CHAPTER

TEN

RECURRENCE RELATIONS AND RECURSIVE ALGORITHMS

10.1 INTRODUCTION

Suppose we ask a friend the age of his oldest daughter. He could tell us directly that she is 19 years old. Or he could tell us that she is 6 years older than his second daughter. If we ask for the age of the second daughter, instead of telling us that she is 13 years old, he might tell us that she is 5 years older than his third daughter. In turn, he could tell us that his third daughter is 2 years older than his only son. When he tells us that his only son is 6 years old, we would have no difficulty in figuring out that his third daughter is 8 years old, his second daughter is 13 years old, and his oldest daughter is 19 years old.

Let us consider another example. Suppose we ask for instructions to get from our house to the railroad station. We are told, "Go on Prospect Avenue, then go east on Green Street. After passing the public library, go onto Springfield Avenue and then go north on Neil Street. At the bus depot, turn right at the traffic light onto University Avenue. At the second traffic light, turn left and you'll see the railroad station." This, of course, is a perfectly clear way to instruct someone to go from our house to the railroad station. However, there is an alternative way to give the instruction, which is simply, "*Go to the bus depot* and turn right at the traffic light onto University Avenue. At the second traffic light, turn left and you'll see the railroad station." We note that such an instruction consists of two parts: one part tells us how to go from the bus depot to the railroad station explicitly, and the other simple and concise part makes use of our knowledge of how to get from our house to the bus depot. Suppose we are not sure how to get from our house to the bus depot. In that case, we can be further instructed to

"*Go to the public library* and onto Springfield Avenue. Then go north on Neil Street to the bus depot." Needless to say, either we know how to get from our house to the public library, or we should ask for further instruction.

The reader has undoubtedly realized what we are trying to say. In the first example, instead of telling us the age of his oldest daughter directly, our friend chose to tell us the age of his oldest daughter in terms of the age of his second-oldest daughter. Then, instead of telling us directly the age of his second-oldest daughter, he chose to tell us the age of his second-oldest daughter in terms of the age of his third-oldest daughter, and so on. In the second example, instead of spelling out explicitly all the details of some instruction, we specified the instruction partly in terms of knowledge we already have. We can make several observations about these two examples. First, using our prior knowledge can be a concise way to give information or instruction; for example, in directing someone to the railroad station, a great deal of information can be compressed into the simple statement, "Go to the bus depot." Secondly, we do need to do some work to make use of the knowledge we already have. In the first example, at a certain point, our friend must tell us directly the age of one of his children so that we can determine the ages of the other children. In the second example, we need to find out how to get to the public library before we can use the given instruction on how to get from there to the railroad station. Third, we might try to refer to some prior knowledge in successive steps (the age of our friend's son, that of his third-oldest daughter, and so on, or how to go to the public library, how to go to the bus depot, and so on.) Such a chain of references can only be terminated when we reach a point where we know explicitly what to do without referring to other prior knowledge.† In this chapter, we shall apply what we have just learned first to the specification of discrete numeric functions, and then to the specification of algorithms.

10.2 RECURRENCE RELATIONS

Consider the numeric function $\mathbf{a} = (3^0, 3^1, 3^2, \ldots, 3^r, \ldots)$. Clearly, the function can be specified by a general expression for a_r, namely,

$$a_r = 3^r \qquad r \geqq 0$$

† An American professor has written a paper in English, his native language, and wishes to have it published in a French journal. Since he knows no French, he asked his French colleague, Professor Bestougeff, to translate the paper into French for him. After the translation was completed, the American professor felt that he should include a footnote in the paper to acknowledge his colleague's contribution. He wrote the footnote, "The author wishes to thank Professor Bestougeff for translating this paper into French for him," in English, and asked Professor Bestougeff to translate it into French for him, which the professor gladly did. It then occurred to the American professor that he should also acknowledge Professor Bestougeff for translating the footnote for him. So, he wrote another footnote, "The author wishes to thank Professor Bestougeff for translating the preceding footnote into French for him," he then asked Professor Bestougeff to translate that into French, and copied the French translation *twice* as two additional footnotes. His problem was completely solved!

As was pointed out in Chap. 9, the function can also be specified by its generating function, namely,

$$A(z) = \frac{1}{1-3z}$$

According to our discussion in Sec. 10.1, we note that there is still another way to specify the numeric function. Since the value of a_r is three times the value of a_{r-1}, for all r, once we know the value of a_{r-1} we can compute the value of a_r. The value of a_{r-1} can, in turn, be computed as three times the value of a_{r-2}, which, again, is equal to three times the value of a_{r-3}. Eventually, we need the value of a_0, which is known to be 1. Thus, we note that the relation

$$a_r = 3a_{r-1}$$

together with the information that $a_0 = 1$ also completely specifies the numeric function **a**.

