75 Linear Algebra

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Augmented matrix

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 1 & -1 & 2 \end{array}\right)$$

$R_{ij} \to i^{th}$ & j^{th} rows are interchanged

$cR_i \to i^{th}$ row is multiplied by c.

$R_{ij}(c) \to R_i + cR_j$.

linear Transformation

let V & W be two vector spaces over a field I. A function $T: V \to W$ is said to be a linear transformation if

$$\boxed{T(c\alpha+\beta) = c\,T(\alpha) + T(\beta)}, \quad \forall \alpha, \beta \in V$$
& $c \in I$

Ex: Identity transformation over a vector space V over a field I

$T(x) = x, \quad \forall x \in V$.

Ex: Zero transformation over a V. S. V over a field I.

$0(x) = 0$.

Ex: $P_n(R) \to$ vector space of all real polynomials of degree $\leq n$.

$P_n(x) \in P_n(R)$ will look like.

$P_n(x) = a_0 + a_1 x + \cdots a_n x^n$.

$a_0 \cdots a_n \in R$.

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$T : P_n(R) \to P_n(R)$

defined by $T(f) = \frac{df}{dx}$, $f(x) = a_0 + a_1 x_1 + \cdots + a_n x$

$T(f) = a_1 + 2a_2 x + \cdots n a_n x^{n-1}$.

let $f_1(x) = a_0 + a_1 x + \cdots a_n x^n$.

$f_2(x) = b_0 + b_1 x + \cdots b_n x^n$.

$T(cf_1 + f_2)$, $c \in R$, $f_1, f_2 \in P_n(R)$.

$= T[c(a_0 + a_1 x + \cdots a_n x^n) + (b_0 + b_1 x + \cdots b_n x^n)]$

$= T\{(ca_0 + b_0) + (ca_1 + b_1)x + \cdots (ca_n + b_n)x^n\}$

$= (ca_1 + b_1) + 2(ca_2 + b_2)x + \cdots n(ca_n + b_n) x^{n-1}$.

$= c[a_1 + 2a_2 x + \cdots n a_n x^{n-1}] + (b_1 + b_2 + \cdots n b_n x^{n-1})$

$= cT(f_1) + T(f_2)$

$\therefore$ T is linear transformation on $P_n(R)$

$T : R^m \to R^n$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$A = (a_{ij})_{n \times m}$ → gives matrix.

$X \in R^m$

$T(X) = AX$ | let $\alpha, \beta \in R^m$ & $c \in R$

$T(c\alpha + \beta) = A(c\alpha + \beta) = cA\alpha + A\beta$

$= cT(\alpha) + T(\beta)$

Properties :

1). $T(0) = 0$

$0 + 0 = 0$

$T(0+0) = T(0)$

$\Rightarrow T(0) + T(0) = T(0)$

$\{T(0) + T(0)\} + \{-T(0)\} = T(0) + \{-T(0)\}$

$\{T(0) + T(0)\} + \{-T(0)\} = 0.$

$\Rightarrow T(0) + \{T(0) + (-T(0))\} = 0.$

$\Rightarrow T(0) + 0 = 0 \Rightarrow T(0) = 0.$

Theorem : Let V be a finite dimensional vector space over a field F and T be a linear transformation from V to a vector space W over F.

L-I set,
$\{\alpha_1, \alpha_2 - \alpha_n\}$ such that
for any $\alpha \in V$ $\exists$ such $c_1 c_2 - c_n \in F$.
such that $\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots c_n\alpha_n$.
No. of elements in a basis is called dimension of V.

let $\dim V = n$, $\beta = \{\alpha_1, \alpha_2 \cdots \alpha_n\}$ is an ordered basis of V and $\{\beta_1, \beta_2 \cdots \beta_n\}$ be a set of vectors of W. Then $\exists$ a unique linear Transformation $T \cdot V$ to W

S.T. $T(\alpha_i) = \beta_i$ $(i = 1, 2, \cdots n)$

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proof:

For any vector $\alpha \in V$, $\exists$ a unique set of scalars $\{c_1, c_2 - c_n\}$ such that $\alpha = c_1\alpha_1 + c_2\alpha_2 - c_n\alpha_n$.

Now define a function $= T$ from V to W by.

$$T(\alpha) = c_1\beta_1 + c_2\beta_2 + \dots c_n\beta_n.$$

let $\alpha, \beta \in V$ and $c \in F$.

for linearity of T we need to show.

$$T(c\alpha + \beta) = cT(\alpha) + T(\beta).$$

$\alpha, \beta \in V$, $\exists$ unique set of scalars.

$\{c_1, c_2, \dots c_n\}$ & $\{d_1, d_2, \dots d_n\}$ such that.

$$\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots c_n\alpha_n$$

$$\beta = d_1\alpha_1 + d_2\alpha_2 + \dots d_n\alpha_n$$

$$c\alpha + \beta = c(c_1\alpha_1 + c_2\alpha_2 + \dots c_n\alpha_n) + (d_1\alpha_1 + d_2\alpha_2 + \dots d_n\alpha_n)$$

$$= (cc_1 + d_1)\alpha_1 + (cc_2 + d_2)\alpha_2 + \dots (cc_n + d_n)\alpha_n$$

$$T(c\alpha + \beta) = (cc_1 + d_1)\beta_1 + (cc_2 + d_2)\beta_2 + \dots (cc_n + d_n)\beta_n$$

$$= c[(c_1\beta_1 + c_2\beta_2 + \dots) + (d_1\beta_1 + d_2\beta_2 + \dots d_n\beta_n)]$$

$$= c\,T(\alpha) + T(\beta).$$

$\therefore T$ is linear

Uniqueness: $\exists$

let U be a L.T from V to W s.t.

$U(\alpha_i) = \beta_i \quad (i = 1, 2 \ldots n)$

we have to show $T(\alpha_i) = \beta_i \; (i = 1, 2 \ldots n)$

$\alpha_1 = 1.\alpha_1 + 0.\alpha_2 + \ldots 0.\alpha_n$

$\alpha_2 = 0.\alpha_1 + 1.\alpha_2 + \ldots 0.\alpha_n$

$\vdots$

$\alpha_n = 0.\alpha_1 + 0.\alpha_2 + \ldots 1.\alpha_n$

$T(\alpha_1) = 1.\beta_1 + 0.\beta_2 + \ldots 0.\beta_n = \beta_1$

$T(\alpha_2) = 0.\beta_1 + 1.\beta_2 + \ldots 0.\beta_n = \beta_2$

$\vdots$

$T(\alpha_n) = 0.\beta_1 + \ldots 1.\beta_n = \beta_n$

$\therefore T(\alpha_i) = \beta_i \quad (i = 1, 2, \ldots n)$

let $\alpha \in V$ be any vector $\{c_1, c_2 \ldots c_n\}$ s.t.

$\exists$ unique set of scalars

$\alpha = c_1\alpha_1 + c_2\alpha_2 + \ldots c_n\alpha_n$

$\Rightarrow U(\alpha) = U[c_1\alpha_1 + c_2\alpha_2 + \ldots c_n\alpha_n]$

$= c_1 U(\alpha_1) + c_2 U(\alpha_2) + \ldots c_n U(\alpha_n)$

($\because$ U is L.T)

$= c_1\beta_1 + c_2\beta_2 + \ldots c_n\beta_n$

$[\because U(\alpha_i) = \beta_i, \; i = 1, \ldots n]$

$= T(\alpha)$

$\therefore$ Uniqueness is proved

Prob: Let T be a linear ~~transformed~~ operator on R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$$

a) verify that T is L.T.

b) If (a, b, c) is a vector of R^3. what are the conditions on a, b, c s.t (a, b, c) is in range of T.

c) what are conditions on (a, b, c) s.t it is in the null space of T.

