1.1 Numbers – Rounding

Example 1

Consider the number: 0.04362018

to decimal places	**to significant figures**
to 2 d.p.	to 2 s.f.
to 3 d.p.	to 3 s.f.
to 4 d.p.	to 4 s.f.
to 6 d.p	to 5 s.f.

Example 2

a. Give the scientific form of the numbers:

$x = 100000$ $y = 0.00001$

$z = 4057.52$ $w = 0.00107$

b. Give the standard form of the numbers:

$s = 4.501 \times 10^7$ $t = 4.501 \times 10^{-7}$

Example 3

Consider the numbers:

$x = 3 \times 10^7$ $y = 4 \times 10^7$

Give x + y and xy in scientific form.

Example 4

Consider the numbers:

$x = 3 \times 10^7$ $y = 4 \times 10^9$

Give x + y and xy in scientific form.

1.2 Sequences in General – Series

Example 1

Consider the sequence:

1, 3, 5, 7, 9. 11, …

a. Write the general term.

b. Find the sum of:

4 terms

9 terms

Example 2

$$\sum_{n=1}^{3} 2^n$$

$$\sum_{n=1}^{4} \frac{1}{n}$$

$$\sum_{k=1}^{3} \frac{1}{2^k}$$

$$\sum_{n=1}^{6} (2n+1)$$

Example 3

a. Write $u_n = n^2$ as a sequence of terms.

b. Write $u_n = 2^n$ as a sequence of terms.

Example 4

Write $u_1 = 3,\ u_{n+1} = 2u_n + 5$ as a sequence of terms.

Example 5

Write $u_1 = 1,\ u_2 = 1,\ u_{n+1} = u_n + u_{n-1}$ as a sequence of terms.

1.3 Arithmetic Sequence

Example 1

a. If $u_1 = 1,\ d = 2$ the sequence is…

b. If $u_1 = 2,\ d = 2$ the sequence is…

c. If $u_1 = -10,\ d = 5$ the sequence is…

d. If $u_1 = 10,\ d = -3$ the sequence is…

Example 2

In an arithmetic sequence let $u_1 = 3$ and $d = 5$. Find:

a. the first four terms

b. the 100^{th} term

Example 3

In an arithmetic sequence let $u_1 = 100$ and $u_{16} = 145$. Find u_7.

Example 4

In an arithmetic sequence let $u_{10} = 42$ and $u_{19} = 87$. Find u_{100}.

Example 5

For the arithmetic sequence 3, 5, 7, 9, 11…

a. Find s_3.

b. s_{101}

Example 6

Find 10 + 20 + 30 + … + 200.

Example 7

Show that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Example 8

The 3rd term of an arithmetic sequence is zero while the sum of the first 15 terms is -300. Find the first term and the sum of the first ten terms.

Example 9

Let $x + 1$, $3x$, and $6x – 5$ be consecutive terms of an arithmetic sequence.
Find the value of x.

Example 10

Let a, 10, b, and $a + b$ be consecutive terms of an arithmetic sequence.
Find the values of a and b.

Example 11

Let 100, a, b, c, and 200 be consecutive terms of an arithmetic sequence.
Find the values of a, b, and c.

1.4 Geometric Sequence

Example 1

a. If $u_1 = 1,\ r = 2$ the sequence is…

b. If $u_1 = 5,\ r = 10$ the sequence is…

c. If $u_1 = 1,\ r = -2$ the sequence is…

d. If $u_1 = 1,\ r = \frac{1}{2}$ the sequence is…

e. If $u_1 = 1,\ r = -\frac{1}{2}$ the sequence is…

Example 2

In a geometric sequence let $u_1 = 3$ and $r = 2$. Find:

a. the first four terms.

b. the 100th term.

Example 3

In a geometric sequence let $u_1 = 10$ and $u_{10} = 196830$. Find u_3.

