Coxeter Groups
Why are they linear?

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Overview

Reflecting on Reflections

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More generally, about *hyperplanes* in $\mathbb{R}^n, S^n$ and $\mathbb{H}^n$.

We're often interested in discrete subgroups of their respective isometry groups, generated by reflections.

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Moral: *Geometric* reflections need *mirrors*.

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$G = \langle s, t, u, v, \cdots | s^2, t^2, u^2, v^2, \cdots \rangle$

Are we done? -No.

G better act on a geometric object with mirrors such that reflections exchange "half-spaces".

Pre-Coxeter Systems

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Dihedral Groups

A group generated by two distinct elements of order 2.

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It's easy to classify dihedral groups:

Theorem

Let W be a dihedral group generated by distinct elements s and t.

1. *Let $P = \langle st \rangle$. Then, $P \trianglelefteq W$ and $W \cong P \rtimes C_2$ where $C_2 = \{1, s\}$ and $|W : P| = 2$.*
2. *Let $p = st$ and $m = |p|$ then $W \cong C_m \rtimes C_2$*

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1. $P \trianglelefteq W$:it suffices to observe that $s(st)s = ts = (st)^{-1}$ and $t(st)t = ts = (st)^{-1}$
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Draw.

Pre-Reflection Systems

Pre-Reflection Systems

Pre-reflection System

Let W be a group. A **pre-reflection system** for W consists of - $R \subset W$, an action of W on a **connected, simplicial** graph Ω and a base-point $\nu_0 \in \Omega^0$ such that:

1. **Involutions** For all $r \in R$, $r^2 = 1$.
2. **Conjugation** R is closed under conjugation by elements of W. For all $w \in W, r \in R$ we have $wrw^{-1} \in R$.
3. **Flipping** For each edge in Ω, there is a **unique** element of R that interchanges its endpoints (*flips* the edge). Also, each element of R flips atleast one edge in Ω.
4. $W = \langle R \rangle$

We call each element of R a pre-reflection.

Observe: Flipping helps specify edge paths uniquely in Ω using words in R.

Let $(\nu_0, \cdots, \nu_k)$ be an edge path in Ω.
$r_i \nu_{i-1} = \nu_i$
$\nu_k = r_k \cdots r_1 \nu_0$ So, we have:

Theorem

Suppose a W-action on a connected simplicial graph Ω is a part of a pre-reflection system. Then, the W-action is transitive.

Note: It need not be free though. What happens as soon as it's also free?

Example Every pre-Coxeter system gives a pre-reflection system.
Let (W,S) be a pre-Coxeter system.
Set $R =$ conjugate closure of S $= \{wsw^{-1} | w \in W, s \in S\}$.
$R, \mathrm{Cay}(W, S), 1$ is a pre-reflection system.

Question

What about the other way round?

Answer: Yes it's possible.

$R, W, \Omega, \nu_0 \longrightarrow$ A pre-reflection system Let $S = S(\nu_0)$ =collection of pre-reflections that flip an edge containing ν_0
Aim: S generates W.
Step-1: R is the conjugate closure of S

Step-2: The set of words in S, S^* is in 1-1 correspondence with the set of edge paths starting at ν_0

Step-1: R is the conjugate closure of S

1. Pick any $r \in R$. Then r flips some edge e in Ω.
2. By transitivity of W-action, there is some $w \in W$ s.t. $s\nu_0$ is an endpoint of e.
3. Pulling back e, $w^{-1}e = \{\nu_0, s\nu_0\}$ for some $s \in S$.
4. s flips $w^{-1}e \implies wsw^{-1}$ flips e. $r = wsw^{-1}$

Step-2: The set of words in S, S^* is in 1-1 correspondence with the set of edge paths starting at ν_0

Consider a word $\mathbf{s} = (s_1, \cdots, s_k) \in S^*$.We construct a sequence of words $w_0, \cdots, w_k$ in S. $w_0 = 1$, $w_i = s_1 \cdots s_i$ for $1 \leq i \geq k$. We claim that $\phi(\mathbf{s}) = (\nu_0, \cdots, \nu_k)$ where $v_i = w_i\nu_0$ is an edge path in Ω joining ν_0 and $w(\mathbf{s})\nu_0$.

