CHAP: 10 RECURRENCE RELATIONS & RECURSIVE ALGORITHMS

→ To find the Time Complexity of Algo. we need numeric function of Algo. but it is not easy to determine directly from Algo. So, for recursive algo. initially we write recurrence relation (Also called difference equation - For a numeric function $(a_0, a_1, a_2, \dots, a_n, \dots)$, an equation relating a_n, for any n, to one or more of the a_i's, $i<n$ is called recurrence Relation)

→ We derive characteristic equation & find the roots of that characteristic equ. to find Total Solⁿ. of recurrence relation

Total Solⁿ = numeric function of Algo. = homogeneous Solⁿ. + Particular Solⁿ.

$$\boxed{a_n = a_n^{(h)} + a_n^{(P)}}$$

→ So, in discrete Computation problem, Some times its easy to obtain a specification of Numeric function in terms of recurrence relation ~~term~~ than to obtain a general expression for the value of the numeric function at n OR a closed form expression for its generating function.

Ex: $a_n = (3^0, 3^1, 3^2, 3^3, \dots, 3^n, \dots)$ = numeric function

$$\left.\begin{aligned} A(z) &= a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots \\ &= 3^0 + 3^1 z + 3^2 z^2 + 3^3 z^3 + \dots + 3^n z^n + \dots \end{aligned}\right\} \to \text{generating function of } a_n$$

closed form of $a_n (A(z))$ is.

$$\boxed{A(z) = \frac{1}{1-3z}}$$

VND 10.2

Importance:

Ex: for Fibbonacci Series: 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

numeric function = ? → very difficult to write from the Algorithm.

∴ first write, recurrence relation,

$$a_r = a_{r-1} + a_{r-2}$$

$$\therefore a_r - a_{r-1} - a_{r-2} = 0$$

$$\therefore \boxed{\alpha^2 - \alpha - 1 = 0} \rightarrow \text{Characteristic equation}$$

∴ roots of this characteristic equation will be

$$\alpha = \frac{1 \pm \sqrt{5}}{2}$$

∴ solⁿ of ~~charact.~~ recurrence relation (difference equation) will be,

$$a_r = a_r^{(h)} + a_r^{(p)} = A_1\left(\frac{1+\sqrt{5}}{2}\right)^r + A_2\left(\frac{1-\sqrt{5}}{2}\right)^r + a_r^{(p)}$$

here, $a_r^{(p)} = 0$

$$\boxed{a_r = A_1\left(\frac{1+\sqrt{5}}{2}\right)^r + A_2\left(\frac{1-\sqrt{5}}{2}\right)^r} \text{ —— (I)}$$

→ A_1, A_2 = Constant Co-efficients whose value can be determined using given boundary values, like $a_0 = 1$, $a_1 = 1$.

∴ from equ. (I) $a_0 = 1$

$$A_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + A_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = 1 \Rightarrow A_1 + A_2 = 1$$

$$\Rightarrow A_1 = 1 - A_2 \text{ —— (II)}$$

also, $a_1 = 1$ & using equ. (I) $A_1\left(\frac{1+\sqrt{5}}{2}\right) + A_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$$\Rightarrow (1 - A_2)(1+\sqrt{5}) + A_2(1-\sqrt{5}) = 2 \quad [\because \text{from (II)}]$$

$$\Rightarrow 1 + \sqrt{5} - \cancel{A_2} - A_2\sqrt{5} + \cancel{A_2} - A_2\sqrt{5} = 2$$

$$1 + \sqrt{5} - 2A_2\sqrt{5} = 2$$

$$\therefore 2\sqrt{5}\, A_2 = \sqrt{5} - 1 \Rightarrow$$

$$\boxed{A_2 = \frac{-1+\sqrt{5}}{2\sqrt{5}}}$$

$$\boxed{A_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}}$$

VND 10.3

(*) LINEAR RECURRENCE RELATIONS WITH CONSTANT CO-EFFICIENTS :

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + C_3 a_{r-3} + \dots\dots + C_k a_{r-k} = \begin{cases} 0 & \text{— CASE-I} \\ f(r) & \text{— CASE-II} \\ \beta^r f(r) & \text{— CASE-III} \end{cases} \quad \text{— (1)}$$

→ equation (1) is general form on linear Recurrence Relation (difference equation), which is of k^{th} Order.

Where, $\boxed{C_0, C_k \neq 0}$

→ if k-boundary values will be given than we can determine values of Co-efficients in the numeric function.

