$L = L_1 \oplus L_2 \quad \forall x \in L, \exists!\, x_1 \in L_1, x_2 \in L_2$ a.i. $x = x_1 + x_2$

$p : L \to L \quad p(x) = x_1$ PROIECTIA PE L_1, paralelă cu L_2

$p(L_1) = L_1 \quad L_1$ subsp. invariant

$L_\lambda = \{x \in L \mid T(x) = \lambda x\}$

$T(ax+by) = aT(x) + bT(y) = a(\lambda x) + b(\lambda y) = \lambda(ax+by) \quad \forall x, y \in L_\lambda$
$a, b \in K$

L_λ subsp. vectorial invariant pt T.

$$\lambda e = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda \end{pmatrix} e = \lambda Id(e)$$

$T(e) = \lambda e \iff (T - \lambda id_L) e = 0$ cu $e \neq 0 \iff \det(T - \lambda id_V) = 0$

$\lambda \neq \mu$ $\quad x \in L_\lambda,\ y \in L_\mu$ $\quad ax+by=0 \Rightarrow aT(x)+bT(y)=0 \Rightarrow$

$x, y \neq 0$

$\Rightarrow a\lambda x + b\mu y = 0 \Leftrightarrow a\lambda x = -b\mu y$

$$\left.\begin{array}{l}\lambda \neq 0\\ a \neq 0\end{array}\right| \Rightarrow x = -a^{-1}\lambda^{-1} b\mu y \Leftrightarrow x = \alpha y \quad \alpha \neq 0$$

$$Tx = \lambda x \Leftrightarrow T(\alpha y) = \lambda \alpha y \Leftrightarrow \cancel{\alpha} T(y) = \lambda \cancel{\alpha} y \Rightarrow \left.\begin{array}{l}T(y) = \lambda y\\ T(y) = \mu y\\ y \neq 0\end{array}\right\} \Rightarrow$$

$\lambda = \mu$ contradicție

Vectorii și valorile proprii pt

$$A = \begin{pmatrix} 0 & 2 & -2 \\ -1 & 3 & -2 \\ -1 & 1 & 0 \end{pmatrix} \qquad P_A(\lambda) = \begin{vmatrix} -\lambda & 2 & -2 \\ -1 & 3-\lambda & -2 \\ -1 & 1 & -\lambda \end{vmatrix} =$$

$$= \begin{vmatrix} -\lambda & 2-\lambda & -2+\lambda^2 \\ -1 & 2-\lambda & -2+\lambda \\ -1 & 0 & 0 \end{vmatrix} = -(2-\lambda)\begin{vmatrix} 1 & -2+\lambda^2 \\ 1 & -2+\lambda \end{vmatrix} = (\lambda-2)\left[\lambda - \cancel{2} + \cancel{2} - \lambda^2\right]$$

$= \lambda(\lambda-2)(1-\lambda)$

$\sigma(A) = \{\lambda \in K \mid \lambda$ valoare proprie$\}$ s.n. SPECTRUL lui A.

$\sigma(A) = \{0, 1, 2\}$

L_0: $Ax = 0 \iff \begin{cases} \cdot \quad 2x_2 - 2x_3 = 0 \\ -x_1 + 3x_2 - 2x_3 = 0 \\ -x_1 + x_2 = 0 \end{cases} \iff \begin{cases} -x_1 + x_2 = 0 \\ x_2 - x_3 = 0 \end{cases}$

$\iff \begin{cases} x_1 = x_2 \\ x_3 = x_2 \end{cases}$ $\quad L_0 = \langle (1,1,1) \rangle \quad L_2 = \langle (1, 1, 0) \rangle$

$L_1 = \langle (0, 1, 1) \rangle$

L_1: $Ax = x \iff \begin{cases} \cdot \quad 2x_2 - 2x_3 = x_1 \\ -x_1 + 3x_2 - 2x_3 = x_2 \\ -x_1 + x_2 = x_3 \end{cases} \iff \begin{cases} -x_1 + 2x_2 - 2x_3 = 0 \\ -x_1 + 2x_2 - 2x_3 = 0 \\ -x_1 + x_2 - x_3 = 0 \end{cases}$

$\iff \begin{cases} x_1 - 2x_2 + 2x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \end{cases} \iff \begin{cases} x_1 = 0 \\ x_2 = x_3 \end{cases}$

L_2: $Ax = 2x \iff \begin{cases} 2x_2 - 2x_3 = 2x_1 \\ -x_1 + 3x_2 - 2x_3 = 2x_2 \\ -x_1 + x_2 = 2x_3 \end{cases} \iff \begin{cases} -x_1 + x_2 - x_3 = 0 \\ -x_1 + x_2 - 2x_3 = 0 \\ -x_1 + x_2 - 2x_3 = 0 \end{cases}$

