TIFR VSRP 2020-2021 Report

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In this internship, I am learning the basics of Ergodic theory. I am using Viana-Oliveira's "Foundations of Ergodic Theory" [VO] as a primary reference, with the aim to study its content linearly while solving as many exercises as possible. I have been presenting my studies in weekly meetings with Prof. Mahan Mj. regularly. Presently, I have covered most of the content comprising the first seven chapters of [VO]. I have used [VB],[GF] and [RF] for relevant background in Analysis. In what follows I will briefly describe key notions and results I've learnt in this process, with an emphasis on techniques that I found were fascinating and/or useful. I conclude with a list of questions/ideas and a list of references I am using.

We begin with a dynamical system equipped with an invariant probability measure (f, μ): let M be a measureable space and $f : M \to M$ be a measureable map. M is equipped with a measure μ that is f-invariant: the measure of each measureable subset of M is the same as that of its pre-image under f. Then f-invariant probability measures on M are precisely the fixed points of the map $f_* : \mathcal{M}_1(M) \to \mathcal{M}_1(M)$ where $\mathcal{M}_1(M)$ is the space of all probability measures on M and f_* sends each measure ν to its pushforward $f_*\nu$ under f. The presence of an invariant probability measure immediately influences the distribution of orbits, $\{f^n(x)|n \in \mathbb{N}\}$ of points x in M. For any set $E \subset M$ of positive measure, μ-almost every point x in E returns to E infinitely many times, that is, $f^n(x) \in E$ for infinitely many values of n. This is a version of Poincare recurrence. In fact a stronger version holds: If $f_1, f_2 \cdots, f_p$ is a commuting collection of measureable maps on M, then for any set $E \subset M$ of positive measure, μ-almost every point $x \in E$ returns to E *simultaneaously* infinitely many times, that is, $f_1^n(x), f_2^n(x), \cdots, f_p^n(x)$ lie in E for infinitely many values of n. This result plays a key role in proving Szemeredi's theorem: any set $S \subset \mathbb{Z}$ with positive upper density contains arithmetic progressions of every length.

Dynamical systems equipped with an invariant measure are ubiquitous in Mathematics and Physics. The systems I explored were: n-dimensional torus $\mathbb{T}^n$ equipped with the Lebesgue measure m_n for rotations R_θ, endomorphisms f_A coming from square matrices with integer entries A; bit shift maps in base β, $g_\beta : [0, 1] \to [0, 1]$ defined as $g_\beta(x) = \beta x - [\beta x]$ that sends $x = (0.a_1a_2a_3\cdots)_\beta$ to $(0.a_2a_3a_4\cdots)_\beta$ where the unit interval is equipped with the Lebesgue measure; the Gauss map and Bernoulli shifts. One can ensure the existence of an invariant probability measure μ when M is a compact metric space and f is a continuous map. Equipped with the weak* topology, the set of Borel probability meaures $\mathcal{M}_1(M)$ is metrized by the Levy-Prohorov metric and is in fact compact. This follows from the observation that any sequence of probability measures in $\mathcal{M}_1(M)$ has a convergent subsequence. Indeed, if $\{\mu_n\}$ is a sequence in $\mathcal{M}_1(M)$, consider the positive linear functionals $\phi_n(\psi) = \int \psi d\mu_n$ on the unit ball in $C^0(M)$, which is separable since M is compact. By a weaker version of Arzela-Ascoli's theorem we are guaranteed the existence of a convergent subsequence $\phi_{n_k} \to \phi$. Then by the Riesz-Markov theorem $\phi(\psi) = \int \psi d\mu$ for some $\mu \in \mathcal{M}_1(M)$. It follows that $\mu_{n_k} \to \mu$ in the weak* topology. $f_* : \mathcal{M}_1(M) \to \mathcal{M}_1(M)$ is a continuous linear map. As a consequence, it is easy to obtain an invariant probability measure: we pick any $\nu \in \mathcal{M}_1(M)$ and consider the sequence of cesaro sums of iterates of ν, that is, $\{\frac{1}{n}\sum_{j=0}^{n-1} f_*^j \nu\}$. Every limit point of this sequence will be an invariant measure.

