Assignment 1

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Question 1. 1. There are 5 available elements that an element can be mapped to. Hence, there are total of 5^5 possible mappings for the set S.

2. Since an element can only be mapped to a unique element. The total number of bijections from S to itself is equal to the number of the permutations of the elements in S, which is $5!$.

Question 2. 1. Suppose f and g is injective then for all $a \in A$ we must have a unique $b \in B$ such that $f(a) = b$. Similarly, for all $b \in B$, we have a unique $c \in C$ such that $g(b) = c$. Then, it follows that for all $a \in A$ we have a unique element $c \in C$ such that $c = g(b) = g(f(a))$.

2. Suppose $g \circ f$ is surjective, then for every $c \in C$ there must exist atleast one element $a \in A$ such that $g \circ f(a) = c$. Now, suppose g is injective, then for all $b \in B$ there must exist a unique element $c \in C$ such that $g(b) = c$. However, since $g \circ f$ is onto, every elements $b \in B$ must have atleast an element $a \in A$ such that $f(a) = b$. Hence, f must be surjective.

Question 3. The relation is not an equivalence relation since it doesn't satisfy the transitive property. For example, by definition of the relation, $1 \sim 2$ and $2 \sim 3$ but $1 \not\sim 3$.

Question 4. Suppose $a, b \in S$ and $b \in [a]$. Then, for all $b_i \in [b]$, $b_i \sim b$ and by transitive property, $b_i \sim a$. Hence, all b_i must be elements of $[a]$. Similarly, by transitive property, all $a_i \in a$ are elements of $[b]$. We have $[b] \subset [a]$ and $[a] \subset [b]$. Hence, $[b] = [a]$.

Question 5. For all $a \in S$ we have $f(a) = f(a) \implies a \sim a$. This satisfies the reflexive property. Next, for all $a, b \in S$ that satisfies $f(a) = f(b)$, we must also have $f(b) = f(a)$, hence, $a \sim b$ and $b \sim a$. This satisfies the symmetric property. Now, for all $a, b, c \in S$ that satisfy $f(a) = f(b)$ and $f(b) = f(c)$. We must also have $f(a) = f(c)$. Hence, we have $a \sim b$, $b \sim c$, and $a \sim c$. This satisfies the transitive property. We conclude that the relation $\sim$ is an equivalence relation.

Question 6. The relation $\sim$ is not an equivalence relation since it does not satisfies the transitive property. A counter-example is $1 \sim 3$ and $3 \sim 6$ but $1 \not\sim 6$.

Question 7. Assume that $\forall a, b \in \mathbb{N}, p|ab$ and further assume that p does not divide one of a or b. Without the loss of generality, let's assume that p doesn't divide a. It follows that a and p are co-primes and hence for some intergers r, s, we must have $ar + ps = 1$. Then, we have

$$\begin{aligned} b &= b(ar + ps) \\ b &= abr + ps \end{aligned}$$

Since, p divides ab, there exist an integer k such that

$$\begin{aligned} b &= pkr + ps \\ b &= p(kr + s) \end{aligned}$$

Hence, p must divides b.

Question 8. Example: $3 \times 5 = 3 \times 0$ but $5 \neq 0$. For an equivalence classes $\mathbb{Z}_p$ where p is a prime. Then, for all $a \in \mathbb{Z}_p$, we have $ar + ps = \gcd(a, p) = 1$. It follows that $ar = 1$ modulo p and $ra = 1$ modulo p. Hence, r is an inverse of a. Thus, for any $b, c \in \mathbb{Z}_4$, if $ab = ac$, we have $b = c$.

Question 9. For all $n \in \mathbb{Z}$, we have $7(5n+3) - 5(7n+4) = 1$. It follows that the $\gcd(5n+3, 7n+4) = 1$ and thus, $5n+3$ and $7n+4$ are relatively prime.

Question 10.

1. Let $n = 8$ then both n and $n+1$ are composite.
2. Let $n = 5!$. We have n is composite. Also, $n+1 = 11^2$ is composite. Futhermore, it follows that $n+i, i \in \{2,3,4,5\}$ are divisible by i.