1. $\phi(\mathbf{s})$ is an edge path:
   There is edge bw ν_0 and $s_i\nu_0 \implies$ there is an edge bw $w_{i-1}\nu_0 = \nu_{i-1}$ and $w_{i-1}s_i\nu_0 = w_i\nu_0 = \nu_i$
2. In the above scenario, set $r_i = w_i w_{i-1}^{-1} = s_1 \cdots s_i s_{i-1} \cdots s_1$. r_i's are the unique pre-reflections flipping the edge joining ν_{i-1} and ν_i.
3. ϕ is invertible. Let $(\nu_0, \cdots, \nu_k)$ be any edge path in Ω. Let $r_i \in R$ be the unique one flipping $\{\nu_{i-1}, \nu_i\}$. Set $w_i = r_i \cdots r_1$ and $s_i = w_{i-1}^{-1} r_i w_{i-1}$. It follows that $\mathbf{s} = (s_1, \cdots, s_k) \in S^*$ and $\phi(\mathbf{s}) = (\nu_0, \cdots, \nu_k)$

We denote by $\Phi : S^* \to R^*$, the map $\Phi(\mathbf{s}) = (r_1, \cdots, r_k)$.

Step-3: S generates W

It suffices to show that $R \subset \langle S \rangle$.

1. Pick any $r \in R$ and say it flips some edge e.
2. Ω is connected *implies* We have an edge path $\gamma = (\nu_0, \cdots, \nu_k)$ having final edge as e.
3. Let $\mathbf{s} \in S^*$ be the unique word in S such that $\phi(\mathbf{s}) = \gamma$.
4. By **Flipping** r is the last entry in $\Phi(\mathbf{s}) = (r_1, \cdots, r_k)$. i.e. $r = r_k = s_1 \cdots s_k s_{k-1} \cdots s_1$

Diagram

Now it remains for us to see when the diagram commutes, this, as discussed previously would inevitably require us to impose a "**geometric**" condition on pre-reflection systems, so that they become what we could call **reflection systems**.

Finding Geodesics

Reduced Expressions

A word $\mathbf{s} = (s_1, \cdots, s_k) \in S^*$ is said to be a **reduced expression** if it is a word of minimum length for $w(\mathbf{s})$ i.e. $l(w(\mathbf{s})) = k$.

Theorem (Reducing words)

Suppose $\mathrm{s} = (s_1, \cdots, s_k) \in S^*$ *and* $w = w(\mathrm{s})$. $\Phi(\mathrm{s}) = (r_1, \cdots, r_k)$ *helps in obtaining an edge path in* Ω *joining* ν_0 *to* $w\nu_0$.
If $r_i = r_j$ *for some* $i < j$ *then the sub-word* $\mathrm{s}' = (s_1, \cdots, \hat{s}_i, \cdots, \hat{s}_j, \cdots, s_k)$ *is such that* $\phi(\mathrm{s})$ *and* $\phi(\mathrm{s}')$ *have the same endpoints.*

Proof

Let $r = r_i = r_j$. $\phi(\mathbf{s}) = (\nu_0, \cdots, \nu_k) = (\nu_0, w_1\nu_0 \cdots, w_k\nu_0)$.

1. r flips the edges $\{w_{i-1}\nu_0, w_i\nu_0\}$ and $\{w_{j-1}\nu_0, w_j\nu_0\}$.
2. r maps the portion $(w_i\nu_0, \cdots, w_{j-1}\nu_0)$ to an edge path of the same length $(w_{i-1}\nu_0, w_j\nu_0)$, reducing the total length by 2.
3. Draw
4. $r_i = r_j \implies s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1 = s_1 \cdots s_{j-1} s_j s_{j-1} \cdots s_1 \implies s_i = s_i \cdots s_{j-1} s_j s_{j-1} \cdots s_i \implies s_{i+1} \cdots s_j s_{j-1} \cdots s_i = 1 \implies s_i \cdots s_j = s_{i+1} \cdots s_{j-1}$
5. So deleting s_i, s_j we have $\phi(\mathbf{s}') = (\nu_0', \cdots, \hat{\nu}_i', \cdots, \hat{\nu}_j', \cdots, \nu_k') = (\nu_0, \cdots, \nu_{i-1}, \cdots, \nu_j, \cdots, \nu_k)$.

So if $\mathbf{s} = (s_1, \cdots, s_k)$ is a reduced expression, there are no repetitions in $\Phi(\mathbf{s}) = (r_1, \cdots, r_k)$.

Walls

For each pre-reflection $r \in R$, define the **wall of r** to be $\Omega^r = \{\text{midpoints of edges flipped by r}\} = \mathrm{Fix}(r)$.

The previous result describes *wall crossings*.

Observe:

- An edge path emanating from base-point ν_0 crosses the wall of r iff r appears in $\Phi(\mathbf{s})$.
- If such an edge path **crosses the wall of r more than once** we may obtain a new edge path having same endpoints that crosses the wall 2 less times.

We will now see that walls are candidates for mirrors.

Theorem (Walls are potential mirrors)

For each pre-reflection $r \in R$, $\Omega - \Omega^r$ has 1 or 2 components. If there are 2 components, they are interchanged by r.

