On Classification of Division Algebras and Geometry of $\mathbb{F}$-manifolds

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The problem of visually distinguishing the reals, $\mathbb{R}^2$ from the complex numbers, $\mathbb{C}$ is one that every mathematician faces at some point of time. This is difficult, as most ways of distinguishing them are algebraic in nature. Beginning with the abstraction that the conversion of $\mathbb{R}^2$ into $\mathbb{C}$ is an instance of that of a topological vector space into a topological field, investigations, of an algebraic nature, lead the author to the problem of classification of division algebras over arbitrary fields. Motivated by a need for a theory of analysis for these topological fields, a notion of differentiability for functions valued in the same is defined. This is used to define a new class of geometric objects - finite dimensional smooth $\mathbb{F}-$manifolds and certain sheaves of functions on these objects are defined so as to study the geometry of the former.

I. INTRODUCTION

Consider the 2-dimensional Euclidean space $\mathbb{R}^2$ and the complex numbers $\mathbb{C}$. How does one distinguish between the two? The standard way of distinguishing between two objects in classical mathematics, founded on set theory, is to realise the given objects as sets with structure. Hence distinguishing between the objects involves comparing the sets followed by comparing the structures. $\mathbb{R}$ is the unique (upto isomorphism) ordered field with the least upper bound property and may be constructed by (Cauchy-)completing the rationals $\mathbb{Q}$. $\mathbb{R}^2$ is the 2-dimensional vector space on $\mathbb{R}$ with the Euclidean metric. Now, there are multiple ways of constructing $\mathbb{C}$ from the reals $\mathbb{R}$, all of them realise $\mathbb{C}$ as an algebraic structure, for instance, as the quotient ring $\mathbb{R}[X]/(X^2+1)$. We look for a comparison between the two when considered as geometric structures. $\mathbb{R}^2$ with the Euclidean metric and $\mathbb{C}$ with the usual hermitian metric are trivial instances of Riemannian and Hermitian manifolds respectively. The hermitian metric on $\mathbb{C}$ induces a Riemannian metric which is the same as the Euclidean metric. Hence, both $\mathbb{R}^2$ and $\mathbb{C}$ have the same geodesics. Note, in contrast, that it is possible to distinguish between $\mathbb{R}^2$ and the upper half plane $\mathbb{H}^2$ (as a subset of $\mathbb{C}$), the latter equipped with the poincare metric [1]. The perspective that we take, follows from the observation that appending complex multiplication transforms $\mathbb{R}^2$, considered as a topological vector space, into the topological field $\mathbb{C}$. This leads us to the question:

Question 1. *Is complex multiplication the only binary operation that makes the vector space $\mathbb{R}^2$ into a field?*

II. PRELIMINARIES

We begin by recalling some definitions. The reader may consult [2]-[3] for more details.

A *topological field* X is a field equipped with a topology such that the field operations $(+, \cdot, ()^{-1})$ are continuous.

A *(real) topological n-manifold* is a second-countable, Hausdorff space M that is locally Euclidean of dimension n. An *atlas* on M is a collection $\mathfrak{U}$ of homeomorphisms (charts) $f_\alpha : U_\alpha \to V_\alpha$ of open subsets U_α of M onto open subsets V_α of $\mathbb{R}^n$ such that the domains of the charts cover M. $\mathfrak{U}$ is said to be a *smooth atlas* if all pairs of comprising charts are mutually smoothly compatible. Two smooth atlases are said to be equivalent if their union is smooth atlas.

