Practice Midterm III

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(20pts, 4 each) Consider the Initial Value Problem (IVP)

$$y'(t) = -20y, y(0) = 5, t \in [0,2]$$

1. What do you need to ensure well-posedness of the IVP?
   In order to insure this IVP is well-posed a unique solution $y(t)$ exists. If the Lipschitz condition is met, that is there is a Lipschitz constant and the function is continuous on a a domain D. $D = \{(t, y)|a \leq t \leq b \text{ and } -\infty \leq y \leq \infty\}$

2. Find a Lipschitz constant, L, for this IVP.
   Let $f(t, y) = -20y$, Now using the Lipschitz formula

$$\left|\frac{f(t,y_2) - f(t,y_1)}{y_2 - y_1}\right| \leq \left|\frac{\partial}{\partial y}(-20y)\right|$$

$$\left|\frac{f(t,y_2) - f(t,y_1)}{y_2 - y_1}\right| \leq 20$$

$$|f(t,y_2) - f(t,y_1)| \leq 20|y_2 - y_1|$$

   We have found L to be 20.

3. Solve this IVP.

$$\frac{dy}{dt} = -20y$$

   Seperable differential equation.

$$\frac{dy}{y} = -20dt$$

$$y(t) = e^{-20t}C$$

$$y(0) = 5 = e^{-20(0)}C$$

$$y = 5e^{-20t}$$

4. Is the equation stiff? Justify (no more than two lines) your answer.
   The solution of the IVP is of exponential decay.
   The derivatives have greater magnitude.

5. Given a general N and $\Delta t = \frac{2}{N}$, write Euler's method to approximate this IVP.
   Consider the following

$$y'(t) = -20y, y(0) = 5, t \in [0,2]$$

$$a = 0, b = 2, \alpha = 5, f(t, y) = -20y$$

$$t = 0; w_0 = 5$$

$$w_i = w_{i-1} + \frac{2}{N} f(t_{i-1}, w_{i-1})$$

$$w_1 = 5 + \frac{2}{N} f(0,5)$$

$$t = 0 + i \cdot \frac{2}{N}$$

(20pt, 4each)

1. Give the local truncation error $\tau_j(\Delta t)$ for fourth order Runge-Kutta method.
   The local truncation error is $O(h^4)$
2. Define the stability of a numerical method.
   The solution should not behave erratically, that is, it is does not grow without bound, it should remain bounded.
   "small changes or perturbations in the initial conditions produce.correspondingly small changes in the subsequent approximations." (Page 341 Section 5.10)
3. Define the consistency of a numerical method.
   As we increase the number of stepsize N, the error between our approximation our numerical method solution should and exact solution should decrease as N increases.
4. Define the convergence of a numerical method. Convergence refers to the Numerical Method solution getting close to the actual solution. As the number of meshpoints increases $N \to \infty$ the approximated solution reaches the actual solution.
5. Given a general Δt, give a 3-step explicit method (you can even create one).

(12 pts, 4 each) Consider the differential equation

$$y^{(3)} = e^t + y, y(0) = 5, t \in [0,2]$$

1. Rewrite this problem into a system of first-order ODEs.

$$u_1(t) = y(t), u_1'(t) = u_2(t)$$

$$u_2(t) = y'(t), u_2'(t) = u_3(t)$$

$$u_3(t) = y''(t), u_3'(t) = e^t + u_1$$

   Done

2. Given a general N and $\Delta t = \frac{2}{N}$, write eulers Method to Approximate the obtained system.

$$w_{j+1} = w_j + \Delta t f(t_j, w_j)$$

$$w_0 = 5, h = \frac{b-a}{N} = \frac{2}{N}$$

3. Given a general N and $\Delta t = \frac{2}{N}$, write the second order Taylor's method to approximate the obtained system.

(28 pts, 4 each) Identify all the following numerical methods:

1. Euler's method (a)
$$w_{j+1} = w_j + \Delta t f(t_i, w_j)$$

2. 2-step Adams-Moulton method (g)
$$w_{j+1} = w_j + (5f(t_{j+1}, w_{j+1}) + 8f(t_j, w_j) - f(t_{j-1}, w_{j-1}))$$

3. second order Taylor's method (f)
$$w_{j+1} = w_j + \frac{\Delta t}{2}\Big(f(t_j, w_j) + \frac{\Delta t}{2} f'(t_j, w_j)\Big)$$

4. fourth Runge-Kutta method (d)
$$w_{j+1} = w_j + \frac{\Delta t}{6}\Big(k_1 + 2k_2 + 2k_3 + k_4\Big)$$
$$k_1 = f(t_j, w_j)$$
$$k_2 = f(t_j + \frac{\Delta t}{2}, w_j + \frac{\Delta t}{2} k_1)$$
$$k_3 = f(t_j + \frac{\Delta t}{2}, w_j + \frac{\Delta t}{2} k_2)$$
$$k_4 = f(t_{j+1}, w_j + \Delta t k_3)$$

5. Modified Euler's method (b)
$$w_{j+1} = w_j + \frac{\Delta t}{2}\Big(f(t_j, w_j) + f(t_{j+1}, w_j + \Delta t f(t_j, w_j))\Big)$$

6. second Runge-Kutta method (c)
$$w_{j+1} = w_j + \Delta t f\Big(t_j + \frac{\Delta t}{2}, w_j + \frac{\Delta t}{2} f(t_j, w_j)\Big)$$

7. 2-step Adams-Bashforth method (e)
$$w_{j+1} = w_j + \frac{\Delta t}{2}\Big(3f(t_j, w_j) - f(t_{j-1}, w_{j-1})\Big)$$

(20 pts, 4 each) Place the following numerical methods in the table below (several can belong to the same category).

	1 step	Multi Step
Explicit Method	Euler, Taylor, Runge Kutta	(Adams Bashforth), Adams Moulton
Implicit Method	Euler	N/A