5.1 The Limit and The Derivative

In Mathematics, a limit is defined as **a value that a function approaches (output) for the given input values.**

Example 1

Find $\lim_{x \to 2}(2x + 3)$.

$2(2)+3=7$

$\lim_{x \to 2}(2x+3) = 7$

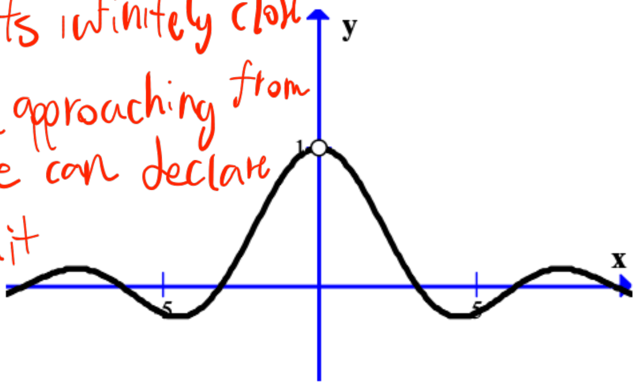
Example 2

Find $\lim_{x \to 0} \frac{\sin x}{x}$.

$\lim_{x \to 0^+} \frac{\sin x}{x} = 1$

$\lim_{x \to 0^-} \frac{\sin x}{x} = 1$

If the output gets infinitely close to given input when approaching from both sides, then we can declare that value the limit for as $x \to$

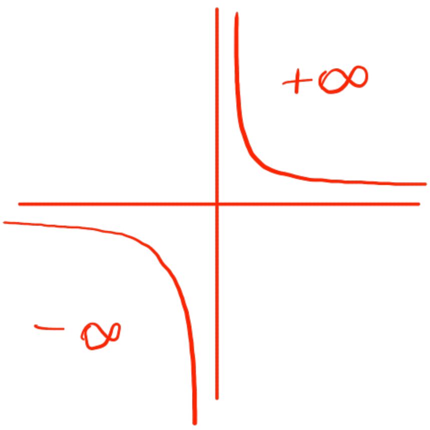


Therefore,

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

Example 3

Find $\lim_{x \to 0} \frac{1}{x}$.



$\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$\lim_{x \to 0^+} \frac{1}{x} = \infty$

Don't match, therefore the limit does not exist; $\lim_{x \to 0} \frac{1}{x}$ does not exist.

Example 4

Find $\lim_{x\to-\infty} \frac{x-3}{x-2}$ and $\lim_{x\to\infty} \frac{x-3}{x-2}$.

$$\lim_{x\to-\infty} \frac{x-3}{x-2} = 2$$

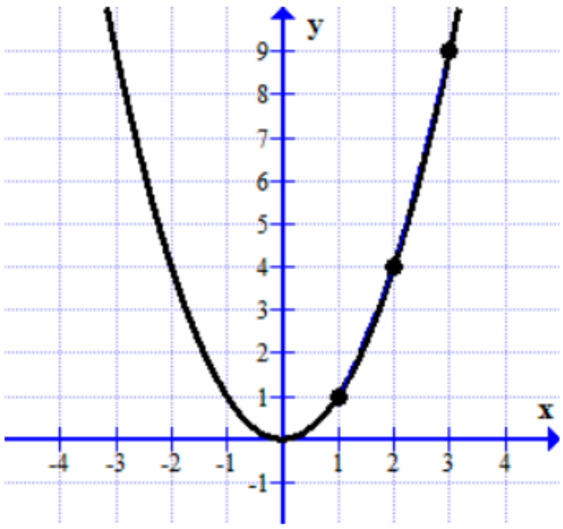
$$\lim_{x\to\infty} \frac{x-3}{x-2} = 2$$

RATE OF CHANGE (OR GRADIENT) IN A CURVE

In a curve which is not a straight line, the rate of change between any two points is not always the same.

Example 5

Consider $f(x) = x^2$.



a. Find the average rate of change from x = 1 to x = 2.

$$m = \frac{4-1}{2-1} = \frac{3}{1} = 3$$

b. Find the average rate of change from x = 1 to x = 3.

$$m = \frac{9-1}{3-1} = \frac{8}{2} = 4$$

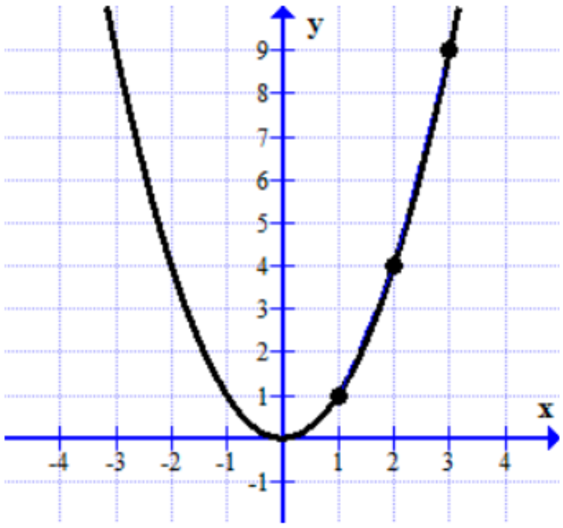
Limit Definition of Derivative (Slope at a Point)

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

What's actually happening when we use the power rule!

Example 6

Consider $f(x) = x^2$.



a. Find the slope at any point x.

$$\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = m$$

$$= \lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$$

$$= \lim_{h\to 0}\frac{2x\cancel{h}+h^{\cancel{2}}}{\cancel{h}}$$

$$= \lim_{h\to 0}(2x+h)$$

After subbing in 0 for h:

$$= 2x+0$$

$$m = 2x$$

- We can't sub in immediately otherwise the limit will be undefined, so we distribute to get the h out of the denominator.

b. Find the slope at x = 1.

$$m_{x=1} = 2(1) = 2$$

$\boxed{f'(x) = \text{DERIVATIVE} = \text{RATE OF CHANGE} = \text{GRADIENT at } x.}$

Trying for Second Derivative w/ formula:

$$\lim_{h\to 0} \frac{2(x+h)-2x}{h} = m$$

$$= \frac{\cancel{2x}+2xh-\cancel{2x}}{h}$$

$$= \frac{2x\cancel{h}}{\cancel{h}}$$

$$= 2x$$

Doesn't work!

Let $f(x) = x^3+3x+2$

$$\lim_{h\to 0} \frac{(x-h)^3+3(x-h)+2}{h} = m$$

$$\lim_{h\to 0} \begin{cases} = \dfrac{(x^2-2xh+h^2)(x-h)+3x-3h+2}{h} \\ = \dfrac{x^3-2x^2h+xh^2-x^2h+2xh^2-h^3+3x-3h+2}{h} \\ = \dfrac{x^3+3x+2}{h} + \dfrac{-2x^2\cancel{h}+xh^{\cancel{2}}-x^2\cancel{h}+2xh^{\cancel{2}}-h^{\cancel{3}\,2}}{\cancel{h}} \\ = \dfrac{x^3+3x+2}{h} - 2x^2+xh-x^2+2xh-h^2 \end{cases}$$

Subsitute zero for h:

$$m = -2x^2-x^2$$

- Either I'm doing something wrong or this formula has certain conditions to work.

5.2 Derivatives of Known Functions

The derivative of a function f(x) is a new function denoted by f′(x). As explained in the preceding section, f′(x) indicates the rate of change, or otherwise the gradient of f(x) at any particular point x.

Power Rule: $f'(x) = nx^{n-1}$.

Example 1

Show that the derivative of $\frac{1}{x^2}$ is $\frac{-2}{x^3}$.

$\frac{1}{x^2} = x^{-2}$ $\quad -2x^{-3} = \frac{-2}{x^3}$

$\frac{dy}{dx}(x^{-2}) = -2x^{-3}$ $\quad$ Therefore, $\frac{dy}{dx}\left(\frac{1}{x^2}\right) = \frac{-2}{x^3}$

Example 2

Show that the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$.

$\sqrt{x} = x^{\frac{1}{2}}$ $\quad$ Therefore, $\frac{dy}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

$\frac{dy}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

Example 3

Let $f(x) = x^7$. Find:

a. $f(0)$, $f(1)$, $f(2)$

$f(0) = 0$

$f(1) = 1$

$f(2) = 128$

b. $f'(x)$

$7x^6$

c. $f'(0), f'(1), f'(2)$

$f'(0) = 0$

$f'(1) = 7$

$f'(2) = 64 \cdot 7 = 420 + 28 = 448$

d. the rate of change of $f(x)$ at x = 2.

$= f'(2) = 448$

e. the gradient of $f(x)$ at x = 2.

$= f'(2) = 448$

- *NOTATION*

If $y=f(x)$, the derivative is denoted by the following symbols

$$y' \quad \text{or} \quad f'(x) \quad \text{or} \quad \frac{dy}{dx} \quad \text{or} \quad \frac{d}{dx}f(x)$$

The derivative at some specific value of x, say x=2, is denoted by

$$f'(2) \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=2}$$

The derivatives of the most common functions are shown below

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
c (constant)	0

Example 4

Find the derivative of each function.

a. $f(x) = x^5 + x^3$

$f'(x) = 5x^4 + 3x^2$

b. $f(x) = x^7 - e^x + \sin x - x + 5$

$f'(x) = 7x^6 - e^x + \cos x - 1$

Example 5

Find the derivative of each function.

a. $f(x) = 3\sin x$

$f'(x) = 3\cos x$

b. $f(x) = 7e^x$

$f'(x) = 7e^x$

c. $f(x) = 5x^3$

$f'(x) = 15x^2$

Example 6

Find the derivative of each function.

a. $f(x) = 2x^3 - 3x^2 + 7x + 5$

$f'(x) = 8x^2 - 6x + 7$

b. $f(x) = 5x^7 + 3\ln x - 7\cos x$

$f'(x) = 30x^6 + \frac{3}{x} + 7\sin x$

Example 7

Find the derivative of $f(x) = x^5 \sin x$.

$f'(x) = 5x^4 \sin x + x^5 \cos x$

Derivative for Multiplied Functions: $a' \cdot b + a \cdot b'$

Example 8

Find the derivative of $f(x) = \frac{x^3}{\sin x}$.

$$f'(x) = \frac{\sin x \cdot 3x^2 - x^3 \cos x}{\sin^2 x}$$

Derivative for Divided Functions: $\frac{\text{Low} \cdot d(\text{high}) - \text{high} \cdot d(\text{low})}{\text{low}^2}$

Example 9

Find the derivative of $f(x) = \frac{x^3 - 2x + 1}{x}$. $\rightarrow H$ $\rightarrow L$

$\frac{dy}{dx} H = 3x^2 - 2$

$\frac{dy}{dx} L = 1$

$$f'(x) = \frac{x(3x^2 - 2) - (x^3 - 2x + 1)(1)}{x^2}$$

$$f'(x) = \frac{3x^3 - 2x - x^3 - 2x + 1}{x^2}$$

$$f'(x) = \frac{2x^3 - 4x + 1}{x^2}$$

Example 10

Find the $f'^{(x)}, f''(x), f'''(x)$ for each function.

a. $f(x) = x^5$

$f'(x) = 5x^4$

$f''(x) = 20x^3$

$f'''(x) = 60x^2$

b. $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

c. $f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$f'''(x) = e^x$

Alternative notation:

$f''(x)$ can also be written as $\frac{d^2y}{dx^2}$ or $\frac{d^2}{dx^2}f(x)$

$f'''(x)$ can also be written as $\frac{d^3y}{dx^3}$ or $\frac{d^3}{dx^3}f(x)$

More Functions (HL only)

	$f(x)$	$f'(x)$
1	a^x	$a^x \cdot \ln a$
2	$\log_a x$	$\dfrac{1}{x \ln a}$

	$f(x)$	$f'(x)$
3	$\tan x$	$\dfrac{1}{\cos^2 x} = \sec^2 x$
4	$\sec x$	$\sec x \tan x$
5	$\csc x$	$-\csc x \cot x$
6	$\cot x$	$-\csc^2 x$

	$f(x)$	$f'(x)$
7	$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
8	$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}$
9	$\arctan x$	$\dfrac{1}{1+x^2}$

Example 11

Find the derivative of $f(x) = 3^x$ and $g(x) = \log_5 x$.

$$\frac{df}{dx} = 3^x \ln 3$$

$$\frac{dg}{dx} = \frac{1}{x \ln 5}$$

$$a^x = e^{\ln a^x}$$ ①

$$= e^{x\ln a}$$

$$\frac{d}{dx}\left(e^{x\ln a}\right) = e^{x\ln a} \cdot (\ln a) = e^{x\ln a} \cdot \ln a$$

$= a^x \cdot \ln a$. Therefore, $\frac{d}{dx}(a^x) = a^x \ln a$

$$\log_a x = \frac{\log_e x}{\log_e a} = \frac{\ln x}{\ln a}$$ ②

$$\frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{\ln a\left(\frac{1}{x}\right) - \ln x\left(\frac{1}{a}\right)}{(\ln a)^2} = \frac{\frac{1}{x}\ln a - \frac{1}{a}\ln x}{(\ln a)^2}$$

Find the Derivative of each of the following:

a. $f(x) = 3^x \tan x$

$(3^x \ln 3)(\tan x) + (3^x)(\sec^2 x)$

$= 3^x \ln 3 \tan x + 3^x \sec^2 x$

b. $g(x) = \dfrac{\arcsin x}{2x+3}$

$$= \frac{(2x+3)\left(1 - \frac{1}{\sqrt{1-x^2}}\right) - \arcsin x(2)}{(2x+3)^2}$$

c. $h(x) = \sec 2x$

$= \sec 2x \tan 2x$

d. $k(x) = \arctan(x^2)$

$$= \frac{1}{1+(x^2)^2} = \frac{1}{1+x^4}$$

Show that the derivative of $f(x) = x^x$ is $f'(x) = x^x(\ln x + 1)$.

