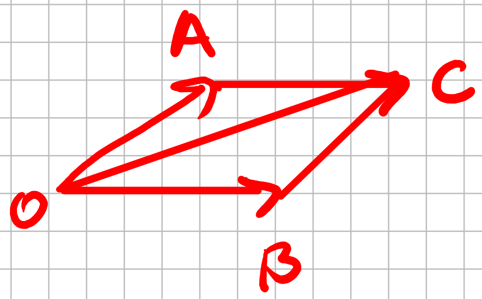
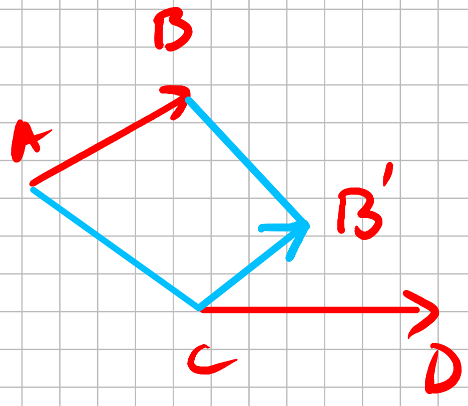
O în plan $\{\vec{OA} \mid A \text{ punct}\} = V_0$



$\vec{AB}$, $\vec{CD}$



$\vec{AB}+\vec{CD}=\vec{CB'}$

$\vec{AB}\sim\vec{A'B'}$ $ABB'A'$ paralelogram

$K^n=\{(x_1,\dots,x_n)^* \mid x_1,\dots,x_n\in K\}$ A^* = transpusa lui A

$C([a,b])=\{f:[a,b]\to\mathbb{R} \mid f$ continuă$\}$

$(f+g)(x)=f(x)+g(x)$, $(\alpha f)(x)=\alpha f(x)$

$K[x] = \{P(x)$ polinom$\}$

$P(x) = a_n x^n + \dots + a_0$

$K = \mathbb{Z}_2$ $\quad P_1(x) = x^2 + x + 1 \quad P_1(0) = 1, \; P_1(1) = 1$

$P_2(x) = x^3 + x + 1 \quad P_2(0) = 1, \; P_2(1) = 1$

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

$Q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_0$

$P(x) + Q(x) = a_0 + b_0 + (a_1 + b_1) x + \dots +$

$\alpha P(x) = (\alpha a_n) x^n + (\alpha a_{n-1}) x^{n-1} + \dots + \alpha a_0.$

$L' \subset L$ ește subsp. vect. $\Leftrightarrow$ $\forall \alpha, \beta \in K, \; x, y \in L'$ $\quad \alpha x + \beta y \in L'$

$L' = \{x \in K^n \mid A \cdot x = 0\}$, unde $A \in M_{m,n}(K)$, $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \}$ m ori

$x, y \in L'$, $\alpha, \beta \in K$ $\quad A(\alpha x + \beta y) = \alpha \underbrace{Ax}_{=0} + \beta \underbrace{Ay}_{=0} = \alpha \cdot 0 + \beta \cdot 0 = 0$

$\Rightarrow \alpha x + \beta y \in L'$

$L_1, L_2 \subset L$ subspații $\quad L_1 \cap L_2$ subspațiu

$x, y \in L_1 \cap L_2$, $\alpha, \beta \in K$ $\quad \alpha x \in L_1 \cap L_2$, $\beta y \in L_1 \cap L_2$

$\alpha x \in L_1$, $(x \in L_1)$, $\alpha x \in L_2$ $(x \in L_2)$

$\alpha x + \beta y \in L_1$, $\alpha x + \beta y \in L_2 \Rightarrow \alpha x + \beta y \in L_1 \cap L_2$

$L_1 \cup L_2$ este subsp $\Leftrightarrow L_1 \subset L_2$ sau $L_2 \subset L_1$

Exemple de subspații

$L = K[X]$, $L'_n = \{ P(X) \in K[X] \mid \text{grad } P(X) \le n \}$.

