Solutions to Quiz II

From phase portraits to introductory vector calculus

✦ True or false

1. A planet moves with the maximum speed at its aphelion (i.e. at the maximum distance from the sun).

   Solution: False. As angular momentum L is a constant and the velocity is perpendicular to the radial vector at aphelion $r_{\rm max}$ and perihelion $r_{\rm min}$, we have

   $$L = \mu\, r_{\rm max}\, v_{\rm min} = \mu\, r_{\rm min}\, v_{\rm max},$$

   where μ is the reduced mass, while $v_{\rm min}$ and $v_{\rm max}$ are the minimum and maximum speeds. Therefore, the particle will be moving at its minimum speed ($v_{\rm min}$) at the aphelion ($r_{\rm max}$).

2. An extended object has a mass density that is independent of the cylindrical polar coordinate ϕ. The moment of inertia of the object will be the same about the x, y and z-axes.

   Solution: False. Clearly, the moment of inertia about the z-axis will be distinct when compared with the moment of inertia about the other two axes.

3. The electrostatic potential in a region is given by $V(\rho,\phi) = \rho^2 + 4\,\rho\cos\phi + 4$, where ρ and ϕ denote the plane polar coordinates. The lines of constant positive V are circles.

   Solution: True. Note that

   $$V(\rho,\phi) = \rho^2 + 4\,\rho\cos\phi + 4 = x^2 + y^2 + 4\,x + 4 = (x+2)^2 + y^2,$$

   which, evidently, describes a circle if V is a positive constant.

✦ Multiple choice questions

4. A particle moves only under the gravitational attraction of an infinitely long massive wire that has uniform density and lies along the z-axis. Which of the following sets contains **ALL** the conserved kinematic quantities of the particle? (Note that, E, $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{L}}$ denote the energy, momentum and angular momentum of the particle, respectively.)

   [A] E **[B]** $\vec{\boldsymbol{L}},\ p_z$ **[C]** $L_z,\ \vec{\boldsymbol{p}}$ **[D]** $E,\ L_z,\ p_z$

   Solution: D. The system is independent of time, so E is conserved. The system is invariant under translations in the z-direction, so p_z is conserved. The system is invariant under rotations about the z-axis and hence L_z is conserved.

5. The orbit of a particle in a central force is a circle which passes through the origin described by $r = r_0\cos\phi$, where r_0 is a constant. The central force $\vec{\boldsymbol{F}}(r)$ is proportional to

   [A] $-\vec{\boldsymbol{r}}/r^6$ **[B]** $-\vec{\boldsymbol{r}}/r^5$ **[C]** $-\vec{\boldsymbol{r}}/r^4$ **[D]** $-\vec{\boldsymbol{r}}/r^3$

   Solution: A. Recall that

   $$E = \frac{\mu}{2}\,\dot{r}^2 + \frac{L^2}{2\,\mu\,r^2} + U(r),$$

   so that on differentiation, we obtain that

   $$\frac{\mu}{2}\,2\,\dot{r}\,\ddot{r} + \frac{{\rm d}U}{{\rm d}r}\,\dot{r} - \frac{2\,L^2}{2\,\mu\,r^3}\,\dot{r} = 0$$

   and, since, $F_r = -{\rm d}U/{\rm d}r$, we have

   $$F_r = \mu\,\ddot{r} - \frac{L^2}{\mu\,r^3}.$$

As $r = r_0 \cos\phi$ and $\dot{\phi} = L/(\mu\, r^2)$, we have

$$\dot{r} = -r_0 \sin\phi\, \dot{\phi}, \quad \ddot{r} = -r_0 \cos\phi\, \dot{\phi}^2 - r_0 \sin\phi\, \ddot{\phi}$$

and

$$\ddot{\phi} = -\frac{2\,L}{\mu\, r^2}\, \dot{r} = \frac{2\,L\, r_0}{\mu\, r^3} \sin\phi\, \dot{\phi},$$

so that

$$\begin{aligned} F_r &= -\mu\, r_0 \cos\phi\, \dot{\phi}^2 - \mu\, r_0 \sin\phi\, \ddot{\phi} - \frac{L^2}{\mu\, r^3} = -\frac{L^2}{\mu\, r^3} - \frac{2\,L^2}{\mu\, r^5}\, r_0^2 \sin^2\!\phi - \frac{L^2}{\mu\, r^3} \\ &= -\frac{2\,L^2}{\mu\, r^3} - \frac{2\,L^2}{\mu\, r^5}\, (r_0^2 - r^2) = -\frac{2\,L^2\, r_0^2}{\mu\, r^5}, \end{aligned}$$

which is the required result.

