# Mini-Test 3

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1. (a) Energy $(E) = \frac{1}{2} m v_0^2 - \frac{GMm}{R+h}$

$$\simeq \frac{GMm}{R}(\alpha^2) - \frac{GMm}{R} \simeq \frac{GMm}{R}(\alpha^2 - 1)$$

Angular Momentum $(L) = m v_0 (R+h) \simeq m V_0 R = m R \alpha \sqrt{\frac{2GM}{R}} = m\alpha\sqrt{2GMR}$

(b) Energy $(E) := \frac{1}{2} m (\dot{r})^2 + \frac{L^2}{2mr^2} + U(r)$

$$\Rightarrow U_{eff}(r) = \frac{L^2}{2mr^2} + U(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

Let $r_0$ be the radius of the circular orbit. In this case, $E = E_{min}$ and $\dot{r} = 0$ since separation is constant.
$\Rightarrow$ Since 'E' has a minima, it means $U_{eff}$ also has a minima and

$$\left.\frac{dU_{eff}}{dr}\right|_{r=r_0} = 0 \Rightarrow \frac{-L^2(\cancel{2})}{\cancel{2}mr_0^3} + \frac{GMm}{r_0^2} = 0$$

$$\Rightarrow r_0 = \frac{L^2}{GMm^2} = \frac{\cancel{m^2}\alpha^2(2G\cancel{M}R)}{\cancel{GMm^2}} = 2R\alpha^2$$

$$\Rightarrow \boxed{r_0 = 2R\alpha^2}$$

Energy of the circular orbit $(E_0) = \frac{-mc^2}{2L^2}$ (where $c = GMm$)

$$= -\frac{m(G^{\cancel{2}}M^{\cancel{2}}m^{\cancel{2}})}{2(\cancel{m^2}\alpha^2)(2G\cancel{M}R)}$$

$$= \frac{-GMm}{4\alpha^2 R}$$

$$\Rightarrow \boxed{E_0 = \frac{-GMm}{4\alpha^2 R}}$$

(c) Using the relation $\varepsilon^2 = 1 - \frac{E}{E_0}$, where $\varepsilon$ is the eccentricity of the orbit, we get

$$\varepsilon^2 = 1 - \left[\frac{\frac{G\cancel{Mm}}{\cancel{R}}(\alpha^2-1)}{\frac{-G\cancel{Mm}}{4\alpha^2\cancel{R}}}\right] = 1 + 4\alpha^4 - 4\alpha^2 \quad \left[\because E = \frac{GMm}{R}(\alpha^2-1),\ E_0 = \frac{-GMm}{4\alpha^2 R}\right]$$

$$\Rightarrow \varepsilon^2 = (2\alpha^2-1)^2 \Rightarrow \boxed{\varepsilon = |2\alpha^2 - 1|}$$

d) For the rocket to perform bounded motion (when it is a satellite), $E < 0$

$\Rightarrow \frac{GMm}{R}(\alpha^2-1) < 0 \Rightarrow \alpha^2 - 1 < 0 \Rightarrow |\alpha| < 1$

Since '$\alpha$' is a positive quantity, $\boxed{\alpha \in (0,1)}$.

The initial point should be perigee, this implies that initial velocity is greater than or equal to orbital velocity. [It should be perigee so that it does not crash into Earth].

$\Rightarrow V_0 = \alpha\sqrt{\frac{2GM}{R}} \geq \sqrt{\frac{GM}{R}} \Rightarrow \boxed{\alpha \geq \frac{1}{\sqrt{2}}}$

Based on the conditions/values for '$\alpha$', the common set of values is given by

$\Rightarrow \boxed{\alpha \in \left[\frac{1}{\sqrt{2}}, 1\right)}$ (when $\alpha = \frac{1}{\sqrt{2}}$, rocket performs circular motion)

