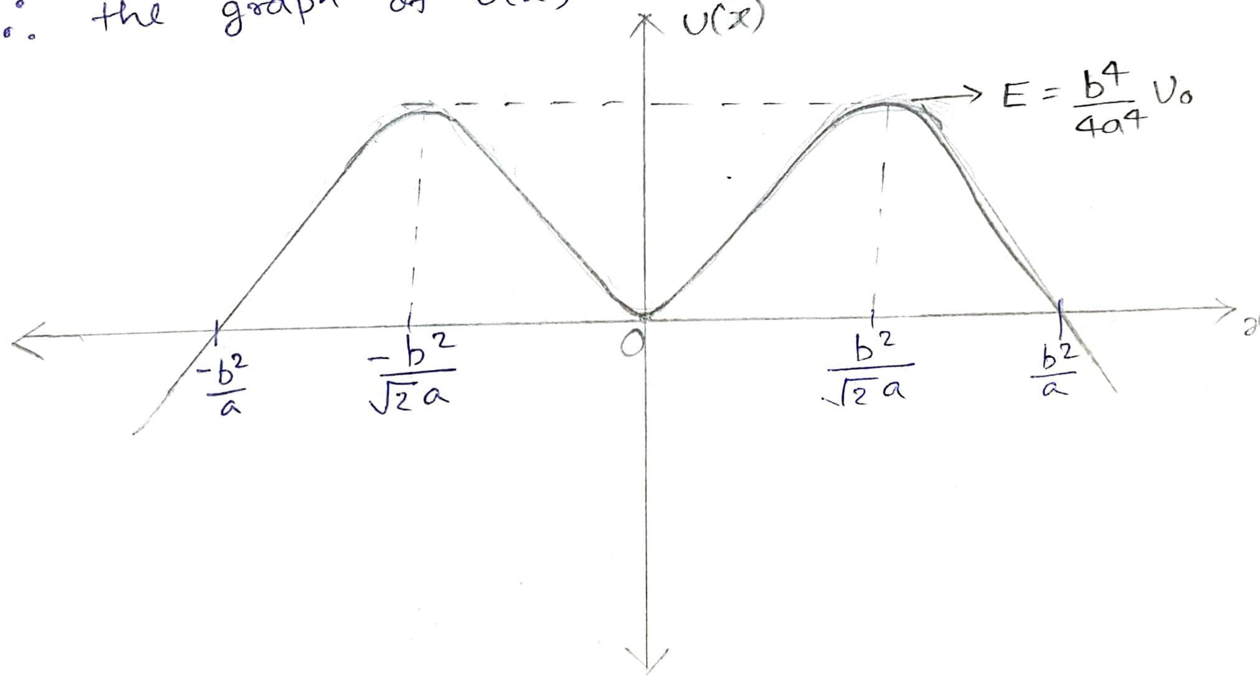
$x = \pm\frac{b^2}{\sqrt{2}a}$ are both points of maxima.

(b) At $x = \pm\frac{b^2}{\sqrt{2}a}$, $U(x)$ is maximum.

$\therefore U_{max} = \frac{b^4}{4a^4}U_0$

at $x=0$, $U(x)=0$

$\therefore$ the graph of $U(x)$ versus x is as follows.