This can also be presented in a simpler fashion. We can write

$$E = \frac{\mu}{2}\left(\dot{r}^2 + r^2\, \dot{\phi}^2\right) + U(r) = \frac{\mu}{2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + r^2\right] \dot{\phi}^2 + U(r),$$

and as $r = r_0 \cos\phi$ and $\dot{\phi} = L/(\mu\, r^2)$, we have

$$E = \frac{\mu\, r_0^2}{2}\, \dot{\phi}^2 + U(r) = \frac{L^2\, r_0^2}{2\,\mu\, r^4} + U(r),$$

which, upon differentiation, leads to

$$F_r = -\frac{\mathrm{d}U}{\mathrm{d}r} = -\frac{2\,L^2\, r_0^2}{\mu\, r^5}.$$

6. A satellite moving in a circular orbit of radius R is given a forward thrust leading to an elliptical orbit of eccentricity $\epsilon = \frac{1}{2}$. The maximum distance $r_{\rm max}$ reached by the satellite will be

   [**A**] $2\,R$ [**B**] $3\,R$ [**C**] $4\,R$ [**D**] $R/2$

   Solution: B. We have, for the elliptical orbit,

   $$r_{\rm min} = a\,(1 - \epsilon), \quad r_{\rm max} = a\,(1 + \epsilon),$$

   and, as $\epsilon = \frac{1}{2}$, we obtain that

   $$r_{\rm min} = \frac{a}{2} = R, \quad r_{\rm max} = \frac{3\,a}{2} = 3\,R.$$

✦ Fill in the blanks

7. A particle moves in a plane along the logarithmic spiral $\rho = \mathrm{e}^{\phi}$, where ρ and ϕ denote the plane polar coordinates. What is the angle between the position and the velocity vectors of the particle?

   Solution We have

   $$\vec{\boldsymbol{r}} = \mathrm{e}^{\phi}\, \hat{\boldsymbol{\rho}}, \quad \vec{\boldsymbol{v}} = \frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} = \mathrm{e}^{\phi}\, \dot{\phi}\, \hat{\boldsymbol{\rho}} + \mathrm{e}^{\phi}\, \frac{\mathrm{d}\hat{\boldsymbol{\rho}}}{\mathrm{d}\phi}\, \dot{\phi} = \mathrm{e}^{\phi}\, \dot{\phi}\, (\hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\phi}}),$$

   since $\mathrm{d}\hat{\boldsymbol{\rho}}/\mathrm{d}\phi = \hat{\boldsymbol{\phi}}$. Clearly, the angle between $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{v}}$ of the particle is $45°$.

8. A particle moving under the influence of the central force $U(r) = \alpha\, r^2$, where $\alpha > 0$, starts with the initial conditions $(x, y) = (x_0, y_0)$ with $(v_x, v_y) = (0, 0)$. Express the trajectory of the particle in the x-y-plane as the function $y(x)$.

   Solution: The solutions along the x and y-directions can be expressed as

$$x(t) = x_0 \cos(\omega\, t) + \frac{v_{x0}}{\omega} \sin(\omega\, t), \quad y(t) = y_0 \cos(\omega\, t) + \frac{v_{y0}}{\omega} \sin(\omega\, t),$$

   where $\omega = (2\,\alpha/\mu)^{1/2}$, with μ being the reduced mass of the particle. Since, $(v_{x0}, v_{y0}) = (0, 0)$, we have $y(x) = (y_0/x_0)\, x$.

9. A mountaineer is attempting to climb Mount Normal. The height of Mount Normal is given by $h(x, y) = h_0 \exp-\left[(x-1)^2 + \frac{1}{4}(y-2)^2 + x\,y + 1\right]$, where h_0 is a constant and (x, y) are the positions with respect to, say, the base camp. The mountaineer is at the position $(x, y) = (1, 1)$. In what direction in the x-y-plane should she move to climb the steepest slope?

