Department of Physics, IIT Madras

PH1010 Physics I — Quiz I — 09.09.2014

Time: 8.00-8.50 am — *Answer all questions* — Max. marks: 20

Name	Roll No.	Old Roll No.(if any)
Instructions: You must write the answers **only** in the allotted box. There are **15** boxes in all. Vectors must be indicated clearly using arrows. You may use the reverse side of all pages for rough work. All symbols have their usual meaning unless stated otherwise. You must use only black or blue ink for writing the answers. Calculators, cell phones or any internet connectable device must not be in your possession during the examination.		Exam Hall No.
		Total Marks

1. In the boxes provided, clearly indicate your answers to each of the questions below. Each box is worth 1 **mark**.

   (a) Evaluate $\epsilon_{ijk}\epsilon_{ijk}$. *(Note that the Einstein summation convention is implied here.)* [1 mark]

   $\boxed{6}$

   (b) A particle with charge q and mass m, moving with a velocity $\vec{v}$, is subjected to an electric field $\vec{E}$ and magnetic field $\vec{B}$. The force experienced by the particle is $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$. Give the expression for F_i, the i^{th} component of the force, using index notation. [1 mark]

   $\boxed{F_i = q(E_i + \varepsilon_{ijk} v_j B_k)}$

   (c) A particle of unit mass moves in a potential given by $V(x) = -x^2 \exp(-x^2/2)$.

   i. The points of stable equilibrium are [1 mark]

   $\boxed{\pm\sqrt{2}}$

   ii. The points of unstable equilibrium are [1 mark]

   $\boxed{0}$

   iii. The frequency of small oscillations around the points of stable equilibrium is given by [1 mark]

   $\boxed{\omega = \frac{2}{\sqrt{e}} \text{ or } \sqrt{\frac{4}{e}} \text{ or } 2e^{-1/2}}$

2. In the box provided, indicate whether the following statements are **True** or **False**, (no explanations need to be provided). [$5 \times 1 = 5$ marks]

   (a) "*Given two vectors $\vec{u}$, and $\vec{v}$ with components (u_1, u_2, u_3), and (v_1, v_2, v_3) respectively, the combination $u_i v_i$ also transforms like a vector.*" $\boxed{F}$

(b) "*Under a parity transformation the Cartesian co-ordinates transform (x, y, z) to $(x', y', z') = (-x, -y, -z)$. Under the same transformation the cylindrical polar co-ordinates (ρ, φ, z) transform into $(\rho', \varphi', z') = (\rho, -\varphi, -z)$.*" $\boxed{F}$

(c) "*In the absence of external forces, the total angular momentum for a collection of many particles exerting mutually equal and opposite, but* **non-central**, *forces is constant.*" $\boxed{F}$

(d) "*The work done by a magnetic force on a charged particle is always zero.*" $\boxed{T}$

(e) "*The phase lag for a driven harmonic oscillator at zero damping (i.e., $\gamma = 0$) is given by $\varphi = 0$ or π when the system is not at resonance.*" $\boxed{T}$

Part B

3. Consider an **underdamped** oscillator that is subject to a driving force, given by the function

$$f(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ at & \text{for} \quad 0 < t < \infty. \end{cases}$$

so that it obeys the differential equation

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = f(t).$$

The complete solution to the above equation has the form $x(t) = x_h(t) + x_p(t)$, where $x_h(t)$ is the general (transient) solution to the corresponding homogeneous equation (i.e., $\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0$) and $x_p(t)$ is a particular solution to the complete equation.

