$$F(c) = a_1 c_1 + \dots + a_n c_n = (a_1, \dots, a_n) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} =: a \cdot c$$

$$F(c+d) = (a_1, , a_n) \left[\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \right] = (a_1 \dots a_n) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} +$$

$$+ (a_1, , a_n) \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = F(c) + F(d)$$

„$\Leftarrow$"

Pret. că $F(\alpha c + \beta d) = \alpha F(c) + \beta F(d)$

$$c = c_1 (1, 0, , 0) + c_2 (0, 1, 0, \dots, 0) + \dots + c_n (0, \dots, 0, 1)$$

$$F(c) = c_1 F((1, 0, , 0)) + c_2 F(0, 1, \dots, 0)) + \dots + c_n F((0, \dots, 0, 1)) =$$

$$= a_1 c_1 + \dots + a_n c_n \quad , \quad a_1 = F((1, 0, , 0)), \dots, a_n = F((0, \dots, 0, 1))$$

Exemple

1. $\begin{cases} x_1 + x_2 - x_3 = 5 \\ 3x_1 + x_2 - 2x_3 = 4 \\ x_1 - x_2 + x_3 = 6 \end{cases} \overset{(-3)E_1+E_2}{\underset{(-1)E_1+E_3}{\Longleftrightarrow}} \begin{cases} x_1 + x_2 - x_3 = 5 \\ -2x_2 + x_3 = -11 \\ -2x_2 + 2x_3 = 1 \end{cases}$

$\overset{E_3-E_2}{\Longleftrightarrow} \begin{cases} x_1 + x_2 - x_3 = 5 \\ -2x_2 + x_3 = -11 \\ x_3 = 12 \end{cases} \Longleftrightarrow \begin{cases} x_3 = 12 \\ -2x_2 = -23 \\ x_1 + x_2 = 17 \end{cases} \Longleftrightarrow$

$\Longleftrightarrow \begin{cases} x_3 = 12 \\ x_2 = \frac{23}{2} \\ x_1 = 12 - \frac{23}{2} = \frac{1}{2} \end{cases}$

Sistem compat. determinat

2. $\begin{cases} x_1+x_2-x_3=4 \\ 2x_1+3x_2-2x_3=6 \\ 3x_1+4x_2-3x_3=10 \end{cases}$ $\begin{matrix} E_2-2E_1 \\ \Leftrightarrow \\ E_3-3E_1 \end{matrix}$ $\begin{cases} x_1+x_2-x_3=4 \\ x_2=-2 \\ x_2=-2 \end{cases}$

$\Leftrightarrow \begin{cases} x_1+x_2-x_3=4 \\ x_2=-2 \\ 0=0 \end{cases}$ Sist. compat. nedet.

$\begin{cases} x_2=-2 \\ x_1=6+x_3 \end{cases}$ $Sol=\{(x_3+6,-2,x_3) \mid x_3\in\mathbb{R}\}$

3. $\begin{cases} x_1+x_2-x_3=4 \\ 2x_1+3x_2-2x_3=6 \\ 3x_1+4x_2-3x_3=5 \end{cases} \iff \begin{cases} x_1+x_2-x_3=4 \\ x_2=-2 \\ x_2=-7 \end{cases}$

$\iff \begin{cases} x_1+x_2-x_3=4 \\ x_2=-2 \\ 0=-5 \end{cases} \Rightarrow$ sistem incompatibil

4. $\begin{cases} x_1+x_2-3x_3+x_4=1 \\ 3x_1-x_2+13x_3-2x_4=5 \\ 2x_1+2x_2-3x_3+x_4=7 \end{cases} \underset{E_3-2E_1}{\overset{E_2-3E_1}{\iff}} \begin{cases} x_1+x_2-3x_3+x_4=1 \\ -4x_2+12x_3-5x_4=2 \\ -3x_3-x_4=5 \end{cases}$

x_4 nec. secundară $\Rightarrow$ $\begin{cases} x_3=-\dfrac{5+x_4}{3} \\ -4x_2+12x_3=2+5x_4 \\ x_1+x_2-x_3=1-x_4 \end{cases}$

$$\begin{cases} x_3 = -\dfrac{5+x_4}{3} \\ -4x_2 = 2+5x_4+4(5+x_4) = 22+9x_4 \end{cases} \iff \begin{cases} x_3 = -\dfrac{5+x_4}{3} \\ x_2 = -\dfrac{22+9x_4}{4}. \end{cases}$$

$$x_1 + x_2 - x_3 = 1 - x_4$$

$$x_1 = 1 - x_4 - x_2 + x_3 = 1 - x_4 + \frac{22+9x_4}{4} - \frac{5+x_4}{3} = \frac{58+11x_4}{12}$$

$$12 + 66 - 20 =$$

$$-12x_4 + 27x_4 - 4x_4$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \qquad c_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \ldots, c_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix}$$