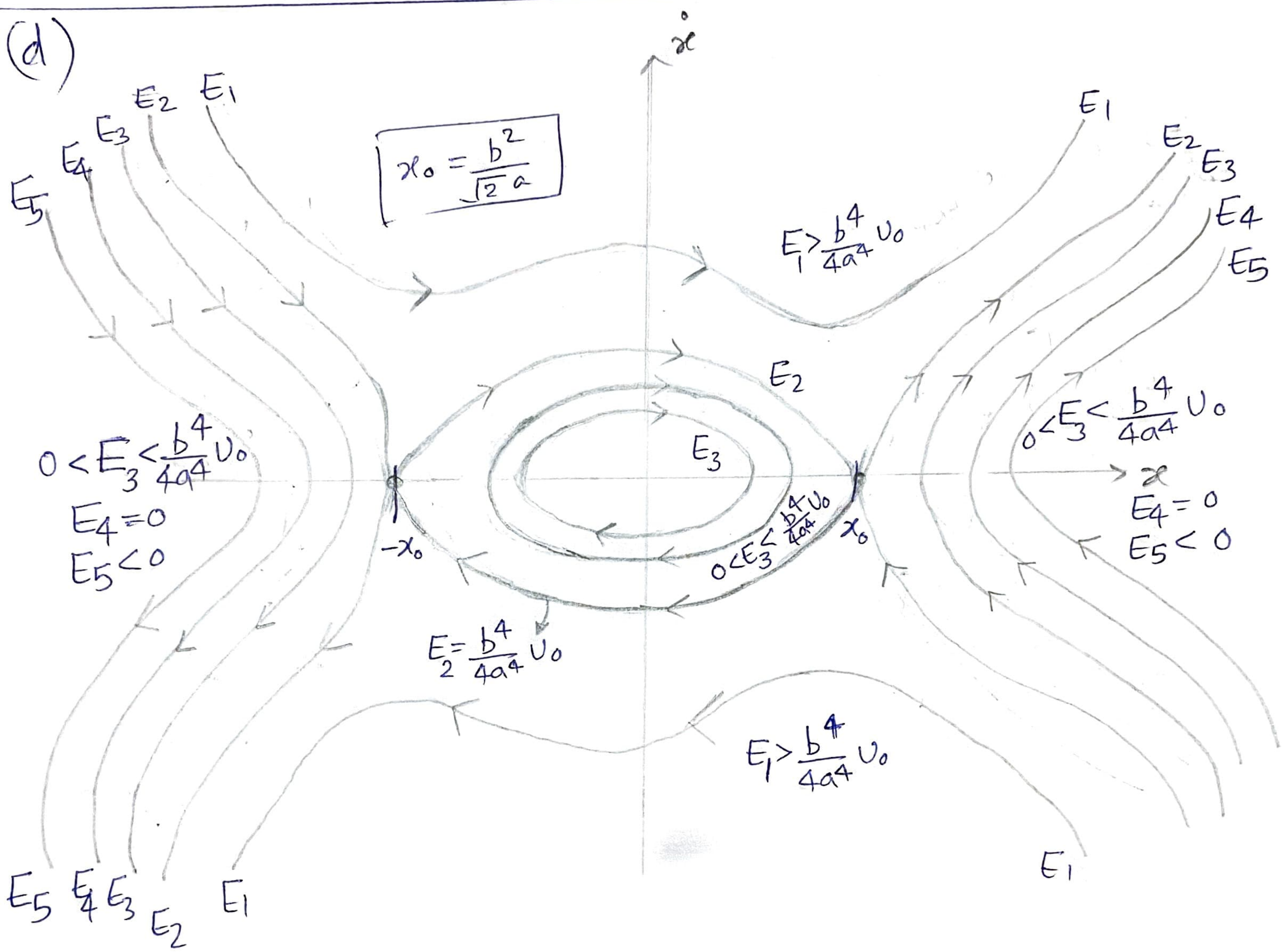


(c) In the above graph drawn in part (b), bounded motion is seen there are two values of x for which we get the same E in the upward parabola between $x = -\frac{b^2}{\sqrt{2}a}$ and $x = +\frac{b^2}{\sqrt{2}a}$, so we can expect bounded motion for values of energy ranging from 0 to U_{max}.

$U_{max} = U_0\left(\frac{b^4}{4a^4}\right)$ (from (b) part)

$\Rightarrow$ we can see that the particle performs bounded motion for the energy values between $E_1 = 0$ and $E_2 = U_0\left(\frac{b^4}{4a^4}\right)$, i.e $0 < E < U_0\left(\frac{b^4}{4a^4}\right)$

(d)



e) At separatrix, $E = \frac{b^4}{4a^4} U_0$ (U_{max})