As another example, consider the sequence of numbers† known as the *Fibonacci sequence of numbers*. The sequence starts with the two numbers 1, 1 and contains numbers that are equal to the sum of their two immediate predecessors. A portion of the sequence is

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

It is quite difficult in this case to obtain a general expression for the rth number in the sequence by observation, which, incidentally, is

$$a_r = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{r+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{r+1}$$

Nor is it obvious what the generating function for the numeric function is. Incidentally, it is

$$A(z) = \frac{1}{1-z-z^2}$$

On the other hand, the sequence can be described by the relation

$$a_r = a_{r-1} + a_{r-2}$$

together with the information $a_0 = 1$ and $a_1 = 1$.

For a numeric function $(a_0, a_1, a_2, \ldots, a_r, \ldots)$, an equation relating a_r, for any r, to one or more of the a_i's, $i < r$, is called a *recurrence relation*. A recurrence relation is also called a *difference equation*, and those two terms will be used interchangeably. In many discrete computation problems, it is sometimes easier to obtain a specification of a numeric function in terms of a recurrence relation than to obtain a general expression for the value of the numeric function at r or a closed-form expression for its generating function. It is clear that according to the recurrence relation, we can carry out a step-by-step computation to determine a_r from $a_{r-1}, a_{r-2}, \ldots$, to determine a_{r+1} from $a_r, a_{r-1}, \ldots$, and so on, provided that the value of the function at one or more points is given so that the computation can be initiated. These given values of the function are called *boundary conditions*. In the first example above, the boundary condition is $a_0 = 1$, and in the second example above, the boundary conditions are $a_0 = 1$ and $a_1 = 1$. We thus conclude that a numeric function can be described by a recurrence relation together with an appropriate set of boundary conditions. The numeric function is also referred to as the *solution of the recurrence relation*.

† Clearly, a numeric function can be viewed simply as a sequence of real numbers, and conversely.

One step beyond determining the values of a numeric function in a step-by-step computation according to a given recurrence relation is to obtain from the recurrence relation either a general expression for the solution or a closed-form expression for its generating function. Unfortunately, no general method of solution for handling all recurrence relations is known. In the following, we shall study a class of recurrence relations known as *linear recurrence relations with constant coefficients*.

10.3 LINEAR RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

A recurrence relation of the form

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r) \tag{10.1}$$

where C_i's are constants, is called a *linear recurrence relation with constant coefficients*. The recurrence relation in (10.1) is known as a *kth-order recurrence relation*, provided that both C_0 and C_k are nonzero. For example,

$$2a_r + 3a_{r-1} = 2^r$$

is a first-order linear recurrence relation with constant coefficients. Also, both

$$3a_r - 5a_{r-1} + 2a_{r-2} = r^2 + 5 \tag{10.2}$$

and

$$a_r + 7a_{r-2} = 0$$

are second-order linear recurrence relations with constant coefficients. In this chapter we shall restrict our discussion to linear recurrence relations with constant coefficients both because we frequently encounter this class of recurrence relations and know how to handle them quite well.

Consider the recurrence relation in (10.2). Suppose we are given that $a_3 = 3$ and $a_4 = 6$, we can compute a_5 as

$$a_5 = \frac{-1}{3}\left[-5 \times 6 + 2 \times 3 - (5^2 + 5)\right] = 18$$

we can then compute a_6 as

$$a_6 = \frac{-1}{3}[-5 \times 18 + 2 \times 6 - (6^2 + 5)] = \frac{119}{3}$$

and so on. Also, we can compute

$$a_2 = \frac{-1}{2}[3 \times 6 - 5 \times 3 - (4^2 + 5)] = 9$$

$$a_1 = \frac{-1}{2}[3 \times 3 - 5 \times 9 - (3^2 + 5)] = 25$$

$$a_0 = \frac{-1}{2}[3 \times 9 - 5 \times 25 - (2^2 + 5)] = \frac{107}{2}$$

and so on. We conclude that (10.2), together with the values $a_3 = 3$ and $a_4 = 6$, completely specifies the discrete numeric function **a**.

