Assignment 3

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Question 1. Let

$$h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then, for $n \in \mathbb{Z}$ we have

$$h^n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

and the determinant of h^n is 1. Futhermore, the identity matrix is simply h^0. Thus, $H = \langle h \rangle \leqslant \mathrm{GL}_2$

Question 2.

1. The positive divisors of 20 are $\{1, 2, 4, 5, 10, 20\}$, hence $\mathbb{Z}$ has 6 subgroups ; $\{0\}, \langle 1 \rangle, \langle 2 \rangle, \langle 4 \rangle, \langle 5 \rangle, \langle 10 \rangle$
2. The positive divisors of 21 are $\{1, 3, 7, 21\}$, hence $\mathbb{Z}$ has 4 subgroups ; $\{e\}, \langle g \rangle, \langle g^3 \rangle, \langle g^7 \rangle$ where e is the identity in G.

Question 4. $\langle 10 \rangle \cap \langle 21 \rangle$ is simply $\{g : g = [\mathrm{lcm}(10, 21) \bmod(24)]^n, n \in \mathbb{Z}\}$ since the elements in the intersection must be divisible by 10 and 21. Thus, $\langle 10 \rangle \cap \langle 24 \rangle = \langle 18 \rangle$

Question 5. The cyclic subgroups of $U(15)$ are $\langle 1 \rangle, \langle 2 \rangle, \langle 4 \rangle, \langle 7 \rangle, \langle 11 \rangle, \langle 14 \rangle$

Question 6. We will prove by contraposition. The converse of the proposition is *if G is a non-cylic group then there exists non-trivial proper subgroup of G.*

First, assume that G is a non-cyclic subgroup. Then, there must exist two elements $a, b \in G$ such that for any $n \in \mathbb{Z}$, $a \neq b^n$ and $b \neq a^n$. Now, we can let a be a generator of cyclic subgroup of G; $\langle a \rangle < G$. This subgroup is proper since $b \notin \langle a \rangle$. Thus, the original proposition must be true.

Question 7. Let $\mu = o(a)$ and $\nu = o(b)$. Now, since $ab = ba$, we can distribute $(ab)^{\mu\nu} = a^{\mu\nu} b^{\mu\nu} = e^{\nu} e^{\mu} = e$. It follows that $\mu\nu$ must be an integer multiple of the order of ab. Thus, $o(ab) | o(a)o(b)$.

Question 8. By the Fundamental Theorem of Cyclic Groups, there is exactly one subgroup of $G = \langle g \rangle$ of order d, namely, $\langle g^k \rangle$ where $n = dk$. Now, the order of this group is clearly d. Thus, by a lemma, there are $\phi(d)$ elements that can generate $\langle g^k \rangle$.

Question 9. By the Fundamental Theorem of Cyclic Groups, the divisors of n are clearly $\{1, 7, n\}$. The only positive integer with this set of divisors is 49. Generally, for any prime number ,p , we will have $n = p^2$ since the divisors of p^2 are $\{1, p, p^2\}$.