   Solution: The direction of the gradient gives the steepest slope. The gradient of $h(x, y)$ is

$$\vec{\boldsymbol{\nabla}} h = \hat{\boldsymbol{x}}\,\frac{\partial h}{\partial x} + \hat{\boldsymbol{y}}\,\frac{\partial h}{\partial y} = -h_0 \left[(2\,x + y - 2)\,\hat{\boldsymbol{x}} + \left(\frac{y}{2} + x - 1\right)\hat{\boldsymbol{y}}\right] \mathrm{e}^{-\left[(x-1)^2 + \frac{1}{4}(y-2)^2 + x\,y + 1\right]}.$$

   The value of the $\vec{\boldsymbol{\nabla}} h$ at $(x, y) = (1, 1)$ is

$$\vec{\boldsymbol{\nabla}} h\Big|_{(1,1)} = -h_0 \left(\hat{\boldsymbol{x}} + \frac{1}{2}\,\hat{\boldsymbol{y}}\right) \mathrm{e}^{-\frac{9}{4}}.$$

   So the unit vector in the direction of the steepest slope at $(x, y) = (1, 1)$ is

$$\frac{\vec{\boldsymbol{\nabla}} h\Big|_{(1,1)}}{\left|\vec{\boldsymbol{\nabla}} h\Big|_{(1,1)}\right|} = -\frac{2}{\sqrt{5}} \left(\hat{\boldsymbol{x}} + \frac{1}{2}\,\hat{\boldsymbol{y}}\right).$$

   Any vector proportional to this is fine.

10. Given the force $\vec{\boldsymbol{F}} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}}$, evaluate the work done to move a particle along the line $y = -x + 1$ from the point $(x, y) = (1, 0)$ to the point $(0, 1)$.

   Solution: Note that

$$\vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} = F_x\, \mathrm{d}x + F_y\, \mathrm{d}y$$

   and, since, in our case, $(F_x, F_y) = (x, y$, we obtain that

$$\vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} = x\, \mathrm{d}x + y\, \mathrm{d}y.$$

   Also, on the line $y = -x + 1$, we have $\mathrm{d}y = -\mathrm{d}x$, so that

$$\int_{(1,0)}^{(0,1)} \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} = \int_{(1,0)}^{(0,1)} (x\, \mathrm{d}x + y\, \mathrm{d}y) = \int_1^0 [x\, \mathrm{d}x + (-x + 1)(-\mathrm{d}x)] = \int_1^0 \mathrm{d}x\, (2\,x - 1) = 0.$$

✦ Questions with detailed answers

11. *Angular velocity of a laminar sheet:* A square laminar sheet of mass M has its corners at $\left(0, -\frac{a}{2}, 0\right)$, $\left(a, -\frac{a}{2}, 0\right)$, $\left(a, \frac{a}{2}, 0\right)$ and $\left(0, \frac{a}{2}, 0\right)$. Its mass per unit area is given by $\sigma(x) = \sigma_0 \left(1 - \frac{x}{a}\right)$. (a) Determine the moment of inertia of the sheet about the y-axis in terms of M and a. (b) The sheet is initially stationary but can rotate about the y-axis. A very small piece of putty with mass $\frac{M}{2}$ is fired at the sheet with velocity $\vec{\boldsymbol{v}} = \frac{v_0}{\sqrt{2}}\,(\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}})$ from $(0, 0, \frac{a}{2})$, where v_0 denotes the speed of the

putty. On impact with the sheet, the putty sticks and the sheet begins to rotate with angular velocity $\omega\,\hat{\boldsymbol{y}}$? Evaluate ω in terms of v_0, M and a. (**Ignore gravity and air resistance.**)

Solution: The contribution to the moment of inertia I of an infinitesimal region a distance x from the y axis is

$$\mathrm{d}I = \sigma(x)\, x^2\, \mathrm{d}x\, \mathrm{d}y,$$

which we can then integrate over the sheet to obtain

$$I = \int_{-\frac{a}{2}}^{\frac{a}{2}} \mathrm{d}y \int_0^a \mathrm{d}x\, \sigma_0 \left(1 - \frac{x}{a}\right) x^2 = a\,\sigma_0 \left(\frac{x^3}{3} - \frac{x^4}{4\,a}\right)_0^a = \frac{1}{12}\,\sigma_0\, a^4.$$

