INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

MAA

EXERCISES [MAA 5.16-5.17]

CONTINUITY – DIFFERENTIABILITY - L'HÔPITAL

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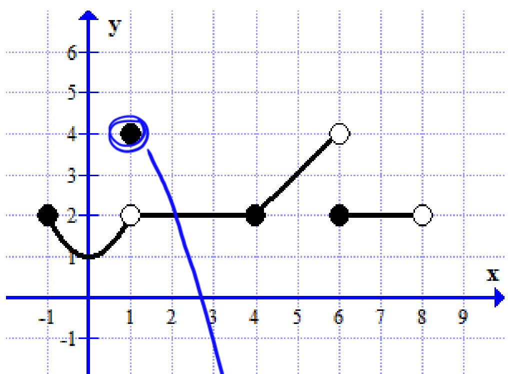
LIMITS AND CONTINUITY

O. Practice questions

1. [Maximum mark: 17] ***[without GDC]***

The graph of the function f is shown below



(a) Write down the domain and the range of the function.

Domain of f: $-1 \le x < 8$	Range of f: $1 \le y \le 4$

[2]

(b) Complete the following table of values or state "not defined".

$f(-1) =$ 2	$f(1) =$ 4	$f(4) =$ 2	$f(6) =$ 2	$f(8) =$ DNE

[3]

(c) Complete the following table of limits or state "doesn't exist"

$\lim_{x\to 1^-} f(x) =$ 2	$\lim_{x\to 1^+} f(x) =$ 2	$\lim_{x\to 1} f(x) =$ 2
$\lim_{x\to 4^-} f(x) =$ 2	$\lim_{x\to 4^+} f(x) =$ 2	$\lim_{x\to 4} f(x) =$ 2
$\lim_{x\to 6^-} f(x) =$ 4	$\lim_{x\to 6^+} f(x) =$ 2	$\lim_{x\to 6} f(x) =$ DNE

[6]

(d) Determine whether the function is continuous at $x = 4$, $x = 1$, $x = 6$. Justify.

continuity at $x = 4$	Yes; limits match from both sides
continuity at $x = 1$	Yes; limits match from both sides
continuity at $x = 6$	No; limits are different from each side

[6]

2. [Maximum mark: 20] ***[without GDC]***

Let

$$f(x)=\begin{cases} x^2+1 & -1\le x<1 \\ 4 & x=1 \\ 2 & 1<x\le 4 \\ x-2 & 4<x<6 \\ 2 & 6\le x<8 \end{cases}$$

(a) Write down the domain of the function.

Domain of f: $-1\le x<8$

[1]

(b) Complete the following table of values or state "not defined".

$f(-1)=$ 2	$f(1)=$ 4	$f(4)=$ 2	$f(6)=$ 2	$f(8)=$ ND

[5]

(c) Complete the following table of limits or state "doesn't exist"

$\lim_{x\to1^-} f(x)=\lim_{x\to1^-}(x^2+1)=2$
$\lim_{x\to1^+} f(x)=\lim_{x\to1^+}(x^2+1)=4$
$\lim_{x\to1} f(x)=$ Doesn't exist
$\lim_{x\to4^-} f(x)=\lim_{x\to4^-}(2)=2$
$\lim_{x\to4^+} f(x)=\lim_{x\to4^+}(x-2)=2$
$\lim_{x\to4} f(x)=4$
$\lim_{x\to6^-} f(x)=\lim_{x\to6^-}(x-2)=4$
$\lim_{x\to6^+} f(x)=\lim_{x\to6^+}(2)=2$
$\lim_{x\to6} f(x)=$ Doesn't exist

[8]

(d) Determine whether the function is continuous at $x=4$, $x=1$, $x=6$. Justify.

