Practice Midterm II

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Formulas:

F1 Composite Trapezoidal

$$h = \frac{b-a}{n}$$

$$\frac{h}{2}\left[f(a) + 2\cdot\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12}h^2 f^{(2)}(\xi)$$

F2 Composite Simpsons

$$S = \frac{h}{3}\left[f(a) + 2\cdot\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{n}{2}} f(X_{2j-1}) + f(b)\right] - \frac{b-a}{180}h^4 f^{(4)}(\xi)$$

F2 Midpoint Formula

$$h = \frac{b-a}{n+2}$$

$$2\cdot h\left[2\sum_{j=0}^{n/2} f(x_{2j})\right] - \frac{b-a}{6}h^2 f^{(2)}(\xi)$$

1. (20 pts, 10 each)

   (a) Given $x_0 = 2, x_1 = 3, f(x_0) = 2, f(x_1) = 3$, construct the linear Lagrange Polynomial interpolant of f that passes through the points $((x_0, f(x_0)), (x_1, f(x_1))$. To get full credit, you must give the Lagrange polynomials you need.

   We must find the interpolation polynomial of degree 1, given that it is linear and we are given 2 nodal points. To find degree $n = 1$, we need $n+1$ points.

   i. Recall that the Lagrange interpolating polynomial is as follows.

   $$P(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

   ii. We know that $y_0 = f(x_0) = 2, y_1 = f(x_1) = 3$

   iii. find $L_{n,k}$

   $$L_{1,0} = \frac{x-x_1}{x_0-x_1}, L_{1,1} = \frac{x-x_0}{x_1-x_0}$$

   $$L_{1,0} = \frac{x-3}{2-3} = -(x-3), L_{1,1} = \frac{x-2}{3-2} = (x-2)$$

iv. $P_1(x) = y_0 L_{1,0} + y_1 L_{1,1}$

$$P_1(x) = 2 \cdot (-(x-3)) + 3 \cdot (x-2) = x$$

We only need one Lagrange polynomial, we are only given two nodal points, making it linear Lagrange interpolation

(b) Use Newton's divided difference formula to construct an interpolating polynomial of degree two of f for the following data: $(x_0, f(x_0) = (-1, 1), (x_1, f(x_1)) = (0, .5), (x_2, f(x_2)) = (1, 3)$.

We want to find the interpolating polynomial of degree 2.

i. Consider the following form of Newton's Divided Difference:

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots x_k](x - x_0)\ldots(x - x_{k-1})$$

Let n = 2, we want to find the following.

$$P_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

ii. Find $f[x_0, x_1]$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

iii. Find $f[x_0, x_1, x_2]$
Recall that $f[x_0, x_1]$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

Recall that $f[x_1, x_2]$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$$

Therefore

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

iv. Consider three nodal points, that is x_0, x_1, x_2, if n = 2, then we need $n + 1$ points.

$$x_0 = -1, x_1 = 0, x_2 = 1$$

v. We define values of the nodal points at f. It is already given

$$f[x_0] = 1, f[x_1] = .5, f[x_2] = 3$$

vi. Using the form of Newton's divided difference. Substituting previous findings.

$$f[x_0, x_1] = \frac{.5 - 1}{0 - (-1)} = -.5$$

$$f[x_1, x_2] = \frac{3 - .5}{1 - 0} = 2.5$$

$$f[x_0, x_1, x_2] = \frac{2.5 - .5}{1 - (-1)} = 1$$

vii. Finding P_2. Substituting the values we found.

$$P_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$P_2(x) = 1 + .5(x + 1) + 1(x + 1)(x - 0)$$

$$P_2(x) = 1.5x + 1.5 + x^2$$

2. (20pts, 10 each)

Come Back To This: WARNING!! Long

Given a function f defined on $[a, b]$ and set of nodes $a = x_0 < x_1, \cdots < x_n = b$. Define a cubic spline interpolant $S(x)$ of $f(x)$ (Hint: List the conditions $s(x)$ has to satisfy).

Consider the following formula

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$

We list the conditions $s(x)$ has to satisfy. From the Book.

(a) i. $S(x)$ is a cubic polynomial, denoted $S_j(x)$, on the subinterval $[x_j, x_{j+1}]$;
We know that $a = x_0$ and $b = x_n$. Let $x_0, x_{n-1}, x_n, f(x_0) = x_0, f(x_{n-1}) = x_{n-1}, f(x_n) = x_n$

$$S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3, x_0 \le x \le x_{n-1}$$

$$S_1(x) = a_1 + b_1(x - x_{n-1}) + c_0(x - x_{n-1}) + d_0(x - x_{n-1}), x_{n-1} \le x \le x_n$$

