Kähler Geometry

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Overview

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$$\mathcal{O}_X\big|_U = \varphi^*(\mathcal{O}_{\mathbb{C}^n}\big|_V)$$

where $\mathcal{O}_{\mathbb{C}^n}$ is the sheaf of holomorphic functions on $\mathbb{C}^n$.

Complex Manifolds

This is equivalent to equipping a topological $2n$-manifold X with a **holomorphic atlas** $\{U_i, \varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{C}^n\}$ such that the transition maps $\varphi_{ij} := \varphi_i \circ \varphi_j\big|_{\varphi_j(U_i \cap U_j)}^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ are holomorphic for all i, j.

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Examples: $\mathbb{CP}^n$ (Projective space), $\mathbb{C}^n/\Gamma$ (Complex torii), Quotients of the form X/G (properly discontinuous action)

Goal

1. Doing geometry on complex manifolds
2. Realize that Kähler manifolds are natural sites for seeking relationships between **topological, complex** and **geometric** structures on a space.
3. Actually see instances of **2**

A General theme

Real manifolds versus complex manifolds

Meromorphic functions

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A meromorphic function on $U \subset_{open} X$ is a map

$$f : U \to \coprod_{x \in U} Q(\mathcal{O}_{X,x})$$

Sheaf of meromorphic functions: $\mathcal{K}_X$
Space of global sections: $\mathrm{K}(X)$

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Theorem

(Siegel) *Let X be a compact, connected complex n-manifold. Then,*

$$a(X) \leq n$$

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Question: Is the converse true?

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$$\begin{tikzcd} U \times \mathbb{C}^r \arrow[rr, "\psi_i^{-1}"] \arrow[dr, "\pi_U"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

and the induced map $\pi^{-1}(x) \cong \mathbb{C}^r$ is $\mathbb{C}$-linear.

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$$E \oplus F,\ E \otimes F,\ E^*,\ \bigwedge^i E,\ \det(E) \cdots$$

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- $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$
Follows from two natural isomorphisms:
$$\mathrm{Pic}(X) \cong \check{H}^1(X, \mathcal{O}_X^*) \text{ and } \check{H}^1(X, \mathcal{O}_X^*) \cong H^1(X, \mathcal{O}_X^*)$$

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3. $I\Big|_{T^{1,0}X} \overset{\xi}{\cong} i \cdot id$ and $I\Big|_{T^{0,1}X} \overset{\eta}{\cong} -i \cdot id$
4. $\mathcal{A}_X^{p,q}(E)$, the sheaf $U \mapsto \mathcal{A}_X^{p,q}(E)(U) := \Gamma(U, \bigwedge^{p,q} X \otimes E)$

Is the converse to 1 true?

Hermitian Structures and Hermitan Vector Bundles

A Riemannian metric g on (X, I) is an **hermitian structure** on X if for any point $x \in X$ the scalar product g_x on T_xX is compatible with the almost complex structure I_x. The induced real (1,1) form $\omega := g(I(-), (-))$ is called the **fundamental form**. Locally,

$$\omega = (i/2) \sum_{i,j=1}^{n} h_{ij} dz^i \wedge d\bar{z}^j$$

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Suitably generalizing this from TX to arbitrary complex vector bundles:

Hermitian vector bundle (E, h): A complex vector bundle E on a smooth manifold M, E_x is endowed with a a hermitian inner product h_x, smooth in x.

Connections and Curvature

A **connection** on a complex vector bundle E is a $\mathbb{C}$-linear sheaf homomorphism $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ which satisifies the Leibniz rule:

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The **curvature** F_∇ of a connection ∇ on a vector bundle E is defined as the composition:

$$F_\nabla := \nabla \circ \nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E) \to \mathcal{A}^2(E)$$

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Theorem

Let (E, h) be a holomorphic vector bundle. Then there exists a **unique hermitian connection** ∇ *that is* **compatible with the holomorphic structure**, *i.e* $\nabla^{1,0} = \bar{\partial}$. *This connection is called the* **Chern connection** *on* (E, h).

Riemannian vs Complex Geometry

A **Kähler structure** is a hermitian structure g for which the fundamental form ω is closed, i.e. $d\omega = 0$. In this case, ω is called a Kähler form on X.

Examples: $\mathbb{C}\mathbf{P}^n$ with the Fubini-Study metric, (in the standard chart U_i) $\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}log(\sum_{l=0}^{n} |\frac{z_l}{z_i}|^2)$, Complex torii, $\mathbb{C}^n$,

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Theorem

Let (X, g) be a Kähler manifold. Then under the isomorphism $\xi : TX \cong T^{1,0}X$, the Chern connnection ∇ on the holomorphic tangent space $T^{1,0}X$ corresponds to the Levi-Civita connection D.

Answers to Some Questions Raised so Far

Theorem

[Moishe] *A Moishezon space is a projective algebraic variety if and only if it admits a Kähler metric.*

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A real (1,1)-form α is called (semi)-positive if for all non-zero holomorphic tangent vector $v \in T^{1,0}X$, one has $-i\alpha(v, \bar{v}) > 0$ (resp.$\geq$0).

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- The **first Chern class** $c_1(L)$ of a holomorphic line bundle $L \in Pic(X)$ is its image under the boundary map:

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Conjecture Calabi

Every real (1,1)-form in $c_1(X)$ is the Ricci for some Kähler form.

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- Equivalent to the equation $Ric(\omega_g) = \mathrm{sgn}(\chi(S)) \cdot \omega_g$
- Generalizing to Kähler manifolds, a Kähler metric ω on X is said to be **Kähler-Einstein** is $Ric(\omega) = \lambda\omega$.

Theorem

*(**Aubin,Yau**) If $c_1(X) = 0$(resp.< 0) then for any $\alpha \in \mathcal{C}_X$, there exists an $\omega \in \alpha$ such that $Ric(\omega) = 0$ (resp. < 0).*

Some More Questions

- Using the homomorphism $Aut(X) \to Aut(\mathrm{Pic}(X))$ to study action of elements of $Aut(X)$ on geometric objects on X.
- Deformation of vector bundles and limiting geometries

The End