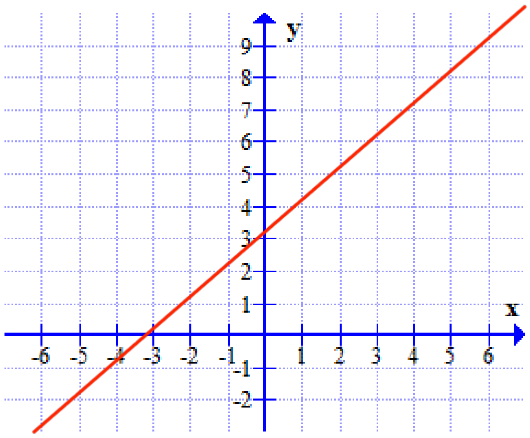
continuity at $x=4$	No; Limit and function output differ.
continuity at $x=1$	No; Limit and function output differ.
continuity at $x=6$	No; Limit doesn't exist but function output does.

[6]

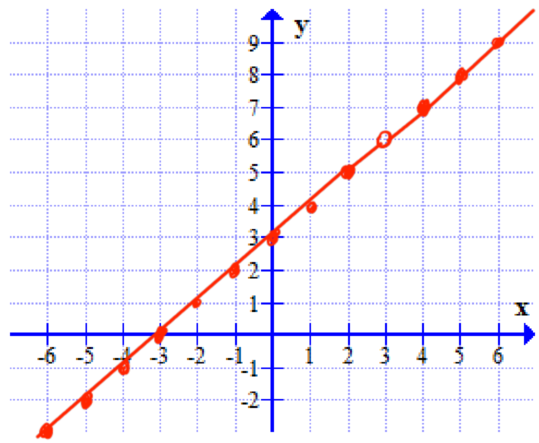
3. [Maximum mark: 10] ***[without GDC]***

(a) Sketch the graphs of the functions $f(x) = x + 3$ and $g(x) = \dfrac{x^2 - 9}{x - 3}$.

$y = f(x)$



$y = g(x)$



[4]

(b) Find the limits

(i) $\lim\limits_{x \to 3} f(x)$ (ii) $\lim\limits_{x \to 3} g(x)$. [2]

(c) Is $f(x)$ continuous at $x = 3$? Justify your answer. [2]

(d) Is $g(x)$ continuous at $x = 3$? Justify your answer. [2]

(b)(i) $\lim\limits_{x \to 3} f(x) = f(3) = 6$ (ii) $\lim\limits_{x \to 3} f(x) =$ UNDEFINED

(c) Yes, the limit and the function output at $x=3$ match.

(d) No, the limit and the function output at $x=3$ <u>don't</u> match.

CONTINUITY AND DIFFERENTIABILITY

Notice: Continuity and differentiability will not be examined.

Thus, the **next 5 questions** are just for further practice on these notions.

O. Practice questions

4. [Maximum mark: 8] ***[without GDC]***

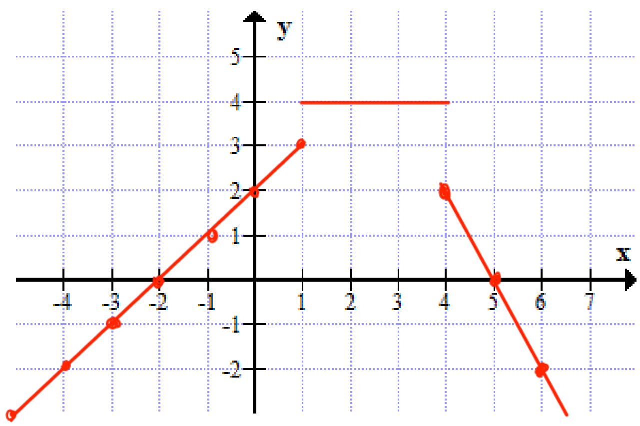
Let

$$f(x)=\begin{cases} x+2 & x\le 1 \\ 4 & 1<x\le 3 \\ -2x+10 & x>3 \end{cases}$$

(a) Sketch the graph of f on the diagram below. [4]

(b) Explain why f is **neither** continuous **nor** differentiable at $x=1$. [2]

(c) Explain why f is continuous **but not** differentiable at $x=3$. [2]



(b) Because the limit value doesn't exist at x=1.

(C) The function is continuous at x=3 because $\lim_{x\to 3} f(x)$ exists and is equal to f(3), but not differentiable because the derivatives of each part don't equal each other; $1\neq 0\neq -2$.

