Department of Physics
Indian Institute of Technology Madras

# End-semester Examination

Date: November 19, 2018 Time: 09:00 a.m – 12.00 noon Total Marks: 50

**Name** | | **Roll No:** | | **Batch** | |

## Instructions

1. Please write your name, roll number and batch number.
2. This booklet should contain **14** single-sided pages (please check now).
3. **The answers have to be written in the boxes provided. Answers written elsewhere will not be evaluated.**
4. **Write the answers and sketch your plots with a blue or black pen only.**
5. **For questions 16 to 23, answers without detailed steps will NOT be awarded full marks.**
6. You can use the empty reverse sides for rough work. No extra sheets will be provided.
7. You are not allowed to use a calculator or any other electronic device during the quiz.

**For use by examiners(Do not write in this space)**

| Q 1-10 | Q 11-15 | Q 16-20 | Q 21 | Q 22 | Q 23 |
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**TOTAL MARKS**

✦ **State whether true or false**(write **True** OR **False** in the box.) [**1 x 10 = 10**]

1. For a particle executing bounded motion in one-dimension, the time period is always independent of its energy.

2. The time interval between successive minima of the displacement of an underdamped (lightly damped) oscillator is less than the time period of the same oscillator without damping.

3. The position vector of a particle moving in a certain force field is given by $\mathbf{r}(t) = ae^{\alpha t}\hat{\mathbf{i}} + be^{-\alpha t}\hat{\mathbf{j}}$, where $a, b, \alpha$ are positive constants and $a \neq b$. The potential energy has a maximum at the origin.

4. The phase space trajectories of a particle moving under the influence of a constant (non-zero) force in one dimension are parabolas.

5. The position vector of a particle in the spherical polar coordinate system is $\mathbf{r} = r\hat{\mathbf{r}} + r\theta\hat{\boldsymbol{\theta}} + r\sin\theta\hat{\boldsymbol{\phi}}$.

6. A particle of mass $m$ is moving in a central force field with potential energy $U(r) = -A/r^2$ where $A > \ell^2/2m$ and $\ell$ is the angular momentum with respect to the centre of the force. The particle will eventually fall into the centre of force.

7. The distance from the Sun to the perihelion in a parabolic Kepler orbit is greater than the same for a hyperbolic orbit, if the angular momentum of the particle (mass $m$) is the same for both orbits.

8. Consider a closed surface with total surface area $S$, and let $\hat{n}$ be the unit outward normal vector at a point on the surface. Then $|\oint \hat{n} dS| = S$.

9. The force $\mathbf{F}(\mathbf{r}) = \hat{\boldsymbol{\phi}}/r$ is not conservative.

10. The flow of an incompressible and homogeneous fluid is always characterised by $\nabla \times \mathbf{v} = 0$.

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|---|---|---|---|---|---|---|---|---|---|
| 1. | | 2. | | 3. | | 4. | | 5. | |
| 6. | | 7. | | 8. | | 9. | | 10. | |

✦ **Give the final answer in the box provided, no derivation to be shown.** **[2 x 5 = 10]**

11. Two satellites orbit the Earth. The height of **satellite-A** at perigee is 3000 km above the Earth's surface, while it is 64000 km above the Earth's surface at apogee. The height of **satellite-B** at perigee is 1500 km above Earth's surface, while it is 5500 km above the Earth's surface at apogee. Take the radius of Earth to be 6500 km. The ratio of the time-period of the two satellites, $T_B/T_A$ is:

12. The scalar potential $U(r)$ corresponding to the conservative force-field $\mathbf{F} = k\mathbf{r}/r^2$ is

13. A two-dimensional velocity field is given by $\mathbf{v} = -kf(x,y)\hat{\mathbf{i}} + k(x^2 + y^2)\hat{\mathbf{j}}$ in the $x - y$ plane. Here, $k$ is a constant of appropriate dimensions. For this velocity-field to represent the incompressible flow of a homogeneous fluid, the function $f(x,y)$ should be

14. A particle of mass $m$ moves in the one-dimensional region $0 < x < \infty$, where its potential energy is $U(x) = A/x^2 - B/x$, with $A, B > 0$. The position of the stable equilibrium point $x_0$, and the angular frequency $\omega$ of small oscillations about this point are given by:

$x_0 =$ $\qquad\qquad$ $\omega =$

15. The flux of the vector-field $\mathbf{F} = k\mathbf{r}/(r^2 + a^2)^{3/2}$ over the surface of the sphere of radius $\sqrt{3}a$ and centre at the origin is given by

✦ **Answer in detail** (write the calculations and answers within the boxes provided)

16. Consider the flow of a fluid in the $x - y$ plane, with velocity field given by $\mathbf{v} = \omega_0(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}})$ where $\omega_0 > 0$.

    (a) Compute the stream function $\psi(x, y)$ of the flow.

    (b) Determine $\nabla\psi$ and compute $\mathbf{v} \cdot \nabla\psi$.

    (c) Determine the vorticity $\mathbf{\Omega} = \nabla \times \mathbf{v}$ of the flow.