$\iff \begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 - x_2 + 2x_3 = 0 \end{cases} \iff \begin{cases} x_3 = 0 \\ x_1 = x_2 \end{cases}$

$T : \mathbb{R}^3 \to \mathbb{R}^3 \quad T(x) = A \cdot x \qquad A = [T]_{can.}$

$B = \{(1,1,1)^*, (0,1,1)^*, (1,1,0)^*\}$ bază in $\mathbb{R}^3$

$= \{f_1, f_2, f_3\}$

$T(f_1) = 0, \quad T(f_2) = f_2, \quad T(f_3) = 2f_3$

$[T]_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad [T]_B = P^{-1} \cdot A \cdot P, \quad P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

P^{-1} a Gauss-Jordan
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$[T]_B = P^{-1} A P \Rightarrow A = P [T]_B \cdot P^{-1}$$

$$A^3 = ! \qquad A^3 = (P [T]_B P^{-1})^3 = P [T]_B^3 P^{-1} =$$
$$= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 8 & -8 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 8 & -8 \\ -1 & 9 & -8 \\ -1 & 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \qquad t \in \mathbb{R} \qquad A \in M_2(\mathbb{R})$$

$$P_A(\lambda) = \begin{vmatrix} \cos t - \lambda & \sin t \\ -\sin t & \cos t - \lambda \end{vmatrix} = \cos^2 t - 2(\cos t)\lambda + \lambda^2 + \sin^2 t = \lambda^2 - 2\lambda \cos t + 1$$

$$\Delta = 4\cos^2 t - 4 = 4(\cos^2 t - 1) \qquad t \neq k\pi \Rightarrow \Delta < 0 \Rightarrow \sigma(A) = \emptyset$$

$A \in M_n(\mathbb{R})$ $\quad \lambda \in \mathbb{C} \setminus \mathbb{R}$ $\quad P_A(\lambda) = 0$ $\quad v \in \mathbb{C}$ a.î. $A \cdot v = \lambda v$

$\bar{\lambda}$ este valoare proprie

$A\bar{v} = \bar{\lambda}\bar{v}$ $\qquad v = v_1 + i \cdot v_2$ $\quad v_1, v_2 \in \mathbb{R}$, $\lambda = a + ib$

$$A(v_1+iv_2) = (a+ib)(v_1+iv_2)$$

$$Av_1 + iAv_2 = av_1 - bv_2 + i(av_2+bv_1)$$

$$Av_1 = av_1 - bv_2$$

$$Av_2 = bv_1 + av_2$$

$L = \langle v_1, v_2 \rangle$ este invariant pt A $\quad [A|_L] = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

L este subsp. invariant <u>ireductibil</u> pt $T:\mathbb{R}^n \to \mathbb{R}^n$, atunci $\dim L = 1$ sau 2.

$$\varphi(x) = x_1^2 + 3x_1x_2 + 2x_1x_3 + x_2^2 + 4x_2x_3 + x_3^2$$

$\psi(x,y)$ a.i. $\varphi(x) = \psi(x,x)$

$$A = \begin{pmatrix} 1 & \frac{3}{2} & 1 \\ \frac{3}{2} & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$\psi(x,y) = x_1y_1 + \frac{3}{2}x_1y_2 + \frac{3}{2}x_2y_1 + x_2y_2 + 2x_2y_3 + 2x_3y_2 + x_3y_3 + x_1y_3 + x_3y_1$$

$$\psi(x,y) = x_1y_1 + 3x_1y_2 + 2x_1y_3 + x_2y_2 + 2x_2y_3 + x_3y_3.$$

$$\psi(x,y) = \frac{1}{2}\big(\varphi(x+y) - \varphi(x) - \varphi(y)\big)$$

$$\varphi(x) = \sum_{i=1}^{n} a_{ii}x_i^2 + \sum_{i \neq j} 2a_{ij}x_ix_j$$

$\psi(x, e_j) = 0 \quad \forall j$, $x = \sum x_i e_i$ $\quad \sum a_{ij}x_i = 0$

$$\psi(x,y) = \sum a_{ij}x_iy_j$$

$\psi(x, e_1) = 0 \Rightarrow \sum a_{i1} x_i = 0$

$\psi(x, e_j) = 0 \ \forall j \Rightarrow \sum a_{ij} x_i = 0 \quad \forall j$

Sistema lineare con matrice A^t $\Leftrightarrow$ $\dim \mathrm{Rad}(\psi) = n - \mathrm{rg} A^t = n - \mathrm{rg} A$