By conservation of Energy principle

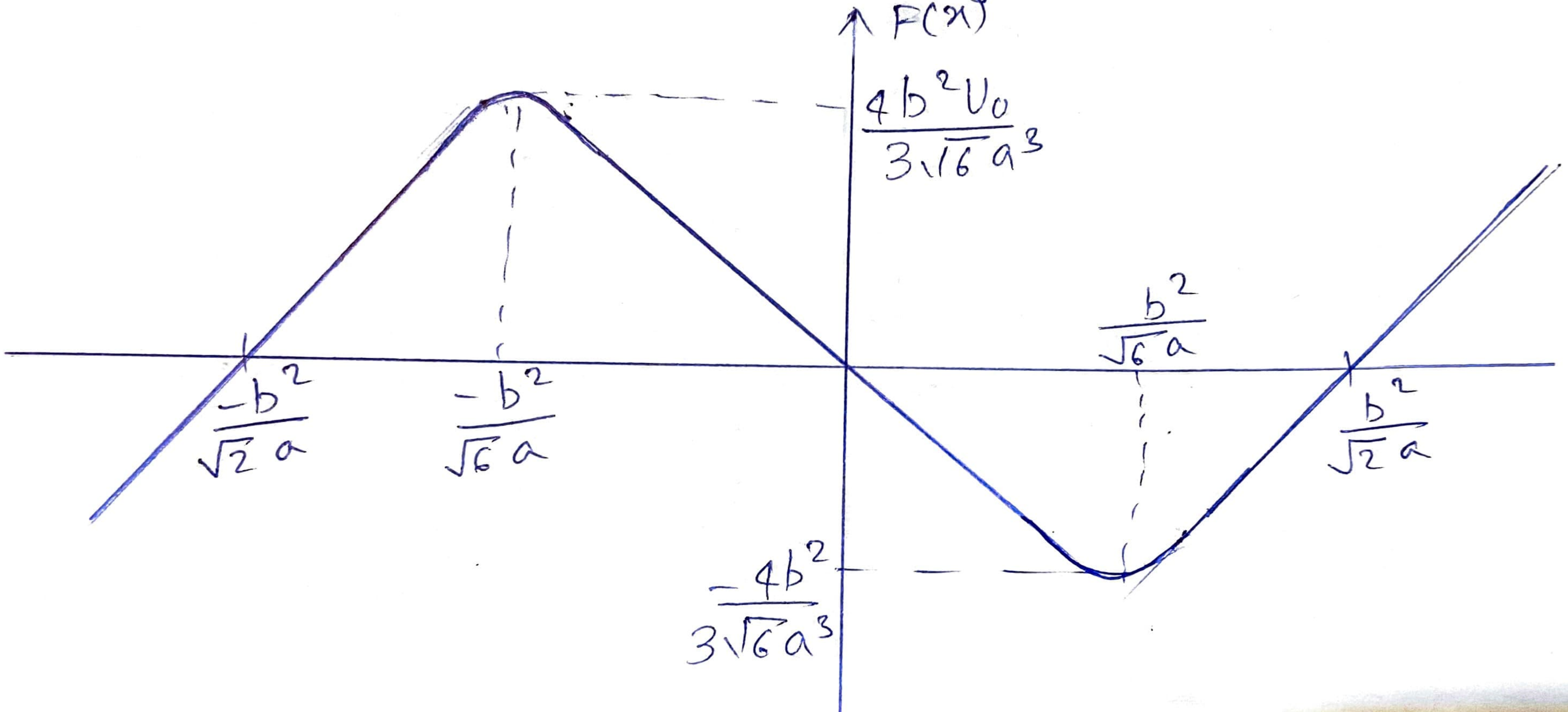
$\Rightarrow KE + U(x) = E$

$$\Rightarrow \boxed{\frac{1}{2}m(\dot{x})^2 + U_0\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{b}\right)^4\right] = \frac{b^4}{4a^4} U_0} \quad (1)$$

Equation (1) is the equation for the separatrix curve.

(f)

$$F = -\frac{dU(x)}{dx} = U_0\left[\frac{4x^3}{b^4} - \frac{2x}{a^2}\right]$$



2.

(a) r_1: radial coordinate of the engine

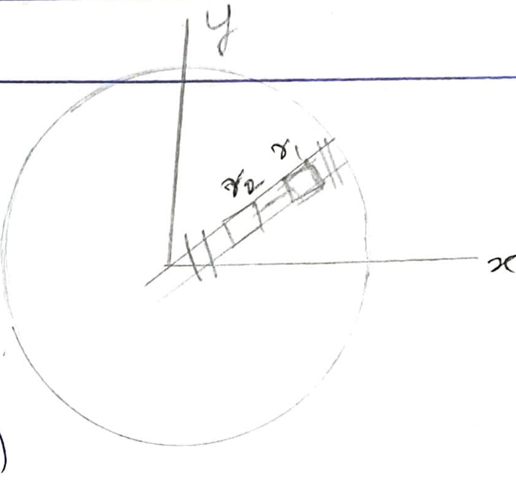
r_2: radial coordinate of the wagon

For the engine:-

$$m(\ddot{r}_1 - r_1 w^2) = -K(r_1 - r_2) \Rightarrow \boxed{\ddot{r}_1 = \omega^2 r_1 - \frac{K(r_1 - r_2)}{m}}$$