$x^x = e^{\ln x^x} = e^{x\ln x}$

$$\frac{d}{dx}\left(e^{x\ln x}\right) = e^{x\ln x} \cdot \left[\left(x \cdot \frac{1}{x}\right) + (1)\ln x\right] = e^{x\ln x} \cdot (1 + \ln x) = x^x(\ln x + 1)$$

Find the equation of the tangent to the curve of $f(x) = x\arctan x$ at the point $(1, \frac{\pi}{4})$. Express the answer in the form $y = mx + c$.

$f'(x) = x\left(\frac{1}{1+x^2}\right) + \arctan x = \frac{x}{1+x^2} + \arctan x$

$f'(1) = \frac{1}{1+1^2} + \arctan(1) = \frac{1}{2} + \frac{\pi}{4} = \frac{2+\pi}{4} = \frac{\pi+2}{4}$

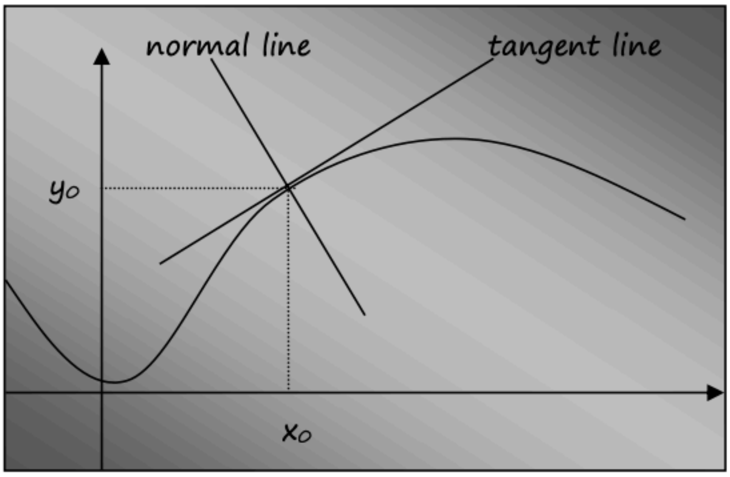
$f(\frac{\pi}{4}) = \frac{\pi}{4}\arctan(\frac{\pi}{4}) = \frac{\pi}{4}$

$y - \frac{\pi}{4} = \frac{\pi+2}{4}(x-1)$

5.3 Tangent and Normal Lines

TANGENT LINE at x_0:
the line with gradient m_T which passes through (x_0, y_0)

NORMAL LINE at x_0:
the perpendicular line to the tangent at (x_0, y_0).
Its gradient is $m_N = -\frac{1}{m_T}$



Example 1 The derivate is the slope of the tangent at any input of a function on a graph.

Consider the function $f(x) = x^2$.

Find the equations of the tangent line and the normal line at x = 3.

$\frac{df}{dx} = 2x$ $\quad f(3) = 3^2 = 9$

$2(3) = 6$ $\quad$ Tangent $= y - 9 = 6(x-3)$

$n = -\frac{1}{6}$ $\quad$ Normal $= y - 9 = -\frac{1}{6}(x-3)$

Example 2

Consider the function $f(x) = 5x^3 - 2x + 1$.

Find the equations of the tangent lines to the curve which are parallel to the line $y = 13x + 8$.

$\frac{df}{dx} = 15x^2 - 2$ → Slope

$f(1) = 5 - 2 + 1 = 4$

$f(-1) = -5 + 2 + 1 = -2$

Tangent $= y - 4 = 13(x-1)$

Normal $= y + 2 = 13(x+1)$

$15x^2 - 2 = 13$

$15x^2 - 15 = 0$

$15x^2 = 15, \; x^2 = 1, \; x = \pm 1$

Example 3

Consider the function $f(x) = x^2 - 4x + 5$.

Find the equations of the tangent line and the normal line at x = 2.

$\frac{dy}{dx} = 2x - 4$ $\quad f(2) = 2^2 - 4(2) + 5 = 1 \rightarrow (2,1)$

$2(2) - 4 = 0$ $\quad$ Tangent: $y = 1$

$m = 0$ $\quad$ Normal: $x = 2$

Example 4

The line $y = mx - 3$ is tangent to the curve $f(x) = x^4 - x$. Find m.

$\frac{df}{dx} = 4x^3 - 1$

$4x^{\cancel{3}} - 1 = m\cancel{x} - 3$

$4x^2 + 2 = m$

Example 5

Consider the function $f(x) = x^4 - x$. Find the tangent lines passing through the point (0, -3).

$\frac{df}{dx} = 4x^3 - 1$

$4(0)^3 - 1 = -1$

Tangent Line passing through Point $(0,3) = m = -1$

5.4 The Chain Rule

Example 1

Find the derivative of $f(x) = \sin(2x^2 + 3)$.

Example 2

Find the derivative of $f(x) = e^{5x+3}$.

Example 3

Find the derivative of $f(x) = \ln(x^2 + 4)$.

Example 4

Find the derivative of $f(x) = \sqrt{3x^2 + 5x + 2}$.

Example 5

Find the derivative of $f(x) = (2x^2 + 3)^2$.

Example 6

Find the derivative of $f(x) = \cos^5 x$ and $g(x) = \frac{1}{\sin x}$.

Example 7

Find the derivative of each of the following:

a. $f(x) = \ln(\sin(3x + 1))$

b. $f(x) = 3\sin^5(x^2 + 1)$

c. $f(x) = e^{\sin 3x}$

d. $f(x) = \sqrt{\sin^2 x + \sin 2x}$

Example 8

Find the derivative of $f(x) = e^{2x} \sin 3x$.

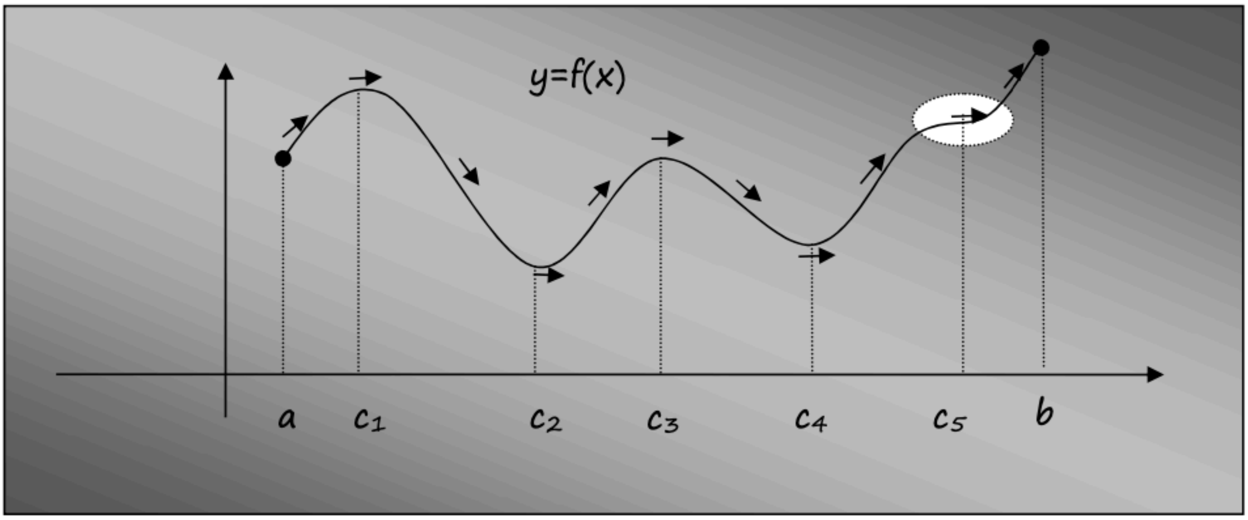
Example 9

Find the derivative of $f(x) = e^{x^2 \sin x}$.

5.5 Monotony (Maximum and Minimum)

- INCREASING – DECREASING FUNCTIONS (MONOTONY)

Consider the following graph



Let us make some observations:

- The domain of the graph is the interval $[a,b]$
  The points $x=a$ and $x=b$ are called **endpoints**
- We say that we have a **local max (**or just **max)** at points:
  $$x=c_1,\ x=c_3,\ x=b$$
  (can you explain why?)
- We say that we have a **local min** (or just **min**) an points:
  $$x=a,\ x=c_2,\ x=c_4$$
  (can you explain why?)
  All these points (max and min) are called ***turning points*** or ***extreme values***
- Notice that $x=c_5$ is not a turning point (neither max nor min), as near $f(c_5)$ you can find smaller as well as larger values.
- The function is **increasing** (goes up) in the interval (a,c_1)
- The function is **decreasing** (goes down) in the interval (c_1,c_2)
- The function is **increasing** (goes up) in the interval (c_2,c_3)
- The function is **decreasing** (goes down) in the interval (c_3,c_4)
- The function is **increasing** (goes up) in the interval (c_4,b)

Remember though that
A +tive gradient means that the function is increasing (goes up)
A –tive gradient means that the function is decreasing (goes down).

But we know that

derivative = gradient

In other words

If $f'(x)>0$ then f is increasing (↗)
If $f'(x)<0$ then f is decreasing (↘)

Notice: The increasing or decreasing behavior of a function is also known as **monotony**!

- TURNING POINTS: MAX – MIN

How can we find the turning points (max or min) of a function?

First, the end points are extreme values (see a and b above).

As far as the interior points is concerned, observe that the gradient at any turning point is 0 (the tangent lines at those values are horizontal!).

PROPOSITION:
If $f(x)$ has a turning point (max or min) at some interior point c and $f'(c)$ exists, then

$$f'(c)=0$$

Notice that $f'(x)=0$ at c_1, c_2, c_3, c_4 and c_5
We have a local max at c_1, c_3 and a local min at c_2, c_4.
However, c_5 is not a turning point.
Hence the inverse proposition is not true.

Therefore, apart from the endpoints, the possible turning points (max/min) are the following

- points x where $f'(x)=0$ (called **stationary points**)
- points where $f'(x)$ does not exist

All these points are also called **critical points.**

Here we only deal with <u>**stationary points**</u> (points where $f'(x)=0$).

To verify whether such a stationary point $x=c$ is a turning point (max or min) we perform the following test

---

<u>**FIRST DERIVATIVE TEST for c**</u>

Check the sign of $f'(x)$ to verify if the function is increasing or decreasing just before and after c:

<table>
<tr><th>x</th><th colspan="2">c</th></tr>
<tr><th>f ′(x)</th><th>+</th><th>−</th></tr>
<tr><td rowspan="2">Conclusion for f</td><td>↗</td><td>↘</td></tr>
<tr><td colspan="2">max</td></tr>
</table>

<table>
<tr><th>x</th><th colspan="2">c</th></tr>
<tr><th>f ′(x)</th><th>−</th><th>+</th></tr>
<tr><td rowspan="2">Conclusion for f</td><td>↘</td><td>↗</td></tr>
<tr><td colspan="2">min</td></tr>
</table>

If the sign does not change, we have neither a max nor a min.

♦ METHODOLOGY

Given $y=f(x)$

Step 1 we find $f'(x)$

Step 2 we solve $f'(x)=0$ (say that roots are a,b,c)

Step 3 we construct a table as follows to perform the first derivative test

<table>
<tr><th>x</th><th colspan="4">a b c</th></tr>
<tr><th>f ′(x)</th><th>+</th><th>−</th><th>+</th><th>+</th></tr>
<tr><td rowspan="2">Conclusion for f</td><td>↗</td><td>↘</td><td>↗</td><td>↗</td></tr>
<tr><td colspan="4">max min nothing</td></tr>
</table>

Example 1

Find the minimum and maximum values of $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 5$ using the first derivative test.

$$\frac{df(x)}{dx} = x^2 - 4x + 3$$

$$0 = (x-3)(x-1)$$

$x = 3, x = 1$

If $x < 1$: $(0-3)(0-1) = 3$ $+$

If $1 < x < 3$: $(2-3)(2-1) = -1$ $-$

If $x > 3$: $(10-3)(10-1) = 63$ $+$

Max: $x = 1$, Min: $x = 3$

An alternative way to check if a stationary point is a max or a min is the following:

SECOND DERIVATIVE TEST for c

Find $f''(x)$ (if it exists!)

If $f''(c) > 0$ then c is a min

If $f''(c) < 0$ then c is a max

If $f''(c) = 0$ we don't get an answer. We go back to the first derivative test.

