DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY MADRAS

PH1101 Physics I End-sem Examination 20/11/2015 1:00–4:00 PM Max. Marks 50

- This is the question paper for the end-semester examination. The answer booklet is provided to you separately.
- **You must write all the answers only in the answer booklet.**
- **First read carefully all the instructions given in the answer booklet. Not following them may result in the paper not being evaluated.**

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1. Indicate whether the following statements are **true** or **false** (in the answer booklet) : [$10 \times 1 = 10$ marks]

   (a) Consider a vector $\boldsymbol{a}$ that depends on time and has a constant magnitude. In such a case, the vector $\dot{\boldsymbol{a}}$ is orthogonal to the original vector $\boldsymbol{a}$.

   (b) $\hat{\varphi}$ is dependent on r and θ in the spherical polar coordinate system.

   (c) The time average of the kinetic and potential energies of a simple harmonic oscillator over one period are equal.

   (d) A particle is exhibiting bounded motion in the potential $U(x) = \alpha\, x^4$, where $\alpha > 0$. The time period of the particle is independent of its total energy.

   (e) The amplitude of an oscillator driven by a sinusoidal force in the absence of damping increases linearly with time.

   (f) The Hohmann transfer orbit between Earth and Mars is a parabolic curve.

   (g) Kepler's second law, viz. that a planet sweeps equal areas in equal intervals of time, is actually applicable to any trajectory in any central potential.

   (h) For a particle moving under the attractive central force $\mathbf{F} = -\hat{r}\, k/r^2$, the centre of the elliptical orbit coincides with the centre of force.

   (i) The surface integral of $\nabla \times \mathbf{F}$, where $\mathbf{F}(\mathbf{r})$ is a vector field, over a closed surface is zero.

   (j) Consider a fluid flowing along a narrow tube. The velocity of the fluid increases with distance along the tube. The divergence of the velocity associated with such a flow is non-zero.

2. Fill-in the blanks (in the appropriate boxes given in the answer booklet) [$10\times 1 = 10$ marks]

   (a) The trajectory of a particle under the influence of certain force $\mathbf{F}$ is given by $x = x_0 \cos(\omega_1 t), y = y_0 \sin(\omega_2 t)$, with x_0, y_0, ω_1 and ω_2 being constants. The condition for which the force $\mathbf{F}$ is central is given by ....

   (b) The amplitude of an underdamped oscillator drops to $1/e$ of its initial value after four cycles. The ratio β/ω_0 of the damping coefficient β and the natural frequency of the oscillator ω_0 is ....

(c) A pendulum consists of a mass m at the end of a string of length l. The speed with which the mass moves at the lowest point so that it is able to just move in a full circle is ....

(d) Assume that the Moon (with mass m) travels around the Earth in a circular orbit of radius R and speed v. The work done on the Moon in one complete revolution is

....

(e) The unit vector perpendicular to the vectors $\boldsymbol{a} = \hat{x} + \hat{y} - \hat{z}$ and $\boldsymbol{b} = 2\hat{x} + \hat{y} - 3\hat{z}$ is

....

(f) The unit vector normal to the ellipsoidal surface $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point $(a/\sqrt{2}, b/\sqrt{2}, 0)$ is ....

(g) The work done by a force $\mathbf{F} = (y+z)\hat{x} - (x+z)\hat{y} + (x+y)\hat{z}$ in taking a particle around a unit circle centred at the origin in the $x - y$ plane in the counter-clockwise direction is ....

(h) A falling raindrop is subject to the frictional force $\boldsymbol{F} = -\alpha\, \boldsymbol{v}$, where α is a constant and $\boldsymbol{v}$ is the velocity of the raindrop. The terminal velocity of a raindrop falling vertically under the action of Earth's gravity is ....

(i) A particle is confined to move on the surface of a cylinder of radius R. The kinetic energy of the particle can be written in terms of the time derivatives of the cylindrical polar coordinates, say, (ρ, φ, z), as ....

(j) Let S be a surface enclosing a volume V. If $\boldsymbol{r}$ denotes the position vector and $\hat{n}$ is the outward unit normal to the surface, then $\int\int_S dS\, \hat{n} \cdot \boldsymbol{r} =$ ....