For the wagon:-

$$m(\ddot{r}_2 - r_2 w^2) = K(r_1 - r_2) \Rightarrow \boxed{\ddot{r}_2 = w^2 r_2 + \frac{K(r_1 - r_2)}{m}}$$



(b) $$R(t) = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} = \frac{r_1 + r_2}{2}$$

$$\Rightarrow \frac{dR(t)}{dt} = \frac{d}{dt}\left(\frac{r_1 + r_2}{2}\right)$$

$$\Rightarrow \frac{d^2R(t)}{dt} = \frac{1}{2}\left[\frac{d^2 r_1}{dt^2} + \frac{1}{2}\frac{d^2 r_2}{dt^2}\right]$$

$$= \frac{1}{2}\left[w^2(r_1 + r_2)\right] = \frac{w^2(r_1 + r_2)}{2}$$

$$= w^2 R(t)$$

$$\Rightarrow \boxed{\frac{d^2R}{dt^2} = w^2 R}$$

Solution for this equation is of the form $R(t) = Ae^{wt} + Be^{-wt}$

$\Rightarrow R(0) = 0 \Rightarrow A + B = 0 \Rightarrow \boxed{A = -B}$

$\dot{R}(0) = \frac{V_0}{2}$ $\left[\text{Using } V_{cm} = \frac{m_1 V_1 + m_2 V_2}{m_1 + m_2} = \frac{m V_0}{2m} = \frac{V_0}{2}\right]$

$\Rightarrow Aw - Bw = \frac{V_0}{2} \Rightarrow 2Aw = \frac{V_0}{2} \Rightarrow \boxed{A = \frac{V_0}{4w}}$

$$\Rightarrow \boxed{R(t) = \frac{V_0}{4w}\left(e^{wt} - e^{-wt}\right)}$$

(c) $\Rightarrow r = r_1 - r_2$

$\Rightarrow \ddot{r} = \ddot{r}_1 - \ddot{r}_2 \Rightarrow \ddot{r} = w^2(r_1 - r_2) - \frac{2K}{m}(r_1 - r_2)$

$\Rightarrow \ddot{r} = \left(w^2 - \frac{2K}{m}\right) r$

If $w^2 < \frac{2K}{m}$, let $\frac{2K}{m} - w^2 = (w')^2$

$\Rightarrow \boxed{\ddot{r} = -(w')^2 r}$

Solution for this differential equation is of the form,

$$\Rightarrow \boxed{r = Ae^{iw't} + Be^{-iw't}}$$

$\Rightarrow r(0) = 0 \Rightarrow A + B = 0 \Rightarrow \boxed{A = -B}$

$\dot{r}(0) = V_0 \Rightarrow (A - B)\, i w' = V_0 \Rightarrow 2Aiw' = V_0 \Rightarrow \boxed{w' = \frac{V_0}{2Ai}} \rightarrow \boxed{A = \frac{V_0}{2w'i}}$

$\Rightarrow r = \frac{V_0}{2w'i}\left[e^{iw't} - e^{-iw't}\right] = \frac{V_0}{2w'i}\left[\cancel{2}\sin(w't)\right]$

$\Rightarrow \boxed{r = \frac{V_0}{w'}\sin(w't)}$ where $\boxed{w' = \sqrt{\frac{2K}{m} - w^2}}$ ☆

(d) $\ddot{r} = \left(w^2 - \frac{2K}{m}\right) r$ [from part (c)]

if $w^2 > \frac{2K}{m}$, let $w^2 - \frac{2K}{m} = (w')^2$

$\Rightarrow \boxed{\ddot{r} = (w')^2 r}$ this differential equation is of the form

Solution for this differential equation is of the form

$\boxed{r = Ae^{w't} + Be^{-w't}}$

$\Rightarrow r(0) = 0 \Rightarrow A = -B$

$\dot{r}(0) = V_0 \Rightarrow 2Aw' = V_0 \Rightarrow A = \frac{V_0}{2w'}$

$\Rightarrow \boxed{r = \frac{V_0}{2w'}\left(e^{w't} - e^{-w't}\right)}$ (where $(w')^2 = w^2 - \frac{2K}{m}$)

$\frac{dr}{dt} = \frac{V_0}{2\cancel{w'}} \times \cancel{w'}\left(e^{w't} + e^{-w't}\right) = \frac{V_0}{2}\underbrace{\left(e^{w't} + e^{-w't}\right)}_{\text{this term is always positive.}}$

$\Rightarrow r(t)$ is strictly increasing with time, so since the spring is real, it will break at some point.

