Assignment 8

ICMA424 Abstract Algebra

Instructions. This assignment will be graded for completeness and correctness. Three questions will be selected at random for careful grading. Solutions that are unreadable or incoherent will receive no credit. Assignment is due by **Monday, 28 November 2022 at 23:59**.

1. Let $\theta : G \longrightarrow H$ be a group homomorphism. Prove that θ is injective if and only if $\ker \theta = \{1\}$.

2. Let H and K be finite subgroups of a group G. Show that

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

   Hint: There are $|H||K|$ products in $|HK|$ but some of these may be the same elements. That is, we may have $hk = h_1k_1$ where $h \neq h_1$ and $k \neq k_1$. We must find the extent to which this can happen. Certainly for every $t \in H \cap K$ we have $hk = (ht)(t^{-1}k)$. This shows that each element in HK can be represented in at least $|H \cap K|$ products from elements in H and K. Now show that it is also at most.

3. In class, we showed that the kernel of a homomorphism from G is a normal subgroup of G. Show that every normal subgroup of G arises in this way. That is, if $N \lhd G$. Show that N is the kernel of some homomorphism from G.

4. 1) Show that the map $f : \mathbb{Z}_6 \longrightarrow \mathbb{Z}_9$ given by $f(x) = 4x$ is not well-defined.

   2) Determine all homomorphisms $\theta : \mathbb{Z}_6 \longrightarrow \mathbb{Z}_9$.

5. LetG, G_1 be groups and let $K \lhd G$ and $K_1 \lhd G_1$. Show that $K \times K_1 \lhd G \times G_1$ and

$$(G \times G_1)/(K \times K_1) \cong (G/K) \times (G_1/K_1).$$

   Hint: Use the first isomorphism theorem.

6. The ring $\{0, 2, 4, 6, 8\}$ under addition and multiplication modulo 10 has a unity. Find it.

7. Let S, T be subrings of a ring R. Prove that $S \cap T$ is a subring of R.

8. Let a belong to a ring R. Let $S = \{x \in R : ax = 0\}$. Show that S is a subring of R.

9. Let $M_2(\mathbb{Z})$ be the ring of all 2×2 matrices over the integers and let

$$R = \left\{ \begin{pmatrix} a & a-b \\ a-b & b \end{pmatrix} : a, b \in \mathbb{Z} \right\}.$$

   Prove or disprove that R is a subring.

10. Show that $4x^2 + 6x + 3$ is a unit in $\mathbb{Z}_8[x]$.