$\text{grad}(P(X) + Q(X)) \le \max(\text{grad } P(X), \text{grad } Q(X))$

$L = M_n(K)$, $L_1 = \{ A \in L \mid A = A^* \}$, $L_2 = \{ A \in L \mid A = -A^* \}$

L_1, L_2 subspații și $L = L_1 \oplus L_2$

$A, B \in L_1$, $\alpha, \beta \in K$ $\quad (\alpha A + \beta B)^* = \alpha A^* + \beta B^* = \alpha A + \beta B$

$\Rightarrow \alpha A + \beta B \in L_1$

Avem nevoie: $\forall A \in L \ \exists!\ A_1 \in L_1,\ A_2 \in L_2$ a.î. $A = A_1 + A_2$

$A = A_1 + A_2$, $\quad A^* = A_1^* + A_2^* = A_1 - A_2 \quad \Rightarrow$

$A_1 = \frac{1}{2}(A + A^*)$

$A_2 = \frac{1}{2}(A - A^*)$

Unicità. Pr. che $A = B_1 + B_2$ $B_j \in L_j$

$\Rightarrow B_1 + B_2 = A_1 + A_2 \Rightarrow$

$\Rightarrow \underset{\substack{\uparrow \\ L_1}}{B_1 - A_1} = \underset{\substack{\uparrow \\ L_2}}{A_2 - B_2} = C \in L_1 \cap L_2$

$C^* = C = -C \Rightarrow 2C = 0 \Rightarrow C = 0$ $B_1 = A_1$, $B_2 = A_2$

$L = k[x]$ $L_1 = \{ P(x) \in L \mid P(-x) = P(x) \}$

$L_2 = \{ P(x) \in L \mid P(-x) = -P(x) \}$

$L = L_1 \oplus L_2$ $L_1 \cap L_2 \ni P(x)$

$\left. \begin{array}{l} P(-x) = P(x) \\ \quad \shortparallel \\ -P(x) \end{array} \right\} \Rightarrow P(x) = -P(x) \Rightarrow P(x) = 0$

$P(x) = P_1(x) + P_2(x)$

$P(-x) = P_1(-x) + P_2(-x) = P_1(x) - P_2(x)$

$\Rightarrow \quad P_1(x) = \dfrac{P(x) + P(-x)}{2}, \quad P_2 = \dfrac{P(x) - P(-x)}{2}$

$L = \{ x \in \mathbb{R}^3 \mid x_1 + x_2 = 0 \}$

Căutăm $L' \subset \mathbb{R}^3$ subsp. a.i. $L \oplus L' = \mathbb{R}^3$

$L = \{ (a, -a, b) \mid a, b \in \mathbb{R} \}$

Luăm $v \in \mathbb{R}^3$ a.i. $v \notin L$, $v = (1, 1, 0)$

$L' = \{ (c, c, 0) \mid c \in \mathbb{R} \}$ $\quad L'$ s.n. un complement al lui L

$x \in \mathbb{R}^3 \quad x = (x_1, x_2, x_3) = (a, -a, b) + (c, c, 0)$

$$\Rightarrow \begin{cases} a + c = x_1 \\ -a + c = x_2 \\ b = x_3 \end{cases} \Rightarrow \begin{cases} b = x_3 \\ c = \dfrac{x_1 + x_2}{2} \\ a = \dfrac{x_1 - x_2}{2} \end{cases}$$

$$x = \left(\frac{x_1 - x_2}{2}, \frac{x_2 - x_1}{2}, x_3 \right) + \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, 0 \right)$$

$\{x_1, .., x_n\}$ liniar-dependenți

$\exists\, \alpha_1, .., \alpha_n$, nu toți $= 0$ a.î. $\alpha_1 x_1 + ... + \alpha_n x_n = 0$

Pres. $\alpha_1 \neq 0 \Rightarrow \alpha_1 \left(x_1 + \frac{\alpha_2}{\alpha_1} x_2 + ... + \frac{\alpha_n}{\alpha_1} x_n \right) = 0$

$\Rightarrow x_1 + \frac{\alpha_2}{\alpha_1} x_2 + ... + \frac{\alpha_n}{\alpha_1} x_n = 0 \Rightarrow x_1 = -\frac{\alpha_2}{\alpha_1} x_2 - ... - \frac{\alpha_n}{\alpha_1} x_n$