In general, for a kth-order linear recurrence relation with constant coefficients as shown in (10.1), if k consecutive values of the numeric function **a**, a_{m-k}, $a_{m-k+1}, \ldots, a_{m-1}$ are known for some m, the value of a_m can be calculated according to (10.1), namely,

$$a_m = -\frac{1}{C_0}[C_1 a_{m-1} + C_2 a_{m-2} + \cdots + C_k a_{m-k} - f(m)]$$

Furthermore, the value of a_{m+1} can be computed as

$$a_{m+1} = -\frac{1}{C_0}[C_1 a_m + C_2 a_{m-1} + \cdots + C_k a_{m-k+1} - f(m+1)]$$

and the values of $a_{m+2}, a_{m+3}, \ldots$ can be computed in a similar manner. Also, the value of a_{m-k-1} can be computed as

$$a_{m-k-1} = -\frac{1}{C_k}[C_0 a_{m-1} + C_1 a_{m-2} + \cdots + C_{k-1} a_{m-k} - f(m-1)]$$

and the value of a_{m-k-2} can be computed as

$$a_{m-k-2} = -\frac{1}{C_k}[C_0 a_{m-2} + C_1 a_{m-3} + \cdots + C_{k-1} a_{m-k-1} - f(m-2)]$$

The values of $a_{m-k-3}, a_{m-k-4}, \ldots$ can be computed in a similar manner. Indeed, for a kth-order linear recurrence relation, the values of k consecutive a_i's are always sufficient to determine the numeric function **a** uniquely. In other words, the values of k consecutive a_i's constitute an appropriate set of boundary conditions.

On the other hand, for a kth-order linear recurrence relation with constant coefficients, fewer than k values of the numeric function will not be sufficient to determine the numeric function uniquely. For example, let

$$a_r + a_{r-1} + a_{r-2} = 4 \tag{10.3}$$

...e are given that $a_0 = 2$, we can find many numeric functions that will satisfy ...e recurrence relation as well as the given boundary condition. Thus,

$$2, 0, 2, 2, 0, 2, 2, 0, 2, 2, 0, \ldots$$

$$2, 2, 0, 2, 2, 0, 2, 2, 0, 2, 2, \ldots$$

$$2, 5, -3, 2, 5, -3, 2, 5, -3, 2, \ldots$$

...e all possibilities. Yet, more than k values of the numeric function might make ...impossible for the existence of a numeric function that satisfies the recurrence ...elation and the given boundary conditions. For example, for the recurrence ...elation in (10.3), if we were given that

$$a_0 = 2 \qquad a_1 = 2 \qquad a_2 = 2$$

...hen obviously a_0, a_1, and a_2 do not satisfy the recurrence relation. Consequently, no **a** can satisfy (10.3) and the boundary conditions.

The values of k nonconsecutive a_i's might or might not constitute an appropriate set of boundary conditions, depending on the specific recurrence relation we have. We shall not study the problem of what constitutes an appropriate set of boundary conditions here, since it is not a significantly important one. See, however, Prob. 10.8.

We should point out that if a kth-order recurrence relation is not a linear recurrence relation with constant coefficients, k consecutive values of the numeric functions might not specify uniquely a solution. For example, consider the recurrence relation

$$a_r^2 + a_{r-1} = 5$$

Given that $a_0 = 1$, we note that

$$1, 2, \sqrt{3}, \ldots$$

$$1, 2, -\sqrt{3}, \ldots$$

$$1, -2, \sqrt{7}, \ldots$$

are all solutions to the recurrence relation that satisfy the boundary condition.

We shall restrict our discussion to the solution of linear recurrence relations with constant coefficients. There is a significant advantage to restrict ourselves to this class of recurrence relations. Since we know that for a given set of boundary conditions the solution to a linear recurrence relation with constant coefficients is unique, as long as we are able to find a numeric function that satisfies the recurrence relation as well as the boundary conditions, it is *the* solution we are looking for. Such an argument should remove some of the "mystery" about the solution procedure we are going to present. We shall determine the solution by "guessing" what it will be. The justification for guessing is simply that it works: when the procedure yields a solution to the recurrence relation, it will be the correct solution.

but also is a root of thé derivative equation of (10.5),

$$C_0 r\alpha^{r-1} + C_1(r-1)\alpha^{r-2} + C_2(r-2)\alpha^{r-3} + \cdots + C_k(r-k)\alpha^{r-k-1} = 0 \quad (10.$$

because α_1 is a multiple root of (10.5). Multiplying (10.6) by $A_{m-1}\alpha$ and replac[ing] α by α_1, we obtain

$$C_0 A_{m-1} r\alpha_1^r + C_1 A_{m-1}(r-1)\alpha_1^{r-1}$$
$$+ C_2 A_{m-1}(r-2)\alpha_1^{r-2} + \cdots + C_k A_{m-1}(r-k)\alpha_1^{r-k} = 0$$

which shows that $A_{m-1} r\alpha_1^r$ is indeed a homogeneous solution.

The fact that α_1 satisfies the second, third, ..., $(m-1)$st derivative equati[ons] of (10.5) enables us to prove that $A_{m-2}r^2\alpha_1^r$, $A_{m-3}r^3\alpha_1^r$, ..., $A_1 r^{m-1}\alpha_1^r$ are al[so] homogeneous solutions in a similar manner.