Example 2

Find the minimum and maximum values of $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 5$ using the second derivative test.

$$\frac{df(x)}{dx} = x^2 - 4x + 3 \longrightarrow 0 = (x-3)(x-1)$$

$x = 3, x = 1$

$$\frac{d^2f(x)}{dx^2} = 2x - 4$$

$2(3) - 4 = 2$; $2(1) - 4 = -2$

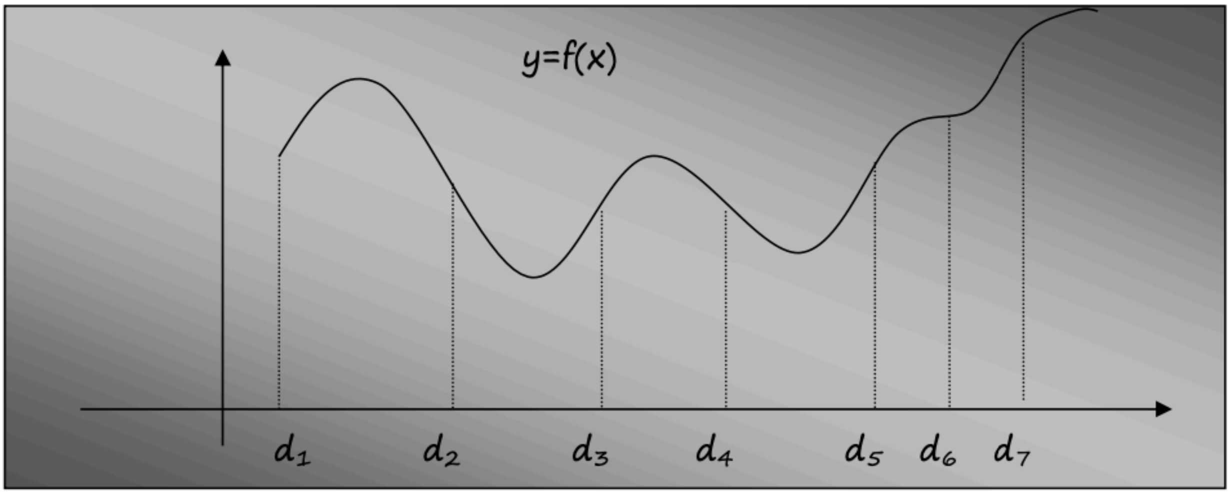
$2 > 0$ so min when $x = 3$

-2 is < 0 so max when $x = 1$

5.6 Concavity (Points of Inflection)

- *CONCAVITY*

Consider again the graph of the preceding section



Our concern now is different! It is to investigate the intervals where the curve

looks like (∪) : we say that the function is ***concave up***

looks like (∩) : we say that the function is ***concave down***[2]

We observe that:

- *The function is* ***concave down*** *(∩) in the interval* (d_1,d_2)
- *The function is* ***concave up*** *(∪) in the interval* (d_2,d_3)
- *The function is* ***concave down*** *(∩) in the interval* (d_3,d_4)
- *The function is* ***concave up*** *(∪) in the interval* (d_4,d_5)
- *The function is* ***concave down*** *(∩) in the interval* (d_5,d_6)
- *The function is* ***concave up*** *(∪) in the interval* (d_6,d_7)

- *The concavity changes at the points* $x=d_2, d_3, d_4, d_5, d_6, d_7$

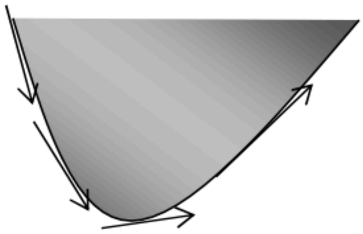
These points are called ***points of inflexion*** *or* ***inflexion points***

It is easy to verify the concavity by using the second derivative

If $f''(x)>0$	then	f is concave up (∪)
If $f''(x)<0$	then	f is concave down (∩)

Short explanation (mainly for HL)

Look at the curve of the following concave up function f(x)



The gradient is -tive in the beginning, it is "less" -tive as we move forward, it sometimes becomes 0 and then becomes +tive and "more" +tive as we move forward. In other words the gradient increases. That is, the function of the gradient $f'(x)$ is increasing. But we know that the derivative of an increasing function is +tive. Hence, the derivative of $f'(x)$, that is $f''(x)$ is +tive!

Similarly, if the function f(x) is concave down, the second derivative must be -tive!

- POINTS OF INFLEXION

How can we find the points of inflexion?

Since the concavity changes at such a point, the sign of $f''(x)$ changes from + to – or vice-versa. Therefore, the second derivative at any point of inflexion must be 0.

PROPOSITION:

If f(x) has a point of inflexion at some point d and $f''(d)$ exists, then

$$f''(d)=0$$

Notice again that the equation $f''(x)=0$ gives us the possible points of inflexion. To verify if x=d is indeed a point of inflexion we must check the sign of $f''(x)$ just before and after that point.

- *METHODOLOGY*

Given	$y=f(x)$
Step 1	we find $f'(x)$ and $f''(x)$
Step 2	we solve $f''(x)=0$ (say that roots are a,b,c)
Step 3	we construct a table as follows

x		a		b		c	
$f''(x)$	+		−		+		+
Conclusion for f	∪		∩		∪		∪
		i.p.		i.p.		nothing	

Example 1

Consider again $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 5$. Find the point of inflection.

$f'(x) = x^2 - 4x + 3$

$f''(x) = 2x - 4$

$0 = 2x - 4$

$x = 2$

If $x < 2$:

$2(0) - 4$

$= -4$ $-$

If $x > 2$:

$2(10) - 4 = 16$

$+$

Concave up

Concavity changes at $x=2$, therefore $x=2$ is an inflection point

Example 2

Consider the function $f(x) = xe^x$. Find the possible maximum, minimum, and points of inflection.

None exists

$f'(x) = xe^x + e^x = e^x(x+1)$

$0 = e^x(x+1)$, $x = -1$

$f''(x) = xe^x + e^x + e^x = xe^x + 2e^x = e^x(x+2)$

$e^{(-1)}(-1+2) = \frac{1}{e}$

$\frac{1}{e} > 0$, therefore, min @ $x = -1$.

$0 = e^x(x+2)$

$x = -2$

If $x < -2$:

$e^{-3}(-3+2) = \frac{-1}{e^3}$ $-$

If $x > -2$:

$e^0(0+2) = 2$ $+$

POI @ $x = -2$;

Concave up.

5.7 Optimization

In problems of optimization, we have to construct a function in terms of some variable x, and then we use derivatives to find the "optimum" solution, that is the maximum or the minimum value of the function.

Example 1

Among all the rectangles of perimeter 20, find the one of the maximum area.

$P = 2L + 2W$

$L = \frac{1}{2}(P - 2W)$

$A = LW = (\frac{1}{2}P - W)W$

$A = (\frac{1}{2}(20) - W)W$

$A = 10W - W^2$

$\frac{dA}{dW} = 10 - 2W$

$0 = 10 - 2W$

$2W = 10;\ W = 5.$

Max area when $L = 5$ & $W = 5$.

Example 2

Among all the rectangles of area 25, find the one of the minimum perimeter.

$P = 2W + 2L$

$A = LW$

$W = \frac{A}{L}$

$P = 2(\frac{A}{L}) + 2L$

$P = 2(\frac{25}{L}) + 2L$

$P = \frac{50}{L} + 2L$

$\frac{dP}{dL} = \frac{-50}{L^2} + 2$

$0 = \frac{-50}{L^2} + 2$

$\frac{50}{L^2} = \frac{2}{1}$

$2L^2 = 50$

$L^2 = 25$

$L = 5$

$W = \frac{25}{5} = 5$

Min perimeter when $L = 5$ & $W = 5$.

• Should normally confirm that it's the min and not a max or something.

Example 3

We want to construct a rectangle fence for an area of 24 m², but the cost for the material of the front side is \$10 per meter while the cost for the material of the other 3 sides is \$5 per meter. Find the cheapest solution.

$15L + 10W = C$

$A = LW$

$W = \frac{A}{L}$

$15L + 10(\frac{24}{L}) = C$

$15L + \frac{240}{L} = C$

$\frac{dC}{dL} = 15 - \frac{240}{L^2}$

$0 = 15 - \frac{240}{L^2}$

$\frac{240}{L^2} = \frac{15}{1};\ 15L^2 = 240$

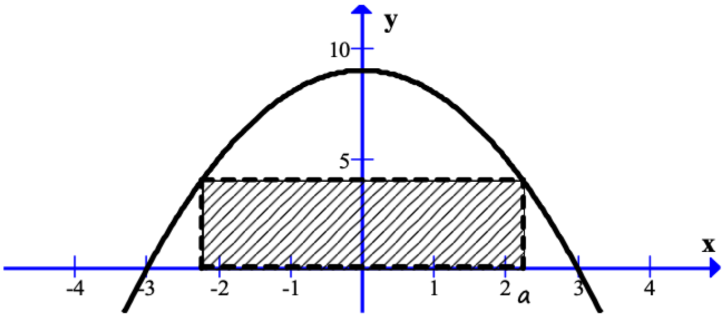
$L = 4$

$W = \frac{24}{4} = 8.$

Cheapest cost when $L = 4$ & $W = 8$.

Example 4

Consider the region enclosed by $y = 9 - x^2$ and the x-axis. Find the rectangle of the largest area inscribed within that region.



$A=(2a)(9-a^2)$

$A=18a-2a^3$

$\frac{dA}{da}=-6a^2+18$

$0=-6a^2+18$

$6a^2=18$

$a^2=3$

$a=\sqrt{3}$

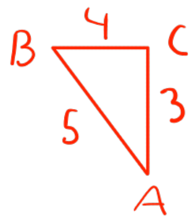
$A=[2(\sqrt{3})](9-(\sqrt{3})^2)$

$A=2\sqrt{3}(6)$

$A_{max}=12\sqrt{3}$

Example 5

A swimmer is at point A inside the sea, 3 km away from the beach. She wants to go to point B at the beach, which is 5 km away. When she swims, she covers 1 km in 30 minutes. When she runs, she covers 1km in 15 minutes. Find:



a) the time she spends when she swims directly to B;

$5 \cdot 30 = 150$ minutes

b) the time she spends when she swims to C and then runs to B;

$3 \cdot 30 + 4 \cdot 15 = 60 + 60 = 120$ minutes

c) the minimum time she can achieve if she swims first to some point D between B and C and then runs to B

5.8 The Indefinite Integral

In general,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \qquad (\text{if } n \neq -1)$$

If we remember the derivatives of the basic functions we obtain the following results:

$f(x)$	$\int f(x)dx$	
x^n	$\frac{x^{n+1}}{n+1}$	+ c
a	ax	
e^x	e^x	
$\sin x$	$-\cos x$	
$\cos x$	$\sin x$	
$\frac{1}{x}$	$\ln x$	

NOTICE: The operation of finding the integral is called ***integration***.

- REMARK FOR $\int \frac{1}{x} dx$ (**only for HL**)

In fact $\boxed{\int \frac{1}{x} dx = \ln|x| + c}$.

Example 1

a. $\int(3x^2 + 5e^x - 2\cos x \, dx)$

b. $\int(2x^4 + 8x^3 - 5x^2 + 7x + 2)\, dx$

c. $\int\left(\frac{2}{x^4} + \frac{8}{x^3} - \frac{5}{x^2} + 2\right) dx$

Example 2

Let $f'(x) = 6x^2 - 4x + 5$. Find $f(x)$ given that $f(1) = 8$.

$\int 6x^2 - 4x + 5 \, dx$

$f(x) = 2x^3 - 2x^2 + 5x + C$

$8 = \cancel{2(1)^3} - \cancel{2(1)^2} + 5(1) + C$

$C = 3$

$f(x) = 2x^3 - 2x^2 + 5x + 3$

- TWO MORE BASIC INTEGRALS (**only for HL**)

In the IB Math HL formula booklet you will also find the formulas

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\frac{x}{a}+c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a}+c$$

Example 3

a. $\int \frac{1}{1+x^2}dx$

$\frac{1}{1}\arctan\frac{x}{1}+C$

$\arctan x+C$

b. $\int \frac{1}{4+x^2}dx$

$\frac{1}{2}\arctan\frac{x}{2}+C$

c. $\int \frac{5}{13+x^2}dx$

$5\int \frac{1}{13+x^2}dx$

$5\left(\frac{1}{\sqrt{13}}\arctan\frac{x}{\sqrt{13}}\right)$

$\frac{5}{\sqrt{13}}\arctan\frac{x}{\sqrt{13}}+C$

d. $\int \frac{5}{9+4x^2}dx$

$\frac{5}{4}\int \frac{1}{\frac{9}{4}+x^2}dx$

$\frac{5}{4}\left(\frac{2}{3}\arctan\frac{2x}{3}\right)=\frac{5}{6}\arctan\frac{2x}{3}+C$

e. $\int \frac{1}{\sqrt{1-x^2}}dx$

$\arcsin x+C$

f. $\int \frac{1}{\sqrt{4-x^2}}dx$

$\arcsin\frac{x}{2}+C$

g. $\int \frac{5}{\sqrt{13-x^2}}dx$

$5\arcsin\frac{x}{\sqrt{13}}+C$

h. $\int \frac{5}{\sqrt{9-4x^2}}dx$ Factor out $\sqrt{4}$, not 4.

$\frac{5}{4}\arcsin\frac{4x}{9}+C$

$\frac{5}{2}\arcsin\frac{2x}{3}+C$

5.9 Integration by Substitution

Example 1

a. $\int e^{3x+2} dx$

b. $\int \sin(2x+1)\,dx$

c. $\int \cos(5-x)\,dx$

d. $\int \frac{1}{7x+3}\,dx$

e. $\int (3x+5)^5 dx$

f. $\int \frac{1}{(3x+5)^5}\,dx$

g. $\int \sqrt{3x+5}\,dx$

Example 2

Find $\int \cos^2 x \, dx$ by using the double angle formula.

Example 3

Find $\int \cos(3x + 5) \, dx$.

Example 4

Find $\int 3x^2(x^3 + 5)^7 dx$.

Example 5

Find $\int x\sqrt{x^2 + 3}dx$.

Example 6

Find $\int \frac{2x + \cos x}{x^2 + \sin x}\,dx$.

Example 7

Find $\int \frac{(\ln x)^2}{x}\,dx$.

Example 8

Find $\int \sin x e^{\cos x}\,dx$.

Example 9

Find $\int x \cos(3x^2 + 1)\,dx$.

Example 10

Find $\int \frac{5x^2}{x^3 + 1}\,dx$.