$\{x_1, .., x_n\} \subset L$ liniar-independenți de cardinal maxim

$\forall x \notin \{x_1, .., x_n\} \Rightarrow \{x_1, .., x_n, x\}$ liniar-dependenți $\Rightarrow$

$x = \alpha_1 x_1 + ... + \alpha_n x_n$

$K[x]$ $\{1, x, x^2, .., x^n, ...\}$ liniar-indep. $\Rightarrow \dim K[x] = \infty$

$K_n[x] = \{P(x) \in K[x] \mid \text{grad } P(x) \leq n\} \Rightarrow \{1, x, .., x^n\}$ și deci

$\dim K_n[x] = n+1$

Exemple:

- baza canonică din K^n: $e_1=(1,0,0),\dots,e_n=(0,0,\dots,0,1)$

$$(x_1,\dots,x_n) = x_1e_1+\dots+x_ne_n$$

- În $\mathbb{R}^3$ $B=\{(1,1,0),(2,0,1),(3,2,5)\} = \{f_1,f_2,f_3\}$

B este bază în $\mathbb{R}^3$ $\forall\, x \in \mathbb{R}^3\ \exists!\, x_1,x_2,x_3$ a.î. $x = x_1f_1+x_2f_2+x_3f_3$

$x=(a,b,c) \iff (a,b,c) = x_1(1,1,0)+x_2(2,0,1)+x_3(3,2,5)$

$$\iff \begin{cases} x_1+2x_2+3x_3 = a \\ x_1 \qquad\quad +2x_3 = b \\ \qquad x_2+5x_3 = c \end{cases}$$

Matricea sistemului $=(f_1|f_2|f_3) = A$

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -1 \\ 0 & 1 & 5 \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ 1 & 5 \end{vmatrix} = -9 \neq 0 \Rightarrow$$

Sist. compat. det.

$\{f_1,f_2,f_3\}$ bază $\iff \det(f_1|f_2|f_3) \neq 0$

$L = M_n(K)$ Baza canonică $\{E_{ij} \mid i,j = \overline{1,n}\}$

$$E_{ij} = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{c} \; j \; \\ \begin{pmatrix} 0 & \_ & 0 \\ 0 \cdots & 1 & \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ \leftarrow i \\ \\ \end{array} \qquad \dim L = n^2$$

$L_1 = \{A \in M_n(K) \mid A = A^t\}$, $L = \{A \in M_n(K) \mid A^t = -A\}$

$\Downarrow$

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{12} & a_{22} \cdots & a_{2n} \\ \vdots & & \\ a_{1n} & a_{2n} \cdots & a_{nn} \end{pmatrix} = a_{11}E_{11} + a_{12}(\underbrace{E_{12} + E_{21}}_{F_{12}}) + \cdots + a_{nn}E_{nn}$$

$$= \sum_{1 \le i \le j \le n} a_{ij} F_{ij}, \quad F_{ii} = E_{ii}$$

$$F_{ij} = (E_{ij} + E_{ji})$$

$$\dim L_1 = n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}$$

$A \in L_2 \quad A = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & \cdots & a_{2n} \\ \vdots & & & \\ -a_{1n} & -a_{2n} & & 0 \end{pmatrix} = a_{12}(E_{12} - E_{21}) + \dots + a_{n-1\,n}(E_{n-1\,n} - E_{n\,n-1}) = \sum_{1 \le i < j \le n} a_{ij} H_{ij}$

$H_{ij} = E_{ij} - E_{ji}$

$\dim L_2 = (n-1) + (n-2) + \dots + 1 = \frac{n(n-1)}{2}$

$\frac{n(n+1)}{2} + \frac{n(n-1)}{2} = n^2$

$L = \langle (1, 1, -2, 3), (3, 4, 1, 5), (2, 3, 3, 2) \rangle$

$\dim L = ?$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 4 & 3 \\ -2 & 1 & 3 \\ 3 & 5 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 7 & 7 \\ 0 & -4 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} \underline{1} & 0 & -1 \\ 0 & \underline{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\operatorname{Rg} A = 2 \Rightarrow \dim L = 2$, bază în $L = \{(1,1,-2,3), (3,4,1,5)\}$

