*INTERNATIONAL BACCALAUREATE*

***Mathematics: analysis and approaches***

# *MAA*

***EXERCISES [MAA 5.21]***

***FURTHER AREAS AND VOLUMES***

*Compiled by Christos Nikolaidis*

## O. Practice questions

**1.** [Maximum mark: 12] ***[without GDC]***

(a) Find (i) $\int \ln x \mathrm{d}x$ (ii) $\int (\ln x)^2 \mathrm{d}x$. [6]

Let R be the region under the curve $y = \ln x$ and above $x-$axis, from $x = 1$ and $x = \mathrm{e}$.

(b) Find the area of the region R. [3]

(c) Find the volume of the solid generated when the region R is rotated through $2\pi$ radians about $x-$axis. [3]

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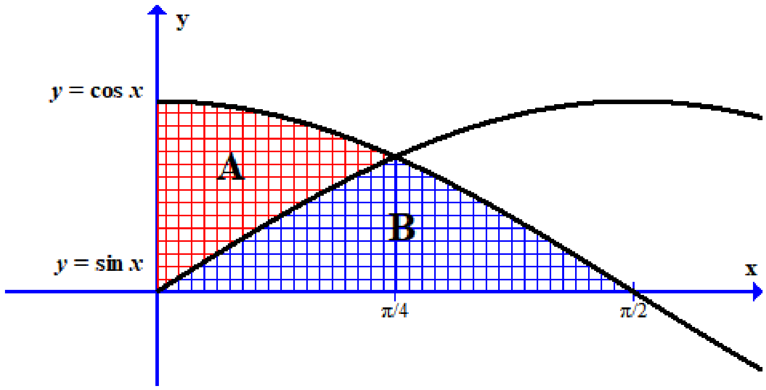
**2.** [Maximum mark: 13] ***[without GDC]***

Part of the functions $y = \sin x$ and $y = \cos x$ are shown on the diagram below




(a) Find the value of the area A. [3]

(b) Find the value of the area B. [4]

(c) The region A is rotated through $2\pi$ about $x$-axis. Find the volume of the solid generated. [4]

(d) The region B is rotated through $2\pi$ about $x$-axis. Write down an expression which represents the volume of the solid generated (do not find its value). [2]

$\int_0^{\pi/2} \cos x \, dx = \sin(\frac{\pi}{2}) - \sin(0) = 1 = A + B$

$\int_0^{\pi/4} \sin x \, dx = [-\cos(x)]_0^{\pi/4} = -\cos\frac{\pi}{4} + \cos 0 = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2} = \frac{1}{2}B$

(a) $B = 2 - \sqrt{2}$

$1 = A + 2 - \sqrt{2}$

(b) $A = \sqrt{2} - 1$

(c) $V_A = \pi \int_0^{\frac{\pi}{4}} (\cos x - \sin x)^2 \, dx$

$\pi \int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x) \, dx$ $= \pi \int_0^{\frac{\pi}{4}} \cos 2x \, dx$

$\pi(\cos^2(\frac{\pi}{4}) - \sin^2(\frac{\pi}{4}) - \cos^2(0) + \sin^2(0))$ $= \pi(\frac{1}{2}\sin 2x)]_0^{\frac{\pi}{4}}$

$\pi(\cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} - 1 + 0)$

$= -\pi$

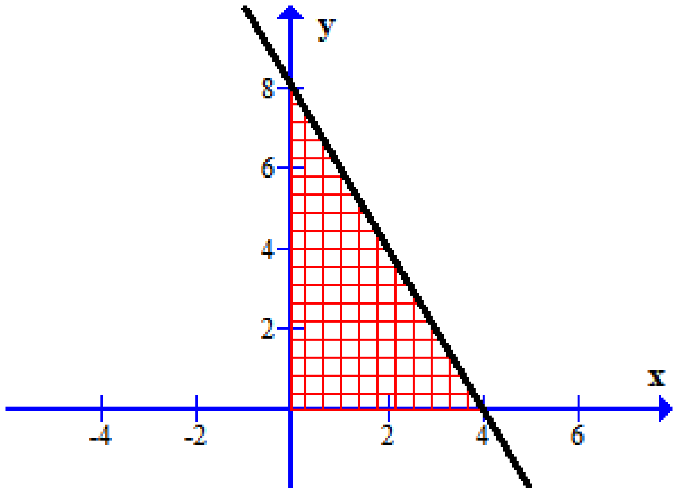
$= \pi(\frac{1}{2}\sin(\frac{\pi}{2}) + \frac{1}{2}\sin(0))$