Example 4

A geometric sequence has a fifth term of 3 and a seventh term of 0.75. Find:

a. the first term u_1 and the common ratio r.

b. the tenth term.

Example 5

Find the sum $2 + 2^2 + 2^3 + \cdots + 2^{10}$.

Example 6

Find the sum $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{10}}$.

Example 7

In a geometric sequence $u_2 = -30$ and $s_2 = -15$. Find u_1 and r.

Example 8

Show that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$

Example 9

Show that $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=2$

Example 10

Show that $0.3333\ldots=\frac{1}{3}$.

Example 11

Show that $0.9999\ldots=1$.

1.5 Percentage Change – Financial Applications

Example 1

a. An amount of 2000 euros is invested at 8% per year. What is the amount returned after 10 years?

b. An amount of 2000 euros is depreciated by 8% every year. What is the amount returned after 10 years?

Example 2

There are ten boxes in a row. The first box contains 100€ and any subsequent box contains 10% more than the previous one. What is the amount in the 10^{th} box?

Example 3

An amount of 2000 euros is invested at 8% per year. After how many complete years does the amount exceed 5000?

Example 4

The current population of a city is 800,000. The population increases by 5.2% every year. Find:

a. the population of the city after 7 years.

b. the population of the 7 years ago.

c. after how many complete years the population of the city doubles.

Example 5

An initial amount of 1000 euros and then an extra amount of 1000 euros at the end of each year is invested with an interest rate of 12% compounded yearly. Find the value of the investment after 7 years.

Example 6

An amount of 1000 euros is invested with an interest rate 12% compounded yearly. An extra payment of 300 euros is added at the end of every year. Find the value of the investment after 7 years.

Example 7

An amount of 1000 euros is invested with an interest rate 12% compounded monthly. An extra payment of 300 euros is added at the end of each year. Find the value of the investment after 7 years.

Example 8

An amount of 1000 euros is invested with an interest rate 12% compounded monthly. An extra payment of 300 euros is added at the end of every month. Find the value of the investment after 7 years.

Example 9

An amount of 1000 euros is invested with an interest rate 12% compounded monthly. A withdrawal of 150 euros is made at the end of each year. Find the value of the investment after 7 years.

1.6 The Binomial Theorem – $(a + b)^n$

The coefficients may be obtained by Pascal's Triangle below:

1		
1 1	→	coefficients of $(a+b)^1$
1 2 1	→	coefficients of $(a+b)^2$
1 3 3 1	→	coefficients of $(a+b)^3$
1 4 6 4 1	→	coefficients of $(a+b)^4$
1 5 10 10 5 1	→	coefficients of $(a+b)^5$
	etc	

Example 1

Find the expansion of $(2x+3)^3$ and $(2x-3)^3$.

Example 2

Expand $(2x-3)^4$.

Example 3

In the expansion $(2x-3)^4$ find the term of x^3.

Example 4

Find the term of x^5 in the expansion of $(2x-3)^7$.

Example 5

In the expansion of $(2x^2+1)^8$ find the coefficient of x^{10}.

Example 6

In the expansion of $\left(2x + \frac{1}{x}\right)^6$ find:

a. the coefficient of x^2.

b. the constant term.

Example 7

Find the constant term in the expansion of $\left(2x^2 - \frac{3}{x}\right)^{12}$.

Example 8

Find the term of x^5 in the expansion of $(2x+3)(4x+1)^7$.

Example 9

a. Verify that $2x^2-3x-2=(2x+1)(x-2)$.

b. Find the coefficient of x^2 in the expansion of $(2x^2-3x-2)^5$.

1.7 Deductive Proof

Example 1

Prove $(a+b)^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3$.

Example 2

Prove $(2a+b)^2 - 4a^2 - b^2 \equiv (a+b)^2 - (a-b)^2$.

Example 3

Given that $a \neq \pm b$, show that $\frac{a-2b}{a-b} \equiv \frac{a^2-ab-2b^2}{a^2-b^2}$.

