Kähler Geometry
Project Report
MTH 393A

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Declaration

I hereby declare that the work presented in the project report entitled **"Kähler Geometry"** contains my own ideas in my own words. At places where ideas and words are borrowed from other sources, proper references, as applicable have been cited. To the best of my knowledge this work does not emanate from or resemble other work created by person(s) other than mentioned herein.

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Abstract

Beginning with some sheaf theoretic preliminaries, we endow topological manifolds with complex structures in two different ways: by equipping it with an equivalence class of holomorphic atlases and by equipping the real tangent bundle with an endomorphism, called an almost complex structure. A comparison between smooth manifolds and complex manifolds shows that complex manifolds are in a sense, "rigid". In an attempt to classify compact complex manifolds, certain invariants are defined, namely the algebraic dimension and the kodaira dimension. In order to investigate the geometry of complex manifolds, holomorphic vector bundles are defined. The holomorphic line bundles on a complex manifold X turn out to be parametrized by the cohomology group $H^1(X, \mathcal{O}_X^*)$. Equipping complex manifolds with a riemannian metric induces hermitian structures. Connections are defined on complex vector bundles, hermitian bundles and holomorphic bundles following which Kähler manifolds are seen as sites where Riemannian geometry and Complex geometry are compatible. Finally a curvature form on complex vector bundles with connection is defined. With results due to Demailly J.P, Siu Y.T. and Moishezon B.G, the algebraic dimension and complex dimension of a complex manifold are related to Kähler structures and existence of almost positive line bundles.

Contents

1 Preliminaries

1.1 Sheaves and Smooth Manifolds

What does it mean to do Geometry? Where can one one do Geometry? What are the ways in which one can do Geometry?

Such questions have fascinated mathematicians and physicists alike for centuries. As it turns out, key to trying to answer such questions are the following observations:

1. Classes of maps between topological spaces encode interesting information, and that

2. Such classes of maps have some properties in common: all constituent elements are continuous and for a continuous map to belong to a class, it must satisfy a condition that is of a local nature.

While **2** is evident, in support of **1**, consider the space $\mathcal{B}U(1) = \mathbf{CP}^\infty$ which is a model of the Eilenberg-Maclane space $K(\mathbb{Z}, 2)$. Homotopy classes of continuous maps from any other space X into $\mathcal{B}U(1)$ represent isomorphism classes of circle bundles on X. Moreover, the set of homotopy classes of maps, $[X, \mathcal{B}U(1)]$ comes with the structure of an abelian group and is isomorphic to the second integral cohomology group of X, $H^2(X, \mathbb{Z})$. Thus, we have:

$$[X, \mathcal{B}U(1)] \cong \{\text{circle bundles on X}\}/\sim \cong H^2(X, \mathbb{Z})$$

Similar such cohomology groups, in particular, $H^{n+1}(X, \mathbb{Z})$ classify certain other geometric structures on X. So, such cohomology groups contain a lot of interesting information about a topological space, yet there is some useful information that they fail to capture. This happens, for instance, when our space of interest X, carries extra structure, say, smooth structure. Then one is now naturally interested in smooth circle bundles on X, which we require to be equipped with connection ∇. There is no smooth manifold M such that $[X, M] \cong \{\text{smooth circle bundles on X with connection}\}/\sim$. The notion of a sheaf is the appropriate generalization of the notion of a space that does fulfill this. We begin by axiomatising **1** and **2** above.

Definition 1.1. Let X be a topological space. An assignment to every open subset U of X, of a set $\mathcal{F}(U)$ and to every pair of open sets U, V with $V \subset U$, of a map, to be called a restriction map, $\mathrm{res}_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ satisfying:

$$\mathrm{res}_{VW} \circ \mathrm{res}_{UV} = \mathrm{res}_{UW}$$

for every triple $W \subset V \subset U$ of open sets, is called a **presheaf** of sets. More generally, a $\mathcal{C}$-valued presheaf is a functor $\mathcal{F} : \mathrm{Op}(X)^{op} \to \mathcal{C}$, where $\mathrm{Op}(X)$ is the poset of open subsets of X and $\mathcal{C}$ is a category of choice.

Definition 1.2. A presheaf $\mathcal{F} : \mathrm{Op}(X)^{op} \to \mathcal{C}$ is said to be **sheaf** if it satisfies the following additional conditions. Let $U = \bigcup_{i \in I} U_i$ be any open covering of an open set U. Then:

$\mathbf{S}_1$: Two elements $s, t \in \mathcal{F}(U)$ are equal if $\mathrm{res}_{UU_i}(s) = \mathrm{res}_{UU_i}(t)$ for all $i \in I$.
$\mathbf{S}_2$: If $s_i \in \mathcal{F}(U_i)$ satisfy $\mathrm{res}_{U_i U_i \cap U_j}(s_i) = \mathrm{res}_{U_j U_i \cap U_j}(s_j)$ for all $i, j \in I$, then there exists an element $s \in \mathcal{F}(U)$ with $\mathrm{res}_{UU_i}(s) = s_i$ for all $i \in I$. We assume that $\mathcal{F}(\emptyset)$ consists of a single point.