$= \pi(\frac{1}{2} + \frac{1}{2}) = \pi$

(d) $V_B = \pi \int_0^{\frac{\pi}{4}} \sin^2 x \, dx + \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 x \, dx$

**3.** [Maximum mark: 13] ***[without GDC]***

Consider the triangular region R enclosed by the line $y = 8 - 2x$ and the two axes.




Let $A$ denote the area of the triangular region R.

When R is rotated about $x$-axis a cone of volume $V_x$ is generated.

When R is rotated about $y$-axis a cone of volume $V_y$ is generated.

(a) Complete the table below (use formulas for Area of a triangle - Volume of a cone)

| **Area** $A = \frac{b \times h}{2}$ | **Volume** $V_x = \frac{1}{3}\pi r^2 h$ | **Volume** $V_y = \frac{1}{3}\pi r^2 h$ |
|---|---|---|
| | | |

[5]

(b) Write down definite integrals that represent the following

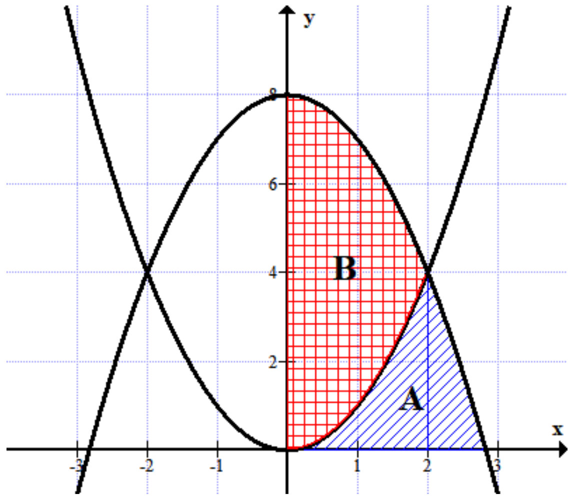
| **Area** $A$ **using** $A_x = \int_a^b y\mathrm{d}x$ | **Area** $A$ **using** $A_y = \int_a^b x\mathrm{d}y$ |
|---|---|
| | |

| **Volume** $V_x = \pi\int_a^b y^2\mathrm{d}x$ | **Volume** $V_y = \pi\int_a^b x^2\mathrm{d}y$ |
|---|---|
| | |

*[**Notice**: Use your GDC to check if the results in (b) agree with those in (a)]*

**4.** [Maximum mark: 24] ***[with GDC]***

Consider the two curves $y = x^2$, $y = 8 - x^2$




(a) Write down definite integrals that represent the following areas; find their values.

| Area | Integral expression | Value |
|---|---|---|
| $A$ using $A_x = \int_a^b y\mathrm{d}x$ | | |
| $A$ using $A_y = \int_a^b x\mathrm{d}y$ | | |
| $B$ using $A_x = \int_a^b y\mathrm{d}x$ | | |
| $B$ using $A_y = \int_a^b x\mathrm{d}y$ | | |

[12]

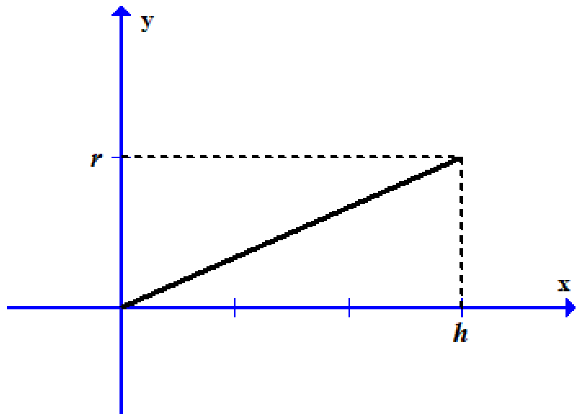
(b) Write down definite integrals that represent the volumes of the solids when the following $2\pi$–rotations occur; find their values.