The Poincare recurrence results suggest that it is legitimate to ask if there are points in sets of positive measure that end up spending some non-zero *time* in it. More precisely, fixing some positive measure set $A \subset M$, we ask if the mean sojourn time $\tau(A,x) = \lim_{n\to\infty} \frac{1}{n}\#\{0 \leq j \leq n-1 | f^j(x) \in A\} = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} U_f^j \chi_A(x)$ exists for $x \in A$, where U_f is the Koopman operator for $L^1(M,\mu)$. This has an affirmative answer due to ergodic theorems of Birkhoff and von Neumann. Birkhoff's ergodic theorem entails the existence of time averages $\tilde{\phi} = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} U_f^j \phi$ for any $\phi \in L^p(M,\mu)$ μ-almost everywere. I learnt how to use Kingman's subadditive ergodic theorem to prove Birkhoff's ergodic theorem and saw how Kingman's theorem follows from Fekete's lemma for subadditive sequences of real numbers, coupled with an estimate associated with orbital decomposition for points in M. Yuval Peres' exposition [YP] was very beneficial in understanding the motivation for the estimate. On another note, observing that f_* is the dual of the Koopman operator U_f, it may be seen that when M is compact and f is continuous, von Neumann's theorem for $C^0(M)$is in a sense dual to the existence of an invariant probability measure on M, due to Banach-Alaoglu's theorem.

(f,μ) is said to be an ergodic system when $\tau(A,x) = \mu(A)$ for μ-almost every point x in any measurable set $A \subset M$. Equivalently, an ergodic measure is dynamically *indivisible*, that is, the measure of any invariant set is either 0 or 1. Alternatively, an ergodic measure is one for which, the time average of any integrable function ϕ, that is, $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{j=n-1} \phi \circ f^j(x)$ is constant and equal to $\int \phi d\mu$ for μ-almost every point $x \in M$. Furthermore, noting that $\mathcal{M}_1(M)$ is a convex set that is partially ordered under absolute continuity "$<<$", ergodic measures are the minimal elements of $\mathcal{M}_1(f) \subset \mathcal{M}_1(M)$, the subspace of f-invariant probability measures. The bit shift map for base 10, g_{10} on the unit interval equipped with Lebesgue measure is ergodic and so is the Gauss map. A crucial technique used to establish their ergodicity is to first find a sequence of successively finer partitions $\mathcal{P}_n = \{P_{k,n}\}$ of a full measure subset of $[0,1]$ such that f^k maps each $P_{k,n}$ bijectively to $(0,1)$ and secondly, a control on the *distortion* of measureable subsets of $P_{k,n}$, that is, one looks for a way to bound the quotient $\mu(f^k(A_1))/\mu(f^k(A_2))$ by $\mu(A_1)/\mu(A_2)$ upto a scaling factor where $A_1, A_2 \subset P_{k,n}$.

On the other hand, for tori $\mathbb{T}^n$, rationally independent rotations are ergodic while rationally dependent rotations are not ergodic for the Lebesgue measure. Linear endomorphisms f_A are ergodic wrt to the Lebesgue measure if and only if no eigenvalue of A is a root of unity. In particular, for establishing the ergodicity of invertible linear endomorphisms f_A which are hyperbolic, that is $A \in SL(n,\mathbb{Z})$ such that A has no eigenvalue of modulus 1, we can use a technique used by Hopf to show that geodesic flows on compact surfaces with negative curvature are ergodic. As A

is hyperbolic, $\mathbb{R}^n$ can be written as the direct sum of two invariant subspaces E^u and E^s, spanned by eigenvectors corresponding to eigenvalues of A having modulus larger(and resp. smaller) than 1. Consequently, we can find positive constants C and $\lambda < 1$ such that A contracts vectors in E^s and expands those in E^u: for $k \geq 1$, $||A^k v_s|| \leq C\lambda^k ||v_s||$ and $||A^{-k} v_u|| \leq C\lambda^k ||v_u||$ for $v_s \in E^s, v_u \in E^u$. Translates of E^s and E^u in $\mathbb{R}^n$ drop down under the canonical projection $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ to give a foliation of $\mathbb{T}^n$ by stable leaves $\mathcal{W}^s$ and unstable leaves $\mathcal{W}^u$. Fixing a continuous function $\phi : \mathbb{T}^n \to \mathbb{R}$, it follows from Birkhoff's ergodic theorem that the *future* time averages $\phi^+(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi \circ f_A^j(x)$ and the *past* time averages $\phi^-(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi \circ f_A^{-j}(x)$ are defined on a full measure subset of $\mathbb{T}^n$. Moreover, ϕ^+ is constant on each stable leaf $\mathcal{W}^s$ while ϕ^- is constant on each unstable leaf $\mathcal{W}^u$. A key property of this foliation is that the leaves $\mathcal{W}^s$ and $\mathcal{W}^u$ are *transversal* and *absolutely continuous*. This is used to match the values ϕ^+ and ϕ^-, so that $\phi^+ = \phi^- =$ const. on a full measure subset of $\mathbb{T}^n$, thereby implying that f_A is ergodic.