A presheaf on a topological space allows one to naturally localise relevant objects (elements of the value of a presheaf on some open subset of the space). Storing this local data allows one to associate a sheaf to every presheaf. This association in fact shows that every sheaf arises in a canonical way.

Let $\mathcal{F}$ be a presheaf on a topological space X. Then the **stalk** $\mathcal{F}_x$ of $\mathcal{F}$ at a point $x \in X$ is the quotient set:

$$\{(U, s) | x \in U, U \in \mathrm{Op}(X), s \in \mathcal{F}(U)\}/\sim$$

where $\sim$ is an equivalence relation defined as: $(U, s) \sim (V, t)$ if and only if there exists a neighbourhood W of x contained in $U \cap V$ such that $\mathrm{res}_{UW}(s) = \mathrm{res}_{VW}(t)$. For any $x \in X$ and a neighbourhood U of x, every element $s \in \mathcal{F}(U)$ has an image in $\mathcal{F}_x$ under the canonical projection onto $\mathcal{F}_x$, which we call the **germ** of s at x, denoted s_x.

Let $E(\mathcal{F}) = \coprod_{x \in X} \mathcal{F}_x$, that is the disjoint union of all stalks. Then $E(\mathcal{F})$ can be equipped with a natural topology. This is the coarsest topology on $E(\mathcal{F})$ such that:

- The map $\pi : E(\mathcal{F}) \to X$ which maps all of $\mathcal{F}_x$ to x is continuous and
- If $s \in \mathcal{F}(U)$ then the section $\tilde{s} : U \to E(\mathcal{F})$ defined by setting $\tilde{s}(x) = s_x$ is continuous.

$E(\mathcal{F})$ is called the **étale space** and the topology described above is called the **étale topology** on $E(\mathcal{F})$. The sheaf of sections of $\pi : E(\mathcal{F}) \to X$ is called the **sheaf associated to the presheaf** $\mathcal{F}$.

Definition 1.3. If $\mathcal{F}_1, \mathcal{F}_2$ are presheaves on a topological space X, then a **homomorphism** $f : \mathcal{F}_1 \to \mathcal{F}_2$ is an association to each open subset U of X, a homomorphism $f(U) : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}_1(U) \arrow[r, "f(U)"] \arrow[d, "res_{UV}^{\mathcal{F}_1}"'] & \mathcal{F}_2(U) \arrow[d, "res_{UV}^{\mathcal{F}_2}"] \\
\mathcal{F}_1(V) \arrow[r, "f(V)"] & \mathcal{F}_2(V)
\end{tikzcd}$$

For any presheaf $\mathcal{F}$ on a topological space X, there is a canonical homomorphism between $\mathcal{F}$ and the sheaf associated to $\mathcal{F}$, $\tilde{\mathcal{F}}$, defined by the homomorphisms $\mathcal{F}(U) \to \tilde{\mathcal{F}}(U)$ mapping each element $s \in \mathcal{F}(U)$ to the section $\tilde{s}$. Whether $\mathcal{F}$ is a sheaf or not can be characterised using this natural homomorphism:

Theorem 1.1.

1. *A presheaf $\mathcal{F}$ satisfies $\boldsymbol{S}_1$ if and only if the induced map $\mathcal{F}(U) \to \tilde{\mathcal{F}}(U)$ is injective, for every open subset U of X.*
2. *A presheaf $\mathcal{F}$ is a sheaf if and only if the natural map $\mathcal{F}(U) \to \tilde{\mathcal{F}}(U)$ is an isomorphism, for every open subset U of X.*

The relationship between the $\mathbf{S}_2$ axiom and the homomorphism $\mathcal{F} \to \tilde{\mathcal{F}}$ is more subtle. We recall that a topological space X is said to be a **shrinking** space if every open cover of X admits a shrinking.

Definition 1.4. Let $\mathcal{U}$ be an open cover of a topological space X. Then $\mathcal{U}$ is said to be **shrinkable** if there is an open cover $\mathcal{V} = \{V_U | U \in \mathcal{U}\}$ such that $\bar{V}_U \subset U$ for each $U \in \mathcal{U}$. In this case, $\mathcal{V}$ is said to be a shrinking of $\mathcal{U}$.

