$(a^x)'$:
$a^x = e^{\ln a^x} = e^{x\ln a}$
$(e^{x\ln a}) = e^{x\ln a} \cdot \ln a$
$= a^x \cdot \ln a$

INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

MAA

EXERCISES [MAA 5.13]

MORE DERIVATIVES

Compiled by Christos Nikolaidis

O. Practice questions

1. [Maximum mark: 18] ***[without GDC]***

Find the derivatives of the following functions

$y = f(x)$	$\frac{dy}{dx}$
$y = x^5 + 5^x$	$y' = 4x^4 + 5^x \cdot \ln 5$
$y = \ln x + \log_5 x$	$y' = \frac{1}{x} + \frac{1}{x\ln 5}$
$y = 5^x \log_5 x$	$y' = 5^x \ln 5(\log_5 x) + 5^x \cdot \frac{1}{x\ln 5}$ $y' = 5^x(\ln x + \frac{1}{x\ln 5})$
$y = 2\tan x + 3\cot x$	$y' = 2\sec^2 x - 3\operatorname{cosec}^2 x$
$y = 2\sec x + 3\operatorname{cosec} x$	$y' = 2\sec x\tan x - 3\operatorname{cosec} x\cot x$
$y = 5\arcsin x + 3\arccos x$	$y' = \frac{5}{\sqrt{1-x^2}} - \frac{3}{\sqrt{1-x^2}}$
$y = x\arctan x$	$y' = \frac{x}{1+x^2}$
$y = \sqrt{x}\arcsin x$	$y' = \frac{\sqrt{x}}{\sqrt{1-x^2}} = \frac{x}{1-x^2}$
$y = \ln(x^2+1) + \log_2(x^2+1)$	$y' = \frac{1}{x^2+1} \cdot 2x + \frac{1}{(x^2+1)(\ln 2)} \cdot 2x$

$y' = \frac{2x}{x^2+1} + \frac{2x}{x^2+1} \cdot \frac{1}{\ln 2}$

$\ln 5 = z$
$e^z = 5$
$\log_5 x = w$
$5^x = w$
$5 = \sqrt[x]{w}$
$e^z = \sqrt[x]{w}$
$e^{zx} = w$
$\ln w = zx$
$\cancel{\ln}(\log_5 x) = x\cancel{\ln 5}$
$\log_5 x = 5x$

$5^x \cdot \ln 5 \cdot \log_5 x$
$\log_5 x = \frac{\ln x}{\ln 5}$
$\cancel{\ln 5} \cdot \frac{\ln x}{\cancel{\ln 5}}$
$= \ln x$
$= 5^x \ln x$

2. [Maximum mark: 28] ***[without GDC]***

Find the derivatives of the following functions

$y = f(x)$	$\frac{\mathrm{d}y}{\mathrm{d}x}$
$y = \sin(3x+5)$	$y' = -\cos(3x+5) \cdot 3 = -3\cos(3x+5)$
$y = \tan(3x+5)$	$y' = 3\sec^2(3x+5)$
$y = \sec(3x+5)$	$y' = 3\sec(3x+5)\tan(3x+5)$
$y = \operatorname{cosec}(3x+5)$	$y' = -3\operatorname{cosec}(3x+5)\tan(3x+5)$
$y = \cot(3x+5)$	$y' = -3\operatorname{cosec}^2(3x+5)$
$y = 2^{3x+5}$	$y' = 3\left[2^{3x+5} \cdot \ln(3x+5)\right]$
$y = \log_2(3x+5)$	$y' = \frac{3}{(3x+5)\ln 2}$
$y = \arcsin(3x+5)$	$y' = \frac{3}{\sqrt{1-(3x+5)^2}}$
$y = \arccos(3x+5)$	$y' = \frac{3}{-\sqrt{1-(3x+5)^2}}$
$y = \arctan(3x+5)$	$y' = \frac{3}{1+(3x+5)^2} = \frac{3}{9x^2+30x+26}$
$y = \arcsin(x^2)$	$y' = \frac{2x}{\sqrt{1-x^4}}$
$y = \arccos(x^2)$	$y' = \frac{-2x}{\sqrt{1-x^4}}$
$y = \arctan(x^2)$	$y' = \frac{2x}{1+x^4}$
$y = \sqrt{\arctan x}$	