ψ degenere $\Leftrightarrow$ $\mathrm{Rad}(\psi) \neq \{0\}$

$\psi \neq 0 \Rightarrow \exists e_1$ t.c. $\psi(e_1, e_1) \neq 0 \quad \psi(e_1, e_1) = \lambda_1$

$L = \langle e_1 \rangle \oplus \langle e_1 \rangle^{\perp_\psi}$

$\psi(x, y) = x_1 y_1 + \frac{3}{2} x_1 y_2 + \frac{3}{2} x_2 y_1 + x_2 y_2$
$+ 2 x_2 y_3 + 2 x_3 y_2 + x_3 y_3 + x_3 y_1 + x_1 y_3$

$A = \begin{pmatrix} 1 & \frac{3}{2} & 1 \\ \frac{3}{2} & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \quad \Delta_1 = 1, \ \Delta_2 = \begin{vmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 1 \end{vmatrix} = 1 - \frac{9}{4} = -\frac{5}{4} < 0$

$\Rightarrow \psi$ indefinita $\Rightarrow \exists x \neq 0$ t.c. $\varphi(x) = 0$

Procedeul Gram-Schmidt:

$B = \{f_1, \dots, f_n\}$ bază în L

Construim $B' = \{g_1, \dots, g_n\}$ a.î.

- $g_i \cdot g_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

- $g_k \in \langle f_1, \dots, f_k \rangle$
- Matricea de trecere Λ are $\det(\Lambda) > 0$

$g_1 = \dfrac{f_1}{|f_1|}$, $g_2' = a g_1 + f_1$ a.î. $g_2' \cdot g_1 = 0 \Rightarrow (a g_1 + f_1) \cdot g_1 =$
$= a + f_1 \cdot g_1 = 0 \Rightarrow a = -f_1 \cdot g_1$, $g_2 = \dfrac{g_2'}{|g_2|}$ etc.

Ex.

$B = \{\underset{f_1}{(1,1,1)}, \underset{f_2}{(1,1,0)}, \underset{f_3}{(0,1,1)}\} \longrightarrow B'$ ortonormată

$$|f_1| = \sqrt{1+1+1} = \sqrt{3}$$

$$g_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \quad g_2 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \quad g_3 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$g_2' = a \cdot g_1 + f_2 \qquad a + f_2 \cdot g_1 = 0 \Rightarrow a = -f_2 \cdot g_1 = -\frac{2}{\sqrt{3}}$$

$$g_2' = \left(-\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\right) + (1, 1, 0) = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$$

$$|g_2'| = \sqrt{\frac{1}{9} + \frac{1}{9} + \frac{4}{9}} = \frac{\sqrt{6}}{3} = \sqrt{\frac{2}{3}}$$

$$g_2 = \sqrt{\frac{3}{2}} \cdot \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$$

$$g_3' = a g_1 + b g_2 + f_3 \qquad g_3' \cdot g_1 = 0, \quad g_3' \cdot g_2 = 0$$

$$a + f_3 \cdot g_1 = 0 \qquad b + f_3 \cdot g_2 = 0 \quad \Rightarrow a = -\frac{2}{\sqrt{3}}, \quad b = \frac{1}{\sqrt{6}}$$

$$g_3' = \left(-\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\right) + \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right) + (0, 1, 1) =$$

$$= \left(-\frac{1}{2}, -\frac{1}{2}, -1\right) + (0, 1, 1) = \left(-\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$|g_3| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} \Rightarrow g_3 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\Lambda_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \end{pmatrix}$$

$$\Lambda_2 = P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\det \Lambda_1 = \frac{1}{6} \begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{vmatrix} = \frac{1}{6} \begin{vmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & -3 & +1 \end{vmatrix}$$

$$\det \Lambda_1 = -\frac{1}{3} \begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} = 1$$

$$\det P = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = 1.$$

$$\det P^{-1} = 1 \qquad B \xrightarrow{P^{-1}A} B' \qquad \det(P^{-1}A) = 1 > 0.$$

