Categorical Structure in the Theory of Classifying Spaces and an Introduction to Classifying Topoi

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Abstract

The purpose of this article is to realise the natural emergence of a general notion of classification, in a cascading fashion, beginning with that of certain topological spaces, satisfying rather mild conditions and more importantly, realising and precisely describing the inherent categorical structure in the same. Such realisation will motivate a powerful generalisation of ideas spanning the fields of Algebraic Topology, Differential Topology, Singular Homology and Cohomology Theory of topological spaces culminating in a grand unification: the notion of a classifying topos.

Any investigation of knowledge and associated truths necessarily requires an entity that is capable of *observation*, *intellectual query*, *inference-making* and *data management/organisation*. Calling such an entity, an "intellectual", it is immediately clear that for an intellectual to exist, a means of comparison is necessary. After all, any observation involves *identification* and *characterisation*. Hence, an intellectual, if one exists, is necessarily capable of doing the same with the help of some mechanism. It is now of considerable interest as to how an intellectual, left to its own, manages to *acquire* knowledge, *realise* and *question* truths. In particular, consider a situation in which an intellectual is a priori able to identify an object and thereby at least capable of *distinguishing* the said object with its *surroundings*. What then are some natural methods of *knowing* about the object? Intuitively, such an intellectual should be interested in studying *interactions*. Interactions of various *orders* (in the sense of increasing complexity) would eventually help realise different characterisations of parts of the object. This gives rise to a notion of *scale* at which investigations are being carried out and moreover, one is also interested in *gluing* information lying at some scale, *scaling* "up" and "down", through the same. Thus a notion of *classification* proves to be fundamental to observation and query. Keeping this in mind:

We begin with the notion of a covering space of a topological space.

Definition 1. Covering Map and Covering Space
Let X be a topological space and $p : E \to X$ be a surjective and continuous map. Then, p is said to be a *covering map* iff every point x of X admits a neighbourhood U in X such that the inverse image $p^{-1}(U)$ can be written as the union of disjoint sets V_α open in E such that for each α, $p\big|_{V_\alpha} :\xrightarrow{\sim} U$ is a homeomorphism. In this case, E is called a *covering space* of X.

Therefore, intuitively a covering map onto a space, when it exists, associates a disjoint collection or "stack" of homeomorphic "slices" to an appropriate neighbourhood of each point in it and the action of the map is to "collapse" the "stack" onto the associated neihbourhood. In algebraic topology, finding such covering maps plays a pivotal role in computing and the fundamental group $\pi_1(X, x_0)$ relative to some base point x_0.

Now one is naturally compelled to ask the following questions:

- Given a topological space X what are the necessary and sufficient conditions for the existence of a covering map and a covering space?
- What can be said about the uniqueness of a covering space/covering map?

In order to answer the aove, it is now necessary for a natural notion for *equivalence of covering maps* or *equivalence of covering spaces.*

Definition 2. Equivalence of Covering Maps/Spaces Let $p : E \to X$ and $p' : E' \to X$ be covering maps. Then p and p' are said to be *equivalent* if there exists a homeomorphism $h : E \xrightarrow{\sim} E'$ such that the following diagram commutes.

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

The homeomorphism h is called an *equivalence of covering maps* or an *equivalence of covering spaces.*

Having defined such a notion we know consider the following without proofs [cf. [5] for proofs]

Theorem 1. *Let $p : E \to X$ be a covering map where E and X are locally path connected and path connected. Then, furthermore, if E is simply connected, for any other covering map $r : F \to X$, there is a covering map $q : E \to F$ such that the following diagram commutes.*

$$\begin{tikzcd} E \arrow[dd, "p"'] \arrow[dr, "q"] & \\ & F \arrow[dl, "r"] \\ X & \end{tikzcd}$$

Theorem 2. *A topological space X has a universal covering space* iff *X is path connected, locally path connected and semilocally simply connected.*

Definition 3. Properly Discontinuous Action If G is a group of homeomorphisms of a topological space X, the action of G on X is said to be *properly discontinuous* iff for $\forall x \in X$ $\exists$ a neighbourhood U of x such that $\forall g \in G$, $[g \neq 1] \implies [g(U) \cap U = \varnothing]$

Theorem 3. *Let X be path connected and locally path connected. If G is a group of homeomorphisms of X, the quotient map $\pi : X \to X/G$ is a covering map* iff *the action of G is properly discontinuous. In this case, the π is regular and G is its group of covering transformations.*

Theorem 4. *If $p : E \to X$ is a regular covering map and G is its group of covering transformations, then there is a homeomorphism $h : E/G \to X$ such that the following diagram is commutative.*

$$\begin{tikzcd} E \arrow[r, equal] \arrow[d, "\pi"'] & E \arrow[d, "p"] \\ E/G \arrow[r, "h"] & X \end{tikzcd}$$

We note the following:
Theorem **2** answers the first question precisely, by giving a set of necessary *and* suffiicient conditions for the existence of a covering space of a topological space while Theorem **1** speaks about the existence of a unique covering map having a universal property. Therefore, in conjunction, theorems **1** and **2** along with defintion **2** discuss the existence and uniqueness of covering spaces of a topological space. Definition **3** and theorems **3-4** on the other hand give precise means to compute covering maps. Moreover, it motivates a deeper look into the connections between topological and algebraic properties.

In particular, the idea of considering group actions on topological spaces will be familiar to the reader who is acquainted with some foundations of differential geometry.