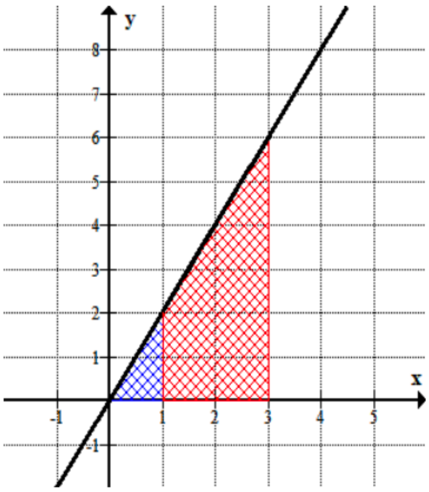
5.10 The Definite Integral and Area Between Curves

Example 1

a. $\int_{-1}^{1}(8x^3+12x^2-6x+3)dx$ b. $\int_{2}^{3}\frac{8}{x^3}dx$

- *GEOMETRICAL INTERPRETATION*

Consider the graph of the straight line f(x)=2x



The area under the line between 0 and 1 (blue triangle) is $\frac{1\cdot 2}{2}=1$.

but also $\int_0^1 2xdx=\left[x^2\right]_0^1=(1)-(0)=1$.

The area under the line between 0 and 3 (big triangle) is $\frac{3\cdot 6}{2}=9$.

but also $\int_0^3 2xdx=\left[x^2\right]_0^3=(9)-(0)=9$.

The area under the line between 1 and 3 (red region) is 9-1=8.

but also $\int_1^3 2xdx=\left[x^2\right]_1^3=(9)-(1)=8$.

This is not an accident!

In general, if y=f(x) is a "continuous" curve above the x-axis, then the area between the curve and the x-axis, from x=a to x=b is equal to the definite integral

$$\boxed{\int_a^b f(x)dx}$$

Example 2

Suppose that $\int_0^5 f(x)dx = 10$. It is also given that $f(0) = 15$ and $f(5) = 3$.

Find the following:

a. $\int_0^5 2f(x)dx$

b. $\int_5^0 f(x)dx$

c. $\int_5^0 \frac{1}{2} f(x)dx$

d. $\int_0^5 (f(x) + 4x)dx$

e. $\int_0^5 (2f(x) + 1)dx$

f. $\int_0^2 f(x)dx + \int_2^5 f(x)dx$

g. $\int_0^5 f'(x)dx$

Example 3

Find $\int_0^2 \frac{x}{x^2+4}\,dx$.

Example 4

Suppose that $\int_0^5 f(x)dx = 10$. Find:

a. $\int_3^8 f(x-3)dx$

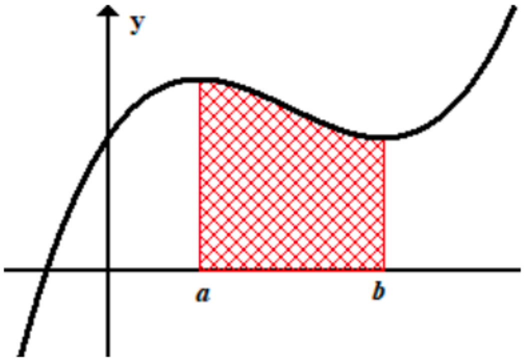
b. $\int_0^{2.5} f(2x)dx$

♦ AREA BETWEEN CURVES

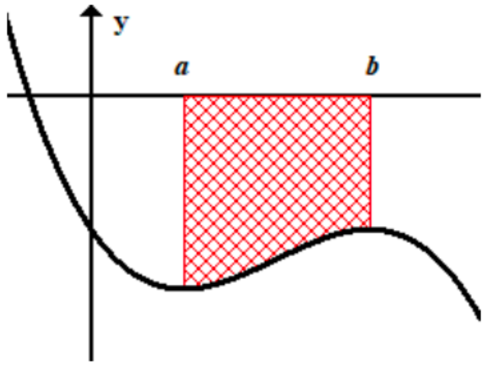
In fact, the definite integral $\int_a^b f(x)dx$ is either positive or negative:

If y=f(x) is **above** the x-axis the result is **positive**.

If y=f(x) is **below** the x-axis the result is **negative**



$\int_a^b f(x)dx$ = **(red area)**



$\int_a^b f(x)dx$ = **– (red area)**

Example 5

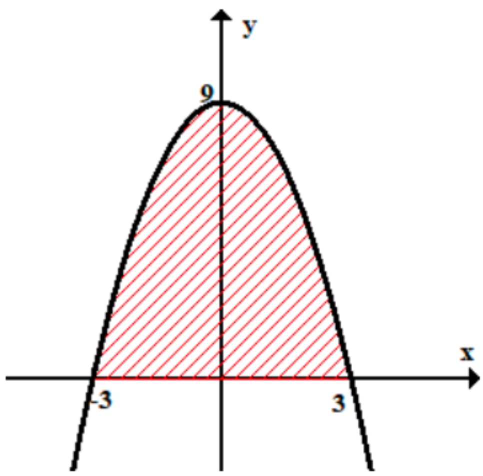
Find the area of the region between the curve $y = 9 - x^2$ and the x-axis.

$\int_{-3}^{3}(9-x^2)dx = \left[9x - \frac{1}{3}x^3\right]_{-3}^{3}$

$\left[9(3) - \frac{1}{3}(3)^3\right] - \left[9(-3) - \frac{1}{3}(-3)^3\right]$

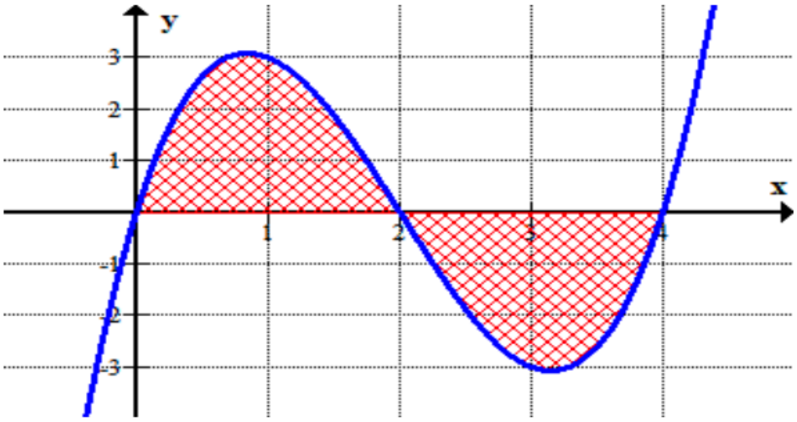
$27 - 9 + 27 - 9$

$18 + 18 = \boxed{36}$



Example 6

Consider the function $f(x) = x^3 - 6x^2 + 8x$.



Find:

a. $\int_0^2 f(x)dx$

$\left[\frac{1}{4}x^4 - 2x^3 + 4x^2\right]_0^2$

$\frac{1}{4}(2)^4 - 2(2)^3 + 4(2)^2 - (0)$

$4 - \cancel{16} + \cancel{16} = \boxed{4}$

b. $\int_2^4 f(x)dx$

$\left[\frac{1}{4}x^4 - 2x^3 + 4x^2\right]_2^4$

$\cancel{\frac{1}{4}}(4)^{\cancel{4}\,3} - 2(4)^3 + 4(4)^2 - \left(\frac{1}{4}(2)^4 - 2(2)^3 + 4(2)^2\right)$

$\cancel{64} - \cancel{128} + \cancel{64} - 4 + \cancel{16} - \cancel{16}$

$= \boxed{-4}$

c. $\int_0^4 f(x)dx$

$0 - 0 = 0$

d. the total area between the curve and the x-axis within [0, 4].

abs

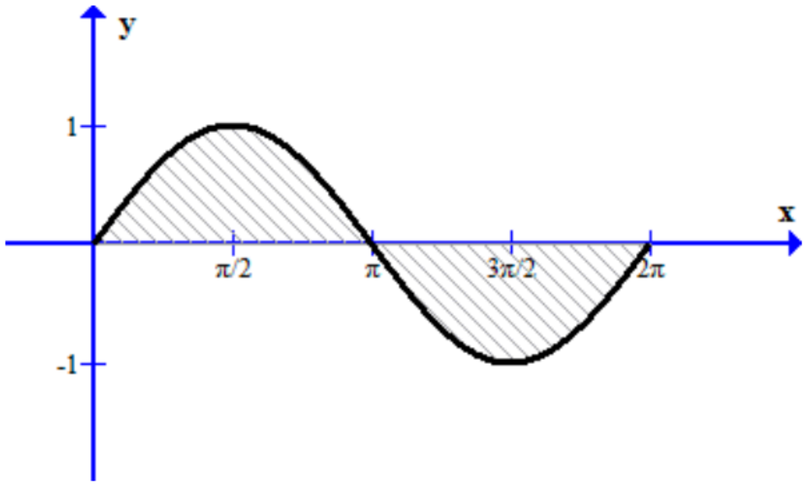
$\left|\int_0^2 f(x)dx\right| + \left|\int_2^4 f(x)dx\right|$

$|4| + |-4|$

$4 + 4 = \boxed{8}$

Example 7

Consider the function $f(x) = \sin x\,, 0 \leq x \leq 2\pi$.



Find:

a. $\int_0^{\pi} f(x)dx$

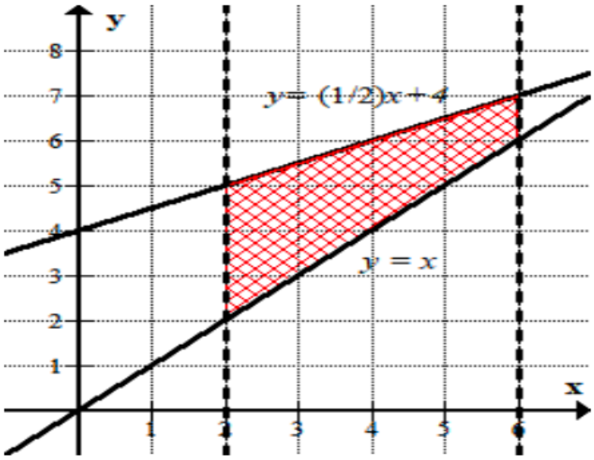
b. $\int_{\pi}^{2\pi} f(x)dx$

c. $\int_0^{2\pi} f(x)dx$

d. the total area between the curve and the x-axis within [0, 4].

Example 8

Find the shaded area below:

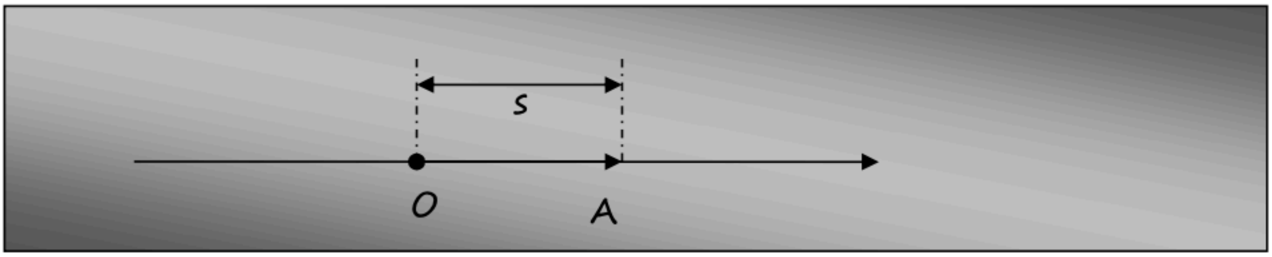


Example 9

Find the area enclosed by the graphs $f(x) = x^2$ and $g(x) = x + 2$.

5.11 Kinematics (Displacement, Velocity, Acceleration)

Consider a body moving along a straight line. The body is free to move forwards and backwards. The **displacement s** is the distance $|OA|$ from a fixed point O (the origin).



The displacement s is given as a function of time. For example

$$s=t^2-4t+3$$

means that

at time t=0 the displacement is 3 units from the fixed point O
at time t=1 the displacement is 0, (the body goes back to point O)
at time t=2 the displacement is -1, (the body is before point O)
at time t=3 the displacement is 0, (at point O again)
at time t=4 the displacement is 3, (it is moving forward)

Then

Velocity = rate of change of displacement: $\boxed{v=\frac{ds}{dt}}$

Acceleration = rate of change of velocity: $\boxed{a=\frac{dv}{dt}}$

Notice also that **a** is the second derivative of s. $\boxed{a=\frac{d^2s}{dt^2}}$

Displacement	Velocity	Acceleration
s	**v**	**a**

$$\xrightarrow{\textbf{derivative}}$$

Example 1

Consider $s = t^3 - 12t + 15$ representing the motion of a particle along a straight line, where displacement s is given in meters and the time t is given in seconds.

Find displacement, velocity, and acceleration at t = 1.

NOTICE

If s>0, the body is to the right of the fixed point O.

If s<0, the body is to the left of the fixed point O.

If s=0, the body is at the fixed point O.

If v>0, the body is moving to the right

If v<0, the body is moving to the left

If v=0, the body is stationary (it changes direction)

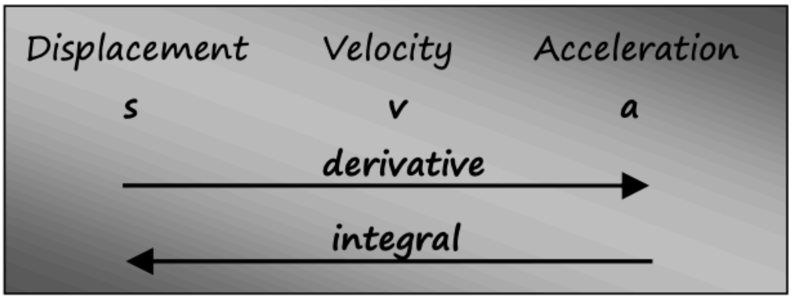
If a>0, the body accelerates

If a<0, the body decelerates

If a=0, the velocity is stationary (probably maximum speed)

♦ *GOING BACKWARDS (BY USING INTEGRATION)*

If we are given the acceleration of a moving body we can find the velocity and the displacement by using integration.