$T : L \to L$ ste simetric, $x, y \neq 0$ $\quad x \in L_\lambda,\ y \in L_\mu \quad \lambda \neq \mu$

$$\begin{array}{ccccc} Tx\cdot y & = & (\lambda x)\cdot y & = & \lambda(x\cdot y) \\ \| & & & & \| \\ x\cdot Ty & = & x\cdot(\mu y) & = & \mu(x\cdot y) \end{array} \Rightarrow \begin{array}{c} \lambda(x\cdot y) = \mu(x\cdot y) \\ \lambda \neq \mu \end{array} \Rightarrow x\cdot y = 0.$$

$$\varphi(x) = x_1^2 + 3x_1x_2 + 2x_1x_3 + x_2^2 + 4x_2x_3 + x_3^2$$

$$A = \begin{pmatrix} 1 & \frac{3}{2} & 1 \\ \frac{3}{2} & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$T: L \to L$ simetrică $\quad \varphi(x,y) = Tx \cdot y \quad$ formă biliniară simetrică

$$\varphi(y,x) = Ty \cdot x = y \cdot Tx = Tx \cdot y$$

Reciproc $\varphi(x,y)$ formă biliniară simetrică $\Rightarrow \exists\, T: L \to L$ a.î.

$\varphi(x,y) = Tx \cdot y$.

Fie forma $\varphi(x) = x_1^2 + 2x_1x_2 + 4x_1x_3 - 3x_2^2 + 6x_2x_3 + x_3^2$

Să se signatura lui φ

$$\varphi(x) = (x_1^2 + 2x_1x_2 + 4x_1x_3 + x_2^2 + 4x_3^2 + 4x_2x_3) - x_2^2 - 4x_3^2 - 4x_1x_3$$
$$-3x_2^2 + 6x_2x_3 + x_3^2 = (x_1 + x_2 + 2x_3)^2 - 4x_2^2 + 2x_2x_3 - 3x_3^2 =$$
$$= (x_1 + x_2 + 2x_3)^2 - 4\left(x_2^2 - \frac{1}{2}x_2x_3 + \frac{1}{16}x_3^2\right) + \frac{1}{4}x_3^2 - 3x_3^2 =$$

$2b = \frac{1}{2} \Rightarrow b = \frac{1}{4}$

$$= (x_1+x_2+x_3)^2 - 4\left(x_2 - \frac{1}{4}x_3\right)^2 - \frac{11}{4}x_3^2 = y_1^2 - y_2^2 - y_3^2$$

Indicele de inerție este 1

Signatura $(1, 2)$

$y_1 = x_1 + x_2 + x_3$

$y_2 = 2\left(x_2 - \frac{1}{4}x_3\right)$, $y_3 = \frac{\sqrt{11}}{2}x_3$

$$A = \begin{pmatrix} 0 & 2 & -2 \\ -1 & 3 & -2 \\ -1 & 1 & 0 \end{pmatrix}$$

$\Gamma_1(A)$ are centrul 0 și raza 4

$\Gamma_2(A)$ — " — 3 și raza 3

$\Gamma_3(A)$ — " — 0 și raza $= 2$

$r_1(A) = 2 + |-2| = 4$

$r_2(A) = |-1| + |-2| = 3$

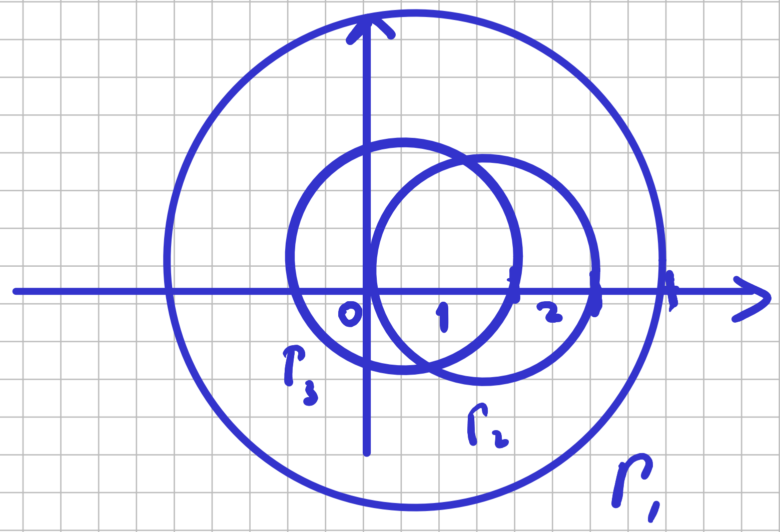
$r_3(A) = |-1| + |1| = 2$

$\sigma(A) = \{0, 1, 2\}$

$0 \in \Gamma_1(A)$ $\quad 2 \in \Gamma_2(A)$

$1 \in \Gamma_3(A)$

$\Gamma_i(A) = \{x \in R^2 \mid |a_{ii} - x| < r_i(A)\}$


0
1
2
Γ_3
Γ_2
Γ_1