A *smooth structure* on a topological n-manifold M is an equivalence class of smooth atlases on M. It is easy to see that this class is uniquely determined by its maximal element. A topological n-manifold equipped with a smooth structure is called a *smooth n-manifold.* A smooth n-manifold equipped with a *Riemannian metric*, a smoothly varying inner product on the tangent space at each point of the manifold, is called a *Riemannian manifold.*

A *complex n-manifold* is defined analogously, by equipping a topological n-manifold with a *complex structure.* A complex n-manifold equipped with a *hermitian metric*, a smoothly varying hermitian inner product on the (complex) tangent space at each point of the manifold, is called a *hermitian n-manifold.*

A *pre-sheaf* $\mathcal{F}$ of rings on a topological space X, is a functor from the category of open subsets of X into **Rng**. So, essentially, $\mathcal{F}$ is comprised of two pieces of data:

- Object data: For each open subset U of X, a ring $\mathcal{F}(U)$.
- Morphism data: For every inclusion U$\subset$V, where U and V are open subsets of X, a ring homomorphism (called a *restriction map*) $\rho_{U,V} : \mathcal{F}(V) \to \mathcal{F}(U)$ such that, whenever U,V and W are open subsets of

X such that U⊂V⊂W, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(W) \arrow[rr, "\rho_{V,W}"] \arrow[dr, "\rho_{U,W}"'] & & \mathcal{F}(V) \arrow[dl, "\rho_{U,V}"] \\ & \mathcal{F}(U) & \end{tikzcd}$$

A pre-sheaf F of rings on a topological space is said to be a *sheaf* if it satisfies conditions (M) (Matching condition) and (G) (Gluing condition):

- **M**: Let $\{U_\alpha\}$ be a collection of open subsets of X with $U = \cup_\alpha U_\alpha$. If $s, s' \in \mathcal{F}(U)$ such that $\rho_{U_\alpha,U}(s) = \rho_{U_\alpha,U}(s')$ for every α, then $s = s'$.
- **G**: Let $\{U_\alpha\}$ be a collection of open subsets of X with $U = \cup_\alpha U_\alpha$. If $\{s_\alpha\}$ is a collection of elements $s_\alpha \in \mathcal{F}(U_\alpha)$ such that $\forall \alpha, \beta$ $\rho_{U_\alpha \cap U_\beta, U_\alpha}(s_\alpha) = \rho_{U_\alpha \cap U_\beta, U_\beta}(s_\beta)$, then there exists an element $s \in \mathcal{F}(U)$ such that $\forall \alpha$ $\rho_{U_\alpha,U}(s) = s_\alpha$.

III. CLASSIFICATION OF DIVISION ALGEBRAS

Returning to **Question 1**, we observe that it admits an immediate generalization:

Question 2. *Given an n-dimensional vector space over the field $\mathbb{F}$, when can one give it a field structure? Moreover, if it does admit a field structure, what are all the (non-isomorphic) field structures that it admits?*

Note that an answer to **Question 2** for $\mathbb{F} = \mathbb{R}$ and $n = 2$ is the answer to **Question 1**. Going ahead, if one has found a field (or a class of fields) $\mathbb{F}$ such that $\mathbb{F}^n$ admits a field structure for known values of n, there is an immediate need for investigating the class of functions valued in $\mathbb{F}^n$, equipped with field structure. Since we began with the case $\mathbb{R}^2 \xrightarrow{\text{field conversion}} \mathbb{C}$, it is also going to be of interest to see if the theory of analysis that we develop for $\mathbb{F}^n$ has generalized versions of results from complex analysis.

It can be unambiguously asserted here that most of complex analysis relies on, and therefore begins with, the notion of differentiability of functions from some open subset of $\mathbb{C}$ into $\mathbb{C}$ - the notion of holomorphic functions. Therefore, a theory of analysis developed for the field $\mathbb{F}^n$ will evidently need a notion of differentiability. To arrive at this notion, recall that a function $f : \Omega(\subset \mathbb{R}^2) \to \mathbb{R}^2$ is said to be differentiable at a point $p \in \Omega$ if there exists an $\epsilon > 0$, a $\mathbb{R}-$linear map $T_p : \mathbb{R}^2 \to \mathbb{R}^2$ and a function $R_p : B(p, \epsilon) \to \mathbb{R}^2$ such that $B(p, \epsilon) \subset \Omega$ and for all $x \in B(p, \epsilon)$,

