INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

MAA

EXERCISES [MAA 5.22]

DIFFERENTIAL EQUATIONS

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O. Practice questions

1. [Maximum mark: 5] ***[without GDC]***

(a) Find the **general** solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}=4x+1$. [2]

(b) Given that $y(3)=1$, find the **particular** solution of the differential equation above. [3]

(a) $dy=(4x+1)dx$

$\int dy=\int(4x+1)dx$

$y=2x^2+x+C$

(b) $1=2(3)^2+(3)+C$

$1=18+3+C$ $\quad y=2x^2+x-20$

$C=-20$

2. [Maximum mark: 7] ***[without GDC]***

(a) Find the **general** solution of the d.e. $\frac{\mathrm{d}y}{\mathrm{d}x}=(4x+1)y^2$ in the form $y=f(x)$, [4]

(b) Given that $y(3)=1$, find the **particular** solution of the differential equation above. [3]

(a) $\int\frac{1}{y^2}dy=\int(4x+1)dx$

$\frac{-1}{y}=2x^2+x+C$

$y=\frac{-1}{2x^2+x+C}$

(b) $1=\frac{-1}{2(3)^2+3+C}$

$1=\frac{-1}{21+C}$ $\quad y=\frac{-1}{2x^2+x+22}$

$21+C=-1,\ C=22$

Inverse general solution yields a particular solution that operates with the negative denominator.

3. [Maximum mark: 8] ***[without GDC]***

The equation $x^2 + y^2 = 1$ defines the unit circle with its centre at the origin. The circle passes through the point $(1, 0)$.

(a) Use implicit differentiation on $x^2 + y^2 = 1$ to show that $\frac{dy}{dx} = -\frac{x}{y}$. [3]

(b) Solve the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ given that the curve passes through $(1, 0)$ to derive the equation of the unit circle. [5]

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(a) $2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{2y}$

$\frac{dy}{dx} = \frac{-x}{y}$

(b) $\frac{dy}{dx} = \frac{-1}{\odot}$

$\int y\,dy = \int -x\,dx$

$\int y\,dy = -\int x\,dx$

$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$

$y = \sqrt{2C - 2x^2}$

$0 = \sqrt{2c - 2(1)}$

$0 = 2c - 2$

$2 = 2c$

$c = 1$

$\Rightarrow y = (-2x^2 + 2)^{\frac{1}{2}}$

4. [Maximum mark: 10] ***[without GDC]***

The **rate of change** of the variable y is proportional to the **square** of y.

(a) Write down a differential equation to represent this information. [2]

(b) Find the **general** solution of the differential equation in (a). [3]

(c) Given that $y(0)=1$ and $y(1)=\frac{1}{3}$, express y in terms of x. [5]

(a) $\frac{dy}{dx}=ky^2$

(b) $\int\frac{1}{y^2}dy=\int k\,dx$

$-\frac{1}{y}=kx+C$

$-y=\frac{1}{kx+C}$

$y=\frac{-1}{kx+C}$

(c) $1=\frac{-1}{C}$

$C=-1$

$\frac{1}{3}=\frac{-1}{k(1)-1}$

$k-1=-3$

$k=-2$

$y=\frac{-1}{-2x-1}$

$y=\frac{1}{2x+1}$

5. [Maximum mark: 18] ***[with GDC]***

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}$, with $y(1) = 2$. ← Homogeneous

(a) Solve the equation

(i) as a **D.E of separable variables**

(ii) as a **homogeneous D.E.**

(iii) as **a linear D.E. of first order** (using integrating factor). [12]

(b) Find the actual value of $y(2)$. [1]

(c) Use Euler's method to find two approximations for $y(2)$:

(i) use a step of $h = 0.2$ and complete a table of three columns (n, x_n, y_n);

(ii) use a step of $h = 0.01$; give only the approximation of $y(2)$. [5]

(a)

(i) $dy = -\frac{y}{x}dx$

$\int \frac{1}{y}dy = -\int \frac{1}{x}dx$

$\ln y = -\ln x + c$

Combine

$y = e^{(-\ln x + c)} = \frac{1}{e^{(\ln x + c)}}$

$2 = \frac{1}{e^{(\ln(1)+c)}}$

$2 = \frac{1}{e^{c}}$

$2e^{c} = 1$

$e^{c} = \frac{1}{2}$

$c = \ln\frac{1}{2}$

$y = \frac{1}{e^{(\cancel{\ln} x + \cancel{\ln}\frac{1}{2})}}$

$y = \frac{1}{x + \frac{1}{2}}$

(i) $\ln y = -\ln x + c$

$\ln y + \ln x = c$

$\ln yx = c$

$yx = e^{c}$

$y = \frac{e^{c}}{x}$

$y = \frac{C}{x}$

(ii) Let $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = -\frac{v\cancel{x}}{\cancel{x}}$

