SOLUTION

Department of Physics, IIT Madras

PH1010 Physics I
Time:

Quiz II
Answer all questions

16.10.2014
Max. marks: 20

Name	Roll No.	Old Roll No.(if any)
Instructions: You must write the answers **only** in the allotted box. There are **12** boxes in all. Vectors must be indicated clearly using arrows. You may use the reverse side of all pages for rough work. All symbols have their usual meaning unless stated otherwise. You must use only black or blue ink for writing the answers. Calculators, cell phones or any internet connectable device must not be in your possession during the examination.		Exam Hall No. Total Marks

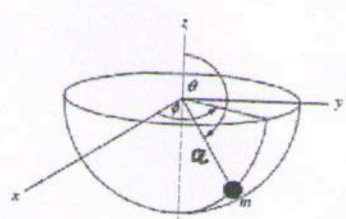
1. In the boxes provided clearly indicate your answers to each of the questions below.

   (a) A particle of mass m moves on the surface of a sphere of radius R. The kinetic energy of the particle written in terms of spherical polar coordinates is given by [1 mark]

   $$\boxed{\frac{1}{2}m\left(R^2\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\varphi}^2\right)}$$

   If $\dot{r}$ is included, "0 mark".

   (b) A particle of mass m is set moving on the inner surface of a hemispherical bowl of radius a under the influence of gravity. The system is described by means of the spherical coordinate system (r, θ, ϕ). Among the three generalized momenta (p_r, p_θ, p_ϕ), which are conserved ? [1 mark]

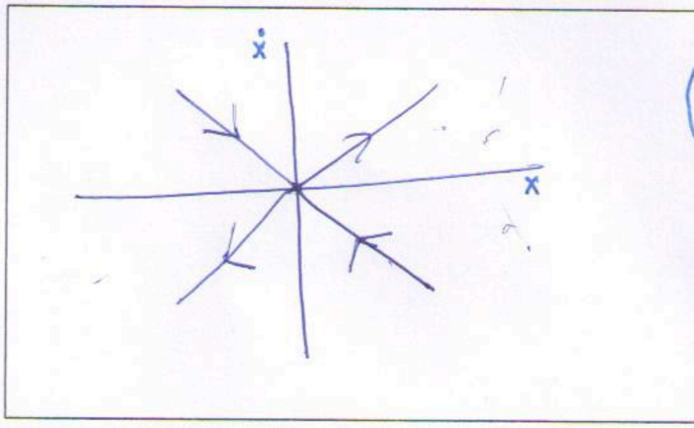


   p_φ (Give zero if more are given)

   (c) Starting from the point $(1, 1)$, specify the unit vector characterizing the direction in which the function $\phi(x, y) = x^2 - y^2 + 2xy$ **increases** most rapidly. [1 mark]

   $\hat{e}_x$ OR $\hat{x}$ OR $\hat{i}$ OR $4\hat{e}_x$

   (d) Plot the phase trajectory for a particle of mass m moving with energy $E = 0$ in an inverted harmonic oscillator potential given by $U(x) = -(k/2)x^2$, with $k > 0$ (**Important: do not forget the time-arrows**). [2 marks]



NO ARROWS $\Rightarrow -1$

CURVED LINES $\Rightarrow 0$

2. In the box provided, indicate whether the following statements are **True** or **False**, (no explanations need to be provided).

[$5 \times 1 = 5$ marks]

(a) $\vec{F} = \hat{e}_r \dfrac{\exp(-\lambda r)}{r^2}$ is a conservative force.

T

(b) "The vector field given by $\vec{F} = 2xz\hat{e}_x + 2yz^2\hat{e}_y + (x^2 + 2y^2z - 1)\hat{e}_z$ is not conservative."

F

(c) "The line element in spherical polar coordinate system is given by $\vec{\delta r} = \delta r\hat{e}_r + r\delta\theta\hat{e}_\theta + r\sin\theta\delta\phi\hat{e}_\phi$ "

T

(d) " Phase trajectories can intersect each other when the force is time-independent."

F

(e) " The Lagrangian for a simple pendulum of length ℓ with a bob of mass m suspended at the origin and made to oscillate in the $x - y$ plane may be written as

$\mathcal{L} = (m/2)\ell^2\dot{\phi}^2 + mg\ell\cos\phi$+const."

T

Part B

3. A vector field is given by $\vec{F} = (3x^2(y+z) + y^3 + z^3)\hat{e}_x + (3y^2(z+x) + z^3 + x^3)\hat{e}_y + (3z^2(x+y) + x^3 + y^3)\hat{e}_z$.

(a) Show explicitly, by calculating its curl, that $\vec{F}$ is conservative [1 mark].

(b) Find the corresponding potential function $\phi(x, y, z)$ such that $\vec{F} = \vec{\nabla}\phi$ [3 marks].