3. Potential energy is given as:-

$U(r) = -U_0, \; |r| \leq a$

$U(r) = 0, \; |r| > a$

Initial velocity $= -v_0 \hat{i}$, $E = \frac{m v_0^2}{2}$, $L = l\hat{k}$

(a) Energy is conserved since the potential does not vary with time. When the particle enters the potential well, there is some momentary impulse due to the radial force (which acts since the potential varies along radial direction). But this radial force does not produce any torque about the origin. (no external torque) Hence, angular momentum is also conserved.

(b) $\vec{l} = m(\vec{r} \times \vec{v})$ $[\vec{r} = x\hat{i} + y\hat{j}]$

$l\hat{k} = m[(x\hat{i} + y\hat{j}) \times (-v_0\hat{i})]$

$= m y v_0 \hat{k}$

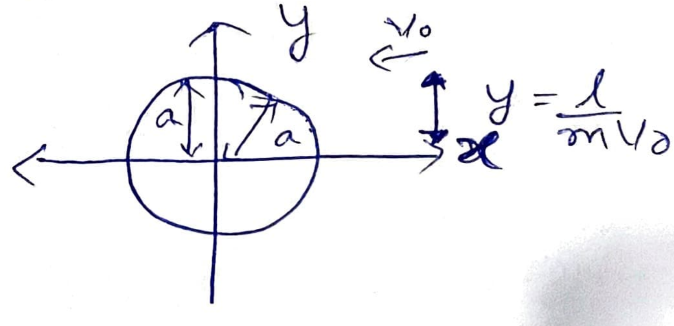
$\therefore \boxed{y = \frac{l}{m v_0}}$

→ For particle to enter well at sometime, (consider $t=0$)

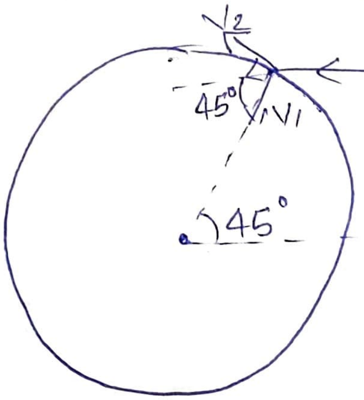
$\Rightarrow y \leq a$

$\Rightarrow \frac{l}{m v_0} \leq a$

$\Rightarrow \boxed{l \leq m v_0 a}$



4. Given, $l = \frac{m v_0 a}{\sqrt{2}}$, $U_0 = \frac{E}{2}$



(a) Let the velocity just after entering the well be

$$\vec{v} = -v_1 \hat{r} + v_2 \hat{\theta}$$

using energy conservation,

$$\frac{1}{2} m v_0^2 = -U_0 + \frac{1}{2} m (v_1^2 + v_2^2)$$

$$= -\frac{1}{4} m v_0^2 + \frac{1}{2} m (v_1^2 + v_2^2)$$

$$\Rightarrow \boxed{v_1^2 + v_2^2 = \frac{3 v_0^2}{2}}$$

Using angular momentum conservation,

$$L_i = \frac{m v_0 a}{\sqrt{2}}, \quad L_f = m \left[\frac{a}{\sqrt{2}}\hat{i} + \frac{a}{\sqrt{2}}\hat{j}\right] \times \left[\frac{v_1 + v_2}{\sqrt{2}}(-\hat{i}) + \frac{v_2 - v_1}{\sqrt{2}}\hat{j}\right]$$

$$= m v_2 a = \frac{m v_0 a}{\sqrt{2}}$$

$$\boxed{v_2 = \frac{v_0}{\sqrt{2}}} \Rightarrow \boxed{v_1 = v_0}$$

So, the velocity of the particle, immediately after it enters the well is $\vec{v} = -v_1 \hat{r} + v_2 \hat{\theta} = -v_0 \hat{r} + \frac{v_0}{\sqrt{2}} \hat{\theta}$

(b) Using energy conservation, we get

$$\Rightarrow \frac{1}{2} m v_0^2 = \frac{1}{2} m (\dot{r})^2 + \frac{1}{2} m (r\dot{\theta})^2 - U_0$$

$$\Rightarrow (\dot{r})^2 + (r\dot{\theta})^2 = v_0^2 + \frac{2U_0}{m} = \frac{3 v_0^2}{2} \Rightarrow \boxed{(\dot{r})^2 + (r\dot{\theta})^2 = \frac{3 v_0^2}{2}} \quad (1)$$