5. [Maximum mark: 8] ***[without GDC]***

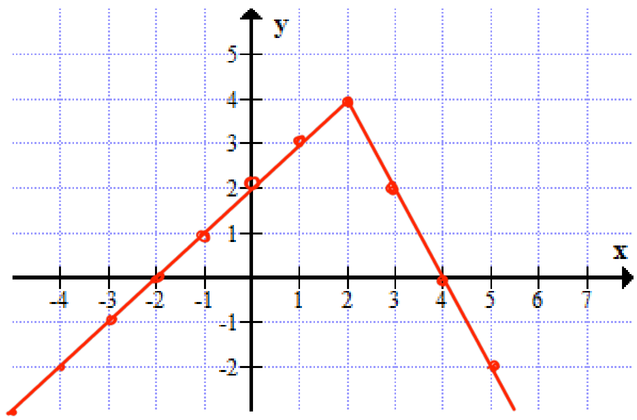
Consider the continuous function

$$f(x)=\begin{cases} x+a & x<2 \\ 2a & x=2 \\ bx+8 & x>2 \end{cases}$$

(a) Find the value (i) of a. (ii) of b. [4]

(b) Sketch the graph of f on the diagram below. [3]

(c) Explain why the function is not differentiable at $x=2$. [1]



(a) $x+a=2a=bx+8$

$x=a$ | $2(2)=bx+8$

(i) $a=2$ | $4=2b+8$

(ii) $b=-2$

(c) Slope changes from 1 to -2 at $x=2$. Therefore the derivative is not continuous, so it's not differentiable.

6. [Maximum mark: 14] ***[without GDC]***

Let

$$f(x)=\begin{cases} x^2+2 & x\le 0 \\ 2 & 0<x\le 3 \\ x-1 & x>3 \end{cases}$$

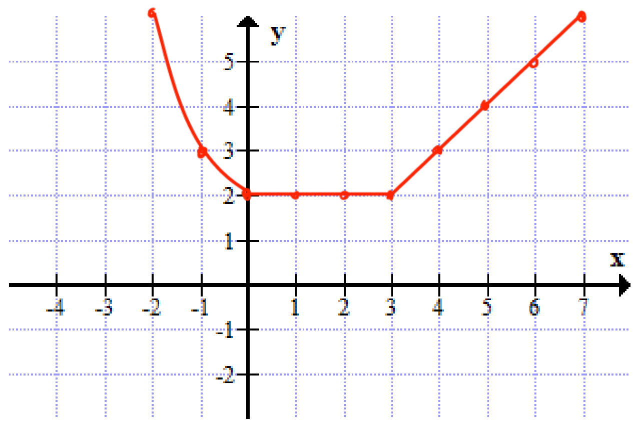
(a) Sketch the graph of f on the diagram below. [5]

(b) Show that f is continuous. [3]

(c) Explain why f is differentiable at $x=0$. [2]

(d) Write down the equation of the tangent line at $x=0$. [2]

(e) Explain why f is not differentiable at $x=3$. [2]



(b) $\lim_{x\to 0^-} f(x)=0+2=2$ $\qquad$ $\lim_{x\to 3^-} f(x)=2$

$\lim_{x\to 0^+} f(x)=2$ $\qquad$ $\lim_{x\to 3^+} =3-1=2$

$\lim_{x\to 0} f(x)=2$, $\lim_{x\to 3} f(x)=2$, all limits are equal from both sides, therefore the graph is continuous.

(c) Because the slope doesn't change – it's 2.

$\lim_{x\to 0^-} = \lim_{x\to 0^+}$

(d) $y=2x+2$

(e) Because the slope changes crossing $x\le 3$ to $x>3$. $0\ne 1$.

7. [Maximum mark: 12] ***[without GDC]***

?