Example 10.2 Consider the difference equation:

$$a_r + 6a_{r-1} + 12a_{r-2} + 8a_{r-3} = 0$$

The characteristic equation is

$$\alpha^3 + 6\alpha^2 + 12\alpha + 8 = 0$$

Thus,

$$a_r^{(h)} = (A_1 r^2 + A_2 r + A_3)(-2)^r$$

is a homogeneous solution since -2 is a triple characteristic root.

Example 10.3 Consider the difference equation

$$4a_r - 20a_{r-1} + 17a_{r-2} - 4a_{r-3} = 0$$

The characteristic equation is

$$4\alpha^3 - 20\alpha^2 + 17\alpha - 4 = 0$$

and the characteristic roots are $\frac{1}{2}$, $\frac{1}{2}$, 4. Consequently, the homogeneous so[lu]tion is

$$a_r^{(h)} = (A_1 r + A_2)(\tfrac{1}{2})^r + A_3(4)^r$$

10.5 PARTICULAR SOLUTIONS

There is no general procedure for determining the particular solution of a diffe[r]ence equation. However, in simple cases, this solution can be obtained by th[e] method of inspection. As will be demonstrated in the examples in this section, w[e] first set up the general form of the particular solution according to the form o[f] $f(r)$, and then determine the exact solution according to the given differen[ce] equation. Consider the difference equation

$$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 \quad (10.7)$$

We assume that the general form of the particular solution is

$$P_1 r^2 + P_2 r + P_3 \quad (10.8)$$

where P_1, P_2, and P_3 are constants to be determined. Substituting the expression in (10.8) into the left-hand side of (10.7), we obtain

$$P_1 r^2 + P_2 r + P_3 + 5P_1(r-1)^2 + 5P_2(r-1) + 5P_3$$
$$+ 6P_1(r-2)^2 + 6P_2(r-2) + 6P_3$$

which simplifies to

$$12P_1 r^2 - (34P_1 - 12P_2)r + (29P_1 - 17P_2 + 12P_3) \quad (10.9)$$

Comparing (10.9) with the right-hand side of (10.7), we obtain the equations

$$12P_1 = 3$$
$$34P_1 - 12P_2 = 0$$
$$29P_1 - 17P_2 + 12P_3 = 0$$

which yield

$$P_1 = \tfrac{1}{4} \qquad P_2 = \tfrac{17}{24} \qquad P_3 = \tfrac{115}{288}$$

Therefore, the particular solution is

$$a_r^{(p)} = \tfrac{1}{4}r^2 + \tfrac{17}{24}r + \tfrac{115}{288}$$

In general, when $f(r)$ is of the form of a polynomial of degree t in r

$$F_1 r^t + F_2 r^{t-1} + \cdots + F_t r + F_{t+1}$$

the corresponding particular solution will be of the form

$$P_1 r^t + P_2 r^{t-1} + \cdots + P_t r + P_{t+1}$$

Example 10.4 Consider the difference equation

$$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1 \quad (10.10)$$

The particular solution is of the form

$$P_1 r^2 + P_2 r + P_3 \quad (10.11)$$

Substituting (10.11) into (10.10), we obtain

$$P_1 r^2 + P_2 r + P_3 + 5P_1(r-1)^2 + 5P_2(r-1) + 5P_3 = 3r^2 - 2r + 1$$
$$+ 6P_1(r-2)^2 + 6P_2(r-2) + 6P_3$$

which simplifies to

$$12P_1 r^2 - (34P_1 - 12P_2)r + (29_1 - 17P_2 + 12P_3) = 3r^2 - 2r + 1 \quad (10.12)$$

Comparing the two sides of (10.12), we obtain the equations

$$12P_1 = 3$$

$$34P_1 - 12P_2 = 2$$

$$29P_1 - 17P_2 + 12P_3 = 1$$

which yield

$$P_1 = \tfrac{1}{4} \qquad P_2 = \tfrac{13}{24} \qquad P_3 = \tfrac{71}{288}$$

Therefore, the particular solution is

$$a_r^{(p)} = \tfrac{1}{4}r^2 + \tfrac{13}{24}r + \tfrac{71}{288}$$

Example 10.5 Consider the difference equation

$$a_r - 5a_{r-1} + 6a_{r-2} = 1 \qquad (10.13)$$

Since $f(r)$ is a constant, the particular solution will also be a constant P. Substituting P into (10.13), we obtain

$$P - 5P + 6P = 1$$

That is,

$$2P = 1$$

or

$$a_r^{(p)} = \tfrac{1}{2}$$

Ex-10.5.1

As another example, consider the difference equation

$$a_r + 5a_{r-1} + 6a_{r-2} = 42 \cdot 4^r \qquad (10.14)$$

We assume that the general form of the particular solution is

$$P4^r \qquad (10.15)$$

Substituting the expression in (10.15) into the left-hand side of (10.14), we obtain