Step-1: It suffices to check this for a smaller set, namely, S

- Pick any $r \in R$. $r = wsw^{-1}$ for some $w \in W$ and $s \in S$.
- $w\Omega^s = \Omega^{wsw^{-1}}$
- So since action by w is a homeo, w maps $\Omega - \Omega^s$ homeomorphically to $\Omega - \Omega^r$

Step-2: For every $\nu \in \Omega^0$, either ν or $s\nu$ lies in the same component as ν_0. Draw.

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Choose an edge path γ of minimum length joining ν_0 and ν in Ω. Let $\mathbf{s} = (s_1, \cdots, s_k)$ be the corresponding word inducing the path.

- **Case-1** s does not appear in $\Phi(\mathbf{s})$. $\implies$ γ does not cross wall of s $\implies$ ν_0 and ν lie in same component of $\Omega - \Omega^s$.
- **Case-2** s occurs in $\Phi(\mathbf{s})$. Then $\exists! i$ s.t. $s = s_i$. Append s in front to make $\mathbf{s}' = (s, \mathbf{s})$. This has exactly 2 s's. Delete them to get a sub-word $\mathbf{s}'' = (s_1, \cdots, \hat{s}_i, \cdots, s_k)$. Now $\gamma' = \phi(\mathbf{s}'')$ does not cross the wall of s. So ν_0 and $s\nu$ are in the same component.

Reflection System

A pre-reflection system (Ω, ν_0) for a group W is said to be a **reflection system** if for each $s \in S$, $\Omega - \Omega^s$ has 2 components.

If (Ω, ν_0) is a reflection system, elements of R, the pre-reflections are promoted to *reflections* and the ones in S are called fundamental reflections.

Half-Space

Closure of a component of $\Omega - \Omega^r$ is called a half-space bounded by the wall of r. The half-space containing ν_0 is called the positive half-space.

Let $n(r, \mathbf{s}) =$number of times the edge path $\phi(\mathbf{s})$ crosses the wall of r.

$(-1)^n(r, \mathbf{s})$ is like the intersection number mod 2 of $\phi(\mathbf{s})$ and Ω^r when Ω^r separates Ω.

Aim: To show

$$(\Omega, \nu_0)\text{is a reflection system}$$

$$\rightsquigarrow$$

$$\nu_0, \nu \text{ lie in the same half-space iff any edge path has even wall crossings}$$

$$\rightsquigarrow$$

$$W \text{ acts on } \Omega^0 \text{ freely}$$

$$\rightsquigarrow$$

$$\Omega \text{ is } W\text{-isomorphic to } \mathrm{Cay}(W, S)$$

Theorem

(Ω, ν_0) is a reflection system for W $\implies$ $W \curvearrowright \Omega^0$ freely.

- Suppose $w\nu_0 = \nu_0$ for some $w \in W$. Choose an edge path joining them γ, i.e. an edge loop at ν of shortest length and $\gamma = \phi(s_1, \cdots, s_k)$.
- First wall that γ crosses is Ω^{s_1}. But then s_1 must appear again in $\Phi(\mathbf{s})$, contradicting choice of shortest length word.

Coxeter matrices

A **Coxeter matrix** $M = (m_{st})$ on a set S is an $S \times S$ symmetric matrix with entries valued in $\mathbb{N} \cup \{\infty\}$ such that

$$m_{st} = \begin{cases} 1 & s = t \\ \geq 2 & o.w. \end{cases}$$

Observe:

1. Pre-Coxeter Systems spew out Coxeter matrices
   Suppose (W, S) is a pre-Coxeter system. Set $m_{st} = |st|$ for all $s, t \in S$.
2. Coxeter matrices give groups with presentations.
   Let $\tilde{S} = \{\tilde{s} | s \in S\}$, $I = \{(s, t) \in S \times S | m_{st} \neq \infty\}$ and $\mathcal{R} = \{(\tilde{s}\tilde{t})^{m_{st}} | (s, t) \in I\}$.
   We have a group $G = \langle \tilde{S} | \mathcal{R} \rangle$.

$$
\begin{tikzcd}
\text{Coxeter Matrices} \arrow[r, rightsquigarrow] & \text{Groups} \arrow[d, equal] \\
\text{Pre-Coxeter Systems} \arrow[u, rightsquigarrow] & \text{Coxeter Systems} \arrow[l, hook']
\end{tikzcd}
$$

Coxeter System/group

A pre-Coxeter system (W, S) is called a **Coxeter system** if the epimorphism $\tilde{W} \to W$ obtained from $\tilde{s} \mapsto s$ is an isomorphism. W is then called a Coxeter group and S is a fundamental set of generators.