The prime objects of study in differential geometry are *differentiable manifolds* and *differentiable maps* between such objects. Precisely, a *differentiable manifold* (of class C^r) is a Haussdorff space with a fixed complete atlas compatible with $\Gamma^r(\mathbb{R}^n)$, the pseudo-group of transformations of class C^r of $\mathbb{R}^n$. In fact, in this context, one'd immediately remember the following [cf. [11]]:

Theorem 5. *Let G be a properly discontinuous group of differentiable transformations acting freely on a differentiable manifold $\mathcal{M}$. then the quotient space $\mathcal{M}/G$ has a structure of a differentiable manifold such that the projection $\pi : \mathcal{M} \to \mathcal{M}/G$ is differentiable.*

Theorem **5** is certainly reminiscent of **3**. This compels us to proceed further into the fundamental notions in the study of differential geometry with a fervent hope of applying the the basic techniques and principles of algebraic topology to the study of differentiable manifolds.

Definition 4. Principal G-Bundles Let G be a topological group and X be a topological space. Then, a *principal G-bundle* over X is a topological space E equipped with a *continuous* map $p : E \to X$ and a *continuous* left action of G on E, $\mu : G \times E \to E$, (written $\mu(g, y) = g \cdot y, \forall g \times y \in G \times E$) such that:

- μ **preserves fibers.** $p(g \cdot y) = p(y)$ and therefore $p(g \cdot p^{-1}(x)) \subseteq p^{-1}(x)$

- **p is locally trivial.** There exists a an open covering $\{U\}_{\alpha\in J}$ of X such that, $\forall\alpha \in J$ there is a homeomorphism, $\phi_\alpha : G\times U_\alpha \xrightarrow{\sim} p^{-1}(U_\alpha)$, called a *local trivialisation* which *respects the G-action*

$$p(\phi_\alpha(g,x)) = x$$
$$\phi_\alpha(g\cdot h,x) = g\cdot\phi_\alpha(h,x),\ \forall g\times h \in G\times G, \forall x\in U_\alpha$$

[cf. 8-12]

Lemma 1. *If $p : E \to X$ is a principal G-bundle with associated action μ and open covering $\{U\}_{\alpha\in J}$ of X admitting a family of local trivialisations $\{\phi_\alpha : G\times U_\alpha \xrightarrow{\sim} p^{-1}(U_\alpha)\}_{\alpha\in J}$, then p is surjective, μ acts on E* freely *and* transitively.

Proof. Before proceeding with the proof we note the following:

- The action $\mu : G\times E\to E$ is *free* $\iff$ $[\forall y\in E, [g\cdot y = y] \implies [g = 1]]$
- The action $\mu : G\times E\to E$ is *transitive* $\iff$ $[\forall x \in X, \forall y, y' \in p^{-1}(x), \exists g\in G, such\ that, g\cdot y = y'\ ]$

Therefore, we have:

- That $p : E\to X$ is surjective follows from the fact that $\bigcup_{\alpha\in J} U_\alpha = X$ and that $\{\phi_\alpha : G\times U_\alpha \xrightarrow{\sim} p^{-1}(U_\alpha)\}_{\alpha\in J}$ are homeomorphisms.
- For any $y \in E$, $p(y) \in X$. So, since $\bigcup_{\alpha\in J} U_\alpha = X$, there is atleast one index $\alpha\in J$ such that $p(y)\in U_\alpha$. Then, due to the corresponding local trivialisation $\phi_\alpha : G\times U_\alpha \xrightarrow{\sim} p^{-1}(U_\alpha)$, $\exists! h\times x \in G\times U_\alpha$ such that $\phi_\alpha(h,x) = y$. But, $p(\phi_\alpha(h,x)) = x$ so $x = p(y)$. Therefore, $[g\cdot y = y] \implies [g\cdot\phi_\alpha(h,p(y)) = \phi_\alpha(g\cdot h,p(y)) = g\cdot y = y]$. That is, $\phi_\alpha(g\cdot h,p(y)) = \phi_\alpha(h,p(y)) = y$. But, h is unique, so, $g\cdot h = h$ and therefore $g = 1$.
- For any $x\in X$ and $y,y' \in p^{-1}(x)$, since $\bigcup_{\alpha\in J} U_\alpha = X$, there is atleast one index $\alpha \in J$ such that $x\in U_\alpha$. Then, due to the corresponding local trivialisation $\phi_\alpha : G\times U_\alpha \xrightarrow{\sim} p^{-1}(U_\alpha)$, $\exists! g\times z, g'\times z' \in G\times U_\alpha$, such that, $\phi_\alpha(g,z) = y$ and $\phi_\alpha(g',z') = y'$. Then, since, $p(\phi_\alpha(g,z)) = z = p(y) = x$ and $p(\phi_\alpha(g',z')) = z' = p(y') = x$, $z = z' = x$. Consider $h = g'g^{-1}$. Acting on y by h, $[h\cdot\phi_\alpha(g,x) = h\cdot y] \implies [\cdot y = \phi_\alpha(h\cdot g,x) = \phi_\alpha(g',x) = y']$. Hence, $h\cdot y = y'$.

■

A connection between the principal G-bundle(s) and the covering space(s) of a topological space will now be given.

Theorem 6. *Let E and X be locally path connected and path connected topological spaces. If $p : E \to X$ is a regular covering map, then p is a principal G-bundle of X where $G = \pi_1(X)/p_*(\pi_1(E))$ acts on the fibres of p via the monodromy action.*

Proof. $p : E \to X$ is a regular covering map $\implies$ $[p_*(\pi_1(E)) \trianglelefteq \pi_1(X)]$
Now, it can be shown that $[p_*(\pi_1(E)) \trianglelefteq \pi_1(X)] \iff [\mathcal{C}(E, p, X) \cong \pi_1(X)/p_*(\pi_1(E))]$ [cf. [1]]. Hence, by theorem **4**, the following diagram is commutative:

$$\begin{tikzcd} E \arrow[r, equal] \arrow[d, "\pi"'] & E \arrow[d, "p"] \\ E/\pi_1(X)/p_*(\pi_1(E)) \arrow[r, "h"] & X \end{tikzcd}$$