| Volume | Integral expression | Value |
|---|---|---|
| $A$ rotated about $x$-axis | | |
| $A$ rotated about $y$-axis | | |
| $B$ rotated about $x$-axis | | |
| $B$ rotated about $y$-axis | | |

[12]

**5.** [Maximum mark: 6] ***[without GDC]***

Part of a line passing through the origin and the point $(h, r)$ is shown below.




(a) Show that the equation of the line is $y = \frac{r}{h}x$. [2]

The region between the line and and $x$-axis is rotated through $2\pi$ about $x$-axis and a cone of base radius $r$ and height $h$ is created.

(b) Prove the formula for the volume of the cone using integration. [4]

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**6.** [Maximum mark: 20] ***[with GDC]***

Consider the three curves $y = \mathrm{e}^{x}$, $y = 4$, $y = 20 - 4x$

(a) Sketch a graph and shade the region R in the first quadrant enclosed by the three curves and the $x$-axis. Indicate any intercepts and points of intersection. [5]

(b) Find the **exact value** of the area of the region R. [5]

(c) Find the **exact value** of the volume of the solid generated when the region R is rotated $2\pi$ radians in $x$-axis [5]

(d) Write down an expression for the volume of the solid generated when the region R is rotated $2\pi$ radians in $y$-axis and hence find its value. [5]

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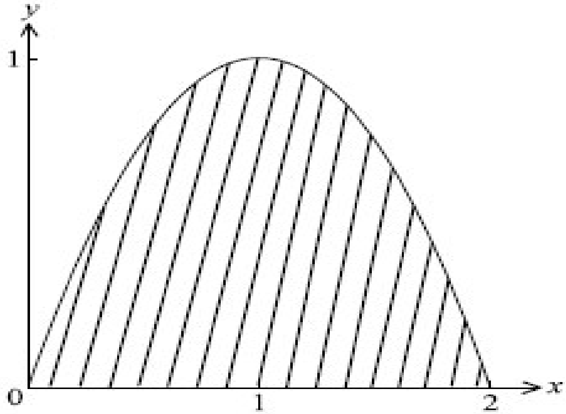
## A. Exam style questions (SHORT)

**7.** [Maximum mark: 4] ***[with / without GDC]***

A part of the graph of $y = 2x - x^2$ is given in the diagram below.




The shaded region is revolved through $360°$ about the $x$-axis.

(a) Write down an expression for this volume of revolution. [2]

(b) Calculate this volume. [2]

(a) $\left[\pi\int(2x-x^2)\,dx\right]_0^2$

(b) $\left[\pi\left(x^2-\frac{1}{3}x^3\right)\right]_0^2$

$\pi\left(2^2-\frac{1}{3}(2)^3\right)-\pi(1)$

$4\pi-\frac{8}{3}\pi-\pi$

$3\pi-\frac{8}{3}\pi$

$=\left(3-\frac{8}{3}\right)\pi$

(a) $\pi\int_0^2(2x-x^2)^2\,dx$

(b) $\pi\int_0^2(4x^2-4x^3+x^4)\,dx$

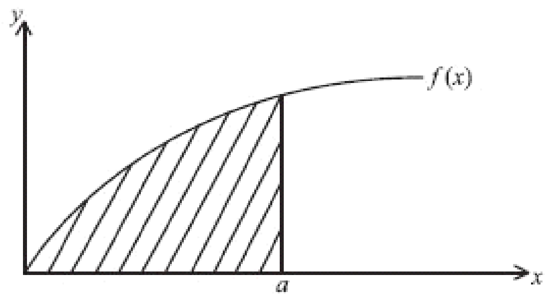
$\pi\left[\frac{4}{3}x^3-x^4+\frac{1}{5}x^5\right]_0^2$

$\pi\left(\frac{4}{3}(2)^3-(2)^4+\frac{1}{5}(2)^5\right)$

$\frac{32}{3}\pi-16\pi+\frac{32}{5}\pi$

**8.** [Maximum mark: 5] ***[with GDC]***

The shaded region in the diagram below is bounded by $f(x)=\sqrt{x}$, the line $x=a$ and the $x$-axis. The shaded region is revolved around the $x$-axis through $360°$. The volume of the solid formed is $0.845\pi$. Find the value of $\alpha$.