(a) Write down the general homogeneous solution $x_h(t)$ (**no need to show derivation**) [1 mark]

$$x_h(t) = e^{-\gamma t}\left(A e^{i\tilde{\omega} t} + B e^{-i\tilde{\omega} t}\right) \qquad \tilde{\omega} = \sqrt{\omega^2 - \gamma^2}$$

or

$$x_h(t) = e^{\gamma t}\left(A\cos\tilde{\omega} t + B\sin\tilde{\omega} t\right) \quad \text{or} \quad = e^{-\gamma t} C\cos(\tilde{\omega} t + \varphi)$$

— 1 mark

(b) Find the particular solution $x_p(t)$ for the above equation by initial guess and subsequent verification (hint: try a polynomial expression in t). [2 marks]

(c) Write down the complete solution $x(t)$ for the above problem, using the specific initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ and the known form for $x_h(t)$. [2 marks]
Continue answers to parts (b) and (c) in the next page

Q. 3 Your answer here

Try a solution of the form

$x_p(t) = \tilde{A}t + \tilde{B}$ — (1 mark)

$\Rightarrow \ddot{x}_p = 0$; $\dot{x}_p = A$ (Substituting in differential eqn).

$\therefore 2\gamma\tilde{A} + \omega^2(\tilde{A}t + \tilde{B}) = at$

Comparing co-efficients

$\omega^2 \tilde{A} = a$

$\tilde{A} = \frac{a}{\omega^2}$

$\therefore \frac{2\gamma}{\omega^2} a + \omega^2 \tilde{B} = 0$

$\Rightarrow \tilde{B} = -\frac{2\gamma}{\omega^4} a$

Both of these connect $\rightarrow$ $\boxed{1 \text{ mark}}$

$x(t) = \left(\frac{a}{\omega^2} t - \frac{2\gamma}{\omega^4} a\right) + e^{-\gamma t}(A\cos\tilde{\omega}t + B\sin\tilde{\omega}t)$

use $x(0) = 0$

$\Rightarrow A - \frac{2\gamma}{\omega^4} a = 0 \quad \therefore A = \frac{2\gamma a}{\omega^4}$ — $\boxed{(1) \text{ mark}}$

$\dot{x}(t) = \frac{a}{\omega^2} + e^{-\gamma t}\tilde{\omega}(-A\sin\tilde{\omega}t + B\cos\tilde{\omega}t)$

$- \gamma e^{-\gamma t}(A\cos\tilde{\omega}t + B\sin\tilde{\omega}t)$

$\dot{x}(0) = \frac{a}{\omega^2} + \tilde{\omega}B - \frac{2\gamma^2}{\omega^4} a = 0$

$\therefore B = \frac{a}{\tilde{\omega}\omega^2}\left[\frac{2\gamma^2}{\omega^2} - 1\right]$ — $\boxed{(1 \text{ mark})}$

Q. 3 Continue your answer here

$$\therefore x(t) = \frac{a}{\omega^2}\left[\frac{2\gamma}{\omega^2} e^{-\gamma t} \cos \tilde{\omega} t + \frac{e^{-\gamma t}}{\tilde{\omega}}\left[\frac{2\gamma^2}{\omega^2} - 1\right] \sin \tilde{\omega} t - \frac{2\gamma + t}{\omega^2}\right]$$

Alternate formulation

(c) $$x(t) = \underbrace{\left(\frac{at}{\omega^2} - \frac{2\gamma a}{\omega^4}\right)}_{x_p(t)} + \underbrace{A \cos(\tilde{\omega} t + \phi)}_{x_h(t)} e^{-\gamma t}$$

$$x(0) = A\cos\phi - \frac{2\gamma}{\omega^4} = 0 \Rightarrow A\cos\phi = \frac{2\gamma a}{\omega^4} \quad -(1)$$

$$\dot{x}(0) = \frac{a}{\omega^2} - A(\gamma\cos\phi + \tilde{\omega}\sin\phi) = 0 \Rightarrow A\sin\phi = \frac{a}{\tilde{\omega}\omega^2}\left[1 - \frac{2\gamma^2}{\omega^2}\right] \quad -(2)$$

