Toposes and Physics? On Categorical Structure in the Theory of Classifying Spaces and an Introduction to Classifying Topoi

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Overview

Dualities

- What are they? What do we know about them?
- Depending on the context, there could be several meanings associated.
- Lets look at some familiar examples:
  - Dual of a Vector Space
  - Wave-particle duality
  - AdS/CFT
  - Adjunctions [We'll see what they mean soon]
- So, is there a unifying idea capturing these instances? Yes. Sometimes.
- For a physicist, many a times, a non-trivial duality between theories arises when they give same (or approximately same) results in some physical scenario.

Dualities; Categories and Adjunctions

In general, we note that:

- An equivalence (non-trivial) between two QFTs between two models is a duality.
- One also finds many large classes of dualities in String theory.
- But such an equivalence is but a special case of an **adjunction**!

Let's look at some basic categoric notions to understand this. A **category C** comprises of the following data:

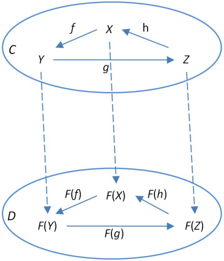
- a collection of **objects**: a,b,c...
- a collection of **morphisms/arrows**: f,g,h...

such that:

- Every arrow has a **domain** and **codomain**. So $f : a \to b$ represents an arrow having domain a and co-domian b. Two arrows with matching dom, co-dom can be **composed**.
- Every object a comes with an **identity arrow** $1_a : a \to a$.

Categories and Adjunctions

satisfying the two axioms:



1. For every arrow $f : a \to b$, $1_b f = f = f 1_a$.
2. For composable f,g,h, associativity holds.

Given two categories **C**,**D**, a **functor** $F : \mathbf{C} \to \mathbf{D}$ comprises of correspondences: $a \mapsto Fa, f \mapsto Ff$, such that $F1_a = 1_{Fa}$ and $F(gh) = FgFh$.

A pair of functors $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$, are said to comprise an **adjunction**, if for every pair of objects c in **C**, d in **D**, there is a bijective correspondence $\mathbf{C}[c, Gd] \cong \mathbf{D}[Fc, d]$ that are natural in c and d.

Back to Dualities

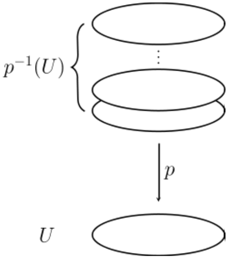
In continuation,

- It turns out that the **mathematical formalizations** of key string-theoretic dualities, such as Topological T-duality and the duality between M-theory and type IIA string theory, can be formulated using **adjunctions**.
- At the heart of such dualities is a "**double dimensional reduction**".
- This is in fact, formalized by one of the most fundamental adjunctions in category theory: **a base change along a point inclusion into a classifying space**!
- But what is a classifying space?

Covering Maps, Covering Spaces

Definition 1

Let X be a top. space and $p : E \to X$ be a surjective and continuous map. Then, p is said to be a *covering map* iff every point x of X admits a neighbourhood U in X such that the inverse image $p^{-1}(U)$ can be written as the union of disjoint sets V_α open in E such that for each α, $p\big|_{V_\alpha} : V_\alpha \xrightarrow{\sim} U$ is a homeomorphism. E is called a *covering space* of X.



Covering Maps, Covering Spaces

In algebraic topology, finding such covering maps plays a pivotal role in computing and the fundamental group $\pi_1(X, x_0)$ relative to some base point x_0.

Now one is naturally compelled to ask the following questions:

- Given a topological space X what are the necessary and sufficient conditions for the existence of a covering map and a covering space?
- What can be said about the uniqueness of a covering space/covering map?
- What are all covering maps?

In order to answer the above, it is now necessary for a natural notion for *equivalence of covering maps* or *equivalence of covering spaces*.

Covering Maps, Covering Spaces

Definition 2

Let $p : E \to X$ and $p' : E' \to X$ be covering maps. Then p and p' are said to be *equivalent* if there exists a homeomorphism $h : E \xrightarrow{\sim} E'$ such that the following diagram commutes.

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

In order to proceed, we restrict ourselves to some *nice* topological spaces [locally connected, locally path-connected, semi-locally simply connected]. For such spaces, one can show that a **universal** covering space exists. In particular, **all** covering maps can be found if one knows the universal covering space: it *characterizes* all covering maps for the chosen base space.