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**9.** [Maximum mark: 4] ***[with GDC]***

Consider the function $f(x)=e^{2x-1}+\dfrac{5}{2x-1}$, $x\neq\dfrac{1}{2}$.

The region between the curve and the $x$-axis between $x=1$ and $x=1.5$ is rotated through 360° about the $x$-axis. Let *V* be the volume formed.

(a) Write down an expression to represent *V*. [3]

(b) Hence write down the value of *V*. [1]

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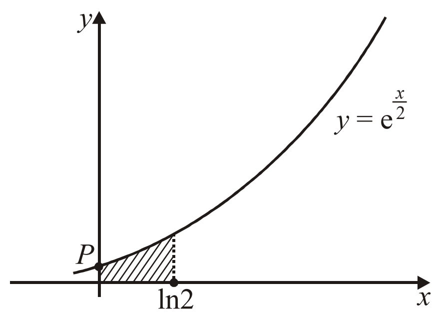
**10.** [Maximum mark: 7] ***[without GDC]***

The diagram shows part of the graph of $y = e^{\frac{x}{2}}$.




(a) Find the coordinates of the point *P*, where the graph meets the $y$-axis. [2]

The shaded region between the graph and the $x$-axis, bounded by $x = 0$ and $x = \ln 2$, is rotated through 360° about the $x$-axis.

(b) Write down an integral which represents the volume of the solid obtained. [2]

(c) Show that this volume is $\pi$. [3]

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**11.** [Maximum mark: 6] ***[without GDC]***

(a) Find $\int 3\sin^2 x \cos x dx$ [3]

(b) Let $g(x) = \sqrt{3}\sin x(\cos x)^{\frac{1}{2}}$ for $0 \le x \le \frac{\pi}{2}$. Find the volume generated when the curve of $g$ is revolved through $2\pi$ about the $x$-axis. [3]

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**12.** [Maximum mark: 6] ***[with GDC]***

The graph of $y = \sin(3x)$ for $0 \le x \le \frac{\pi}{4}$ is rotated through $2\pi$ radians about the $x$-axis. Find the volume of the solid of revolution formed.

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**13.** [Maximum mark: 9] ***[with GDC]***

The function $f$ is defined by $f(x)=\frac{\ln x}{x^3}$, $x \geq 1$.

The region enclosed by the $x$-axis, the graph of $f$ and the line $x=3$ is denoted by *R*.

(a) Find the volume of the solid of revolution obtained when *R* is rotated through 360° about the $x$-axis. [3]

(b) Show that the area of *R* is $\frac{1}{18}(4-\ln 3)$ [6]

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**14.** [Maximum mark: 4] ***[without GDC]***

The area between the graph of $y = e^x$ and the $x$-axis from $x = 0$ to $x = k$ ($k > 0$) is rotated through 360° about the $x$-axis. Find, in terms of $k$ and e, the volume of the solid generated.

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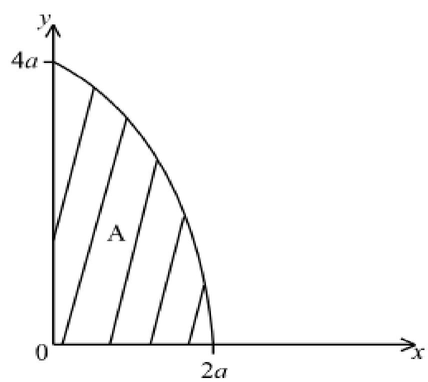
**15.** [Maximum mark: 6] ***[without GDC]***

The diagram below shows the shaded region *A* which is bounded by the axes and part of the curve $y^2 = 8a(2a - x)$, $a > 0$. Find in terms of $a$ the volume of the solid formed when *A* is rotated through 360° around the $x$-axis.