So, the quotient map π is covering and by theorem **3**, the action of $\pi_1(X)/p_*(\pi_1(E))$ on E is properly discontinuous. Now, it remains to show that this properly discontinuous action makes p into a principal G-bundle, which is essentially trivial and has therefore been omitted. ■

In particular, we have the following:

Theorem 7. *The universal covering map $\tilde{p} : \tilde{X} \to X$, when it exists, is a principal bundle of X for the group $\pi_1(X)$.*

Proof. Since E is simply connected along with being locally path connected and path connected, $\pi_1(E)$ is trivial. Therefore, by theorem **6**, $\tilde{p}$ is a principal bundle of X for the group $\pi_1(X)$. ■

It is evident now that the notion of a covering map onto a topological space and that of a principal bundle of the same are intimately connected. To be precise, the natural questions one can ask for covering spaces of topological spaces are very similar to those one can ask for principal bundles in the topological spaces. This is in itself, we realise is *not very surprising*. After all, one is asking questions about certain *subcategories* of $\underset{\sim}{\mathbf{Top}}/X$, namely: $\underset{\sim}{\mathbf{TopCov}}(X)$ with objects as covering spaces, and $\underset{\sim}{\mathcal{P}_G}(X)$ with objects as principal G-bundles, where in both cases one has appropriate morphisms. Therefore what we are interested in, is, the *interaction* as shown below:

$$
\begin{tikzcd}
\utilde{\mathbf{TopCov}}(X) \arrow[rr, leftrightarrow, bend left] \arrow[rr, phantom, "?"] \arrow[dr, hook] & & \utilde{\mathcal{P}_G}(X) \arrow[dl, hook'] \\
& \utilde{\mathbf{Top}}/X &
\end{tikzcd}
$$

We fiil in some aesthetic gaps:

Definition 5. Morphism of Principal G-bundles Let X be a topological space and $(p : E \to X, \mu : G \times E \to E), (q : F \to X, \nu : G \times F \to F)$ be principal G-bundles of X. Then, a *morphism of principal G-bundles* from E to F, is a continuous map $r : E \to F$ such that the following diagrams commute:

$$
\begin{tikzcd}
E \arrow[rr, "r"] \arrow[dr, "p"'] & & E' \arrow[dl, "q"] \\
& X &
\end{tikzcd}
$$

$$
\begin{tikzcd}
G \times E \arrow[r, "id_G \times r"] \arrow[d, "\mu"'] & G \times F \arrow[d, "\nu"] \\
X \arrow[r, equal] & X
\end{tikzcd}
$$

I keeping with definition **2**, we trivially note the following [cf. 8,12]:

Lemma 2. *Any morphism of principal G-bundles of a topological space X is an isomorphism. Therefore, $\utilde{\mathcal{P}_G}(X)$ is a* set *and in fact, a collection of isolated groups (a category with a single object), contained as subcategories.*

Proof. Trivially follows from local triviality and the fact that such a morphism respects the group actions. ■

An immediate generalisation of the notions considered above would be to consider morphism(s) of prinicipal-G bundles of *different* topological spaces. In this regard, the following is observed:

Lemma 3. *Let X be a topological space and $(p : E \to X, \mu : G \times E \to E)$ be a principal G-bundle of X. If $\zeta : Y \to X$ is a continuous map, then the pullback of p along ζ in* $\underset{\sim}{\textbf{Top}}$ *inherits a principal G-bundle structure. Precisely,*

$$\begin{tikzcd} E \times_Y Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ Y \arrow[r, "\zeta"] & X \end{tikzcd}$$

π_2 a principal G-bundle of Y.

Proof. In order to show that $\pi_2 : E \times_X Y \to Y$ is a principal G-bundle of Y, we note the following:

- Continuity of π_2 is trivial.

- If $\{U\}_{\alpha \in J}$ is the associated open covering of X, $\{\zeta^{-1}(U)_{\alpha \in J}\}$ is an open covering of Y. This is because, from continuity of ζ, $\zeta^{(}U_\alpha)$ is open in Y $\forall \alpha \in J$ and that $[Y = \zeta^{-1}(X)] \iff [Y = \zeta^{-1}(\bigcup_{\alpha \in J} U_\alpha)] \iff [Y = \bigcup_{\alpha \in J} \zeta^{-1}(U_\alpha)]$.

- Next, we define the local trivialisations for Y. So, in defining such a local trivialisation, $\tilde{\phi}_\alpha : G \times \zeta^{-1}(U_\alpha) \to \pi_2^{-1}(\zeta^{-1}(U_\alpha))$, we note that since $\zeta \circ \pi_2 = p \circ \pi_1, \pi_2^{-1}(\zeta^{-1}(U_\alpha)) = \pi_1^{-1}(p^{-1}(U_\alpha))$. Now, since the above diagram is a pullback square, we define $\tilde{\phi}_\alpha(g, x) = (\phi_\alpha(g, \zeta(x)), x), \forall g \times x \in G \times \zeta^{-1}(U_\alpha)$ so that it is evident that this map is also a homeomorphism.

- Lastly, the action of G on $E \times_X Y, \mu' : G \times (E \times_X Y) \to E \times_X Y$ must be definied keeping compatibility with the local trivialisations derived previously. and naturally we have that $g \cdot (e \times y) = (g \cdot e) \times y$. It may now be checked easily that μ' is compatibile with $\{\tilde{\phi}_\alpha : G \times \zeta^{-1}(U_\alpha) \to \pi_1^{-1}(p^{-1}(U_\alpha))\}_{\alpha \in J}$ and makes π_2 a principal G-bundle.

