EXERCISES [MAA 5.15]

RELATED RATES

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O. Practice questions

1. [Maximum mark: 12] ***[without GDC]***

The quantities A and B increase at constant rates $\frac{\mathrm{d}A}{\mathrm{d}t}=3$ and $\frac{\mathrm{d}B}{\mathrm{d}t}=2$ respectively.

(a) Given that $C=2A^3+1$, find the rate of change of C, at the instant when $A=2$. [3]

(b) Given that $\ln D=\frac{3}{A}$, find the rate of change of D, at the instant when $D=\mathrm{e}$. [5]

(c) Given that $P=AB$, find the rate of change of P, at the instant when $A=B=4$. [4]

(a) $\frac{dC}{dt}=3A^2\frac{dA}{dt}$

$\frac{dC}{dt}@(A=2)=3(2)^2(3)$

$=36$

(b) $e^{(3/A)}=D=\sqrt[A]{e^3}$

$\frac{dD}{dt}=e^{(3/A)}\cdot\left[\frac{A-3\frac{dA}{dt}}{A^2}\right]=e^{(3/A)}\cdot\frac{A-9}{A^2}$

$A=\frac{3}{D}$

$\frac{dD}{dt}=e^{D}\cdot\frac{\left(\frac{3}{D}\right)-9}{\left(\frac{3}{D}\right)^2}$

$\frac{dD}{dt}@(D=e)=e^e\cdot\frac{\frac{3}{\cancel{e}}-9}{1}\cdot\frac{e^{\cancel{2}}}{9}$

$=e^e\cdot\frac{3e-9e}{9}=e^e\cdot\frac{-2e}{3}$

2. [Maximum mark: 24] ***[without GDC]***

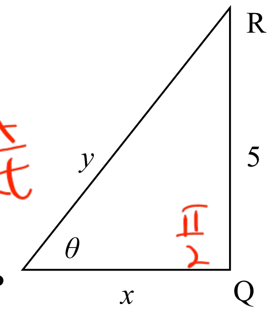
The diagram shows a right-angled triangle PQR, with $\hat{Q} = \frac{\pi}{2}$ and QR = 5 m (constant).

The sides PQ = x and PR = y are variable, as P can be moved horizontally on the line (PQ).

Hence the angle $\theta = R\hat{P}Q$, the area A and the perimeter P of the triangle are also variable.

just realized;

$\frac{d}{dt}(A) = (1)\left(\frac{dA}{dt}\right) = \frac{dA}{dt}$



(a) Complete the following table

Variables	Relation between the **variables**	Relation between the corresponding **rates of change**
x and θ	$\tan\theta = \frac{5}{x}$	$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = -\frac{5}{x^2}\frac{dx}{dt}$
y and θ	$\sin\theta = \frac{5}{y}$	$\cos\theta\frac{d\theta}{dt} = -\frac{5}{y^2}\frac{dy}{dt}$
x and y	$x^2+25=y^2$	$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$ / $x\,dx = y\,dy$
A and x	$A = \frac{5}{2}x$	$\frac{dA}{dt} = \frac{5}{2}\frac{dx}{dt}$ / $dA = \frac{5}{2}dx$
P, x and y	$P = 5+x+y$	$\frac{dP}{dt} = \frac{dx}{dt}+\frac{dy}{dt}$ / $dP = dx+dy$

[12]

(b) At the instant when $x = 5$ m, write down the values of

(i) θ (ii) y (iii) A (iv) P [4]

(c) Given that P is moving to the left by 0.5 m per second, find the rate of change of the following at the instant when $x = 5$ m

(i) x (ii) y (iii) A (iv) P [8]

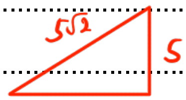
(b) (i) $\theta = \arctan\left(\frac{5}{5}\right) = \frac{\pi}{4}$ (ii) $y = \frac{5}{\sin(\frac{\pi}{4})} = \frac{5}{1}\cdot\frac{2}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$

