*INTERNATIONAL BACCALAUREATE*

***Mathematics: analysis and approaches***

# *MAA*

***EXERCISES [MAA 4.12]***

***CONTINUOUS DISTRIBUTIONS IN GENERAL***

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## O. Practice questions

**1.** [Maximum mark: 10] ***[without GDC]***

The continuous random variable $X$ has probability density function

Area under a curve from and to the same value = 0 because they subtract each other out.

$$f(x)=\begin{cases}\frac{1}{9}x^2, & \text{for } 0\le x\le 3\\ 0, & \text{otherwise}\end{cases}$$

• P of an exact x in continuous curves is always equal to zero.

(a) Confirm that $f(x)$ is a pdf. [3]

(b) Find the values of (i) $\mathrm{P}(X=2)$ (ii) $\mathrm{P}(X\le 2)$ (iii) $\mathrm{P}(X<2)$ [4]

(c) Find the mean $\mu=\mathrm{E}(X)$. [3]

(a) $\int_0^3 \frac{1}{9}x^2\,dx = \frac{1}{27}x^3\Big|_0^3$ • $P(x\le y)=P(x<y)$; Always, only in continuous.

$\frac{1}{27}(3)^3=\frac{27}{27}=1$. Therefore it is a pdf because the whole area of the curve $=1$.

(b)

(i) $P(x=2)=0$

(ii) $P(x\le 2)=\int_0^2 \frac{1}{9}x^2\,dx$

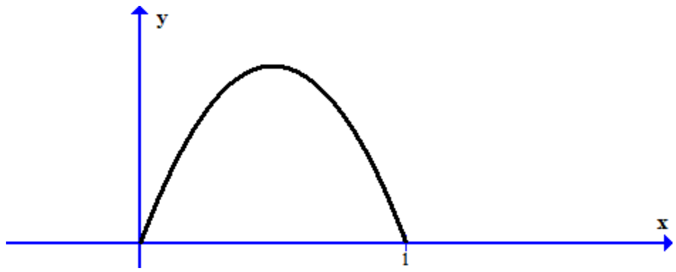
$=\frac{1}{27}(2)^2=4/27$

(iii) $P(x<2)=4/27=P(x\le 2)$ Rule!

**2.** [Maximum mark: 12] ***[with / without GDC]***

The continuous random variable $X$ has probability density function

$$f(x)=\begin{cases} 6x-6x^2, & \text{for } 0\le x\le 1 \\ 0, & \text{otherwise} \end{cases}$$




(a) Show that $\int_0^1 f(x)\mathrm{d}x=1$. [3]

(b) Find $\mathrm{P}\left(X\le\frac{1}{2}\right)$. [2]

(c) Write down the mean $\mu=\mathrm{E}(X)$, the median and the mode. [3]

(d) Find $\mathrm{E}(X^2)$. [2]

(e) Find $\mathrm{Var}(X)$. [2]

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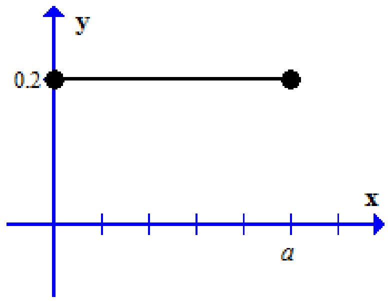
**3.** [Maximum mark: 11] ***[without GDC]***

The continuous random variable $X$ has probability density function

$$f(x)=\begin{cases}0.2, & \text{for } 0\le x\le a\\ 0, & \text{otherwise}\end{cases}$$

which is shown in the diagram below (it is known as uniform distribution).




(a) Find the value of $a$. [2]

(b) Find $\mathrm{P}(X=2)$ and $\mathrm{P}(X<2)$. [3]

(c) Find the mean of $X$. [2]

(d) Find $\mathrm{E}(X^2)$ and hence $\mathrm{Var}(X)$. [4]

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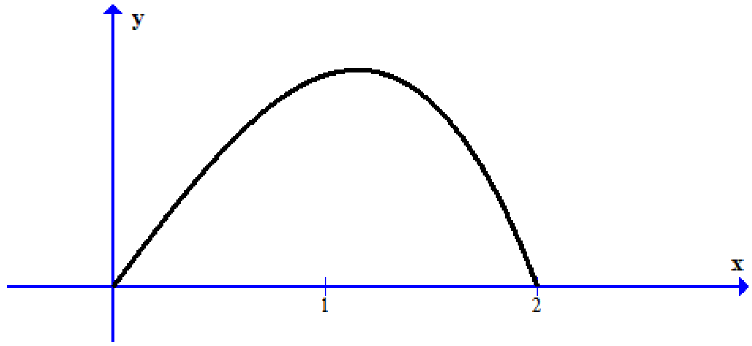
**4.** [Maximum mark: 25] ***[with GDC]***

The continuous random variable $X$ has probability density function

$$f(x)=\begin{cases} x-ax^3, & \text{for } 0\le x\le 2 \\ 0, & \text{otherwise} \end{cases}$$

The graph of $f$ is shown in the diagram below.