$\div$ (2) by (1) $\Rightarrow$

$$\boxed{\tan\phi = \frac{\omega^2}{2\gamma\tilde{\omega}}\left[1 - \frac{2\gamma^2}{\omega^2}\right] = \frac{\omega^2 - 2\gamma^2}{2\gamma\tilde{\omega}}}$$ (either form or equivalent) $\rightarrow$ $\boxed{1 \text{ mark}}$

A can be obtained by squaring and adding (1) and (2), or substituting for $\cos\phi$ in (1).

$$\boxed{A = \frac{2\gamma a}{\omega^4 \cos\phi} \quad \text{where} \quad \cos\phi = \frac{2\gamma\tilde{\omega}}{\sqrt{(\omega^2 - 2\gamma^2)^2 + 4\gamma^2\tilde{\omega}^2}}}$$ — $\boxed{1 \text{ mark}}$

OR

$$\boxed{A = \frac{a}{\tilde{\omega}\omega^2} \quad \text{after completing the calculation}}$$

4. Consider a particle that feels an angular force only, of the form $F_\theta = 3m\dot{r}\dot{\theta}$.

(a) Show that $\dot{r} = \pm\sqrt{Ar^4 + B}$, where A and B are constants of integration, determined by the initial conditions. [3 marks]

(b) Assume that the particle starts its motion with initial conditions $\dot{\theta}(0) = \omega_0 \neq 0$, $r(0) = r_0 > 0$ and $\dot{r}(0) = v_0 > 0$. Derive an integral expression for T_∞, which is the time taken for the particle to reach $r = \infty$ (**no need to evaluate the integral**). [1 mark]

(c) Show that T_∞ is finite. [1 mark]

Q. 4 Write your answer here

$$\ddot{r} - r\dot{\theta}^2 = 0 \quad (1) \rightarrow r \text{ eqn.}$$

$$r\ddot{\theta} = \dot{r}\dot{\theta} \quad (2) \rightarrow \theta \text{ eqn}$$

$$\frac{d\dot{\theta}}{dt} = \frac{1}{r}\frac{dr}{dt}\frac{d\theta}{dt}$$

$$\frac{d\dot{\theta}}{\dot{\theta}} = \frac{dr}{r}$$

$$\therefore \dot{\theta} = Ar \quad \boxed{\text{1mark}}$$

Now Eqn.(1) can be re-written as

$$\frac{d^2 r}{dt^2} = A^2 r^3 \Rightarrow \frac{d\dot{r}}{dt} = A^2 r^3$$

$$\text{Since } \frac{d\dot{r}^2}{dt} = 2\dot{r}\frac{d\dot{r}}{dt} \quad (3)$$

$$\Rightarrow \frac{1}{2}\frac{d\dot{r}^2}{dt} = A^2 r^3 \frac{dr}{dt}$$ (multiplying the above by $\dot{r}$ & using (3)) $\boxed{\text{1mark}}$

$$\text{Integrating} \quad \frac{\dot{r}^2}{2} = A^2\frac{r^4}{4} + C$$

Q. 4 Continue your answer here

$$\therefore \dot{r} = \pm\sqrt{\frac{A^2}{2} r^4 + 2C}$$

put $\frac{A^2}{2} = A$ & $2C = B$

$$\dot{r} = \pm\sqrt{Ar^4 + B} \quad \text{—— (1)}$$ [Give $\boxed{1 \text{ mark}}$ if they do the integration properly]

we are given that $\dot{r} > 0$

Now,

$$\int_0^T dt = \int_{r_0}^{r} \frac{dr'}{\sqrt{Ar'^4 + B}} = T$$ $\boxed{1 \text{ mark}}$

$$\therefore T_\infty = \int_{r_0}^{\infty} \frac{dr'}{\sqrt{Ar'^4 + B}}$$

For large r the above integral

$$T_\infty < \frac{1}{\sqrt{A}} \int_{r_0}^{\infty} \frac{dr}{r^2} \sim \frac{1}{\sqrt{A}} \frac{1}{r_0} < \infty$$ —— $\boxed{1 \text{ mark}}$

for the argument