(*) CASE-I: Homogeneous Solutions:

→ for any linear recurrence relation,

Total Sol^n. = homogeneous Sol^n + Particular Sol^n

$$\therefore \boxed{a_r = a_r^{(h)} + a_r^{(P)}}$$

→ Sol^n of homogeneous linear recurrence relation (CASE-I in equ (1)) will be in the form of $A_i \alpha_i^r$ where, A_i = Co-efficient

α_i = Characteristic roots

→ Based on root α get simplified characteristic equation of recurrence relation,

$$\boxed{C_0 \alpha^k + C_1 \alpha^{k-1} + C_2 \alpha^{k-2} + \dots\dots + C_k \alpha^{k-k} = 0}$$

→ if all the characteristic roots will be distincts, then,

$$\boxed{a_r^{(h)} = \sum_{i=1}^{k} A_i \alpha_i^r} \Rightarrow \text{homogeneous Sol}^n$$

VND 10.4

Ex: 10.2 5 Solve the difference equation is

$$a_r + 6a_{r-1} + 12a_{r-2} + 8a_{r-3} = 0$$

Solⁿ:

Characteristic equation for given difference equation,

$$\alpha^3 + 6\alpha^2 + 12\alpha + 8 = 0 \quad [\because k = 3]$$

$$\therefore (\alpha+2)(\alpha^2 + 4\alpha + 4) = 0$$

$$\therefore (\alpha+2)(\alpha+2)^2 = 0$$

$$\therefore \boxed{\alpha_1, \alpha_2, \alpha_3 = -2} \rightarrow \text{all three roots are common.}$$

$\therefore$ homogeneous solⁿ. $\boxed{a_r^{(h)} = (A_1 r^2 + A_2 r + A_3)(-2)^r}$

Ex: 10.3: Solve the difference equation: $4a_r - 20a_{r-1} + 17a_{r-2} - 4a_{r-3} = 0$

Solⁿ: Characteristic equ.,

$$4\alpha^3 - 20\alpha^2 + 17\alpha - 4 = 0 \quad [\because k = 3]$$

$$\therefore (\alpha-4)(4\alpha^2 - 4\alpha + 1) = 0$$

$$\therefore (\alpha-4)(2\alpha-1)^2 = 0$$

$$\therefore \alpha_1 = 4, \ \alpha_2 = \alpha_3 = 1/2$$

$\therefore$ homogeneous solⁿ. $\boxed{a_r^{(h)} = A_1 4^r + (A_2 r + A_3)\left(\frac{1}{2}\right)^r}$

(*) CASE-II: if $C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \dots + C_k a_{r-k} = f(r)$ then.

Total solⁿ $a_r = a_r^{(h)} + a_r^{(p)}$

$a_r^{(p)}$ → particular solⁿ.

$a_r^{(h)}$ → homogeneous solⁿ.

→ When $f(r)$ is of the form of a polynomial of degree t in r

$$\boxed{F_1 r^t + F_2 r^{t-1} + F_3 r^{t-2} + \dots + F_t r + F_{t+1}}$$

the corresponding particular solⁿ will be of the form,

$$\boxed{P_1 r^t + P_2 r^{t-1} + P_3 r^{t-2} + \dots + P_t r + P_{t+1}}$$

Example: difference equation is $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$

$\therefore$ total sol^n $a_n = a_n^{(h)} + a_n^{(p)}$

$\Rightarrow$: for $a_n^{(h)}$, Characteristic equation will be

$$\alpha^2 + 5\alpha + 6 = 0 \qquad [\because k = 2]$$

$$\therefore (\alpha+2)(\alpha+3) = 0$$

$$\therefore \alpha_1 = -2,\ \alpha_2 = -3$$

$$\therefore \boxed{a_n^{(h)} = A_1(-2)^n + A_2(-3)^n}$$

$\Rightarrow$ for $a_n^{(p)}$, assume the general form of particular sol^n.

$$P_1 n^2 + P_2 n + P_3 \qquad \text{where } P_1, P_2, P_3 = \text{Constants.}$$

$\sim$ Substituting the expression in L.H.S. in difference equation, we get,

$$(P_1 n^2 + P_2 n + P_3) + 5[P_1(n-1)^2 + P_2(n-1) + P_3] + 6[P_1(n-2)^2 + P_2(n-2) + P_3] = 3n^2$$

by simplifying we get,

$$12P_1 n^2 - (34P_1 - 12P_2)n + (29P_1 - 17P_2 + 12P_3) = 3n^2$$

$$\therefore 12P_1 = 3$$

$$\therefore \boxed{P_1 = \frac{1}{4}}$$

$$-(34P_1 - 12P_2) = 0$$

$$\therefore 34P_1 = 12P_2$$

$$\therefore P_2 = \frac{\overset{17}{\cancel{34}}\left(\frac{1}{\cancel{4}_2}\right)}{12}$$