∎

Finally, we immediately observe the following:

- From Lemma **2** and **3**, $\underset{\sim}{\mathcal{P}_G}(-)$ seems to be a contravariant functor from some subcategory of $\underset{\sim}{\textbf{Top}}$ to $\underset{\sim}{\textbf{Set}}$

At this point, there is an uncanny feeling of an apparent familiarity with the general nature of ideas that we've discussed. Hence, we now ask, once again, the canonical questions, just as we had done in studying Covering spaces:

- Given a topological group G and a topological space X, what are the necessary and sufficient conditions for the existence of a prinicipal G-bundle of X?
- What can be said about the uniqueness of such a prinicipal bundle?
- What are *all G-bundles*?

Having asked the above, one can perhaps naiively expect a set of conditions resembling theorems **1-4** to be found here. And indeed, what follows is a revealing experience not very far from our speculation. Before we begin, the notion of a CW complex is defined. A CW complex is an extremely attractive and rich structure that perhaps captivates any person who realises its existence. It is worth a special mention that being a combinatorial object, CW complexes play a singular role in algebraic topology [cf. 1] perhaps does best in introducing the same and we borrow the same from him:

Definition 6. CW Complex Consider the following inductive procedure:

1. Start with a discrete set X^0 whose points are regarded as **0-cells**.
2. Inductively, form the **n-skeleton** X^n from X^{n-1} by attaching **n-cells** e^n_α via maps $\phi_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \coprod_\alpha D^n_\alpha$ with a collection of n-disks D^n_α under the identifications $x \sim \phi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus, as a *set*, $X^n = X^{n-1} \coprod_\alpha e^n_\alpha$ where each n-cell e^n_α is an open n-disk.
3. One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n \in \mathbb{Z}^+$, or or one can continue indefinitely, setting $X = \bigcup_n X_n$. In the latter case X is given the weak topology: $[A \subset X$ is open (or closed) in $X] \iff [A \cap X^n$ is open (or closed) in X^n $\forall n \in \mathbb{Z}^+ \cup \{0\}]$

A space X constructed in this way is called a *cell complex* or *CW complex.*

And thus we begin. Again, proofs have been omitted in favour of a discussion on the fundamental ideas required for the same which will follow subsequently. For interesting as well as aesthetic intricacies, cf. [8,11,12].

Theorem 8. *Let X be an arbitrary space and P be a principle G-bundle over X. If B is a CW complex and $f, g : B \to X$ are homotopic maps, then the pullbacks f^*P, g^*P are isomorphic as principal G-bundles over B.*

$$\begin{tikzcd}
f^*P \arrow[r, "\pi_1^f"] \arrow[d, "\pi_2^f"'] \arrow[rr, bend left, "\sim"] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "p"] & g^*P \arrow[l, "\pi_1^g"'] \arrow[d, "\pi_2^g"] \arrow[dl, phantom, "\llcorner", very near start] \\
B \arrow[r, "f"'] & X & B \arrow[l, "g"]
\end{tikzcd}$$

Theorem 9. *Let E and B be topological spaces such that E is weakly contractible. If $p : E \to B$ is a principal G-bundle, then for all CW complexes X there is a natural bijection $\phi : [X, B] \to \underset{\sim}{\mathcal{P}_G}(X)$ defined as $(f : X \to B) \mapsto (f^*(p) : f^*P \to X)$.*
B is then called the classifying space *for G and $p : E \to B$ is called the* universal G-bundle*.*

Theorem 10. *For any topological group G, there exists a classifying space BG. Moreover, if a universal G-bundle $p : EG \to BG$ exists then:*

1. *BG can be constructed as a CW complex.*
2. *a CW-classifying space BG is unique up to canonical homotopy equivalence.*
3. *EG is unique up to G-homotopy equivalence.*

Theorems **8-10** manifestly answer queries made previously and in a matter of a paragraph offer a picture that is extremely vivid and fundamental to our interests. A leap is in progress which is evidently clear from the cohomological aspects that have been pulled in a matter of a paragraph. Precisely, this comes in through the important notion of a Serre fibration as we quickly define below:

Definition 7. Serre Fibration [cf. 3] A sequence $\mathcal{F} \xrightarrow{i} \mathcal{E} \xrightarrow{\pi} \mathcal{B}$ is called a *Serre fibration* if $F = \pi^{-1}(\star_{\mathcal{B}})$ and if π has the following homotopy lifting property:
If $\mathcal{P}$ is any finite polyhedron, i.e a finite simplicial set, and $g : \mathcal{P} \to \mathcal{E}$ is a continuous map such that $H : \mathcal{P} \times I \to \mathcal{E}$ is a homotopy between $\pi \circ g = H(-,0)$ and $h_1 = H(-,1)$, then there is a homotopy $G : \mathcal{P} \times I \to \mathcal{E}$ between g and a map $g_1 = G(-,1)$ which *lifts* H in the sense that $\pi \circ G = H$.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "i"] & \mathcal{E} \arrow[r, "\pi"] & \mathcal{B} \\ & \mathcal{P} \arrow[u, "g"] \arrow[r, hook, "id_P \times 0"'] & \mathcal{P} \times I \arrow[u, "H"'] \arrow[ul, dashed, "G"] \end{tikzcd}$$

The spaces $\mathcal{F}$, $\mathcal{E}$, and $\mathcal{B}$ are called the *Fiber*, *total space* (Espace totale for Leray), and *Base space*, respectively.

The importance of Serre fibrations lies in the fact (proven in Serre's thesis) that associated to each fibration is a long exact sequence of homotopy groups:

$$...\pi_{n+1}(\mathcal{B}) \xrightarrow{\partial} \pi_n(\mathcal{F}) \longrightarrow \pi_n(\mathcal{E}) \longrightarrow \pi_n(\mathcal{B}) \xrightarrow{\partial} ...$$

This now allows us to talk about the singular cohomology of a topological space X and it'll be evident how all the notions introduced above are compactly presented in this frame work.