Range of $T = \{T(\alpha) : \alpha \in R^3\}$

Null space of T, $N(T) = \{\alpha \in V \mid T(\alpha) = 0\}$

Sol: a) let. $\alpha = (x_1, x_2, x_3)$ & $\beta = (y_1, y_2, y_3) \in R^3$.

& $c \in R$.

we have to check.

$$T(c\alpha + \beta) = cT(\alpha) + T(\beta)$$

$$T(c\alpha + \beta) = T[c(x_1, x_2, x_3) + (y_1, y_2, y_3)]$$

$$= T[(cx_1 + y_1) + (cx_2 + y_2) + (cx_3 + y_3)]$$

$$= [3(cx_1 + y_1), (cx_1 + y_1 - cx_2 - y_2), (2(cx_1 + y_1) + cx_2 + y_2 + cx_3 + y_3)]$$

$$= c(3x_1, x_1 - x_2, 2x_1 + x_2 + x_3) + (3y_1, y_1 - y_2, 2y_1 + y_2 + y_3)$$

$$= cT(\alpha) + T(\beta)$$

$\therefore$ T is linear transformation.

b). let $a, b, c \in \operatorname{Im} T$.

$\exists$ at lest one $\alpha = (x, y, z)$ s.t.

$T(\alpha) = (a, b, c)$

$\Rightarrow T(x, y, z) = (a, b, c)$

$\Rightarrow (3x, x-y, 2x+y+z) = (a, b, c)$

$\Rightarrow 3x = a$

$x - y = b$

$2x + y + z = c$.

<u>Argumented matrix:</u>

$$\begin{pmatrix} 3 & 0 & 0 & a \\ 1 & -1 & 0 & b \\ 2 & 1 & 1 & c \end{pmatrix} \xrightarrow{1/3 R_1} \begin{pmatrix} 1 & 0 & 0 & a/3 \\ 1 & -1 & 0 & b \\ 2 & 1 & 1 & c \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{pmatrix} 1 & 0 & 0 & a/3 \\ 0 & -1 & 0 & b - a/3 \\ 0 & 1 & 1 & c - 2a/3 \end{pmatrix}$$

$$\xrightarrow{-R_2} \begin{pmatrix} 1 & 0 & 0 & a/3 \\ 0 & 1 & 0 & a/3 - b \\ 0 & 1 & 1 & c - 2a/3 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 0 & 0 & a/3 \\ 0 & 1 & 0 & a/3 - b \\ 0 & 0 & 1 & c - (a+b) \end{pmatrix}$$

There is no condition on (a, b, c) st: it is in $\operatorname{im} T$.

ⓒ $N(T) = \{ (x, y, z) : T(x, y, z) = (0, 0, 0) \}$

If $(a, b, c) \in N(T)$

$T(a, b, c) = (0, 0, 0)$

$\Rightarrow (3a, a-b, 2a+b+c) = (0, 0, 0)$

$$\left.\begin{aligned} 3a &= 0 \\ a - b &= 0 \\ 2a + b + c &= 0 \end{aligned}\right\} \Rightarrow a = b = c = 0.$$

Rank & Nullity:

let T be a linear transformation from a vector space V to V.S W over the field T.

Image T = im T.

$= \{ T(\alpha) : \alpha \in V \}$

dim (im T) = rank T

Null space of T, $N(T) = \{ \alpha \in V \mid T(\alpha) = 0 \}$.

Rank - Nullity theorem:

let $T : V \to W$ be a linear transformation, where V and W are V.S over the field T.

let dim V = n, Then.

rank T + Nullity T = dim V.

proof: let Nullity of $T = k (\leq n)$

let $\{ \alpha_1, \alpha_2 -- \alpha_n \}$ is a basis of N(T).

Extension theorem:

let V be a vector space over a field F let S be a linearly independent sets of vector of V. Then either S is a basis of V or it can be extended to a basis of V.

Now by extension theorem the set $\{ \alpha_1, \alpha_2 -- \alpha_n \}$ is extended to a basis of V which is given by $\{\alpha_1, \alpha_2 - \alpha_k, \alpha_{k+1} --- \alpha_n\}$

Now the set $\{T(\alpha_1), T(\alpha_2), \cdots T(\alpha_k), T(\alpha_{k+1})$
$\cdots T(\alpha_n)\}$.

spans the image of space of T i.e im T

let $\alpha \in V$.

$\alpha = c_1\alpha_1 + c_2\alpha_2 + -- c_n\alpha_n$.

$T(\alpha) = T(c_1\alpha_1 + c_2\alpha_2 + -- c_n\alpha_n)$.

$= c_1T(\alpha_1) + c_2T(\alpha_2) + --- c_nT(\alpha_n)$.

more over we observe that.

$T(\alpha_i) = 0 \quad (i = 1, 2 --- n)$

$\therefore$ im T is spammed by the set.

$\{T(\alpha_{k+1}), T(\alpha_{k+2}) -- T(\alpha_n)\}$

let $c_{k+1}T(\alpha_{k+1}) + c_{k+2}T(\alpha_{k+2}) + -- c_nT(\alpha_n) = 0$.

$\Rightarrow T(c_{k+1}\alpha_{k+1} + c_{k+2}\alpha_{k+2} + -- c_n\alpha_n) = 0$.

$\Rightarrow c_{k+1}\alpha_{k+1} + c_{k+2}\alpha_{k+2} + -- c_n\alpha_n \in N(T)$.

$\therefore \exists$ scalars $d_1, d_2 -- d_n$ such that.

$c_{k+1}\alpha_{k+1} + c_{k+2}\alpha_{k+2} + --- c_n\alpha_n$.

$= d_1\alpha_1 + d_2\alpha_2 + --- d_n\alpha_n$.

$\Rightarrow d_1\alpha_1 + d_2\alpha_2 + -- d_n\alpha_n - c_{k+1}\alpha_{k+1} - c_{k+2}\alpha_{k+2}$
$--- c_n\alpha_n = 0$.

$\Rightarrow d_1 = d_2 = \text{---} = d_n = 0$. $\left[\because \text{The set } \{\alpha_1, \alpha_2 \ldots \alpha_n\} \text{ is L.I}\right]$

$\Rightarrow c_{k+1} = c_{k+2} = c_n = 0$.

The set $\{T(\alpha_{n+1}), T(\alpha_{n+2}), \ldots T(\alpha_n)\}$.

is linearly independent.

$\therefore$ It is a basis of im T.

$\therefore$ Rank $T = n-k$
Nullity $T = k$ $\Big\}$ rank T + nullity T
$= n-k+k = n$.
$= \dim V$.

$$\boxed{\therefore \text{rank } T + \text{Nullity } T = \dim V}$$

Prob: consider L.T on R given by

$T(x_1, x_2, x_3) = [x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x_1 - 2x_2 + 2x_3]$

find nullity T and prove rank - nullity theorem to find it.

Sol. $N(T) = \{(x_1, x_2, x_3) : T(x_1, x_2, x_3) = (0,0,0)\}$

$$\left.\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 &= 0 \\ -x_1 - 2x_2 + 2x_3 &= 0 \end{aligned}\right\} \quad (1)$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix} \xrightarrow[R_3 + R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{3}R_2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1+R_2} \begin{pmatrix} 1 & 0 & 2/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{pmatrix}$$

The system equivalent to ① is.

$$\left.\begin{aligned} x_1 + \tfrac{2}{3}x_3 &= 0 \\ x_2 - \tfrac{4}{3}x_3 &= 0 \end{aligned}\right) \quad \begin{aligned} x_3 &= c \\ x_1 &= -\tfrac{2}{3}c \\ x_2 &= 4/3\, c \end{aligned}$$

$$N(T) = L\left\{ \begin{pmatrix} -2/3 \\ 4/3 \\ 1 \end{pmatrix} \right\} = L\left\{ \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} \right\}$$

Nullity $T = 1$.