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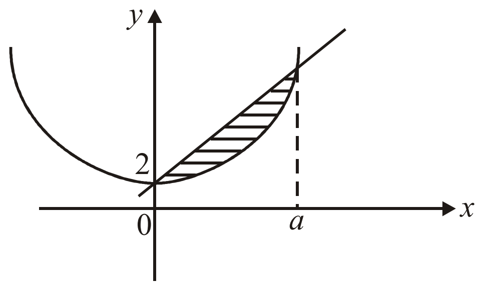
**16.** [Maximum mark: 10] ***[without GDC]***

The area of the enclosed region shown in the diagram is defined by $y \geq x^2 + 2$ and $y \leq ax + 2$, where $a > 0$.




(a) Find the area of the region. [3]

(b) This region is rotated 360° about the $x$-axis to form a solid of revolution. Find, in terms of $a$, the volume of this solid of revolution. [4]

(c) This region is rotated 360° about the $y$-axis to form a solid of revolution. Write down an expression that represents the volume of this solid of revolution. [3]

(a) $\int_0^a ax+2\,dx - \int_0^a x^2+2\,dx = A$

$A = \left[\frac{a}{2}x^2 + 2x\right]_0^a - \left[\frac{1}{3}x^3 + 2x\right]_0^a$

$A = \frac{1}{2}a^3 + \cancel{2a} - \frac{1}{3}a^3 - \cancel{2a}$

$\frac{3}{6}a^3 - \frac{2}{6}a^3 = \frac{1}{6}a^3 \text{ units}^2$

(b) $V = \pi\int_0^a (ax+2)^2\,dx - \pi\int_0^a (x^2+2)^2\,dx$

$V = \pi\int_0^a (a^2x^2 + 4ax + 4)\,dx - \pi\int_0^a (x^4 + 4x^2 + 4)\,dx$

$V = \pi\int_0^a (a^2x^2 + 4ax + \cancel{4} - x^4 - 4x^2 - \cancel{4})\,dx$

$V = \pi\int_0^a (-x^4 - 4x^2 + a^2x^2 + 4ax)\,dx$

$V = \pi\left[-\frac{1}{5}x^5 - \frac{4}{3}x^3 + \frac{1}{3}a^2x^3 + 2ax^2\right]_0^a$

$V = \pi\left(-\frac{1}{5}a^5 - \frac{4}{3}a^3 + \frac{1}{3}a^5 + 2a^3\right)$

$V = \pi\left(-\frac{3}{15}a^5 + \frac{5}{15}a^5 - \frac{4}{3}a^3 + \frac{6}{3}a^3\right)$

$V = \pi\left(\frac{2}{15}a^5 + \frac{2}{3}a^3\right) \text{ units}^3$

(c) $V = \pi\int$

**17.** [Maximum mark: 6] ***[without GDC]***

The region bounded by the curve $y = \frac{\ln(x)}{x}$ and the lines $x = 1$, $x = \mathrm{e}$, $y = 0$ is rotated through $2\pi$ radians about the $x$-axis. Find the volume of the solid generated.

**18.** [Maximum mark: 6] ***[without GDC]***

The region enclosed by the curves $y^2 = kx$ and $x^2 = ky$, where $k > 0$, is denoted by *R*. Given that the area of *R* is 12, find the value of *k*.

## B. Exam style questions (LONG)

**19.** [Maximum mark: 12] ***[without GDC]***

Consider functions of the form $y = e^{-kx}$.

(a) Show that $\int_0^1 e^{-kx}dx = \frac{1}{k}(1-e^{-k})$. [3]

(b) Let $k = 0.5$

(i) Sketch the graph of $y = e^{-0.5x}$, for $-1 \leq x \leq 3$, indicating the $y$-intercept.

(ii) Shade the region enclosed by this graph, $x$-axis, $y$-axis and line $x = 1$.

(iii) Find the area of this region. [5]

The region enclosed by the graph of $y = e^{-kx}$, $x$-axis, $y$-axis and line $x = 1$ is rotated through $2\pi$ about $x$-axis.