Let $f(x)=\begin{cases} x^2-4x+3 & x<3 \\ c & x=3 \\ ax+b & x>3 \end{cases}$

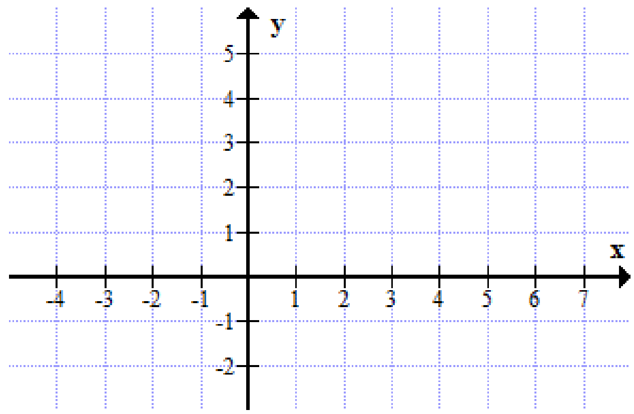
(a) Given that the function is continuous

(i) Show that $c=2$. (ii) Find a linear relation between a and b. [3]

(b) Given that the function is differentiable, show that $a=2$ and find the value of b. [3]

(c) Sketch the graph of f on the diagram below. [4]

(d) Write down the values of $\lim_{x\to 3} f(x)$ and $f'(3)$. [2]



(a) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = c$

$\lim_{x\to 3^-} f(x) = 0$

$0 = c = 3a+b$

$\lim_{x\to 3^+} f(x) = 3a+b$ $c=0$

8. [Maximum mark: 8] ***[with GDC]***

Let $f(x)=\begin{cases} -2x^3 & x<1 \\ x^2+ax+b & x\geq 1 \end{cases}$

The function is continuous and differentiable.

(a) Find the value of a and of b. [5]

(b) Find the range of $f(x)$. [1]

(c) Find the equation of the tangent line to the curve $y=f(x)$ at $x=1$. [2]

(a) $-2x^3=x^2+ax+b$ $\quad -6(1)^2=2(1)+a$

$-6x^2=2x+a$ $\quad -6=2+a$

$a=-8$

$-2(1)^3=(1)^2-8(1)+b$

$-2=1-8+b$

$b=5$

(b) If $x<1$: $\quad$ If $x\geq 1$:

$-6x^2=0$ $\quad 2x-8=0$

$x=0,\ -2(0)^3=0,\ y_{min}=0$ $\quad x=4,\ 4^2-8\cdot 4+5=-11$

$y_{min}=-11$

$f(x)\geq -11$

(c) $2(1)-8=-6=m_{x=1}$

Tangent Line: $y+2=-6(x-1)$

DIFFERENTIATION FROM FIRST PRINCIPLES

O. Practice questions

9. [Maximum mark: 5] ***[without GDC]***

Show by using first principles that the derivative of $f(x) = x^4$ is $f'(x) = 4x^3$

$$\lim_{h \to 0} \frac{(x+h)^4 - x^4}{h} = f'(x)$$

$$= \lim_{h \to 0} \frac{\cancel{x^4} + 4x^3\cancel{h} + 6x^2h^{\cancel{2}\,1} + 4xh^{\cancel{3}\,2} + h^{\cancel{4}\,3} - \cancel{x^4}}{\cancel{h}}$$

$$= \lim_{h \to 0} -4x^3 + 6x^2h + 4xh^2 + h^3$$

$$f'(x) = 4x^3 + \cancel{6x^2(0)} + \cancel{4x(0)^2} + \cancel{(0)^3}$$

$$f'(x) = 4x^3$$

10. [Maximum mark: 5] ***[without GDC]***

Let $f(x) = x^4$. Show by using first principles that $f'(1) = 4$.

$$f'(x) = \lim_{h \to 0} \frac{\cancel{x^4} + 4x^3\cancel{h} + 6x^2h^{\cancel{2}} + 4xh^{\cancel{3}\,2} + h^{\cancel{4}\,3} - \cancel{x^4}}{\cancel{h}}$$

$$f'(x) = \lim_{h \to 0} 4x^3 + 6x^2 + 4xh^2 + h^3$$

$$f'(x) = 4x^3$$

$$f'(1) = 4(1)^3 = 4$$

A. Exam style questions (SHORT)

11. [Maximum mark: 6] ***[without GDC]***

Find from first principles the derivative of $f(x) = x^3 + 2x + 1$.