(iii) $A = \frac{5}{2}(5) = 5$, (iv) $P = 5+5+5\sqrt{2} = 10+5\sqrt{2}$

(c)(i) $\frac{dx}{dt} = 0.5$

$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = -\frac{5}{(5)^2}(0.5)$

$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{-1}{10}$



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$\frac{1}{\cos^2(\frac{\pi}{4})} = \frac{1}{(\frac{2}{4})} = \frac{4}{2} = \frac{1}{2}$ $\quad \frac{d\theta}{dt} = \frac{-1}{10} \cdot = \frac{-1}{5}$

(iii) $\frac{dA}{dt} = \frac{5}{2}\left(\frac{1}{2}\right) = \frac{5}{4}$

(ii) $2(5)(0.5) = 2(5\sqrt{2})\frac{dy}{dt}$

$\cancel{5} = 10\sqrt{2}\frac{dy}{dt}$

$1 = 2\sqrt{2}\frac{dy}{dt}$

$\frac{dy}{dt} = \frac{1}{2\sqrt{2}}$

(iv) $\frac{dD}{dt} = 0.5 + \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{2}}{2\sqrt{2}}$

A. Exam style questions (SHORT)

3. [Maximum mark: 5] ***[without GDC]***

The quantities A and B increase at constant rates $\frac{\mathrm{d}A}{\mathrm{d}t}=3$ and $\frac{\mathrm{d}B}{\mathrm{d}t}=2$ respectively.

Given that $F=2\mathrm{A}^2B+2B^3$, find the rate of change of F, when $A=B=1$.

$$\frac{dF}{dt}=4A\frac{dA}{dt}B+2A^2\frac{dB}{dt}+6B^2\frac{dB}{dt}$$

$$\frac{dF}{dt}=4(1)(3)(1)+2(1)^2(2)+6(1)^2(2)$$

$$\frac{dF}{dt}=12+4+12=28$$

4. [Maximum mark: 6] ***[without GDC]***

The quantities A and B increase at constant rates $\frac{\mathrm{d}A}{\mathrm{d}t}=3$ and $\frac{\mathrm{d}B}{dt}=2$ respectively.

Given that $F^2=2\mathrm{A}^2B$, find the rate of change of F, when $A=B=1$.

5. [Maximum mark: 6] ***[without GDC]***

Air is pumped into a spherical ball which expands at a rate of 8 cm^3 per second (8 $cm^3 s^{-1}$).

Find the **exact** rate of increase of the radius of the ball when the radius is 2 cm.

$V=\frac{4}{3}\pi r^3$

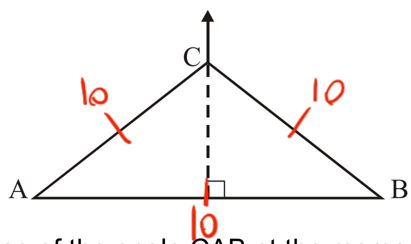
$8 = 4\pi r^2 \frac{dr}{dt}$

$8 = 4\pi(2)^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{8}{16\pi} = \frac{1}{2\pi}$

6. [Maximum mark: 6] ***[without GDC]***

The following diagram shows an isosceles triangle ABC with AB = 10 cm and AC = BC. The vertex C is moving in a direction perpendicular to (AB) with speed 2 cm per second.



Calculate the rate of increase of the angle CAB at the moment the triangle is equilateral.

$10^2 = 10^2 + 10^2 - 2(10)(10)\cos\hat{A}$

$100 = 200 - 200\cos\hat{A}$

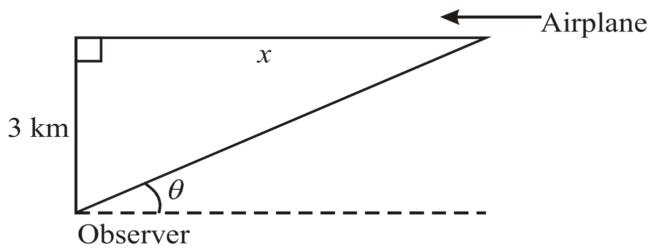
$-100 = -200\cos\hat{A}$

$1 = 2\cos\hat{A}$

$\hat{A} = \arccos\left(\frac{1}{2}\right) =$

7. [Maximum mark: 6] ***[without GDC]***

An airplane is flying at a constant speed at a constant altitude of 3 km in a straight line that will take it directly over an observer at ground level. At a given instant the observer notes that the angle θ is $\frac{1}{3}\pi$ radians and is increasing at $\frac{1}{60}$ radians per second.