(c) Show that the volume of the solid generated is $\frac{\pi}{2k}(1-e^{-2k})$ [4]

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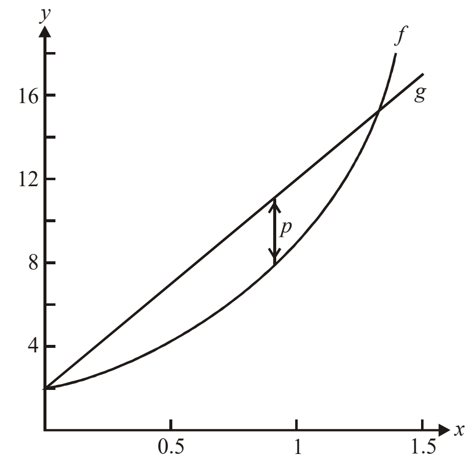
**20.** [Maximum mark: 15] ***[with GDC]***

The diagram below shows the graphs of $f(x)=1+e^{2x}$, $g(x)=10x+2$, $0 \le x \le 1.5$.




(a) (i) The graphs of $f$ and $g$ intersect at $x=0$ and $x=a$. Find the value of $a$.

(ii) Find an expression for the vertical distance $p$ between the two graphs.

(iii) Given that $p$ has a maximum value for $0 \le x \le a$, find the value of $x$ at which this occurs.

(iv) Hence find the maximum value of $p$. [8]

Let R be the region enclosed by enclosed by the graphs of $f$ and $g$.

(b) Find the area of the region R. [3]

(c) Find the volume of the solid generated when R Is rotated through $2\pi$ in $x$-axis. [4]

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**21.** [Maximum mark: 21] ***[with GDC]***

The function $f$ is defined on the domain $x \ge 1$ by $f(x) = \dfrac{\ln x}{x}$.

(a) (i) Show, by considering the first and second derivatives of $f$, that there is one *l.* maximum point on the graph of $f$.

(ii) State the **exact** coordinates of this point. [9]

(b) The graph of $f$ has a point of inflexion at P. Find the $x$-coordinate of P. [3]

Let *R* be the region enclosed by the graph of $f$, the $x$-axis and the line $x = 5$.

(c) Find the **exact** value of the area of *R*. [6]

(d) The region R is rotated through an angle $2\pi$ about the $x$-axis. Find the volume of the solid of revolution generated. [3]

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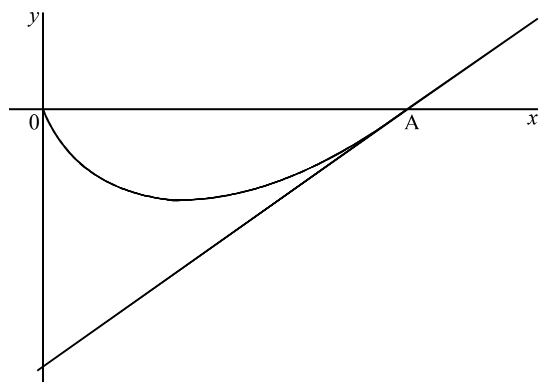
**22.** [Maximum mark: 20] ***[with GDC]***

Consider the function $f_k(x)=\begin{cases} x\ln x - kx, & x>0 \\ 0, & x=0 \end{cases}$, where $k \in \mathbb{N}$.

(a) Find the derivative of $f_k(x)$, $x>0$. [2]

(b) Find the interval over which $f_k(x)$ is increasing. [2]

The graph of the function $f_k(x)$ is shown below.




(c) (i) Show that the stationary point of $f_k(x)$ is at $x=\mathrm{e}^{k-1}$.

(ii) One $x$-intercept is at (0, 0). Find the coordinates of the other $x$-intercept. [4]

(d) Find the area enclosed by the curve and the $x$-axis. [5]

(e) Find the equation of the tangent to the curve at A. [2]

(f) Show that the area of the triangular region created by the tangent and the coordinate axes is twice the area enclosed by the curve and the $x$-axis. [2]

(g) Show that the $x$-intercepts of $f_k(x)$ for consecutive values of $k$ form a geometric sequence. [3]

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**23.** [Maximum mark: 16] ***[with GDC]***

Let $f(x) = \ln\left|x^5 - 3x^2\right|$, $-0.5 < x < 2$, $x \neq a$, $x \neq b$;

($a, b$ are values of $x$ for which $f(x)$ is not defined).