A. Exam style questions (SHORT)

3. [Maximum mark: 8] ***[without GDC]***

Consider the function $y = \tan x - 8\sin x$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

(b) Find the value of $\cos x$ for which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. [3]

(c) Solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. for $-\pi \le x \le 2\pi$ [3]

(a) $\frac{dy}{dx} = \sec x \tan x - 8\cos x$

(b) $0 = \sec x \tan x - 8\cos x$

$8\cos x = \sec x \tan x$

$8\cos x = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$

$8\cos x = \frac{\sin x}{\cos^2 x}$

$8\cos x = \frac{\sin x}{1-\sin^2 x}$

$\cos x = \frac{\sin x}{8 - 8\sin^2 x}$

(a) $\frac{dy}{dx} = \sec^2 x - 8\cos x$

(b) $\frac{dy}{dx} = \sec^2 x - 8\cos x$

$\frac{dy}{dx} = \sec^2 x$

$\frac{dy}{dx} = \frac{1}{\cos^2 x} - 8\cos x$

$0 = \frac{1 - 8\cos^3 x}{\cos^2 x}$

$8\cos^3 x = 1$

$\cos^3 x = \frac{1}{8}$

$\cos x = \frac{1}{2}$

(c) $x = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$

• Cube root retains sign, no plus-or-minus.

?

4. [Maximum mark: 6] ***[without GDC]***

Consider the function $f(t) = 3\sec^2 t + 5t$.

(a) Find $f'(t)$ [3]

(b) Find (i) $f(\pi)$ (ii) $f'(\pi)$ [3]

$f'(t) =$

(a) $6(\sec t)\sec t \cdot \tan t$

$f'(t) = 6\sec^2 t \tan t + 5$

$f'(t) = 6\sec^2 t \cdot \tan x + 5$

(b.i) $f(\pi) = 3 + 5\pi$

(b.ii) $f'(\pi) = 6(0) + 5 = 5$

?

5. [Maximum mark: 5] ***[with / without GDC]***

The function f is defined by $f : x \mapsto 3^x$. Find the solution of the equation $f''(x) = 2$.

$f(x) = 3^x$

$f'(x) = 3^x \ln 3$

$f''(x)\, 3^x \ln 3 \cdot \ln 3$ ← Not product rule. ln is a constant technically.

$f''(x) = 3^x \ln^2 3$ IF it doesn't have a Variable then it

$2 = 3^x \ln^2 3$ is by definition a constant, because it

$\frac{2}{\ln^2 3} = 3^x$ won't change with different inputs.

$x = \log_3\left(\frac{2}{\ln^2 3}\right)$

$x = 0.4560$

6. [Maximum mark: 4] ***[without GDC]***

Differentiate $y = \arccos(1-2x^2)$ with respect to x, and simplify your answer.

$y' = \dfrac{1}{\sqrt{1-(1-2x^2)^2}} \cdot -4x$

$y' = \dfrac{-4x}{(1-(1-4x^2+4x^4))^{\frac{1}{2}}}$

$y' = \dfrac{-4x}{(\cancel{1}-\cancel{1}+4x^2-4x^4)^{\frac{1}{2}}}$

$y' = \dfrac{-4x}{\sqrt{4x^2-4x^4}}$

7. [Maximum mark: 6] ***[without GDC]***

Let $y = x\arcsin x$, $x \in]-1,1[$. Show that $\dfrac{d^2y}{dx^2} = \dfrac{2-x^2}{(1-x^2)^{\frac{3}{2}}}$.