Theorem 1.2. ***Paracompact spaces are shrinking*** *A topological space X is paracompact if and only if every open cover $\mathcal{U}$ of X has a locally finite refinement that is also a shrinking of $\mathcal{U}$.*

Using Theorem 1.2, one can show that:

Theorem 1.3. *If every open subset U of X is paracompact and $\mathcal{F}$ is a presheaf on X that satisfies the $\boldsymbol{S}_2$ axiom, then the induced map $\mathcal{F}(U) \to \tilde{\mathcal{F}}(U)$ is surjective, for every open subset U of X.*

A sheaf $\mathcal{G}$ is said to be a **subsheaf** of a sheaf $\mathcal{F}$ if there is a homomorphism $f : \mathcal{G} \to \mathcal{F}$ such that the homomorphisms $f(U) : \mathcal{G}(U) \to \mathcal{F}(U)$ are all injective. This is in fact equivalent to requiring the induced **stalk maps** $f_x : \mathcal{G}_x \to \mathcal{F}_x$ to be injective for every $x \in X$.

1.1.1 Natural Operations on Sheaves

Let X and Y be topological spaces and $f : X \to Y$ be a continuous map between them. If Y is equipped with a sheaf $\mathcal{F}$, we can naturally pull this back so as to equip X with a sheaf too. More precisely, consider the following pullback diagram:

$$\begin{tikzcd} f^*(E(\mathcal{F})) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & E(\mathcal{F}) \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

The sheaf sections of $\pi_2 : f^*(E(\mathcal{F})) \to X$ is called the **inverse image** of $\mathcal{F}$ by f. When X happens to be a subspace of Y we call the inverse image of $\mathcal{F}$ under the inclusion, the restriction of $\mathcal{F}$ to X and denote it as $\mathcal{F}\big|_X$.

Consider the case when we have sheaves $\{\mathcal{F}_i\}_{i\in I}$ on an open covering $\{U_i\}_{i\in I}$ of a topological space X. We wish to glue these sheaves together into a sheaf $\mathcal{F}$ on X such that the restriction of $\mathcal{F}$ to each U_I gives back $\mathcal{F}_i$.

Theorem 1.4. *Let $\{U_i\}_{i\in I}$ be an open covering of a topological space X and let $\mathcal{F}_i$ be a sheaf on U_i for each $i \in I$, such that:*

- *For every $i, j \in I$ there is an isomorphism $m_{ij} : \mathcal{F}_i\big|_{U_i\cap U_j} \to \mathcal{F}_j\big|_{U_i\cap U_j}$ such that*
- *For every $i, j, k \in I$, $m_{jk} \circ m_{ij} = m_{ik}$ when restricted to $U_i \cap U_j \cap U_k$.*

Then, there is a unique sheaf $\mathcal{F}$ on X such that $\mathcal{F}\big|_{U_i}$ is isomorphic to $\mathcal{F}_i$ for every $i \in I$.

We now have a situation complementary to the one when we were pulling back sheaf structure. Say $f : X \to Y$ is a continuous map and X is equipped with a sheaf $\mathcal{F}$. In this case one can push the sheaf structure onto Y. We define a sheaf on Y called the **direct image** of $\mathcal{F}$ by f, by associating to every open subset U of X, the object $\mathcal{F}(f^{-1}(U))$, denoting this by $f_*(\mathcal{F})$. As one might expect, inverse image and direct image functors, induced by a continuous map $f : X \to Y$ are adjoint to each other.

Let $\mathcal{A}$ be a sheaf of rings over a topological space X and $\mathcal{M}$ be a sheaf of abelian groups. We say that $\mathcal{M}$ is an $\mathcal{A}-$module if for every open subset U of X, $\mathcal{M}(U)$ is an $\mathcal{A}(U)$-module and the restriction maps respect the module structures, i.e., whenever we have open sets V, U such that $V \subset U$, $\mathrm{res}_{UV}(f \cdot s) = \mathrm{res}_{UV}(f) \cdot \mathrm{res}_{UV}(s)$, for all $f \in \mathcal{A}(U)$ and $s \in \mathcal{M}(U)$. An $\mathcal{A}$-homomorphism of an $\mathcal{A}-$module $\mathcal{M}_1$ into another $\mathcal{A}-$module $\mathcal{M}_2$, is a homomorphism of sheaves $f : \mathcal{M}_1 \to \mathcal{M}_2$ such that for every open subset U of X, the homomorphism $f(U) : \mathcal{M}_1(U) \to \mathcal{M}_2(U)$ is $\mathcal{A}(U)$-linear.

1.1.2 Locally Free Sheaves and Vector Bundles

Given the framework we have developed so far, a **smooth manifold of dimension n** M is a Hausdorff, second countable topological space equipped with a subsheaf $\mathcal{A}^M$ of the sheaf of continuous $\mathbb{R}$-valued functions on M, such that for any $x \in M$ there is a neighbourhood U of x in M such and a homeomorphism φ of U with an open subset V of $\mathbb{R}^n$ such that $\mathcal{A}^M\big|_U = \varphi^*(C^\infty\big|_V)$, where C^∞ is the sheaf of smooth functions on $\mathbb{R}^n$.