$v + x\frac{dv}{dx} = -v$

$x\frac{dv}{dx} = -2v$

$\frac{1}{v}dv = -2\frac{1}{x}dx$

$\int \frac{1}{v}dv = -2\int \frac{1}{x}dx$

$\ln(v) = -2\ln x + c$

$\ln(\frac{y}{x}) + 2\ln x = C$

$\ln(\frac{y}{\cancel{x}}) + \ln x^{\cancel{2}} = C$

$\ln(yx) = C$

$e^{c} = yx;\ y = \frac{e^{c}}{x} = \frac{C}{x}$

(iii) $\underbrace{\frac{dy}{dx} + \frac{1}{x}}_{P(x)} y = \underbrace{0}_{Q(x)}$

$I = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$Iy = \int IQ \, dx$

$xy = \frac{1}{2}x^2(0) + C$

$xy = C$

$y = \frac{C}{x}$

(b)

$2 = \frac{C}{1}$

$C = 2$

$y = \frac{2}{x}$

$y = \frac{2}{2} = 1$

$y(2) = 1$

A. Exam style questions (SHORT)

D.E. OF SEPARABLE VARIABLES

6. [Maximum mark: 5] ***[without GDC]***

Given that $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x - 2x$ and $y = 3$ when $x = 0$, find an expression for y in terms of x.

$\int (1)\,dy = \int (e^x - 2x)\,dx$

$y = e^x - x^2 + C$

$3 = e^0 - (0)^2 + C$

$3 = 1 + C$

$C = 2$

$y = -x^2 + e^x + 2$

7. [Maximum mark: 6] ***[without GDC]***

Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2$ given that $y = 1$ when $x = 0$. Give your answer in the form $y = f(x)$.

$\int \frac{1}{y^2}\,dy = 2\int x\,dx$

$-\frac{1}{y} = \cancel{2}\left(\frac{1}{\cancel{2}}x^2\right) + C$

$-\frac{1}{y} = x^2 + C$

$y = \frac{-1}{x^2 + C}$

$1 = -\frac{1}{C}$

$C = -1$

$y = \frac{-1}{x^2 - 1} = \frac{1}{1 - x^2}$

8. [Maximum mark: 6] ***[without GDC]***

Solve the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = 1$, given that $y = 0$ when $x = 2$. Give your answer in the form $y = f(x)$.

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9. [Maximum mark: 6] ***[without GDC]***

Solve the differential equation $xy\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2$, given that $y = 0$ when $x = 2$.

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10. [Maximum mark: 6] ***[without GDC]***

(a) Show that $\dfrac{1}{x(5-x)} = \dfrac{1}{5}\left(\dfrac{1}{x} + \dfrac{1}{5-x}\right)$

(b) Find the general solution of the differential equation $\dfrac{dx}{dt} = kx(5-x)$

where $0 < x < 5$, and k is a constant.

(a) RHS:

$\dfrac{1}{5x} + \dfrac{1}{5(5-x)}$

$\dfrac{5(5-x)}{5x(5(5-x))} + \dfrac{5x}{5x(5(5-x))} = \dfrac{25-\cancel{5x}+\cancel{5x}}{5x(25-5x)}$

(b) $\dfrac{dt}{dx} = \dfrac{1}{k} \cdot \dfrac{1}{x(5-x)}$

$\dfrac{dt}{dx} = \dfrac{1}{k}\left(\dfrac{1}{5}\left(\dfrac{1}{x} + \dfrac{1}{5-x}\right)\right)$

$\dfrac{dt}{dx} = \dfrac{1}{k}\left(\dfrac{1}{5x} + \dfrac{1}{5(5-x)}\right)$

$dt = \dfrac{1}{k}\left(\dfrac{1}{5}\int\dfrac{1}{x}\,dx + \dfrac{1}{5}\int\dfrac{1}{(5-x)}\,dx\right)$

$u = 5-x$

$du = -dx$

$dx = -du$

$-\int \dfrac{1}{u}\,du$

$t = \dfrac{1}{k}\left(\dfrac{1}{5}\ln x - \dfrac{1}{5}\ln(5-x)\right)$

$t = \dfrac{1}{5k}\ln\left(\dfrac{x}{5-x}\right)$

• No C because a domain restriction of $0<x<5$ is given.

$\ln\left(\dfrac{x}{5-x}\right) = 5kt$

$\left(\dfrac{x}{5-x}\right) = e^{5kt}$

11. [Maximum mark: 6] ***[without GDC]***

Solve the differential equation $(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} - xy = 0$ where $x > 0,\ y > 0$, given that $y = 1$ when $x = 1$.

