INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

MAA

EXERCISES [MAA 5.20]

INTEGRATION BY PARTS

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O. Practice questions

1. [Maximum mark: 16] ***[without GDC]***

Find $I_1 = \int 2x e^x \mathrm{d}x$, $I_2 = \int 2x\cos x \mathrm{d}x$, $I_3 = \int 2x\sin x \mathrm{d}x$, $I_4 = \int 2x\ln x \mathrm{d}x$

$u = 2x \mid v = e^x$

$du = 2 \mid dv = e^x$

$I_1 = 2xe^x - \int e^x 2\,dx$

$I_1 = 2xe^x - 2\underbrace{\int e^x dx}_{\text{Integratable}}$

$I_1 = 2xe^x - 2e^x + C$

$I_2 = 2x\sin x - 2\int \sin x\, dx$

$I_2 = 2x\sin x + 2\cos x + C$

$I_3 = -2x\cos x + 2\int \cos x\, dx$

$I_3 = -2x\cos x + 2\sin x$

$I_4 = \ln x \cdot x^2 - \int x^2 \frac{1}{x}\, dx$

$I_4 = x^2 \ln x - \int x\, dx$

$I_4 = x^2 \ln x - \frac{1}{2}x^2 + C$

2. [Maximum mark: 10] ***[without GDC]***

Find $I_1 = \int (3x^2 + 4x + 1)\cos x\,dx$, $\quad I_2 = \int (3x^2 + 4x + 1)\ln x\,dx$

$I_1 = \int 3x^2 \cos x\,dx + \int 4x\cos x\,dx + \int \cos x\,dx$

$3x^2 \sin x - \int \sin x\, 6x\,dx$

$3x^2 \sin x + 6x\cos x + 6\int \cos x\,dx$

$= 3x^2 \sin x + 6x\cos x + 6\sin x$

Wasting ur time

$I_1 = (3x^2+4x+1)\sin x - \int \sin x(6x+4)\,dx$

$I_1 = (3x^2+4x+1)\sin x - (-(6x+4)\cos x + \int \cos x(6)\,dx)$

$I_1 = (3x^2+4x+1)\sin x + (6x+4)\cos x - 6\sin x + C$

$I_1 = (3x^2+4x-5)\sin x + (6x+4)\cos x + C$

$I_2 = \ln x(x^3+2x^2+x) - \int (x^3+2x^2+x)\frac{1}{x}\,dx$

$I_2 = (x^3+2x^2+x)\ln x - \int x^2+2x+1\,dx$

$I_2 = (x^3+2x^2+x)\ln x - \frac{1}{3}x^3 - x^2 - x + C$

$\frac{d}{dx}\sin(2x) = \cos(2x)2 = 2\cos(2x)$; $\frac{d}{dy}\cos 2x = -\sin 2x \cdot 2 = -2\sin 2x$

3. [Maximum mark: 6] ***[without GDC]***

Find $I = \int e^{2x}\cos 2x\,dx$ by using integration by parts in two different ways:

METHOD A: by integrating e^{2x} first.

$I = \cos 2x\left(\frac{1}{2}e^{2x}\right) + \frac{1}{\cancel{2}}\int e^{2x}(\cancel{2}\sin 2x)\,dx$

$I = \frac{1}{2}\cos 2x\, e^{2x}\ \int e^{2x}\sin 2x\,dx$

$I = \frac{1}{2}\cos 2x e^{2x} + \frac{1}{2}\sin 2x e^{2x} - \frac{1}{2}\int e^{2x}(2\cos 2x)\,dx$

$I = \frac{1}{2}e^{2x}(\sin 2x + \cos 2x) - \int e^{2x}\cos 2x\,dx$

$I = \frac{1}{2}e^{2x}(\sin 2x + \cos 2x) - I$

$2I = \frac{1}{2}e^{2x}(\sin 2x + \cos 2x)$

$I = \frac{1}{4}e^{2x}(\sin 2x + \cos 2x) + C$

METHOD B: by integrating $\cos 2x$ first.