(e) given, $\alpha = 3/4 \Rightarrow V_0 = \frac{3}{4}\sqrt{\frac{2GM}{R}}$

$\Rightarrow r_{max} = \frac{r_0}{1-\epsilon} \Rightarrow$ since $\alpha \geq \frac{1}{\sqrt{2}}$, $\epsilon = 2\alpha^2 - 1$

$\Rightarrow r_{max} = \frac{r_0}{1-(2\alpha^2-1)} = \frac{\cancel{2}R\alpha^2}{\cancel{2}(1-\alpha^2)} = \frac{R\alpha^2}{1-\alpha^2} = \frac{R\left(\frac{9}{16}\right)}{7/16} = \frac{9R}{7}$

So, let the maximum height be $h_{max}$.

$\Rightarrow r_{max} = h_{max} + R \Rightarrow \boxed{h_{max} = \frac{2R}{7}}$

$\therefore$ The maximum height reached by the rocket, with respect to the Earth's surface, is $\frac{2R}{7}$, where $R$ is the radius of the Earth.

(f) For $\alpha = 1$, $E = \frac{GMm}{R}(1^2-1) = 0$, eccentricity $(\epsilon) = 2\alpha^2 - 1 = 1$.

So, there is some minimum separation beyond which particles cannot come closer, and the particle moves away to infinity. Since $\epsilon = 1$, $E = 0$, we can say that its trajectory is parabolic (and motion is unbound).

(1. (f) The equation of trajectory is :-

$$(1-\varepsilon^2)x^2 - 2r_0\varepsilon x + y^2 = r_0^2$$ (where $\varepsilon$ is eccentricity)

$\varepsilon = 1$

$$\Rightarrow \quad -2r_0 x + y^2 = r_0^2 \quad \Rightarrow \boxed{y^2 = r_0^2 + 2r_0 x} \longrightarrow$$ This represents a parabola.

[2.] (a)(i) surface area element for curved surface of cylinder, $(S_1)$

$$\vec{dS_1} = a\,d\phi\,dz\,\hat{\rho}$$

(ii) Surface area element for bottom surface of cylinder, $(S_2)$

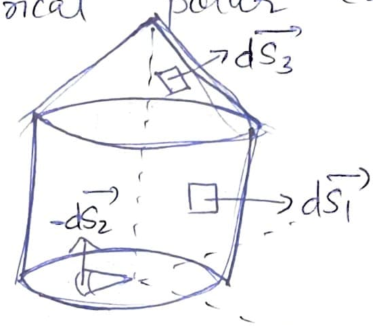
$$\vec{dS_2} = -\rho\,d\rho\,d\phi\,\hat{z}$$

(iii) surface area element for the cone's curved surface, is $(S_3)$ $\Rightarrow \vec{dS_3} = \rho\,d\phi\,dz(\hat{\rho}+\hat{z})$

$\Rightarrow \vec{dS_3} = \sqrt{2}\,\rho\,d\phi\,dz\dfrac{(\hat{\rho}+\hat{z})}{\sqrt{2}}$, where $0<z<2a$ and $\rho = 2a-z$, $\dfrac{\hat{\rho}+\hat{z}}{\sqrt{2}} \rightarrow$ unit vector along normal

volume element, $dV$ in cylindrical polar coordinates is given by

$$\boxed{dV = \rho\,d\rho\,d\phi\,dz}$$




(b)(i) $\text{Flux} = \iint_{S_1} \vec{F}\cdot\vec{dS_1}$ $\qquad \vec{F} = \rho\sin^2\phi\,\hat{\rho} + \rho\sin\phi\cos\phi\,\hat{\phi} + 4z\hat{z}$

$$= \iint(\rho\sin^2\phi)(a\,d\phi\,dz) \quad [\rho = a]$$

$$= a^2\int_0^a\int_0^{2\pi}\sin^2\phi\,d\phi\,dz$$

$$= a^2\int_0^a\int_0^{2\pi}\left(\frac{1-\cos 2\phi}{2}\right)d\phi\,dz = a^2\int_0^a \pi\,dz$$

$$= \underline{\underline{\pi a^3}}$$

(ii) $\text{Flux} = \iint_{S_2}\vec{F}\cdot\vec{dS_2} = \iint_{S_2}(-\rho\,d\rho\,d\phi)(4z)$ since $z=0$ for the bottom surface

$$\Rightarrow \text{Flux} = \iint_{S_2}\vec{F}\cdot\vec{dS_2} = \underline{\underline{0}}$$