Consider an abelian group π and subsequently the singular cohomology theory of the topological space X. In particular, we will be interested in doing so for some CW complex. Then the n^{th} degree cohomological group of X with coefficients in π is the image of a functor $H^n(-,\pi)$from $\utilde{\textbf{HTop}}$ to $\utilde{\textbf{Ab}}$. Hence, for a CW complex with a fundamental group $\mathcal{G}$, it follows from the serre fibration for: $\mathcal{G} \xrightarrow{i} \mathcal{E}\mathcal{G} \xrightarrow{\pi} \mathcal{B}\mathcal{G}$ that $\pi_i(\mathcal{G}) = \mathcal{G}$ and non-zero $\iff i = 1$. Then, by definition, it follows that $\mathcal{B}\mathcal{G}$ is the Eilenberg-Maclane space of $\mathcal{G}$ if degree 1, $K(\mathcal{G}, 1)$. In particular, for any n and in general any abelian group π, we have the Eilenberg-Maclane space of π of degree n, $K(\pi, n)$ such that $H^n(K(\pi, n), \pi) \cong Aut(\pi)$ such that there is a natural bijection $\theta_X : H^n(X, \pi) \cong [X, K(\pi, n)]$ with with each n-dimensional cohomology class $c_n \mapsto f : X \to K(\pi, n)$ such that $f^*\gamma_n = c_n$ where γ_n is the unique n dimensional cohomology class corresponding to the identity homomorphism $1_\pi \in Aut(\pi)$.

$K(\pi, n)$ is then a classifying space of nth degree for cohomology.[cf. 3,4,7,8]

Note that:

- Theorem **9** implies that $\underset{\sim}{\mathcal{P}_G}(-) : \underset{\sim}{\mathbf{HTop}}_{CW} \to \underset{\sim}{\mathbf{Set}}$ is a representable functor from the full subcategory of CW complexes in $\underset{\sim}{\mathbf{Top}}$ to $\underset{\sim}{\mathbf{Set}}$.

- In particular, we note that CW complexes are the cofibrant objects in the classical model structure on topological spaces. This means in particular that every topological space is weakly homotopy equivalent to a CW-complex.[cf. 7]

- Recalling the notion of a subobject classifier, we note that in this context then, that is, in $\underset{\sim}{\mathbf{HTop}}_{CW}$, BG is a "prinicipal G-bundle classifier" and the universal bundle $p : EG \to BG$ is a "true map" which "classifies" every principal G-bundle of any object in $\underset{\sim}{\mathbf{HTop}}_{CW}$ with the help of continuous "characteristic maps" uniquely determined upto homotopy.

Needless to say, we are **compelled** to investigate the notion of classification in a good category - an elementary topos.

In what follows, the principal G-bundles, that have proven to be a natural and more importantly, foundational construct in Algebraic Topology, Differential Topology and Cohomology Theory of Topological spaces with coefficients in an abelian group, will be elegantly generalised in steps.

Recall the figure in which we were concerned about the interactions in $\underset{\sim}{\mathbf{Top}}/X$ between $\underset{\sim}{\mathbf{TopCov}}(X)$ and $\underset{\sim}{\mathcal{P}_G}(X)$. It is observed that when the topological group G is discrete, there is a merger and the principal G-bundles over X are covering maps.

This follows from the observation that the associated local trivialisations $\{\phi_\alpha : G \times U_\alpha \xrightarrow{\sim} p^{-1}(U_\alpha)\}_{\alpha \in J}$ can also be written as $\{\tilde{\phi}_\alpha : \Sigma_{g\in G} U_\alpha \xrightarrow{\sim}$

$p^{-1}(U_\alpha)\}_{\alpha\in J}$ due to the homeomorphism $\Sigma_{g\in G}U_\alpha \cong G\times U_\alpha$. Since a covering map is étalé, we have the following:

Theorem 11. *A principal G-bundle over X is an étalé map $p : E \to X$ with a continuous action $\mu : G\times E \to E$ such that:*

- *p is surjective.*
- *The restriction of the action to each fiber E_x, $\mu\big|_{G\times E_x}$ is free and transitive.*

The proof of theorem **11** follows trivially and hence has been omitted. In turn we have:

Definition 8. G-torsor A *G-torsor* over the space X for a discrete group G is a surjective étalé map $p : E \to X$ equipped with a continuous group action on E that is free and transitive when restricted to its fibers.

Any étalé map onto a topological space X, $p : E \to X$ defines an étalé space (E, p) and the functor $\Gamma : \underset{\sim}{\mathbf{Sh}}(X) \to \widehat{O(X)}$ gives a sheaf $\Gamma E : O(X)^{op} \to \underset{\sim}{\mathbf{Set}}$. Therefore, it seems natural to be able to push the action that comes with a G-torsor on to the sheaf so obtained, in the following manner:

1. A collection of left actions $\{\mu_U : G\times\Gamma E(U) \to \Gamma E(U)\}_{U\in O(X)}$ are defined as: for any $U \in O(X)$, $\mu_U(g,\sigma) := g\cdot\sigma : U \to X = \mu\circ\sigma$ $\forall g \in G, \sigma \in \Gamma E(U)$. Then for any $V \subseteq U$, the following diagram is trivially commutative:

$$\begin{array}{ccc} G\times\Gamma E(U) & \xrightarrow{\mu_U} & \Gamma E(U) \\ {\scriptstyle id_G\times\Gamma\rho^U_V}\big\downarrow & & \big\downarrow{\scriptstyle \rho^U_V} \\ G\times\Gamma E(V) & \xrightarrow{\mu_V} & \Gamma E(V) \end{array}$$

2. Therefore, the action also induces one on stalks evaluated at any point $x \in X$, $\mu_x : G\times \varinjlim_{x\in U}\Gamma E(U) \to \varinjlim_{x\in U}\Gamma E(U) = \Gamma E_x$

A better look at **1-2** above immediately suggests a way [cf.2]to simplify our characterisation of a $G - torsor$ even further:

1. Each map $\mu_U : G \times \Gamma E(U) \to \Gamma E(U)$ when transposed gives a map $\eta^g_E : G \to \Gamma E(U)^{\Gamma E(U)}$. Collecting such maps gives $\{\eta^g_E : G \to \Gamma E(U)^{\Gamma E(U)}\}_{g \in G, U \in O(X)}$. This collection can evidently be organised into a collection of natural transformations $\{\eta^g : \Gamma E \to \Gamma E\}_{g \in G}$ and subsequently into a morphism of monoids [cf. 9]: $\tilde{\mu} : G \to Hom(\Gamma E, \Gamma E)$ Therefore, in turn, a G-torsor is uniquely characterised by the following commutative diagrams:

$$\begin{tikzcd} 1 \arrow[r, "id"] \arrow[d, "e"] & Hom(F,F) \\ G \arrow[ur, "\tilde{\mu}"] & \end{tikzcd}$$

$$\begin{tikzcd} G \times G \arrow[r, "\tilde{\mu} \times \tilde{\mu}"] \arrow[d, "\cdot"'] & Hom(\Gamma E, \Gamma E) \times Hom(\Gamma E, \Gamma E) \arrow[d, "\circ"] \\ G \arrow[r, "\tilde{\mu}"] & Hom(\Gamma E, \Gamma E) \end{tikzcd}$$

2. The Geometric morphism $\Delta : \underset{\sim}{\mathbf{Set}} \to \underset{\sim}{\mathbf{Sh}}(X) : \Gamma$ where Δ is the constant sheaf functor which is left adjoint to Γ, the global sections functor, both of which are left exact. In any case, observe that $Hom(\Gamma E, \Gamma E) = \Gamma E^{\Gamma E}(X) = \Gamma(\Gamma E^{\Gamma E})$, so that, transposing $\tilde{\mu} : G \to Hom(\Gamma E, \Gamma E) = \Gamma(\Gamma E^{\Gamma E})$ gives a map $\Delta(G) \to \Gamma E^{\Gamma E}$ of sheaves and yet another transposition gives a map $\widehat{\mu} : \Delta(G) \times \Gamma E \to \Gamma E$ In particular, writing ΓE as F, the commutative diagrams for the monoid homomorphism upon transposition become:

$$\begin{tikzcd} 1 \times F \arrow[r, "\sim"] \arrow[d, "e \times 1_F"] & F \\ \Delta(G) \times F \arrow[ur, "\widehat{\mu}"] & \end{tikzcd}$$

$$\begin{tikzcd}
\Delta(G) \times \Delta(G) \times F \arrow[r, "1_{\Delta(G)} \times \widehat{\mu}"] \arrow[d, "m \times 1_F"'] & \Delta(G) \times F \arrow[d, "\widehat{\mu}"] \\
\Delta(G) \times F \arrow[r, "\widehat{\mu}"] & F
\end{tikzcd}$$

3. Since Δ preserves finite limits, the commutative diagrams imply that $\Delta(G)$ along with the unit and multiplication maps, $e : 1 \to \Delta(G)$ and $m : \Delta(G) \times \Delta(G) \to \Delta(G)$ respectively, is precisely a group object in $\underset{\sim}{\mathbf{Sh}}(X)$ and therefore the action of $\Delta(G)$ on the sheaf F is the same as the action of G on F in $\underset{\sim}{\mathbf{Sh}}(X)$.

Therfore, we can finally characterise a G-torsor as follows:

Lemma 4. *If G is a discrete group, then a G-torsor on a topological space X is an action μ of G on a sheaf F over X such that:*

1. *The map $F \to 1$ is an epimorphism.*

2. *Thea action $\widehat{\mu} : \Delta(G) \times F \to F$ induces an isomorphism $\widehat{\mu} \times \pi_2 : \Delta(G) \times F \xrightarrow{\sim} F \times F$ of sheaves.*

Lemma **4** therefore formally establishes a natural bijective correspondence between principal G-bundles over X where G is discrete and G-torsors over X. Note that the characterisation given above depends only on a *geometric morphism* $\gamma : \underset{\sim}{\mathbf{Sh}}(X) \to \underset{\sim}{\mathbf{Set}}$. Hence, it admits an immediate generalisation to any co-complete topos $\mathcal{E}$:

Definition 9. Let G be a discrete group, while $\gamma : \mathcal{E} \to \underset{\sim}{\mathbf{Set}}$ is a geometric morphism. Then a G-torsor over $\mathcal{E}$ is an object T of $\mathcal{E}$ equipped with a left action $\mu : \gamma^*(G) \times T \to T$ by the group object $\gamma^*(G)$ for which

1. The canonical map $T \to 1$ is an epimorphism.

2. Thea action μ induces an isomorphism $\mu \times \pi_2 : \gamma^*(G) \times T \xrightarrow{\sim} T \times T$.

Having generalised the notion of a principal G-bundle over a topological space X to a G-torsor over a co-complete elementary topos, in keeping with the progression of ideas previously, one can again ask the canonical set of questions. Without restating them, we proceed by firstly noting that the topos $\mathbf{B}G$ for the discrete topological group G is of considerable interest. In particular, one is interested in the relationship between the classifying space BG and $\mathbf{B}G$, if at all there are non-trivial aspects regarding the same. The following is a relevant commentary which has been borrowed from [cf. 4] mutatis mutandis:

"The topos $\mathbf{B}G$ has the same properties as the space BG, for **tautological reasons**: *the cohomology of the topos* $\boldsymbol{B}G$ *is the group cohomology of* G, because the definitions of topos cohomology and group cohomology are verbally the same in this case.