Definition 1.5. A **homogeneous first order differential operator** on an smooth manifold $(M, \mathcal{A})$ is a sheaf homomorphism $\mathcal{A} \to \mathcal{A}$ which satisfies the Leibniz rule (that is, it is a sheaf derivation).

For every open subset U of M, let $\mathcal{T}(U)$ denote the set of all homogeneous first order differential operators on U. Then, each $\mathcal{T}(U)$ is a $\mathbb{R}-$vector space and an $\mathcal{A}(U)-$module. Further, defining $[,] : \mathcal{T}(U) \times \mathcal{T}(U) \to \mathcal{T}(U)$ by setting $[D_1, D_2](f) = D_1(D_2(f)) - D_2(D_1(f))$, we observe that $\mathcal{T}(U)$ is an $\mathbb{R}-$Lie algebra. This gives rise to a sheaf of Lie alebras on M, called the **tangent sheaf** on M.

There are other canonical $\mathcal{A}-$modules on a smooth manifold $(M, \mathcal{A})$.

- The direct sum $\mathcal{A}^r$ of $\mathcal{A}$ with itself r times is an $\mathcal{A}-$module.
- If Z is a closed subset of M, then the subsheaf of $\mathcal{A}$ given by associating, with every open subset U of M, the set $\mathcal{I}_Z(U)$ of all elements of $\mathcal{A}(U)$ that vanish on $Z \cap U$, is called the **ideal sheaf** $\mathcal{I}_Z$ of Z on M. The name comes from the fact that $\mathcal{I}_Z(U)$ is an ideal of the ring $\mathcal{A}(U)$ for every open subset U of X. $\mathcal{I}_Z$ is also an $\mathcal{A}-$module.

Definition 1.6. Let $\mathcal{A}$ be a sheaf of rings on a topological space X. An $\mathcal{A}-$module $\mathcal{M}$ is said to be **locally free of rank r** if every $x \in X$ has a neighbourhood U such that $\mathcal{M}\big|_U$ is isomorphic to $\mathcal{A}^r\big|_U$ as an $\mathcal{A}(U)-$ module.

For instance, the tangent sheaf $\mathcal{T}$ is a locally free $\mathcal{A}-$ module of rank dim(M). If Z is closed submanifold of M co-dimension r, then the ideal sheaf $\mathcal{I}_Z$ is a locally free $\mathcal{A}-$module of rank r.

Let $(M, \mathcal{A})$ be a smooth manifold and let $\mathcal{E}$ be a sheaf of abelian groups on M such that $\mathcal{E}$ is an $\mathcal{A}$-module. We are interested in understanding the structure of $\mathcal{E}$ when it is locally free. The simplest example of such an $\mathcal{A}$-module is $\mathcal{A}$ itself. Let $\mathcal{M}_x = \{f_x \in \mathcal{A}_x | f(x) = 0\}$. $\mathcal{A}_x$ and $\mathcal{M}_x$ are $\mathbb{R}$-algebras and the latter is an ideal of the former. In fact, we have a natural map $ev : \mathcal{A}_x \to \mathbb{R}$ defined by setting $ev(f_x) = f(x)$. ev is surjective linear map and $ker(ev) = \mathcal{M}_x$. Therefore, the quotient $\mathcal{A}_x/\mathcal{M}_x$ is isomorphic to $\mathbb{R}$.

The next simplest $\mathcal{A}$-module is $\mathcal{A}^r$. Consider the map $ev : \mathcal{A}_x^r \to \mathbb{R}^r$ defined as $ev(f_{1x}, \cdots, f_{rx}) = (f_1(x), \cdots, f_r(x))$. Here, $ker(ev) = \mathcal{M}_x\mathcal{A}_x^r$. So, $\mathcal{A}_x^r/\mathcal{M}_x\mathcal{A}_x^r \cong \mathbb{R}^r$. Therefore, for $\mathcal{E}$, we have $E_x = \mathcal{E}_x/\mathcal{M}_x\mathcal{E}_x \cong_\varphi \mathcal{A}_x^r/\mathcal{M}_x\mathcal{A}_x^r \cong \mathbb{R}^r$. But unlike the simpler cases, E_x is not canonically isomorphic to $\mathbb{R}^r$ as indicated by the subscript φ which denotes the isomorphism of $\mathcal{A}(U)$-modules between $\mathcal{E}\big|_U$ and $\mathcal{A}^r\big|_U$. Let $E = \bigcup_{x\in M} E_x$. This comes with a natural projection map $\pi : E \to M$ mapping all of (and exactly that) E_x to x. One may now observe the following:

1. E is a smooth manifold of dimension $n + r$.
2. $\pi : E \to M$ is a smooth map.
3. For every $x \in M$, there exists a neighbourhood U of x and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that $\pi \circ \varphi^{-1} : U \times \mathbb{R}^r \to U$ is the standard projection.
4. $\pi^{-1}(x)$ has a vector space structure for each $x \in M$ and $\varphi\big|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \mathbb{R}^r$ is an isomorphism.