However, in this opposite direction we must be given some extra information in order to estimate any emerging constant c.

Example 2

Let $v = 12t^2 - 2t$. Find the acceleration and the displacement, given that the initial displacement is 5 m.

Example 3

Let $a = 12t$. Find the displacement, given that the moving body starts from rest and the initial displacement is 5 m.

♦ DISPLACEMENT vs DISTANCE TRAVELLED

Suppose that the velocity v is given in terms of t. Then

Displacement from O	$s = \int v\,dt$
Displacement from t_1 to t_2	$S = \int_{t_1}^{t_2} v\,dt$
Distance travelled from t_1 to t_2	$d = \int_{t_1}^{t_2} \lvert v \rvert\,dt$

Example 4

The velocity of a moving body is given in ms^{-1} by $v = 12 - 3t^2$. The initial displacement from a fixed point 0 is 1 m. Find:

a. the displacement from 0.

b. the displacement in the first 2 seconds.

c. the distance travelled in the first 3 seconds.

Example 5

Let $v = 4t - t^2$. Given that the initial displacement is 10 m, find the displacement and the distance travelled:

a. during the first 3 seconds.

b. during the first 6 seconds.

5.12 Continuity and Differentiability (HL only)

We say that a function is **continuous** at x=a, when

- The value f(a) exists;
- The limit $\lim_{x \to a} f(x)$ exists;
- $\lim_{x \to a} f(x) = f(a)$

∘ Find function value
∘ Find limit value
∘ If they match, continuous, if not, no.

For the continuity at any particular point we must check all three presuppositions.

Example 1

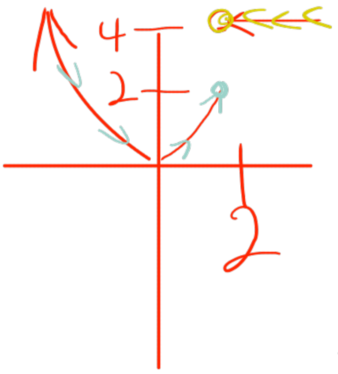
Determine if the function $f(x) = \begin{cases} 2x+1 & if\ x \neq 2 \\ 7 & if\ x = 2 \end{cases}$ is continuous.

$2(2)+1=5$, not 7 ∴ (2, 7) is not part of the line.

$f(2)=7$
$\lim_{x\to 2} f(x) = 5$ } Therefore, f(x) is discontinuous because the function value doesn't match the limit value.

Example 2

Determine if the function $f(x) = \begin{cases} x^2 & if\ x \leq 2 \\ 5 & if\ x > 2 \end{cases}$ is continuous.



$f(2) = (2)^2 = 4$

$\lim_{x\to 2^+} f(x) = 5$

$\lim_{x\to 2^-} f(x) = 4$

Example 3

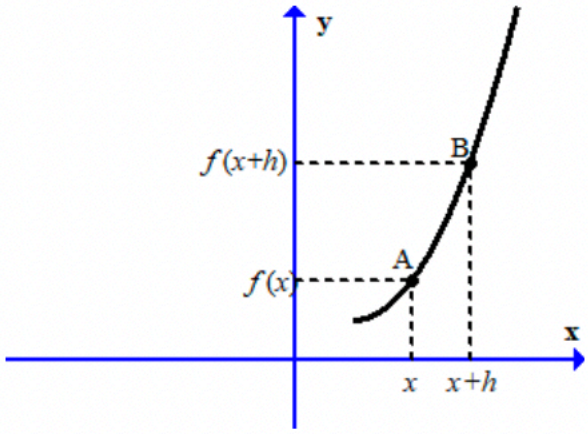
Determine if the function $f(x) = \begin{cases} x^2 & if\ x \leq 2 \\ 4 & if\ x > 2 \end{cases}$ is continuous.

$f(2) = (2)^2 = 4$ } Limit and function output match at $x=2$, therefore the graph is continuous at $x=2$.

$\lim_{x\to 2^+} f(x) = 4$
$\lim_{x\to 2^-} f(x) = 4$ } Approaches the same from both sides, so the limit exists.

- THE FORMAL DEFINITION OF THE DERIVATIVE

Let $y=f(x)$ be a continuous curve and $A(x,f(x))$ some point on it:



We select a neighboring point B with

x-coordinate = $x+h$ (where h is very small)

y-coordinate $=f(x+h)$

As we move from point A to point B, the rate of change is

$$\frac{\Delta y}{\Delta x}=\frac{f(x+h)-f(x)}{h}$$

If we let h become very small, that is $h\to 0$, the result will be the rate of change at point A, that is the derivative $f'(x)$.

$$\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

Example 4

Show from first principles that the derivative of the function $f(x)=x^3+2x$ is $f(x)=3x^2+2$.

$$f'(x)=\lim_{h\to 0}\frac{[(x+h)^3+2(x+h)]-(x^3+2x)}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{x^3}+3x^2\cancel{h}+3xh^{\cancel{2}}+h^{\overset{2}{\cancel{3}}}+\cancel{2x}+2\cancel{h}-\cancel{x^3}-\cancel{2x}}{\cancel{h}}$$

$$=\lim_{h\to 0}3x^2+3xh+h^2+2$$

$$=3x^2+2$$

Example 5 Differentiable if $\frac{df}{dx}$ from the left $= \frac{df}{dx}$ from the right.

Determine if the function $f(x) = \begin{cases} x^2 & if\ x \leq 2 \\ 4x - 4 & if\ x > 2 \end{cases}$ is differentiable at $x = 2$.

$f(2) = (2)^2 = \underline{4}$

$\lim_{x \to 2} f(x) = 4(2) - 4 = \underline{\underline{4}}$

Continuous!

If $x \leq 2$:

$f'(x) = 2x$

$f'(2) = 4$

If $x > 2$:

$f'(x) = 4$

Match, therefore differentiable

Example 6

Consider the function $f(x) = \begin{cases} x^2 & if\ x < 2 \\ a & if\ x = 2 \\ bx + c & if\ x > 2 \end{cases}$.

Find a, b, and c, given that the function is continuous and differentiable.

$x^2 = bx + c$

$(2)^2 = 2b + c$

$4 = 2b + c$

$4 = 2(4) + c$

$\underline{\underline{c = -4}}$

$2x = b$

$2(2) = b$

$\underline{\underline{b = 4}}$

$x^2 = a = bx + c$

$(2)^2 = a = 4(2) - 4$

$4 = a = 4$

$\underline{\underline{a = 4}}$

5.13 L'Hopital's Rule (HL only)

- **A FIRST DISCUSSION**

We know that

$$\frac{0}{5}=0 \quad \text{and} \quad \frac{5}{0} \text{ is not defined}$$

However, when x tends to 0,

$$\frac{x}{5} \text{ tends to } 0 \quad \text{and} \quad \frac{5}{x} \text{ tends either to } +\infty \text{ or } -\infty$$

But what about

$$\frac{0}{0}=?$$

Consider a function of the form $\frac{f(x)}{g(x)}$

if $f(x)\to 0$, $g(x)\to 5$	then $\frac{f(x)}{g(x)}$ tends to 0	e.g $\lim\limits_{x\to 5}\frac{x-5}{x}=\frac{0}{5}=0$

if $f(x)\to 5$, $g(x)\to 0$	then $\frac{f(x)}{g(x)}$ tends to $+\infty$ or $-\infty$	e.g $\lim\limits_{x\to 5}\frac{x}{(x-5)^2}=+\infty$

But again, what happens when both expressions tend to 0:

$$f(x)\to 0$$
$$g(x)\to 0$$

The result could be anything:

$$0 \quad \text{or} \quad +\infty \quad \text{or} \quad -\infty \quad \text{or} \quad \text{any real number!}$$

For example, when $x\to 0$, all the functions below have the form $\frac{0}{0}$, however

$$\lim_{x\to 0}\frac{x^3}{x}=0, \quad \lim_{x\to 0}\frac{x}{x^3}=+\infty, \quad \lim_{x\to 0}\frac{\sin x}{x}=1, \quad \lim_{x\to 0}\frac{2\sin x}{x}=2 \quad \text{etc}$$

(check these functions on your GDC near x=0).

That is why we say that: $\frac{0}{0}$ is an **indeterminate form**.

In the same sense: $\frac{\infty}{\infty}$ is also an **indeterminate form**.

L' Hôpital's rule:

If	$f(x)\to 0$ $g(x)\to 0$	or	$f(x)\to\pm\infty$ $g(x)\to\pm\infty$	$\lim \dfrac{f(x)}{g(x)} = \lim \dfrac{f'(x)}{g'(x)}$

provided that $\lim \dfrac{f'(x)}{g'(x)}$ exists.

Sometimes we need to apply L'Hôpital's rule more than once.

Example 1

$\lim\limits_{x\to 0} \frac{e^x - x - 1}{x^2}$

Indeterminate form

$$\frac{e^0 - 0 - 1}{(0)^2} = \frac{0}{0}$$

With l'hopital's rule:

$$\lim_{x\to 0} \frac{e^x - x - 1}{x^2} \overset{(\frac{0}{0})}{=} \frac{e^x - 1}{2x} \overset{(\frac{0}{0})}{=} \frac{e^x}{2} = \frac{1}{2}$$

Example 2

Find the horizontal asymptotes of $f(x) = \frac{3e^{2x}+2}{e^{2x}-1}$.

$$\lim_{x\to\infty} f(x) = \frac{3e^{2(\infty)} + 2}{e^{2(\infty)} - 1} \begin{array}{l} \} \text{Heading for } \infty \\ \} \text{Also heading for } \infty \end{array}$$

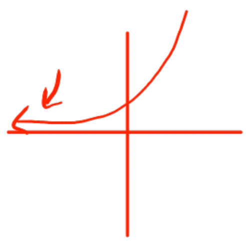
$$\lim_{x\to -\infty} f(x) = \frac{3e^{2x}+2}{e^{2x}-1} = 3$$

Therefore, $\lim\limits_{x\to\infty} f(x) = \frac{\infty}{\infty}$ } Indeterminate, right to use l'hopital's unlocked!

$$\lim_{x\to\infty} f(x) \overset{(\frac{\infty}{\infty})}{=} \frac{6\cancel{e^{2x}}}{2\cancel{e^{2x}}} = \frac{6}{2} = 3$$

$HA^{+}: y = 3$

$e^{-\infty}$ Looks like



$$\lim_{x\to -\infty} e^x = 0$$

Therefore, we can evaluate $\frac{3e^{-2\infty}+2}{e^{-2\infty}-1}$

as $\frac{0+2}{0-1} = -2$. $HA^{-} = -2$.

- **OTHER INDETERMINATE FORMS**

The following are also indeterminate forms

$$0\cdot\infty,\quad \infty-\infty,\quad 1^{\infty}$$

For example, look at the following $0\cdot\infty$ forms:

$$\lim_{x\to 0}\left(x\cdot\frac{1}{x}\right)=\lim_{x\to 0}1=1$$

$$\lim_{x\to 0}\left(x^2\cdot\frac{1}{x}\right)=\lim_{x\to 0}x=0$$

$$\lim_{x\to 0}\left(x\cdot\frac{1}{x^3}\right)=\lim_{x\to 0}\frac{1}{x^2}=+\infty$$

More complicated forms can be transformed to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and thus answered by using L'Hôpital's rule.

Example 3

$\lim_{x\to 0}(x\ln x)$ $=0\ln 0$

$\ln 0$ bi2areb 3ala $-\infty$.

Yani, $\lim_{x\to 0}=0\cdot-\infty$ ☆

$x\ln x=\dfrac{\ln x}{\frac{1}{x}}$

$\lim_{x\to 0}\dfrac{\ln x}{\frac{1}{x}}=\dfrac{\ln 0}{\frac{1}{0}}$ } ltnaynton bi2arèboun 3ala ∞.

$=\dfrac{-\infty}{\infty}$ ☆

$\lim_{x\to 0}\left[\dfrac{\frac{1}{x^2}-\frac{-\ln x}{x^2}}{\frac{1}{x^2}}\right]=\lim_{x\to 0}1+\ln x$

• L'hopital's needs a fraction, and using it you replace the tops and bottoms with their derivatives; it is <u>not</u> quotient rule.

Example 4

$$\lim_{x\to+\infty}(\sqrt{x+1}-\sqrt{2x})$$

$$\frac{\sqrt{x+1}-\sqrt{2x}}{1}\frac{(\sqrt{x+1}+\sqrt{2x})}{(\sqrt{x+1}+\sqrt{2x})}$$

$$\frac{x+1+\cancel{\sqrt{2x^2+2x}}-\cancel{\sqrt{2x^2+2x}}-2x}{\sqrt{x+1}+\sqrt{2x}}$$

$$=\frac{1-x}{\sqrt{x+1}+\sqrt{2x}}$$

$$\lim_{x\to+\infty}\left(\frac{1-x}{\sqrt{x+1}+\sqrt{2x}}\right)=\frac{1-\infty}{\sqrt{\infty+1}+\sqrt{2\infty}}=\frac{\infty}{\infty}$$

$$\lim_{x\to+\infty}\frac{-1}{\frac{1}{2\sqrt{x+1}}+\frac{1}{2\sqrt{2x}}}\begin{matrix}\}-1\\ \}\infty\end{matrix}\quad\boxed{=-\infty}$$

Example 5

Show that $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x=e.$

5.14 Implicit Differentiation – More Kinematics (HL only)

Show your work on this topic!