$v_1, .., v_m \in K^n \qquad \dim \langle v_1, .., v_m \rangle = \operatorname{rg}(v_1 | \dots | v_m)$

și o bază este dată de acei vectori pe care se sprijină minorul principal (care conțin pivoți)

$$\delta_x(y) = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}$$

$$\sum_{x \in M} a_x \delta_x = 0 \iff \left(\sum a_x \delta_x\right)(y) = 0 \ \forall y \iff a_x = 0 \ \forall x$$

$f : M \to K$

M finită

$$f = \sum_{x \in M} a_x \delta_x \qquad a_x = f(x)$$

$T: K^n \longrightarrow K^m \qquad T(x) = A \cdot x \qquad A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$

Concret. $T(x) = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 3 \\ 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (x_1 + x_2 + 2x_3, 3x_1 + x_2 + 3x_3, 2x_1 + x_2 + 4x_3)^t$

$T(x) = (x_1 - x_2, x_1 + x_2, x_1 + 3x_2)^t, \; T: \mathbb{R}^2 \to \mathbb{R}^3$

$T(x) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$T: K^n \to K^m$ este liniară $\Leftrightarrow$ $\exists A \in M_{m,n}(K)$ a.î.

$T(x) = A \cdot x$

Fie $T: \mathbb{R}^2 \to \mathbb{R}^3$ $T(x) = (x_1 - x_2, x_1 + x_2, x_1 + 3x_2)$

$B_1 = \{(1,1), (1,-1)\}$ bază în $\mathbb{R}^2$

$B_2 = \{(1,0,1), (1,1,0), (0,1,1)\}$ bază în $\mathbb{R}^3$

Matricea lui T în rap. cu bazele B_1, B_2 $=: [T]_{B_1}^{B_2}$

$T(1,1) = (0,2,4)$

$T(1,-1) = (2,0,-2)$

$$\left(\begin{array}{ccc|cc} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 4 & -2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cc} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & -1 & 1 & 4 & -4 \end{array}\right) \longrightarrow$$

$$\longrightarrow \left(\begin{array}{ccc|cc} 1 & 0 & -1 & -2 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 & -4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cc} 1 & 0 & -1 & -2 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & -2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 3 & -2 \end{array}\right)$$

$$T(1,1) = 1\cdot(1,0,1) + (-1)(1,1,0) + 3(0,1,1)$$

$$T(1,-1) = 0\cdot(1,0,1) + 2(1,1,0) - 2(0,1,1)$$

$$[T]_{B_1}^{B_2} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 3 & -2 \end{pmatrix}$$

$$[T] = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{pmatrix}$$

$T_1 : L_1 \to L_2 \quad T_2 : L_2 \to L_3 \qquad T = T_2 \circ T_1 : L_1 \to L_3$

$B_1 = \{e_1, \dots, e_k\}$ bază în L_1

$B_2 = \{f_1, \dots, f_m\}$ bază în L_2

$B_3 = \{h_1, \dots, h_n\}$ bază în L_3

$$T_1(e_j) = \sum_{s=1}^{m} a_{sj} f_s \qquad T(e_j) = \sum_{s=1}^{n} c_{sj} h_s$$

$$T_2(f_j) = \sum_{s=1}^{n} b_{sj} h_s$$

$$T(e_j) = T_2(T_1(e_j)) = T_2\left(\sum_{s=1}^{m} a_{sj} f_s\right) = \sum_{s=1}^{n} a_{sj} T_2(f_s)$$

$$= \sum_{s=1}^{m} a_{sj} \sum_{t=1}^{n} b_{ts} h_t = \sum_{r=1}^{n} c_{rj} h_r$$

$$\sum_{r=1}^{n}\left(\sum_{s=1}^{m} a_{sj}\, b_{rs}\right) h_r = \sum_{r=1}^{n} c_{rj}\, h_r$$

$$c_{rj} = \sum_{s=1}^{m} b_{rs}\, a_{sj} \quad \Rightarrow [T] = [T_2]\cdot[T_1]$$

Fie $B_1 = \{(1,3,2),(2,1,1),(1,1,2)\}$ Scrieți matricea de schimbare a bazei de la baza canonică la baza B_1 și invers.

