Department of Physics
Indian Institute of Technology Madras

End of semester examination

PH1010 Physics I 2019 **20.11.2019**

Time: 9:00 AM to 12:00 Noon **Max. Marks: 50**

Name	Roll No.	Batch No. and Teacher's name

Instructions

1. Begin by completing the information requested above. Please write your complete name, your roll number, the name of your instructor, and your batch number.
2. This question paper cum answer sheet booklet contains **32** pages. Please check right away that all the pages are present. You are expected to answer all the questions. There are no negative marks.
3. Write the answers only in the allotted box. All vectors must be indicated clearly. **All expressions in your answers should be reduced to their simplest mathematical forms and in terms of the given parameters.**
4. You can use the empty pages assigned for rough work. No extra sheets will be provided.
5. All symbols have their usual meanings unless stated otherwise. **All constants are of appropriate dimensions. Further, assume that unless otherwise stated all constants are positive**.
6. You must use only black or blue ink for writing the answers. Pencil must not be used even in sketches.
7. Calculators, cell phones or any internet connectable device must not be in your possession during the examination.

For use by examiners (Do not write in this space)

Q1	Q2	Q3	Q4		**Total Marks**
Q5	Q6	Q7	Q8	Q9	

Useful Formulae

(a) Del operator in Cylindrical coordinate system

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial \rho} + \frac{\hat{\phi}}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

(b) Divergence and Curl of a vector function in spherical coordinate system

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\vec{\nabla}\times\vec{A} = \frac{\hat{r}}{r\sin\theta}\left[\frac{\partial}{\partial \theta}\left(A_\phi \sin\theta\right) - \frac{\partial A_\theta}{\partial \phi}\right] + \frac{\hat{\theta}}{r\sin\theta}\left[\frac{\partial A_r}{\partial \phi} - \sin\theta\frac{\partial}{\partial r}\left(rA_\phi\right)\right] + \frac{\hat{\phi}}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial \theta}\right]$$

(c) Divergence and Curl of a vector function in cylindrical coordinate systems

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(A_\rho \rho\right) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla}\times\vec{A} = \hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \hat{z}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(A_\phi \rho\right) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right)$$

(d) Vector Identities

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$$

1. Indicate whether the following statement is true or false (write **True** or **False** in the box provided.) [**1 × 9 = 9 Marks**]

(i) The force $F(x)$ corresponding to the potential energy $U(x) = \frac{x^2}{1+x^4}$ has two maxima and two minima.

(ii) If a particle of mass *m* moves in such a way that its position vector is given by $\vec{r}(t) = \hat{x}\, a\, cos\, \omega t + \hat{y}\, b\, sin\, \omega t + \hat{z} v_0 t$ (where a, b, and v_0 are constants, and $a \neq b$), then the resultant trajectory of the particle is an elliptical helix.

(iii) If particle of unit mass moves in a one-dimensional region with potential energy $U(x) = \frac{x^2}{2} - \frac{x^2}{3}$, then the time period of small oscillations is $T = 4\pi$ in relevant units.

(iv) The trajectory of a particle moving under the influence of a central force with potential energy $U(r) = \frac{k}{r}$ is circular (where $k > 0$).

(v) If $\vec{r}$ is the position vector and $\vec{B}$ represents a uniform magnetic field, then $\vec{\nabla} \times (\vec{B} \times \vec{r}) = \vec{B}$.

(vi) If S is the surface enclosing the volume V and $\hat{n}$ is the outward unit vector normal to the surface S, then $\iint \vec{r}\,.\,\hat{n} dS = \frac{V}{3}$.

(vii) $\vec{\nabla}.\vec{v} = 0$ for an incompressible fluid flowing with a velocity $\vec{v}$.

(viii) The streamlines associated with the two-dimensional incompressible fluid flow with velocity potential $\phi\,(x, y) = k \tan^{-1}\left(\frac{x}{y}\right)$ are hyperbolic.

(ix) The streamlines and equipotential lines in two-dimensional incompressible and irrotational fluid flow are mutually perpendicular to each other.

2. Write the final answer in the box provided: **Derivation is NOT required.** [**1 × 11 = 11 Marks**]

(i) Sand drops vertically (from a negligible height) at a rate σ kg/s onto a moving conveyor belt. A force is applied to the belt in order to keep it moving at a constant speed v. What is the work done per unit time?

(ii) What is the directional derivative of $f(x, y, z) = f_o\, e^{-[x^2+y^2+z^2]}$ at (0, 1, 1) in the direction $-\frac{1}{4}(\hat{y} + \hat{z})$?

(iii) Find the potential energy corresponding to the central force

$$\vec{F}(\vec{r}) = -\frac{k}{r^2}\left(1 + \frac{r}{\beta}\right) e^{-\frac{r}{\beta}}\, \hat{r}$$, where $k,\ \beta > 0$.

(iv) A particle of mass m is subjected to a one-dimensional force $F(x) = F_0(-\alpha x + \beta x^3)$, where $\alpha,\ \beta > 0$. Find the time period of small oscillations around the stable equilibrium point.

(v) A critically damped oscillator with natural frequency of oscillation ω_0 is found to have position $x(0) = 1$ and speed $v(0) = 0$ at time $t = 0$. The expression for its position at $t > 0$ is given by

(vi) A particle of mass m and angular momentum L moves in a central force of the form $\vec{F}(r) = -\frac{k}{r^4}\hat{r}$, where $k > 0$. Find the maximum value of the effective potential energy.