$\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1-x^2}} + \arcsin x$

$\dfrac{dy}{dx} = x(1-x^2)^{-\frac{1}{2}} + \arcsin x$

$\dfrac{d^2y}{dx^2} = x\left[-\frac{1}{2}(1-x^2)^{-\frac{3}{2}} \cdot -2x\right] + (1-x^2)^{-\frac{1}{2}} + \dfrac{1}{(1-x^2)^{\frac{1}{2}}}$

$\dfrac{d^2y}{dx^2} = \dfrac{\cancel{2}x^2}{\cancel{2}(1-x^2)^{3/2}} + \dfrac{2}{(1-x^2)^{\frac{1}{2}}}$

$\dfrac{d^2y}{dx^2} = \dfrac{x^2}{(1-x^2)^{3/2}} + \dfrac{2-2x^2}{(1-x^2)^{3/2}}$

$\dfrac{d^2y}{dx^2} = \dfrac{2-x^2}{(1-x^2)^{3/2}}$

8. [Maximum mark: 7] ***[without GDC]***

Find the equations of the tangent and the normal lines to the curve $y = \tan^2 x$ at $x = \frac{\pi}{4}$.

$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x(\cos x) - \sin(-\sin x)}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$\cos^2 x + \sin^2 x = 1$

$= \frac{1}{\cos^2 x} = \sec^2 x$

$\frac{dy}{dx} = 2\tan x \sec^2 x$

$\frac{dy}{dx} @ \frac{\pi}{4} : 2(1) \cdot \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} = \frac{2}{\frac{2}{4}} = \frac{\cancel{2}}{1} \cdot \frac{4}{\cancel{2}} = 4$

$y = \tan^2\left(\frac{\pi}{4}\right), y = 1$

Tangent Line: $y = 4x - \pi + 1$

Normal Line: $y = -\frac{1}{4} + \frac{\pi}{16} + 1$

9. [Maximum mark: 6] ***[without GDC]***

A normal to the graph of $y = \arctan(x-1)$, for $x > 0$, has equation $y = -2x + c$, where $c \in \mathbb{R}$. Find the value of c.

$\frac{dy}{dx} = \frac{1}{1+(x-1)^2} = \frac{1}{1+x^2-2x+1} = \frac{1}{x^2-2x+2}$

• Slope -2 is equal to reciprocal of derivative at x.

$-2 = -(x^2 - 2x + 2)$

$\cancel{-2} = -x^2 + 2x \cancel{-2}$

$0 = -x^2 + 2x$

$0 = (-x+2)x$

$x = 0$, $\boxed{x = 2}$

False, x must be gt 0.

$y = \arctan(1) = \frac{\pi}{4}$

$\frac{\pi}{4} = -2(2) + c$

$c = \frac{\pi}{4} + 4$

10. [Maximum mark: 8] ***[without GDC]***

Consider the curve $y = x^3 + 4x^2 + x - 6$

(a) Find the equation of the tangent line at the point where $x = -1$.

(b) Find the coordinates of the point where this tangent meets the curve again.

(a) $\frac{dy}{dx} = 3x^2 + 8x + 1$

$y = (-1)^3 + 4(-1)^2 - 1 - 6$

$y = -4$

$\frac{dy}{dx}$ @ $-1 = 3(-1)^2 + 8(-1) + 1 = -4$

Tangent Line: $y + 4 = -4(x+1)$

$y = -4x - 8$

(b) $-4x - 8 = x^3 + 4x^2 + x - 6$

$0 = x^3 + 4x^2 + 5x + 2$

$$\begin{array}{r l}
 & x^2 + 3x + 2 \\
x+1 \,\big) & \cancel{x^3} + 4x^2 + 5x + 2 \\
 & -(\cancel{x^3} + x^2) \\
 & 3\cancel{x^2} + 5x + 2 \\
 & -(3\cancel{x^2} + 3x) \\
 & \cancel{2x + 2} \\
 & -(2x + 2) \\
 & 0
\end{array}$$

Synthetic Division

$$\begin{array}{r|rrrr}
-1 & 1 & 4 & 5 & 2 \\
 & & -1 & -3 & -2 \\
\hline
 & 1 & 3 & 2 & 0
\end{array}$$

$= x^2 + 3x + 2$

Also works and is way faster

$x^3 + 4x^2 + 5x + 2 \equiv (x+1)(x^2 + 3x + 2) \equiv (x+1)^2(x+2)$

$x = -2$

$y = (-2)^3 + 4(-2)^2 - 2 - 6 = -8 + 16 - 8 = 16 - 16 = 0$

P of Reintersection $= (-2, 0)$

11. [Maximum mark: 5] ***[without GDC]***

The function *f* is given by $f : x \mapsto e^{(1+\sin \pi x)}$, $x \geq 0$.