Theorem

Let (W, S) be a pre-Coxeter system. TFAE:-

1. *(W, S) is a Coxeter system*
2. *$Cay(W, S)$ is a reflection system*
3. *(W, S) satisfies the (D) condition.*

Geometric Interpretation: Let $\Omega =$Cay(W,S). For each distinct pair $\{s, t\}$ of elements in S, let $W_{s,t}$ be the dihedral group they generate. Let Λ be the 2-complex of W formed by gluing in $2m_{st}$-gons for each coset of $W_{s,t}$ in W.

(1) $\implies$ Λ is simply connected and is therefore the Cayley 2-complex of W. (2) $\implies$ for each $s \in S$, $Fix(s)$ separates Ω.

We will prove (1)*implies*(2).

Theorem (Intersection number mod 2 is well-defined)

Suppose (W, S) is a Coxeter system. Then:

- *For any word $\mathrm{s} \in S^*$ with $w = w(\mathrm{s})$ and any $r \in R$, the number $(-1)^{n(r,\mathrm{s})}$ depends only on w. Denote this number by $\eta(r, w)$.*
- *There is a homomorphism, $W \to Aut(R \times \{\pm\})$ sending $w \mapsto \varphi_w$ where $\varphi_w(r, \epsilon) = (wrw^{-1}, \eta(r, w^{-1})\epsilon)$*

Proof Idea

1. Suppose Ω =Cay(W,S)is a reflection system where $\nu_0 = 1$. So each wall separates Ω into 2 half-spaces: Ω^r_+ and Ω^r_-.
2. Given $w \in W$ we can compute
$$\eta(r,w) = \begin{cases} +1 & 1, w \text{ are on same side} \\ -1 & 1, w \text{ are on opp. side} \end{cases}$$
3. The set of half-spaces is indexed by $R \times \{\pm\}$
4. W acts on $R \times \{\pm\}$.
5. **What is the action?**
$w \in W$ maps Ω^r_+ to $\Omega^{wrw^{-1}}_\epsilon$. [Since $w\Omega^r = \Omega^{wrw^{-1}}$]
6. $\epsilon = 1 \iff w\Omega^r_+ = \Omega^{wrw^{-1}}_+ \iff$
$1, w$ are on same side of $\Omega^{wrw^{-1}} \iff$
$1, w^{-1}$ are on same side of $\Omega^r \iff \eta(r, w^{-1}) = 1$.

Proof

Step-1: Defining the homomorphism on generators
For each $s \in S$ define $\varphi_s(r, \epsilon) = (srs, (-1)^{\delta(r,s)}\epsilon)$. Clearly, $\varphi_s^2 = id_{R\times\{\pm\}}$. So φ_s is a bijection.

Step-2: Extending to all of S^*
$\mathbf{s} = (s_1, \cdots, s_2)$ let $\nu = s_k \cdots s_1$ and $\varphi_{\mathbf{s}} = \varphi_{s_k} \circ \cdots \circ \varphi_{s_1}$.
Claim: $\varphi_{\mathbf{s}}(r, \epsilon) = (\nu r \nu^{-1}, (-1)^{n(r,\mathbf{s})}\epsilon)$

We prove by induction on k. We've done $k = 1$ already. Let $k > 1$ and supposed the claim holds for words of length $k - 1$.

Let $\mathbf{s}' = (s_1, \cdots, s_{k-1})$ and $u = s_{k-1} \cdots s_1$, so that $\mathbf{s} = (\mathbf{s}', s_k)$ and $\nu = s_k u$.

$$\begin{aligned}
\varphi_{\mathbf{s}}(r, \epsilon) &= \varphi_{s_k} \circ \varphi_{\mathbf{s}'}(r, \epsilon) \\
&= \varphi_{s_k}(uru^{-1}, (-1)^{n(r,\mathbf{s}')}\epsilon) \\
&= (s_k uru^{-1} s_k, (-1)^{n(r,\mathbf{s}')+\delta(s_k, uru^{-1})}\epsilon) \\
&= (\nu r \nu^{-1}, (-1)^{n(r,\mathbf{s})}\epsilon)
\end{aligned}$$

Because, $n(r, \mathbf{s}') + \delta(s_k, uru^{-1}) = n(r, \mathbf{s})$.

Step-3: Checking that s $\mapsto \varphi_s$ takes each relation to the identity

- $\varphi_s^2 = 1$ for each $s \in S$
- (W,S) is a Coxeter system $\implies$ other relations are of the form $(st)^m = 1$ where $m = m_{st}$.
  $(\varphi_s \circ \varphi_t)^m = 1 \iff (-1)^{n(r,\mathbf{s})} = 1$ for all $r \in R$ where $\mathbf{s} = (s, t, \cdots, s, t)$.