The total mass of the sheet is

$$M = \int_{-\frac{a}{2}}^{\frac{a}{2}} \mathrm{d}y \int_0^a \mathrm{d}x\, \sigma_0 \left(1 - \frac{x}{a}\right) = \frac{1}{2}\,\sigma_0\, a^2.$$

(This can also be stated by saying the average σ over the interval 0 to a is $\frac{\sigma_0}{2}$, due to the linear variation from σ_0 to 0.) Therefore, we can write **[1 mark]**

$$I = \frac{M\,a^2}{6}.$$

The angular momentum before and after the putty sticks is conserved. The initial angular momentum is solely due to the putty, which can be calculated from the initial position $\vec{\boldsymbol{r}}_0 = \left(0, 0, \frac{a}{2}\right)$ and momentum $\vec{\boldsymbol{p}} = \frac{M}{2}\vec{\boldsymbol{v}} = \frac{M\,v_0}{2\sqrt{2}}\,(\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}})$: **[0.5 mark]**

$$\vec{\boldsymbol{L}}_{\text{initial}} = \vec{\boldsymbol{r}}_0 \times \vec{\boldsymbol{p}} = \frac{M\,v_0}{2\sqrt{2}} \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 0 & 0 & \frac{a}{2} \\ 1 & 0 & -1 \end{vmatrix} = \frac{M\,a\,v_0}{4\sqrt{2}}\,\hat{\boldsymbol{y}}.$$

The putty intersects with the centre of the sheet so once stuck it has a moment of inertia $I_{\text{putty}} = \frac{M\,a^2}{8}$ about the y-axis. Therefore, the angular momentum after collision is **[0.5 mark]**

$$\vec{\boldsymbol{L}}_{\text{final}} = (I + I_{\text{putty}})\,\omega\,\hat{\boldsymbol{y}} = \frac{7\,M\,a^2\,\omega}{24}\,\hat{\boldsymbol{y}}.$$

Equating the initial and final angular momenta, we get

$$\frac{7\,M\,a^2\,\omega}{24} = \frac{M\,a\,v_0}{4\sqrt{2}},$$

which implies that **[1 mark]**

$$\omega = \frac{6\,v_0}{7\sqrt{2}\,a}.$$

12. *Kepler problem in velocity space:* Recall that the orbit of a particle moving under the influence of the central force $U(r) = -\alpha/r$, where $\alpha > 0$, is given by $r(\phi) = r_0/(\epsilon\cos\phi + 1)$, where $r_0 = L^2/(\mu\,\alpha)$ and $\epsilon = \sqrt{1 + 2\,E\,L^2/(\mu\,\alpha^2)}$, where μ, E and L denote the reduced mass, energy and angular momentum of the particle. (a) Express the velocities v_x and v_y of the particle in terms of ϕ. (b) Show that the particle describes a circle in the (v_x, v_y) space.

Solution: Since

$$r = \frac{r_0}{1 + \epsilon\cos\phi},$$

we have

$$x = r\cos\phi = \frac{r_0\cos\phi}{1+\epsilon\cos\phi}, \quad y = r\sin\phi = \frac{r_0\sin\phi}{1+\epsilon\cos\phi},$$

so that

$$v_x = \dot{x} = -\frac{r_0\sin\phi\,\dot{\phi}}{(1+\epsilon\cos\phi)^2}, \quad v_y = \dot{y} = \frac{r_0\,\dot{\phi}}{(1+\epsilon\cos\phi)^2}\,(\epsilon+\cos\phi)$$