$$f'(x) = \lim_{h \to 0} \frac{[(x+h)^3 + 2(x+h) + 1] - x^3 - 2x - 1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2\cancel{h} + 3xh^{\cancel{2}} + h^{\cancel{3}2} + \cancel{2x} + 2\cancel{h} + \cancel{1} - \cancel{x^3} - \cancel{2x} - \cancel{1}}{\cancel{h}}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 + 2$$

$$= 3x^2 + 2$$

12. [Maximum mark: 4] ***[without GDC]***

Find the derivative of the function $f(x) = mx + c$ from first principles.

$$y = \frac{m(x+h) + c - (mx + c)}{h}$$

$$y = \cancel{mx} + mh + \cancel{c} - \cancel{mx} \cancel{-c}$$

$$y = \frac{m\cancel{h}}{\cancel{h}}$$

$$y = m$$

• This proves that the derivative of any function is it's slope, whether it's linear or curved.

13*. [Maximum mark: 6] ***[without GDC]***

Show that the derivative of $f(x)=\sqrt{x}$ is $f(x)=\dfrac{1}{2\sqrt{x}}$ from first principles.

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = f'(x)$$

$$= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

14. [Maximum mark: 6] ***[without GDC]***

Show that the derivative of $f(x)=\dfrac{1}{x^2}$ is $f'(x)=-\dfrac{2}{x^3}$ from first principles.

Broken Sign Somewhere

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{(x-h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x^2}{x^2(x-h)^2} - \frac{(x-h)^2}{x^2(x-h)^2}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\cancel{x^2}-(\cancel{x^2}-2hx+h^2)}{x^2(x^2-2hx+h^2)}}{h}$$

$$= \lim_{h\to 0} \frac{2hx-h^2}{x^4-2hx^3+h^2x^2} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{2hx-h^2}{hx^4-2h^2x^3+h^3x^2}$$

$$= \lim_{h\to 0} \frac{2x-h}{x^4-2hx^3+h^2x^2}$$

$$= \frac{2x}{x^4}$$

$$= \frac{2}{x^3}$$

LIMITS AND L'HÔPITAL's RULE

O. Practice questions

15. [Maximum mark: 6] ***[without GDC]***

Find the limits

$\lim\limits_{x\to 1}\frac{x^2+1}{(x-1)^2}=$ $\frac{2}{0} \equiv +\infty$
$\lim\limits_{x\to 1^+}\frac{x^2-1}{(x-1)^2}=$ $+\infty$
$\lim\limits_{x\to 1^-}\frac{x^2-1}{(x-1)^2}=$ $\frac{<0}{0} \equiv -\infty$

16. [Maximum mark: 8] ***[without GDC]***

Find the limits

(a) $\lim\limits_{x\to 0}\frac{e^{2x}-1}{2x}$ $=\frac{0}{0}$ $\cancel{A}$ [3]

(b) $\lim\limits_{x\to 0}\frac{e^{2x}-1-2x}{x^2}$ $=\frac{0}{0}$ $\cancel{A}$ [5]

(a) $\lim\limits_{x\to 0}\frac{\cancel{2}e^{2x}}{\cancel{2}} \to e^{2(0)}=1$

(b) $\lim\limits_{x\to 0}\frac{\cancel{2}e^{2x}-\cancel{2}}{\cancel{2}x}=\frac{e^{2x}-1}{x} \to \frac{e^{2(0)}-1}{0}=\frac{0}{0}$ $\cancel{A}$

$\lim\limits_{x\to 0} 2e^{2x} \to 2e^{2(0)}=2$

17. [Maximum mark: 9] ***[without GDC]***

Find the limits

$\lim_{x\to 0}\frac{3e^{2x}+5}{4e^{2x}+6}=\frac{8}{10}=\frac{4}{5}$
$\lim_{x\to -\infty}\frac{3e^{2x}+5}{4e^{2x}+6}=$ $3e^{-2\infty}=\frac{1}{3e^{2\infty}}=\frac{1}{\infty}=0$; $\lim_{x\to-\infty}\frac{5}{6}$
$\lim_{x\to +\infty}\frac{3e^{2x}+5}{4e^{2x}+6}=\frac{+\infty}{+\infty}$ $\lim_{x\to+\infty}\frac{6\cancel{e^{2x}}}{8\cancel{e^{2x}}}=\frac{6}{8}=\frac{3}{4}$

