PH 1010-2020 MINI-TEST 1 SOLUTION KEY

1. (i) *Complex exponential method:*

Put $f(t) = \text{Re } f_0 e^{i\omega t}$ and $x_p(t) = \text{Re } A e^{i(\omega t+\phi)}$, say.

Substitute in Eq: 1, which gives

$$(\omega_0^2 - \omega^2 + 2i\omega\omega_0)Ae^{i\phi} = f_0 \implies (\omega_0^2 + \omega^2)Ae^{i(\theta+\phi)} = f_0$$

Since RHS is real, LHS too should be, which leads to

$$\phi = -\theta = \tan^{-1}\left(\frac{2\omega\omega_0}{\omega^2 - \omega_0^2}\right)$$ **(0.5 MARK)** and $$A = \frac{f_0}{\omega_0^2 + \omega^2}$$ **(0.5 MARK)**

Equivalent forms:

$$\phi = \cos^{-1}\left(\frac{\omega_0^2 - \omega^2}{\omega_0^2 + \omega^2}\right) = -\sin^{-1}\left(\frac{2\omega\omega_0}{\omega_0^2 + \omega^2}\right)$$

(ii) Put $x = te^{-\beta t}$, then

$$\dot{x} = -\beta x + e^{-\beta t} \quad ; \quad \ddot{x} = \beta^2 x - 2\beta e^{-\beta t}$$ **(0.5 MARK)**

Substitute in Eq. 1: after some rearrangement of terms, we get

$$(\beta - \omega_0)\{(\beta - \omega_0)t - 2\}e^{-\beta t} = 0$$

Since the condition has to be valid at all t, we conclude $\beta = \omega_0$. **(0.5 MARK)**

(iii) Complete solution: $x(t) = e^{-\omega_0 t}(c_1 + c_2 t) + A\cos(\omega t + \phi)$

After using initial conditions in the complete solution, we get the relations

$$c_1 = -A\cos\phi \quad ; \quad c_2 = \omega_0 c_1 + A\omega \sin\phi$$ **(0.5 MARK)**

After using A and ϕ from (i), we get the final results

$$c_1 = f_0 \frac{\omega^2 - \omega_0^2}{(\omega^2 + \omega_0^2)^2} \quad ; \quad c_2 = -f_0 \frac{\omega_0}{\omega^2 + \omega_0^2}$$ **(0.5 MARK)**

OR

$$x(t) = \frac{f_0}{(\omega^2 + \omega_0^2)}\left\{\frac{\omega^2 - \omega_0^2}{(\omega_0^2 + \omega^2)}e^{-\omega_0 t} - \omega_0 t e^{-\omega_0 t} + \cos(\omega t + \phi)\right\}$$

2. (i) $\mathbf{F} = -\nabla U$ **(0.5 MARK)**

ZERO for $F = -\dfrac{dU}{dr}$.

$$\mathbf{F} = -U_0 \left\{ e^{-\mathbf{p}\cdot\mathbf{r}} \nabla \left(\frac{1}{r}\right) + \frac{1}{r} \nabla e^{-\mathbf{p}\cdot\mathbf{r}} \right\} = U_0 \frac{e^{-\mathbf{p}\cdot\mathbf{r}}}{r} \left\{ \frac{\hat{r}}{r} + \mathbf{p} \right\}$$ **(0.5 MARK)**

(ii) Both $\hat{r}$ and $\mathbf{p}$ are along y-axis, hence force is along y-axis. The particle is initially at rest, hence the velocity is also along y-axis.

$$\mathbf{v}_\infty = v_\infty \mathbf{j}$$

Conservation of energy gives

$$\frac{m v_\infty^2}{2} = U(0,1/p,0) = \frac{pU_0}{e}$$ **(0.5 MARK)**

Since m=1 unit as given, we find

$$\mathbf{v}_\infty = \sqrt{\frac{2pU_0}{e}}\mathbf{j} \qquad \text{OR} \qquad \mathbf{v}_\infty = \sqrt{\frac{2pU_0}{me}}\mathbf{j}$$ **(0.5 MARK)**

3. (i) $v_x(t) = \omega R(1 - \cos \omega t)$; $v_y(t) = \omega R \sin \omega t$

$$v = \sqrt{v_x^2 + v_y^2} = \omega R \sqrt{2(1 - \cos \omega t)} = 2\omega R |\sin(\omega t/2)|$$ **(0.5 MARK)**

NOTE: Either expression is ok .

$$v_{\max} = 2\omega R$$ **(0.5 MARK)**

(ii) $\overline{v_x} = \omega R$; $\overline{v_y} = 0$ **(1 MARK)**