The minimality of ergodic measures in $\mathcal{M}_1(f)$ implies that ergodic measures for f are precisely the extremal elements of $\mathcal{M}_1(f)$. Therefore, it makes sense to enquire if $\mathcal{M}_1(f)$ is the convex hull of the set of ergodic measures wrt f, that is, we ask if any invariant measure on M can be written as a possibly uncountable convex combination of ergoidc measures. This has a positive answer when M is a complete, separable metric space and f is a continuous map, due to the ergodic decomposition theorem. *In the sequel, M is a metric space equipped with the Borel measure and f is continuous.* Any collection of ergodic measures on M is mutually singular when M is second countable. So we expect to be able to see an invariant probability measure *disintegrate* into ergodic ones. Formally, if $\mathcal{P}$ is a partition of M into measurable subsets, then a disintegration of a probability measure μ on M wrt $\mathcal{P}$, is a collection of probability measures $\{\mu_P | P \in \mathcal{P}\}$ such that $\mu_P(P) = 1$, the map $P \mapsto \mu_P$ is measureable and $\mu(A) = \int \mu_P(A) d\hat{\mu}(P)$, where $\mathcal{P}$ is given the quotient σ-algebra and $\hat{\mu}$ is the quotient measure induced via the canonical projection $\pi : M \to \mathcal{P}$. The *ergodic decomposition theorem* states that if M is complete and separable, and μ is an invariant probability measure on M, then there exists a full measure set $M_0 \subset M$ and a *measurable* partition $\mathcal{P}$ of M_0 such that μ admits a disintegration wrt to $\mathcal{P}$ into ergodic measures. Moreover, any two such disintegrations are $\hat{\mu}$-almost identical. A partition $\mathcal{P}$ is said to be measureable if it can be written as the join $\vee_{n=1}^{\infty} \mathcal{P}_n$ of an ascending sequence of countable partitions $\mathcal{P}_n$.

If (f, μ) is an ergodic system, the fact that time/orbital averages of real-valued integrable functions coincide with their space averages almost everywhere, suggests that we may expect to find points whose orbits are dense in the ambient space. With this in mind, we call a set closed invariant set $\Lambda \subset M$ *minimal* if it contains no proper closed invariant subset. This is equivalent to requiring that the orbit $\{f^n(x) | n \in \mathbb{N}\}$ of every point $x \in \Lambda$ is dense in Λ. If f is *uniquely ergodic*, that is, it admits a unique invariant probability measure, then the support of μ is a minimal set. However, the converse is not true. In [Fur61], Furstenberg shows the existence of a real analytic diffeomorphism of the torus $\mathbb{T}^2$ that preserves the Lebesgue measure on $\mathbb{T}^2$, is minimal, but not ergodic and in particular, not uniquely ergodic. The primary technique used is to show the ergodic equivalence of a blatantly non-ergodic map with a minimal one. Given this result, one may expect to find a converse, or more loosely, a relation between the distribution of orbits and ergodicity, if one changes the quantifier used in defining a minimal set. f is said to be *transitive* if there is some point $x \in M$ whose orbit is dense in M. The status quo of unique ergodicity, minimality, transitivity is changed by a theorem due to Haar. Let G be a locally compact topological group. Then, Haar's theorem says that there exists a Borel measure μ_G on G that is finite on all compact subsets, positive on all non-empty open subsets and is invariant under all left translations $L_g : G \to G$, $x \mapsto g{\cdot}x$. Moreover, μ_G is *irreducible*: if η is any measure invariant under all left translations and finite on all compact subsets, then $\eta = c\mu_G$ for some $c > 0$. It is the the local compactness of G that gives μ_G its irreducibility. This can be seen, by observing that if we have such a μ_G and η, then $\eta << \mu_G$ and c must be the Radon-Nikodym derivative of η wrt μ_G. On the other hand, if $\eta << \mu_G$ then due to the local compactness of G, $\frac{d\eta}{d\mu_G}$ is locally integrable and by Lebesgue's differentiation theorem, $\frac{d\eta}{d\mu_G}(x) = \lim_{r\to 0} \nu(B(x,r))/\mu_G(B(x,r))$, which is independent of x. It is therefore crucial to find a finite quotient $\nu(B(e,\rho))/\mu_G(B(e,\rho))$, where e is the identity element, which follows from local compactness. This quantity is used with Vitali's covering theorem to show $\eta << \mu_G$. Finally, if G is a compact metrizable topological group, then G admits a unique probability measure μ_G, as given by Haar's theorem, called haar measure. G can be metrized by a metric that is invariant under all left and right translations. Using this metric, it can bee seen that for any $g \in G$, the left translation L_g is uniquely ergodic if and only if the orbit $\{g^n | n \in \mathbb{Z}^+\}$ is dense in G, or equivlently L_g is transitive.

