INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

MAA

EXERCISES [MAA 5.23]

MACLAURIN SERIES – EXTENSION OF BINOMIAL THEOREM

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O. Practice questions

1. [Maximum mark: 7] ***[without GDC]***

Find the Maclaurin series of the function $f(x) = e^{2x}$ up to and including the term in x^4

(a) by using the formula of the Maclaurin series $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$ [5]

(b) by using the Maclaurin series of $e^x = 1 + x + \frac{x^2}{2!} + \cdots$ [2]

$f(x) = e^{2x}$ $f(0) = 1$ (a) $f(x) = 1 + 2x + x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$

$f'(x) = 2e^{2x}$ $f'(0) = 2$

$f''(x) = 4e^{2x}$ $f''(0) = 4$

$f'''(x) = 8e^{2x}$ $f'''(0) = 8$

$f^4(x) = 16e^{2x}$ $f^4(0) = 16$

(b)

2. [Maximum mark: 8] ***[without GDC]***

Find the Maclaurin series of the function $f(x)=(2+x)^3$

(a) by using the formula of the Maclaurin series $f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots$ [5]

(b) by using the expansion of the binomial theorem. [3]

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3. [Maximum mark: 10] ***[without GDC]***

Find the Maclaurin series of the function $f(x) = \frac{1}{(2+x)^2} = (2+x)^{-2}$, up to x^2

(a) By using the formula of the Maclaurin series. [4]

(b) By using the extended version of the binomial theorem. [4]

(c) Write down the values of x for which the series converges. [2]

4. [Maximum mark: 9] ***[without GDC]***

Find the Maclaurin series of the function $f(x)=\sqrt{2+x}=(2+x)^{\frac{1}{2}}$, up to x^2

(a) by using the formula of the Maclaurin series. [5]

(b) by using the extended version of the binomial theorem. [4]

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5. [Maximum mark: 8] ***[without GDC]***

(a) Find the Maclaurin series of the function $f(x)=\sqrt{x^2+1}$, up to and including the term in x^6 by using the extended version of the binomial theorem. [4]

(b) Given that $f'(x)=\dfrac{x}{\sqrt{x^2+1}}$ find the Maclaurin series of $g(x)=\dfrac{x}{\sqrt{x^2+1}}$ up to and including the term in x^5. [4]

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6. [Maximum mark: 17] ***[without GDC]***

By using the Maclaurin series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, find the Maclaurin series of

(a) $f(x) = e^{-x}$ up to and including the term in x^3. [2]

(b) $f(x) = e^{-x^2}$ up to and including the term in x^6. [3]

(c) $f(x) = xe^x$ up to and including the term in x^4. [2]

(d) $f(x) = x^2e^{-x}$ up to and including the term in x^5. [3]

(e) $f(x) = e^x - e^{-x}$ up to and including the term in x^5. [3]

(f) $f(x) = (x+1)e^{4x}$ up to and including the term in x^3. [4]

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A. Exam style questions (SHORT)

7. [Maximum mark: 10] ***[without GDC]***

(a) Find the first three terms of the Maclaurin series for $\ln(1+e^x)$. [6]

(b) Hence, or otherwise, determine the value of $\lim_{x\to 0}\frac{2\ln(1+e^x)-x-\ln 4}{x^2}$. [4]

8. [Maximum mark: 10] ***[without GDC]***

The function f is defined by $f(x) = \ln\left(\frac{1}{1-x}\right)$.

(a) Write down the value of the constant term in the Maclaurin series for $f(x)$. [1]

(b) Find the first three derivatives of $f(x)$ and hence show that the Maclaurin series for $f(x)$ up to and including the x^3 term is $x + \frac{x^2}{2} + \frac{x^3}{3}$. [6]

(c) Use this series to find an approximate value for $\ln 2$. [3]

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9. [Maximum mark: 10] ***[without GDC]***

Let $f(x) = \ln(1+\sin x)$.

(a) Show that $f''(x) = \dfrac{-1}{1+\sin x}$

(b) Find the third and the fourth derivatives of f

(c) Hence, find the Maclaurin series, up to the term in x^4, for $f(x)$.

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10. [Maximum mark: 7] ***[without GDC]***

The variables x and y are related by the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - y\tan x = \cos x$.

(a) Find the Maclaurin series for y up to and including the term in x^2 given that $y = -\frac{\pi}{2}$ when $x = 0$.

(b) Show that an approximation for y when $x = 0.1$ is $y \cong 0.1 - \frac{201\pi}{400}$

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11. [Maximum mark: 7] ***[without GDC]***

(a) Using the extended version of the binomial theorem, find the Maclaurin series of

$$f(x)=\frac{1}{3x+5}$$

up to the term in x^3; [5]

(b) Find the values of x for which the series converges. [2]

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12. [Maximum mark: 8] ***[without GDC]***

(a) Using the extended version of the binomial theorem, find the Maclaurin series of

$$f(x)=\sqrt{3x+5}$$

up to the term in x^3; [6]

(b) Find the values of x for which the series converges. [2]

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B. Exam style questions (LONG)

13*. [Maximum mark: 22] ***[without GDC]***

It is given that the Maclaurin series of the function $f(x)=\ln(1+\sin x)$, up to the term in x^4 is

$$f(x)=x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{12}x^4+\ldots$$

(a) Deduce the Maclaurin series, up to and including the term in x^4 for

(i) $y=\ln(1-\sin x)$;

(ii) $y=\ln\cos x$;

(iii) $y=\ln\sec x$; [8]

(b) By differentiating the Maclaurin series of $y=\ln\cos x$, deduce the Maclaurin series of $y=\tan x$ [4]

(c) Hence calculate the limits (i) $\lim\limits_{x\to 0}\dfrac{\ln\sec x}{x\sqrt{x}}$. (ii) $\lim\limits_{x\to 0}\left(\dfrac{\tan(x^2)}{\ln\cos x}\right)$ [6]

(d) By considering the difference of the two series of

$y=\ln(1+\sin x)$ and $y=\ln(1-\sin x)$

deduce that $\ln 3\approx\dfrac{\pi}{3}\left(1+\dfrac{\pi^2}{216}\right)$. [4]

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14. ([Maximum mark: 14] ***[with GDC]***

(a) Given that $y = \ln\cos x$, show that the first two non-zero terms of the Maclaurin series for y are $-\frac{x^2}{2} - \frac{x^4}{12}$. [8]

(b) Use this series to find an approximation in terms of π for $\ln 2$. [6]

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