8 unknowns, $a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1$

ii. $S_j(x_j) = f(x_j)$ and $S_j(X_{j+1}) = f(x_{j+1})$ for each $j = 0, 1, \ldots, n-1$;

$$S_0(x_0) = f(x_0)$$

$$S_0(x_0) = a_0 = f(x_0)$$

The second part

$$S_0(x_1) = a_0 + b_0(x_1 - x_0) + c_0(x_1 - x_0)^2 + d_0(x_1 - x_0)^3 = f(x_1)$$

iii. $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$;

$$S_1(x_1) = a_1 + b_1(x_1 - x_{n-1}) + c_1(x_1 - x_{n-1})^2 + d_1(x_1 - x_{n-1})^3$$

$$S_0(x_1) = a_0 + b_0(x_1 - x_0) + c_0(x_1 - x_0)^2 + d_0(x_1 - x_0)^3$$

iv. $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$;

$$S'_1(x_1) = b_1(x_1 - x_{n-1}) + 2c_1(x_1 - x_{n-1}) + 3d_1(x_1 - x_{n-1})^2$$

v. $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$;

vi. One of the following sets of boundary conditions is satisfied:

A. $S''(x_0) = S''(x_n) = 0$ (**natural** (or free) **boundary**;

B. $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (**clamped boundary**)

(b) Construct the natural cubic spline interpolant for the following data:

x	$f(x)$
0	1
.5	2.72

i. We know that
$x_0 = 0, x_1 = .5$
$f(x_0) = 1, f(x) = (x_1) = 2.72$

ii. The cubic spline for the interval $[x_0, x_1]$ is as follows

$$S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3$$

for $x_0 = 0$

$$S_0(x) = a_0 + b_0(x - 0) + c_0(x - 0)^2 + d_0(x - 0)^3$$

iii. $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1})$ Show it meets such conditions.

$S_j(x_j) = f(x_j)$

$$S_0(0) = f(0)$$

$$S_0(0) = a_0 + b_0(0 - 0) + c_0(0 - 0)^2 + d_0(0 - 0)^3 = a_0$$

$$S_0(0) = f(0) = a_0 = 1$$

$s_j(x_{j+1}) = f(x_{j+1})$ Recall $a_0 = 1$

$$S_0(.5) = f(.5)$$

$$S_0(.5) = a_0 + b_0(.5 - 0) + c_0(.5 - 0)^2 + d_0(.5 - 0)^3$$

$$S_0(.5) = 1 + (.5)b_0 + (.25)c_0 + (.1250)d_0 = 2.72$$

$$S_0(.5) = (.5)b_0 + (.25)c_0 + (.1250)d_0 = 2.72 - 1$$

iv. One of the following sets of boundary conditions is satisfied:

A. $S''(x_0) = S''(x_n) = 0$ (**natural** (or free) **boundary**;

B. $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (**clamped boundary**)

We do the following, since we where asked to find the **Natural cubic spline**

$$S''(x_0) = S''(x_n) = 0$$

$$S'(x) = b_0 + 2c_0(x) + 3d_0(x)^2$$

$$S''(x) = 2c_0 + 6d_0(x) = 0$$

Remember that we want $S''(x)$ to be zero, therefore $c_0 = 0$, $2c_0 + 6d_0(x) = 0$ in order for this to be true.

$$S''(0) = 2c_0 + 6d_0(0) = 0 \to d_0 = 0$$

$$S''(.5) = 2c_0 + 6d_0(.5) = 0 \to c_0 = 0$$

We have found c_0, d_0

$$S_0(.5) = (.5)b_0 + (.25)c_0 + (.1250)d_0 = 2.72 - 1$$

$$S_0(.5) = b_0 = \frac{2.72 - 1}{.5}$$

Therefore we have found all the necessary values.

$$S_0(x) = 1 + 3.5(x - 0) = 1 + 3.5x$$

3. (20pt, 5,15)

(a) What is the order of the error bound while using an (n+ 1)-point formula approximating $f'(x_0)$?

i. The erorr bound while using $n+1$ point formula is as follows
The order is $O(h^3)$

(b) Use the most accurate 3 points formula to determine each missing entry in the table below:

x	$f(x)$	$f'(x)$
7.1	1	35
7.2	3	5
7.3	2	- 25

We recall some formulas

$$f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h} - \frac{hf''(\xi)}{2}$$
$$f'(x_0) = \frac{f(x_0+h)-f(x_0-h)}{2h} - \frac{h^2 f'''(\xi)}{6}$$
$$f'(x_0) = \frac{3f(x_0)-4f(x_0+h)+f(x_0-2h)}{2h} - \frac{h^2 f^{(3)}(\xi)}{3}$$
$$f'(x_0) = \frac{-3f(x_0)+4f(x_0+h)-f(x_0+2h)}{2h} - \frac{h^2 f''(\xi)}{3}$$

The following formulas apply here.

i. Three-Point Endpoint Formula

$$\bullet f'(x_0) = \frac{1}{2h}[-3f(x_0)+4f(x_0+h)-f(x_0+2h)] + \frac{h^2}{3}f^{(3)}(\xi_0)$$

Where ξ_0 lies between x_0 and x_0+2h

ii. Three-Point Midpoint Formula

$$\bullet f'(x_0) = \frac{1}{2h}[f(x_0+h)-f(x_0-h)] - \frac{h^2}{6}f^{(3)}(\xi_1)$$

Where ξ_1 lies between x_0-h and x_0+h

$x = 7.1$

A. Consider the above formulas. We will ignore the error for the moment.

B. We are given 3 points, We will use more than one formula depending on the value, to calculate endpoints we can only use endpoint formulas, do not use midpoints.