$$\therefore \boxed{P_2 = \frac{17}{24}}$$

$$29P_1 - 17P_2 + 12P_3 = 0$$

$$\therefore P_3 = \frac{-29\left(\frac{1}{4}\right) + 17\left(\frac{17}{24}\right)}{12}$$

$$\boxed{P_3 = \frac{115}{288}}$$

$$\therefore a_n^{(p)} = P_1 n^2 + P_2 n + P_3$$

$$\boxed{a_n^{(p)} = \frac{1}{4}n^2 + \frac{17}{24}n + \frac{115}{288}}$$

$$\therefore \boxed{a_n = a_n^{(h)} + a_n^{(p)}}$$

Ex: 10.4 : Solve the difference equation: $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1$

Sol^n: homogeneous sol^n $\boxed{a_r^{(h)} = A_1(-2)^r + A_2(-3)^r}$ ——— (I)

⇒ Now, for particular sol^n $a_r^{(P)}$, general sol^n will be $P_1r^2 + P_2r + P_3$

$\therefore 12P_1r^2 - (34P_1 - 12P_2)r + (29P_1 - 17P_2 + 12P_3) = 3r^2 - 2r + 1$

$\therefore 12P_1 = 3$

$\therefore P_1 = 1/4$

$34P_1 - 12P_2 = 2$

$\therefore P_2 = \dfrac{34(1/4) - 2}{12}$

$\boxed{P_2 = \dfrac{13}{24}}$

$29P_1 - 17P_2 + 12P_3 = 1$

$\therefore \boxed{P_3 = \dfrac{71}{288}}$

$\therefore \boxed{a_r^{(P)} = \dfrac{1}{4}r^2 + \dfrac{13}{24}r + \dfrac{71}{288}}$ ——— (II)

(*) CASE-III: INHOMOGENEOUS LINEAR RECURRENCE RELATION :

→ if $C_0a_r + C_1a_{r-1} + C_2a_{r-2} + \ldots + C_ka_{r-k} = \beta^r f(r)$ ——— (I)

where, β = constant

$f(r)$ = polynomial of degree t

the corresponding particular sol^n will be,

$$\boxed{(P_1r^t + P_2r^{t-1} + P_3r^{t-2} + \ldots + P_tr + P_{t+1})\cdot\beta^r}$$

⇒ CASE-III: Sub Case: if for equ. (I) β is characteristic root in the multiplicity m then the corresponding particular sol^n will be,

$$\boxed{r^m(P_1r^t + P_2r^{t-1} + P_3r^{t-2} + \ldots + P_tr + P_{t+1})\beta^r}$$

$\therefore$ in such case $(\alpha - \beta)^m$ will be the factors of the characteristic equation.

VAD 10.7

Ex: 10.5 Solve the difference equation: $a_r - 5a_{r-1} + 6a_{r-2} = 1$

Sol^n: Here, Characteristic equ. will be

$$\alpha^2 - 5\alpha + 6 = 0$$

$$\therefore (\alpha-2)(\alpha-3) = 0$$

$$\therefore \alpha_1 = 2, \alpha_2 = 3$$

$$\therefore \boxed{a_r^{(h)} = A_1 2^r + A_2 3^r} \quad \text{--- (I)}$$

$\Rightarrow$ To find particular sol^n.

$f(r) = 1 \quad \therefore$ degree is zero.

$\therefore$ sol^n for particular sol^n is

$$a_r^{(P)} = P$$

$\therefore$ by putting particular sol^n in given difference equation.

$$P - 5P + 6P = 1$$

$$\cancel{2P} = \cancel{1} \quad 2P = 1$$

$$\therefore \boxed{P = 1/2}$$

$$\therefore \boxed{a_r^{(P)} = 1/2} \quad \text{--- (II)}$$

$\therefore$ total sol^n from (I) & (II)

$$\therefore \boxed{a_r = A_1 2^r + A_2 3^r + 1/2}$$

Continue... Ex: $\cancel{105}$: $a_r + 5a_{r-1} + 6a_{r-2} = 42\,4^r$

$\Rightarrow$ To find particular sol^n general sol^n will be

$$a_r^{(P)} = P\,4^r$$

$42 = f(r) \longrightarrow$ of degree zero

$4^r = \beta^r$

$\therefore$ from given difference equation,

$$P4^r + 5P4^{r-1} + 6P4^{r-2} = 42 \cdot 4^r$$

$$\therefore P + 5P4^{-1} + 6P4^{-2} = 42$$

$$\therefore \left(P + \frac{5P}{4} + \frac{6P}{16}\right) = 42 \Rightarrow \frac{P}{16}(16 + 20 + 6) = 42$$