and, since, $\dot{\phi} = L/(\mu\,r^2)$, we can write

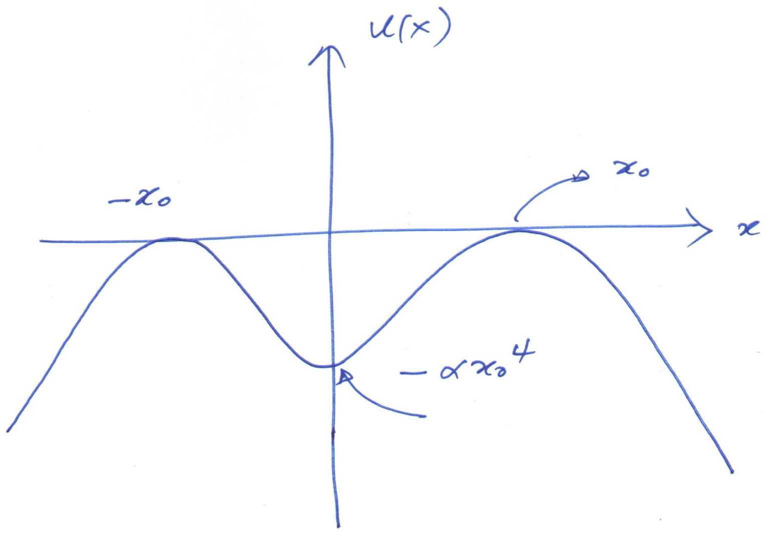
$$v_x = -\frac{L}{\mu\,r_0}\sin\phi, \quad v_y = \frac{L}{\mu\,r_0}\,(\epsilon+\cos\phi)\,,$$

leading to

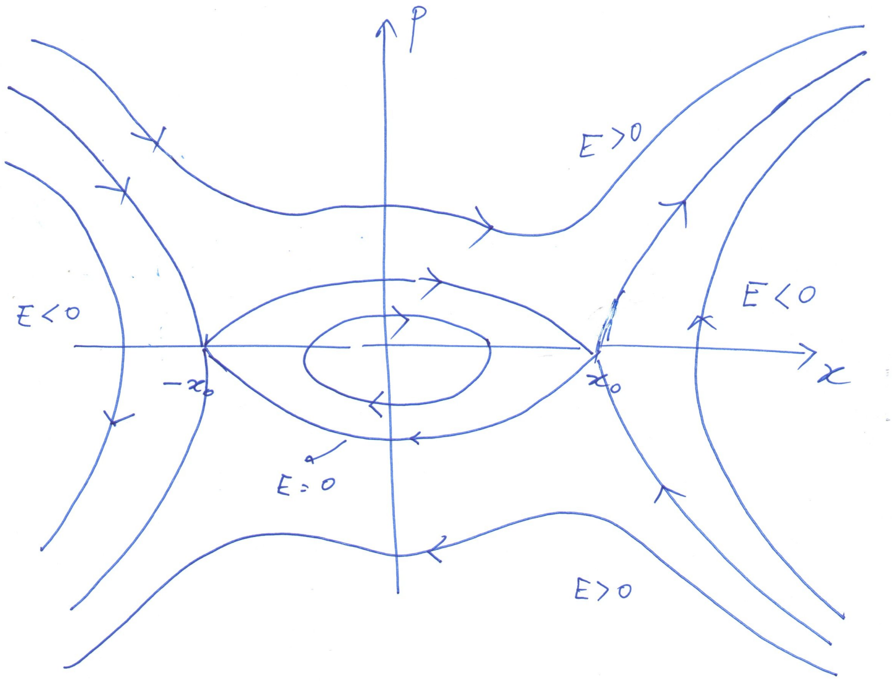
$$v_x^2 + \left(v_y - \frac{\epsilon\,L}{\mu\,r_0}\right)^2 = \frac{L^2}{\mu^2\,r_0^2}.$$

13. *Trajectories in phase space:* Consider a particle moving in the following one-dimensional potential: $U(x) = -\alpha\,(x^2 - x_0^2)^2$, where $\alpha > 0$. (a) Draw the potential $U(x)$, specifically marking the values of the extrema. (b) Determine the range of energy for which the system can exhibit bounded motion. (c) Draw the following phase space trajectories indicating the direction of motion with arrows: (i) bounded motion and (ii) unbounded motion for a positive as well as a negative value of energy. $\boxed{\text{1.5+0.5+2 marks}}$

<u>*Solution:*</u> The potential $U(x)$ is illustrated in the figure below.



The particle exhibits bounded motion if starts between $-x_0$ and x_0 with energy $-\alpha\,x_0^4 < E < 0$. The resulting phase trajectories are illustrated in the figure below.


p
E>0
E<0
E<0
$-x_0$
x_0
x
E=0
E>0