INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

MAA

EXERCISES [MAA 5.14]

IMPLICIT DIFFERENTIATION – MORE KINEMATICS

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O. Practice questions

1. [Maximum mark: 8] ***[without GDC]***

Consider the equation $xy + x^2y^2 = e^x - e^y + 2$

(a) Confirm that $P(1,1)$ lies in the curve. [1]

(b) Find (i) $\frac{dy}{dx}$ in terms of x and y (ii) $\frac{dy}{dx}$ at P. [5]

(c) Find the equations of the tangent and normal lines in the form $y - y_1 = m(x - x_1)$ [2]

(a) $1 + 1 = \cancel{e} - \cancel{e} + 2$

$2 = 2$, Point lies in the curve.

(b)

(i) $y + x\frac{dy}{dx} + 2xy^2 + 2x^2y\frac{dy}{dx} = e^x - e^y\frac{dy}{dx}$

$x\frac{dy}{dx} + 2x^2y\frac{dy}{dx} + e^y\frac{dy}{dx} = -y - 2xy^2 + e^x$

$\frac{dy}{dx}(x + 2x^2y + e^y) = e^x - 2xy^2 - y$

$\frac{dy}{dx} = \frac{e^x - 2xy^2 - y}{e^y + 2x^2y + x}$

(ii)

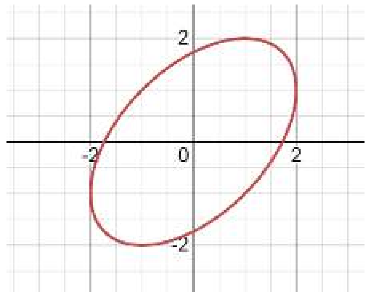
$\left.\frac{dy}{dx}\right|_P = \frac{e - 2 - 1}{e + 2 + 1} = \frac{e-3}{e+3}$

(c) Tangent Line: $y - 1 = \frac{e-3}{e+3}(x-1)$

Normal Line: $y - 1 = \frac{-e-3}{-e+3}(x-1)$

2. [Maximum mark: 15] ***[without GDC]***

The curve of the equation $x^2 - xy + y^2 = 3$ is shown below.



(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x - y}{x - 2y}$ [3]

(b) Find the coordinates of the points where the tangent lines are parallel to

(i) the line $y = x$ (ii) the x-axis (iii) y-axis [12]

(a) $2x - \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0$

$2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$

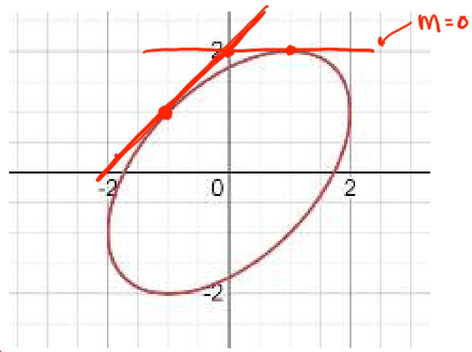
$2x - y = (x - 2y)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{2x - y}{x - 2y}$

(b)(i)

3*. [Maximum mark: 10] ***[without GDC]***

The curve of the equation $x^2 - xy + y^2 = 3$ is shown below.



The line $y = mx + 2$ is tangent to the curve. B

(a) Find the values of m. $\frac{dy}{dx} = 0$ [8]

(b) Draw the corresponding tangent lines on the diagram above. [2]

(a) $2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(2y - x) = y - 2x$

$\frac{dy}{dx} = \frac{y-2x}{2y-x}$

$m = \frac{y-2x}{2y-x}$

$m = \frac{2}{4}$

$m = \frac{1}{2}$

$\frac{1}{2} = \frac{y-2x}{2y-x}$

$\cancel{2y} - x = \cancel{2y} - 4x$

$x = 0$

$(0)^2 - (0)(y) + y^2 = 3$

$y^2 = 3$

$y = \pm\sqrt{3}$

A. Exam style questions (SHORT)

4. [Maximum mark: 5] ***[without GDC]***

Find the gradient of the tangent to the curve $3x^2 + 4y^2 = 7$ at the point where $x = 1$ and $y > 0$.

$6x + 8y\left(\frac{dy}{dx}\right) = 0$

$\frac{dy}{dx} 8y = -6x$

$\frac{dy}{dx} = \frac{-6x}{8y}$

$3(1)^2 + 4y^2 = 7$

$4y^2 = 4$

$y^2 = 1$

$y = \pm 1$

Answer $= \frac{-6(1)}{8(1)} = -\frac{3}{4}$

5. [Maximum mark: 5] ***[without GDC]***

If $2x^2 - 3y^2 = 2$, find the two values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 5$.