Questions and Ideas:

- Let M be a compact metric space and $f : M \to M$ be a continuous map. Then, $f_* : \mathcal{M}_1(M) \to \mathcal{M}_1(M)$ is a continuous linear operator, when $\mathcal{M}_1(M)$ is equipped with the weak* topology. I wish to understand the dynamics of f_*. In particular, I wish to understand the behaviour of f_* near its fixed points - $\mathcal{M}_1(f)$. A more specific question here is: Fix $\nu \in \mathcal{M}_1(M)$. Then the limit points of $\{\frac{1}{n}\sum_{j=0}^{n-1} f_*^j \nu\}$ lie in $\mathcal{M}_1(f)$. Consider such a limit point μ, so that we have a subsequence $\{n_k\}$ such that $\frac{1}{n_k}\sum_{j=0}^{n_k-1} f_*^j \nu \to \mu$. I wish to understand the stability of this convergence.

- Consider $\mathbb{T}^2$ with the Lebesgue measure m. We know that rationally independent rotations and all hyperbolic linear endomorphisms are ergodic. Now, it follows from the Oxtoby-Ulam theorem that there exists a residual subset $\mathcal{R}$ of the group of volume-preserving homeomorphism of $\mathbb{T}^2$, $Homeo_{vol}(\mathbb{T}^2)$ such that each map in $\mathcal{R}$ is ergodic. Is it feasible to describe all ergodic transformations in $Homeo_{vol}(\mathbb{T}^2)$ wrt to m with the collection of all rationally independent rotations and hyperbolic linear endomorphisms? Prof. Mahan suggested that a more accessible question would be to look for an answer in $Sympl(\mathbb{T}^2)$, the group of symplectomorphisms of $\mathbb{T}^2$.

- (Inspired by Weyl's theorem) Consider $\mathbb{H}^2$ and $f \in \mathrm{PSL}(2,\mathbb{R})$. Let $U\mathbb{H}^2$ be the unit tangent bundle of $\mathbb{H}^2$. Then df acts on $U\mathbb{H}^2$, $(x,v) \mapsto (f(x), df_x v))$. As $U\mathbb{H}^2$ is diffeomorphic to $\mathbb{H}^2 \times \mathrm{S}^1$, fixing some point $x \in \mathbb{H}^2$ and a unit vector $v \in T_x\mathbb{H}^2$, I wish to work out how the sequence $\{df_{f^n(x)} v\}$ is distributed in the unit circle $\{z \in \mathbb{C} | |z| = 1\}$ for various f.

I would like to thank Prof. Mahan for guiding me. With his guidance, in the weekly meetings, I was able to fix glitches in my understanding and discuss my interpretations/ideas. By helping me realize the significance of investing time in constructing

one's own bag of examples for interesting results, I was also able to learn better. With every meeting, I have been able to work on my communication skills. The internship has helped me begin preparing a background for exploring new facets of geometry.

References

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