Step 1: Calculation of Eigenvalues

$$\boxed{\det(\lambda I - A) = 0}$$

Step 2: With λ, find v using substitution method to avoid $v=0$ solution.

Solved Example

Find the Eigenvalues and Eigenvectors of the square matrix A.

$$A = \begin{bmatrix} -2 & 2 & 2 \\ 0.2 & -3.6 & -0.6 \\ -0.2 & -0.4 & -3.4 \end{bmatrix}$$

Step 1: Calculation of Eigenvalues

$$\det(\lambda I - A) = 0 \Rightarrow \begin{vmatrix} \lambda+2 & -2 & -2 \\ -0.2 & \lambda+3.6 & +0.6 \\ 0.2 & 0.4 & \lambda+3.4 \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 + 9\lambda^2 + 26\lambda + 24 = 0$$

$$\Rightarrow \lambda_1 = -2, \ \lambda_2 = -3, \ \lambda_3 = -4$$

Step 2

(a) Eigenvectors for $\lambda_1 = -2$

$$\Rightarrow |\lambda_1 I - A| v_1 = 0$$

$$\Rightarrow \begin{bmatrix} 0 & -2 & -2 \\ -0.2 & 1.6 & 0.6 \\ 0.2 & 0.4 & 1.4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0$$

$$-2\alpha_2 - 2\alpha_3 = 0 \Rightarrow \alpha_2 = -\alpha_3$$

$$-0.2\alpha_1 + 1.6\alpha_2 + 0.6\alpha_3 = 0$$

$$0.2\alpha_1 + 0.4\alpha_2 + 1.4\alpha_3 = 0$$

$\{\alpha_1, \alpha_2, \alpha_3\} \neq 0$ trivial.

(Any one of $\alpha_1, \alpha_2, \alpha_3$ as 1)

By substitution method. Take $\alpha_3 = 1$.

Then $v_1 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix}$

b) Eigenvector for $\lambda_2 = -3$

$$\Rightarrow |\lambda_2 I - A| v_2 = 0$$

$$\Rightarrow \begin{bmatrix} -1 & -2 & -2 \\ -0.2 & 0.6 & 0.6 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = 0$$

$$-\beta_1 - 2\beta_2 - 2\beta_3 = 0$$

$$-0.2\beta_1 + 0.6\beta_2 + 0.6\beta_3 = 0$$

$$0.2\beta_1 + 0.4\beta_2 + 0.4\beta_3 = 0$$

By substitution method. Take $\beta_3 = 1$.

Then

$$v_2 = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

c) Eigenvector for $\lambda_3 = -3$

$$|\lambda_3 I - A| v_3 = 0$$

$$\begin{bmatrix} -2 & -2 & -2 \\ -0.2 & -0.4 & 0.6 \\ 0.2 & 0.4 & -0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-2x_1 - 2x_2 - 2x_3 = 0$$

$$-0.2x_1 - 0.4x_2 + 0.6x_3 = 0$$

$$0.2x_1 + 0.4x_2 - 0.6x_3 = 0$$

By substitution method. Take $x_3 = 1$

Then

$$v_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$$

Eigenvector matrix;
$$V = \begin{bmatrix} -5 & 0 & -5 \\ -1 & -1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

Properties of Eigenvalues and Eigenvectors

$$\boxed{Av = \lambda v}$$

Property 1

$$A^2 v = A \cdot A v = A \cdot \lambda v = \lambda A v = \lambda \lambda v = \lambda^2 v$$

$$\Rightarrow \quad \boxed{A^n v = \lambda^n v}$$

Conclusion: For n^{th} power of matrix A

a) Eigenvectors will remain the same of A.

b) Eigenvalues will be n^{th} power of Eigenvalues of A.

Property 2

$$(A + cI)v = Av + cIv = Av + cv$$
$$= (A + c)v$$

Conclusion: If a constant c is added to the matrix A.

a) Eigenvectors will remain the same of A.

b) Eigenvalues will be $(\lambda + c)$

Properties of Similarity Transformation

Two $n \times n$ matrices, A and $\hat{A}$ are called similar, if there exists an invertible $n \times n$ matrix T, such that

$$\boxed{\hat{A} = T^{-1} A T}$$

$$\boxed{A = T \hat{A} T^{-1}}$$

$T \rightarrow$ Transformation matrix.

Property 1

A and $\hat{A}$ have the same determinant.

$$|\hat{A}| = |T^{-1} A T|$$

$$= |T^{-1}|\,|A|\,|T|$$

$$= \frac{1}{\cancel{|T|}} \cdot |A| \cdot \cancel{|T|}$$

(Determinant is scalar)

$$= |A|$$

$$|A \cdot B| = |A| \cdot |B|$$

$$|A^{-1}| = |A|^{-1} = \frac{1}{|A|}$$

Property 2

A and $\hat{A}$ have the same characteristic equation and hence the eigenvalues.

$$Av = \lambda v$$

$$\Rightarrow (A - \lambda I)v = 0$$

$$v \neq 0$$

So, ($|A - \lambda I| = 0$ ← Characteristic equation of A.

$$\Rightarrow |T\hat{A}T^{-1} - \lambda I| = 0$$

$$\Rightarrow |T\hat{A}T^{-1} - \lambda TT^{-1}| = 0$$

$$\Rightarrow |T(\hat{A} - \lambda I)T^{-1}| = 0$$

$$\Rightarrow \cancel{|T|} \cdot |\hat{A} - \lambda I| \cdot \frac{1}{\cancel{|T|}} = 0$$

$\Rightarrow |\lambda I - \hat{A}| = 0$ ← Characteristic equation of $\hat{A}$.