C. Let $x_0 = 7.1, x_1 = 7.2, x_2 = 7.3$, x_0 is an endpoint, and we know that $h = 7.2-7.1 = .1$. Consider the endpoint formula.

$$f'(x_0) = \frac{-3f(x_0)+4f(x_0+h)-f(x_0+2h)}{2h} - \frac{h^2 f''(\xi)}{3}$$

Plug in h and x_0

$$f'(x_0) = \frac{-3f(7.1)+4f(7.1+.1)-f(7.1+2(.1))}{2(.1)} - \frac{(.1)^2 f''(\xi)}{3}$$
$$f'(x_0) = \frac{-3f(7.1)+4f(7.2)-f(7.3)}{.2} - \frac{(.1)^2 f''(\xi)}{3}$$

Now consider the given values of $f(x_0), f(x_1), f(x_1)$ given by the tables, as one can see this endpoint formula has all values defined.

$$f'(x_0) = \frac{-3(1)+4(3)-2}{.2} - \frac{(.1)^2 f''(\xi)}{3}$$
$$f'(x_0) = \frac{7}{.2} = 35$$

$x = 7.2$ A. As one can see $x = 7.2$ is not an endpoint but a midpoint, using an endpoint formula would not be reasonable as we would need $f(7.4)$ which is not given.

$$f'(x_0) = \frac{1}{2h}[f(x_0+h) - f(x_0-h)] - \frac{h^2}{6}f^{(3)}(\xi_1)$$

B. Once again determine our $h = 7.3 - 7.1 = .1$, it is equally spaced so it is not surprising that h is the same as previously.

C. We now compute. Ignore the error.

$$f'(x_0) = \frac{1}{2(.1)}[f(7.2+.1) - f(7.2-.1)] - \frac{h^2}{6}f^{(3)}(\xi_1)$$

$$f'(x_0) = \frac{1}{2(.1)}[f(7.2+.1) - f(7.2-.1)]$$

$$f'(7.2) = \frac{1}{2(.1)}[f(7.3) - f(7.1)]$$

$$f'(7.2) = \frac{1}{.2}[2-1] = 5$$

$x = 7.3$ A. $x = 7.3$ is an endpoint, we use the endpoint formula. Next $h = 7.2 - 7.3 = -.1$, take not of this.

B. Next we compute. Ignore the error.

$$f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)] + \frac{h^2}{3}f^{(3)}(\xi_0)$$

$$f'(7.3) = \frac{1}{2(-.1)}[-3f(7.3) + 4f(7.3+(-.1)) - f(7.3+2(-.1)]$$

$$f'(7.3) = \frac{1}{-.2}[-3(2) + 4(3) - (1)]$$

$$f'(7.3) = \frac{1}{-.2}[5] = -25$$

4. (20 pts, 5 each)

(a) Give the general expression of the Lagrange polynomials $L_{n,k}(x)$

$$L_{n,k}(x) = \prod_{i=0;j\neq k}^{n} \frac{x - x_i}{x_k - x_i}$$

(b) What is the name of the polynomial interpolant $P(x)$ of f such that $P(x_i) = f(xi), i = 0,...,n$ and $P'(x_i) = f'(x_i), i = 0,...,n$?
Hermite polynomials

(c) What kind of polynomials are used for Gaussian quadratures ?

Legendre polynomials

(d) Explain adaptive quadrature methods in no more than two lines.
We are given a tolerance, when the function behaves badly that is it is largely curved at many points we use more number of points in our method, when that is not the case we use less points.

5. (20pts, 16,4)Below is a MATLAB implementation of the trapezoid rule with some code missing.

(a)
```
function [I] = trapezoid_rule(f,a,b,N)
%function to approximate $\int_a^b f(x)dx$ using the trapezoid rule
%INPUTS %f is the function at hand
%a is the lower bound of the interval
%b is the upper bound of the interval
% N is the number of panels used
%OUTPUTS:
% I is the approximate integral
    I = 0;
    h = (b - a)/N;
    c = a:h:b;
    for j = 0: N
        X_j = a + j*h;
        X_jp1 = (h/2)*(f(c(j)) + f(c(j + 1)));
        I = I + x_jp1;
    end
end
```

(b) Explain the differences between the Trapezoid rule and the Simpson's rule (number of points needed,order of the error bound, etc.).

derivation
- A. Trapezoid consists of the summation of trapezoid, that is there is a rectangle and a triangle, a slanted line.
- B. Simpson's rule uses a rectangle and a parabola, making it slightly more accurate.

order
- A. Rate of Trapezoid Rule: Standard Trapezoid $O(h^3)$, Composite $O(h^2)$
- B. Rate of Simpsons Rule: Standard Simpsons $O(h^5)$, Composite $O(h^4)$

error bound
- A. Composite Simpson's rule error $\frac{b-a}{180}h^4 f^4(\mu)$