Example 4

Prove $\frac{1}{m+1} + \frac{1}{m^2+m} \equiv \frac{1}{m}$.

Example 5

Show that $2x^2 - 12x + 19 \equiv 2(x-3)^2 + 1$.

Example 6

Given that $2x^2 - 12x + 19 \equiv 2(x-b)^2 + c$ is an identity, find the values of a, b, and c.

1.8 Methods of Proof (HL only)

♦ **THREE METHODS OF PROOF**

We will use a simple example to demonstrate three kinds of proof.

Let X be the name of a city

- ***Deductive proof***

The usual process of reasoning is to start from the hypothesis and reach the result by using logical steps. We have already discussed this method in the previous paragraph.

Show that:

If X is a Greek city, then X is a European city

Proof. Assume that X is a Greek city.
But Greece is part of Europe.
Then X is a Europe city.

- ***Proof by a counterexample***

We use a counterexample to establish that a statement **is not true** in general. The converse of the statement above is

If X is a European city, then X is a Greek city.

Show that this statement is not true.

Proof. Select Rome. It is a European city but not in Greece! Hence the statement is not true in general.

- ***Proof by contradiction***

The contrapositive of the original statement is true:

If X is a non-European city, then it is not a Greek city.

Proof. Let X be a non-European city.
Assume that the result is false, i.e. X is a Greek city.
But then X would be a European city! Contradiction.
Thus, the result is true: X **is not a Greek city**.

The principle of contradiction is based on the fact that any statement is equivalent to its contrapositive statement.

Example 1

Deductive Proof:

If a is even, then a^2 is also even.

Example 2

Proof by Contradiction:

If a^2 is even, then a is even.

Example 3

Deductive Proof:

If a is a multiple of 4, then a^2 is also a multiple of 4.

Example 4

Proof by Counterexample:

a^2 is a multiple of 4 does not imply that a is a multiple of 4.

Example 5

Proof by Contradiction:

Show that $\sqrt{2}$ is irrational.

Example 6

Proof by Contradiction:

The equation $6x + 15y = 100$ has no integer solutions.

1.9 Mathematical Induction (HL only)

Induction:

1. We **show** that the statement is true for n=1
2. We **assume** that the statement is true for n=k (some k)
3. We **prove** that the statement is true for n=k+1 based on the assumption of step 2

Example 1

Prove by mathematical induction that $1+2+3+\cdots+n=\frac{n(n+1)}{2}$ for any $n\geq 1$.

Example 2

Prove by mathematical induction that the number $6^n - 1$ is divisible by 5 for any $n \geq 1$.

Example 3

Prove by mathematical induction that $n! > 2^n$ for any $n \geq 4$.

Example 4

Consider the sequence $u_1 = 0$, $u_{n+1} = 2u_n + 2$.

a. Find the first six terms of the sequence. Is it an arithmetic or a geometric sequence?

b. Compare the results with the first powers of 2. What do you notice? Can you guess a general formula for u_n in terms of n?

c. Prove your guess is true by mathematical induction.

NOTICE

Sometimes, we must assume two preceding steps in order to prove to the next step. These proofs look like

Induction:

1. *We **show** that the statement is true for n=1 and n=2*
2. *We **assume** that the statement is true for n=k and n=k+1*
3. *We **prove** that the statement is true for n=k+2 based on the assumptions of step 2.*

It will be very clear whether we have to follow this proof.

Example 5

Consider the Fibonacci sequence $u_1 = 1$, $u_2 = 1$, $u_{n+2} = u_n + u_{n+1}$.

Prove by induction that $u_n < 2^n$ for any $n \geq 1$.