  Case-1:$r \notin \langle s, t\rangle$ Then $n(r, \mathbf{s}) = 0$ [r does not appear in $\Phi(\mathbf{s})$]
  Case-2:$r \in \langle s, t\rangle$. then $n(r, \mathbf{s}) = 2$ (Draw Cayley Graph!)

The Exchange Condition

Suppose we have a pre-Coxeter system (W, S). This is said to satisfy the Exchange condition, if:
Given a reduced expression $\mathbf{s} = (s_1, \cdots, s_k)$ for $w \in W$ and $s \in S$. Then either $l(sw) = k + 1$ or there is an index i such that $w = ss_1 \cdots \hat{s}_i \cdots s_k$

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The Word problem Suppose $G = \langle S | \mathcal{R} \rangle$. Given a word $\mathbf{s}$ in $S \cup S^{-1}$, is there an algorithm for evaluating whether the value $g(\mathbf{s})$ is equal to the identity?

Tits solved this for Coxeter groups.

An elementary **M-operation** on a word in S is one of the following:

1. **Delete** a subword of the form (s, s)
2. **Replace** an alternating word of the form $(s, t, \cdots)$ of length m_{st} by the alternating word $(t, s, \cdots)$, of same length.

A word is said to be **M-reduced** if it cannot be shortened using a sequence of M-operations.

Tits's soln

Suppose (W, S) satisfies the Exchange condition. Then, two reduced expressions **s** and **t** represent the same element of W iff one can be transformed into the other by a sequence of elementary M-operations of type (2).

Recap

We essentially had an algebraic question, namely:-

I have $W = \langle S \rangle$ where $s^2 = 1$ for all $s \in S$. When can I say that $W = \langle S | \mathcal{R} \rangle$ where $\mathcal{R} = \{(st)^{m_{st}} | m_{st} = |st|\}$?

And we had a geometric answer for this.

$$
\begin{tikzcd}
 & \text{Pre-Coxeter} \arrow[dl, bend right, "a"'] \arrow[dr, bend left, "b"] & \\
\text{Coxeter} \arrow[ur, bend right, "c"'] & & \text{Pre-Reflection} \arrow[ul, bend left, "d"]
\end{tikzcd}
$$

In order to figure out **a** we look at **b**.

- **b**: Given a Coxeter system, (W, S), set $R =$ conjugate closure of S in W. Then, $(R, \text{Cay}(W, S), 1)$ is a pre-reflection system.
- **But** $W \curvearrowright \text{Cay}(W, S)$ freely, sth that pre-reflection systems don't enjoy a priori.

So, we ask:

Pre-reflection system (W, R, Ω, ν_0) + (X)? $\implies$ $W \curvearrowright \Omega$ freely or equivalently, $\Omega \cong$Cay(W, S)?

X= Boost pre-reflections to reflections or equivalently, enable the wall-crossing lemma to hold externally, which leads to conditions (D) and (E).

GOAL

To show that Coxeter groups are linear.

The guiding principle here is that, roughly speaking, the following determine each other:

Geometric Coxeter groups, certain (invariant) symmetric bilinear forms, simplices and more generally convex polytopes in X^n.

Simplices in $\mathbb{X}^n$ are totally specified by their Gram matrices upto isometry (mutatis mutandis)

The cosine matrix, associated to any Coxeter system via its Coxeter matrix, is a similar object which serves the dual role of:

- Helping identify when the underlying Coxeter group can be represented by geometric reflections
- Help build the canonical representation of a Coxeter system

Cones and all that

Let's recall a few objects, starting with cones.

Fix a real fin. dim. vector space V.

Convex Polyhedral Cone

A **cone** in V is the intersection of a finite number of linear half-spaces in V.

Essential cone

A cone C in V is called **essential** if it doesn't contain any line.

Simplicial cone

An essential cone C is called **simplicial** if any m of its codimension 1 faces intersect in a codimension m face.

Cones and all that

This gives us a way to talk about cones and in particular simplices in $\mathbb{X}^n$

Polyhedral Cone in $\mathbb{X}^n$

Image of a cone in $T_x\mathbb{X}^n$ under the exponential map $exp_x : T_x\mathbb{X}^n \to \mathbb{X}^n$

Sticking to vector spaces for the moment, let's also recall what a pseudo-reflection is:

Pseudo-reflection

A **pseudo-reflection** on a vector space V is a linear operator r on V, such that $1 - r$ has rank 1.