$$P4^r + 5P4^{r-1} + 6P4^{r-2}$$

which simplifies to

$$\tfrac{21}{8}P4^r \qquad (10.16)$$

Comparing (10.16) with the right-hand side of (10.14), we obtain

$$P = 16$$

Therefore, the particular solution is

$$a_r^{(p)} = 16 \cdot 4^r$$

In general, when $f(r)$ is of the form β^r, the corresponding particular solution is of the form $P\beta^r$, if β is not a characteristic root of the difference equation. Furthermore, when $f(r)$ is of the form

$$(F_1r^t + F_2r^{t-1} + \cdots + F_tr + F_{t+1})\beta^r$$

the corresponding particular solution is of the form

$$(P_1r^t + P_2r^{t-1} + \cdots + P_tr + P_{t+1})\beta^r$$

if β is not a characteristic root of the difference equation. Consider the following example.

Example 10.6 Consider the difference equation

$$a_r + a_{r-1} = 3r2^r \qquad (10.17)$$

The general form of the particular solution is

$$(P_1r + P_2)2^r \qquad (10.18)$$

Substituting (10.18) into (10.17), we obtain

$$(P_1r + P_2)2^r + [P_1(r-1) + P_2]2^{r-1} = 3r2^r$$

which simplifies to

$$\tfrac{3}{2}P_1r2^r + (-\tfrac{1}{2}P_1 + \tfrac{3}{2}P_2)2^r = 3r2^r \qquad (10.19)$$

Comparing the two sides of (10.19), we obtain the equations

$$\tfrac{3}{2}P_1 = 3$$

$$-\tfrac{1}{2}P_1 + \tfrac{3}{2}P_2 = 0$$

Thus,

$$P_1 = 2 \qquad P_2 = \tfrac{2}{3}$$

and the particular solution is

$$a_r^{(p)} = (2r + \tfrac{2}{3})2^r$$

□

For the case that β is a characteristic root of multiplicity $m-1$, when $f(r)$ is of the form

$$(F_1r^t + F_2r^{t-1} + \cdots + F_tr + F_{t+1})\beta^r$$

the corresponding particular solution is of the form

$$r^{m-1}(P_1r^t + P_2r^{t-1} + \cdots + P_tr + P_{t+1})\beta^r$$

Let us examine the following examples.

Example 10.7 Consider the difference equation

$$a_r - 2a_{r-1} = 3 \cdot 2^r \qquad (10.20)$$

Because 2 is a characteristic root (of multiplicity 1), the general form of the particular solution is

$$Pr2^r \tag{10.21}$$

Substituting (10.21) into (10.20), we obtain

$$Pr2^r - 2P(r-1)2^{r-1} = 3 \cdot 2^r$$

that is, Divide by 2^{r-1} $\Rightarrow 2Pr - 2(Pr - P) =$

$$P2^r = 3 \cdot 2^r$$

$\therefore -Pr + 2P = 6$

or

$$P = 3$$

Thus, the particular solution is

sol: $= cf + PI$

$$a_r^{(p)} = 3r2^r$$

□

Example 10.8 For the difference equation

$$a_r - 4a_{r-1} + 4a_{r-2} = (r+1)2^r \tag{10.22}$$

since 2 is a double characteristic root, the general form of the particular solution is

$$\boxed{r^2(P_1 r + P_2)2^r} \tag{10.23}$$

Substituting (10.23) into (10.22), we obtain, after simplification:

$$6P_1 r2^r = r2^r$$

$$(-6P_1 + 2P_2)2^r = 2^r$$

which yield

$$P_1 = \tfrac{1}{6} \qquad P_2 = 1$$

Thus, the particular solution is

$$a_r^{(p)} = r^2\left(\frac{r}{6} + 1\right)2^r$$

□

Example 10.9 Consider the difference equation

most

$$a_r = a_{r-1} + 7 \tag{10.24}$$

Since 1 is a characteristic root of the difference equation and 7 can be written as $7 \cdot 1^r$, the general form of the particular solution is Pr. (The reader should find out what happens if we assume the general form of the particular solution to be P instead.) Substituting $a_r^{(p)} = Pr$ into (10.24), we obtain

$$Pr = P(r-1) + 7$$

that is,

$$P = 7$$

□

Example 10.10 For the difference equation

$$a_r - 2a_{r-1} + a_{r-2} = 7 \quad 1^r$$

we let $a_r^{(p)} = Pr^2$. We ask the reader to carry out the substitution to confirm that $P = \frac{7}{2}$. □