$B = \{e_1, e_2, e_3\}$ baza canonică

$B_1 = \{f_1, f_2, f_3\}$ $\quad x = x_1 f_1 + x_2 f_2 + x_3 f_3 = x_1(1,3,2) +$

$+ x_2(2,1,1) + x_3(1,1,2) = (x_1 + 2x_2 + x_3,\ 3x_1 + x_2 + x_3,\ 2x_1 + x_2 + 2x_3)$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$B_1 \underset{(f_1|f_2|f_3)^{-1}}{\overset{(f_1|f_2|f_3)}{\rightleftarrows}} B$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -3 & 1 & 0 \\ 0 & -3 & 0 & -2 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & -5 & -2 & -3 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -\frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & -2 & \frac{1}{3} & 1 & \frac{5}{3} \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -\frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{6} & -\frac{1}{2} & -\frac{5}{6} \end{array}\right) \longrightarrow \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\;\middle|\;\underbrace{\begin{matrix} -\frac{1}{6} & \frac{1}{2} & \frac{3}{2} \\ \frac{2}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{2} & -\frac{5}{6} \end{matrix}}_{(f_1 | f_2 | f_3)^{-1}}\right)$$

$$L = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0, \; x_1 + x_2 = 0\}$$

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 + x_2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 + x_2 - x_3 = 0 \\ x_3 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_3 = 0 \\ x_1 = -x_2 \end{cases} \qquad L = \{(-x_2, x_2, 0) \mid x_2 \in \mathbb{R}\} = \langle (-1, 1, 0) \rangle$$

$$\dim L = 1 \qquad \operatorname{rg} A = 2, \quad \dim L = 3 - 2$$

$T : L \to M$ izomorfism $\Rightarrow$ T injectiv $\qquad T(x) = T(y) \Rightarrow x = y$

$$\left.\begin{array}{l} T(x) = 0 \\ T(0) = 0 \end{array}\right\} \Rightarrow x = 0 \qquad \operatorname{Ker} T = \{0\}$$

$T : L \to M \qquad \operatorname{Ker} T = \{0\} \Leftrightarrow T(x) = 0 \Leftrightarrow x = 0$

$T(x) = T(y) \Rightarrow T(x-y) = 0 \Rightarrow x - y = 0 \Leftrightarrow x = y$

$\operatorname{Ker} T = \{0\} \Rightarrow T$ injectivă

$\dim L = \dim M$, T injectivă

$\{e_1, , e_n\}$ bază în L $\Rightarrow \{T(e_1), , T(e_n)\}$ bază în M

$\operatorname{Im} T := T(L)$ $\quad \dim \operatorname{Im} T =: \operatorname{rg}(T)$

$T : K^n \to K^m$ $\quad T(x) = A \cdot x$, $A = (c_1 | \ldots | c_n)$

$\{e_1, , e_n\}$ baza canonică $\quad c_j = T(e_j) \ \forall j = \overline{1,n}$

$\operatorname{Im} T = \langle T(e_1), , T(e_n) \rangle = \langle c_1, , c_n \rangle$

Imaginea unei aplicații liniare este generată de coloanele matricei sale în raport cu baza canonică.

Fie $T: \mathbb{R}^3 \to \mathbb{R}^4$ $T(x) = (x_1 + x_2 - x_3, x_1 - 2x_2 + x_3, 2x_1 - x_2, 3x_2 - 2x_3)$ $[T] = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & -1 & 0 \\ 0 & 3 & -2 \end{pmatrix}$

Să det. $\operatorname{Ker} T$ și $\operatorname{Im} T$.