Definition 1.7. A smooth map $\pi : E \to M$ satisfying (1)-(4) above is called a **smooth vector bundle of rank r**.

Definition 1.8. A **homomorphism** of a vector bundle E into another vector bundle F is a smooth map $f : E \to F$ such that the following diagram commutes and the induced maps on the fibres, $f\big|_{\pi_1^{-1}(x)} : E_x \to F_x$ are linear.

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & F \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd}$$

Theorem 1.5. *Let $(M, \mathcal{A})$ be a smooth manifold. Then we have a categorical equivalence:*

$$\begin{tikzcd} LocFreeSh(M) \arrow[rr, bend left, "Vec(-)"] & & VecBund(M) \arrow[ll, bend left, "Sh(-)"] \end{tikzcd}$$

where, $Vec(-)$ is the functor described previously that associates a vector bundle to any locally free $\mathcal{A}$-module, while $Sh(-)$ is the sheaf of sections functor.

We end with a question:

Q1 Let $\mathcal{A}$ be a sheaf of rings over a topological space X and $\mathcal{M}$ be a sheaf of abelian groups. We say that $\mathcal{M}$ is a Lie algebra over $\mathcal{A}$ if for every open subset U of X, $\mathcal{M}(U)$ has a bracket $[,]_U$ such that it is a Lie algebra with respect to $\mathcal{A}(U)$ and the restriction maps respect the module and Lie-algebra structures, i.e., in addition, whenever we have open sets V, U such that $V \subset U$, $\mathrm{res}_{UV}(fX) = \mathrm{res}_{UV}(f)\mathrm{res}_{UV}(X)$ and $\mathrm{res}_{UV}([fX, gY]) = [\mathrm{res}_{UV}(f)\mathrm{res}_{UV}X, \mathrm{res}_{UV}(g)\mathrm{res}_{UV}(Y)]$ for all $f, g \in \mathcal{A}(U)$ and $X, Y \in \mathcal{M}(U)$.Clearly then, for any $\mathcal{A}$-module $\mathcal{E}$, $Hom(\mathcal{E}, \mathcal{E})$ is a Lie-algebra over $\mathcal{A}$. How does one classify such Lie algebras?

1.2 Complex Manifolds and their Geometry

A **complex manifold** X, of dimension n, is a real $2n$ dimensional manifold equipped with a complex structure, that is, an equivalence class of holomorphic atlases on X (which is uniquely represented by its maximal element).

Definition 1.9. A **holomorphic atlas** on a smooth manifold is an atlas $\{U_i, \varphi_i\}$ of the form $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{C}^n$ such that the transition maps $\varphi_{ij} := \varphi_i \circ \varphi_j\big|^{-1}_{\varphi_j(U_i \cap U_j)}$: $\varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ are holomorphic for all i, j. Two holomorphic atlases $\{U_i, \varphi_i\}$ and $\{V_j, \psi_j\}$ are said to be equivalent if for all i, j the maps $\varphi_i \circ \psi_j\big|^{-1}_{\psi_j(U_i \cap V_j)}$: $\psi_j(U_i \cap V_j) \to \varphi_i(U_i \cap V_j)$ are holomorphic.

For a complex manifold X, we denote by $\mathcal{O}_X$, the sheaf of holomorphic functions on X. For any point $x \in X$, we have that the stalk $\mathcal{O}_{X,x}$ is isomorphic to $\mathcal{O}_{\mathbb{C}^n,0}$, where the (stalk) isomorphism is induced by a holomorphic chart centered at x. We denote the quotient field of $\mathcal{O}_{X,x}$ by $Q(\mathcal{O}_{X,x})$. A typical element of $Q(\mathcal{O}_{X,x})$ is of the form g_x/h_x where $g, h \in \mathcal{O}_X(U)$, U being a connected neighbourhood of x with $h \not\equiv 0$. We begin by noting some global differences between smooth and complex manifolds.

Theorem 1.6. *Let X be a compact, connected, complex manifold. Then, any global holomorphic function on X is constant, so that, $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$.*

Existence of partitions of unity on smooth manifolds allows us to construct plenty of non-constant smooth functions on any smooth manifold. In fact, more is true: one can recover X as a smooth manifold from the ring $C^\infty(X)$. Hartog's theorem allows us to weaken the compactness condition:

Theorem 1.7. *Let X be a compact manifold of dimension atleast 2 and let $x \in X$. Then, $\Gamma(X - \{x\}, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$. If X is in addition compact and connected (so that X-$\{x\}$ is not compact), $\Gamma(X - \{x\}, \mathcal{O}_X) = \mathbb{C}$.*

At this point, one could ask: "Does one even have examples of compact complex manifolds?"". We shall answer this question shortly. A broader class of functions on a complex manifold are meromorphic functions. We recall that for 2-dimensional complex manifolds, that is, Riemann surfaces, one has:

Definition 1.10. Let X be a Riemann surface and U be an open subset of X. A meromorphic function $f : U \to \mathbb{C}$ is a holomorphic function $f : U' \to \mathbb{C}$ such that:

- $U - U'$ contains only isolated points.
- For every point $x_0 \in U - U'$, $\lim_{x \to x_0} |f(x)| = \infty$.