By Rank - nullity theorem

rank T + Nullity $T = \dim R^3$

rank $T + 1 = 3$

$\therefore$ rank $T = 2$

Algebra of Linear transformation:

V, W over a field F

$(T_1 + T_2)\alpha = T_1(\alpha) + T_2(\alpha), \;\forall\, \alpha \in V.$

$(cT_1)(\alpha) = cT(\alpha).$

prove that $T_1 + T_2$ is linear if T_1, T_2 are linear.

$T_1 + T_2 : V \to W$

let $\alpha, \beta \in V$ & $c \in F$.

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$(T_1+T_2)(c\alpha+\beta) = T_1(c\alpha+\beta) + T_2(c\alpha+\beta)$

$= cT_1(\alpha) + T_1(\beta) + cT_2(\alpha) + T_2(\beta)$

(using linearity of T_1 & T_2)

$= c[T_1(\alpha) + T_2(\alpha)] + T_1(\beta) + T_2(\beta)$

$= c(T_1+T_2)(\alpha) + (T_1+T_2)(\beta)$

$\Rightarrow$ $T_1 + T_2$ is also linear.

$cT_1 : V \to W$.

let $\alpha, \beta \in V$ & $c' \in F$.

$cT_1(c'\alpha+\beta) = cT_1(c'\alpha+\beta)$

$= c\{c'T_1(\alpha) + T_1(\beta)\}$

By linearity of T_1

$= cc'T_1(\alpha) + cT_1(\beta)$

$= c'cT(\alpha) + cT_1(\beta)$

$= c'(cT_1)(\alpha) + (cT_1)(\beta)$

$\therefore$ cT_1 is also linear.

let $T_1: V \to W$ and $T_2: W \to Z$ be two linear transformations, where V, W, Z are vector space over field F.

The composition $T_2 T_1$ is well defined and it is a f^n = from $V \to Z$.

Prove that : $T_2 . T_1 : V \to Z$ is also L.T.

let α, β and $c \in F$.

$$T_2 . T_1 \; (c\alpha + \beta) = c\, T_2 T_1 (\alpha) + T_2 T_1 (\beta).$$

Non-singular Transformation :

let T be a linear transformation from a vector space V to a vector space W over a field F. Then T is said to be non-singular iff,

$$N(T) = \{0\} \quad \text{ie} \quad T(\alpha) = 0 \text{ iff } \alpha = 0.$$

Theorem : Let T be a non-singular transformation from a V.S V to a V.S W over a field F, then T carries every L.I if V to a L.I set of W.

$\{\alpha_1, \alpha_2 \ldots \alpha_n\}$ L.I set of V.

then $\{T(\alpha_1), T(\alpha_2), \ldots T(\alpha_n)\}$ is Linearly independent (L.I).

proof: let $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a L.I subset of V.

we ~~wish~~ need to show that, the set $\{T(\alpha_1), T(\alpha_2) \ldots T(\alpha_n)\}$ is L.I, consider the relation.

$$c_1 T(\alpha_1) + c_2 T(\alpha_2) + \ldots c_n T(\alpha_n) = 0.$$

$\Rightarrow T(c_1\alpha_1 + c_2\alpha_2 + \ldots c_n\alpha_n) = 0.$ ($\because$ T is linear)

$\Rightarrow c_1\alpha_1 + c_2\alpha_2 + \ldots c_n\alpha_n = 0$ ($\because$ T is non singular)

$\Rightarrow c_1 = c_2 = \ldots = c_n = 0.$ [$\because$ set $\{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is L.I]

$\Rightarrow$ The set $\{T(\alpha_1), T(\alpha_2), \ldots T(\alpha_n)\}$ is L.I

Isomorphism :

Let V and W be two vector spaces over a field F.
Then V is said to be isomorphic to W iff $\exists$ an one-to-one and onto linear transformation from V to W.

V is isomorphic to W & W is isomorphic to Z.

V is isomorphic to Z.

$V \sim W$, $W \sim V$

Theorem: If V and W are isomorphic, where V and W are two finite dimensional vector spaces over a field F, then $\dim V = \dim W$.

Proof: let $\dim V = \dim W = n$ (say)

we need to prove V is ~~iso~~ isomorphic to W.

Assume, $B = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ and $B' = \{\beta_1, \beta_2 \ldots \beta_n\}$ be two ordered bases of V and W respectively.

Let us ~~define~~ consider linear ~~a~~ transformation T ~~from~~

$T: V \to W$ defined by

$T(\alpha_i) = \beta_i \quad (i = 1, 2, \ldots n)$ —(i)

let $T(\alpha) = 0$.

$\therefore \exists$ unique set of scalars $c_1, c_2, \ldots c_n$

s.t. $\alpha = c_1\alpha_1 + c_2\alpha_2 + \ldots c_n\alpha_n$.

$T(\alpha) = 0.$

$\Rightarrow T(c_1\alpha_1 + c_2\alpha_2 + \cdots c_n\alpha_n) = 0$

$\Rightarrow c_1 T(\alpha_1) + c_2 T(\alpha_2) + \cdots c_n T(\alpha_n) = 0.$

$\Rightarrow c_1 \beta_1 + c_2 \beta_2 + \cdots c_n \beta_n = 0.$ [using (i)]

$\Rightarrow c_1 = c_2 = \cdots = c_n = 0.$

($\because$ The set B' is L.I)

$\Rightarrow \alpha = 0$ $\quad N(T) = \{0\}$

$\therefore$ Nullity of $T = 0.$

As V is finite dimensional, T is a bijection

$\Rightarrow V$ is isomorphic to $W.$

$\Rightarrow V$ is isomorphic to W

Now let V is isomorphic to $W.$

we need to show that $\dim V = \dim W.$

As V is isomorphic to W, $\exists$ a L.T, $T: V \to W$

which is one to one as well as onto.

let $\dim V = n$, $\dim W = m.$

let $B = \{\alpha_1, \alpha_2, \cdots \alpha_n\}$ is a basis of V

As T is a bijection, T is one-to-one

$\Rightarrow T(\alpha) = 0 \Rightarrow \alpha = 0.$

$N(T) = \{0\}.$

$\Rightarrow T$ is non-singular

$\Rightarrow$ The set $\{T(\alpha_1), T(\alpha_2), \cdots T(\alpha_n)\}$ is L.I.

Now the $\operatorname{Im} T = W = \operatorname{span}\{T(\alpha_1), T(\alpha_2), \cdots T(\alpha_n)\}$

$\operatorname{Im} T = \{T(\alpha) : \alpha \in V\}$

$\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$

$T(\alpha) = T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n)$

$= c_1T(\alpha_1) + c_2T(\alpha_2) + \cdots + c_nT(\alpha_n)$

$\Rightarrow$ The set $\{T(\alpha_1), T(\alpha_2), \ldots T(\alpha_n)\}$ is a basis of W.

dim $W = n$.

$\Rightarrow$ dim V = dim W.

Let V be a vector space over a field F and dim $V = n$.

Prove that V is isomorphic to F^n.

Let $\{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is a basis of V.

Let $\alpha \in V$ be any vector.

$\exists$ unique set of scalars $\{c_1, c_2 \ldots c_n\}$

s.t. $\alpha = c_1\alpha_1 + c_2\alpha_2 \cdots c_n\alpha_n$.

$[\alpha]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ is called the co-ordinate of α w.r.t. B.

$[\alpha]_B \in F^n$.

Define a f^n:

$T: V \to F^n$ by

$T(\alpha) = [\alpha]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$, $\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots c_n\alpha_n$.