(c) Calculate the work done $W = \int \vec{F} \cdot d\vec{r}$ along any line connecting the point A at $(1, -1, 1)$ to B at $(2, 1, 2)$ [1 mark]:

Q.3. Write your answer below

a) $\begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$

1-mark

$(\nabla \times F)_x = \hat{e}_x \left(\frac{\partial}{\partial y} F_z - \frac{\partial F_y}{\partial z} \right)$

$= \hat{e}_x \left(\frac{\partial}{\partial y} 3z^2(x+y) + x^3 + y^3 \right) - \frac{\partial}{\partial z} \left(3y^2(z+x) + z^3 + x^3 \right) \Big)$

$= \hat{e}_x (3z^2 + 3y^2 - 3y^2 - 3z^2)$

$= 0$

$(\nabla \times F)_y = \hat{e}_y \left(\frac{\partial}{\partial x} F_z - \frac{\partial}{\partial z} F_x \right)$

$= \hat{e}_y \left(\frac{\partial}{\partial x} \left(3z^2(x+y) + x^3 + y^3 \right) - \frac{\partial}{\partial z} \left(3x^2(y+z) + y^3 + z^3 \right) \right)$

$= \hat{e}_y (3z^2 + 3x^2 - 3x^2 - 3z^2) = 0$

$(\nabla \times F)_z = \hat{e}_z \left(\frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x \right)$

$= \hat{e}_z \left(\frac{\partial}{\partial x} \left(3y^2(z+x) + z^3 + x^3 \right) - \frac{\partial}{\partial y} \left(3x^2(y+z) + z^3 + y^3 \right) \right)$

$= \hat{e}_z (3x^2 + 3y^2 - 3x^2 - 3y^2)$

$= 0$

Q. 3 Continue your answer here

b) $\dfrac{\partial U}{\partial x} = 3x^2(y+z) + y^3 + z^3$

$U = x^3(y+z) + y^3 x + z^3 x + f(y,z)$ — (1) $\boxed{1\text{ mark}}$

Putting this in the eqn for $\dfrac{\partial U}{\partial y}$ gives

$\cancel{x^3} + 3y^2\cancel{x} + \dfrac{\partial f}{\partial y} = 3y^2(z+\cancel{x}) + z^3 + \cancel{x^3}$

$\Rightarrow \quad \dfrac{\partial f}{\partial y} = 3y^2 z + z^3$

~~Put $U(x) = x^3(y+z) + y^3 x + z^3 x + 3y^2 z + z^3 + f(z)$ into the eqn for $\frac{\partial U}{\partial z}$ gives~~

$\cancel{x^3 + 3z^2 x + y^3 + 3z^2 x} + y^3 z + z^3 y + f(z)$

$f(y,z) = y^3 z + z^3 y + f(z)$

$\therefore U = x^3(y+z) + y^3 x + z^3 x + y^3 z + z^3 y + f(z)$ — (2) $\boxed{-1\text{ mark}}$

Substititing (2) into

$\dfrac{\partial U}{\partial z} = 3z^2(x+y) + x^3 + y^3$

OR

$U = y^3(z+x) + z^3 + x^3 + g(x,z)$

Q. 3 Continue your answer here

$\Rightarrow$'s

$\cancel{x^3} + \cancel{3z^2x} + \cancel{y^3} + \cancel{3z^2y} + \frac{\partial f}{\partial z} =$

$\cancel{3z^2x} + \cancel{3z^2y} + \cancel{x^3} + \cancel{y^3}$

$\Rightarrow \quad f = \text{Const}$ $\boxed{\text{— 1mark}}$

OR

$\therefore U(x,y,z) = x^3(y+z) + y^3(x+z) + z^3(x+y) + c$

ALSO SEE P-9

c) F is conservative

$$\int F \cdot dr = \int_A^B \nabla \phi \cdot dl = \phi(B) - \phi(A)$$

$$= \phi(2,1,2) - \phi(1,-1,1)$$

$$= \big(8(3) + 1(4) + 8(3)\big) - \big(1\times(0) + (-1)\times(2) + 1\times(0)\big)$$

$$= 54$$

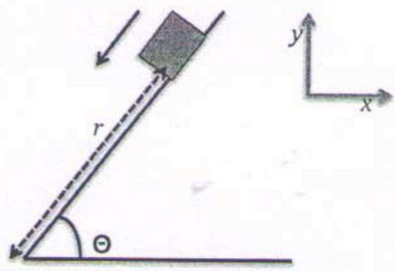
$\boxed{\text{— 1mark}}$

4. A particle of mass m rests on a smooth plane at position $r = r_0$ at $t = 0$(see figure). The plane is raised to an angle θ at a constant rate α, (assume $\theta = 0$ at $t = 0$), causing the particle to move down the plane under gravity.