Example 10.11 Consider the difference equation

$$a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r$$

The general form of the particular solution is

$$P_1 r2^r + P_2 r + P_3$$

(Note that 2 is a characteristic root of the difference equation.) Substitution and comparison will yield

$$P_1 = -2 \qquad P_2 = \tfrac{1}{2} \qquad P_3 = \tfrac{7}{4}$$

and

$$a_r^{(p)} = -r2^{r+1} + \tfrac{1}{2}r + \tfrac{7}{4}$$

□

10.6 TOTAL SOLUTIONS

Finally, we must combine the homogeneous solution and the particular solution and determine the undetermined coefficients in the homogeneous solution. For a kth-order difference equation, the k undetermined coefficients $A_1, A_2, \ldots, A_k$ in the homogeneous solution can be determined by the boundary conditions, a_{r_0}, $a_{r_0+1}, \ldots, a_{r_0+k-1}$, for any r_0. Suppose the characteristic roots of the difference equation are all distinct. The total solution is of the form

$$a_r = A_1\alpha_1^r + A_2\alpha_2^r + \cdots + A_k\alpha_k^r + p(r)$$

where $p(r)$ is the particular solution. Thus, for $r = r_0, r_0+1, \ldots, r_0+k-1$, we have the system of linear equations:

$$a_{r_0} = A_1\alpha_1^{r_0} + A_2\alpha_2^{r_0} + \cdots + A_k\alpha_k^{r_0} + p(r_0)$$

$$a_{r_0+1} = A_1\alpha_1^{r_0+1} + A_2\alpha_2^{r_0+1} + \cdots + A_k\alpha_k^{r_0+1} + p(r_0+1) \tag{10.25}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{r_0+k-1} = A_1\alpha_1^{r_0+k-1} + A_2\alpha_2^{r_0+k-1} + \cdots + A_k\alpha_k^{r_0+k-1} + p(r_0+k-1)$$

These k linear equations can be solved for $A_1, A_2, \ldots, A_k$. For example, for the difference equation in (10.14), the total solution is

$$a_r = A_1(-2)^r + A_2(-3)^r + 16 \cdot 4^r$$

Suppose we are given the boundary conditions $a_2 = 278$ and $a_3 = 962$. Solv[ing] the equations

$$278 = 4A_1 + 9A_2 + 256$$

$$962 = -8A_1 - 27A_2 + 1024$$

we obtain

$$A_1 = 1 \qquad A_2 = 2$$

Thus,

$$a_r = (-2)^r + 2(-3)^r + 16 \cdot 4^r$$

is the total solution of the difference equation.

One might question how we can be sure that solutions of the k equations (10.25) are always unique. It can be shown mathematically that this is indeed [the] case.† However, in Sec. 10.2, recall that we demonstrated the uniqueness of [the] solution of a kth-order linear recurrence relation with constant coefficients [for] any given boundary conditions consisting of k consecutive values $a_{r0}, a_{r0+1}, \ldots, a_{r0+k-1}$. Consequently, the uniqueness of the solution of the recurrence rela[tion] guarantees the uniqueness of the solutions of the k equations in (10.25). On [the] other hand, if we are given the value of the numeric function at k not necess[arily] consecutive points, although we can set up k equations for the undeterm[ined] coefficients $A_1, A_2, \ldots, A_k$ similar to that in (10.25), since the solution of [the] recurrence relation might not be uniquely specified by such boundary condit[ions] it is not always the case that these equations can be solved uniquely.

When the characteristic roots of the difference equation are not all distin[ct a] derivation similar to the foregoing can be carried out. Again, the undeterm[ined] coefficients in the homogeneous solution can be determined uniquely by the v[alues] of the numeric function at k consecutive points.

10.7 SOLUTION BY THE METHOD OF GENERATING FUNCTIONS

Instead of solving a difference equation for an expression for the value [of a] numeric function as we did above, we can also determine the generating func[tion] of the numeric function from the difference equation. In many cases, once [the] generating function is determined, an expression for the value of the nu[meric] function can easily be obtained.

Consider the recurrence relation

$$a_r = 3a_{r-1} + 2 \qquad r \geq 1 \tag{10.26}$$

with the boundary condition $a_0 = 1$. Let us point out that in (10.26) we [write] down explicitly (for the first time in this chapter) that the recurrence relat[ion]

† See, for example, chap. 3 of Liu [5].

accordance with the outcome of any particular comparison step. (2) No additional registers for storing intermediate results is needed. We note that in a nonadaptive algorithm, some of the comparison steps can be carried out simultaneously. [For example, $A(x_1, x_2)$ and $A(x_3, x_4)$ can be carried out simultaneously while $A(x_1, x_2)$ and $A(x_1, x_3)$ cannot be.] Consequently, there is the possibility of speeding up the computation by parallel processing. The advantage of algorithms that do not use additional storage registers is obvious when the number of items to be sorted is large. For a most complete discussion on sorting algorithms and related topics, see chap. 5 of Knuth [3].