Let $\alpha, \beta \in V$ be two vectors & $c \in F$

$T(c\alpha + \beta) = [c\alpha + \beta]_B = c[\alpha]_B + [\beta]_B$.

Let $\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots c_n\alpha_n$

$[\alpha]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$

$\beta = d_1\alpha_1 + d_2\alpha_2 + \cdots d_n\alpha_n$

$$[\beta]_B = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

Now, $c\alpha+\beta = c(c_1\alpha_1 + c_2\alpha_2 + \cdots c_n\alpha_n)$

$+ (d_1\alpha_1 + d_2\alpha_2 + \cdots d_n\alpha_n)$

$= (cc_1+d_1)\alpha_1 + (cc_2+d_2)\alpha_2 + \cdots (cc_n+d_n)\alpha_n$

$$[c\alpha+\beta]_B = \begin{pmatrix} cc_1+d_1 \\ cc_2+d_2 \\ \vdots \\ cc_n+d_n \end{pmatrix}$$

$$= c\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = c[\alpha]_B + [\beta]_B$$

$T(c\alpha+\beta) = [c\alpha+\beta]_B$

$= c[\alpha]_B + [\beta]_B$

$= cT(\alpha) + T(\beta)$

$\Rightarrow$ T is a L.T.

let $T(\alpha) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$[\alpha]_B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\Rightarrow \alpha = 0$

$\therefore N(T) = \{0\} \Rightarrow$ Nullity $T = 0$.

$\Rightarrow$ T is a bijection.

$\therefore V$ & F^n are isomorphic.

s of W.

dim V = n.

α

$c_n\alpha_n$.

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Matrix Representation:

Let V be a finite dimensional vector space and $\dim V = n$.

$B = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ & $B' = \{\alpha_1', \alpha_2', \dots \alpha_n'\}$ be two ordered basis of V.

let $\alpha \in V$.

$\exists$ scalars $c_1, c_2, \dots c_n$ & $d_1, d_2, \dots d_n$

s.t. $\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots c_n\alpha_n$

$= d_1\alpha_1' + d_2\alpha_2' + \dots d_n\alpha_n'$.

$\alpha_1' = A_{11}\alpha_1 + A_{21}\alpha_2 + \dots A_{n1}\alpha_n$.

$\alpha_2' = A_{12}\alpha_1 + A_{22}\alpha_2 + \dots A_{n2}\alpha_n$.

$\vdots$

$\alpha_n' = A_{1n}\alpha_1 + A_{2n}\alpha_2 + \dots A_{nn}\alpha_n$.

$\alpha = d_1\alpha_1' + d_2\alpha_2' + \dots d_n\alpha_n'$.

$= d_1(A_{11}\alpha_1 + A_{21}\alpha_2 + \dots A_{n1}\alpha_n)$

$+ d_2(A_{12}\alpha_1 + A_{22}\alpha_2 + \dots A_{n2}\alpha_n) + \dots$

$+ d_n(A_{1n}\alpha_1 + A_{2n}\alpha_2 + \dots A_{nn}\alpha_n)$

$= (A_{11}d_1 + A_{12}d_2 + \dots A_{1n}d_n)\alpha_1$

$+ (A_{21}d_1 + A_{22}d_2 + \dots A_{2n}d_n)\alpha_2$

$+ \dots$

$+ (A_{n1}d_1 + A_{n2}d_2 + \dots A_{nn}d_n)\alpha_n$.

$\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots c_n\alpha_n$.

As representation of α w.r.t the ordered basis B is unique, we get,

$c_1 = A_{11}d_1 + A_{12}d_2 + \dots A_{1n}d_n$

$c_2 = A_{21}d_1 + A_{22}d_2 + \dots A_{2n}d_n$

$c_n = A_{n1}d_1 + A_{n2}d_2 + \cdots A_{nn}d_n.$

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

$$[\alpha]_B = \underset{\substack{\downarrow \\ \text{matrix of transformation.}}}{A_{n\times n}} [\alpha]_{B'}$$

The matrix A is invertible (can be proved)

$$[\alpha]_{B'} = A^{-1}[\alpha]_B$$

Let T be a linear transformation for a finite dimensional vector space V to a finite dimensional vector space V over a common field F.

let $\dim V = n$ & $\dim W = m$.

$B = \{\alpha_1, \alpha_2, \cdots \alpha_n\}$ & $B' = \{\beta_1, \beta_2, \cdots \beta_n\}$ be two ordered basis of V & W respectively.

$T(\alpha_1) = A_{11}\beta_1 + A_{21}\beta_2 + \cdots A_{m1}\beta_m$

$T(\alpha_2) = A_{12}\beta_1 + A_{22}\beta_2 + \cdots A_{m2}\beta_m.$

$\cdots$

$T(\alpha_n) = A_{1n}\beta_1 + A_{2n}\beta_2 + \cdots A_{mn}\beta_m.$

The matrix $A_{m\times n} = (A_{ij})_{m\times n}$ is called the matrix of the L.T, T w.r.t the ordered basis B & B'.

It is denoted by $[T]_B^{B'}$

In particular if we consider a linear operator on V. Then we normally consider only one ordered basis of V. The matrix of transformation is written as $[T]_B$.

Ex: consider linear operator

$T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by.

$T(x,y) = (3x-y, -2x+5y)$

find $[T]_B$ and $[T]_{B'}$, where

$B = \{(1,0), (0,1)\}$, $B' = \{(1,1); (1,-1)\}$.

$T(1,0) = (3,-2) = 3(1,0) + (-2)(0,1)$

$T(0,1) = (-1,5) = (-1)(1,0) + 5(0,1)$.

$\therefore [T]_B = \begin{bmatrix} 3 & -1 \\ -2 & 5 \end{bmatrix}$

$T(1,1) = (2,3) = A_{11}(1,1) + A_{21}(1,-1)$

$T(1,-1) = (4,-7) = A_{12}(1,1) + A_{22}(1,-1)$

$A_{11} + A_{21} = 2$
$A_{11} - A_{21} = 3$

$2A_{11} = 5/2$
$A_{11} = 2.4$

$\Rightarrow A_{11} = 5/2$
$A_{21} = -1/2$.

$\left.\begin{array}{l} A_{12} + A_{22} = 4 \\ A_{12} - A_{22} = -7 \end{array}\right\} \Rightarrow \begin{array}{l} A_{12} = -3/2 \\ A_{22} = 11/2. \end{array}$

$\therefore [T]_{B'} = \begin{bmatrix} 5/2 & -3/2 \\ -1/2 & 11/2 \end{bmatrix}$

$T: P^2(\mathbb{R}) \to P^2(\mathbb{R})$

$P^2(\mathbb{R}) = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$.

$T(f(x)) = (x+1).f'(x) - f(0)$.

Find the matrix of T w.r.t the standard ordered basis B of $P^2(\mathbb{R})$.

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$B = \{1, x, x^2\}$. (P.T the set is L.I)

$f(x) \in P^2(R)$

$f(x) = a_0 + a_1x + a_2x^2$.

$T(1) = (x+1)\cdot 0 - 1 = -1 = (-1)\cdot 1 + 0\cdot x + 0\cdot x^2$.

$T(x) = (x+1)\,1 - 0 = x+1 \Rightarrow 1\cdot 1 + 1\cdot x + 0\cdot x^2$

$T(x^2) = (1+x)\,2x - 0 = 2x+2x^2 \Rightarrow 0\cdot 1 + 2\cdot x + 2\cdot x^2$.

$$[T]_B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Ex: $T: R^3 \to R^2$ defined by.

$T(x,y,z) = (x+y,\ 2z-x)$

Consider the ~~two~~ ordered basis.