18. [Maximum mark: 10] ***[without GDC]***

(a) By using l'Hopital, find the limits

(i)	$\lim_{x\to\pm\infty}\frac{ax+b}{dx+e}=\frac{\pm\infty}{\pm\infty}=\lim_{x\to\pm\infty}\frac{a}{d}=\frac{a}{d}$
(ii)	$\lim_{x\to\pm\infty}\frac{ax^2+bx+c}{dx^2+ex+f}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\pm\infty}\frac{2ax+b}{2dx+e}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\pm\infty}\frac{2a}{2d}=\frac{a}{d}$
(iii)	$\lim_{x\to\pm\infty}\frac{ax+b}{dx^2+ex+f}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\pm\infty}\frac{a}{2dx+e}=\frac{a}{\infty}=0$

[8]

(b) What is the geometric interpretation of these results for each rational function? [2]

They depict the horizontal asymptote for each line. for (i) and (ii) HA: $y=\frac{a}{d}$, and for (iii) HA: $y=0$.

19. [Maximum mark: 10] ***[without GDC]***

Find the limits

$\lim_{x\to+\infty}\frac{3x^2+x+e^x}{x^2+5x+3}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to+\infty}\frac{6x+e^x}{2x+5}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to+\infty}\frac{e^x+6}{2}=+\infty$
$\lim_{x\to-\infty}\frac{3x^2+x+e^x}{x^2+5x+3}\overset{\frac{-\infty}{-\infty}}{=}\lim_{x\to-\infty}\frac{6x+e^x}{2x+5}\overset{\frac{-\infty}{-\infty}}{=}\lim_{x\to-\infty}\frac{\overset{0}{e^x}+6}{2}=\frac{6}{2}=3$

20*. [Maximum mark: 33] ***[without GDC]***

By using only the result $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (without using l'Hopital's rule), find the following limits

(a) $\lim_{x \to 0} \frac{\sin 3x}{3x}$	1
(b) $\lim_{x \to 0} \frac{\sin 2x}{x}$	2
(c) $\lim_{x \to 0} \frac{\sin 5x}{3x}$	$\frac{5}{3}$
(d) $\lim_{x \to 2} \frac{\sin(x-2)}{x-2}$	1
(e) $\lim_{x \to 0} \frac{3x + 2\sin x}{x}$	$\frac{3\cancel{x}}{\cancel{x}} + 2\frac{\sin x}{x} = 3+2=5$
(f) $\lim_{x \to 0^+} \frac{\sin x}{x^2}$	$\frac{\sin x}{x} \circ \frac{1}{x} = 1 \circ +\infty = +\infty$
(g) $\lim_{x \to 0^+} \frac{\sin x}{\sqrt{x}}$	$\frac{\sqrt{x}\sin x}{x} = \frac{\sqrt{x}}{1} \circ 1 = 0$
(h) $\lim_{x \to 0} \frac{\sin 2x}{x \cos x}$	$\frac{2\sin x \cos x}{x \cos x} = 2(1) \circ 1 = 2$
(i) $\lim_{n \to +\infty} n \sin \frac{1}{n}$	$\frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$
(j) $\lim_{n \to +\infty} n^2 \sin \frac{1}{n}$	$+\infty \circ 1 = +\infty$
(k) $\lim_{n \to +\infty} n \sin \frac{1}{n^2}$	$\frac{1}{n} \circ \sin\left(\frac{1}{n} \cdot \frac{1}{n}\right)$

