EXERCISE 1: CONVEX SETS

31099/61099, APPLIED OPTIMIZATION

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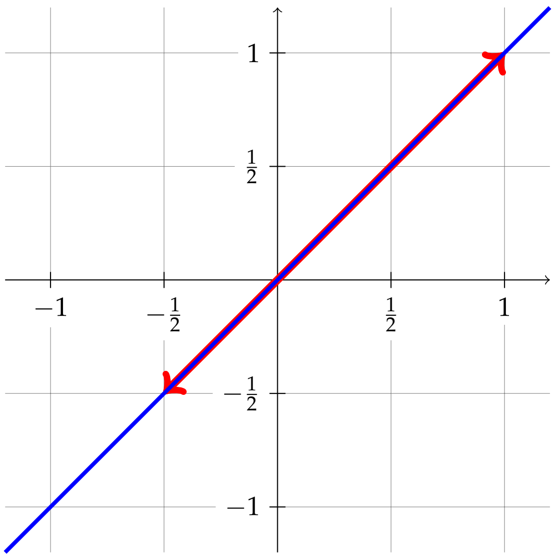
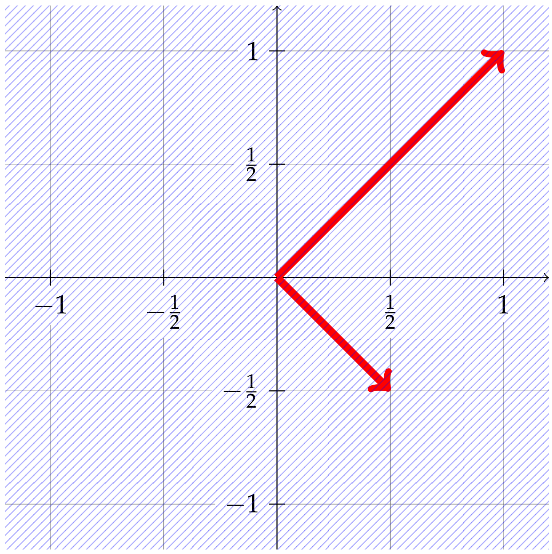
1 Example sets (2 pt)

Sketch the following sets in R2

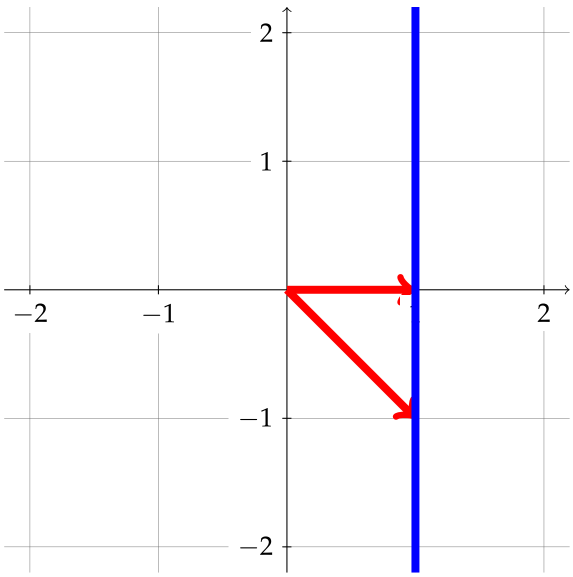
1.1 span $\left\{ \binom{1}{1}, \binom{-0.5}{-0.5} \right\}$



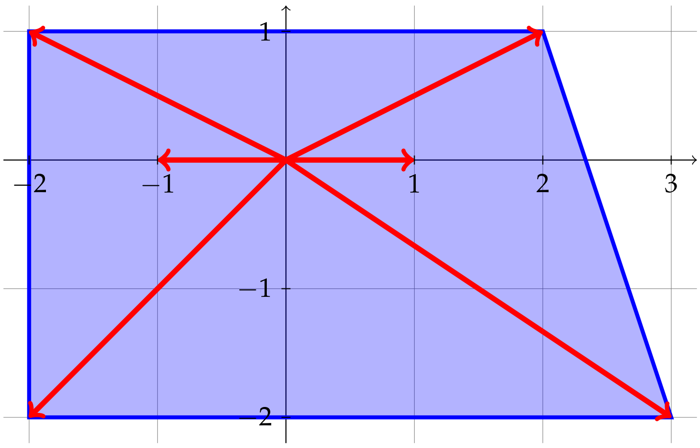
1.2 span $\left\{\binom{1}{1}, \binom{0.5}{-0.5}\right\}$



1.3 aff $\left\{\binom{1}{0}, \binom{1}{-1}\right\}$



1.4 conv $\left\{ \binom{1}{0}, \binom{2}{1}, \binom{3}{-2}, \binom{-1}{0}, \binom{-2}{1}, \binom{-2}{-2} \right\}$



2 Convexity (1 pt)

Let $C \in \mathbb{R}^n$ be a convex set, with $x_1, ..., x_k \in C$, and let $\theta_1, ..., \theta_k \in \mathbb{R}$ satisfy $\theta_i \geq 0$, $\theta_1 + ... + \theta_k = 1$. Show that $\theta_1 x_1 + ... + \theta_k x_k \in C$.