To compare the classifying space BG and the (classifying! as we'll see later) topos $\mathbf{B}G$ of G-sets, one first has to put these two objects in one and the same category. For this reason, we replace the space BG by its topos $\underset{\sim}{\mathbf{Sh}}(BG)$ of all sheaves (of sets) on BG.

The topos $\underset{\sim}{\mathbf{Sh}}(X)$ of sheaves on X contains *basically the same information* as the space X itself, and should be viewed simply as tile space X disguised as a topos. This view is supported by the fact that for two spaces X and Y, continuous mappings between them correspond to topos mappings (logical morphisms) between $\underset{\sim}{\mathbf{Sh}}(X)$ and $\underset{\sim}{\mathbf{Sh}}(Y)$. Moreover, for a sufficiently good space X (e.g., a CW-complex), the cohomology groups of the space X are the same as those of the topos $\underset{\sim}{\mathbf{Sh}}(X)$.

To come back to the comparison between the space BG and the topos $\mathbf{B}G$ of G-sets, we note that after having replaced BG by its topos $\underset{\sim}{\mathbf{Sh}}(BG)$, the two can be related by a mapping $\underset{\sim}{\mathbf{Sh}}(BG) \to \mathbf{B}G$. This topos map is a weak homotopy equivalence, although $\mathbf{B}G$ is a much smaller and simpler topos than $\underset{\sim}{\mathbf{Sh}}(BG)$. The known isomorphisms between the cohomology and homotopy groups of the space BG and those of the topos $\mathbf{B}G$ are induced by this map $\underset{\sim}{\mathbf{Sh}}(BG) \to \mathbf{B}G$."

Lemma 5. *For any discrete group* G*, when viewed as a right* G*-set with the underlying binary operation as the right action, denoted* $\tilde{G}$ *is a G-torsor over* $\boldsymbol{B}G$ *and is therefore a canonical construction.*

Proof. In order to show that $\tilde{G}$ is a G-torsor, we first define a left action on the same, $\mu : \gamma^*(G) \times \tilde{G} \to \tilde{G}$ as follows:

The inverse image functor $\gamma^* : \underset{\sim}{\mathbf{Set}} \to \mathbf{B}G$ is such that $\gamma^*(S) = (S, \star)$ where $\star$ is the trivial action. Therefore, in particular, $\gamma^*(G) = (G, \star)$. So, that the map $\mu : \gamma^*(G) \times \tilde{G} \to \tilde{G}$ can be chosen to be the underlying binary operation itself. μ is then continuous trivially. Now,

- The canonical map from $\tilde{G} \to 1$ is certainly epi as 1 will map to the identity element under any homomorphism.
- The induced map, $(\mu, \pi_2) : \gamma^*(G) \times \tilde{G} \to \tilde{G} \times \tilde{G}$ is again trivially an isomorphism as $(g, h) \mapsto (g \cdot h, h)$, $\forall g \times H \in \gamma^*(G) \times \tilde{G}$.

∎

Subsequently, we have the following result [cf. 10] which establishes that $\mathbf{B}G$ is the classifying topos for G-torsors over any cocomplete elementary topos $\mathcal{E}$:

Theorem 12. *For each discrete group G and each topos $\mathcal{E}$ over $\underset{\sim}{\boldsymbol{Set}}$ there is an equivalence of categories $\underset{\sim}{\boldsymbol{Hom}}(\mathcal{E}, \boldsymbol{B}G) \xrightarrow{\sim} \underset{\sim}{\boldsymbol{Tor}}(\mathcal{E}, G)$ between geometric morphisms $\mathcal{E} \to \boldsymbol{B}G$ and G-torsors over $\mathcal{E}$. Moreover, this equivalence is natural in $\mathcal{E}$ in the sense that, for any geometric morphism $f : \mathcal{F} \to \mathcal{E}$ over $\underset{\sim}{\boldsymbol{Set}}$, the following diagram commutes upto natural isomorphism:*

$$\begin{tikzcd}
\underset{\sim}{\mathbf{Hom}}(\mathcal{E}, \mathbf{B}G) \arrow[r, "\sim"] \arrow[d, "\underset{\sim}{\mathbf{Hom}}(f{,}\mathbf{B}G)"'] & \underset{\sim}{\mathbf{Tor}}(\mathcal{E}, G) \arrow[d, "f^*"] \\
\underset{\sim}{\mathbf{Hom}}(\mathcal{F}, \mathbf{B}G) \arrow[r, "\sim"] & \underset{\sim}{\mathbf{Tor}}(\mathcal{F}, G)
\end{tikzcd}$$

The compelling need for a context in which, a generalisation of the various notions of classifications we've encountered here so far, in a roughly progressing order of well-motivated abstraction:

- Given a suitably *nice* topological space X: Existence of a **universal** cover; The group of covering transformations **classifies** all covering maps onto X.
- Given any topological group G: Existence of a **universal** G-bundle $\pi : EG \to BG$; BG **classifies** all principal G-bundles over all CW complexes.

- Given any abelian group π: Existence of Eilenberg-Maclane spaces $K(\pi, n)$ and **universal** cohomology class $\forall n \in \mathbb{Z}^+$; $K(\pi, n)$ is a **classifying** space of n^{th} degree of (singular) cohomology.

- Given any discrete group X: Existence of a **universal** G-torsor $\tilde{G}$; $\mathbf{B}G$ **classifies** all G-torsors over any cocomplete elementary topos $\mathcal{E}$.

Therefore, given **structures** of a certain **kind**, we axiomatize them into a **theory, T,** and call these **structures** $\mathcal{M}$ in a category $\mathcal{E}$ or rather the **models** of **T** as **T-models** in $\mathcal{E}$. $\utilde{\mathbf{Mod}}(\mathcal{E}, \mathbf{T})$ is then defined as the category of all these models with suitable model (homo) morphisms between them.

Moreover, it is assumed that the inverse image functor of a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ will carry an **T-model** $\mathcal{M}$ in $\mathcal{E}$ to a **T-model** $f^*\mathcal{M}$ in $\mathcal{F}$.