The matrix multiplication algorithm presented in Sec. 10.9 is due to Strassen [8]. See Prob. 10.38 for an algorithm that uses 7 multiplication operations and 15 addition operations. Note, however, that the time complexity of such an algorithm would still be $\Theta(n^{2.81})$. To reduce the exponent of n from 2.81 to a smaller number, one's immediate reaction would be to search for a multiplication algorithm for 3×3 matrices that uses 21 or fewer multiplication operations. However, no such algorithm has so far been discovered. Surprisingly, multiplication algorithms for larger matrices that lead to improvement on the exponent of n have been discovered. For example, Pan [7] shows that we can multiply two 48×48 matrices using 47,216 multiplication operations, which reduces the exponent to 2.78.

1. Batcher, K. E.: Sorting Networks and Their Applications, *AFIPS Proc. of the 1968 SJCS*, **32**: 307–314 (1968).
2. Bose, R. C., and R. J. Nelson: A Sorting Problem, *Journal of ACM*, **9**: 282–296 (1962).
3. Knuth, D. E.: "The Art of Computer Programming, Vol. 3, Sorting and Searching," Addison-Wesley Publishing Company, Reading, Mass., 1973.
4. Levy, H., and F. Lessman: "Finite Difference Equations," The Macmillan Company, New York, 1961.
5. Liu, C. L.: "Introduction to Combinatorial Mathematics," McGraw-Hill Book Company, New York, 1968.
6. Liu, C. L., and J. W. S. Liu: "Linear Systems Analysis," McGraw-Hill Book Company, New York, 1975.
7. Pan, V. Ya.: Strassen's Algorithm Is Not Optimal, *Proc. 19th Annual Symposium on Foundations of Computer Science*, 166–176 (1978).
8. Strassen, V.: Gaussian Elimination Is Not Optimal, *Numerische Mathematik*, **13**: 354–356 (1969).

PROBLEMS

10.1 Solve the following recurrence relations:
(a) $a_r - 7a_{r-1} + 10a_{r-2} = 0$, given that $a_0 = 0$ and $a_1 = 3$.
(b) $a_r - 4a_{r-1} + 4a_{r-2} = 0$, given that $a_0 = 1$ and $a_1 = 6$.

10.2 Solve the following recurrence relations:
(a) $a_r - 7a_{r-1} + 10a_{r-2} = 3^r$, given that $a_0 = 0$ and $a_1 = 1$.
(b) $a_r + 6a_{r-1} + 9a_{r-2} = 3$, given that $a_0 = 0$ and $a_1 = 1$.
(c) $a_r + a_{r-1} + a_{r-2} = 0$, given that $a_0 = 0$ and $a_1 = 2$.

10.3 Solve the following recurrence relations:
(a) $a_r - a_{r-1} - a_{r-2} = 0$, given that $a_0 = 1$ and $a_1 = 1$.
(b) $a_r - 2a_{r-1} + 2a_{r-2} - a_{r-3} = 0$, given that $a_0 = 2$, $a_1 = 1$, and $a_2 = 1$.

10.4 Given that $a_0 = 0$, $a_1 = 1$, $a_2 = 4$, and $a_3 = 12$ satisfy the recurrence relation

$$a_r + C_1 a_{r-1} + C_2 a_{r-2} = 0$$

determine a_r.

10.5 The solution of the recurrence relation

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} = f(r)$$

is

$$3^r + 4^r + 2$$

Given that $f(r) = 6$ for all r, determine C_0, C_1, and C_2.

10.6 The solution of the recurrence relation

$$a_r = A a_{r-1} + B 3^r \qquad r \geq 1$$

is

$$a_r = C 2^r + D 3^{r+1} \qquad r \geq 0$$

Given that $a_0 = 19$ and $a_1 = 50$, determine the constants A, B, C, and D.

10.7 Let

$$4a_r + C_1 a_{r-1} + C_2 a_{r-2} = f(r) \qquad r \geq 2$$

be a second-order linear recurrence with constant coefficients. For some boundary conditions a_0, a_1, the solution of the recurrence is

$$1 - 2r + 3 \cdot 2^r$$

Determine a_0, a_1, C_1, C_2, and $f(r)$. (The solution is not unique.)

10.8 Consider the recurrence relation

$$a_r = a_{r-1} - a_{r-2}$$

(*a*) Solve the recurrence relation, given that $a_1 = 1$ and $a_2 = 0$.
(*b*) Can you solve the recurrence relation if it is given that $a_0 = 0$ and $a_3 = 0$?
(*c*) Repeat part (*b*) if it is given that $a_0 = 1$ and $a_3 = 2$.