(a) Show that $a=\dfrac{1}{4}$. [3]

(b) Find $\mathrm{P}(X\le 1)$ and $\mathrm{P}(X>1)$. [3]

(c) Find the mean of $X$. [2]

(d) Find $\mathrm{E}(X^2)$ and hence $\mathrm{Var}(X)$. [3]

(e) Find (i) the median. (ii) $Q_1$ (iii) $Q_3$ [6]

(f) Show that the mode is $\dfrac{2\sqrt{3}}{3}$. [4]

(g) Find $\mathrm{E}(2X+3)$. [2]

(h) Find $\mathrm{E}(X^2+1)$. [2]

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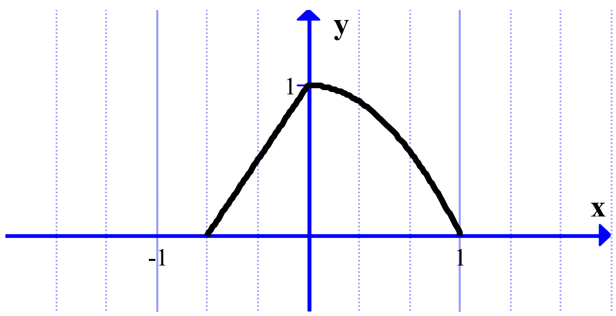
**5.** [Maximum mark: 23] ***[with GDC]***

The continuous random variable $X$ has probability density function

$$f(x) = \begin{cases} \frac{3}{2}x + 1, & \text{for } -\frac{2}{3} \le x \le 0 \\ 1 - x^2, & \text{for } \quad 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

which is shown in the diagram below.




(a) Justify that $f$ is a pdf. [3]

(b) Find $\mathrm{P}(X > 0.5)$. [2]

(c) Find the mean of $X$. [3]

(d) Find $\mathrm{E}(X^2)$ and hence $\mathrm{Var}(X)$. [5]

(e) Find $\mathrm{E}(2X+1)$. [2]

(f) Write down the mode. [1]

(g) Find the median and the quartiles $Q_1$ and $Q_3$. [7]

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## A. Exam style questions (SHORT)

**6.** [Maximum mark: 12] ***[with GDC]***

The continuous random variable $X$ has probability density function

$$f(x)=\frac{1}{6}x(1+x^2), \quad \text{for } 0\le x\le 2$$
$$f(x)=0 \quad \text{otherwise.}$$

(a) Sketch the graph of *f* for $0\le x\le 2$. [2]

(b) Write down the mode of $X$. [1]

(c) Find the mean of $X$. [4]

(d) Find the median of $X$. [5]

(a) $f'(x)=\frac{x}{6}(2x)+\frac{1}{6}(1+x^2)$

$f'(x)=\frac{1}{3}x^2+\frac{1}{6}x^2+1=\frac{1}{2}x^2+1$

$0=\frac{1}{2}x^2+1$

$\frac{1}{2}x^2=-1$ Graph it.

$x^2=-2$

(d) $\int_0^2$

**7.** [Maximum mark: 8] ***[with / without GDC]***

Let $f(x)$ be the probability density function for a random variable $X$, where

$$f(x)=\begin{cases} kx^2, & \text{for } 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that $k=\dfrac{3}{8}$. [2]

(b) Calculate (i) $\mathrm{E}(X)$; (ii) the median of $X$. [6]

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**8.** [Maximum mark: 6] ***[with GDC]***

A continuous random variable $X$ has probability density function

$$f(x)=\begin{cases} 12x^2(1-x), & \text{for } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability that $X$ lies between the mean and the mode.

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**9.** [Maximum mark: 7] ***[with GDC]***

A continuous random variable $X$ has a probability density function given by

$$f(x)=\begin{cases}\dfrac{(x+1)^3}{60} & \text{for } 1\le x\le 3\\ 0, & \text{otherwise}\end{cases}$$

(a) Find $\mathrm{P}(1.5\le X\le 2.5)$; [2]

(b) Find $\mathrm{E}(X)$; [2]

(c) Find the median of $X$. [3]

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**10.** [Maximum mark: 6] ***[without GDC]***

A continuous random variable $X$ has probability density function $f$ defined by

$$f(x)=\begin{cases}\mathrm{e}^x & \text{for } 0\le x\le \ln 2\\ 0, & \text{otherwise}\end{cases}$$

Find the **exact** value of $\mathrm{E}(X)$.

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**11.** [Maximum mark: 6] ***[without GDC]***

A continuous random variable $X$ has probability density function $f$ given by

$$f(x)=\begin{cases}\dfrac{8}{\pi(x^2+4)}, & \text{for } 0\le x\le 2\\ 0, & \text{otherwise}\end{cases}$$

(a) State the mode of $X$. [1]

(b) Find the **exact** value of $\mathrm{E}(X)$. [5]

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**12.** [Maximum mark: 6] ***[without GDC]***

Let $f(x)$ be as above [see question 11]

(a) Justify that $f$ is a pdf

(b) Find $\mathrm{E}(X^2)$.