$\frac{d}{dx}$: $4x - 6y\frac{dy}{dx} = 0$

$6y\frac{dy}{dx} = 4x$

$\frac{dy}{dx} = \frac{4x}{6y}$

$2(5)^2 - 3y^2 = 2$

$50 - 3y^2 = 2$

$-3y^2 = -48$

$y^2 = 16$

$y = \pm 4$

$\frac{dy}{dx} = \frac{\cancel{4}(5)}{16(\cancel{4})} = \boxed{\frac{5}{16}}$

$\frac{dy}{dx} = \frac{\cancel{4}(5)}{16(-\cancel{4})} = \boxed{-\frac{5}{16}}$

6. [Maximum mark: 6] ***[without GDC]***

Consider the equation $2xy^2 = x^2 y + 3$.

(a) Find y and $y < 0$ when $x = 1$ and $y < 0$.

(b) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 1$ and $y < 0$.

(a) $2(1)y^2 = (1)^2 y + 3$

$2y^2 = y + 3$

$2y^2 - y - 3 = 0$

$2y^2 - 3y + 2y - 3 = 0$

$y(2y-3) + 1(2y-3) = 0$

$(2y-3)(y+1) = 0$

$y = \frac{3}{2}, \underline{\underline{y = -1}}$

FALSE, $y < 0$

(b) $\frac{d}{dx}$: $2x\left(2y\frac{dy}{dx}\right) + 2y^2 = x^2\frac{dy}{dx} + 2xy$

$4xy\frac{dy}{dx} - x^2\frac{dy}{dx} = 2xy - 2y^2$

$\frac{dy}{dx}(4xy - x^2) = 2xy - 2y^2$

$\frac{dy}{dx} = \frac{2xy - 2y^2}{4xy - x^2}$

$\frac{dy}{dx} = \frac{2(1)(-1) - 2(-1)^2}{4(1)(-1) - (-1)^2} = \frac{-2-2}{-4-1}$

$= \frac{-4}{-5}$

$\boxed{= \frac{4}{5}}$

7. [Maximum mark: 6] ***[without GDC]***

The point $\mathrm{P}(1, p)$, where $p > 0$, lies on the curve $2x^2 y + 3y^2 = 16$.

(a) Calculate the value of p.

(b) Calculate the gradient of the tangent to the curve at P.

8. [Maximum mark: 6] ***[without GDC]***

Find the gradient of the normal to the curve $3x^2y + 2xy^2 = 2$ at the point (1, –2).

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9. [Maximum mark: 6] ***[without GDC]***

A curve has equation $xy^3 + 2x^2y = 3$. Find the equation of the tangent to this curve at the point (1, 1).

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10. [Maximum mark: 6] ***[without GDC]***

A curve has equation $x^3y^2 = 8$. Find the equation of the normal to the curve at the point (2, 1).

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11. [Maximum mark: 7] ***[without GDC]***

Find the equation of the normal to the curve $5xy^2 - 2x^2 = 18$ at the point (1, 2).

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12. [Maximum mark: 6] ***[without GDC]***

Find the equation of the normal to the curve $x^3 + y^3 - 9xy = 0$ at the point (2, 4).

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13. [Maximum mark: 5] ***[without GDC]***

The tangent to the curve $y^2 = x^3$ at the point P(1, 1) meets the x-axis at Q and the y-axis at R. Find the ratio PQ : QR.

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14. [Maximum mark: 7] ***[without GDC]***

Consider the curve with equation $x^2 + xy + y^2 = 3$

(a) Find in terms of k, the gradient of the curve at the point $(-1, k)$. [5]

(b) Given that the tangent to the curve is parallel to the x-axis at this point, find the value of k. [2]

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15. [Maximum mark: 6] ***[without GDC]***

Find the gradient of the tangent to the curve $x^3y^2 = \cos(\pi y)$ at the point (–1, 1).

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16. [Maximum mark: 6] ***[without GDC]***

Given that $3^{x+y} = x^3 + 3y$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

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17. [Maximum mark: 6] ***[without GDC]***

Given that $\mathrm{e}^{xy} - y^2 \ln x = \mathrm{e}$ for $x \geq 1$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point (1, 1).

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18. [Maximum mark: 9] ***[without GDC]***

Consider the curve $x^3 + y^2 = xy$, $x \neq 0$, $y \neq 0$

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [3]

(b) Find the coordinates of the point on the curve where the tangent line is parallel to x-axis. [3]

(c) Find the coordinates of the point on the curve where the tangent line is parallel to y-axis. [3]

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19*. [Maximum mark: 7] ***[without GDC]***

If $y = \ln\left(\frac{1}{3}(1+\mathrm{e}^{-2x})\right)$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{3}(\mathrm{e}^{-y} - 3)$.