10.9 (*a*) Determine the particular solution for the difference equation

$$a_r - 3a_{r-1} + 2a_{r-2} = 2^r$$

(*b*) Determine the particular solution for the difference equation

$$a_r - 4a_{r-1} + 4a_{r-2} = 2^r$$

10.10 (*a*) Determine the particular solution for the difference equation

$$a_r - 2a_{r-1} = f(r)$$

where $f(r) = 7r$.
(*b*) Repeat part (*a*) for $f(r) = 7r^2$.
(*c*) Determine the particular solution for the difference equation

$$a_r - a_{r-1} = 7r$$

(*d*) Repeat part (*c*) if $f(r) = 7r^2$.
(*e*) Let

$$C_0 a_r + C_1 a_{r-1} + \cdots + C_k a_{r-k} = f(r)$$

be a difference equation with a characteristic root 1. Let $f(r) = r^l$. What can be said about the form of the particular solution $a_r^{(p)}$?

10.11 (*a*) Solve the recurrence relation

$$a_r + 3a_{r-1} + 2a_{r-2} = f(r)$$

$a_2 = 1, a_3 = -3, a_4 = 7$
$= 0, r \geq 3$
$a_r = (-1)^{r-1} - (-2)^r$

where

$$f(r) = \begin{cases} 1 & r = 2 \\ 0 & \text{otherwise} \end{cases}$$

with the boundary condition $a_0 = a_1 = 0$.

$a_r = \begin{cases} (-1)^{r-1} - (-2)^r & r \geq 2 \\ 0 & \text{else} \end{cases}$

(*b*) Repeat part (*a*) for

$a_3 = a_4 = 0$, $a_5 = 1$, $a_6 = -3$, $a_7 =$

$$f(r) = \begin{cases} 1 & r = 5 \\ 0 & \text{otherwise} \end{cases}$$

$a_r = (-1)^{r-4} - (-2)^{r-}$

(*c*) Consider the recurrence relation

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r)$$

Let $\hat{a}_r$ denote the solution of the recurrence relation for $f(r) = \hat{f}(r)$ with the boundary conditions $\hat{a}_0 = \hat{a}_1 = \hat{a}_2 = \cdots = \hat{a}_{k-1} = 0$. Let $\tilde{a}_r$ denote the solution of the recurrence relation for $f(r) = \tilde{f}(r)$ with the boundary conditions $\tilde{a}_0 = \tilde{a}_1 = \tilde{a}_2 = \cdots = \tilde{a}_{k-1} = 0$. Given that $\hat{f}(r) = 0$ for $r < k$, and

$$\tilde{f}(r) = \begin{cases} 0 & 0 \leq r \leq l - 1 \\ \hat{f}(r - l) & r \geq l \end{cases}$$

for some fixed l, what can we conclude about $\hat{a}_r$ and $\tilde{a}_r$?

10.12 (*a*) Consider the recurrence relation

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r)$$

Let $\hat{a}_r$ denote the solution of the recurrence relation for $f(r) = \hat{f}(r)$ with the boundary conditions $\hat{a}_0 = \hat{a}_1 = \hat{a}_2 = \cdots = \hat{a}_{k-1} = 0$. Let $\tilde{a}_r$ denote the solution of the recurrence relation for $f(r) = \tilde{f}(r)$ with the boundary conditions $\tilde{a}_0 = \tilde{a}_1 = \tilde{a}_2 = \cdots = \tilde{a}_{k-1} = 0$. Show that $\bar{a}_r = \hat{a}_r + \tilde{a}_r$ is the solution of the recurrence relation for $f(r) = \hat{f}(r) + \tilde{f}(r)$ with the boundary conditions $\bar{a}_0 = \bar{a}_1 = \bar{a}_2 = \cdots = \bar{a}_{k-1} = 0$, provided that $\hat{f}(r) = \tilde{f}(r) = 0$ for $r < k$.
(*b*) Solve the recurrence equation

$$a_r + 5a_{r-1} + 6a_{r-2} = f(r)$$

where

$$f(r) = \begin{cases} 0 & r = 0, 1, 5 \\ 6 & \text{otherwise} \end{cases}$$

given that $a_0 = a_1 = 0$.

10.13 Gossip is spread among r people via telephone. Specifically, in a telephone conversation between A and B, A tells B all the gossip he has heard, and B reciprocates. Let a_r denote the minimum number of telephone calls among r people so that all gossip will be known to everyone.
(*a*) Show that $a_2 = 1$, $a_3 = 3$, and $a_4 = 4$.
(*b*) Show that

$$a_r \leq a_{r-1} + 2$$

(*c*) Show that

$$a_r \leq 2r - 4 \qquad \text{for } r \geq 4$$

[Indeed, it can be shown that $a_r = 2r - 4$. See B. Baker and R. Shostak, Gossips and Telephones, *Discrete Mathematics*, **2**: 191–193, (1972).]