   (a) Write down the Lagrangian of the particle in appropriate generalized coordinates. [2 marks]

   (b) Write down the Euler-Lagrange equations of motion for this system. [1 mark]

   (c) Solve the equation(s) of motion to determine the motion of the particle. [2 marks]



$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\sin\theta$$ $\boxed{\text{2 marks}}$

$\boxed{\text{1 mark}}$ for the correct K.E

$\boxed{\text{1 mark}}$ for the correct P.E

OR

$\theta = \alpha t \qquad \dot{\theta} = \alpha.$

$$\boxed{L = \frac{1}{2} m(\dot{r}^2 + r^2\alpha^2) - mgr\sin\alpha t}$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

$$\frac{\partial L}{\partial r} = mr\alpha^2 - mg\sin\alpha t$$

$$\cancel{m}\ddot{r} - \cancel{m}\alpha^2 r + \cancel{m}g\sin\alpha t = 0$$

Q. 4 Continue your answer here

$$\boxed{\ddot{r} - \alpha^2 r + g\sin\alpha t = 0}$$ — $\boxed{1 \text{ mark}}$

— (2)

c) Inhomogenous differential eqn
solve the homogenous eqn first

$$\ddot{r} = \alpha^2 r$$

sol$\underline{n}$ is

$$r = Ae^{\alpha t} + Be^{-\alpha t}$$

Now look at a particular soln of the form

$$r_p = C\sin\alpha t$$

Substituting in (2) $\Rightarrow$'s

$$-C\alpha^2\sin\alpha t - \alpha^2 C\sin\alpha t = -g\sin\alpha t$$

$$\therefore C = \frac{g}{2\alpha^2}$$

Thus ~~gen~~ solution is

$$\underline{r(t) = r_p(t) + r_h(t)}$$

$$\boxed{r(t) = Ae^{\alpha t} + Be^{-\alpha t} + \frac{g}{2\alpha^2}\sin\alpha t}$$ — 1 mark

Q. 4 Continue your answer here

$r(0) = r_0$

$\dot{r}(0) = 0$

$r_0 = A + B$

$0 = \alpha(A - B) + \frac{g}{2\alpha}$

$\therefore A + B = r_0 \quad \therefore A = \frac{1}{2}\left(r_0 - \frac{g}{2\alpha^2}\right)$

$A - B = -\frac{g}{2\alpha^2}$

$B = \frac{1}{2}\left(r_0 + \frac{g}{2\alpha^2}\right)$

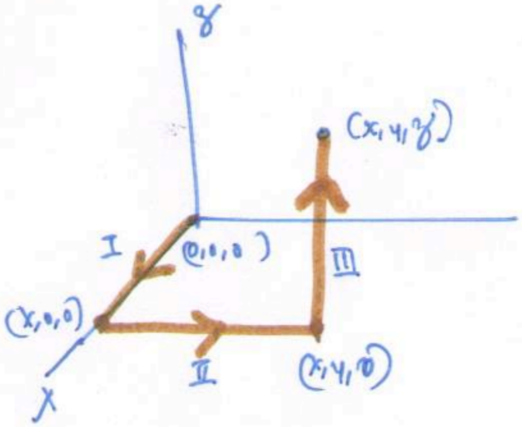
$$\therefore \boxed{r(t) = \frac{1}{2}\left(r_0 - \frac{g}{2\alpha^2}\right)e^{\alpha t} + \frac{1}{2}\left(r_0 + \frac{g}{2\alpha^2}\right)e^{-\alpha t} + \frac{g}{2\alpha^2}\sin\alpha t}$$

— 1 mark

OR

$$\boxed{r(t) = r_0 \cosh\alpha t - \frac{g}{2\alpha^2}\sinh\alpha t + \frac{g}{2\alpha^2}\sin\alpha t}$$

Q3: Alternative method to find $U(x, y, z)$



Choose the path as given in the figure (any path would do, of course):

Along I, $d\vec{r} = dx\,\hat{e}_x$, along II, $d\vec{r} = dy\,\hat{e}_y$ and along III, $d\vec{r} = dz\,\hat{e}_z$

$$\phi(x,y,z) - \phi(0,0,0) = \int \vec{F}\cdot d\vec{r} = \int_{I} F_x\,dx + \int_{II} F_y\,dy + \int_{III} F_z\,dz$$

$$= W_I + W_{II} + W_{III}$$

$W_I = 0$, since $y = z = 0$ along I.

$$W_{II} = x\int_0^y 3y'^2\,dy' + x^3\int_0^y dy = xy^3 + x^3 y$$

$$W_{III} = 3(x+y)\int_0^z z'^2\,dz' + (x^3+y^3)z = (x+y)z^3 + z(x^3+y^3)$$

Thus, $U(x,y,z) = W_{II} + W_{III} = x(y^3+z^3) + y(x^3+z^3) + z(x^3+y^3) + \text{const.}$