Cones and all that

So what do pseudo-reflections look like?
$1 - r$ has rank 1, so $1 - r = \langle -, \alpha \rangle e$ where e is some non-zero vector in range of $1 - r$ and α is a non-zero form. In fact, α is determined by the hyperplane it annihilates - $\ker(1 - r)$ and the value $(1 - r)(e)$.

So if r has finite order, the eigenvalue corresponding to e is a root of unity.
Note: All linear reflections are pseudo-reflections.

Linear Reflections

A linear operator $r : V \to V$ is called a reflection on V if $r^2 = 1$ and $Fix(r)$ is a hyperplane.

Let H be the hyperplane r fixes. If α is any non-zero linear form on V that annihilates H, then $r(v) = v - \frac{2\alpha(v)}{\alpha(e)} e$ where e is an eigenvector corresponding to -1.

So the set of reflections is parametrized by $Gr(V, n-1)$

Spherical simplices

Consider $\mathbb{R}^n$ with the standard inner product. Then,
$S^n = \{v \in \mathbb{R}^{n+1} | \langle v, v \rangle = 1\}$.
For any $u \in S^n$, $T_u S^n = u^\perp$
Hyperplanes are great subspheres and hyperspaces are hemispheres
Reflection about hyperplane normal to u: $r(v) = v - 2\langle v, u \rangle u$.

Another way of obtaining convex polytopes in S^n - intersect an essential cone C with S^n, and similarly for spherical n-simplexes.

Consider a spherical n-simplex $\sigma \subset S^n \subset \mathbb{R}^{n+1}$.

Describing σ:

- σ has $n+1$ codimension 1 faces, label them $\sigma_0, \cdots, \sigma_n$.
- Let u_i be the **inward pointing** unit vector normal to σ_i.
- Note that σ_i determines a hyperplane H_i in S^n. u_i is s.t. $H_i = \{x \in \mathbb{R}^{n+1} | \langle u_i, x \rangle = 0\} \cap S^n$.
- inward pointing $\rightarrow$ u_i and σ lie on the same side of the half-space bounded by the linear subspace through H_i.
- So $\sigma = \{x \in S^n | \langle u_i, x \rangle \geq 0\}$

Gram matrix of σ

The Gram matrix of σ, is defined as a $(n+1) \times (n+1)$ matrix $c_{ij}(\sigma) = \langle u_i, u_j \rangle$.

Observe:

- Writing $U = [u_0 \cdots u_n]$, we have $c(\sigma) = U^T U$.
- $(c_{ij}(\sigma))$ is a positive definite matrix with all diagonal entries equal to 1.

Dihedral angles

The dihedral angle between σ_i and σ_j, denoted by $\theta_{ij} = \pi - cos^{-1}(\langle u_i, u_j \rangle)$. By convention, $\theta_{ii} = \pi$

Theorem

A spherical simplex is determined upto isometry by its Gram matrix.

Take any two n-simplices σ and σ' in S^n. Label their faces and corresponding unit normals as $\sigma_0, \cdots, \sigma_{n+1}, \sigma'_0, \cdots, \sigma'_{n+1}$ and $u_0, \cdots, u_{n+1}, u'_0, \cdots, u'_{n+1}$.
We have $\langle u_i, u_j \rangle = \langle u'_i, u'_j \rangle$ for all i,j. Let U be the unique linear operator mapping $u_i \mapsto u'_i$. Then by construction, U is an isometry.

(When) can we find a spherical simplex with dihedral angles prescribed beforehand?

More precisely, suppose we have a $(n+1) \times (n+1)$ symmetric matrix carrying values $\theta_{ij} \in (0, \pi)$ in off-diagonal positions and π in diagonal positions.
Do we have a spherical simplex σ with faces $\sigma_0, \cdots, \sigma_n$ such that the dihedral angle between σ_i and σ_j is θ_{ij}?

Answer: Yes. if the Gram matrix $A_{ij} = -cos(\theta_{ij})$ is positive definite.

- If A is positive definite, we have $A = P^{-1}DP$ for a unitary matrix P and a diagonal matrix D.
- $A = P^{-1}\sqrt{D}\sqrt{D}P = U^T U$, where $U = \sqrt{D}P$. Let the columns of U be $\{u_i\}_{0 \leq i \leq n+1}$. Then the required simplex $P^n = \bigcap_{i=0}^{n+1} \{x | \langle x, u_i \rangle \geq 0\}$

Cosine Matrix

Let $M = (m_{ij})$ be a Coxeter matrix on a set I. The **cosine matrix** associated to M is an $I \times I$ matrix $(c_{ij}) = -cos(\frac{\pi}{m_{ij}})$.