1.10 Systems of Linear Equations

Example 1

Solve each of the systems.

a. $\begin{cases} 2x + 3y = 9 \\ 4x + 7y = 19 \end{cases}$

b. $\begin{cases} x + 2y = 1 \\ 2x + 4y = 10 \end{cases}$

c. $\begin{cases} x + 2y = 1 \\ 2x + 4y = 2 \end{cases}$

Example 2

Solve each of the systems.

a. $\begin{cases} 5x + 11y - 21z = -22 \\ x + 2y - 4z = -4 \\ 3x - 2y + 3z = 11 \end{cases}$

b. $\begin{cases} 2x + 3y + 3z = 3 \\ x + y - 2z = 4 \\ 5x + 7y + 4z = 5 \end{cases}$

c. $\begin{cases} 2x + 3y + 3z = 3 \\ x + y - 2z = 4 \\ 5x + 7y + 4z = 10 \end{cases}$

1.11 Complex Numbers – Basic Operations

$\boxed{i^2 = -1}$

- THE DEFINITION

A number z of the form $\boxed{z = x + yi}$

where x, y $\in$R, is called a **complex number**. We also say,

the **real part** of z is x:	Re(z)=x
the **imaginary part** of z is y:	Im(z)=y

The set of all complex numbers is denoted by C. A real number x is also complex of the form x+0i (it has no imaginary part).

Every complex
√ root comes in
± form.

- THE CONJUGATE $\bar{z}$

The **conjugate** complex number of $z = x+yi$ is given by $\bar{z}=x-yi$

(Sometimes, the conjugate number of z is denoted by z^*)

- THE MODULUS $|z|$

The **modulus** of z=x+yi is defined by: $|z| = \sqrt{x^2+y^2}$

Distance from the origin (0,0).

Notice

z	$x+yi$
$\bar{z}$	$x-yi$
$-z$	$-x-yi$
$-\bar{z}$	$-x+yi$

all have the same modulus $\sqrt{x^2+y^2}$

- EQUALITY: $z_1 = z_2$

Two complex numbers are equal if they have equal real parts and equal imaginary parts: Let $z_1=x_1+y_1i$ and $z_2=x_2+y_2i$

$$z_1 = z_2 \quad \Leftrightarrow \quad \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases}$$

Thus, the equation of complex number must be thought as a system of two simultaneous equations.

Example 1

Let $z_1 = 3 + 4i$ and $z_2 = a + (3b - 2)i$. Find a and b if $z_1 = z_2$.

$3 = a$

$4 = 3b - 2$

$6 = 3b$

$b = 2$

You set the imaginary parts equal without the i, since it would cancel out anyways, like this:

$4\cancel{i} = 3b\cancel{i} - 2\cancel{i}$

Example 2

Consider the two complex numbers $z = 7 + 4i$ and $w = 2 + 3i$.

a. $z + w$

$7 + 4i + 2 + 3i$

$= 7i + 9$

b. $z - w$

$7 + 4i - 2 - 3i$

$= 5 + i$

c. zw

$= 14 \cdot 12i^2$

$= 14 \cdot (-12)$

$= -144 - 24$

$= -168$

Just foil, not real · real times imaginary · imaginary

$(7+4i)(2+3i)$

$14 + 21i + 8i - 12$

$= 2 + 29i$

d. $\frac{z}{w}$

$$\frac{(7+4i)(2-3i)}{(2+3i)(2-3i)}$$

$$= \frac{14 - 21i + 8i - 12i^2}{4 - \cancel{6i} + \cancel{6i} - 9i^2}$$

$$= \frac{14 - 13i - 12(-1)}{4 - 9(-1)}$$

$$= \frac{\left(\frac{26 - 13i}{13}\right)}{13} = 2 - i$$

Example 3

Calculate:

a. $z = (2+i)^3$

$\binom{3}{0}(2)^3(i)^0 + \binom{3}{1}(2)^2(i)^1$
$+ \binom{3}{2}(2)^1(i)^2 + \binom{3}{3}(2)^0(i)^3$

$= 8 + 12i + 6i^2 + i^3$
$= 8 + 12i - 6 + 1$
$= 3 + 12i$

b. $w = \frac{(2+i)^3}{1-i}$

This is a 2, not a 3.

