# Assignment 2

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**Question 1.** We know that $a + b \in \mathbb{R}$. Obviously, $a + b - 1$ is also in $\mathbb{R}$. This satifies the closure property. Next, let $a, b, c \in \mathbb{R}$ then, $a*(b*c) = a+(b+c-1)-1 = a+b+c-1-1 = a+b-1+c-1 = (a*b)*c$, hence, the associative property is satisfied. The identity of this operation is 1 since for all $a \in \mathbb{R}$ we have $1 * a = 1 + a - 1 = a = a + 1 - 1 = a * 1$. The inverse of $a \in \mathbb{R}$ is $-a + 2 \in \mathbb{R}$ because $(-a+2)*a = -a+2+a-1 = 1 = a-a+2-1 = a*(-a+2)$. Lastly, to show that the operation $*$ is commutative, for all $a, b \in \mathbb{R}$, $a*b = a+b-1 = b+a-1 = b*a$. Thus, $\{G, *\}$ is an abelian group.

**Question 2.** The closure and associative properties follow from matrix multiplication. The identity matrix, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,is clearly in the group. Next, for any

$$A = \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}$$

in the group, we have its multiplicative inverse

$$A^{-1} = \frac{1}{m} \begin{bmatrix} 1 & -b \\ 0 & m \end{bmatrix} = \begin{bmatrix} 1/m & -b/m \\ 0 & 1 \end{bmatrix}$$

which is also an element of the group since $m \neq 0$

**Question 3.** Let $a, b \in G$, since $b \neq 1$ we must have $\ln(b) \neq 0$. Next, since $a > 0$, it follows that $a^{\ln(b)} > 0$ and $a \neq 1$. This satisfies the closure property. Now, for any $c \in G$, we have $a * (b * c) = a^{\ln\left(b^{\ln(c)}\right)} = a^{\ln(c)\ln(b)} = a^{\ln\left(c^{\ln(b)}\right)} = (a * b) * c$. This satifies the associative property. Let $i \in \mathbb{R}$ such that $i * a = i^{\ln(a)} = a$ and $a * i = a^{\ln(i)} = a$, then, it follows that

$$\ln(a)\ln(i) = \ln(a)$$
$$i = e$$

Clearly, $i = e \in G$. Now, suppose we have $a^{-1} \in \mathbb{R}$ such that $a * a^{-1} = a^{-1} * a = e$. Then,

$$\ln(a)\ln\left(a^{-1}\right) = \ln(e)$$
$$a^{-1} = e^{\frac{1}{\ln(a)}}$$

and for all $a \in G$, $a^{-1} \neq 1, a^{-1} > 0$, thus satifying as the inverse of $a$ in the group. Then, the inverse of $e^2$ is $e^{\frac{1}{\ln(e^2)}} = e^{1/2}$

**Question 4.** The inverse of each element in $U(7)$ and the Cayley table for the group $U(7)$ is shown in table 1 and 2.

| elements in $U(7)$ | Inverse |
|---|---|
| 1 | 1 |
| 2 | 4 |
| 3 | 5 |
| 4 | 2 |
| 5 | 3 |
| 6 | 6 |

Table 1: Inverse of elements in $U(7)$

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 2 | 4 | 6 | 1 | 3 | 5 |
| 3 | 3 | 6 | 2 | 5 | 1 | 4 |
| 4 | 4 | 1 | 5 | 2 | 6 | 3 |
| 5 | 5 | 3 | 1 | 6 | 4 | 2 |
| 6 | 6 | 5 | 4 | 3 | 2 | 1 |

Table 2: Cayley table for group $U(7)$

**Question 5.** Obviously, the identity matrix is an element of $H$. Next, for any diagonal matrix $A, B \in H$ we have its product;

$$A \cdot B = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & b_n \end{bmatrix} = \begin{bmatrix} a_1 b_1 & 0 & \dots & 0 \\ 0 & a_2 b_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_2 b_n \end{bmatrix}$$

is a diagonal matrix as well. Hence, the operation is closed within $H$. Lastly, each elements have a non-zero determinant. It follows that for every elements in $H$ are invertible. Furthermore, inverse of a diagonal matrix is also a diagonal matrix. Thus, for every element $h$ in $H$, its inverse $h^{-1}$ also exists within $H$. We conclude that $H \leqslant G$.

**Question 6.** Obviously, we have the identity $e \in G$ satifying $e^2 = e$, then $e \in H$ and $H$ is not empty. Now, let $h, g \in H$, we have

$$\begin{aligned} hg &= f \in G \\ f^2 &= hghg \\ f^2 &= hggh \\ f^2 &= heh \\ f^2 &= hh = e \end{aligned}$$

Thus, $hg = f \in H$ and the operation is closed in $H$. Lastly, for any $g \in H$, $gg = e$ so clearly the inverse of $g$ is $g$ itself.

When $G = U(7)$, we have $H = \{1, 6\}$ and when $G = U(8)$, we have $G = \{1, 3, 5\}$.

**Question 7.** Let $H = \{a \in G : 2a = 0\}$. We can rewrite this in multiplicative notation to get $H = \{a \in G : a^2 = e\}$ then the proof of this can be found in the previous question.

When $G = \mathbb{Z}_{12}$, we have $H = \{0, 6\}$. When $G = \mathbb{Z}_{13}$, we have $H = \{0\}$

**Quesiton 8.** The identity, $e \in G$ satifies $ea = ae = a$ hence $e \in C(a)$. Next, let $x, y \in C(a)$ and their product, $xy = z \in G$. Now, $za = xya = xay = axy = az$ so $z \in C(a)$. Lastly, for any $x \in C(a)$ we have its inverse $x^{-1} \in G$ such that

$$\begin{aligned} x^{-1}x &= e \\ x^{-1}xa &= a \\ x^{-1}ax &= a \\ x^{-1}axx^{-1} &= ax^{-1} \\ x^{-1}a &= ax^{-1} \end{aligned}$$

So $x^{-1} \in C(a)$.