$x \in \operatorname{Ker} T$ $\begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 - 2x_2 + x_3 = 0 \\ 2x_1 - x_2 = 0 \\ 3x_2 - 2x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 + x_2 - x_3 = 0 \\ -3x_2 + 2x_3 = 0 \\ -3x_2 + 2x_3 = 0 \\ 3x_2 - 2x_3 = 0 \end{cases}$

$\Leftrightarrow \begin{cases} x_1 + x_2 - x_3 = 0 \\ 3x_2 - 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = \frac{2}{3} x_3 \\ x_1 = \frac{1}{3} x_3 \end{cases}$

$\operatorname{Ker} T = \{ (\frac{1}{3} x_3, \frac{2}{3} x_3, x_3) \mid x_3 \in \mathbb{R} \} = \langle (1,2,3) \rangle$

$\dim \operatorname{Ker} T = 1$

$\mathrm{Im}\, T = \langle (1,1,2,0), (1,-2,-1,3), (-1,1,0,-2) \rangle$

$\dim \mathrm{Im}\, T$:
$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & -1 & 0 \\ 0 & 3 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & 2 \\ 0 & -3 & 2 \\ 0 & 3 & -2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \underline{1} & 1 & -1 \\ 0 & \underline{-3} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow \dim \mathrm{Im}\, T = 2 = \mathrm{rg}\, A$$

$\mathrm{Im}\, T = \langle (1,1,2,0), (1,-2,-1,3) \rangle$

$\dim \mathrm{Ker}\, T = 3 - \mathrm{rg}\, T$

T: di dimensioni ; $T : L \to M$ $\quad \dim L = \dim \mathrm{Ker}\, T + \dim \mathrm{Im}\, T$.

$\mathbb{R}^3 = \langle (1,1,1), (1,0,1) \rangle \oplus \langle (1,2,1) \rangle =: L_1 \oplus L_2$

Să se scrie matricea pe L_1 paralelă cu L_2 în raport cu baza canonică

$x \in \mathbb{R}^3 \qquad x = v_1 + v_2 \qquad , v_1 = a(1,1,1) + b(1,0,1)$

$v_2 = c(1,2,1)$

$P(x) = v_1 \qquad v_1 = x - v_2(x)$

Pasul 1 $\{(1,1,1), (3,0,2), (2,1,1)\}$ bază în $\mathbb{R}^3$

$$\begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & 2 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & -1 & -1 \end{vmatrix} = 2 \neq 0$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & x_1 \\ 1 & 0 & 2 & x_2 \\ 1 & 1 & 1 & x_3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & x_1 \\ 0 & -3 & -1 & x_2-3x_1 \\ 0 & -1 & -1 & x_3-2x_1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & x_1 \\ 0 & -1 & -1 & x_3-3x_1 \\ 0 & -3 & -1 & x_3-2x_1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & x_3-2x_1 \\ 0 & -1 & -1 & x_3-3x_1 \\ 0 & 0 & 2 & -2x_3+7x_1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & x_3-2x_1 \\ 0 & -1 & -1 & x_3-3x_1 \\ 0 & 0 & 1 & \frac{-2x_3+7x_1}{2} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & x_3-2x_1 \\ 0 & +1 & 0 & -\frac{x_1}{2} \\ 0 & 0 & 1 & \frac{-2x_3+7x_1}{2} \end{array}\right)$$

$$a = -2x_1+x_3, \quad b = -\frac{x_1}{2}, \quad c = \frac{7x_1-2x_3}{2}$$

$$p(x) = (-2x_1+x_3)(1,1,1) + \left(-\frac{x_1}{2}\right)(1,0,1) =$$

$$= \left(-\frac{5}{2}x_1 + x_3\ ,\ -2x_1 + x_3\ ,\ -\frac{5}{2}x_1 + x_3\right)$$

$$[p] = \begin{pmatrix} -\frac{5}{2} & 0 & 1 \\ -2 & 0 & 1 \\ -\frac{5}{2} & 0 & 1 \end{pmatrix}$$

Baza duală $\{e_1^*, \dots, e_n^*\}$ $\quad$ e_j^* este proiecția pe $\langle e_j \rangle$

$F : L \to L^{**} \qquad F(x) \in L^{**}\ ,\ F(x) : L^* \to K\ ,\ F(x)(f) = f(x)$

$F(x)(f) =: (f, x)$

$T : L \to M\ ,\ T^* : M^* \to L^*$

$T^*(f) \in L^* \qquad T^*(f)(x) = f(T(x)) \quad \forall\, x \in L$

$$T^*(f) = f \circ T$$

$$[T^*] = [T]^t$$