Example 1

Find $y' = \frac{dy}{dx}$ given that $2x^2 + x^2y^3 = x + y^2 + 3$.

$$4x + 2xy^3 + x^2 \cdot 3y^2 \frac{dy}{dx} = 1 + 2y\frac{dy}{dx}$$

$$4x + 2xy^3 - 1 = 2y\frac{dy}{dx} - x^2 3y^2 \frac{dy}{dx}$$

$$4x + 2xy^3 - 1 = (2y - x^2 3y^2)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{4x + 2xy^3 - 1}{2y - x^2 3y^2}$$

Example 2

Consider the equation of the circle $x^2 + y^2 = 1$.

Find the tangent lines:

Slope $m \rightarrow \frac{dy}{dx}$

point

a. at the point $(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

$$2x + 2y\frac{dy}{dx} = 0$$

$$\cancel{2}y\frac{dy}{dx} = -\cancel{2}x$$

$$y\frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Tangent Line: $y - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$

@$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$$\frac{dy}{dx} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{-1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{3}} = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

b. at the points of the circle with $x = 0$.

$$0^2 + y^2 = 1$$

$$y = \pm 1$$

$(0, 1), (0, -1)$

Slope $= 0$

$$\frac{dy}{dx} = \frac{0}{1} = 0$$

$$\frac{dy}{dx} = \frac{0}{-1} = 0$$

$y - 1 = 0$	$y + 1 = 0$
$y = 1$	$y = -1$

Tangent Lines

Example 3

Let $x^2 + y + x\cos y = \pi$.

find the tangent and the normal lines at $x = 0$.

$$2x + \frac{dy}{dx} + \left(x - \sin y \frac{dy}{dx}\right) + \cos y = 0$$

$$0^2 + y = \pi$$

$$y = \pi$$

$$\frac{dy}{dx}(1 - x\sin y) = -\cos y - 2x$$

$$\frac{dy}{dx} = \frac{2x + \cos y}{x\sin y - 1}$$

$$\frac{dy}{dx} @ (0, \pi) = \frac{2(0) + \cos\pi}{0\sin\pi - 1}$$

$$= \frac{\cos\pi}{-1}$$

$$= \frac{-1}{-1}$$

$$\boxed{= 1}$$

Tangent Line: $y - \pi = x$

Normal Line: $y - \pi = -x$

Example 4

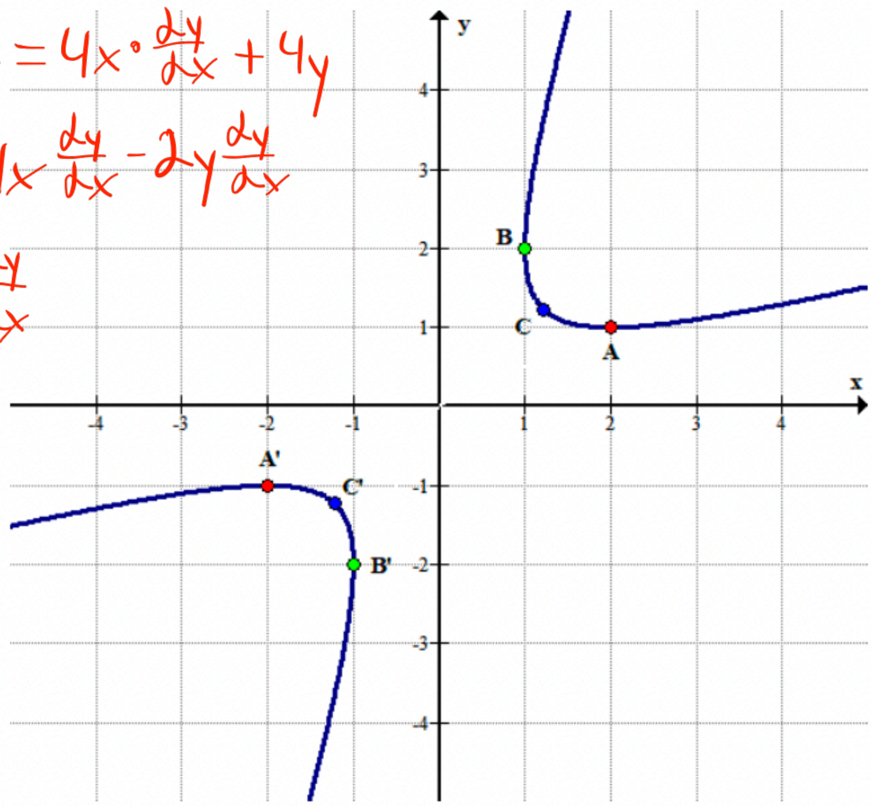
Power Rule

The diagram below show the graph of the relation $x^2 + y^2 = 4xy - 3$.

$2x + 2y\frac{dy}{dx} = 4x \cdot \frac{dy}{dx} + 4y$

$2x - 4y = 4x\frac{dy}{dx} - 2y\frac{dy}{dx}$

$\frac{x-2y}{2x-y} = \frac{dy}{dx}$



Find the points on the curve where the tangent line is parallel to:

a. x-axis $m = 0$

$(-2, -1)$

$(2, 1)$

$\frac{x-2y}{2x-y} = 0$

$x = 2y$

$(2y)^2 + y^2 = 8y^2 - 3$

$-3y^2 = -3$

$y^2 = 1,\ y = \pm 1$

b. y-axis $m =$ UNDEFINED

$(-1, -2)$

$(1, 2)$

c. the line $y = -x$ $m = -1$

$\frac{x-2y}{2x-y} = -1$

$x - 2y = -2x + y$

$3x = 3y$

$x = y$

$x^2 + x^2 = 4x^2 - 3$

$-2x^2 = -3$

$x = \pm\sqrt{\frac{3}{2}}$

$y = \pm\sqrt{\frac{3}{2}}$ as $y = x$.

$P's = \left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right), \left(-\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}\right)$

- MORE ON KINEMATICS

Remember the following scheme

Displacement	Velocity	Acceleration
s	v	a

$$s \xrightarrow{\text{derivative}} v \xrightarrow{\text{derivative}} a$$

$$s \xleftarrow{\text{integral}} v \xleftarrow{\text{integral}} a$$

In this situation, s,v,a are all given in terms of time t.

For example, given that

$$v=6t^2$$

then

$$a=\frac{dv}{dt}=12t$$

However, the velocity (v) is sometimes given as a function of s, instead of t. For example, $v=3s^2$

The following formula for acceleration may be derived (using the chain rule)

$$\boxed{a=\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}=\frac{dv}{ds}v}$$

Concentrate on

$$\boxed{a=v\frac{dv}{ds}}$$

Example 5

Let $v=3s^2$. Find the acceleration.

$a=6s\frac{ds}{dt}$

$a=6sv$

$a=6s(3s^2)$

$a=18s^3$

5.15 Rate of Change Problems (HL only)

Example 1

Consider a square object which is expanding. If the side of the object increases in a constant rate of 2 ms^{-1} find the rate of change of its area, at the instant when the side is 10m.

Example 2

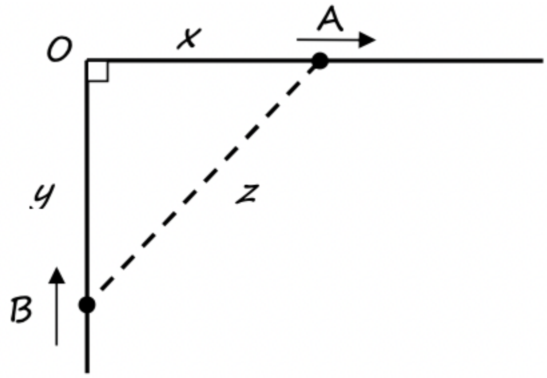
Consider an expanding sphere. If the volume increases in rate 5 cm^3/sec, find the rate of change of its radius r:

a. when $r = 3$ cm .

b. when the volume reaches 36π cm^3.

Example 3

Two cars, A and B, are traveling at 50 km/h and 70 km/h respectively, on straight roads, as shown in the diagram below.



At a given instant both cars are 5 km away from O. Find, at that instant, the rates of change:

a. of the distance between the cars.

b. of the angle $\theta = O\hat{A}B$ in radians per minute.

Example 4

It is given that $A = \frac{1}{3}r^2h + 2r^3$. Find the rate of change of h when r = 3 and h = 6, under two circumstances:

a. when h is always double of r and $\frac{dA}{dt} = 30.$

b. when $\frac{dA}{dt} = 30$ and $\frac{dh}{dt} = 8.$

5.16 Further Integration by Substitution (HL only)

Example 1

Find:

a. $\int \frac{\cos x}{\sin x + 1} dx$

$\cos x \int \frac{1}{\sin x + 1} dx$

$\cancel{\cos x}\left(\frac{1}{\cancel{\cos x}} \arctan \frac{1}{\cos x}\right) + C$

$= \arctan(\sec x) + C$

b. $\int \frac{x}{x^2+1} dx$

$x \int \frac{1}{x^2+1} dx$

$\cancel{x}\left(\frac{1}{\cancel{x}} \arctan \frac{1}{x}\right) + C$

$= \arctan\left(\frac{1}{x}\right) + C$

c. $\int \sin x \, e^{\cos x} dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$dx = -\frac{du}{\sin x}$

$\int \cancel{\sin x}\, e^u \left(-\frac{du}{\cancel{\sin x}}\right)$

$= -\int e^u du$

$= -\ln(\cos x) + C$

d. $\int x^2 e^{x^3+5} dx$

$u = x^3 + 5$

$\frac{du}{dx} = 2x^2$

$dx = \frac{du}{2x^2}$

$\int \cancel{x^2} e^u \frac{du}{2\cancel{x^2}}$

$= \frac{1}{2}\int e^u du$

$= \frac{1}{2} e^{x^3+5} + C$

e. $\int x \sin(7x^2+3)\, dx$

$u = 7x^2 + 3$

$\frac{du}{dx} = 14x$

$dx = \frac{du}{14x}$

$\int \cancel{x} \sin(u) \frac{du}{14\cancel{x}}$

$\int \sin u \, du$

$= -\cos(7x^2+3) + C$

f. $\int 5x(x^2+3)^5 dx$

g. $\int 3x\sqrt{x^2+3}dx$

h. $\int \frac{(\ln x)^3}{2x} dx$

Example 2

Find:

a. $\int x^3\sqrt{x^2+3}\,dx$

b. $\int \frac{x^2}{x+2}\,dx$

$u = x+2 \to x = u-2$

$du = dx \qquad x^2 = (u-2)^2$

$x^2 = u^2-4u+4$

$\int \frac{x^2}{u}\,du$

$\int \frac{u^2-4u+4}{u}\,du$

$\int \frac{u^{\cancel{2}}}{\cancel{u}}\,du - \int \frac{4\cancel{u}}{\cancel{u}}\,du + 4\int \frac{1}{u}\,du$

$\frac{1}{2}(x+2)^2 - 4(x+2) + 4\ln(x+2) + C$

Example 3

Find:

a. $\int \frac{e^x}{e^{2x}+4}\,dx$

$u = e^x$

$\frac{du}{dx} = e^x$

$dx = \frac{du}{e^x}$

$\int \frac{\cancel{u}}{u^2+4}\,\frac{du}{\cancel{u}}$

$= \int \frac{1}{u^2+4}\,du$

$= \frac{1}{2}\arctan\left(\frac{e^x}{2}\right) + C$

b. $\int \frac{e^{2x}}{e^x+4}\,dx$

$u = e^x$

$\frac{du}{dx} = e^x$

$dx = \frac{du}{e^x}$

$\int \frac{u^{\cancel{2}}}{u+4}\cdot\frac{du}{\cancel{u}}$

$\int \frac{u}{u+4}\,du$

$\int \frac{u+4-4}{u+4}\,du$

$\int \frac{u+4}{u+4}\,du + \int \frac{-4}{u+4}\,du$

$= e^x - 4\ln(e^x+4) + C$

Example 4

see expression	use substitution
$a^2 + x^2$	$x = a\tan\theta$
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$

Find:

a. $\int \frac{dx}{x^2+4}\,dx$

b. $\int \frac{dx}{\sqrt{4-x^2}}\,dx$

Example 5

Find $\int \sqrt{16 - x^2}dx.$

$a = 4$ $\quad dx = 4\cos\theta\, d\theta$

$x = 4\sin\theta \rightarrow \theta = \arcsin(\frac{x}{4})$

$\boxed{\begin{array}{l}\cos 2\theta = 2\cos^2\theta - 1 \\ \cos^2\theta = \dfrac{\cos 2\theta + 1}{2}\end{array}}$

$\int \sqrt{16 - 16\sin^2\theta} \cdot 4\cos\theta\, d\theta$

$\int \sqrt{16(1-\sin^2\theta)} \cdot 4\cos\theta\, d\theta$

$\int \sqrt{16} \cdot \sqrt{(1-\sin^2\theta)} \cdot 4\cos\theta\, d\theta$

$4\int \sqrt{\cos^2\theta} \cdot 4\cos\theta\, d\theta$

$16\int \sqrt{\cos^2\theta} \cdot \cos\theta\, d\theta \rightarrow 16\int \cos^2\theta\, d\theta$

$16\int \frac{\cos 2\theta + 1}{2}$

$8\int \cos 2\theta\, d\theta + 8\int d\theta$

$u = 2\theta$

$du = 2d\theta$

$d\theta = \frac{du}{2}$

$4\int \cos u\, du + 8\int du$

$4\sin 2\theta + 8\theta + C$

$8\sin\theta\cos\theta + 8\theta + C$

Example 6

Find:

a. $\int \frac{2}{x^2 - 4x + 3}dx$

$\int \frac{2}{(x-3)(x-1)}dx$

$\frac{A}{(x-3)} + \frac{B}{(x-1)} = \frac{2}{(x-3)(x-1)}$

$A(x-1) + B(x-3) = 2$

If $(x=1)$: $B = -1$ | If $(x=3)$: $A = 1$

$= \int \left[\frac{1}{(x-3)} - \frac{1}{(x-1)}\right]dx$

$= \ln(x-3) - \ln(x-1) + C$

b. $\int \frac{2}{x^2 - 4x + 4}dx$

$2\int \frac{1}{(x-2)^2}dx$

$2\int (x-2)^{-2}dx$

$= \frac{-2}{(x-2)} + C$

c. $\int \frac{2}{x^2 - 4x + 5}dx$

$2\int \frac{1}{(x-2)^2+1}dx$

$= 2\arctan(x-2) + C$

Example 7

Find $\int \frac{2}{\sqrt{-x^2 + 4x - 3}}dx$

5.17 Integration by Parts (HL only)

- ***DISCUSSION***

In this paragraph we study integrals of the form

$$I=\int (f\cdot g)dx$$

Let's make it clear from the very beginning that there is no "product rule" for integrals. It is sometimes very difficult, and more often impossible, to find the indefinite integral of a product. However, we exploit the product rule for differentiation

$$(f\cdot g)'=f'\cdot g+f\cdot g'$$

which gives

$$f'\cdot g=(f\cdot g)'-f\cdot g'$$

If we apply integration on both sides we obtain the so-called ***Integration by parts formula***

$$\boxed{\int f'\cdot g dx=f\cdot g-\int f\cdot g' dx}$$

This formula does not give an answer for any product but in some cases the integral in the RHS is much easier than the original and thus we obtain a result.