In particular, the Identity theorem also holds for meromorphic functions on Riemann surfaces, so that any meromorphic function that is not identicall zero, has only isolated singularities, thus turning the set of all meromorphic functions into a field. Generalizing this, we define:

Definition 1.11. A **meromorphic function** on a complex manifold X is a map $f : X \to \coprod_{x \in X} Q(\mathcal{O}_{X,x})$ which associates to every $x \in X$, an element $f(x) \in Q(\mathcal{O}_{X,x})$ such that for any $x_0 \in X$ there is a neighbourhood U of x_0 and two holomorphic functions $g, h : U \to \mathbb{C}$ such that $f(x) = g/h$ for all $x \in U$. We denote the **sheaf of meromorphic functions** on X by $\mathcal{K}_X$. In particular, its space of global sections is denoted by $K(X)$.

Similarly, since the Identity theorem also holds for complex manifolds of arbitrary dimension, when X is connected, $K(X)$ is a field, which we call the **function field** of X. There is a natural inclusion $\mathcal{O}_X \subset \mathcal{K}_X$ giving a field extension, for which, it is natural to try and estimate the size of $K(X)$ by its transcendence degree. However, in general, this will only give a coarse size, as we comment later.

Theorem 1.8. ***(Siegel)*** *Let X be a compact, connected, complex manifold of dimension n. Then, $trdeg_{\mathbb{C}}K(X) \leq n$.*

Definition 1.12. The **algebraic dimension** of a compact, connected, complex manifold is $a(X) := trdeg_{\mathbb{C}}K(X)$.

Definition 1.13. Let X be a complex manifold of complex dimension n and $Y \subset X$ be a real submanifold of dimension $2k$. Then Y is a **complex submanifold** if there exists a holomorphic atlas $\{U_i, \varphi_i\}$(chosen from the complex structure) of X such that $\varphi_i : U_i \cap Y \cong \varphi(U_i) \cap \mathbb{C}^k$ where $\mathbb{C}^k$ is considered under the canonical immersion into $\mathbb{C}^n$.

In particular, X is said to be **projective** if it is isomorphic to a closed submanifold of some projective space $\mathbb{CP}^n$. In practice, it frequently happens that submanifolds become *singular*. So it is easy to construct submanifolds through **analytic subvarieties** as we define below:

Definition 1.14. Let X be a complex manifold. An analytic subvariety of X is a closed subset Y of X, such that for any point $x \in X$ there exists a neighbourhood U of x in X such that $Y \cap U$ is the zero set of finitely many holomorphic functions $f_1, \cdots, f_k \in \mathcal{O}(U)$.

Zero sets of polynomial on $\mathbb{C}^n$ give plenty of examples. For an analytic subvariety Y of X, a point $x \in Y$ is a smooth or regular point of Y if the functions $f_1, \cdots, f_k$ can be chosen such that $\varphi(x)\varphi(U)$ is a regular point the map $f := (f_1 \circ \varphi^{-1}, \cdots, f_k \circ \varphi^{-1}) : \varphi(U) \to \mathbb{C}^k$, where (U, φ) is a local chart around x. Results from differential topology such as Morse's lemma, can be used to determine local structure around (degenerate) critical (i.e. non-regular) points.

1.2.1 Holomorphic Vector Bundles

A **holomorphic vector bundle** of rank r on a complex manifold X is a complex manifold E together with a holomorphic map $\pi : E \to X$ and the structure of an r-dimensional complex vector space on every fibre $E_x = \pi^{-1}(x)$ satisfying the following condition: There exists an open covering $X = \bigcup U_i$ and biholomorphic maps $\psi_i : \pi^{-1}(U_i) \cong U_i \times \mathbb{C}^r$ commuting with the projections to U_i such that the induced map $\pi^{-1}(x) \cong \mathbb{C}^r$ is $\mathbb{C}$-linear. A holomorphic vector bundle of rank one is called a **holomorphic line bundle**. We observe that for the $\{\psi_i\}$ above, the transition functions $\psi_{ij} := (\psi_i \circ \psi_j^{-1})(x, -) : \mathbb{C}^r \to \mathbb{C}^r$ are all $\mathbb{C}$-linear.