20*. [Maximum mark: 7] ***[without GDC]***

Find the gradient of the curve $2\sin(xy)=1$ when $y=\frac{1}{2}$ and $\pi<x<2\pi$.

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21. [Maximum mark: 5] ***[without GDC]***

Take the logarithm $\ln()$ on both sides and then use implicit differentiation to find the derivative of the function $y=x^x$ in terms of x.

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22. [Maximum mark: 5] ***[without GDC]***

Take the logarithm $\ln()$ on both sides and then use implicit differentiation to find the derivative of the function $y=(x^2+1)^x$ in terms of x.

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23. [Maximum mark: 5] ***[without GDC]***

Take the logarithm $\ln()$ on both sides and then use implicit differentiation to find the derivative of the function $y=x^{x^2+1}$ in terms of x.

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THE DERIVATIVE OF THE INVERSE FUNCTION

O. Practice questions

24. [Maximum mark: 6] ***[without GDC]***

Consider the curve $x^3 + y^2 = xy$, $x \neq 0$, $y \neq 0$.

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}x}{\mathrm{d}y}$ to verify that $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1}$

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25. [Maximum mark: 5] ***[without GDC]***

Given that the derivative of $f(x) = \mathrm{e}^x$ is known, use the property $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1}$ to show that the derivative of its inverse function $y = \ln x$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$

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26. [Maximum mark: 5] ***[without GDC]***

Given that the derivative of $f(x)=x^2$ is known, use the property $\frac{\mathrm{d}y}{\mathrm{d}x}=\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1}$ to show that the derivative of its inverse function $y=\sqrt{x}$ is $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2\sqrt{x}}$

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27. [Maximum mark: 5] ***[without GDC]***

For $y=f(x)$ and a point $(a,\,b)$ on the curve,

the property $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$ implies that $f'(a)=\frac{1}{f^{-1}{}'(b)}$

Confirm the result for the function $f(x)=x^2, x\geq 0$ at the point (3,9).

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A. Exam style questions (SHORT)

28. [Maximum mark: 6] ***[without GDC]***

Given that the derivative of $f(x)=\sin x$ is known, use the property $\frac{\mathrm{d}y}{\mathrm{d}x}=\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1}$ to show that the derivative of its inverse function $y=\arcsin x$ is $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\sqrt{1-x^2}}$

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29. [Maximum mark: 6] ***[without GDC]***

Given that the derivative of $f(x)=\cos x$ is known, use the property $\frac{\mathrm{d}y}{\mathrm{d}x}=\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1}$ to find the derivative of its inverse function $y=\arccos x$.

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30. [Maximum mark: 7] ***[without GDC]***

(a) Use the quotient rule to show that the derivative of $\tan x$ is $\sec^2 x$. [2]

(b) Hence by using $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1}$, show that the derivative of $\arctan x$ is $\frac{1}{1+x^2}$. [5]

MORE KINEMATICS

Notice:

If the velocity is given in terms of time t then

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}$$

If the velocity is given in terms of the displacement s, then

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}s}$$

In fact, in the second case we use implicit differentiation with respect to t.

For example,

If $v = t^2$, then $a = \frac{\mathrm{d}v}{\mathrm{d}t} = 2t$

If $v = s^2$, then $a = \frac{\mathrm{d}v}{\mathrm{d}t} = 2s\frac{\mathrm{d}s}{\mathrm{d}t} = 2s \times s^2 = 2s^3$ (since $\frac{\mathrm{d}s}{\mathrm{d}t} = v$)

O. Practice questions

31. [Maximum mark: 5] ***[without GDC]***

(a) A particle moves along a straight line and its velocity at time t is given by

$$v = 5 + 10t.$$

Find its acceleration. [2]

(b) A particle moves along a straight line and its velocity is given by

$$v = 5 + 10s.$$

where s is the displacement. Find its acceleration in terms of s. [3]

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A. Exam style questions (SHORT)

32. [Maximum mark: 6] ***[without GDC]***

(a) Given that $v = t^2 + 10\mathrm{e}^t$, find the acceleration when $t = 0$. [3]

(b) Given that $v = s^2 + 10\mathrm{e}^s$, find the acceleration when $s = 0$. [3]

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33. [Maximum mark: 6] ***[without GDC]***

A particle moves along a straight line. When it is in distance s from a fixed point, where $s > 1$, the velocity is given by $v = \dfrac{(3s+2)}{(2s-1)}$. Find the acceleration when $s = 2$.

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