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$. $B' = \{(1,0), (1,-1)\}$

of R^3 & R^2.

find $[T]_B^{B'}$.

Sol: $T(1,0,0) = (1,-1) = 0\cdot(1,0) + 1\,(1,-1)$

$T(0,1,0) = (1,0) = 1\cdot(1,0) + 0\cdot(1,-1)$

$T(0,0,1) = (0,2) = \underset{2}{c_1}(1,0) + \underset{-2}{c_2}(1,-1)$

$$[T]_B^{B'} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$$

$[T(\alpha)]_{B'}$

$T(\alpha) = x_1\beta_1 + x_2\beta_2 + \cdots\ x_m\beta_m$.

$\therefore [T(\alpha)]_{B'} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$\alpha \in V$.

$\exists$ scalars $y_1, y_2, \ldots y_n$ s.t.

$\alpha = y_1\alpha_1 + y_2\alpha_2 + \cdots y_n\alpha_n$

$T(\alpha_1) = A_{11}\beta_1 + A_{21}\beta_2 + \cdots + A_{m1}\beta_m.$

$T(\alpha_2) = A_{12}\beta_1 + A_{22}\beta_2 + \cdots + A_{m2}\beta_m$

$\cdots$

$T(\alpha_n) = A_{1n}\beta_1 + A_{2n}\beta_2 + \cdots A_{mn}\beta_m.$

$\therefore T(\alpha) = T(y_1\alpha_1 + y_2\alpha_2 + \cdots y_n\alpha_n)$

$= y_1 T(\alpha_1) + y_2 T(\alpha_2) + \cdots + y_n T(\alpha_n)$

$= y_1(A_{11}\beta_1 + A_{21}\beta_2 + \cdots A_{m1}\beta_m)$

$+ y_2(A_{12}\beta_2 + A_{22}\beta_2 + \cdots A_{m2}\beta_m)$

$\cdots$

$+ y_n(A_{1n}\beta_1 + A_{2n}\beta_2 + \cdots A_{mn}\beta_m)$

$= (A_{11}y_1 + A_{12}y_2 + \cdots A_{1n}y_n)\beta_1 + (A_{21}y_1 + A_{22}y_2 + \cdots A_{2n}y_n)\beta_2$

$+ \cdots + (A_{m1}y_1 + A_{m2}y_2 + \cdots A_{mn}y_n)\beta_m.$

$T(\alpha) = x_1\beta_1 + x_2\beta_2 + \cdots x_n\beta_n.$

As the representation of $T(\alpha)$ w.r.t B' is unique.

$\Rightarrow x_1 = A_{11}y_1 + A_{12}y_2 + \cdots A_{1n}y_n$

$x_2 = A_{21}y_1 + A_{22}y_2 + \cdots A_{2n}y_n.$

$\cdots$

$x_m = A_{m1}y_1 + A_{m2}y_2 + \cdots A_{mn}y_n.$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$\boxed{[T(\alpha)]_{B'} = [T]_B^{B'} [\alpha]_B}$$

$\dim V = n$.

$T : V \to V \qquad B = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$.

$[T]_B \to$ has all information related to L.O.

rank T = rank $[T]_B$

nullity $T = n -$ rank T

$[T(\alpha)]_B = [T]_B [\alpha]_B$

let $\dim V = n$

& $B = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$

)β_2 & $B' = \{\beta_1, \beta_2 \ldots \beta_n\}$ be two ordered bases

of V and T be a linear operator on V.

$\exists$ a non-singular matrix P s.t

$[\alpha]_B = P[\alpha]_{B'}$. — ① $\Rightarrow$ $[\alpha]_{B'} = P^{-1}[\alpha]_B$

Now $[T(\alpha)]_B = P[T(\alpha)]_{B'}$

$\Rightarrow [T(\alpha)]_{B'} = P^{-1}[T(\alpha)]_B$

$= P^{-1}[T]_B [\alpha]_B$

$\Rightarrow \boxed{[T(\alpha)]_{B'} = P^{-1}[T]_B \cdot P[\alpha]_{B'}}$

$\therefore [T]_{B'} = P^{-1}[T]_B P$

$\Rightarrow [T]_B$ & $[T]_{B'}$ are similar.

$$[T]_B = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & d_n \end{bmatrix}$$

$d_i \neq 0 \quad (i = 1, 2 \ldots k)$

$d_i = 0 \quad (i = k+1, \ldots n)$

rank $T = k$, nullity $T = n - k$

$B = \{ v_1, v_2 \ldots v_n \}$

$T(v_1) = d_1 v_1 + 0.v_2 + \ldots 0.v_n = d_1 v_1$

$T(v_2) = 0.v_1 + d_2.v_2 + \ldots 0.v_n = d_2 v_2.$

$T(v_n) = d_n v_n.$

$(T - d_1 I)(v_1) = 0$ $\quad\quad$ $Ax = 0$, $\det A = 0$.

$\Rightarrow \det [T - d_1 I]_B = 0$

Def^n: let T be linear operator on a finite dimensional vector space V over a field F. Then a scalar $c \in F$ is called an eigen value of T. if $\exists$ a non zero vector $\alpha \in V$ s.t.

$(T - cI)(\alpha) = 0$

or $T(\alpha) = c\alpha$

i.e. $\det [T - cI]_B = 0$

$\Rightarrow \det [[T]_B - cI] = 0.$

let us consider another basis B' of V.

$[T]_{B'} = P^{-1} [T]_B P$ for some non-singular matrix P.

ch. eq^n for new matrix. is given by

$\det [[T]_{B'} - cI] = 0.$

$\Rightarrow \det [P^{-1} [T]_B P - cP^{-1} I P] = 0.$

$\Rightarrow \det \left[P^{-1}\left([T]_B - cI \right) P \right] = 0.$

$\Rightarrow \det P^{-1} \det \left([T]_B - cI \right) \det P = 0.$

$\Rightarrow \det P^{-1} \det P \left([T]_B - cI \right) = 0.$

$\Rightarrow \det (P^{-1}P) \det \left([T]_B - cI \right) = 0.$

$\Rightarrow \det \left([T]_B - cI \right) = 0.$

$$[T]_B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 1-c & 0 & 0 \\ 2 & 1-c & 1 \\ 3 & -1 & 0-c \end{vmatrix} = 0$$

$\Rightarrow c = c_1, c_2, c_3$

$[T]_B = A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\lambda^2 + 1 = 0$

$\Rightarrow \lambda = \pm i$

$T(v_i) = d_i v_i \quad (i = 1, 2, \ldots, n)$

$\Rightarrow (T - d_i I)(v_i) = 0$

$N(T - d_i I) = E_{d_i}$ (Eign space corresponding to the eign value d_i)

If we are able to find a set of n L.I. eign vectors and from a basis B' of V. then $[T]_{B'}$ will be a diagnal matrix.

$T: P^2(\mathbb{R}) \longrightarrow P^2(\mathbb{R})$

$T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2.$

find the eign vector of T.

$B = \{1, x, x^2\}$

$T(1) = 1 = 1.1 + 0.x + 0.x^2.$

$T(x) = 1 + x + x^2$

$T(x^2) = 1 + 2x + 2x^2$

$= 1.1 + 0.x + 2.x^2$

$$A = [T]_B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\text{ch. eq}^n = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0.$$

$\Rightarrow \lambda = 1, 1, 2.$

$T(v_i) = d_i v_i$

$\Rightarrow (T - d_i I)(v_i) = 0$ $\qquad N(T) = \{x : T(x) = 0\}.$

$\det(T - \lambda I) = 0$

$N(T - d_i I) = \{x : (T - d_i I)(x) = 0\}.$

$= E_{d_i}$, eigen space corresponding to eigen value d_i.

for $\lambda = 1$:

$$A - I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$E_1 = \left\{ x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} : (A - I)x = 0 \right\}.$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : y + z = 0 \right\}$$

$$= \{ (c, d, -d) : c, d \in R \}$$

$$= \{ c(1,0,0) + d(0,1,-1) : c, d \in R \}$$

$$= L\{ (1,0,0), (0,1,-1) \}$$

(A.m) Algebraic multiplicity of $\lambda = 1$
$= 2$

Geometric multiplicity (G.m) of $\lambda = 1$
is $= 2$.