Canonical representation

- Start with a **Coxeter** matrix $M = (m_{ij})_{(i,j)\in I\times I}$ in a set I. Take a set of symbols indexed by I, $S = \{s_i | i \in I\}$. Let $W = \langle S|\mathcal{R}\rangle$ be the associated group.
- Let (c_{ij}) be the **cosine** matrix for M.
- Let B_M be the bilinear symmetric form on $\mathbb{R}^I$ with basis $\{e_i\}_{i\in I}$, defined as $B_M(e_i, e_j) = c_{ij}$.

GOAL

Construct a representation $\rho : W \to GL(\mathbb{R}^I)$ such that B_M is W-invariant.

Theorem

Let V be a finite dimensional vector space and $\rho : G \to Gl(V)$ be an irreducible representation. Suppose $\exists r \in G$ s.t. $\rho(r)$ is a pseudo-reflection. Then:

1. *If $u \in End(V)$ that commutes with $\rho(G)$, then u is a homothety.*
2. *Any non-zero G-invariant bilinear form on V is non-degenerate. Moreover, such a form is either symmetric or skew-symmetric.*
3. *Any two such G-invariant bilinear forms are proportional.*

(1) **"u is a homothety on a non-trivial linear subspace and we show that this subspace blows up"**

- Let $\tau = 1 - \rho(r)$ and $D = \tau(V)$. Then, dim(D)=1.
- Choose non-zero vectors $x, y \in V$ s.t. $\tau(y) = x$.
- $u\tau = \tau u \implies u(\tau(y)) = \tau(u(y)) \implies u(x) \in D \implies u(D) \subset D$
- So, $u\big|_D = c.1\big|_D$ for some scalar c.
- Let $N = ker(u - c.1)$ and pick any $v \in \rho(G)$. $u - c.1$ commutes with v, so $(u - c.1)(v(N)) = 0$. $v(N) \subset N$.
- But ρ is irreducible and $N \neq 0$, so $N = V$.

(3)

$$\begin{tikzcd} V \arrow[rr, "u"] \arrow[dr, "y\mapsto B'(-,y)"'] & & V \arrow[dl, "y\mapsto B(-,y)"] \\ & V^* & \end{tikzcd}$$

As B, B' are non-degenerate, there is an isomorphism u on V s.t. $B'(x, y) = B(u(x), y)$ for all $x, y \in V$. But B, B' are G-invariant, so u commutes with $\rho(G)$. Hence, u is a homothety so that B and B' are proportional.
(2) Define $N = \{x \in V | B(x, y) = 0 \text{ for all } y\}$
$N' = \{y \in V | B(x, y) = 0 \text{ for all } x\}$. N, N' are $\rho(G)$-stable subspaces. As ρ is irreducible and B is non-zero, $N = N' = 0$.
Finally, take $B'(x, y) = B(y, x)$. Then by (3),
$B(x, y) = \lambda B(y, x) = \lambda^2 B(x, y)$, so $\lambda \in \{\pm\}$.

Constructing the Canonical Representation

Step-1: Define on generators

- For each $s \in S$, let H_i be the hyperplane in $\mathbb{R}^l$

$$H_i = \{x \in \mathbb{R}^l | B_M(e_i, x) = 0\}$$

- Define the reflection $\rho_i : \mathbb{R}^l \to \mathbb{R}^l$ about H_i by $\rho_i(x) = x - 2B_M(e_i, x)e_i$

Step-2: Computing orders of $\rho_i \rho_j$

- Let $W_{ij} = \langle \rho_i, \rho_j \rangle$ be the dihedral group generated by ρ_i, ρ_j. Let $E_{ij} = \mathbb{R}e_i + \mathbb{R}e_j$.
- E_{ij} is W_{ij}-stable [ρ_i flips e_i fixes e_j and similarly].

Write $m = m_{ij}$. $\mathrm{B}_M\big|_{E_{ij}} = \begin{pmatrix} 1 & -cos(\frac{\pi}{m}) \\ -cos(\frac{\pi}{m}) & 1 \end{pmatrix}$

Case-1: $m \neq \infty$

- $B_M\big|_{E_{ij}}$ is then positive definite on E_{ij} and we can identify it with $\mathbb{R}^2$.
- Let L_i, L_j be lines in E_{ij} orthogonal to e_i and e_j respectively.
- Computing in e_i, e_j basis, L_i is the $\mathbb{R}$-span of $v_i = (1, -\alpha)$ and L_j is the $\mathbb{R}$-span of $v_j = (-\alpha, 1)$
- $B_M\big|_{E_{ij}}(v_i, v_i) = 1 - \alpha^2 = B_M\big|_{E_{ij}}(v_j, v_j)$
- $B_M\big|_{E_{ij}}(v_i, v_j) = -\alpha(1 - \alpha^2)$
- $cos(\theta_{ij}) = B_M\big|_{E_{ij}}(v_i, v_j) / \sqrt{B_M\big|_{E_{ij}}(v_i, v_i) B_M\big|_{E_{ij}}(v_j, v_j)} = -\alpha$ where θ_{ij} is the angle bw L_i, L_j and $\alpha = -cos(\pi/m)$
- So the W_{ij}-action on E_{ij} is isomorphic to a D_{2m}-action on $\mathbb{R}^2$. $|\rho_i \rho_j| = m$.