Definition 10. Classifying Topos A *classifying topos* for **T-models** is a topos $\mathcal{B}(\mathbf{T})$ over $\utilde{\mathbf{Set}}$ such that for every cocomplete topos $\mathcal{E}$ there is an equivalence of categories $c_{\mathcal{E}} : \utilde{\mathbf{Mod}}(\mathcal{E}, \mathbf{T}) \xrightarrow{\sim} \utilde{\mathbf{Hom}}(\mathcal{E}, \mathcal{B}(\mathbf{T}))$ which is natural in $\mathcal{E}$, that is, the following diagram is commutative:

$$\begin{tikzcd}
\utilde{\mathbf{Mod}}(\mathcal{E}, \mathbf{T}) \arrow[r, "\sim"] \arrow[d, "f^*"'] & \utilde{\mathbf{Hom}}(\mathcal{E}, \mathcal{B}(\mathbf{T})) \arrow[d, "\utilde{\mathbf{Hom}}(f, \mathcal{B}(\mathbf{T}))"] \\
\utilde{\mathbf{Mod}}(\mathcal{F}, \mathbf{T}) \arrow[r, "\sim"] & \utilde{\mathbf{Hom}}(\mathcal{F}, \mathcal{B}(\mathbf{T}))
\end{tikzcd}$$

Therefore, there exists a *universal or generic* **T-model** $U_{\mathbf{T}}$ in $\mathcal{B}(\mathbf{T})$, namely the model corresponding to the identity on $\mathcal{B}(\mathbf{T})$ under the aforementioned equivalence.

The following follows trivially from the above:

Lemma 6. *For any cocomplete topos $\mathcal{E}$ and any* ***T-model*** *$\mathcal{M}$ in $\mathcal{E}$, there exists a geometric morphism $f : \mathcal{E} \to \mathcal{B}(\boldsymbol{T})$, unique upto isomorphism, such that $\mathcal{M} \cong f^*(U_{\boldsymbol{T}})$*

We conclude by giving several examples of classifying topoi for familiar structures.

- **The Object Classifier.** *An object classifier* is a *Grothendieck* topos that classifies all objects of a cocomplete topos $\mathcal{E}$. The topos $\mathcal{S}[U] = \utilde{\mathbf{Set}}^{\utilde{\mathbf{Fin}}}$ is the **object classifier** possessing the **universal object** $U \in \utilde{\mathbf{Set}}^{\utilde{\mathbf{Fin}}}$: the inclusion functor $\utilde{\mathbf{Fin}} \rightarrowtail \utilde{\mathbf{Set}}$. So, for any cocomplete topos $\mathcal{E}$, there is an equivalence of categories, natural in $\mathcal{E}$,

$$\utilde{\mathbf{Hom}}(\mathcal{E}, \mathcal{S}[U]) \xrightarrow{\sim} \mathcal{E}, f \mapsto f^*(U)$$

- **The Classifying Topos for Rings.** If $\utilde{\mathbf{C}}$ is any category with finite limits, a *ring-object* in $\utilde{\mathbf{C}}$ is an object R of $\utilde{\mathbf{C}}$ equipped with morphisms:

$$1 \overset{0}{\underset{1}{\rightrightarrows}} R \overset{\bullet}{\underset{+}{\leftleftarrows}} R \times R$$

such that such that the usual diagrams for commutativity and unit axioms hold. Then, with the evident notion of a morphism between such ring-objects, they form a category $\utilde{\mathbf{Ring}}(\utilde{\mathbf{C}})$. In particular for any topos $\mathcal{E}$ there is a category $\utilde{\mathbf{Ring}\mathbf{C}}$ of ring objects in $\mathcal{E}$. For any geometric morphism $\zeta : \mathcal{F} \to \mathcal{E}$, between topoi $\mathcal{F}$ and $\mathcal{E}$, the inverse image functor ζ^* induces a functor $\zeta^* : \utilde{\mathbf{Ring}}(\utilde{\mathbf{E}}) \to \utilde{\mathbf{Ring}}(\utilde{\mathbf{F}})$.

The presheaf topos $\utilde{\mathbf{Set}}^{\utilde{(\mathbf{fp\text{-}rings})}}$ is a **classifying topos** for ring objects and the **universal ring object** is the ring object in $\utilde{\mathbf{Set}}^{\utilde{(\mathbf{fp\text{-}rings})}}$ given by the inclusion functor from fp-rings into rings. Thus for any cocomplete topos there is an equivalence of categories, natural in $\mathcal{E}$:

$$\utilde{\mathbf{Hom}}(\mathcal{E}, \utilde{\mathbf{Set}}^{\utilde{\mathbf{fp\text{-}rings}}}) \xrightarrow{\sim} \utilde{\mathbf{Ring}}(\mathcal{E})$$

,

$$f \mapsto f^*(R)$$

- The Classifying Topos for Linear Orders. The topos $\utilde{\mathbf{Ssets}}$ of simplicial sets is a **classifying topos** for orders, with **universal order** $V = Hom_{\Delta}(-, [1])$. More explicitly, for any cocomplete topos $\mathcal{E}$ there is an equivalence of categories:

$$\utilde{\mathbf{Orders}}(\mathcal{E}) \xrightarrow{\sim} \utilde{\mathbf{Hom}}(\mathcal{E}, \utilde{\mathbf{Sset}})$$

This equivalence associates with an order $I = (I, R, b, t) \in \mathcal{E}$ the geometric morphism $\mathcal{E} \to \underset{\sim}{\mathbf{Sset}}$ whose inverse image functor is the geometric realisation functor $| - |_I : \underset{\sim}{\mathbf{Sset}} \to \mathcal{E}$, while the direct image functor is the singular complex functor S_I.

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