- ***THE METHOD***

Consider

$$I=\int xe^x dx$$

Since $(e^x)'=e^x$ this integral can be expressed as

$$I=\int (e^x)'x dx$$

The integration by parts formula gives

$$I = e^x x-\int e^x x' dx \;\; = e^x x-\int e^x dx= xe^x- e^x+c$$

[you may easily confirm that the derivative of the result gives xe^x]

In fact, the process is even quicker. We integrate one of the factors and then we differentiate the other factor as follows

$$\int \underline{\underline{f'}} \cdot g\,dx = f \cdot \underline{g} - \int f \cdot g'\,dx$$

derivative

integral

For example, in $I=\int xe^x dx$ we integrate $\underline{\underline{e^x}}$ and then differentiate $\underline{x}$

$$\int x\underline{\underline{e^x}}dx = \underline{x}e^x - \int e^x dx$$

and thus the final result is $xe^x - e^x + c$

Notice

If we try to integrate the other factor, that is $\underline{\underline{x}}$, and then differentiate $\underline{e^x}$, we obtain

$$\int \underline{\underline{x}}e^x dx = \frac{x^2}{2}\underline{e^x} - \int \frac{x^2}{2}e^x dx$$

The result of course is not wrong, but it is not practical! The second integral is worse than the original!

But how do we choose the function we integrate first?
We follow the priority list below

Priority (for integration)

1. e^x, sinx, cosx
2. x^n (or polynomials)
3. lnx, arctanx, arcsinx, arccosx

1: $dAB = dA \circ B + A \circ dB$ 2: Integrate both sides

3: Integral and full derivative $\int dAB = \int dA \circ B + \int A \circ dB$

cancel so you're left with: 4: Take partial:

$AB = \int dA \circ B + \int A \circ dB$ $\int dA \circ B = AB - \int A \circ dB$

Example 1

$\int u\,dv = uv - \int v\,du$

Find:

a. $\int \underbrace{x}_{u} \underbrace{\cos x\, dx}_{dv}$

Say this is the derivative so you force it to work.

$u = x \quad | \quad dv = \cos x\,dx$

$du = 1 \quad | \quad v = \sin x$

$x\sin x - \int \sin x (1)\, dx$

$\boxed{x\sin x + \cos x + C}$

b. $\int \underbrace{e^x}_{dv} \underbrace{(2x+5)}_{u} dx$

$u = 2x+5 \quad | \quad dv = e^x$

$du = 2 \quad | \quad v = e^x$

$\int e^x(2x+5)\,dx = e^x(2x+5) - 2\int e^x\,dx$

$= 5e^x + 2xe^x - 2e^x$

$= 2xe^x + 3e^x$

$= e^x(2x+3) + C$

c. $\int \underbrace{x^2}_{u} \underbrace{e^x dx}_{dv}$

$u = x^2 \quad | \quad dv = e^x dx$

$du = 2x \quad | \quad v = e^x$

$= x^2e^x - 2\int e^x x\,dx$

Derive this by parts again specifically.

$u_2 = x \quad | \quad dv_2 = e^x$

$du_2 = 1 \quad | \quad v_2 = e^x$

$\rightarrow x^2e^x - 2\left[xe^x - \int e^x (1)\,dx\right]$

$= x^2e^x - 2xe^x - e^x$

$\boxed{= e^x(x^2 - 2x - 1)}$

d. $\int x^2 \ln x\, dx$

$u = \ln x \quad | \quad dv = x^2$

$du = \frac{1}{x} \quad | \quad v = \frac{1}{3}x^3$

$= \frac{\ln x}{3}x^3 - \int \frac{1}{3}\cancel{x^3}^{2}\left(\frac{1}{x}\right)dx = \frac{1}{3}x^3\ln x - \frac{1}{3}\left(\frac{1}{3}x^3\right) + C$

$= \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$

e. $\int \ln x\, dx$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x} \quad v = x$

$= x\ln x - \int \cancel{x}\frac{1}{\cancel{x}}\,dx \; \boxed{= x\ln x - x + C}$

Example 2

Find:

a. $\int e^x \sin x\, dx$

$I = \sin x(e^x) - \int e^x(\cos x)\,dx$

$I = e^x \sin x - \cos x(e^x) - \int e^x(\sin x)\,dx = I$

↑ Infinite Loop, use this trick:

Let $I = \int e^x \sin x\,dx$

$I = e^x \sin x - e^x \cos x - I$

$2I = e^x \sin x - e^x \cos x$

$I = \dfrac{e^x \sin x - e^x \cos x}{2} + C$

b. $\int x^2 e^{3x} dx$

$u = x^2 \quad | \quad dv = e^{3x}$

$du = 2x \quad | \quad v = \frac{1}{3}e^{3x}$

$= x^2\left(\frac{1}{3}e^{3x}\right) - \frac{2}{3}\int e^{3x} x\,dx$

$u_2 = x \quad | \quad dv_2 = e^{3x}$

$du_2 = 1 \quad | \quad v_2 = \frac{1}{3}e^{3x}$

$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left[x\left(\frac{1}{3}e^{3x}\right) - \frac{1}{3}\int e^{3x} dx\right]$

$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right] + C$

$= \frac{1}{3}x^2 e^{3x} - \frac{2}{9}xe^{3x} - \frac{2}{27}e^{3x} + C$

- **CASES OF INTEGRATION BY PARTS**

General Form $n \in Z^+$, $a,b \in R$	Examples	Theoretical Questions
$I_n = \int x^n e^x dx$ $I_{n,a} = \int x^n e^{ax} dx$	$\int x^3 e^x dx$, $\int x^2 e^{\frac{x}{2}} dx$	Express I_n in terms of I_{n-1} Hence find $I_0, I_1, I_2, \ldots$
$I_n = \int x^n \cos x dx$ $I_n = \int x^n \sin x dx$	$\int x^2 \cos x dx$	Express I_n in terms of I_{n-2}
$I_{n,a} = \int x^n \cos(ax) dx$ $I_{n,a} = \int x^n \sin(ax) dx$	$\int x^2 \cos 3x dx$	
$I_a = \int x^a \ln x dx$	$\int \sqrt{x} \ln x dx$, $\int \frac{\ln x}{x^5} dx$	Find a general formula for I_a
$I_{a,b} = \int e^{ax} \sin(bx) dx$ $I_{a,b} = \int e^{ax} \cos(bx) dx$	$\int e^{-x} \sin 2x dx$	Find a general formula for $I_{a,b}$
$I_n = \int \cos^n x dx$ $I_n = \int \sin^n x dx$	$\int \cos^2 x dx$, $\int \cos^3 x dx$	Express I_n in terms of I_{n-2} Hence find I_2, I_4 and I_3, I_5
$I_{a,b} = \int \sin(ax)\cos(bx) dx$	$\int \sin 2x \cos 3x dx$	Find a general formula for $I_{a,b}$
$I_n = \int x^n \arctan x dx$ $I_n = \int x^n \arcsin x dx$ $I_n = \int x^n \arccos x dx$	$\int \arctan x dx$, $\int x \arctan x dx$, $\int x^2 \arctan x dx$ $\int \arcsin x dx$, $\int x^2 \arcsin x dx$	
$I_n = \int (\ln x)^n dx$	$\int (\ln x)^2 dx$, $\int (\ln x)^3 dx$	

- ## INTEGRATION BY PARTS FOR DEFINITE INTEGRALS

Again, as in the case of substitution, it would be safe to find the indefinite integral first and then proceed to the definite integral! Otherwise the integration by parts for definite integrals takes the form

$$\boxed{\int_a^b f'\cdot g\,dx=[f\cdot g]_a^b-\int_a^b f\cdot g'\,dx}$$

Example 3

Find $\int_0^2 e^x(2x+5)dx$.

$u=2x+5 \quad | \quad dv=e^x$

$du=2 \quad | \quad v=e^x$

$=\left[(2x+5)(e^x)\right]_0^2-\left[\int e^x(2)\,dx\right]_0^2$

$=\left[2xe^x+5e^x\right]_0^2-\left[2e^x\right]_0^2$

$=\left[(2(2)e^2+5e^2)-5\right]-\left[2e^2-2\right]$

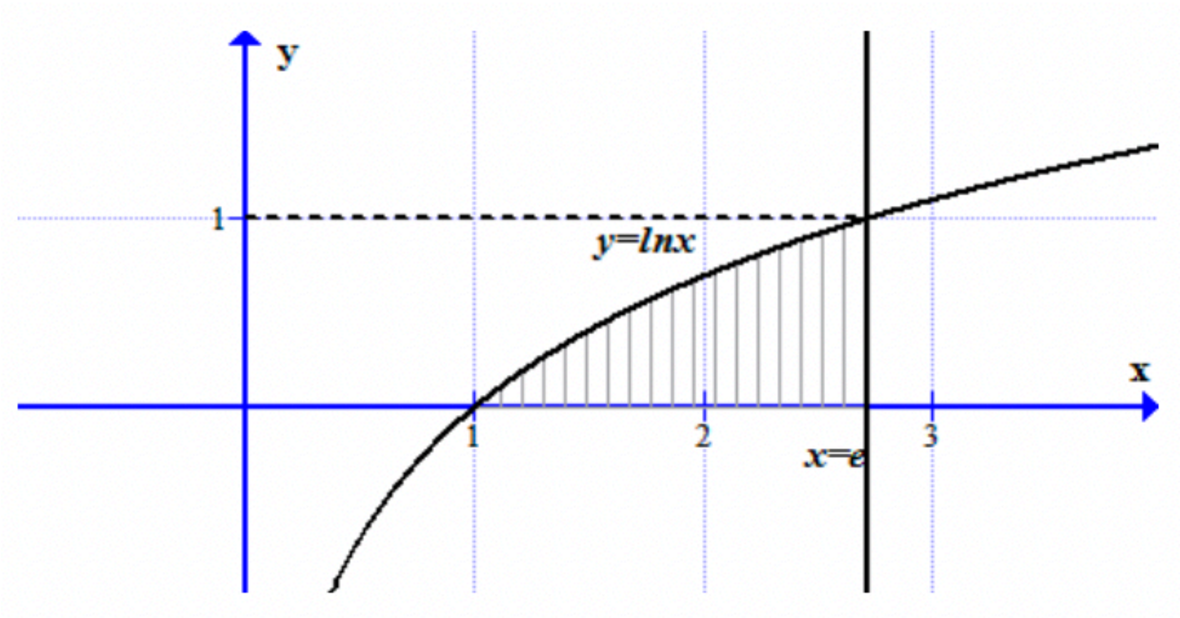
$=4e^2+5e^2-5-2e^2+2$

$=7e^2-3$

5.18 Further Area Between Curves – Volumes (HL only)

Example 1

Find the area among $y = \ln x$, x-axis, and the line $x = e$.



a. about the x-axis.

$A = \int_1^e \ln x \, dx$

$u = \ln x \quad | \quad dv = dx$

$du = \frac{1}{x} \quad | \quad v = x$

$= [x \ln x]_1^e - [\int dx]_1^e$

$= [x \ln x]_1^e - [x]_1^e$

$= [e \ln \cancel{e} - 0] - [e - 1]$

$= \cancel{e} - \cancel{e} + 1$

$\boxed{= 1}$

b. about the y-axis. $y = \ln x$ $e^y = x$

$A = \int_0^1 (e - e^y) \, dy$

$A = [ey - e^y]_0^1$

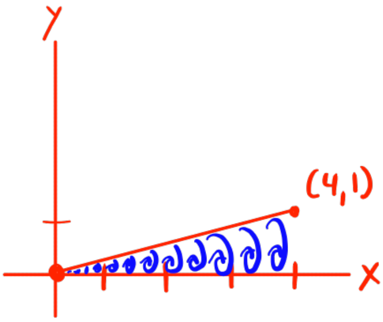
$A = [\cancel{e} - \cancel{e}] - [0 - 1]$

$\boxed{A = 1}$

Example 2

Consider the segment of the straight line $y = \frac{1}{4}x$, where $0 \leq x \leq 4$. Find the volume of the cone generated by a $360°$ rotation of this segment.