Case-2: $m = \infty$ $\mathrm{B}_M\big|_{E_{ij}} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ $ker(\mathrm{B}_M\big|_{E_{ij}}) = \mathbb{R}.(e_1 + e_j)$.

Let $u = e_i + e_j$.

$$\begin{aligned} \rho_i(e_i) &= -e_i & \rho_j(e_i) &= e_i + 2e_j \\ \rho_i(e_j) &= 2e_i + e_j & \rho_j(e_j) &= -e_j \end{aligned}$$

A computation gives: $(\rho_i\rho_j)^n(e_i) = 2nu + e_i$. So $|\rho_i\rho_j| = \infty$

Hence, the map $S \to GL(\mathbb{R}^l)$ defined by $s_i \mapsto \rho_i$ extends to a homomorphism $\rho : W \to GL(\mathbb{R}^l)$, called the **canonical representation.**

In order to prove that ρ is mono, we need to look at the dual, called the **geometric representation** of W.

Computing the Geometric Representation

What does $\rho^* : W^* \to GL(E^*)$ $[E = \mathbb{R}^l]$ look like?

Define $\xi_i = B_M(-, e_i)$. Let $\varphi \in E^*, x \in E$

$$\begin{aligned}\rho_i^*(\varphi)(x) &= \rho^*(s_i)(\varphi)(x)\\ &= \varphi(\rho_{s_i}(x))\\ &= \varphi(x - 2B_M(x, e_i)e_i)\\ &= (\varphi - 2e_i(\varphi)\xi_i)(x)\end{aligned}$$

$\rho_i^*(\varphi) = \varphi - 2e_i(\varphi)\xi_i$ which is indeed a reflection since $e_i(\xi_i) = 1$

Let C be the simplicial cone in E^* defined by $e_i(\varphi) \geq 0$ for all $i \in I$ and let C^0 be its interior.

Theorem

Let $w \in W$. If $wC^o \cap C^o \neq \phi$ then $w = 1$

This implies that ρ^* and therefore ρ is faithful.

Observe:

- Let $H_i = \{\varphi \in E^* | e_i(\varphi) \geq 0\}$. Then, H_i is the linear half-space of E^* containing ξ_i.
- For each i we have $s_i.H_i \cap H_i = \phi$. $C^o = \bigcap_{i=1}^n H_i$
- When can we say that for any $w \in W - \{1\}$, $w.C^o \cap C^o = \phi$?

Tits' Criterion

Tits gave a criterion which helps answer the question, in much more generality. Let's rephrase the question as follows:

- Let W be a group acting on a set X. $B \subset X$ is called **pre-fundamental** for W if whenever $w.B \cap B = \phi$, we have $w = 1$.
- Suppose we have a pre-Coxeter system (W, S) acting on X and for each $s \in S$, $B_s \subset X$ is pre-fundamental for $\langle s \rangle$.

When can we say that $B = \bigcap_{s \in S} B_s$ is pre-fundamental for W?

Answer: Yes, if the data $(W, S, \{B_s\}_{s \in S})$ satisfies property (P).

Property (P)

$(W, S, \{B_s\}_{s \in S})$ is said to satisfy property (P) if for all $w \in W$ and all $s \in S$, either $wB \subset B_s$ or $wB \subset sB_s$. Morever, in case the latter holds, we have $l(sw) = l(w) - 1$.

$(W, S, \{B_s\}_{s \in S})$ satisfies property $(P)] \implies B = \bigcap_{s \in S} B_s$ is pre-fundamental for W.

- Suppose $w \in W - 1$ is s.t. $wB \cap B \neq \phi$.
- We may write $w = sw'$ where $l(w') = l(w) - 1$ for some $s \in S$.
- But by (P), $wB \subset B_s$, so $w'B \subset sB_s$ and $l(w) = l(w') - 1$, a contradiction.

Tits' Lemma

Tits' contribution here is that he reduced checking (P) for W to checking it for dihedral pairs.

Lemma

For each pair of distinct elements $s, t \in S$ if $(W_{s,t}, \{s, t\}, \{B_s, B_t\})$ satisfies (P), then so does $(W, S, \{B_s\}_{s \in S})$.

The End