Using conservation of angular momentum, we get

$$m (r\dot{\theta})(r) = \frac{m v_0 a}{\sqrt{2}} \Rightarrow r^2 \dot{\theta} = \frac{v_0 a}{\sqrt{2}} \Rightarrow \boxed{\dot{\theta} = \frac{v_0 a}{\sqrt{2} r^2}}$$

substituting $\dot{\theta}$ in (1), we get

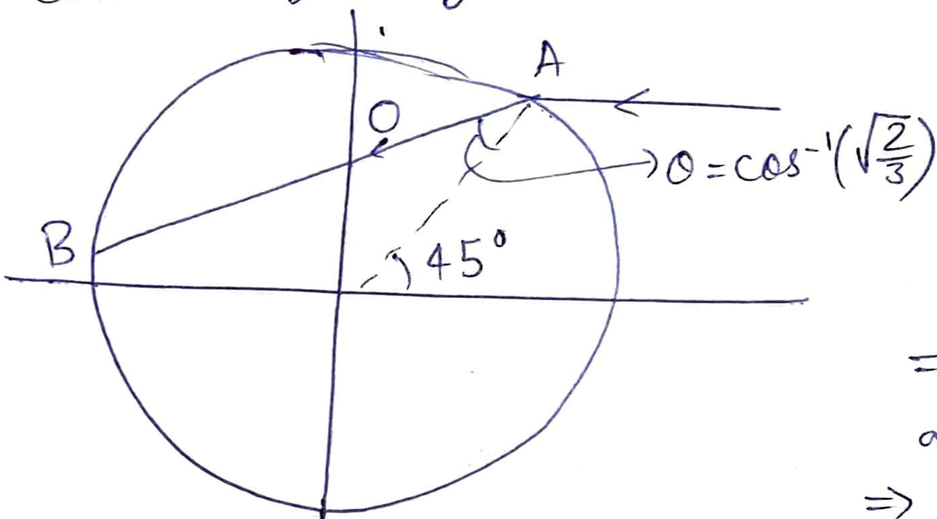
$$\Rightarrow (\dot{r})^2 + \left(\frac{v_0 a}{\sqrt{2} r}\right)^2 = \frac{3 v_0^2}{2} \Rightarrow (\dot{r})^2 = \frac{3 v_0^2}{2} - \frac{v_0^2 a^2}{2 r^2} \Rightarrow (\dot{r})^2 = v_0^2 \left(\frac{3}{2} - \frac{a^2}{2r^2}\right)$$

$$\Rightarrow \boxed{\dot{r} = v_0 \sqrt{\frac{3}{2} - \frac{a^2}{2r^2}}}$$

Hence, we have radial speed:

$$\boxed{\dot{r} = v_0 \sqrt{\frac{3}{2} - \frac{a^2}{2r^2}}}, \text{ for } |r| \leq a$$

(c) So, we can see that time taken from A to O, (O is the point where the particle is radially at rest) is equal to the time taken from O to B, because the net velocity is constant when it moves on the chord joining A and B.



At O, $\dot{r} = 0$

$$\Rightarrow \dot{r} = -V_0 \sqrt{\frac{3}{2}\left(1 - \frac{a^2}{3r^2}\right)} = 0$$

$$\boxed{r = \frac{a}{\sqrt{3}}}$$

Now,

$$\Rightarrow \frac{dr}{dt} = -V_0 \sqrt{\frac{\cancel{3}}{2}\left(\frac{3r^2 - a^2}{\cancel{3}r^2}\right)}$$

$$\Rightarrow \int_a^{a/\sqrt{3}} \frac{r\,dr}{\sqrt{3r^2 - a^2}} = \int_0^t -\frac{V_0}{\sqrt{2}}\,dt$$

Let 't' be the time taken from A to O.

$$\Rightarrow \left[\frac{\sqrt{3r^2 - a^2}}{3}\right]_a^{a/\sqrt{3}} = -\frac{V_0}{\sqrt{2}}t$$

$$\Rightarrow + \frac{\sqrt{2}a}{3} = +\frac{V_0}{\sqrt{2}}t \Rightarrow \boxed{t = \frac{2a}{3V_0}}$$

$\therefore$ Time for which mass (particle) stays inside the well is $T = 2t = \boxed{\frac{4a}{3V_0}}$