(a) Find $f'(x)$. [2]

Let x_n be the value of x where the $(n+1)^{th}$ maximum or minimum point occurs, $n \in \mathbb{N}$ (*i.e.* x_0 is the value of x where the first maximum or minimum occurs, x_1 is the value of x where the second maximum or minimum occurs, *etc*).

(b) Find x_n in terms of n. [3]

(a) $y = e^{(1+\sin\pi x)}$

$\frac{dy}{dx} = e^{(1+\sin\pi x)} \cdot \cos\pi x \cdot \pi$

$0 = e^{(1+\sin\pi x)} \cdot \pi\cos\pi x$

$0 = e^{1+\sin\pi x}$ | $0 = \pi\cos\pi x$

UNDEFINED | $0 = \cos\pi x$

$x = \frac{\arccos(0)}{\pi} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$

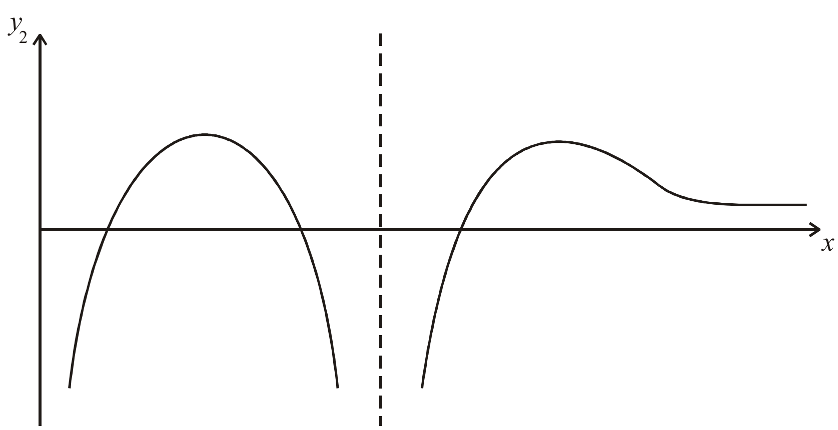
$x_n = \frac{2n+1}{2}$

1

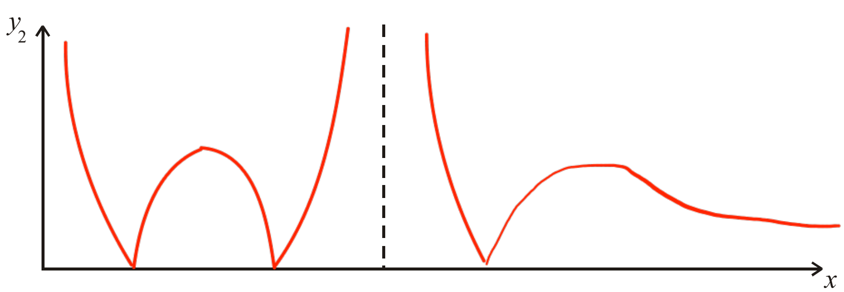
0

12. [Maximum mark: 6] ***[without GDC]***

The diagram below shows the graph of $y_1 = f(x)$.

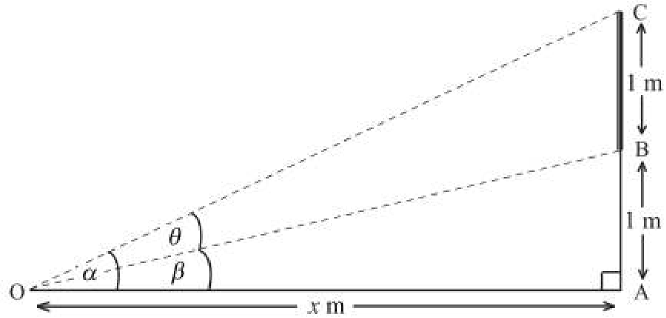


On the axes below, sketch the graph of $y_2 = |f'(x)|$.