$$f(x) = f(p) + T_p(x-p) + \|(x-p)\|\, R_p(x-p)$$

where $R_p(x-p) \to 0$ as $x \to p$. Observe that one gets the notion of holomorphicity for $f : \Omega(\subset \mathbb{C}) \to \mathbb{C}$ at a point $p \in \Omega$ by requiring $T_p : \mathbb{C} \to \mathbb{C}$ to be a $\mathbb{C}-$linear map and replacing $\mathbb{R}^2$ with $\mathbb{C}$ wherever necessary in the above definition. Therefore, we define:

A function $f : \Omega(\subset \mathbb{F}^n) \to \mathbb{F}^n$ is said to be differentiable at a point $p \in \Omega$ if there exists an $\epsilon > 0$, a linear map $T_p : \mathbb{F}^n \to \mathbb{F}^n$ and a function $R_p : B(p, \epsilon) \to \mathbb{F}^n$ such that $B(p, \epsilon) \subset \Omega$ and for all $x \in B(p, \epsilon)$,

$$f(x) = f(p) + T_p(x-p) + \|(x-p)\|\, R_p(x-p)$$

where $R_p(x-p) \to 0$ as $x \to p$

Note that we have required the vector space to be normed. This suggests that we could investigate **Question 2** for the case of normed $\mathbb{F}-$vector spaces. This has an additional merit: the norm induces a topology on the vector space. Yet another basic and important property of $\mathbb{C}$ that is considered along with holomorphic functions, is that of completeness with respect to the metric induced by the norm on $\mathbb{C}$. Therefore, in view of developing a useful theory of analysis for n-dimensional $\mathbb{F}-$ vector spaces admitting a field structure, we have a modified **Question 2**:

Question 3. *Given an n-dimensional banach space over the field $\mathbb{F}$, when can one give it a field structure? Moreover, if it does admit a field structure, what are all the (non-isomorphic) field structures that it admits?*

Question 3 can be answered for $\mathbb{F} = \mathbb{R}$ using Frobenius's theorem which essentially states that [4]:

Theorem 1. *Frobenius's Theorem If V is a finite-dimensional associative division algebra over $\mathbb{R}$, then it isomorphic to either the reals - $\mathbb{R}$, the complex numbers - $\mathbb{C}$ or the quaternions - $\mathbb{H}$.*

A field is a commutative, associative division algebra. Since, $\mathbb{H}$ is non-commutative, we conclude that only finite-dimensional banach spaces over $\mathbb{R}$ are $\mathbb{R}$ and $\mathbb{C}$. A complete answer to **Question 3** would therefore follow from a generalization of Frobenius's theorem that classifies finite-dimensional associative division algebras over arbitrary fields. Before we go any further, we make some remarks:

- If we do not require the division algebras over $\mathbb{R}$ to be associative, then by Hurwitz's theorem, we have a classification for all normed finite-dimensional division algebras over $\mathbb{R}$ given by the sequence:

  $$\mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{O}$$

  where $\mathbb{O}$ are the octonions. Each inclusion following $\mathbb{C}$ "splits" properties. For instance, $\mathbb{H}$ is not commutative but associative while $\mathbb{O}$ is neither commutative nor associative.

- It is encouraging to remark here that our intuition about generalized theories of analysis has not only been fruitful in answering **Question 1** but has actually been worked out for quaternions $\mathbb{H}$ and octonions $\mathbb{O}$. Quaternionic and Octonionic analysis, in particular, the generalized Cauchy integral formulae can be found in A. Sudbery (1977) ([5]) and Xingmin Li-Lizhong Peng (2002) ([6]) respectively.

- A "splitting" similar to the one described above also occurs in the corresponding theories of analysis. After $\mathbb{C}$, the notions of analyticity, conformality and holomorphy do not coincide.