$\int \cos 2x = \frac{1}{2}\sin(2x) + C$

$I = e^{2x}\cdot\frac{1}{2}\sin(2x) - \frac{1}{\cancel{2}}\int \sin(2x)\cdot\cancel{2}e^{2x}\,dx$

$I = \frac{1}{2}e^{2x}\sin(2x) - \underbrace{\int e^{2x}\sin(2x)\,dx}_{I}$

$I = \frac{1}{2}e^{2x}\sin(2x) - \left(-\frac{1}{2}e^{2x}\cos(2x) + \frac{1}{\cancel{2}}\int e^{2x}\cancel{2}\cos(2x)\,dx\right.$

$I = \frac{1}{2}e^{2x}\sin(2x) + \frac{1}{2}e^{2x}\cos(2x) - I$

$2I = \frac{1}{2}e^{2x}(\sin(2x) + \cos(2x))$

$I = \frac{1}{4}e^{2x}(\sin(2x) + \cos(2x))$

4. [Maximum mark: 10] ***[without GDC]***

Let $I_n = \int x^n \mathrm{e}^x \mathrm{d}x$.

(a) Find I_0 . [2]

(b) Express I_n in terms of I_{n-1} by using integration by parts. [2]

(c) Find I_1, I_2 and I_3 by using the recursive relation found above. Express the results in the form $I_n = p(x)\mathrm{e}^x + c$, where $p(x)$ is a polynomial. [6]

(a) $I_0 = \int x^0 e^x dx = \int e^x dx = e^x$

(b) $I_n = x^n e^x - \int e^x n x^{n-1} dx$

$I_n = x^n e^x - n \int x^{n-1} e^x dx$

$I_n = x^n e^x - n I_{n-1}$

(c) $I_3 = x^3 e^x - 3 I_2 = x^3 e^x - 3(x^2 e^x - 2(x e^x - e^x)) =$

$I_2 = x^2 e^x - 2 I_1 = x^2 e^x - 2(x e^x - e^x) = (x^2 - 2x + 2)e^x + c$

$I_1 = x e^x - I_0 = x e^x - e^x = (x-1)e^x + c$

$I_0 = e^x$

$x^3 e^x - 3x^2 e^x + 6x e^x + 6e^x = (x^3 - 3x^2 + 6x + 6)e^x + c$

5. [Maximum mark: 12] ***[without GDC]***

Consider $I = \int 4\sin x \cos x \mathrm{d}x$

(a) Find I by using

(i) the substitution $u = \sin x$.

(ii) the substitution $u = \cos x$.

(iii) the double angle formula for $\sin 2\theta$.

(iv) integration by parts. [10]

(b) Explain the difference in the results. [2]

(a) (iii) $\sin 2x = 2\sin x\cos x$

$I = \int 2\sin 2x \, dx = -\cos 2x + C$

6. [Maximum mark: 12] ***[without GDC]***

Calculate the definite integral $I=\int_0^1 (3x+2)\mathrm{e}^x \mathrm{d}x$

METHOD A: Find the definite integral first and then the definite (preferable!)

METHOD B: Apply integration by parts on the definite integral, keeping the limits.

A. Exam style questions (SHORT)

7. [Maximum mark: 6] ***[without GDC]***

Find $\int (\theta \cos\theta - \theta)\,\mathrm{d}\theta$.

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8. [Maximum mark: 6] ***[without GDC]***

Find $\int \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x$.

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9. [Maximum mark: 6] ***[without GDC]***

Find $\int e^{x} \cos x \, dx$.

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10. [Maximum mark: 6] ***[without GDC]***

Find $\int e^{2x} \sin x \, dx$.

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11*. [Maximum mark: 7] ***[without GDC]***

Let $f(x) = x\ln x - x$, $x > 0$.

(a) Find $f'(x)$. [3]

(b) Using integration by parts, find $\int (\ln x)^2 \mathrm{d}x$. [4]

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12. [Maximum mark: 6] ***[without GDC]***

Find $\int \arctan x \mathrm{d}x$.

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13*. [Maximum mark: 6] ***[without GDC]***

Find $\int 2x \arctan x \mathrm{d}x$.

14*. [Maximum mark: 6] ***[without GDC]***

Find $\int \frac{x^2}{\mathrm{e}^{2x}} \mathrm{d}x$.

15. [Maximum mark: 6] ***[with / without GDC]***

(a) Use integration by parts to find $\int x^2 \ln x \mathrm{d}x$.

(b) Evaluate $\int_1^2 x^2 \ln x \mathrm{d}x$

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16. [Maximum mark: 5] ***[without GDC]***

Calculate the exact value of $\int_1^{\mathrm{e}} x^5 \ln x \mathrm{d}x$

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17. [Maximum mark: 6] ***[without GDC]***

Show that $\int_0^{\frac{\pi}{6}} x\sin 2x\mathrm{d}x = \frac{\sqrt{3}}{8} - \frac{\pi}{24}$.

