Practice Midterm 1

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1. Let $f(x) = -3x^3 + 4x - 2$

   (a) Use Newton's method to find x_1 if $x_0 = 1$.
   Consider that
   $$x_n = x_{n-1} - \frac{f x_{n-1}}{f'(x_{n-1})}$$
   to find x_1
   $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
   We know that $f(1) = -1$, $f'(x) = -9x^2 + 4$, $f'(1) = -5$ therefore
   $$x_1 = 1 + \frac{-1}{5} = \frac{4}{5} = .8$$

   (b) Use the Secant method to find x_2 if $x_0 = 2$ and $x_1 = 1$.
   Consider that
   $$x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-1} - x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}$$
   $$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$
   given that $x_0 = 2, x_1 = 1$, $f(1) = -1, f(2) = -18$
   $$x_2 = 1 - \frac{(-1)(1-2)}{-1+18} = \frac{16}{17} \approx 0.9412$$

2. Let $g(x) = \sqrt{x+6}$

   (a) Show that $c = 3$ is a fixed point of the function $g(x)$
   We know that $g(c) = 3$
   $$g(c) = c = \sqrt{c+6}$$
   $$c^2 - c - 6 = 0$$
   $$(c-3)(c+2) = 0$$
   The graph $g(x)$ has a fixed point at $c = -2$ and $c = 3$

   Optionally, you may show $g(c) = c$ to be true:
   $$g(c) = c$$
   $$g(3) = \sqrt{3+6} = \sqrt{9} = 3$$

(b) Find a bound for $|g'(x)|$ for $x \in [0,4]$
We know that there $c = 3$ is a fixed point, $c \in [0,4]$

$$g'(x) = \frac{1}{2(\sqrt{x+6})}$$
$$|g'(x)| = \frac{1}{2}\left|\frac{1}{\sqrt{x+6}}\right|$$
$$g'(0) = \frac{1}{2\sqrt{6}}, g'(4) = \frac{1}{2\sqrt{10}}$$
$$\frac{1}{2\sqrt{10}} \le g'(x) \le \frac{1}{2\sqrt{6}}$$

(c) Is the fixed-point method converging to the fixed point $c = 3$ in the interval $[0,4]$? If so, what is the rate of convergence?
since $|g'(c)| = \frac{1}{6} < 1$ we know it converges. Since $|g'(c)| \ne 0$ the rate of convergence is linear.
The following shows it : Using Taylors Theorem

$$g(x) = \sqrt{x+6}, g'(x) = \frac{1}{2\sqrt{x+6}}, g''(x) = \frac{-1}{4(x+6)^{\frac{3}{2}}}$$
$$g(x) = \sqrt{c+6} + \frac{1}{2\sqrt{c+6}}(x-c) + (\frac{-1}{4(\xi+6)^{\frac{3}{2}}})(x-c)^2$$

We know that $\xi : [x,c]$, $x_{n+1} = g(x_n)$, we know that $g(c) = c, \sqrt{c+6} = c$

$$x_{n+1} = c + \frac{1}{2\sqrt{c+6}}(x_n - c) + (\frac{-1}{4(\xi+6)^{\frac{3}{2}}})(x_n-c)^2$$
$$x_{n+1} - c = \frac{1}{2\sqrt{c+6}}(x_n - c) + (\frac{-1}{4(\xi+6)^{\frac{3}{2}}})(x_n-c)^2$$
$$\frac{x_{n+1}-c}{x_n-c} = \frac{1}{2\sqrt{c+6}} + (\frac{-1}{4(\xi+6)^{\frac{3}{2}}})(x_n-c)$$
$$\lim_{n\to\infty}\left|\frac{x_{n+1}-c}{x_n-c}\right| = \frac{1}{2\sqrt{c+6}} + (\frac{-1}{4(\xi+6)^{\frac{3}{2}}})(x_n-c)$$
$$\text{consider that } \lim_{n\to\infty}(\frac{-1}{4(\xi+6)^{\frac{3}{2}}})(x_n-c) = 0, (\lim_{n\to\infty}(x_n) = c)$$
$$\lim_{n\to\infty}\left|\frac{x_{n+1}-c}{x_n-c}\right| = \frac{1}{2\sqrt{c+6}}$$
$$\lim_{n\to\infty}\left|\frac{x_{n+1}-c}{x_n-c}\right| = \frac{1}{2c}, c = 3$$
$$\lim_{n\to\infty}\frac{|x_{n+1}-c|}{|x_n-c|^{\alpha}} = \frac{1}{2c} = \frac{1}{6} = \lambda$$