In order to answer **Question 3** in full generality, we would again want to add more conditions to the class of spaces. Indeed, in retrospect, we have repeated this process a number of times here so as to make progress.

IV. $\mathbb{F}-$MANIFOLDS

Associated with $\mathbb{R}$ and $\mathbb{C}$, there exist natural objects which are rich sources of geometry - real smooth manifolds with riemannian geometry and complex analytic manifolds with complex geometry. We intend to construct similar objects and investigate their geometry for the topological fields which comprise the answer to **Question 3**.

We begin by observing that the defintion of a topological n-manifold makes sense if one replaces $\mathbb{R}^n$ with a banach space over $\mathbb{F}$. So, we define:

An n-dimensional *topological* $\mathbb{F}-$*manifold* is a second-countable, Hausdorff space M that is locally homeomorphic to open subsets of $\mathbb{F}^n$. An *atlas* on M is a collection $\mathfrak{U}$ of homeomorphisms (charts) $f_\alpha : U_\alpha \to V_\alpha$ of open subsets U_α of M onto open subsets V_α of $\mathbb{F}^n$ such that the domains of the charts cover M.

We can now use the notion of differentiability for functions $f : \mathbb{F}^n \to \mathbb{F}^n$ to give M, as defined above, a smooth structure.

A *smooth structure* on a n-dimensional topological $\mathbb{F}$-manifold M is an equivalence class of smooth atlases on M. It is easy to see that this class is again uniquely determined by its maximal element. Therefore, a *smooth* $\mathbb{F}-$*manifold* is an n-dimensional topological $\mathbb{F}-$manifold equipped with a smooth structure. Similarly, analytic $\mathbb{F}-$ manifolds can be defined. However, before we can make any further progress, we run into the following question:

Question 4. *Let V be a finite-dimensional banach space over $\mathbb{F}$. When is V Hausdorff? When does V have a non-discrete topology?*

Question 4 gives us the conditions that we were looking for and these must be added to **Question 3**. This brings us to yet another natural construction.

V. CERTAIN SHEAVES

Consider the collection of sets in the topology of $\mathbb{C}$ - $\mathcal{T}$. For each $U \in \mathcal{T}$, let $\mathcal{F}(U)$ be the set of holomorphic functions $f : U \to \mathbb{C}$. Then $\mathcal{F}(U)$ is a ring for each $U \in \mathcal{T}$. For any $U, V \in \mathcal{T}$ such that $U \subset V$, let $\rho_{U,V} : \mathcal{F}(V) \to \mathcal{F}(U)$ be defined by $(V \xrightarrow{f} \mathbb{C}) \mapsto$ (the composition) $(U \hookrightarrow V \xrightarrow{f} \mathbb{C})$. Then, $\rho_{U,V}$ is a ring homomorphism. These assignements make $\mathcal{F}$ into a sheaf. This allows converting statements written purely in terms of open subsets of $\mathbb{C}$ and holomorphic maps between them, into those about rings and homomorphisms. But what about the reverse process? In fact, the construction of $\mathcal{F}$ remains well-defined if we replace $\mathbb{C}$ with a riemannian or a complex analytic or even an $\mathbb{F}-$manifold. Such sheaves of functions should prove to be interesting when one considers $\mathbb{F}-$manifolds with removed points/subsets. As evident, the sheaves provide an alternate means for defining the $\mathbb{F}-$manifolds.

VI. EPILOGUE

Question 3 after the modification in section IV, leads one to the study of Brauer groups ([7]), eventually leading one into algebraic geometry. A generalised theory of analysis for the topological fields derived from banach spaces over $\mathbb{F}$ for nice classes of fields $\mathbb{F}$ remains open. The theory of $\mathbb{F}-$manifolds developed here can be continued for much longer, leading one to associated Lie theory ([8]).

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