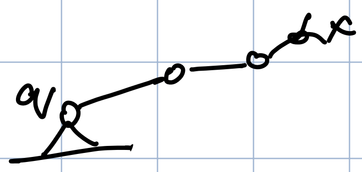

EOM: $M\ddot{q} + C = \hat{\tau}$

$$x = f_k(q)$$

$$\dot{x} = \frac{\partial f_k(q)}{\partial x}\dot{q} = J\dot{q}$$

$$\ddot{x} = J\ddot{q} + \dot{J}\dot{q} \approx J\ddot{q}$$

Inconsistent TSC

$$\ddot{x}_d = \ddot{x}_r + K_P(x - x_r) + K_D(\dot{x} - \dot{x}_r)$$

$$\min_{\ddot{q},\hat{\tau}} \|\ddot{x}_d - \ddot{x}\|_Q$$

s.t. $M\ddot{q} + C = \hat{\tau}$

$$\tau_{min} \leq \hat{\tau} \leq \tau_{max}$$

Consitent TSC

TSC Dynamics & Controller

$\hat{\tau} = J^{T} f$, $\dot{q} = J^{-1}\dot{x}$, $\ddot{q} \approx J^{-1}\ddot{x}$

$$M\ddot{q} + C = \hat{\tau}$$

$$M J^{-1}\ddot{x} + C = J^{T} f$$

$$f = \underbrace{J^{-T} M J^{-1}}_{\Lambda}\ddot{x} + \underbrace{J^{-T} C}_{\mu}$$

$$f = \Lambda \ddot{x} + \mu$$

$$\ddot{x}_d = \ddot{x}_r + K_P(x - x_r) + K_D(\dot{x} - \dot{x}_r)$$

$$f_d = \Lambda \ddot{x} + \mu$$

$$\boxed{\hat{\tau} = J^{T} f_d}$$

Joint Space EOM (State Space form)

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \quad \dot{x} = \underbrace{\begin{bmatrix} \dot{q} \\ -M^{-1}C \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}}_{g(x)} \hat{\tau}$$

u

$$\dot{x} = f(x) + g(x)\, u$$

Control Barrier function

$$h(x) = \|x - x_0\|^2 - r = \|f_K(q) - x_0\|^2 - r$$

$x_0 \equiv$ obstacle

$$L_f h(x) = \frac{\partial h(x)}{\partial x} f(x) \quad , \quad L_g h(x) = \frac{\partial h(x)}{\partial x} g(x)$$

$$\frac{\partial h(x)}{\partial x} = \begin{bmatrix} 2(f_K(q) - x_0)^T J & 0 \end{bmatrix}$$

$$L_f h(x) = 2(f_K(q) - x_0)^T J \underline{\underline{\dot{q}}}$$

$$L_g h(x) = \underline{\underline{0}}$$

first order CBF

$$L_f h(x) + \cancel{L_g h(x)} \geq -\alpha_1 h(x)$$

$$h_1 = L_f h(x) + \underline{\underline{\alpha_1}} \; h(x) \geq 0$$

Second order CBF

$$\mathcal{L}_f^2 h(x) = \frac{\partial h_1}{\partial x} f(x) \quad , \quad \mathcal{L}_g \mathcal{L}_f h(x) = \frac{\partial h_1}{\partial x} g(x)$$

$$\frac{\partial h_1}{\partial x} = \left[\frac{\partial h_1}{\partial q} \qquad \frac{\partial h_1}{\partial \dot{q}} \right]$$

$$\frac{\partial h_1}{\partial q} = \frac{\partial}{\partial q} \mathcal{L}_f h(x) + \alpha_1 \; h(x)$$

$$= \frac{\partial}{\partial q} 2(f_K(q) - x_0)^T J \dot{q} \; + \; \frac{\partial \alpha_1}{\partial q} \left(\|f_K(q) - x_0\|^2 - r \right)$$

$$= 2 J^T J \dot{q} \qquad + \underset{=}{\alpha_1} 2 \; (x - x_0)^T J$$

$$\frac{\partial h_1}{\partial \dot{q}} = \qquad 2 \left(f_K (q) - x_0 \right)^T J$$

$$\mathcal{L}_f^2 h = \left[2 J^T J \dot{q} + \alpha_1 2 \; (x - x_0)^T J \quad 2 \left(f_K(q) - x_0 \right)^T J \right] f(x)$$

$$= 2 \dot{q}^T J^T J \dot{q} + \alpha_1 2 (x - x_0)^T J \dot{q} \; - 2 (x - x_0)^T J M^{-1} C$$

$$L_g L_f h = \left[2J^T J \dot{q} + \alpha_1 \, 2\,(x - x_0)^T J \quad 2\,(f_k(q) - x_0)^T J \right] g(x)$$

$$= \underline{\underline{2\,(x - x_0)^T J M^{-1}}}$$

$$h_2 = L_f^2 h + L_g L_f h \, \underline{\underline{u}} \geq -\alpha_2 h_1(x)$$

Safe constraint on both controllers

$$L_g L_f h \, u \geq -\alpha_2 h_1(x) - L_f^2 h$$

$$\boxed{-L_g L_f h \, \tau \leq \alpha_2 h_1(x) + L_f^2 h}$$

$\downarrow$

Safety constraint

$$\boxed{\begin{aligned} &\min_{\tau_{safe}} \; \| \tau_{safe} - \tau \| \\ &\text{s.t} \quad -L_g L_f \tau \leq \alpha_2 h_1(x) + L_f^2 h \end{aligned}}$$