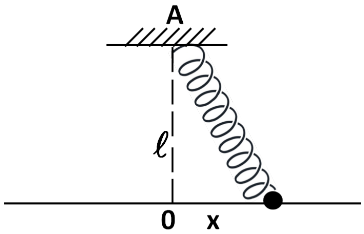
$\boxed{\text{1+1+1=3 marks}}$

17. A particle of mass $m$ is constrained to move along the $x$-axis. It is attached to a spring with spring constant $\kappa$, the other end of which is fixed at a point $A$, whose coordinates are $(0, \ell)$ (see figure below) on the positive $y$-axis. The relaxed length of the spring is $\ell/2$.

(a) Express the potential energy $U(x)$ of the particle as a function of $x$.

(b) For $x \ll \ell$, expand $U(x)$ (using binomial series) in the form $U(x) \simeq U(0) + \beta x^2$. Find $\beta$.

(c) Determine the angular frequency $\omega$ of small oscillations about $x = 0$.



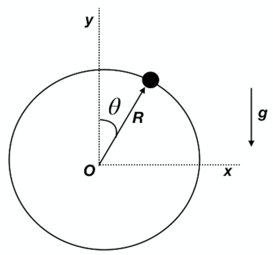

$\boxed{1+1+1=3 \text{ marks}}$

18. A bead with unit mass is initially at rest on top of a fixed vertical hoop of radius $R$. It is given a very gentle push so that it slides down the hoop without friction.

    (a) Write down the equations of motion of the bead in plane polar coordinates, with the origin at the centre of the hoop and the angle $\theta$ measured with respect to the vertical (see figure).

    (b) Determine the normal reaction force $N$ on the bead as function of the angle $\theta$.

$\boxed{1+2=3 \text{ marks}}$



19. A particle of unit mass moving in one dimension is subjected to a linear dissipative force $F = -bv$, where $b > 0$. The position and velocity of the particle are $x(t)$ and $\dot{x} = v(t)$ respectively. Ignore gravity.

(a) Solve the equation of motion to obtain $v(t)$ and $x(t)$, if the initial values for the same (at $t = 0$) are $v_0$ and $x_0$ respectively.

(b) For $x_0 = 1, v_0 = 1, b = 1$ (in appropriate units), plot the phase trajectory ($\dot{x}$ versus $x$).

2+1=3 marks

20. Let $S$ be the surface enclosing a volume $V$, and $\mathbf{A}$ is a vector field defined over this region.

(a) Using Gauss' divergence theorem, prove the relation $\int_V \nabla\phi\, dV = \oint_S \phi\hat{\mathbf{n}}dS$, where $\phi(\mathbf{r})$ is a scalar function. (**hint**: use $\mathbf{A} = \phi\mathbf{c}$, where $\mathbf{c}$ is a constant vector).

(b) Use the result in (b), and choose an appropriate scalar field $\phi(\mathbf{r})$, to express the position vector $\mathbf{R_{CM}}$ of the centre of mass of a body of volume $V$, uniform density and arbitrary shape in terms of an integral over the surface of that body.

1+2=3 marks

21. Consider the vector field $\mathbf{F} = z\hat{\mathbf{i}} + x\hat{\mathbf{j}} - y^2\hat{\mathbf{k}}$.

(a) Compute $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

(b) Evaluate the line integral $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathcal{C}$ is the unit circle centred at the origin in the $x - y$ plane.

(c) Compute the flux $\int_S \mathbf{F} \cdot d\mathbf{S}$, where $S$ is the curved surface of the (upper) hemisphere $x^2 + y^2 + z^2 \leq 1$, $0 \leq z \leq 1$. (**hint**: use divergence theorem to simplify your calculation).

2+1+2=5 marks

22. Two asteroids with identical mass $m$ and angular momentum $\ell$ are moving around the sun (mass $M_s$), one in a circular orbit and the other in a parabolic orbit.

(a) Write down the general orbit equation in 2D polar coordinates. As special cases, write down equations for circular and parabolic orbits (**no derivation needed**).

(b) Find the radial dependence of the asteroid's speed $v_p(r)$ in the parabolic orbit (**hint**: use conservation of angular momentum).

(c) Using your result in (b), find the ratio of the speeds, $v_p/v_c$, in the parabolic orbit and the circular orbit, at a point of intersection of the orbits. The answer should be a number.

2+2+1=5 marks

23. Consider a forced, **undamped** oscillator with natural frequency $\omega_0$ **at resonance**, whose equation of motion is given by

$$\ddot{x} + \omega_0^2 x = f_0 \cos \omega_0 t \tag{1}$$

(a) Write down the general solution to Eq.(1), when $f_0 = 0$.

(b) Determine the particular solution to Eq.(1) when $f_0 > 0$ (**hint**: expect the amplitude to increase linearly with time in the absence of damping; do not forget the phase factor).

(c) Using (a) and (b), and for initial conditions $x(0) = x_0$, $\dot{x}(0) = 0$, determine the complete solution to Eq. (1).

1+2+2=5 marks