18. [Maximum mark: 7] ***[without GDC]***

Find $\int_0^a \arcsin x\mathrm{d}x$, $0 < a < 1$.

B. Exam style questions (LONG)

19. [Maximum mark: 14] ***[without GDC]***

(a) Find $I_n = \int x^n \ln x \mathrm{d}x$ in terms of n, where $n \in R$, $n \neq -1$. [4]

(b) Find $J_n = \int \frac{(\ln x)^n}{x} \mathrm{d}x$ in terms of n, where $n \in R$. [3]

(c) **Hence** find

(i) $\int \sqrt{x} \ln x \mathrm{d}x$, (ii) $\int \frac{\sqrt{\ln x}}{x} \mathrm{d}x$, (iii) $\int \left(\frac{\ln x}{x^2} + \frac{(\ln x)^2}{x} \right) \mathrm{d}x$. [7]

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20. [Maximum mark: 16] ***[without GDC]***

Let $A_n = \int x^n \sin x \mathrm{d}x$ and $B_n = \int x^n \cos x \mathrm{d}x$

(a) Find A_0 and B_0. [2]

(b) Express A_n in terms of B_{n-1} by using integration by parts. [2]

(c) Express B_n in terms of A_{n-1} by using integration by parts. [2]

(d) **Hence** express

(i) A_n in terms of A_{n-2}.

(ii) B_n in terms of B_{n-2}. [5]

(e) Find A_1, A_2 and A_3 by using the results above. [5]

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21. [Maximum mark: 28] ***[without GDC]***

Let $I_n = \int \sin^n x \mathrm{d}x$

(a) Find I_0 and I_1. [4]

(b) Express I_n in terms of I_{n-2} by using integration by parts. [4]

(c) Find I_2, I_3, I_4, I_5 by using the recursive relation found in (b). [4]

(d) Find I_2, I_4 by using the double angle formula $\sin^2 x = \dfrac{1-\cos 2x}{2}$. [8]

(e) Find I_3, I_5 by using the substitution $u = \cos x$ [8]

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NOTICE

The following table is from my lecture notes.

Please make sure that you are able to solve all the examples in column 2 and the theoretical questions in column 3.

General Form	Examples	Theoretical Questions
$I_n=\int x^n e^x dx$ $I_{n,m}=\int x^n e^{mx} dx$	$\int x^3 e^x dx$, $\int x^2 e^{3x} dx$	Express I_n in terms of I_{n-1} Hence find $I_0, I_1, I_2, \dots$
$I_n=\int x^n \cos x dx$ $I_n=\int x^n \sin x dx$	$\int x^2 \cos x dx$	Express I_n in terms of I_{n-2}
$I_{n,m}=\int x^n \cos(mx) dx$ $I_{n,m}=\int x^n \sin(mx) dx$	$\int x^2 \cos 3x dx$	
$I_n=\int x^n \ln x dx$	$\int x^5 \ln x dx$, $\int \frac{\ln x}{x^5} dx$ $\int \sqrt{x} \ln x dx$	Find a general formula for I_n
$I_{n,m}=\int e^{nx}\sin(mx) dx$ $I_{n,m}=\int e^{nx}\cos(mx) dx$	$\int e^{3x}\sin 2x dx$ $\int e^{-x}\sin 2x dx$	Find a general formula for $I_{n,m}$
$I_n=\int \cos^n x dx$ $I_n=\int \sin^n x dx$	$\int \cos^2 x dx$ $\int \cos^3 x dx$	Express I_n in terms of I_{n-2} Hence find I_2, I_4 and I_3, I_5
$I_{n,m}=\int \sin(nx)\cos(mx) dx$	$\int \sin 2x \cos 3x dx$	Find a general formula for $I_{n,m}$
$I_n=\int x^n \arctan x dx$ $I_n=\int x^n \arcsin x dx$ $I_n=\int x^n \arccos x dx$	$\int \arctan x dx$, $\int x \arctan x dx$, $\int x^2 \arctan x dx$ $\int \arcsin x dx$, $\int x^2 \arcsin x dx$	
$I_n=\int (\ln x)^n dx$	$\int (\ln x)^2 dx$, $\int (\ln x)^3 dx$	