A. Exam style questions (SHORT)

21. [Maximum mark: 9] ***[without GDC]***

Find the limits (a) $\lim_{x \to 0}\left(\frac{1}{x^2}-\frac{1}{x}\right)$ (b) $\lim_{x \to 0^+}\left(\frac{1}{x+1}-\frac{1}{x}\right)$ (c) $\lim_{x \to 0^-}\left(\frac{1}{x+1}-\frac{1}{x}\right)$

(a) $\lim_{x \to 0}$

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22. [Maximum mark: 8] ***[without GDC]***

Find the limits (i) $\lim_{x \to 1}\frac{3\ln x}{x-1}$ (ii) $\lim_{x \to 1}\frac{x-1}{3\ln x}$

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23. [Maximum mark: 5] ***[without GDC]***

Find the limit $\lim_{x\to 0}\dfrac{2\cos x-2+5x^2}{x^2}$

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24. [Maximum mark: 5] ***[without GDC]***

Hence, or otherwise, determine the value of $\lim_{x\to 0}\dfrac{2\ln(1+\mathrm{e}^x)-x-\ln 4}{x^2}$.

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25. [Maximum mark: 6] ***[without GDC]***

Find the value of $\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right)$.

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26. [Maximum mark: 4] ***[without GDC]***

Find the value of $\lim_{x\to 1}\left(\frac{\ln x}{\sin 2\pi x}\right)$.

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27. [Maximum mark: 4] ***[without GDC]***

Find $\lim\limits_{x \to 0} \dfrac{\tan x}{x + x^2}$

28. [Maximum mark: 7] ***[without GDC]***

Find $\lim\limits_{x \to 1} \dfrac{1 - x^2 + 2x^2 \ln x}{1 - \sin \dfrac{\pi x}{2}}$.

29. [Maximum mark: 6] ***[without GDC]***

Using l'Hopital's Rule determine the value of $\lim_{x\to 0}\frac{\tan x - x}{1-\cos x}$

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30. [Maximum mark: 5] ***[without GDC]***

Use l'Hopital's Rule to find $\lim_{x\to 0}(\csc x - \cot x)$

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31. [Maximum mark: 4] ***[without GDC]***

Calculate $\lim_{x \to 0} \frac{2^x - 1}{x}$

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32. [Maximum mark: 6] ***[without GDC]***

Calculate $\lim_{x \to 0} \frac{(1+x^2)^{\frac{3}{2}} - 1}{\ln(1+x) - x}$

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33. [Maximum mark: 8] ***[without GDC]***

Let $f(x)=\dfrac{2\tan x-\tan 2x}{\sin 2x-2\sin x}$

(a) The function is not defined at $x=0$. Look at the graph of f in the GDC and find values of $f(x)$ near $x=0$ and thus guess the value of $\lim_{x\to 0} f(x)$. [1]

(b) Confirm your guess by using l'Hôpital's rule. [7]

B. Exam style questions (LONG)

34. [Maximum mark: 13] ***[without GDC]***

Let $f(x) = (2x+3)^2$

(a) Find

(i) $f'(x)$ by using the chain rule (ii) $f'(0)$. [3]

(b) Confirm the result (a) (i) by using first principles. [6]

(c) Confirm the result (a) (ii), by using first principles. [4]

35. [Maximum mark: 10] ***[without GDC]***

(a) Write down the term in x^r in the expansion of $(x+h)^n$, where $0 \le r \le n,\ n \in Z^+$ [1]

(b) Hence differentiate x^n, $n \in Z^+$, from first principles. [5]

(c) Starting from the result $x^n \times x^{-n} = 1$, deduce the derivative of x^{-n}, $n \in Z^+$. [4]

36. [Maximum mark: 12] ***[without GDC]***

(a) Find the limit $\lim_{x\to 0}\frac{\ln(1+ax)}{x}$. [4]

(b) **Hence** find $\lim_{n\to+\infty} n\ln\left(1+\frac{a}{n}\right)$. [2]

(c) **Hence** show that $\lim_{n\to+\infty}\left(1+\frac{a}{n}\right)^n = e^a$. [3]

(d) Given that $FV = PV\left(1+\frac{r}{100k}\right)^{nk}$, find $\lim_{k\to+\infty} FV$. [3]

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