∴ $\lambda = 1$ is a regular eigen value.

for $\lambda = 2$:

$$A - 2I = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

to

$$\xrightarrow{-R_2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_3 - R_2]{R_1 + R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N(A - 2I) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : (A - 2I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x - z = 0 \;\&\; y = 0 \right\}.$$

$$= \{ (c, 0, c) : c \in R \}$$

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$= \{ c(1,0,1) : c \in R \}$

$\therefore E_2 = L\{(1,0,1)\}$

A.M of $\lambda = 2$ is $= 1$

G.M of $\lambda = 2$ is $= 1$

$\therefore \lambda = 2$ is also a regular eigen value.

<u>Theorem</u>: If all the distinct eigen values of a matrix are regular, then the matrix is diagonalizable.

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$\lambda = 1$ $\quad \lambda = 2$

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The vectors in $P^2 R$ associated with the eigen vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$ of R^3 are

$1.1 + 0.x + 0.x^2 = 1$

$0.1 + 1.x + -1.x^2 = x - x^2$

$1.1 + 0.x + 1.x^2 = 1 + x^2$

$\therefore$ The basis $B' = \{1, x - x^2, 1 + x^2\}$ of $P^2(R)$ consisting of eigen vectors of T will diagonalize the operator.

$T(1) = 1 \cdot 1 = 1$

$T(x-x^2) = 1 \cdot (x-x^2)$

$T(1+x^2) = 2(1+x^2)$

Q) $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

ch. eqⁿ: $\det(A-\lambda I) = 0.$

$\Rightarrow \begin{vmatrix} 1-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} = 0. \Rightarrow (1-\lambda)^2 = 0 \Rightarrow \lambda = 1, 1.$

f.

is

A.M of $\lambda = 1$ is 2.

$E_1 = \left\{ A = \begin{pmatrix} x \\ y \\ z \end{pmatrix} : (A-I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$= N(A-I)$

$A-I = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \xrightarrow{1/3 R_1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$E_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y = 0 \right\} = \{ (c, 0) : c \in \mathbb{R} \}$

$= \{ c(1,0) : c \in \mathbb{R} \}$

$= L\{(1,0)\}$

$\dim E_1 = 1$

G.M of $\lambda = 1$ is 1

$\Rightarrow \lambda = 1$ is irregular.

$\therefore$ A is <u>not</u> diagonalizable.

(iii)

Prob: $T: P^2(R) \to P^2(R)$ defined by

$T(f(x)) = f''(x) + 2f'(x) - f(x)$.

Determine whether T is invertible. If yes, find T^{-1}.

Sol: Consider the standard ordered basis

$B = \{1, x, x^2\}$.

$T(1) = -1 \quad = -1.(1) + 0.x + 0.x^2$

$T(x) = 2 - x \quad = 2.1 + (-1).x + 0.x^2$

$T(x^2) = 2 + 4x - x^2 = 2.1 + 4.x + (-1)x^2$

$$\therefore A = [T]_B = \begin{bmatrix} -1 & 2 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

$\det[T]_B = -1 \neq 0$.

$\Rightarrow [T]_B$ is invertible.

Hence T is also invertible i.e. T^{-1} exists.

We know that $[T^{-1}]_B = [T]_B^{-1}$

$$[T^{-1}]_B = \begin{bmatrix} -1 & 2 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -1 & 0 \\ -10 & -4 & -1 \end{bmatrix}$$

We know that $[T^{-1}(f(x))]_B = [T^{-1}]_B [f(x)]_B$

$\because [T(\alpha)]_B = [T]_B [\alpha]_B$.

If $f(x) = c_0 + c_1 x + c_2 x^2$

$$[f(x)]_B = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

$[T^{-1}]_B \cdot [f(x)]_B$

$$= \begin{bmatrix} -1 & 0 & 0 \\ -2 & -1 & 0 \\ -10 & -4 & -1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -c_0 \\ -2c_0 - c_1 \\ -10c_0 - 4c_1 - c_2 \end{bmatrix}$$

$$T^{-1} f(x) = T^{-1}(c_0 + c_1 x + c_2 x^2)$$

$$= (-c_0) \cdot 1 + (-2c_0 - c_1)x + (-10c_0 - 4c_1 - c_2)x^2.$$

ch eqn = of T:

$\det([T]_B - \lambda I) = 0.$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 2 & 2 \\ 0 & -1-\lambda & 4 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0.$$

$\Rightarrow (\lambda+1)^3 = 0 \Rightarrow \lambda = -1, -1, -1.$

$A.M(\lambda = -1) = 3.$

$E_{-1} = N(A+I)$

$$A+I = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_{-1} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 2y + 2z = 0,\ 4z = 0 \right\}$$

$$= \left\{ \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} : c \in \mathbb{R} \right\} = L\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$]_B.$

$\therefore$ Dim $(E_{-1}) = 1$

$\therefore$ G.M of $\lambda = -1$ is 1

$\therefore$ For $\lambda = -1$

AM $\neq$ G.M.

$\Rightarrow$ A is not diagonalizable.

$\Rightarrow$ T is " ".

$\det A = (-1)^3 = -1 \neq 0$.

$\Rightarrow$ A is invertible.

Hence T is also invertible.

Q) $T: P^2(\mathbb{R}) \to P^2(\mathbb{R})$

$$T(f(x)) = f(1) + f'(0)\,x + [f'(0) + f''(0)]\,x^2$$

Test whether T is diagonalizable. If yes, determine a basis of $P_2(\mathbb{R})$ consisting of eigen vectors of T.

Sol.^{n} $\beta = \{1, x, x^2\}$

$T(1) = 1.1 + 0.x + 0.x^2$

$T(x) = 1.\cancel{x} + 1.x + 1.x^2$

$T(x^2) = 1 + 0.x + 2x^2$

$$A = [T]_\beta = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Characteristic eq^n:

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1, 1, 2.$$

for A.M of $\underline{(\lambda = 1)}$ =

$A.M_1 = 2$

$E_1 = N(A - I)$

$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : (A - I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$$A - I = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$E_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : y + z = 0 \right\}$

$E_1 = \left\{ \begin{pmatrix} c \\ d \\ -d \end{pmatrix} : c, d \in R \right\}$

$= \left\{ c\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} : c, d \in R \right\}$

$= L\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

$\therefore$ dim of $E_1 = 2$

G.M of $\lambda = 1$ is 2.

$\therefore$ A.M = G.M

$\underline{\text{For } \lambda = 2}$:

$A.M = 1$

$E_2 = N(A - 2I)$

$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : (A - 2I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$$A - 2I = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow[R_1 + R_2]{R_3 + R_2} \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[-R_2]{-R_1}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

~~characteristic eqn~~

$$E_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x - z = 0,\ y = 0 \right\}$$

$$= \left\{ \begin{pmatrix} c \\ 0 \\ c \end{pmatrix} : c \in \mathbb{R} \right\}$$

$$= L\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \left(\text{Linear span of set}\right)$$

$\dim E_2 = 1$

$\therefore Am = GM$

$\therefore$ Both the eigen values of A are regular

Hence A is diagonalizable.

$\Rightarrow$ T is also diagonalizable.