This shows that the rate of convergence α is $\alpha = 1$

3. (a) Give Taylor's theorem for a function f at a point x_0.
The following is necessary in order to apply Taylor's theorem.
$f \in C^k[a,b], f^{k+1}$ exist on given interval and $x_0 \in [a,b]$ there exists a number $\xi(x)$ between x_0 and x.

$$f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)(x-x_0)^k}{k!} + \frac{f^{(k+1)}(\xi(x))}{(k+1)!}(x-x_0)^{k+1}$$

(b) Using Newton's method, what is the rate of convergence for the sequence error?

The rate of convergence for Newton's method is Quadratic.
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(c) Give the largest interval possible approximation of a number p up to 10^{-5} relative error.
We know that relative error $\frac{|p-p*|}{p}$

$$\frac{|p-p*|}{p} < 10^{-5}$$

$$|p-p*| < p \times 10^{-5}$$

$$-p \times 10^{-5} < p - p* < p \times 10^{-5}$$

$$-p \times 10^{-5} + p < p* < p \times 10^{-5} + p$$

The largest interval $p*$ must lie on is $(p(-10^{-5}+1), p(10^{-5}+1))$

(d) Suppose $p*$ approximates the number p. What are the different types of error that you can use to quantify the accuracy of the approximation.

i. Relative error: $\left|\frac{p-p*}{p}\right|$
ii. Absolute error: $\left|p-p*\right|$
iii. Actual error: $p-p*$

4. Below is a MATLAB implementation of the bisection method with some code missing.

(a) Fill in the blanks to correctly execute the bisection method.

```
function [c,err,n] = bisection(f,a,b,tol,N)
%function to solve f(x) = 0 using the bisection method over [a,b]
%ASSUMPTIONS: we assume f(a)*f(b) < 0
%INPUTS:
%f is a function at hand
% a is the lower bound of the tested interval
% b is the upper bound of the tested interval
% tol is the error tolerance
% N is the maximum number of iterations
%OUTPUTS:
% c is the computed root
% err is the error bound at the end
% n is the last ieration before breaking
    n = 0;
    a_n = a;
    b_n = b;
    err = b_n - a_n;
    while err > tol && n < N
        x_n = %(a_n + b_n)/2;
        if f(x_n)*f(a_n) > 0
            %a_n = x_n;
            %c = a_n;  <--Consider revising to [b_n = b_n] based on previous semester.
        elseif f(x_n)*f(b_n) >= 0
            %b_n = x_n;
            %c = b_n;  <--Consider revising to [a_n = a_n] based on previous semester.
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            end
        err = (b_n - a_n)/2;
        n = n + 1;
    end
    c = (b_n + a_n)/2.0;
end
```

(b) Is the code working if you consider $f(x) = x^4, a = -1, b = 2$? Explain why.

No. The bisection method has been established by the intermediate value theorem which states if the a function $f(x)$ is continuous in a given interval ([a,b]) and if evaluated endpoints $f(a)$ and $f(b)$ have opposite signs that is $f(a) \times f(b) < 0$ there there exists a root inside the interval.

$$f(-1) = 1, f(2) = 2^4, f(-1) \times f(2) \nless 0$$