Assignment 8

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1. The forward implication is trivial; If θ is injective then there would only be one element that maps to the identity in H. Since θ is a homomorphism, it follows that this element must be the identity in G. Next, assume that $\ker\theta = \{1\}$. Then for any $x, y \in G$ such that $\theta(x) = \theta(y)$, we have

$$\begin{aligned} f(x) - f(y) &= e_H \\ f(x-y) &= e_H \\ x - y &= e_G \\ x &= e_G + y \\ x &= y \end{aligned}$$

   Thus, θ is a homomorphism.

2.

3. From the first isomorphism theorem, if $\theta : G \to H$ is a homomorphism, then the mapping from $G/\ker\theta$ to $\theta(G)$ given $g\ker\theta \mapsto \theta(g)$ is an isomorphism.

   Now, I follow the pattern and define a map $\sigma : G/N \to \phi(G)$ where ϕ is the trivial map and $\sigma(gN) = g$. It follows that $N \in \ker\sigma$. Next, consider $\sigma^{-1}(g) = gN$, then $\ker(\sigma^{-1}) = \{g \in G, \sigma^{-1}(g) = 1_{G/N}\}$. From a lemma, $gN = g$ only if $g \in N$, then,$\ker(\sigma^{-1}) = \{g \in N\} = N$

4. 1) We have $f(0) = 0$ and $f(6) = 6$ so $f(0) \neq f(6)$ but $0 = 6$.
   2) Since $1 \in \mathbb{Z}_6$ is the generator, we have $f(g) = f(1 \times g) = f(1) \times f(g)$ for all $g \in \mathbb{Z}_6$and let us denote $f(1) = a$. Then, $f(g) = ga$. We also need $f(0) = 0$, hence, $f(0) = f(6) = 6a$. We can say that $6a = 0 \mod 9$ then

$$\begin{aligned} 6a &= 0 \mod 9 \\ 6a &= 9 \mod 9 \\ 2a &= 3 \mod 3 \\ 2a &= 0 \mod 3 \\ a &= 0 \mod 3 \end{aligned}$$

      Thus, $a \in \{0, 3, 6\}$and there are 3 homomorphisms.

5. For all $k \in K$ and $k_1 \in K_1$ we have $ak = ka, \forall g \in G$ and $g_1k_1 = k_1g_1, \forall g_1 \in G_1$. Then

$$\begin{aligned} (a, a_1)(k, k_1) &= (ak, a_1k_1) \\ &= (ka, k_1a_1) \\ &= (k, k_1)(a, a_1) \end{aligned}$$

   Thus, $K \times K_1$ is normal in $G \times G_1$. Now, define a map $\theta(g, g_1) \mapsto (g, g_1)(K, K_1)$, then $K \times K_1$ is a $\ker\theta$. By the first isomorphism theorem, it follows that $(G \times G_1)/(K \times K_1) \cong \theta(G \times G_1)$. Since,$\theta(G \times G_1) = (G/K \times G_1/K_1)$, thus $(G \times G_1)/(K \times K_1) \cong G/K \times G_1/K_1$

6.