$(x^2+1)\frac{dy}{dx} = xy$

$u = x^2+1$

$du = 2x\,dx$

$(x^2+1)\,dy = xy\,dx$

$dx = \frac{du}{2x}$

$\int \frac{1}{y}\,dy = \int \frac{x}{(x^2+1)}\,dx$

$\int \frac{x}{u}\left(\frac{du}{2x}\right) = \frac{1}{2}\int \frac{1}{u}\,du$

$\ln(y) = \frac{1}{2}\ln(x^2+1) + C$

$\ln\left(\frac{y}{\sqrt{x^2+1}}\right) = C$

$e^C = \frac{y}{\sqrt{x^2+1}}$

$e^C\sqrt{x^2+1} = y$

$e^C\sqrt{1^2+1} = 1$

$e^C\sqrt{2} = 1$

$e^C = \frac{1}{\sqrt{2}}$

$C = \ln\left(\frac{1}{\sqrt{2}}\right)$

$y = \cancel{e}^{\cancel{\ln}\left(\frac{1}{\sqrt{2}}\right)}\sqrt{x^2+1}$

$y = \frac{1}{\sqrt{2}}\sqrt{x^2+1} = \sqrt{\frac{x^2+1}{2}}$

12. [Maximum mark: 6] ***[without GDC]***

Solve the differential equation $(x+2)^2\frac{\mathrm{d}y}{\mathrm{d}x}=4xy$ ($x>-2$), given that $y=1$ when $x=-1$.

$\int \frac{1}{y}\,dy=\int \frac{4x}{(x+2)^2}\,dx$

$(x+2)^2+0$

$u=(x+2)^2$

$du=2(x+2)(1)\,dx$

$du=(2x+4)\,dx$

$dx=\frac{du}{2x+4}$

$\int \frac{4x}{u}\left(\frac{du}{2x+4}\right)$

13. [Maximum mark: 6] ***[without GDC]***

Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+y^2}{1+x^2}$, given that $y = \sqrt{3}$ when $x = \frac{\sqrt{3}}{3}$.

Give your answer in the form $y = \frac{ax + \sqrt{a}}{a - x\sqrt{a}}$ where $a \in Z^+$.

14. [Maximum mark: 6] ***[without GDC]***

The equation of motion of a particle with mass m, subjected to a force kx can be written as $kx = mv\frac{\mathrm{d}v}{\mathrm{d}x}$, where *x* is the displacement and v is the velocity. When $x = 0$, $v = v_0$.

Find v, in terms of m, k and v_0.

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15. [Maximum mark: 6] ***[with GDC]***

A sample of radioactive material decays at a rate which is proportional to the amount of material present in the sample. Find the half-life of the material if 50 grams decay to 48 grams in 10 years.

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16. [Maximum mark: 6] ***[without GDC]***

The tangent to the curve $y = f(x)$ at the point $\mathrm{P}(x, y)$ meets the x-axis at $\mathrm{Q}(x-1, 0)$.

The curve meets the y-axis at $\mathrm{R}(0, 2)$. Find the equation of the curve.

17. [Maximum mark: 6] ***[without GDC]***

When air is released from an inflated balloon it is found that the rate of decrease of the volume of the balloon is proportional to the volume of the balloon. This can be represented by the differential equation $\frac{dv}{dt} = -kv$, where v is the volume, t is the time and k is the constant of proportionality.

(a) If the initial volume of the balloon is v_0, find an expression, in terms of k, for the volume of the balloon at time t.

(b) Find an expression, in terms of k, for the time when the volume is $\frac{v_0}{2}$.

(a) $\int \frac{1}{V} dV = -\int k \, dt$

$\ln V = -tk + c$

$e^{(-tk+c)} = V$

HOMOGENEOUS D.E

18. [Maximum mark: 5] ***[without GDC]***

Show that the solution of the homogeneous differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + 1, \ x > 0,$$

given that $y = 0$ when $x = \mathrm{e}$, is $y = x(\ln x - 1)$.

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19. [Maximum mark: 11] ***[without GDC]***

Solve the differential equation $x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + 3xy + 2x^2$

given that $y = -1$ when $x = 1$. Give your answer in the form $y = f(x)$.

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20. [Maximum mark: 9] ***[without GDC]***

Solve the differential equation $x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + xy + 4x^2$, given that $y = 2$ when $x = 1$. Give your answer in the form $y = f(x)$.

21. [Maximum mark: 13] ***[with GDC]***

Consider the differential equation $\frac{dy}{dx}=\frac{y^2+x^2}{2x^2}$ for which $y=-1$ when $x=1$.