If we define

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \text{ then}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad A = [T]_\beta$$

The basis γ of eigen values of T w.r.t which, the matrix of the T becomes diagonal is given by

$$\gamma = \{1,\ x - x^2,\ 1 + x^2\}$$

$$\therefore [T]_\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solve: The system of ODES.

$\dot{x}_1 = 3x_1 + x_2 + x_3$ $\quad \dot{} = \frac{d}{dt}$.

$\dot{x}_2 = 2x_1 + 4x_2 + 2x_3$

$\dot{x}_3 = -x_1 - x_2 + x_3$

$\dot{x} = Ax, \quad A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\dot{x} = Ax, \ x \in R^n, \quad A \in M_{n\times n}(R)$ —①

Let A is diagonalizable

$\exists$ a non-singular matrix P s.t.

$P^{-1}AP = D$, a diagonal matrix.

$\Rightarrow \quad A = PDP^{-1}$

① $\Rightarrow \quad \dot{x} = PDP^{-1}x$

$\Rightarrow \quad P^{-1}\dot{x} = DP^{-1}x$. , let $P^{-1}x = y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$

$\Rightarrow \quad \dot{y} = Dy$

$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & & 0 \\ \vdots & & & & \lambda_n \\ 0 & 0 & 0 & \cdots & \end{bmatrix}$

$\Rightarrow \left.\begin{aligned} \dot{y}_1 &= \lambda_1 y_1 \\ \dot{y}_2 &= \lambda_2 y_2 \\ &\vdots \\ \dot{y}_n &= \lambda_n y_n \end{aligned}\right\}$ $\quad \frac{dy_1}{dt} = \lambda_1 y_1 \Rightarrow y_1 = c_1 e^{\lambda_1 t}$

$y_2 = c_2 e^{\lambda_2 t}$

$\vdots$

$y_n = c_n e^{\lambda_n t}$.

Now $x = Py$.

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & & & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$$

$$\begin{array}{cc} v_1 & v_n \\ | & | \\ \lambda_1 & \lambda_n \end{array}$$

$\therefore$ General Solution

$$x = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & & & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$\boxed{x = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + \cdots + c_n v_n e^{\lambda_n t}}$$

$\dot{x} = Ax$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

Eigen values of A are $\lambda = 2, 2, 4$.

For $\lambda = 2$:

$$A - 2I = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix} \xrightarrow[R_3 + R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_2 = N(A - 2I)$$
$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z = 0 \right\}$$
$$= \left\{ \begin{pmatrix} c \\ d \\ -c-d \end{pmatrix} : c, d \in \mathbb{R} \right\}$$

$$= \left\{ c\begin{pmatrix}1\\0\\-1\end{pmatrix} + d\begin{pmatrix}0\\1\\-1\end{pmatrix} : c,d \in R \right\}$$

$$= L\left\{ \begin{pmatrix}1\\0\\-1\end{pmatrix} + \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\}$$

$\therefore \dim E_2 = 2.$

$\therefore$ A.M = G.M. for $\lambda = 2$.

<u>for $\lambda = 4$</u>:

$$A - 4I = \begin{pmatrix}-1 & 1 & 1\\ 2 & 0 & 2\\ -1 & -1 & -3\end{pmatrix} \xrightarrow{R_3+R_1} \begin{pmatrix}-1 & 1 & 1\\ 2 & 0 & 2\\ -2 & 0 & -2\end{pmatrix}$$

$$\downarrow \frac{R_3+R_2}{R_2/2}$$

$$\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix} \quad \begin{pmatrix}-2 & 1 & 0\\ 1 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} \xleftarrow{R_1-R_2} \begin{pmatrix}-1 & 1 & 1\\ 1 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$$

$$E_4 = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} : x+z=0,\ y+2z=0 \right\}$$

$$= \left\{ \begin{pmatrix}-c\\-2c\\c\end{pmatrix} : c \in R \right\}$$

$$= \left\{ c\begin{pmatrix}-1\\-2\\1\end{pmatrix} : c \in R \right\}$$

$$= L\left\{ \begin{pmatrix}-1\\-2\\1\end{pmatrix} \right\}$$

$\therefore \dim E_4 = 1.$

$\therefore$ A.M = G.M for $\lambda = 4$.

$\Rightarrow$ A is diagonalizable.

$\therefore$ G.S of system in given $\dot{x} = Ax$ is given by

$$x(t) = c_1\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{2t} + c_3\begin{pmatrix}-1\\-2\\1\end{pmatrix}e^{4t}$$

Check the diagonalizability of the matrices.

$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

ch. eq

$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(1-\lambda) = 0.$

$\lambda = 1, 1$

for $\lambda = 1$,

AM $= 2$

$E_1 = N(A - I)$

$= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ $x \in R$, $y = 0$

$\begin{pmatrix} c \\ 0 \end{pmatrix}, c\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

dim of $E_1 = 1$, GM $= 1$

AM $\neq$ G.M. $\Rightarrow$ A is not diagonalizable.

$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 9 \Rightarrow \lambda - 1 = \pm 3$

$\lambda = 4, -2.$

for $\lambda = 4$

AM $= 1$ $(A - 4I)$

$E_4 = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}$ $x - y = 0$

$\begin{pmatrix} c \\ c \end{pmatrix}, c\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow L\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

dim of $E_4 = 1$

$\therefore$ G.M $= 1$ $\therefore$ AM $=$ GM

for $\lambda = -2$:

$(A + 2I)$ $E_{-2} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$ $x + y = 0.$

$\begin{pmatrix} c \\ -c \end{pmatrix}, c\begin{pmatrix} 1 \\ -1 \end{pmatrix}, L\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$P^{-1}AP = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 0-\lambda & 0 & 1 \\ 1 & 0-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda\left(-\lambda(1-\lambda)+1\right] + 1\left[1+\cancel{\lambda}\right)$$

$$\Rightarrow -\lambda\left[-\lambda+\lambda^2+1\right]+1$$

$$\Rightarrow (\lambda-1)(\lambda^2+1) = 0.$$

$\Rightarrow \lambda = 1,\ \lambda = \pm i$, if field of complex no's is considered.

In this case A is diagonalizable.

for $\lambda = 1$, $A - I$; for $\lambda = +i$, $A - iI$; for $\lambda = -i$, $A + iI$

---

check whether $T : P^2(R) \to P^2(R)$ is diagonalizable,

where $T(g(x)) = -g(x) - g'(x)$.

Sol:- $\beta = \{1, x, x^2\}$

$T(1) = -1 = (-1)(1) + 0 \cdot x + 0 \cdot x^2$

$T(x) = -x - 1 = (-1) \cdot 1 + (-1) \cdot x + 0 \cdot x^2$

$T(x^2) = -x^2 - 2x = 0 \cdot 1 + (-2) \cdot x + (-1)\, x^2$,

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$$A = [T]_\beta = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$

<u>characteristic eqⁿ :</u>

$$\begin{vmatrix} -1-\lambda & -1 & 0 \\ 0 & -1-\lambda & -2 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \quad \therefore \quad (-1-\lambda)^3 = 0$$

$\therefore \lambda = -1, -1, -1.$

<u>for $\lambda = -1.$</u>

$AM = 3.$

$$E_{-1} = N(A+I) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_{-1} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : y = 0,\ z = 0,\ x \in R \right\}$$

$$= \left\{ \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} ; c \in R \right\}$$

$$= \left\{ c \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ; c \in R \right\}$$

$$= L\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

dim of E_{-1} is 1

$G.M = 1.$

$\therefore AM \neq G.M.$

$\therefore$ A is non diagonalizable.