Property 3:

Eigenvectors are related through transformation matrix T.

We know, $$Av = \lambda v$$

$$\hat{A} = T^{-1}AT$$

Consider the eigenvalue equation of $\hat{A}$

$$\hat{A}w = \lambda w$$

where w is the eigenvector of $\hat{A}$ we want to relate to v.

$$\Rightarrow \quad T^{-1}ATw = \lambda w$$

$$\Rightarrow \quad ATw = \lambda Tw \quad \text{(pre-multiply } T\text{)}$$

Take $v = Tw$. Then:

$$Av = \lambda v$$

So, $v = v \Rightarrow w = T^{-1}v$

$w \leftarrow$ Eigenvector of $\hat{A}$

$v \leftarrow$ Eigenvector of A

$$\boxed{w = T^{-1}v}$$

Property 4:

Trace of a matrix remains the same under similarity transformations.

Trace of a matrix : The trace of a square matrix A, & tr (A), is the sum of its diagonal elements.

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

Proof

$$tr(\hat{A}) = tr(T^{-1}AT)$$

Cyclic property of trace

$$tr(XYZ) = tr(YZX) = tr(ZXY)$$

So;

$$tr(\hat{A}) = tr(T^{-1}AT) = tr(A\,T\,T^{-1})$$

$$= tr(A)$$

$$tr(\hat{A}) = tr(A)$$

Property 5:

Rank of matrix remains the same under similarity transformations.

Rank of a matrix: rank(A) represents the maximum number of linearly independent rows or columns in the matrix.

$$\left.\begin{aligned} \text{rank}(TA) &= \text{rank}(AT) = \text{rank}(A) \\ \text{rank}(T^{-1}A) &= \text{rank}(AT^{-1}) = \text{rank}(A) \end{aligned}\right\}$$

for T and T^{-1} invertible.

$\Rightarrow$ The multiplication with T and T^{-1} transforms the basis of the vector space in which A operates. $\rightarrow$ but it does not change the relationships between the rows or coloumns of the matrix.

Proof:

$$\begin{aligned} \text{rank}(\hat{A}) &= \text{rank}(T^{-1}AT) \\ &= \text{rank}((T^{-1}A)\cdot T) \\ &= \text{rank}(T^{-1}A) \\ &= \text{rank}(A). \end{aligned}$$

Diagonalization of Matrix using Similarity Transformation

- Diagonalization is a process that transforms a square matrix A into a diagonal matrix Λ.

Conditions for Diagonalization

A square matrix A must have n linearly independent eigenvectors to be diagonalizable, where n is the size of the matrix. This condition typically satisfies if A has distinct eigenvalues.

Consider $n = 3$. Then eigenvectors corresponding to eigenvalues λ_1, λ_2, and λ_3 are v_1, v_2, and v_3. We can write:

$$\left.\begin{aligned} Av_1 &= \lambda_1 v_1 \\ Av_2 &= \lambda_2 v_2 \\ Av_3 &= \lambda_3 v_3 \end{aligned}\right\} \Rightarrow A\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

$$\Rightarrow A\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\Rightarrow \quad AV = V\Lambda \quad \text{— (1)}$$

where $\quad V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$

Diagonal matrix $\quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Premultiplying (1) with V^{-1}

$$\boxed{\Lambda = V^{-1} A V}$$

Postmultiplying (1) with V^{-1}

$$\boxed{A = V \Lambda V^{-1}}$$

Solved Example

Diagonalize the matrix

$$A = \begin{bmatrix} -2 & 2 & 2 \\ 0.2 & -3.6 & -0.6 \\ -0.2 & -0.4 & -3.4 \end{bmatrix}$$

Step 1

Eigenvalues of $A \Rightarrow \det(\lambda I - A) = 0$

$\lambda_1 = -2 \qquad \lambda_2 = -3 \qquad \lambda_3 = -4$

The eigenvalues are distinct $\Rightarrow$ Matrix A is diagonalisable.

Step 2

Eigenvectors matrix, V

$$V = \begin{bmatrix} -5 & 0 & -5 \\ -1 & -1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

Step 3

$$\Lambda = V^{-1} A V$$

$$V^{-1} = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ -0.2 & -0.4 & 0.6 \\ 0 & 0.2 & 0.2 \end{bmatrix}$$

Diagonal matrix

$$\Lambda = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

Advantages of Diagonal matrix in Computation

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

1. Determinant

$$|\Lambda| = \lambda_1 \times \lambda_2 \times \lambda_3$$

2. Eigenvalues

$$\lambda_1, \lambda_2, \lambda_3$$

3. Matrix Power

$$\Lambda^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

4. Matrix Exponential

$$e^\Lambda = \begin{bmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{bmatrix}$$

5. Matrix Inversion

$$\Lambda^{-1} = \begin{bmatrix} 1/\lambda_1 & 0 & 0 \\ 0 & 1/\lambda_2 & 0 \\ 0 & 0 & 1/\lambda_3 \end{bmatrix}$$

6. Eigenvectors

– are the standard basis vectors.

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

7. Power of A-matrix

$$A^K = V\Lambda^K V^{-1}$$

8. State Transition Matrix $\phi(t)$

$$\phi(t) = e^{At} = Ve^{\Lambda t}V^{-1}$$