(vii) A particle of mass m moves in an elliptical orbit under the influence of an inverse square law attractive central force. If the ratio of the maximum angular speed to the minimum angular speed of the particle in its orbit is β, then express the eccentricity of the orbit in terms of β.

(viii) An infinitely large non-conducting plane in the x-y plane has uniform surface charge density σ. Determine the electric field $\vec{E}$ in the region $z > 0$.

(ix) A current density in spherical polar coordinate system is given by $\vec{J}(\vec{r}) = J_0\, e^{-\lambda r}\,\hat{r}$ in the region $r > 0$. Find the expression for the charge density $\rho(\vec{r}, t)$ in the given region.

(x) A two-dimensional incompressible fluid flow has velocity $\vec{v}(x, y) = k\left(\frac{\hat{x}y - \hat{y}x}{x^2+y^2}\right)$. Find the stream function associated with this velocity.

(xi) A fluid is flowing in the x-y plane with a velocity $\vec{v}(x, y) = k\left(\frac{-\hat{x}y + \hat{y}x}{\sqrt{x^2+y^2}}\right)$, where k is a constant. Determine the vorticity corresponding to the given velocity.

3. An electron of mass m and charge $-q$ is moving under the influence of a uniform electric field, $\vec{E} = E_0\hat{y}$ and a uniform magnetic field, $\vec{B} = B_0\hat{z}$. The initial position of the electron is the origin (0, 0,0) and the initial velocity is $\vec{v} = v_0\hat{x}$.
(i) Write the equations of motion. (ii) Determine the trajectory $\vec{r}(t)$ of the electron at time $t > 0$. **[4 Marks]**

4. Consider a particle of unit mass and total energy E moving in a one dimensional region with potential energy given by $U(x) = -\frac{U_0}{e^x + e^{-x}}$, where $U_0 > 0$.

(i) Sketch $U(x)$ versus x (in the range $-\infty < x < \infty$), suitably marking the values of the extrema.

(ii) Determine the angular frequency ω of small oscillations about the point of stable equilibrium.

(iii) Sketch the corresponding phase trajectories for energies, $E < 0$, $E = 0$, and $E > 0$. Indicate the direction of motion with arrows. **[5 Marks]**

5. An undamped driven oscillator of mass m and natural frequency of oscillation ω_0 is subjected to an external force $F(t) = \mathrm{F}_o\cos^2\omega t$, where F_o and ω ($\omega \neq \omega_0$) are constants. The oscillator satisfies the following equation of motion

$$m\ddot{x} + m\omega_0^2 x = F(t)$$

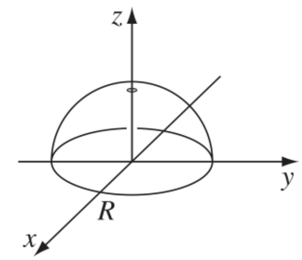
(i) Write down the general solution $x_h(t)$ to the corresponding homogeneous differential equation.

(ii) Find the particular solution $x_p(t)$ to the above equation of motion.

(iii) Determine the complete solution x(t) to the equation of motion using the initial conditions x (t = 0) = 0, $\dot{x}$ (t = 0) = 0. **[5 Marks**]

6. A particle of mass m moves under the influence of gravity in a circular orbit of radius R and a time period T. At a certain instant, the angular momentum of the particle is brought to zero. Find the time required for it to fall into the centre of the force. **[4 Marks]**

7. If S is the surface enclosing a volume V, then Gauss's divergence theorem states that $\iiint \vec{\nabla}.\vec{A}d\tau = \iint \vec{A}.\hat{n}dS$, where $\hat{n}$ is the outward unit vector normal to the surface S.



Verify the divergence theorem by evaluating the volume and surface integrals for the vector function

$$\vec{A}\,(r,\theta,\varphi) = \hat{r}\,(r\,cos\theta) + \hat{\theta}\,(r\,sin\theta) + \hat{\varphi}\,(r\,sin\theta\,cos\varphi),$$

using as the volume, the inverted hemispherical bowl of radius R, resting on the x-y plane and centred at the origin as shown in the adjacent figure. **[4 Marks]**

8. (i) Using Gauss's divergence theorem, prove that $\iint \hat{n} \times \vec{B}\, dS = \iiint \vec{\nabla} \times \vec{B}\, d\tau$.

(ii) A rigid body of volume *V* and surface *S* rotates with angular velocity $\vec{\omega}$**.**

Show that $\vec{\omega} = \frac{1}{2V} \iint \overrightarrow{dS} \times \vec{v}$, where $\vec{v} = \vec{\omega} \times \vec{r}$ is the velocity of the point $\vec{r}$ on the surface *S*.

(**Hint**. Take $\vec{A} = \vec{B} \times \vec{C}$, where $\vec{C}$ is a constant vector. Select the relevant vector identities from those provided in page 2) **[4 Marks]**

9. A force is given by $\vec{F} = \hat{x}\left(\frac{ax-by}{\sqrt{x^2+y^2}}\right) + \hat{y}\left(\frac{ay+bx}{\sqrt{x^2+y^2}}\right)$, where a and b are constants.

(i) Verify whether the force is conservative or not.

(ii) Evaluate the work done by the force in moving a particle around a circle of radius R, centred at the origin in the x-y plane. **[4 Marks]**

Rough work

Rough work

Rough work

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