$$\ell_1 = (a_{11}, \ldots, a_{1n}), \ldots, \ell_n = (a_{n1}, \ldots, a_{nn})$$

$A = (c_1 | \ldots | c_n)$ matricea avind coloanele $c_1, \ldots, c_n$

$$A = \begin{pmatrix} \ell_1 \\ \hline \vdots \\ \hline \ell_n \end{pmatrix}$$

$A = (c + c_1 | \ldots | c_n)$ $\qquad \det A = \det(c | \ldots | c_n) + \det(c_1 | \ldots | c_n)$

$$A = \begin{pmatrix} \ell_1 + \ell \\ \hline \vdots \\ \hline \ell_n \end{pmatrix} \qquad \det A = \det \begin{pmatrix} \ell_1 \\ \hline \vdots \\ \hline \ell_n \end{pmatrix} + \det \begin{pmatrix} \ell \\ \hline \vdots \\ \hline \ell_n \end{pmatrix}$$

$$A = (\alpha C_1 | C_2 | \dots | C_n) \Rightarrow \det A = \alpha \det (C_1 | \dots | C_n)$$

$$\det (C_1 + \beta C_2 | C_2 | \dots | C_n) = \det (C_1 | \dots | C_n)$$

$$\| \qquad\qquad\qquad \|$$

$$\det(C_1 | C_2 | \dots | C_n) + \beta \underbrace{\det(C_2 | C_2 | \dots | C_n)}_{\substack{\| \\ 0}}$$

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -2 \\ 3 & 4 & -3 \end{pmatrix} \qquad \det A = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & -2 \\ 3 & 4 & -3 \end{vmatrix} = 1 \begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix} - 1 \begin{vmatrix} 2 & -2 \\ 3 & -3 \end{vmatrix}$$

$$+ (-1) \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1 + 1 = 0$$

$$\begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & -2 \\ 3 & 4 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\|$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 0 \end{vmatrix} = 0$$

$A = (a_{ij})_{i,j=\overline{1,n}}$ $\quad$ $\operatorname{Per}(a_{11}) = a_{11}$, $\operatorname{Per}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} =$

$= a_{11}a_{22} + a_{12}a_{21}$, $\quad$ $\operatorname{Per} A = a_{11} P_1 + a_{12} P_2 + \dots + a_{1n} P_n$

Rez. sistemul cu metoda Cramer

$$\begin{cases} x_1 + x_2 - x_3 = 5 \\ 3x_1 + x_2 - 2x_3 = 4 \\ x_1 - x_2 + x_3 = 6 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 3 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & -2 & 5 \\ 0 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -3 \end{vmatrix} = 6$$

$$D_1 = \begin{vmatrix} 5 & 1 & -1 \\ 4 & 1 & 2 \\ 6 & -1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 5 & -1 \\ 1 & 4 & 2 \\ -1 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 5 & -1 \\ 0 & -1 & 3 \\ 0 & 11 & 0 \end{vmatrix} =$$

$$= -\begin{vmatrix} 1 & 5 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 33 \end{vmatrix} = +33, \quad D_2 = \begin{vmatrix} 1 & 5 & -1 \\ 3 & 4 & 2 \\ 1 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -1 \\ 0 & -11 & 5 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 5 & -1 \\ 0 & 1 & 2 \\ 0 & -11 & 5 \end{vmatrix} = -\begin{vmatrix} 1 & 5 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 27 \end{vmatrix} = -27$$

$$D_3 = \begin{vmatrix} 1 & 1 & 5 \\ 3 & 1 & 4 \\ 1 & -1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 0 & -2 & -11 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 0 & -2 & -11 \\ 0 & 0 & 12 \end{vmatrix} = -24$$

$$x_1 = \frac{33}{6} = \frac{11}{2}, \quad x_2 = -\frac{27}{6} = -\frac{9}{2}, \quad x_3 = -\frac{24}{6} = -4$$

$$\begin{cases} x_1 + x_2 - x_3 = 4 \\ 2x_1 + 3x_2 - 2x_3 = 6 \\ 3x_1 + 4x_2 - 3x_3 = 10 \end{cases} \qquad A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -2 \\ 3 & 4 & -3 \end{pmatrix}$$

$\det A = 0$

$$\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \neq 0 \Rightarrow \text{rg}\, A = 2$$

$$\bar{A} = \begin{pmatrix} 1 & 1 & -1 & 4 \\ 2 & 3 & -2 & 6 \\ 3 & 2 & -3 & 10 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \neq 0 \qquad \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & -2 \\ 3 & & -3 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 1 & 4 \\ 2 & 3 & 6 \\ 3 & 4 & 10 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 1 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{vmatrix} = 0 \Rightarrow rg\,\bar{A} = 2 = rg\,A \Rightarrow \text{sistema compatibile}$$

$$A = \begin{pmatrix} 1 & 2 & 1 & 5 & 6 \\ 3 & 1 & 3 & 2 & 7 \\ 2 & -1 & 2 & -3 & 1 \\ 4 & 3 & 4 & 7 & 13 \end{pmatrix} \qquad rg\,A = ?$$

$$A \to \begin{pmatrix} 1 & 2 & 1 & 5 & 6 \\ 0 & -5 & 0 & -13 & -11 \\ 0 & -5 & 0 & -13 & -11 \\ 0 & -5 & 0 & -13 & -11 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 5 & 6 \\ 0 & -5 & 0 & -13 & -11 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$Rg\,A = 2$