(a) (i) Sketch the graph of $f(x)$, indicating on your sketch the number of zeros *of* $f(x)$. Show also the position of any asymptotes.

(ii) Find all the zeros of $f(x)$, (that is, solve $f(x) = 0$). [5]

(b) Find the **exact** values of $a$ and $b$. [3]

(c) Find $f'(x)$, and indicate clearly where $f'(x)$ is not defined. [3]

(d) Find the **exact** value of the $x$-coordinate of the local maximum of $f(x)$, for $0 < x < 1.5$. (You may assume that there is no point of inflexion.) [3]

(e) **Write down** the definite integral that represents the area of the region **enclosed** by $f(x)$ and the $x$-axis. (Do **not** evaluate the integral.) [2]

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**24*.** [Maximum mark: 24] ***[without GDC]***

(a) Using the formula for $\cos(A+B)$ prove that $\cos^2\theta = \dfrac{\cos 2\theta + 1}{2}$. [3]

(b) Hence, find $\int \cos^2 x \mathrm{d}x$. [4]

Let $f(x) = 4\cos x$ and $g(x) = \sec x$ for $x \in \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[$.

Let $R$ be the region enclosed by the two functions.

(c) Find the **exact** values of the $x$-coordinates of the points of intersection. [4]

(d) Sketch the functions $f$ and $g$ and clearly shade the region $R$. [3]

The region $R$ is rotated through $2\pi$ about the $x$-axis to generate a solid.

(e) (i) Write down an integral which represents the volume of this solid.

(ii) Hence find the exact value of this volume. [10]

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**25.** [Maximum mark: 21] ***[with GDC]***

The function $f$ is defined by $f(x) = x\sqrt{9-x^2} + 2\arcsin\left(\frac{x}{3}\right)$.

(a) Write down the largest possible domain, for each of the two terms of the function and $f$, and hence state the largest possible domain, $D$, for $f$. [2]

(b) Find the volume generated when the region bounded by the curve $y = f(x)$, the $x$-axis, the $y$-axis and the line $x = 2.8$ is rotated through $2\pi$ radians about the $x$-axis. [3]

(c) Find $f'(x)$ in simplified form. [5]

(d) **Hence** show that $\int_{-p}^{p} \frac{11-2x^2}{\sqrt{9-x^2}}\mathrm{d}x = 2p\sqrt{9-x^2} + 4\arcsin\left(\frac{p}{3}\right)$, where $p \in D$. [2]

(e) Find the value of $p$ which maximises the value of the integral in (d). [2]

(f) (i) Show that $f''(x) = \frac{x(2x^2-25)}{\left(9-x^2\right)^{\frac{3}{2}}}$.

(ii) Hence justify that $f(x)$ has a point of inflexion at $x = 0$, but not at $x = \pm\sqrt{\frac{25}{2}}$. [7]

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**26.** [Maximum mark: 11] ***[without GDC]***

Let $f(x) = x\cos 3x$ .

(a) Use integration by parts to show that $\int f(x)\,\mathrm{d}x = \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + c.$ [3]

(b) Use your answer to part (a) to calculate the exact area enclosed by $f(x)$ and the $x$-axis in each of the following cases. Give your answers in terms of $\pi$

(i) $\frac{\pi}{6} \le x \le \frac{3\pi}{6}$ (ii) $\frac{3\pi}{6} \le x \le \frac{5\pi}{6}$ (iii) $\frac{5\pi}{6} \le x \le \frac{7\pi}{6}$ [4]

(c) Given that the above areas are the first three terms of an arithmetic sequence, find an expression for the total area enclosed by $f(x)$ and the $x$-axis for $\frac{\pi}{6} \le x \le \frac{(2n+1)\pi}{6}$, where $n \in \mathbb{Z}$. Give your answers in terms of $n$ and $\pi$. [4]

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