Pull-backs, subbundles, homomorphisms of vector bundles may be defined analogously. A key observation here is that: A holomorphic rank r vector bundle $\pi : E \to X$ is totally determined by the holomorphic cocycle $\{(U_i, \psi_{ij} : U_i \cap U_j \to Gl(r, \mathbb{C}))\}$. We next turn to some explicit constructions. On $\mathbb{CP}^n$ for instance, there is essentially only one line bundle, called the **tautological line bundle** $\mathcal{O}(1)$. Its dual can be described as follows:

Lemma 1.1. *The set $\mathcal{O}(-1) \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$ that consists of all pairs $(l, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}$ with $z \in l$ forms in a natural way, a holomorphic line bundle over $\mathbb{CP}^n$.*

Remark: The line bundle $\mathcal{O}(1)$ is the dual $\mathcal{O}(-1)^*$. For $k > 0$ let $\mathcal{O}(k)$ be the line bundle $\mathcal{O}(1)^{\otimes k}$. Analogously for $k < 0$ one defines $\mathcal{O}(k) = \mathcal{O}(-k)^*$. If E is any vector bundle on $\mathbb{CP}^n$, we denote $E \otimes \mathcal{O}(k)$ by $E(k)$.

If we let $\mathcal{O}(0)$ denote the trivial bundle then we see that the set of line bundles on $\mathbb{CP}^n$ forms a group isomorphic to $\mathbb{Z}$. This is justified by (2) below.

Lemma 1.2.

1. *The tensor product and the dual endow the set of all isomorphism classes of holomorphic line bundles ona complex manifold X with the structure of an abelian group, called the **Picard group** Pic(X) of X.*

2. *There is a natural isomorphism $Pic(X) \cong H^1(X, \mathcal{O}_X^*)$.*

In particular, the tautological line bundle $\mathcal{O}(1)$ corresponds to the generator in $H^1(\mathbb{CP}^n, \mathcal{O}^*_{\mathbb{CP}^n}) \cong \mathbb{Z}$. Any holomorphic map $f : Y \to X$ induces a group homomorphism $f^* : Pic(X) \to Pic(Y)$, the pull-back map. We have the following natural question

Q2: The above remarks give a natural group homomorphism $A : Aut(X) \to Aut(Pic(X))$. Can we study Aut(X) using this map? In particular, one would wish to classify automorphsims of Riemann surfaces by looking at their images under A.

Now, $H^1(X, \mathcal{O}_X^*)$ isn't readily computable and doesn't immediately answer if there exists a non-trivial line bundle on X. So we use the following exact sequence:

Definition 1.15. The **exponential sequence** on a complex manifold X is the short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

where $\mathbb{Z}$ is the locally constant sheaf and $\mathbb{Z} \to \mathcal{O}_X$ is the natural inclusion. The map $\mathcal{O}_X \to \mathcal{O}_X^*$ is given by the exponential $f \mapsto exp(2\pi i \cdot f)$.

The exponential sequence induces the long exact cohomology sequence:

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$$

which enables, in principle, a computation of Pic(X).

Definition 1.16. The **first Chern class** $c_1(L)$ of a holomorphic line bundle $L \in Pic(X)$ is its image under the boundary map

$$c_1 : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$$

Theorem 1.5 may be readily extended to:

Theorem 1.9. *Associating to a holomorphic vector bundle, its sheaf of sections, defines a canonical bijection between the set of holomorphic vector bundles of rank r and the set of locally free $\mathcal{O}_X$-modules of rank r on X.*

Therefore, for any holomorphic vector bundle on a complex manifold X, we may define $H^q(X, E)$ to be the q-th cohomology of its sheaf of sections.

Lemma 1.3. *For every line bundle L on a complex manifold X, the space:*

$$R(X, L) := \bigoplus_{m>=0} H^0(X, L^{\otimes m})$$

has a natural ring structure. Here, $L^{\otimes 0} = \mathcal{O}_X$. For $L = K_X$, the determinant line bundle, we denote $R(X, L)$ by $R(X)$.

Again, as discussed previously, if X is connected, $R(X)$ is an integral domain.

Definition 1.17. The **kodaira dimension** of a connected complex manifold is denoted by kod(X), is defined to be $-\infty$ when $R(X) = \mathbb{C}$, and $trdeg_{\mathbb{C}}(Q(R(X))) - 1$ otherwise.

1.2.2 Kähler manifolds, Connections and Curvature

Definition 1.18. An **almost complex manifold** is a smooth manifold X equipped with a vector bundle endomorphism $I : TX \to TX$ such that $I \circ I = -id$. I is called an almost complex structure on X.

We note the following:

- An complex manifold X admits a natural almost complex structure.
- Denoting the complexified tangent bundle of X by $T_{\mathbb{C}}X$, we have a direct sum decomposition $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$ which respectively are the eigenspaces of the $\mathbb{C}$-linear extension of I corresponding to values i and $-i$.
- One defines the complex vector bundles $\bigwedge_{\mathbb{C}}^k X := \bigwedge^k (T_{\mathbb{C}}X)^*$ and $\bigwedge_{\mathbb{C}}^{p,q} X := \bigwedge^p (T^{1,0}X)^* \otimes_{\mathbb{C}} \bigwedge^q (T^{0,1}X)^*$. Their sheaves of sections are denoted by $\mathcal{A}^k_{X,\mathbb{C}}$ and $\mathcal{A}^{p,q}_X$ respectively.