13. [Maximum mark: 10] ***[without GDC]***

In Exercises [MAA 3.9], we had seen the following optimization problem:
In the diagram below, we had to find the maximum value of the angle of vision θ.
We expressed the angle θ in terms of x and by using the GDC we found the maximum value of θ. Let us now use differentiation to solve this problem:



(a) Show that $\theta = \arctan\dfrac{2}{x} - \arctan\dfrac{1}{x}$. [4]

(b) Hence, or otherwise, find the **exact** value of x for which θ is a maximum and justify that this value of x gives the maximum value of θ. [5]

(c) Find the maximum value of θ. [1]

(a) $\tan\beta = \dfrac{1}{x}$, $\tan\alpha = \dfrac{2}{x}$

$\alpha = \beta + \theta$

(c) $\theta = \arctan\left(\dfrac{2}{\sqrt{2}}\right) - \arctan\left(\dfrac{1}{\sqrt{2}}\right)$

$\theta = \alpha - \beta$

$\theta = \arctan\dfrac{2}{x} - \arctan\dfrac{1}{x}$

(b) $\dfrac{d\theta}{dx} = \dfrac{1}{1+\left(\frac{2}{x}\right)^2} \cdot \left(\dfrac{-2}{x^2}\right) - \dfrac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(\dfrac{-1}{x^2}\right)$

$\dfrac{d\theta}{dx} = \dfrac{-2}{(1+4x^{-2})(x^2)} + \dfrac{1}{(1+x^{-2})(x^2)}$

$0 = \dfrac{-2}{x^2+4} + \dfrac{1}{x^2+1}$

$\dfrac{2}{x^2+4} = \dfrac{1}{x^2+1}$

$2(x^2+1) = x^2+4$

$2x^2+2 = x^2+4$

$x = \sqrt{2}$

Length must be positive.

If $x=1$:

$\dfrac{d\theta}{dx} = -\dfrac{1}{5} + \dfrac{1}{2} = \dfrac{3}{10}$

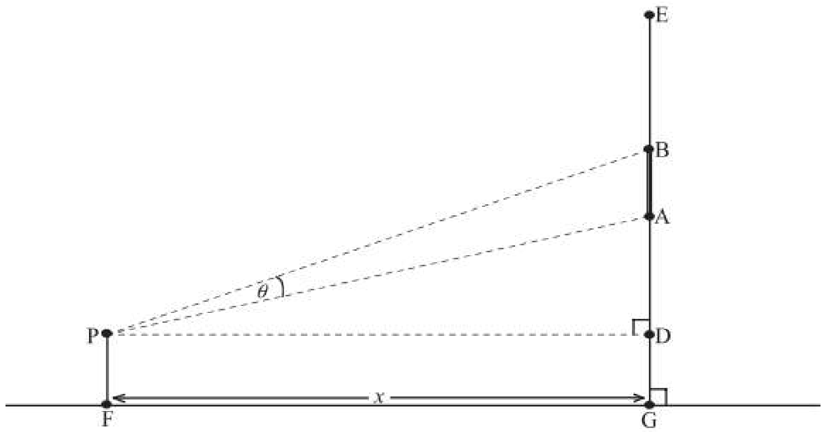
If $x=2$:

$\dfrac{d\theta}{dx} = -\dfrac{1}{4} + \dfrac{1}{5} = -\dfrac{1}{20}$

Changes from increasing to decreasing slope at $x=\sqrt{2}$, therefore it's a maximum point.

14. [Maximum mark: 9] ***[without GDC]***

A man PF is standing on horizontal ground at F at a distance x from the bottom of a vertical wall GE. He looks at a picture AB on the wall. Tha angle BPA is θ.



Let $\mathrm{DA}=a$, $\mathrm{DB}=b$, where angle PDE is a right angle. Find the value of x for which $\tan\theta$ is a maximum, giving your answer in terms of a and b. Justify that this value of x does give a maximum value of $\tan\theta$.