Find the speed, in kilometres per hour, at which the airplane is moving towards the observer.

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8. [Maximum mark: 5] ***[without GDC]***

In the previous problem, find the rate of change of the distance between the observer and the airplane, at the instant when the angle θ is $\frac{1}{3}\pi$ radians and is increasing at $\frac{1}{60}$ radians per second.

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9. [Maximum mark: 6] ***[with GDC]***

A conical tank with vertex down is 8 metres in diamater and 12 meters deep. Water flows into the tank at 10 m^3 per minute. Find the rate of change of the depth of the water at the instant when the water is 6 meters deep.

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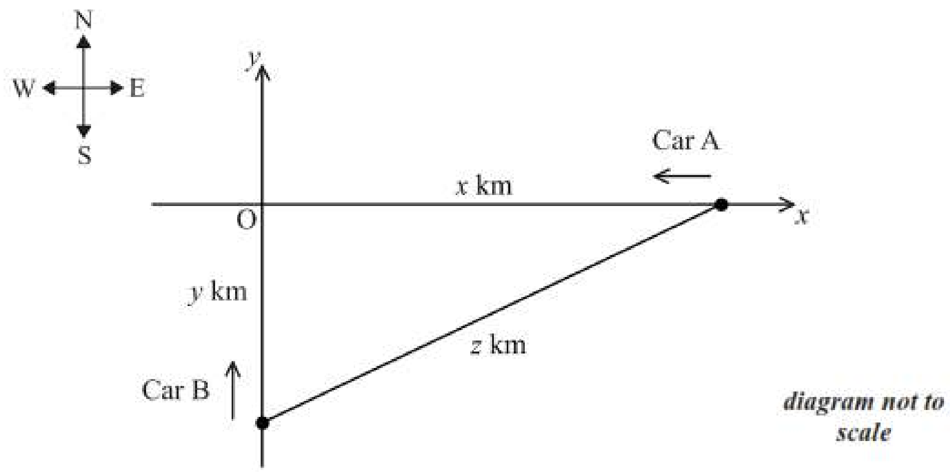
10. [Maximum mark: 6] ***[with GDC]***

Car A is travelling on a straight east-west road in a westernly direction at 60 km h^{-1}.

Car B is travelling on a straight north-soutg road in a northernly direction at 70 km h^{-1}.

The roads intersect at the point O. When Car A is x km east of O, and Car B is y km south of O, the distance between the cars is z km.



Find the rate of change of z when Car A is 0.8 km east of O and Car B is 0.6 km south of O.

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11. [Maximum mark: 5] ***[with GDC]***

In Ex 10, answer the same question if Car A was travelling in an easternly direction.

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12. [Maximum mark: 6] ***[without GDC]***

The volume of a solid is given by $V = \frac{4}{3}\pi r^3 + \pi r^2 h$.

At the time when the radius is 3 cm, the volume is 81π cm^3, the radius is changing at a rate of 2 cm/min and the volume is changing at a rate of 204π cm^3/min. Find the rate of change of the height at this time.

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B. Exam style questions (LONG)

13. [Maximum mark: 12] ***[without GDC]***

The height and the diameter of the base of a cylinder are equal.

(a) Express the volume V of the cylinder in terms of the radius r. [1]

(b) Show that the surface area S of the cylinder is given by $S = 6\pi r^2$. [2]

(c) Find the relations between

(i) the rates $\frac{dV}{dt}$ and $\frac{dr}{dt}$. (ii) the rates $\frac{dS}{dt}$ and $\frac{dr}{dt}$. [4]

(d) **Hence**, find the relation between the rates $\frac{dV}{dt}$ and $\frac{dS}{dt}$. [2]

The volume of the cylinder increases at a constant rate of 6 cm^3 min^{-1}.

(e) Find the rate of change of the surface area at the instant when r = 12 cm. [3]

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