Minorul principal $\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$

$$\begin{cases} x_1 + x_2 - x_3 + x_4 = 1 \\ 3x_1 - 2x_2 + x_3 + 3x_4 = 4 \\ 4x_1 - x_2 + \quad 4x_4 = 5 \end{cases}$$

$$\bar{A} = \begin{pmatrix} 1 & 1 & -1 & 1 & 1 \\ 3 & -2 & 1 & 3 & 4 \\ 4 & -1 & 0 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & -5 & 4 & 0 & 1 \\ 0 & -5 & 4 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & -5 & 4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\operatorname{rg} \bar{A} = 2 = \operatorname{rg} A$, deoarece

$\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}$ minor princ in A și $\bar{A}$

Sist $\Longleftrightarrow \begin{cases} x_1 + x_2 - x_3 + x_4 = 1 \\ -5x_2 + 4x_3 = 1 \end{cases} \Longleftrightarrow \begin{cases} x_2 = \dfrac{4x_3 - 1}{5} \end{cases}$

$$x_1 = 1 + x_3 - x_4 - \frac{4x_3 - 1}{5} = \frac{6 + x_3 - 5x_4}{5}$$

$$Sol = \left\{ \left(\frac{x_3 - 5x_4 + 6}{5}, \frac{4x_3 - 1}{5}, x_3, x_4 \right) \mid x_3, x_4 \in \mathbb{R} \right\}$$

$2 = 4 - 2$, Nr. de nec. secundare este $n - rg A$

$A = M_{m,n}(\mathbb{R})$ $rg A$ maxim şi $= rg \bar{A}$

Dacă $rg A = m \Rightarrow$ Avem $n-m$ nec. secundare

$rg A = n \Rightarrow$ Compatibil determinat

$$A \in M_{m,n}(\mathbb{R})$$

$$p \cdot A = \begin{array}{c} \\ m \end{array} \left(\underbrace{\begin{matrix} p & 0 & \dots & 0 \\ 0 & p & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & p \end{matrix}}_{m} \right) \cdot A = p \cdot I_m \cdot A$$

$$A \in M_{m,n}(\mathbb{R}),\ B \in M_{n,k}(\mathbb{R})$$

$$B = (c_1 | \dots | c_k)$$

$$C = A \cdot B = (\overline{c_1} | \dots | \overline{c_k})$$

$$\overline{c_j} = A \cdot c_j \quad \forall\, j = \overline{1,k}$$

$A \in M_n(\mathbb{R})$

$A^{-1} = (c_1 | \dots | c_n)$ $\quad A \cdot A^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{pmatrix}$

$A \cdot c_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_1, \quad A \cdot c_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = e_2, \quad \dots, \quad A \cdot c_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} = e_n$

Metoda Gauss - Jordan

$$\begin{cases} x_1 + x_2 - x_3 = 1 \\ 2x_1 + 3x_2 + 3x_3 = 3 \\ x_1 - x_2 - 3x_3 = 4 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & 3 & 3 \\ 1 & -1 & -3 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 5 & 1 \\ 0 & -2 & -2 & 3 \end{array}\right) \longrightarrow$$

$$\xrightarrow[L_3+2L_2]{L_1-L_2} \left(\begin{array}{ccc|c} 1 & 0 & -6 & 0 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 8 & 5 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -6 & 0 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 1 & \frac{5}{8} \end{array}\right)$$

$$\xrightarrow[L_2-5L_3]{L_1+6L_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{15}{4} \\ 0 & 1 & 0 & -\frac{17}{8} \\ 0 & 0 & 1 & \frac{5}{8} \end{array}\right)$$

$$Ax = b \quad , x = A^{-1} b$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 3 & 3 & 0 & 1 & 0 \\ 1 & -1 & -3 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 5 & -2 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -6 & 3 & -1 & 0 \\ 0 & 1 & 5 & -2 & 1 & 0 \\ 0 & 0 & 8 & -5 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -6 & 3 & -1 & 0 \\ 0 & 1 & 5 & -2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{8} & \frac{1}{4} & \frac{1}{8} \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{4} & \frac{1}{2} & \frac{3}{4} \\ 0 & 1 & 0 & \frac{9}{8} & -\frac{1}{4} & -\frac{5}{8} \\ 0 & 0 & 1 & -\frac{5}{8} & \frac{1}{4} & \frac{1}{8} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{3}{4} & \frac{1}{2} & \frac{3}{4} \\ \frac{9}{8} & -\frac{1}{4} & -\frac{5}{8} \\ -\frac{5}{8} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{3}{4} & \frac{1}{2} & \frac{3}{4} \\ \frac{9}{8} & -\frac{1}{4} & -\frac{5}{8} \\ -\frac{5}{8} & \frac{1}{4} & \frac{1}{8} \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 3 \\ 1 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$