$$\Rightarrow \boxed{P = 16}$$

$$\therefore \boxed{a_r^{(P)} = 16\,4^r}$$

$\therefore$ Total sol^n. $a_r = a_r^{(h)} + a_r^{(P)}$

$$\boxed{a_r = A_1(-2)^r + A_2(-3)^r + 16\,4^r}$$

VAID 10.8

Ex: 10.6 Solve the difference equation : $a_r + a_{r-1} = 3r \cdot 2^r$

Sol^n. For homogeneous sol^n characteristic equation,

$\alpha + 1 = 0$ $\qquad \therefore a_r^{(h)} = A_1 (-1)^r$ ——— (I)

$\therefore \alpha = -1$

For particular sol^n.

here $\qquad f(r) = 3r \rightarrow$ of degree 1

$\beta^r = 2^r$

$\therefore a_r^{(P)} = (P_1 r + P_2) \cdot 2^r$

→ Putting in given difference equation,

$(P_1 r + P_2) \cdot 2^r + (P_1(r-1) + P_2) \cdot 2^{r-1} = 3r \cdot 2^r$

$\therefore \frac{3}{2} P_1 r \cdot 2^r + \left(-\frac{1}{2} P_1 + \frac{3}{2} P_2\right) 2^r = 3r \cdot 2^r$

$\therefore \frac{\cancel{3}}{2} P_1 = \cancel{3}$

$\therefore \boxed{P_1 = 2}$

$-\frac{1}{2} P_1 + \frac{3}{2} P_2 = 0$

$-\frac{1}{\cancel{2}} \times \cancel{2} + \frac{3}{2} P_2 = 0$

$\therefore \frac{3}{2} P_2 = 1$

$\therefore \boxed{P_2 = \frac{2}{3}}$

$\therefore a_r^{(P)} = \left(2r + \frac{2}{3}\right) \cdot 2^r$ ——— (II)

From (I) & (II), total sol^n will be

$$\boxed{a_r = A_1(-1)^r + \left(2r + \frac{2}{3}\right) 2^r}$$

Ex: 10.7: Solve the difference equation : $a_r - 2a_{r-1} = 3 \cdot 2^r$

Sol^n: For homogeneous sol^n: characteristic equation will be

$\alpha - 2 = 0$

$\therefore \alpha = 2$ → $\beta = 2$ is one of the characteristic root in the multiplicity 1.

$\therefore \boxed{a_r^{(h)} = A_1 2^r}$ ——— (I)

VNID 10.9

Continue... ex:10.7

$\Rightarrow$ For particular Sol.^{n}

as β is one of the characteristic root in the multiplicity 1

particular sol^n will be

$$a_r^{(P)} = P r \cdot 2^r \quad \text{——— (II)}$$

~ substituting (II) in given difference equation, we get,

$$P r \cdot 2^r - 2P(r-1)2^{r-1} = 3 \cdot 2^r$$

$$\cancel{Pr \cdot 2^r} - \cancel{2^r Pr} + 2^r \cdot P = 3 \cdot \cancel{2^r}$$

$$\therefore P = 3$$

$$\therefore a_r^{(P)} = 3r \cdot 2^r \quad \text{——— (III)}$$

$$\therefore a_r = a_r^{(h)} + a_r^{(P)} = A_1 \cdot 2^r + 3r \cdot 2^r \Rightarrow \boxed{a_r = (A_1 + 3r) \cdot 2^r}$$

Ex:10.8 Solve the difference equation:

$$a_r - 4a_{r-1} + 4a_{r-2} = (r+1)2^r$$

Sol^n: $\Rightarrow$ For homogeneous sol^n

$$a_r - 4a_{r-1} + 4a_{r-2} = 0$$

$$\therefore \alpha^2 - 4 + 4 = 0$$

$$\therefore (\alpha - 2)^2 = 0$$

$$\therefore \boxed{\alpha_1 = \alpha_2 = 2} \Rightarrow \boxed{a_r^{(h)} = (A_1 r + A_2)2^r} \quad \text{——— (I)}$$

also, $\beta = 2$

$\Rightarrow \therefore \beta$ is one of the characteristic root in the multiplicity 2