Definition 1.19. Let X be a complex manifold and I be the induced almost complex structure. A Riemannian metric g on X is an **hermitian structure** on X if for any point $x \in X$ the scalar product g_x on T_xX is compatible with the almost complex structure I_x. The induced real (1,1) form $\omega := g(I(-), (-))$ is called the **fundamental form**.

Locally the fundamental form is of the form:

$$\omega = (i/2) \sum_{i,j=1}^n h_{ij} dz_i \wedge dz_j$$

where for any $x \in X$ the matrix $(h_{ij}(x))$ is a positive definite hermitian matrix. We call a complex manifold X endowed with an hermitian structure g as a **hermitian manifold**. In this case, any two elements in the triple (g, I, ω) determine the third.

Definition 1.20. A **Kähler structure** is a hermitian structure g for which the fundamental form ω is closed, i.e. $d\omega = 0$. In this case, ω is called a Kähler form on X.

Definition 1.21. Let E be a complex vector bundle on a real manifold M. An hermitian structure h on $E \to M$ is a hermitian scalar product h_x on each fibre E_x which depends smoothly on x. The pair (E, h) is called a **hermitian vector bundle**.

A **connection** on a complex vector bundle E is a $\mathbb{C}$-linear sheaf homomorphism $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ which satisifies the Leibniz rule:

$$\nabla(f \cdot s) = d(f) \otimes s + f \cdot \nabla(s)$$

for any local function f on M and any local section s of E. A section s of E is called parallel with respect to ∇ on E if $\nabla(s) = 0$.

Let (E, h) be a hermitian vector bundle. A connection ∇ on E is said to be a **hermitian connection** with respect to h if for arbitrary local sections s_1, s_2, one has:

$$d(h(s_1, s_2)) = h(\nabla(s_1), s_2) + h(s_1, \nabla s_2).$$

The case when we have a holomorphic vector bundle E on X is of special interest. In this case, a connection ∇ on E is said to be compatible with the holomorphic structure if $\nabla^{0,1} = \bar{\partial}$.

Theorem 1.10. *Let (E, h) be a holomorphic vector bundle together with a hermitian structure. Then there exists a unique hermitian connection ∇ that is compatible with the holomorphic structure. This connection is called the* ***Chern connection*** *on (E, h).*

The following relates the Levi-Civita connection (which is uniquely defined once a Riemannian metric has been given) and the Chern connection.

Theorem 1.11. *Let (X, g) be a Kähler manifold. Then under the isomorphism $\xi : TX \cong T^{1,0}X$, the Chern connnection ∇ on the holomorphic tangent space $T^{1,0}X$ corresponds to the Levi-Civita connection D.*

If E is a vector bundle on a manifold M endowed with a connection $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$, then we may naturally extend it to Leibniz map $\nabla : \mathcal{A}^k(E) \to \mathcal{A}^{k+1}(E)$ as:

$$\nabla(\alpha \otimes s) = d(\alpha) \otimes s + (-1)^k \alpha \wedge \nabla(s)$$

where α is a local k-form on M and s is a local section of E. We define the **curvature** F_∇ of a connection ∇ on a vector bundle E as the composition:

$$F_\nabla := \nabla \circ \nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E) \to \mathcal{A}^2(E)$$

We conclude by stating two results that relate the algebraic dimension of a complex manifold and its complex dimension.

Definition 1.22. A real (1,1)-form α is called (semi)-positive if for all non-zero holomorphic tangent vector $v \in T^{1,0}X$, one has $-i\alpha(v, \bar{v}) > 0$ (resp.$\geq$0).

Definition 1.23. A compact, connected, complex manifold X is said to be a **Moishezon space** if $a(X) = dim_{\mathbb{C}}(X)$.

Theorem 1.12. *(**[Moishe]**) A Moishezon space is a projective algebraic variety if and only if it admits a Kähler metric.*

We define a line bundle L to be **almost positive** if it carries a hermitian metric whose associated canonical (Chern) connection has semi-positive curvature everywhere and which is positive at some point.

Theorem 1.13. *(**[Siu],[Demailly]**) Let X be a compact manifold carrying an almost positive line bundle. Then X is a Moishezon space.*

Conclusion

- Sheaves and their cohomology turn out to be powerful means to consider local-to-global problems on complex manifolds.
- Kähler manifolds turn out to be a very nice category of spaces to study the interplay of topological, metric and complex structure.
- The Picard group encodes and lot of interesting information about complex manifolds and in particular the author would like to explore **Q2**, which is about the connections between Aut(Pic(X)) and Aut(X).

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