$w = \frac{(3+12i)(1+i)}{(1-i)(1+i)}$

$w = \frac{3+3i+12i-12}{1+1}$

$w = \frac{-9+15i}{2}$

Example 4

Find z if $z(1-i) = 2 + 11i$.

$z = \frac{2+11i}{1-i}$

$z = \frac{(2+11i)(1-i)}{2}$

$z = \frac{2-2i+11i+11}{2}$

$z = \frac{13+9i}{2}$

1.12 Polynomials Over the Complex Field

> ***Fundamental theorem of algebra***
>
> *A polynomial of degree n>1 has exactly n roots (in C)*

Example 1

Find all three roots of the cubic function $f(z) = z^3 - 5z^2 + 9z - 5$ given that $z = 1$ is a real root.

$(z-1)$

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 9 & -5 \\ \hline & 1 & -4 & 5 & 0 \end{array}$$

$f(z) = (z-1)(z^2-4z+5)$

$$z = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$$

$$z = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$z = 1,\ z = 2+i,\ z = 2-i$

Example 2

Find all three roots of the cubic function $f(z) = z^3 - 5z^2 + 9z - 5$ given that $z = 2 + i$ is a complex root.

$(z-2+i)(z-2-i)$

$z^2 - 2z \cancel{-iz} - 2z + 4 \cancel{+2i} \cancel{+iz} \cancel{-2i} - i^2$

$z^2 - 4z + 4 + 1$

$= z^2 - 4z + 5$

$$\begin{array}{r|l} & z - 1 \\ z^2-4z+5 & z^3 - 5z^2 + 9z - 5 \\ & -(z^3 - 4z^2 + 5z) \\ \hline & -z^2 + 4z - 5 \\ & -z^2 + 4z - 5 \\ \hline & 0 \end{array}$$

$z = 2+i,\ z = 2-i,\ z = 1$

Example 3

Consider the cubic function $f(z) = z^3 + az^2 + bz + c$. Given the roots $z = 1$ and $z = 2 + i$, find the coefficients a, b, and c.

$(z-1)(z-2+i)(z-2-i)$

$(z-1)(z^2-4z+5)$

$z^3 + 4z^2 + 5z - z^2 + 4z - 5$

$= z^3 - 5z^2 + 9z - 5$

$a = 3,\ b = 9,\ c = -5$

1.13 The Complex Plane

♦ THE POLAR FORM (MODULUS-ARGUMENT FORM)

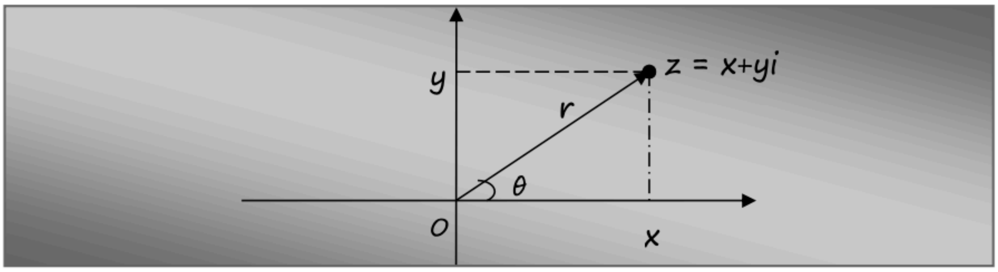
We have just seen that a complex number $z=x+yi$ is represented on the plane by a pair of <u>**Cartesian coordinates**</u> $(\mathbf{x},\mathbf{y})$.