For particular sol^n

$$a_r^{(P)} = (P_1 r + P_2) \cdot 2^r \cdot r^2 \quad \text{——— (II)}$$

here,
$f(r) = (r+1)$ of degree 1
$\beta = 2$

$\therefore$ Substituting (II) in given difference equation, we get

$$(P_1 r + P_2) 2^r \cdot r^2 - 4 \cdot 2^{r-1}(r-1)^2[P_1(r-1) + P_2] + 4 \cdot 2^{r-2} \cdot (r-2)^2[P_1(r-2) + P_2] = 2^r \cdot r + 2^r$$

~ by simplifying

$$6P_1 r \cancel{2^r} = r \cancel{2^r}$$

$$\therefore \boxed{P_1 = 1/6}$$

$$(-12P_1 + 2P_2) \cdot \cancel{2^r} = \cancel{2^r}$$

$$P_2 = \frac{1 + 12(1/6)}{2}$$

$$\therefore \boxed{P_2 = 1}$$

VND 10.10

Continue... ex. 10.8

$$a_r = a_r^{(h)} + a_r^{(p)} = \boxed{(A_1 r + A_2)\cdot 2^r + \left(\frac{1}{6} r + 1\right) 2^r \cdot r^2}$$

Ex: 10.9 Solve the difference equation: $a_r = a_{r-1} + 7$.

Sol^n: $\Rightarrow$ for particular Sol^n

here, $f(r) = 7$ which is of degree 0 and $\beta = 1$ & also β is one of the characteristic root in the multiplicity 1.

$\Rightarrow$ for homogeneous Sol^n

$a_r - a_{r-1} = 0$

$\alpha - 1 = 0 \Rightarrow \alpha = 1$

$\therefore a_r^{(h)} = A_1 \cdot 1^r = \boxed{A_2}$ —— I

$\therefore a_r^{(p)} = P \cdot 1^r \cdot r^1 = Pr$ —— II

$\rightarrow$ equ. II substituting in given difference equation, we get

$Pr - P(r-1) = 7$

$\cancel{Pr} - \cancel{Pr} + P = 7 \Rightarrow P = 7.$

$\therefore a_r^{(p)} = 7r \Rightarrow \boxed{a_r = A_2 + 7r}$

Ex: 10.10: Solve the difference equation: $a_r - 2a_{r-1} + a_{r-2} = 7$

Sol^n $\Rightarrow$ for particular Sol^n.

here $f(r) = 7$ of degree zero $\beta = 1$ which is also characteristic root in the multiplicity of 2

$\therefore a_r^{(p)} = P \cdot 1^r \cdot r^2 = \boxed{Pr^2}$ —— II

$\Rightarrow$ for homogeneous Sol^n

$a_r - 2a_{r-1} + a_{r-2} = 0$

$\therefore \alpha^2 - 2\alpha + 1 = 0 \Rightarrow (\alpha - 1)^2 = 0$

$\therefore \alpha_1 = \alpha_2 = 1$

$\boxed{\therefore a_r^{(h)} = (A_1 r + A_2) 1^r}$ —— I

$\Rightarrow$ Substituting equ. II in given difference equation, we get

$P = 7/2 \quad \therefore a_r^{(p)} = \frac{7}{2} r^2$

$$\therefore \boxed{a_r = A_1 r + A_2 + \frac{7}{2} r^2}$$

Ex: 10.11 Solve the difference equation:

$$a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r \quad \text{—— (I)}$$

Sol^n: ⇒ for particular Sol^n

here, $f(r) = 1$ which is of order zero

$\beta = 2$ & also

β is one of the characteristic root in the multiplicity 1

→ ∴ partial sol^n $P_1 2^r \cdot r$

→ Also here $g(r) = r$

∴ another partial sol^n $= P_2 r + P_3$

$$\therefore \boxed{a_r^{(P)} = P_1 r 2^r + P_2 r + P_3} \quad \text{—— (III)}$$

⇒ for homogeneous Sol^n.

$a_r - 5a_{r-1} + 6a_{r-2} = 0$

$\therefore \alpha^2 - 5\alpha + 6 = 0 \Rightarrow (\alpha - 3)(\alpha - 2) = 0$

$\therefore \alpha_1 = 3, \alpha_2 = 2$

$$\therefore \boxed{a_r^{(h)} = A_1 3^r + A_2 2^r} \quad \text{—— (II)}$$

→ Substituting equ. (III) in given difference equation, we get

$$P_1 = -2, \quad P_2 = \frac{1}{2}, \quad P_3 = \frac{7}{4}$$

$$\therefore a_r = a_r^{(P)} + a_r^{(h)}$$

$$\therefore \boxed{a_r = A_1 3^r + A_2 2^r - r 2^{r+1} + \frac{1}{2} r + \frac{7}{4}} \text{ Ans.}$$