3. Answer all the questions in the appropriate space provided in the answer booklet. [5 × 2 = 10 marks]

(a) An electric dipole of constant dipole moment $\boldsymbol{p}$ is located at the origin. The dipole creates the electric potential

$$\varphi(\boldsymbol{r}) = \frac{\boldsymbol{p} \cdot \boldsymbol{r}}{4\pi\,\epsilon_0\, r^3}.$$

Express the corresponding electric field $\boldsymbol{E}$ in terms of $\boldsymbol{p}$, $\hat{r}$ and r. [Note: The electric field $\boldsymbol{E}$ is given in terms of the electrostatic potential ϕ by the relation $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$.]

(b) Write the most general solution for a critically damped oscillator with natural frequency ω_0. Show that, for arbitrary initial conditions, a critically damped oscillator can never pass through the origin $x = 0$ more than once in finite time.

(c) A particle under the influence of a central force moves in a spiral orbit $r = k\,\varphi^2$, where k is a constant. Derive the expression for the force that gives rise to such a trajectory.

(d) Consider an infinitely long cylinder of radius $2a$ for which volume charge density $\sigma(\mathbf{r})$ is given by (here σ_0 is a constant and $\rho = \sqrt{x^2 + y^2}$):

$$\sigma(\mathbf{r}) = \begin{cases} \sigma_0 & \text{for} \quad 0 \le \rho \le a\,, \\ \sigma_0 \dfrac{\rho^2}{a^2} & \text{for} \quad a \le \rho \le 2a\,. \end{cases}$$

Using Gauss' law, derive the expression for the electric field for $\rho < a$ as well as $a < \rho < 2a$.

(e) Let the vector field $\mathbf{v}(x,y) = (4 - y^2, 0)$ be a velocity field for a fluid in the channel specified by $\{-\infty < x < \infty, -2 \leq y \leq 2\}$. Schematically sketch the velocity field. Indicate with arrows how a paddle wheel (with its axis along $\hat{z}$) would rotate when it is placed at locations $(0,-1)$ and $(0,1)$ respectively.

4. A pendulum consists of a rigid bar of negligible mass and length ℓ, and a bob of mass m at its end. At $t = 0$ it is pulled to the right by an angle $\theta(t = 0) = \theta_0$ and released so that it oscillates in the $x-y$ plane. Assume that the point of suspension is the origin and gravity acts in the $-\hat{y}$ direction. [5 marks]

(a) Write down expressions for kinetic energy and potential energy of the pendulum in terms of θ and $\dot{\theta}$.

(b) Construct the potential $U(\theta)$ and identify the stable and unstable equilibrium points.

(c) Plot two phase trajectories in the $(\theta, \dot{\theta})$ plane, corresponding to $\theta_0 = \pi/4$ and $\theta_0 = \pi$.

5. Consider an underdamped harmonic oscillator of unit mass with natural frequency ω_0 and damping constant β, which satisfies the equation of motion $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$. [5 marks]

(a) Derive the solution $x(t)$ for the initial condition $x(0) = x_0 > 0, \dot{x}(0) = 0$.

(b) Determine the locations of the first and second minima of $x(t)$ along the t-axis.

(c) Assume that the oscillator is driven by a force $f(t) = f_0 e^{\alpha t}$ with $\alpha > 0$. Find the particular solution $x_p(t)$ to the corresponding equation of motion.

6. A particle moves in the repulsive central potential $U(r) = k/r$, where $k > 0$.

Note: Such a potential arises when a positively charged particle with a small mass (say, an α-particle) is scattered by a much more heavier positively charged particle (say, the nucleus of an atom with a large atomic number). [5 marks]

(a) Solve the orbital equation to obtain r as a function of φ.

(b) Express the eccentricity of the orbit in terms of the energy E of the particle and its angular momentum L. What is the allowed range of the eccentricity?

(c) Plot the trajectory of the particle in the x-y-plane, clearly indicating the location of the source of the central potential.

7. The magnetic field due to a current density $\mathbf{j}$ is (here $\rho = \sqrt{x^2 + y^2}$, μ_0 is the permeability of the vacuum and j_0 and a are constants of suitable dimensions) [5 marks]

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0\, j_0}{2}\,(x\,\hat{y} - y\,\hat{x}) & \text{for} \quad 0 \leq \rho \leq a\,, \\[2mm] \dfrac{\mu_0\, j_0\, a^2}{2\rho^2}\,(x\,\hat{y} - y\,\hat{x}) & \text{for} \quad \rho \geq a\,. \end{cases}$$

Compute the circulation of the magnetic field over a circle of radius $R > a$ lying in the xy-plane centred at the origin, in two different ways:

(a) First, explicitly carry out the line integral.

(b) Next, evaluate the integral using Stokes' Theorem.