(iii) $\text{Flux} = \iint_{S_3}\vec{F}\cdot\vec{dS_3} = \iint_{S_3}(\rho\sin^2\phi\,\hat{\rho} + \rho\sin\phi\cos\phi\,\hat{\phi} + 4z\hat{z})\cdot(\rho\,d\phi\,dz(\hat{\rho}+\hat{z}))$

$$\Rightarrow \iint_{S_3} \vec{F}\cdot d\vec{S_3} = \iint \rho^2\sin^2\phi\, d\phi\, dz + \iint 4\rho z\, d\phi\, dz$$

for the curved surface of cone, $\rho = 2a - z$

$$= \int_0^{2\pi}\int_a^{2a} (2a-z)^2\, dz \sin^2\phi\, d\phi + \int_0^{2\pi}\int_a^{2a} 4(2a-z)\, z\, dz\, d\phi$$

$$= \left[-\frac{(2a-z)^3}{3}\right]_a^{2a} \times \int_0^{2\pi}\sin^2\phi\, d\phi + 8\pi\int_a^{2a}(2az-z^2)\, dz$$

$$= \frac{a^3}{3}\left[\int_0^{2\pi}\left(\frac{1-\cos 2\phi}{2}\right)d\phi\right] + 8\pi\left[az^2-\frac{z^3}{3}\right]_a^{2a}$$

$$= \frac{\pi a^3}{3} + 8\pi\left[3a^3 - \frac{7a^3}{3}\right] = \underline{\underline{\frac{17\pi a^3}{3}}}$$

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(c) Now, from (b), we have net flux of the vector field $F = \frac{20\pi a^3}{3}$

Divergence of a vector field $\vec{A} = A_\rho\hat{\rho} + A_\phi\hat{\phi} + A_z\hat{z}$ in cylindrical polar coordinates is given by,

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(A_\phi) + \frac{\partial A_z}{\partial z}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{F} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2\sin^2\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\rho\sin\phi\cos\phi) + \frac{\partial}{\partial z}(4z)$$

$$= \frac{2\cancel{\rho}\sin^2\phi}{\cancel{\rho}} + \frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{\cancel{\rho}\sin 2\phi}{2}\right) + 4$$

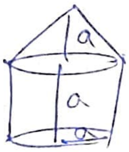
$$= 2\sin^2\phi + \cos 2\phi + 4 = 1 + 4 = 5$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{F} = 5}$$

According to Gauss's Divergence Theorem,

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iiint_V (\vec{\nabla}\cdot\vec{F})\, dV$$

$$= 5\iiint_V dV = 5\,(\text{Total Volume})$$




$$= 5\left(\pi a^2(a) + \frac{1}{3}\pi a^2(a)\right)$$

$$= \boxed{\frac{20\pi a^3}{3}}$$, which is the same as what we got in part (b).

∴ Gauss' divergence Theorem is verified.

$\boxed{3.}$ (a) $m_{eff} = m\left(1+\frac{\rho^2}{4a^2}\right)$, $U_{eff} = \frac{l_z^2}{2m\rho^2} + \frac{mg\rho^2}{4a}$

(b) $\epsilon = 3$

(c) $a = \frac{VT}{\lambda}$

(d) $0 < l < \sqrt{\frac{m\alpha}{\lambda e}}$

(e) FALSE

(f) $\vec{v}(r) = \frac{\mathring{M}}{\rho_m(4\pi r^2)}\hat{r}$

g) TRUE