(i) Solve the differential equation giving your answer in the form $y=f(x)$.

(ii) Find the value of y when $x=2$.

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22. [Maximum mark: 13] ***[without GDC]***

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{x}+\frac{y^2}{x^2} \text{ (where } x>0\text{)}$$

given that $y=2$ when $x=1$. Give your answer in the form $y=f(x)$.

FIRST ORDER LINEAR D.E

23. [Maximum mark: 10] ***[without GDC]***

The variables x and y are related by $\frac{\mathrm{d}y}{\mathrm{d}x} - y\tan x = \cos x$.

Solve the differential equation given that $y = 0$ when $x = \pi$. Give the solution in the form $y = f(x)$.

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24. [Maximum mark: 8] ***[without GDC]***

Solve the differential equation $(\cos x - \sin x)\dfrac{\mathrm{d}y}{\mathrm{d}x} + (\cos x + \sin x)y = (\cos x - \sin x)$

given that $y = -1$ when $x = \dfrac{\pi}{2}$.

25. [Maximum mark: 11] ***[with GDC]***

Given that $\frac{dy}{dx} + 2y\tan x = \sin x$, and $y = 0$ when $x = \frac{\pi}{3}$, find the maximum value of y.

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26. [Maximum mark: 8] ***[without GDC]***

Solve the differential equation $(u+3v^3)\frac{\mathrm{d}v}{\mathrm{d}u}=2v$. Give your answer in the form $u=f(v)$.

27. [Maximum mark: 13] ***[without GDC]***

Consider the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x}-2y=\frac{x^3}{x^2+1}.$$

(a) Find an integrating factor for this differential equation. [5]

(b) Solve the differential equation given that $y=1$ when $x=1$, giving your answer in the form $y=f(x)$. [8]

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28*. [Maximum mark: 9] ***[without GDC]***

A curve that passes through the point $(1,\ 2)$ is defined by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2x\left(1+x^2-y\right).$$

Solve the differential equation giving your answer in the form $y=f(x)$.

29. [Maximum mark: 11] ***[without GDC]***

(a) Express $\dfrac{1}{(x+1)(x+2)}$ in partial fractions. [4]

(b) Solve the following differential equation $(x+1)(x+2)\dfrac{dy}{dx}+y=x+1$ giving your answer in the form $y=f(x)$. [7]

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30. [Maximum mark: 10] ***[with GDC]***

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{xy}{4-x^2}=1$, where $|x|<2$ and $y=1$ when $x=0$.

(i) By first finding an integrating factor, solve this differential equation. Give your answer in the form $y=f(x)$.

(ii) Calculate, correct to two decimal places, the value of y when $x=1$.

EULER'S METHOD

31. [Maximum mark: 6] ***[with GDC]***

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2$ where $y = 1$ when $x = 0$.

Use Euler's method with a step length of 0.1 for the following:

(a) Show the first two steps (for the calculation of y_1 and y_2) analytically. [3]

(b) Find an approximate value of y when $x = 0.4$. [3]

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32. [Maximum mark: 5] ***[with GDC]***

Let $\frac{\mathrm{d}y}{\mathrm{d}x} - 2y^2 = \mathrm{e}^x$ and $y = 1$ when $x = 0$. Use Euler's method with $h = 0.1$ to find an approximation for the value of y when $x = 0.4$. Give all intermediate values in a table.

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33. [Maximum mark: 5] ***[with GDC]***

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2+x^2}{2x^2}$ for which $y=-1$ when $x=1$.

Use Euler's method with $h=0.25$ to find an estimate for the value of y when $x=2$.

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34. [Maximum mark: 6] ***[with GDC]***

A curve that passes through the point (1, 2) is defined by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2x\left(1+x^2-y\right).$$

(i) Use Euler's method to get an approximate value of *y* when $x=1.3$, taking steps of 0.1. Show intermediate steps to four decimal places in a table.

(ii) How can a more accurate answer be obtained using Euler's method?

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35. [Maximum mark: 7] ***[with GDC]***

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}y}+\frac{xy}{4-x^2}$ = 1, where $|x|<2$ and $y=1$ when $x=0$.

(a) Use Euler's method with $h=0.25$, to find an approximate value of y when $x=1$, giving your answer to two decimal places. [4]

The exact solution of this equation is $y=\sqrt{4-x^2}\left(\arcsin\left(\frac{x}{2}\right)+\frac{1}{2}\right)$ and $y(1)=1.77$.

(b) Sketch the graph of $y=f(x)$ for $0\le x\le 1$. Use your sketch to explain why your approximate value of y is greater than the true value of y. [3]

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