$PA^2 = x^2+a^2$

$PA = \sqrt{x^2+a^2}$

$\tan\theta = \dfrac{b-a}{\sqrt{x^2+a^2}}$

$\dfrac{d\tan\theta}{dx} = (b-a)\left[-\frac{1}{2}(x^2+a^2)^{-\frac{3}{2}} \circ 2x\right]$

$\dfrac{d\tan\theta}{dx} = (b-a)\left[\dfrac{-\cancel{2}x}{\cancel{2}(x^2+a^2)^{3/2}}\right]$

$\dfrac{d\tan\theta}{dx} = (b-a)\left[\dfrac{x}{(x^2+a^2)^{3/2}}\right]$

$0 = \dfrac{xb-xa}{(x^2+a^2)^{3/2}}$

B. Exam style questions (LONG)

15. [Maximum mark: 11] ***[without GDC]***

For each of the functions $f(x)=\mathrm{e}^{x}+\mathrm{e}^{-x}$ and $g(x)=\mathrm{e}^{x}-\mathrm{e}^{-x}$

(a) Determine whether it is even or odd. Justify your answer. [3]

(b) Find the stationary points (if any) and determine their nature. [4]

(c) Find the points of inflexion (if any); justify your answer. [4]

16. [Maximum mark: 12] ***[with / without GDC]***

The function *f* is defined by $f(x) = \dfrac{x^2}{2^x}$, for $x > 0$.

(a) (i) Show that $f'(x) = \dfrac{2x - x^2 \ln 2}{2^x}$.

(ii) Obtain an expression for $f''(x)$, simplifying your answer as far as possible. [5]

(b) (i) Find the **exact** value of *x* satisfying the equation $f'(x) = 0$.

(ii) Show that this value gives a maximum value for $f(x)$. [4]

(c) Find the x-coordinates of the two points of inflexion on the graph of f. [3]

17. [Maximum mark: 13] ***[without GDC]***

A family of cubic functions is defined as $f_k(x) = k^2x^3 - kx^2 + x,\ k \in \mathbb{Z}^+$.

(a) Express in terms of k

(i) $f_k'(x)$ and $f_k''(x)$;

(ii) the coordinates of the points of inflexion P_k on the graphs of f_k. [6]

(b) Show that all P_k lie on a straight line and state its equation. [2]

(c) Show that for all values of k, the tangents to the graphs of f_k on P_k are parallel, and find the equation of the tangent lines. [5]

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18. [Maximum mark: 11] ***[without GDC]***

A curve has equation $f(x) = \dfrac{a}{b + \mathrm{e}^{-cx}}$, $a \neq 0$, $b > 0$, $c > 0$.

(a) Show that $f''(x) = \dfrac{ac^2\mathrm{e}^{-cx}\left(\mathrm{e}^{-cx} - b\right)}{\left(b + \mathrm{e}^{-cx}\right)^3}$. [5]

(b) Find the coordinates of the point on the curve where $f''(x) = 0$. [4]

(c) Show that this is a point of inflexion. [2]

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19. [Maximum mark: 18] ***[without GDC]***

Give exact answers in this part of the question.

The temperature $g(t)$ at time *t* of a given point of a heated iron rod is given by

$$g(t) = \frac{\ln t}{\sqrt{t}}, \quad t > 0.$$

(a) Find the interval where $g'(t) > 0$. [4]

(b) Find the interval where $g''(t) > 0$ and the interval where $g''(t) < 0$. [5]

(c) Find the value of t where the graph of $g(t)$ has a point of inflexion. [3]

(d) Let $t*$ be a value of t for which $g'(t*) = 0$ and $g''(g'(t*) = 0) < 0$. Find $t*$. [3]

(e) Find the point where the normal to the graph of $g(t)$ at the point $(t*, g(t*))$ meets the t-axis. [3]

INDUCTION

Please look at also the **DERIVATIVES** section from **EXERCISES**

[MAA 1.9] MATHEMATICAL INDUCTION