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**13.** [Maximum mark: 6] ***[without GDC]***

The probability density function $f(x)$ of the continuous random variable $X$ is defined on the interval $[0, a]$ by

$$f(x) = \begin{cases} \frac{1}{8}x & \text{for } 0 \le x \le 3 \\ \frac{27}{8x^3} & \text{for } 3 \le x \le a \end{cases}$$

Find the value of $a$.

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**14.** [Maximum mark: 6] ***[with GDC]***

Let $f(x)$ be as above [see question 13]

Find the mean and the median of $X$.

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**15.** [Maximum mark: 8] ***[without GDC]***

The random variable $T$ has the probability density function

$$f(t) = \frac{\pi}{4}\cos\left(\frac{\pi t}{2}\right), -1 \le t \le 1.$$

(a) Find $\mathrm{P}(T = 0)$ . [1]

(b) Find the interquartile range. [5]

(c) Write down

(i) the mean (ii) the median [2]

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**16.** [Maximum mark: 10] ***[with / without GDC]***

The continuous random variable $X$ has probability density function

$$f(x)=\begin{cases}\dfrac{x}{1+x^2}, & \text{for } 0\le x\le k\\ 0, & \text{otherwise}\end{cases}$$

(a) Find the **exact** value of $k$. [5]

(b) Find the mode of $X$. [2]

(c) Calculate $\mathrm{P}(1\le X\le 2)$. [3]

**17.** [Maximum mark: 9] ***[with GDC]***

The time, $T$ minutes, required by candidates to answer a question in a mathematics examination has probability density function

$$f(t)=\begin{cases}\dfrac{1}{72}(12t-t^2-20), & \text{for } 4\le t\le 10\\ 0, & \text{otherwise}\end{cases}$$

(a) Find (i) $\mu$, the expected value of $T$ ; (ii) $\sigma^2$, the variance of $T$. [6]

(b) A candidate is chosen at random. Find the probability that the time taken by this candidate to answer the question lies in the interval $[\mu-\sigma,\mu]$. [3]

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**18.** [Maximum mark: 6] ***[with GDC]***

The lifetime of a particular component of a solar cell is $Y$ years, where $Y$ is a continuous random variable with probability density function $f(y)=0.5\mathrm{e}^{-y/2}$, $y\ge 0$.

(a) Find the probability, correct to four significant figures, that a given component fails within six months. [3]

Each solar cell has three components which work independently and the cell will continue to run if at least two of the components continue to work.

(b) Find the probability that a solar cell fails within six months. [3]

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## B. Exam style questions (LONG)

**19.** [Maximum mark: 12] ***[with GDC]***

A continuous random variable $X$ has probability density function defined by

$$f(x)=\begin{cases}\dfrac{c}{4+x^2}, & \text{for } -\dfrac{2}{\sqrt{3}}\le x\le 2\sqrt{3}\\ 0, & \text{otherwise}\end{cases}$$

(a) Find the **exact** value of the constant $c$ in terms of $\pi$. [5]

(b) Sketch the graph of $f(x)$ and hence state the mode of the distribution. [3]

(c) Find the **exact** value of $\mathrm{E}(X)$. [4]

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**20.** [Maximum mark: 20] ***[without GDC]***

The probability density function of the random variable $X$ is given by

$$f(x) = \begin{cases} \dfrac{k}{\sqrt{4-x^2}}, & \text{for } 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant $k$. [5]

(b) Show that $\mathrm{E}(X) = \dfrac{6(2-\sqrt{3})}{\pi}$. [7]

(c) Determine whether the median of $X$ is less than $\dfrac{1}{2}$ or greater than $\dfrac{1}{2}$. [8]

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**21.** [Maximum mark: 17] ***[with GDC]***

The continuous random variable $X$ has probability density function $f(x)$ where

$$f(x)=\begin{cases} \mathrm{e}-k\mathrm{e}^{kx}, & 0\le x\le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that $k=1$. [3]

(b) What is the probability that the random variable $X$ has a value that lies between $\frac{1}{4}$ and $\frac{1}{2}$? Give your answer **exactly**, in terms of $\mathrm{e}$. [2]

(c) Find the mean and variance of the distribution. Give your answers **exactly**, in terms of $\mathrm{e}$. [6]

The random variable $X$ above represents the lifetime, in years, of a certain type of battery.

(d) Find the probability that a battery lasts more than six months. [2]

A calculator is fitted with three of these batteries. Each battery fails independently of the other two. Find the probability that at the end of six months

(e) none of the batteries has failed; [2]

(f) exactly one of the batteries has failed. [2]

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**22.** [Maximum mark: 13] ***[with GDC]***

(a) Use integration by parts to show that

$\int 2x \arctan x dx = (x^2 + 1)\arctan x - x + C$ , where *C* is a constant. [6]

(b) The probability density function of the random variable $X$ is defined by

$$f(x) = \begin{cases} \frac{\pi}{2} - 2x \arctan x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

The value of $a$ is such that $P(X < a) = \frac{3}{4}$.

(i) Show that $a$ satisfies the equation $a(2\pi + 4) = 3 + 4(a^2 + 1)\arctan a$.

(ii) Find the value of $a$. [7]

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