Principal G-Bundles

We encounter them while studying Gauge transformations.

Definition 3

Let G be a top. group and X be a top. space. Then, a *principal G-bundle* over X is a top. space E equipped with a *continuous* map $p : E \to X$ and a *conts.* left action of G on E, $\mu : G \times E \to E$, such that:

- μ **preserves fibers.** $p(g \cdot y) = p(y)$ and therefore $p(g \cdot p^{-1}(x)) \subseteq p^{-1}(x)$
- **p is locally trivial.** There exists a an open covering $\{U_\alpha\}_{\alpha \in J}$ of X such that, $\forall \alpha \in J$ there is a homeomorphism, $\phi_\alpha : G \times U_\alpha \xrightarrow{\sim} p^{-1}(U_\alpha)$, called a *local trivialisation* which *respects the G-action* $p(\phi_\alpha(g, x)) = x$ $\phi_\alpha(g \cdot h, x) = g \cdot \phi_\alpha(h, x)$, $\forall g \times h \in G \times G, \forall x \in U_\alpha$

We can now ask the same questions here mutatis mutandis.

Principal G-Bundles

Surprisingly(?) we get answers that are similar, structurally. This is summarized by the following results:

Theorem

Let E and B be top. spaces such that E is weakly contractible. If $p : E \to B$ is a principal G-bundle, then for all CW complexes X there is a natural bijection $\phi : [X, B] \to \mathcal{P}_G(X)$ defined as $(f : X \to B) \mapsto (f^*(p) : f^*P \to X)$. B is then called the classifying space for G and $p : E \to B$ is called the universal G-bundle. [In fact, B can be constructed as a CW complex.]

The structural similarity mentioned above becomes clear in categorical setup. One can see that this boils down to representability of a functor and conditional property-preserving qualities of pullbacks (in a nice setting).

Towards Classifying Topoi

- Given a suitably *nice* topological space X: Existence of a **universal** cover; The group of covering transformations **classifies** all covering maps onto X.
- Given any topological group G: Existence of a **universal** G-bundle $\pi : EG \to BG$; BG **classifies** all principal G-bundles over all CW complexes.
- Given any discrete group X: Existence of a **universal** G-torsor $\tilde{G}$; $\mathbf{B}G$ **classifies** all G-torsors over any cocomplete elementary topos $\mathcal{E}$.

Therefore, given **structures** of a certain **kind**, we axiomatize them into a **theory, T,** and call these **structures** $\mathcal{M}$ in a category $\mathcal{E}$ or rather the **models** of **T** as **T-models** in $\mathcal{E}$. $\mathbf{Mod}(\mathcal{E}, \mathbf{T})$ is then defined as the category of all these models with suitable model (homo) morphisms between them.
Moreover, it is assumed that the inverse image functor of a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ will carry an **T-model** $\mathcal{M}$ in $\mathcal{E}$ to a **T-model** $f^*\mathcal{M}$ in $\mathcal{F}$.

Classifying Topoi: Conclusion

A *classifying topos* for **T-models** is a topos $\mathcal{B}(\mathbf{T})$ over **Set** such that for every cocomplete topos $\mathcal{E}$ there is an equivalence of categories $c_{\mathcal{E}} : \mathbf{Mod}(\mathcal{E}, \mathbf{T}) \xrightarrow{\sim} \mathbf{Hom}(\mathcal{E}, \mathcal{B}(\mathbf{T}))$ which is natural in $\mathcal{E}$, that is, the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{E}, \mathrm{T}) \arrow[r, "\sim"] \arrow[d, "f^*"'] & \mathrm{Hom}(\mathcal{E}, \mathcal{B}(\mathrm{T})) \arrow[d, "\mathrm{Hom}(f, \mathcal{B}(\mathrm{T}))"] \\
\mathrm{Mod}(\mathcal{F}, \mathrm{T}) \arrow[r, "\sim"] & \mathrm{Hom}(\mathcal{F}, \mathcal{B}(\mathrm{T}))
\end{tikzcd}$$

Therefore, there exists a *universal or generic* **T-model** $U_{\mathbf{T}}$ in $\mathcal{B}(\mathbf{T})$, namely the model corresponding to the identity on $\mathcal{B}(\mathbf{T})$ under the aforementioned equivalence.

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