An alternative way to describe a point on the plane (and thus the position of z) is the so-called <u>**Polar coordinates**</u> $(\mathbf{r},\boldsymbol{\theta})$:

We draw a vector (an arrow) from O to the point and consider

$\mathbf{r}$ = the length of the vector

$\boldsymbol{\theta}$ = the angle between the x-axis and the vector



Notice:

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}, \quad \tan\theta = \frac{y}{x} \qquad (*)$$

For a complex number $z=x+yi$

r is in fact the **modulus** $|z|$

θ is called **argument** of z. We write $\mathbf{arg(z)=\theta}$

The relations (*) above give

$$\mathbf{x=r\cos\theta} \quad \text{and} \quad \mathbf{y=r\sin\theta}$$

Thus, a complex number can be also written as

$$z = x+yi = (r\cos\theta)+(r\sin\theta)i = r\,(\cos\theta+i\sin\theta)$$

The form

$$\boxed{z = r\,(\cos\theta+i\sin\theta)}$$

is known as the **polar form** of the complex number z
(or otherwise **modulus-argument form** or **trigonometric form**)

♦ TRANSFORMATION FROM $\mathbf{z = x+yi}$ TO $\mathbf{z = r\,(\cos\theta+i\sin\theta)}$

Given: $z = x+yi$. We find r and θ by

- $\boxed{r = |z| = \sqrt{x^2+y^2}}$
- $\boxed{\tan\theta = \dfrac{y}{x}}$, having in mind the quadrant of $x+yi$

Example 1

Find the polar form of $z = 1 + \sqrt{3}i$ and $w = 3 + 4i$.

Example 2

Find the polar form of:

a. $z_1 = 1 + i$

b. $z_2 = -1 - i$

c. $z_3 = 1 - i$

d. $z_4 = -1 + i$

- CIS FORM: $z = r\text{cis}\theta$

There is an abbreviation for the polar form

$$z = r\,(\cos\theta + i\sin\theta).$$

It is sometimes written as

$$z = r\text{cis}\theta.$$

- EULER'S FORM: $z = re^{i\theta}$

Another abbreviation is due to Euler.

We define‡

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Consequently, the trigonometric form $z = r(\cos\theta + i\sin\theta)$ obtains the form

$$\boxed{z = re^{i\theta}}$$

Example 5

Write down all of the possible forms of:

a. $z_1 = 1 + i$

b. $z_2 = 3 + 4i$

c. $z_3 = 3 - 4i$

1.14 De Moivre's Theorem

De Moivre's theorem	$[r(\cos\theta + \mathrm{i}\sin\theta)]^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta) = r^n \mathrm{e}^{\mathrm{i}n\theta} = r^n \operatorname{cis} n\theta$

Example 1

Let $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$. Find:

a. z^2

b. z^{-1}

c. z^6

Example 2

Find $(1+i)^{10}$.

1.15 Roots of $z^n = a$

$$z_k = cis\frac{2k\pi}{n} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$$

Example 1

a. Write down the 5th roots of 1.

b. Factor $z^5 - 1$.

c. Use the sum of the roots to show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.

♦ n-th ROOTS OF A COMPLEX NUMBER a

Consider now the equation

$$\boxed{z^n = a}$$

where a is a complex number.

Let $z = r\text{cis}\theta$ be a root. Then $z^n = r^n\text{cis}(n\theta)$

We also express the complex number a in polar form: $a = \rho\text{cis}\phi$.

Then

$$z^n = a \Leftrightarrow r^n\text{cis}(n\theta) = \rho\text{cis}\phi \Leftrightarrow \begin{cases} r^n = \rho \Leftrightarrow r = \sqrt[n]{\rho} \\ n\theta = \phi + 2k\pi \Leftrightarrow \theta = \dfrac{\phi + 2k\pi}{n} \end{cases}$$

For $k=0,1,2,\ldots,n-1$ we obtain the following n roots of a

$$\boxed{z_k = \sqrt[n]{\rho}\,\text{cis}\left(\frac{\phi + 2k\pi}{n}\right)}$$

Example 2

Solve the equation $z^3 = 8i$.