→ about the x-axis.



$V = \pi \int_0^4 \left(\frac{1}{4}x\right)^2 dx$

$V = \frac{\pi}{16} \int_0^4 x^2 dx$

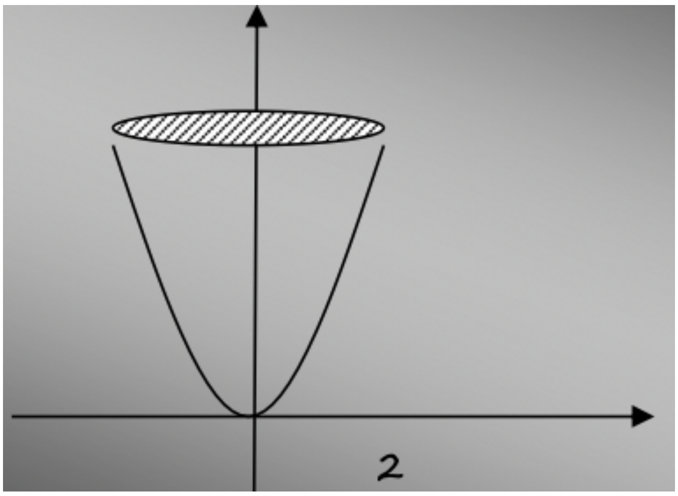
$V = \frac{\pi}{16}\left[\frac{1}{3}(4)^3 - \frac{1}{3}(0)^3\right]$

$V = \frac{\pi}{16}\left[\frac{64}{3}\right]$

$V = \frac{64\pi}{48} = \boxed{\frac{4\pi}{3}}$

Example 3

If we rotate the parabola $y = x^2$ about the y-axis we obtain the 3D shape below.



Find the volume if the height is 4.

$V = \pi \int_0^4 (y^{1/2})\, dy$

$V = \pi\left[\frac{3}{2}(4)^{3/2} - \frac{3}{2}(0)^{3/2}\right]$

$V = \pi\left[\frac{3}{2}\sqrt{64}\right]$

$V = \pi\left(\frac{24}{2}\right)$

$V = 12\pi$

Example 4

Consider the region between the curves of $y = 2x^2$ and $y = 2x$. Find the volume of the cone generated by a 360° rotation of this segment. Find the volume of the solid if the region is rotated:

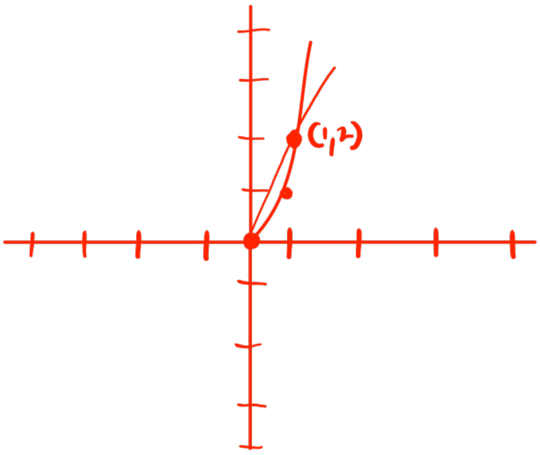
$2x^2 = 2x$

$2x^2 - 2x = 0$

$2x(x-1) = 0$

$x = 0,\ x = 1$

a. about the x-axis.



$\pi\int r_1^2 - \pi\int r_2^2$

$\pi\int_0^1 (2x)^2 dx - \pi\int_0^1 (2x^2)^2 dx$

$\pi\int_0^1 4x^2 dx - \pi\int_0^1 4x^4 dx$

$\pi\left[\frac{4}{3}(1)^3 - \frac{4}{3}(0)^3\right] - \pi\left[\frac{4}{5}(1)^5 - \frac{4}{5}(0)^5\right]$

$\frac{4}{3}\pi - \frac{4}{5}\pi$

$\frac{20}{15}\pi - \frac{12}{15}\pi$

$= \frac{8}{15}\pi$

b. about the y-axis.

curve becomes the big.

$y = 2x^2 \qquad y = 2x$

$x = \sqrt{\frac{y}{2}} \qquad x = \frac{2}{y}$

$\pi\int_0^2 \left(\frac{y}{2}\right)^2 dy - \pi\int_0^2 \left(\frac{2}{y}\right)^2 dy$

$= \pi\int_0^2 \left(\frac{y}{2} - \frac{2}{y}\right)^2 dy$

$= \pi\int_0^2 \left(\frac{1}{2}y - 2y^{-1}\right)^2 dy$

$= \pi\int_0^2 \left(\frac{1}{4}y^2 + 4y^{-2}\right) dy$

$= \pi\left[\frac{1}{4}(2)^2 + 4(2)^{-2}\right]$

$= \pi[1+1]$

$= 2\pi$

5.19 Differential Equations (HL only)

Example 1

Initial Condition

Solve the differential equation $\frac{dy}{dx} = 4xy^2$, given that $y(1) = 2$.

$\frac{1}{y^2}dy = 4x\,dx$

$\int \frac{1}{y^2}dy = \int 4x\,dx$

$-\frac{1}{y} = 2x^2 + C$

$-1 = (2x^2 + C)y$

$y = \frac{-1}{2x^2 + C}$

General Solution: Applies for any value of c regardless of what the initial condition is.

Apply initial condition

$2 = \frac{-1}{2(1)^2 + C}$

$2 = \frac{-1}{2 + C}$

$4 + 2C = -1$

$2C = -5$

$C = -\frac{5}{2}$

Hence, $y = \frac{-1}{2x^2 - \frac{5}{2}} = \frac{-2}{4x^2 - 5}$

Particular Solution: Defines a c value.

Example 2

Solve the differential equation $\frac{dy}{dx} = xy^2 + x$, given that $y(0) = 1$.

$\frac{dy}{dx} = x(y^2 + 1)$

$\frac{dy}{(y^2+1)} = x\,dx$

$\int \frac{1}{y^2+1}dy = \int x\,dx$

$\arctan(y) = \frac{1}{2}x^2 + C$

$y = \tan\left(\frac{1}{2}x^2 + C\right)$

General Solution

$1 = \tan(C)$

$C = \arctan(1)$

$C = \frac{\pi}{4}$

$y = \tan\left(\frac{1}{2}x^2 + \frac{\pi}{4}\right)$

Particular Solution

Example 3

Find the general solution of the differential equation:

$$x^2\frac{dy}{dx} = x^2 - xy + y^2$$

$$\frac{dy}{dx} = \frac{x^2 - xy + y^2}{x^2} = \frac{x^2}{x^2} - \frac{xy}{x^2} + \frac{y^2}{x^2}$$

$$\frac{dy}{dx} = 1 - \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

Let $v = \frac{y}{x}$

$$\frac{dy}{dx} = 1 - v + v^2$$

$$y = xv$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = 1 - v + v^2$$

$$x\frac{dv}{dx} = v^2 - 2v + 1$$

$$x\frac{dv}{dx} = (v-1)^2$$

$$\int \frac{1}{(v-1)^2}\,dv = \int \frac{1}{x}\,dx$$

$$\frac{-1}{v-1} = \ln(x) + C$$

$$\frac{-1}{\ln(x)+C} = v - 1 \Leftrightarrow v = \frac{-1}{\ln(x)+c} + 1$$

$$\frac{y}{x} = \frac{-1}{\ln(x)+C} + 1 \Leftrightarrow y = -\frac{x}{\ln(x)+C}$$

↑ General Solution

Example 4

Find the general solution of the differential equation:

$$\frac{dy}{dx} + \frac{2}{x}y = 5x^2$$

• Integrating Factor

$P(x) = \frac{2}{x}$, $Q(x) = 5x^2$

$$I = e^{\int P(x)dx} = e^{2\int x\,dx} = e^{2\ln x} = \cancel{e}^{\cancel{\ln} x^2} = x^2$$

$$Iy = \int I \cdot Q(x) \cdot dx = \int x^2(5x^2)\,dx = 5\int x^4\,dx$$

$$(x^2)y = \cancel{5}\left(\tfrac{1}{\cancel{5}}x^5\right) + C$$

$$x^2y = x^5 + C$$

$$y = x^3 + \frac{1}{x^2}C$$

Example 5 $y_{n+1} = y_n + h \cdot f(x, y)$; $x_{n+1} = x_n + h$ where h is constant (step value)

Consider the differential equation $\frac{dy}{dx} = xy^2 + x$ given that $y(0) = 1$. Find an approximation of $y(1)$ using step $\underline{h = 0.2}$.

x	Slope	y
0		1
0.2	1+0.2(0)	1
0.4	1+0.2(0.4)	1.08
0.6	1.08+0.2(...)	1.253...
0.8	1.253...+0.2(...)	1.56181...
1	1.56181...+0.2(...)	2.11 (3 sig. figs)

$y_n = y_{n-1} + \text{Step} \cdot (\text{Slope}_{n-1})$

- Builds off of the previous step.

Avoid rounding until the very final approximation.

5.20 Maclaurin Series – Extension of Binomial Theorem (HL only)

Example 1

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots$$

→ More terms = more accurate approximation.

Find that Maclaurin series of the function $f(x) = \sin x$ up to the term in x^5.

$f(x) = \sin x$ $f(0) = \sin(0) = 0$

1

$f'(x) = \cos x$ $f'(0) = \cos(0) = 1$

2

$f''(x) = -\sin x$ $f''(0) = -\sin(0) = 0$

$f'''(x) = -\cos x$ $f'''(0) = -\cos(0) = -1$

$f^4(x) = \sin x$ $f^4(0) = \sin(0) = 0$

$f^5(x) = \cos x$ $f^5(0) = \cos(0) = 1$

3

$f(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + \dots$

$f(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \dots$

- The x in the function and the McLauren Series are two different variables.

Example 2

Find that Maclaurin series of the function $f(x) = e^{x^2}$.

$f(x) = e^{x^2}$ $f(0) = 1$

$f'(x) = 2x \cdot e^{x^2}$ $f'(0) = 0$

$f''(x) = 2e^{x^2} + 4x^2e^{x^2}$ $f''(0) = 2$

$f'''(x) = 4xe^{x^2} + 8xe^{x^2} + 8x^4e^{x^2}$ $f'''(0) = 0$

$f^4(x) = 4e^{x^2} + 8x^3e^{x^2} + 8e^{x^2} + 16x^3e^{x^2} + 32x^3e^{x^2} + 16x^5e^{x^2}$ $f^4(0) = 12$

$f(x) = 1 + 0 + \frac{2}{2!}x^2 + 0 + \frac{12}{4!}x^4 + \dots$

$f(x) = 1 + x^2 + \frac{x^4}{2} + \dots$

Example 3

Find that Maclaurin series of the function $f(x) = e^x \sin x$.

$f(x) = \left(1 + x + \frac{x^2}{2!} + \dots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)$

• Multiply their individual Maclaurin series together by distribution.

$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^2 - \frac{x^4}{3!} + \frac{x^7}{5!} + \frac{x^3}{2!} - \frac{x^5}{2!3!} + \frac{x^7}{2!5!} + \dots$

$f(x) = x + x^2 - \frac{1}{3}x^3 + \dots$

Example 4

Find that Maclaurin series of the function $f(x) = \frac{1}{x^2+1}$.

$\frac{1}{x^2+1} = \frac{d}{dx}(\arctan x)$

• Maclaurin series of arctanx given in databooklet

$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots$

$\frac{1}{x^2+1} = \frac{d}{dx}\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots\right) = 1 - x^2 + x^4 - \dots$

Example 5

Find that Maclaurin series of $\cos x$ by integrating the series of:

• Maclaurin series always start with an x of zero.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$-\cos x = \frac{1}{2}x^2 - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots + C$$

$-\cos(0) = C$

$C = -1$

$$-\cos x = \frac{1}{2}x^2 - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots - 1$$

$$\cos x = -\frac{1}{2}x^2 + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + 1$$

Answer

Example 6

Find that Maclaurin series up to x^2 for the solution of the differential equation $\frac{dy}{dx} = x^2 + y^2$ with $y = 3$ when $x = 0$.

$$\frac{d^2y}{dx^2} = 2x + 2y\frac{dy}{dx}$$

$f(0) = 3$

$f'(0) = 0^2 + 3^2 = 9$

$f''(0) = 2(0) + 2(3)9 = 54$

$$f(x) = 3 + 9x + \frac{54x^2}{2!} + \ldots$$

$$f(x) = 3 + 9x + 27x^2 + \ldots$$

♦ THE EXTENTION OF THE BINOMIAL THEOREM

Remember that the binomial theorem gives

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

If we expand more the coefficients we obtain

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n$$

This version allows us to use negative or even fractional values for index n.

Example 7

Find the Maclaurin series up to x^4 for $f(x) = \frac{x}{(1+x)^3}$.

$f(x) = x(1+3)^{-3} \leftarrow n$

$f(x) = x\left(1 - 3x + \frac{-3(-4)}{2}x^2 + \frac{-3(-4)(-5)}{3!}x^3 + \ldots\right)$

$f(x) = x - 3x^2 + 6x^3 - 10x^4 + \ldots$

Example 8

Find the Maclaurin series up to x^3 for $f(x) = \sqrt{1+x}$ by using the extension of the binomial theorem.

$f(x) = (1+x)^{\frac{1}{2}} \leftarrow n$

$f(x) = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3$

$f(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

Example 9

Find the Maclaurin series up to x^3 for $f(x) = (2x+3)^{-2}$ by using the extension of the binomial theorem.

$f(x) = (2x)^{-2}\left(1 - 2\left(\frac{3}{2x}\right) + \frac{-2(-2-1)}{2!}\left(\frac{3}{2x}\right)\right)$