$$A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix}$$

$$\begin{vmatrix} 3-\lambda & 1 & -2 \\ -1 & 0-\lambda & 5 \\ -1 & -1 & 4-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (3-\lambda)\left[-\lambda(4-\lambda)+5\right] - 1\left[\overset{(\lambda-4)}{\cancel{(4-\lambda)}}+5\right] - 2\left[1-\lambda\right]$$

$$\Rightarrow \cancel{(3-\lambda)\left[\lambda+\lambda^2+5\right] - (\lambda+6) - 2 + 2\lambda}$$

$$\Rightarrow \cancel{3\lambda + 3\lambda^2 + 15 - \lambda^2 - \lambda^3 - 5\lambda - \lambda - 6 - 2 + 2\lambda = 0}$$

$$\Rightarrow \cancel{-\lambda^3 + 2\lambda^2 - \lambda + 7 = 0.}$$

$$\Rightarrow \cancel{\lambda^3 - 2\lambda^2 + \lambda - 7 = 0.}$$

$$\Rightarrow (3-\lambda)\left[-4\lambda+\lambda^2+5\right] - 1\left[\lambda+1\right] - 2(1-\lambda)$$

$$\Rightarrow -12\lambda + 3\lambda^2 + 15 + 4\lambda^2 - \lambda^3 - 5\lambda - \lambda - 1 - 2 + \lambda = 0$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 17\lambda + 12 = 0.$$

$$\lambda = 3, 2, 2.$$

<u>for $\lambda = 3$</u>

$A.M = \cancel{2}\,1$

$$e_3 = N[A-3I] = \begin{pmatrix} 0 & 1 & -2 \\ -1 & -3 & 5 \\ -1 & -1 & 1 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2+3R_1} \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix}$$

$$\xrightarrow{-R_2} \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix} \xrightarrow{R_3+R_2} \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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$$E_3 = N(A-3I)$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : y - 2z = 0,\ x + z = 0 \right\}$$

$$= \left\{ \begin{pmatrix} -c \\ 2c \\ c \end{pmatrix} : c \in R \right\}$$

$$= \left\{ c \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} : c \in R \right\}$$

$$= L\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\} \qquad G.M = 1$$

<u>for $\lambda = 2$</u>

$$AM = 2$$

$$E_2 = (A - 2I) = \begin{pmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2+R_1} \begin{pmatrix} 1 & 1 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & +1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \xleftarrow[-R_2]{R_1+R_2}$$

$$E_2 = (A - 2I)$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + z = 0,\ y - 3z = 0 \right\}$$

$$= \left\{ \begin{pmatrix} -c \\ 3c \\ c \end{pmatrix} : c \in R \right\}$$

$$= \left\{ c \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} : c \in R \right\}$$

$$= L\left\{ \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

$$G.M = 1.$$

for $\lambda = 2$, $AM \neq G.M.$

$\Rightarrow$ A is not diagonalizable.

Jordan Block :

$$A = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ 0 & 0 & \cdots & & A_m \end{pmatrix}$$

$A_1 = \lambda_1$

$A_2 = \lambda_2$

$$A = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \boxed{3} \end{pmatrix}$$

Q) $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

ch. polynomial.

$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = 0$ $\quad \lambda^2 - 4\lambda + 4 = 0$

$\lambda = 2, 2$

Am = 2

$A - 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$

$E_2 = N(A - 2I)$

$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} ; \quad x - y = 0 \right\}$

$= L\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$\therefore$ G.M = 1

A.M = 2

$\therefore$ Am $\neq$ G.M

Consider

$(A - 2I)^2 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

K_2 (generalised eigen space)

$= N(A - 2I)^2$

$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\} = L\left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right\}$

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$$(A-2I)v_1 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\{(A-2I)v_1, v_1\} = \left\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$$P = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow P^{-1}AP = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$(A-2I)v_2 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (A-2I)^2 v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ proof}$$

$$\{(A-2I)v_2, v_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\Rightarrow P^{-1}AP = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Ex $\begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix}$ find Jordan canonical form of.

ch. polynomial.

$$\begin{vmatrix} 3-\lambda & 1 & -2 \\ -1 & 0-\lambda & 5 \\ -1 & -1 & 4-\lambda \end{vmatrix} \Rightarrow (3-\lambda)\left[(-\lambda)(4-\lambda)+5\right] - 1\left[\lambda-4+5\right] - 2\left[1-\lambda\right]$$

$$\lambda = 3, 2, 2.$$

for $\lambda = 2$, A.M $= 2$.

$$(A-2I) = \begin{pmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{pmatrix} \xrightarrow{R_3+R_1} \begin{pmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2+R_1} \begin{pmatrix} 1 & 1 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Downarrow R_1+R_2$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \xleftarrow{-R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

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$$E_2 = N(A-2I)$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x+z=0,\ y-3z=0 \right\}$$

$$= L\left\{ \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \right\} \quad \therefore \text{G.M} = 1$$

A.M $\neq$ G.M.

$\therefore$ A is not diagonalised.

For $\lambda = 2$,

$$(A-2I)^2 = \begin{pmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & -1 \\ -4 & -2 & 2 \\ -2 & -1 & 1 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2+2R_1} \begin{pmatrix} 2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$K_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 2x+y-z=0 \right\}$$

$$= \left\{ \begin{pmatrix} c \\ d \\ 2c+d \end{pmatrix} : c, d \in R \right\}$$

$$= \cancel{B} \left\{ c\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} : c, d \in R \right\}$$

$$K_2 = L\left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{v_2} \right\}$$

$$(A-2I)v_1 = \begin{pmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 9 \\ 3 \end{pmatrix}$$

$$(A-2I)^2(v_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ A cycle of length 2 for the eigenvalue 2 is given by $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 9 \\ 3 \end{pmatrix} \right\}$.

for $\lambda = 3$:-

$$A - 3I = \begin{pmatrix} 0 & 1 & -2 \\ -1 & -3 & 5 \\ -1 & -1 & 1 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2+3R_1} \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xleftarrow{-R_2} \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xleftarrow{R_3 - R_2}$$

$$K_3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : y - 2z = 0,\ x + z = 0 \right\}$$

$$K = L\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

$$P = \begin{pmatrix} -3 & 1 & -1 \\ 9 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

---

c) $T : P^2(R) \to P^2(R)$

$$T(g(x)) = -g(x) - g'(x)$$

Sol:- $B = \{1, x, x^2\}$.

$$T(1) = -1 = -1\cdot 1 + 0\cdot x + 0\cdot x^2$$

$$T(x) = -x - 1 = -1\cdot 1 + -1\cdot x + 0\cdot x^2$$

$$T(x^2) = -x^2 - 2x = 0\cdot 1 + (-2)\cdot x + (-1)\cdot x^2,$$

$$[T]_B = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}.$$

eigen values are $\lambda = -1, -1, -1$.

$$A+I = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_{-1} = N(A+I)$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : y=0, \ z=0, \ x=c \right\}$$

$$= L\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}. \quad \therefore G.M = 1$$

$A.M = 3$

$A.M \neq G.M.$

for $\lambda = -1$

$A.M = 3, \ G.M = 1$

A is not diagonalizable.

To find generalised eigen space

$$K_{-1} = N(A+I)^3$$

consider the matrix

$$(A+I)^3 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$K_{-1} = N(A+I)^3$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x=c, \ y=d, \ z=e \right\}.$$

$$= L\left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$$

$$(A+I)v_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \times$$

$$(A+I)v_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$(A+I)^2 v_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(A+I)(v_3) = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$$

$$(A+I)^2(v_3) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore \{(A+I)^2 v_3 \,;\, (A+I)v_3,\ v_3\}$

forms a cycle of length 3.

let $P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$P^{-1}AP = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Relevant forms of $P^2(R)$ is given by.

$$B = \{2,\ -2x,\ x^2\}$$

$$[T]_B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$