Synoptic :→ Sym ①

Sardar Patel Institute of Technology

Bhavan's Campus, Munshi Nagar, Andheri (West), Mumbai-400058, India
(Autonomous College Affiliated to University of Mumbai)

Mid Semester Examination

March - 2018

Max. Marks: 30
Class: SE(Comp. and IT)
Course Code: BS41
Name of the Course: Applied Mathematics-II

Duration: 1.5 Hrs
Semester: IV
Branch: Comp, IT

Instructions:
(1) All questions are compulsory
(2) Assume suitable data if necessary

Q1 :

Ans Every square matrix satisfies its characteristic eqn is called Cayley-Hamilton's Theorem. — 1 mark

$\therefore |A-\lambda I| = 0$

$$\therefore \begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$$

After simplification, $\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$ — 1 mark

By Theorem Replace λ by A

$$\therefore A^3 - 5A^2 + 9A - I = 0 \quad \text{—— } (1)$$

Now, $A^2 = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$ — 1 mark

Consider LHS of eqn (1) is

$L.H.S. = A^3 - 5A^2 + 9A - I$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= RHS$ — ① mark

Hence C-H Theorem is verified.

Now premultiply by A^{-1} to eqn ①

$\therefore A^2 - 5A + 9I - A^{-1} = 0$

$\therefore A^{-1} = A^2 - 5A + 9I$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Also $A^4 = \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -42 & 13 \end{bmatrix}$ — (02) marks

or

Q1:

Ans The characteristic eqn is $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} -1-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$$ — ① mark

$\Rightarrow \lambda^2 - 9 = 0 \Rightarrow \lambda = 3, -3$

i) For $\lambda = 3$ $\therefore [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $R_2 + \frac{1}{2}R_1$ $\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore -4x_1 + 4x_2 = 0 \Rightarrow x_1 - x_2 = 0$ — ① mark

$\therefore X_1 = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ put $x_2 = t$ $\therefore \boxed{x_1 = t}$

ii) For $\lambda = -3$, $[A - \lambda I]x = 0$ gives

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

After simplifying $X_2 = t \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ — 1 mark

$\therefore M = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ and $|M| = -3$

$\therefore M^{-1} = \dfrac{\operatorname{adj} M}{|M|} = -\dfrac{1}{3} \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$ — 1 mark

Since $f(A) = \tan A$

$\therefore f(D) = \begin{bmatrix} \tan 3 & 0 \\ 0 & \tan(-3) \end{bmatrix} = \begin{bmatrix} \tan 3 & 0 \\ 0 & -\tan 3 \end{bmatrix}$

$\therefore \tan A = M f(D) M^{-1}$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \tan 3 & 0 \\ 0 & -\tan 3 \end{bmatrix} \left(\frac{-1}{3}\right) \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$\tan A = -\frac{1}{3} \begin{bmatrix} \tan 3 & -4\tan 3 \\ -2\tan 3 & -\tan 3 \end{bmatrix}$$

$$\Rightarrow 3\tan A = \begin{bmatrix} -\tan 3 & 4\tan 3 \\ 2\tan 3 & \tan 3 \end{bmatrix}$$

$$= \tan 3 \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix} = \tan 3 \cdot A$$

$\Rightarrow 3\tan A = A \cdot \tan 3$

2 marks

Q2 :→

Ans : step I:

$A'A = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{bmatrix} 25-\lambda & 7 \\ 7 & 25-\lambda \end{bmatrix} = 0 \Rightarrow \lambda = 32, 18$ — 1 mark

Step II : Take square roots of these values & arrange them in descending order to get D

$\therefore \lambda_1 = 32 \quad \therefore \sigma_1 = 4\sqrt{2}$

$\lambda_2 = 18 \quad , \sigma_2 = 3\sqrt{2}$

$$\therefore D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix}$$

Step III :→ for $\lambda = 32 \quad \therefore v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $\lambda = 18 \quad \therefore v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Since $(v_1 \cdot v_2) = (1,1) \cdot (1, -1) = 1 - 1 = 0$

$\therefore v_1, v_2$ are orthogonal

2 marks

we now normalise them by dividing them by their norms

$\therefore \|v_1\| = \sqrt{2} \qquad \|v_2\| = \sqrt{2}$

$\therefore$ The normalised vectors, $v_1 = \dfrac{v_1}{\|v_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

and $v_2 = \dfrac{v_2}{\|v_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$$\therefore V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

1 mark

Step IV :→ The columns of U are

$$u_1 = \frac{1}{\sigma_1} \cdot A v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$u_2 = \frac{1}{\sigma_2} A v_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\therefore U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\therefore A = UDV'$

$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$ — 2 marks

Q3:→

Ans we have $\int_0^1 kx^2(1-x^3)\,dx = 1$

$\therefore \boxed{k=6}$

new p.d.f. is $f(x) = 6x^2(1-x^3)$, $0 \le x \le 1$
$= 0$, otherwise — 2 marks

Now $P\left(0 \le x \le \frac{1}{2}\right) = \int_0^{1/2} f(x)\,dx$

$= \frac{15}{64}$ — 1 mark

$E(x) = \int_0^1 x f(x)\,dx = \frac{6}{14}$

$E(x^2) = \int_0^1 x^2 f(x)\,dx = \frac{9}{20}$

$\therefore Var(x) = E(x^2) - \{E[x]\}^2 = 0.351$ — 3 marks

Q4:→

Ans we know $P(x) = \frac{e^{-m} m^x}{x!}$ — 1

But mean $= m = 1$ and S.D $= \sqrt{m} = 1$ — 1 mark

$\therefore (1) \Rightarrow P(x) = \frac{1}{e}\frac{1}{x!}$

Then prepare the table ———] (2) marks

$\therefore \sum P(x)|x-1| = \frac{1}{e} + 0 + \frac{1}{e}\frac{1}{2!} + \frac{1}{e}\frac{2}{3!} + \frac{1}{e}\frac{3}{4!} + \cdots \frac{1}{e}\frac{x-1}{x!} + \cdots$

$= \frac{1}{e}\left[1 + 0 + \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots \frac{x-1}{x!}\right] + \cdots$

$= \frac{1}{e}\left[1 + \left(\frac{1}{1!} - \frac{1}{1!}\right) + \left(\frac{2}{2!} - \frac{1}{2!}\right) + \left(\frac{3}{3!} - \frac{1}{3!}\right) + \left(\frac{4}{4!} - \frac{1}{4!}\right) + \cdots \left(\frac{x}{x!} - \frac{1}{x!}\right)\right] + \cdots$ — (1 m

$= \frac{1}{e}\left[1 + \frac{1}{1!} + \frac{2}{2!} + \frac{3}{3!} + \frac{4}{4!} + \cdots \frac{x}{x!} + \cdots - \frac{1}{1!} - \frac{1}{2!} - \frac{1}{3!} - \frac{1}{4!} - \cdots - \frac{1}{x!} \cdots\right]$

$= \frac{1}{e}\left(1 + \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \frac{1}{(x-1)!} + \cdots\right) - \left[1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \frac{1}{x!} \cdots\right] + 1\right]$ — (1 mark

$= \frac{1}{e}\left(1 + \cancel{e} - \cancel{e} + 1\right] = \frac{2}{e}(1) = \frac{2}{e}$ S.D.

Q5:

Ans: prepare the table ——— (3) marks

$r = \frac{\sum xy}{\sqrt{\sum x^2 \cdot \sum y^2}}$ ——— (1) mark

But $\sum xy = 97$, $\sum x^2 = 216$, $\sum y^2 = 162$

$\therefore r = 0.5186$] — (2) marks.