PH1010, QUIZ-I Solutions. Sept 11 2013.

PART-A

1

a) 6

b) $F_i = q(E_i + \epsilon_{ijk} v_j B_k)$

OR b) $F_i = q\left(E_i + \sum_{j,k=1}^{3} \epsilon_{ijk} v_j B_k\right)$

c) $v_z = 1$ (any unit of velocity or no units acceptable)

d) $\frac{d\hat{\theta}^2}{dt^2} = -\ddot{\theta}\,\hat{r} - \dot{\theta}^2\hat{\theta}$

2

a) TRUE

b) TRUE

c) FALSE

d) FALSE

e) FALSE

f) TRUE

$\Longrightarrow$ for part B

PART-B

3 (a) $x_1 = e^{-\gamma t}$, $\dot{x}_1 = -\gamma e^{-\gamma t}$, $\ddot{x}_1 = \gamma^2 e^{-\gamma t}$.

$$\ddot{x}_1 + 2\gamma \dot{x}_1 + \gamma^2 x_1 = \gamma^2 e^{-\gamma t} - 2\gamma^2 e^{-\gamma t} + \gamma^2 e^{-\gamma t} = 0$$

$\therefore$ $e^{-\gamma t}$ is a solution.

$$x_2 = t e^{-\gamma t}, \quad \dot{x}_2 = e^{-\gamma t} - \gamma t e^{-\gamma t} = e^{-\gamma t}(1-\gamma t)$$

$$\ddot{x}_2 = -\gamma e^{-\gamma t}(1-\gamma t) - \gamma e^{-\gamma t} = -\gamma e^{-\gamma t}(2-\gamma t).$$

$$\therefore \quad \ddot{x}_2 + 2\gamma \dot{x}_2 + \gamma^2 x_2 = -\gamma e^{-\gamma t}(2-\gamma t) + 2\gamma e^{-\gamma t}(1-\gamma t) + \gamma^2 t e^{-\gamma t}$$

$$= -2\gamma e^{-\gamma t} + \gamma^2 t e^{-\gamma t} + 2\gamma e^{-\gamma t} - 2\gamma^2 t e^{-\gamma t} + \gamma^2 t e^{-\gamma t}$$

$$= 0$$

Hence $e^{-\gamma t}$ and $t e^{-\gamma t}$ are both solutions to the homogeneous equation.

$\boxed{\text{OR}}$

$$\ddot{x} + 2\gamma \dot{x} + \omega^2 x = 0$$

is solved by $x = A e^{\alpha t}$ where

$$A\alpha^2 e^{\alpha t} + 2\gamma \alpha A e^{\alpha t} + \omega^2 A e^{\alpha t} = 0 \Rightarrow \alpha^2 + 2\gamma\alpha + \omega^2 = 0$$

or

$$\alpha_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$$

Two Solutions: $e^{-\gamma t} e^{\pm i\sqrt{\omega^2-\gamma^2}\, t}$ or $e^{-\gamma t}\cos \Omega t$ & $e^{-\gamma t} \sin \Omega t$

Can also Consider $e^{-\gamma t}\cos \Omega t$ & $e^{-\gamma t} \dfrac{\sin \Omega t}{\Omega}$ as two Solutions.

Taking the limit $\Omega \to 0$ gives $e^{-\gamma t}$ & $t e^{-\gamma t}$ as two Solutions to the homogeneous equation $\ddot{x} + 2\gamma \dot{x} + \gamma^2 x = 0$

3b particular Solution.

$$\text{Let } x_p'(t) = c\, e^{i\omega t}$$

Consider

$$\ddot{x}_p' + 2\gamma \dot{x}_p' + \gamma^2 x_p' = f_0 e^{i\omega t}$$

$$\Rightarrow -\omega^2 c + 2i\gamma c\omega + \gamma^2 c = f_0$$

$$\Rightarrow c = \frac{f_0}{(\gamma^2-\omega^2)+2i\gamma\omega} = \frac{f_0 e^{-i\phi}}{\sqrt{(\gamma^2-\omega^2)^2+4\gamma^2\omega^2}}$$

$$= \frac{f_0 e^{-i\phi}}{(\gamma^2+\omega^2)}$$

phase: $\cos\phi = \dfrac{\gamma^2-\omega^2}{\gamma^2+\omega^2}$, $\sin\phi = \dfrac{2\gamma\omega}{\gamma^2+\omega^2}$

or $\tan\phi = \dfrac{2\gamma\omega}{\gamma^2-\omega^2}$

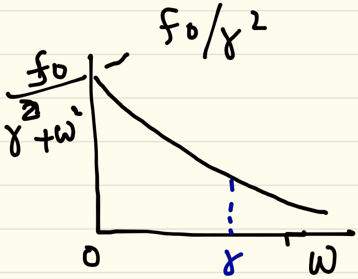
$$\therefore x_p(t) = \mathrm{Re}\, x_p'(t) = \frac{f_0}{\gamma^2+\omega^2}\cos(\omega t-\phi)$$

[acceptable if Re( ) is not explicitly written or $\cos\phi$ & $\sin\phi$ are not written but only $\tan\phi$ is.]

general Solution:

$$x(t) = A e^{-\gamma t} + B t e^{-\gamma t} + \frac{f_0}{\gamma^2+\omega^2}\cdot\cos(\omega t-\phi)$$

3c



No resonance when $\omega = \gamma$

4 $\vec{F} = -k\vec{r}$ $\quad k>0$

(a) $m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} = -kr\hat{r}$

(Newton's law in polar coordinates

Acceptable if simply written as above OR derived)

angular part: $r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$

$$\therefore \frac{d}{dt}(r^2\dot{\theta}) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$$

$$\therefore r^2\dot{\theta} = \text{Constant.} = C$$

(b)

Radial part: $m(\ddot{r} - r\dot{\theta}^2) = -kr$

$$\Rightarrow m\ddot{r} = -kr + mr\frac{C^2}{r^4} = -kr + \frac{mC^2}{r^3}$$

$$V(r) = \frac{kr^2}{2} + \frac{mC^2}{2r^2} \Rightarrow V'(r) = kr - \frac{mC^2}{r^3}$$

$\therefore$ Indeed $m\ddot{r} = -\frac{dV(r)}{dr}$

c $\quad U(x) = \frac{kx^2}{2} + \frac{\alpha}{2x^2}$

$m\ddot{x} = -U'(x)$, leads to

$m\ddot{x} = -kx + \frac{\alpha}{x^3}$, which is same equation

as the r Equation of part b

But $r(t+\frac{\pi}{\omega}) = r(t)$ where $\omega = \sqrt{\frac{k}{m}}$,

[as $r^2(t) = \tilde{x}^2(t) + \tilde{y}^2(t)$ & $\tilde{x}(t)$ & $\tilde{y}(t)$, $\left\{\begin{array}{l}\ddot{\tilde{x}} + \omega^2\tilde{x} = 0 \\ \ddot{\tilde{y}} + \omega^2\tilde{y} = 0\end{array}\right\}$

are both harmonic oscillators with time (Cartesian)

period $\frac{2\pi}{\omega}$. $x^2(t)$ & $y^2(t)$ have the

period $\frac{\pi}{\omega}$, so does $r(t)$.]

But $x(t)$ Satisfies same Equation as $r(t)$

$\therefore x(t+\frac{\pi}{\omega}) = x(t)$ or the $\boxed{\text{period of } x \text{ is } \frac{\pi}{\omega} \text{ or } \pi\sqrt{\frac{k}{m}}}$

Also Small oscillation period about equil. point. But note that π/ω is the period for all amplitude, small or large.