Proof. When $k = 1$, this says that each point of C is a point of C. When $k = 2$, it means whenever $\theta_1 + \theta_2 = 1$ the point $\theta_1 x_1 + \theta_2 x_2$ is in C because $\theta_2 = 1 - \theta_1$ and so the point in question is $\theta_1 x_1 + (1 - \theta_1) x_2$, which is a point on the line between x_1 and x_2.

Now we assume all length-$(k-1)$ combinations are contained in C, and take a length-k combination of points in C:

$$\theta_1 x_1 + \theta_2 x_2 + ... + \theta_k x_k$$

By the inductive hypothesis, we know that

$$y = \frac{\theta_1}{\theta_1 + \theta_2 + ... + \theta_{k-1}} x_1 + \frac{\theta_2}{\theta_1 + \theta_2 + ... + \theta_{k-1}} x_2 + ... + \frac{\theta_{k-1}}{\theta_1 + \theta_2 + ... + \theta_{k-1}} x_{k-1}$$

is in C. (This is only defined if $\theta_1 + ... + \theta_{k-1} \neq 0$; if it's 0, then θ_k is the only nonzero coefficient, so we effectively had a length-1 convex combination to begin with.) If not, the original convex combination can be written as

$$\theta_1 x_1 + \theta_2 x_2 + ... + \theta_k x_k = (\theta_1 + ... + \theta_{k-1}) y + \theta_k x_k$$

which lies on the line segment $[y, x_k]$, and therefore it is in C by the definition of a convex set. Therefore by induction, convex combinations of all size must be contained in C.

□

3 Linear Equations (1 pt)

Show that the solution set of linear equations $\{x | Ax = b\}$ with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{mxn}$ and $b \in \mathbb{R}^m$ is an affine set.

We choose two elements from set: x_0, x_1

$$x_0, x_1 \in R^n$$

$$Ax_0 = b$$

$$Ax_1 = b$$

Affine set $\rightarrow$ $\alpha Ax_0 + (1-\alpha)Ax_1 = \alpha Ax_0 + Ax_1 - \alpha Ax_1 = \alpha b + b - \alpha b = b$
So, the affine combination is also a solution.

4 Linear Inequations (1 pt)

1. Show that the solution set of linear inequations $\{x|Ax \preceq b, Cx = d\}$ with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{mxn}$ and $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{kxn}$ and $d \in \mathbb{R}^k$ is a convex set. Here $\preceq$ means componentwise less or equal.

We choose two elements from set: x_0, x_1

$$Ax_0 \preceq b$$

$$Ax_1 \preceq b$$

$$\alpha, \beta \geq 0$$

$$\alpha + \beta = 1 \rightarrow \alpha = 1 - \beta \geq 0 \rightarrow \beta \leq 1$$

$$\beta \in [0, 1]$$

$$\alpha Ax_0 + \beta Ax_1 = (1-\beta)Ax_0 + \beta Ax_1$$

because $Ax_0 \preceq b$ *and* $Ax_1 \preceq b$ and also $\beta \in [0, 1]$
We can conclude that this equation is also less than b:

$$(1-\beta)Ax_0 + \beta Ax_1 \preceq (1-\beta)b + \beta b$$

$$(1-\beta)Ax_0 + \beta Ax_1 \preceq b$$

Also:

$$Cx_0 = d$$

$$Cx_1 = d$$

$$(1-\beta)Cx_0 + \beta Cx_1 = Cx_0 - \beta Cx_0 + \beta Cx_1 = d - \beta d + \beta d = d$$

2. Is it an affine set?

$$Cx_0 = d$$

$$Cx_1 = d$$

$$(1-\beta)Cx_0 + \beta Cx_1 = Cx_0 - \beta Cx_0 + \beta Cx_1 = d - \beta d + \beta d = d$$

to check $Ax \preceq b$:

$$Ax_0 \preceq b$$

$$Ax_1 \preceq b$$

$$\alpha Ax_0 + (1-\alpha)Ax_1$$

We have $Ax_0 \preceq b$ *and* $Ax_1 \preceq b$ but this time $\alpha \in R$ so, we do not have any limits for α and we cannot conclude that $\alpha Ax_0 + (1-\alpha)Ax_1 \preceq b$
So, it's not affine.

5 Voronoi description of halfspace (1 pt)

Let a and b be distinct points in $\mathbb{R}^n$. Show that the set of all points that are closer (in Euclidean norm) to a than b, i.e., $\{x| \|x-a\|^2 \leq \|x-b\|^2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \leq d$. Draw a picture.

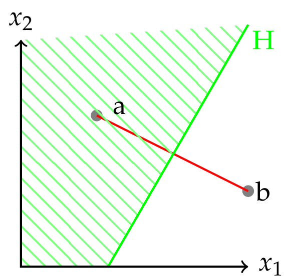
Let a and b be distinct points in $\mathbb{R}$. We need to show that the set of all points that are closer (in Euclidean norm) to a than b, i.e., $\{x \mid \|x-a\|^2 \leq \|x-b\|^2\}$, is a halfspace.
To describe it as an inequality of the form $c^T x \leq d$, we are going to start resolving the inequality $\|x-a\|^2 \leq \|x-b\|^2$ as follows

$$\|x-a\|^2 \leq \|x-b\|^2$$
$$(x-a)(x-a)^T \leq (x-b)(x-b)^T$$

Solving $(x-a)(x-a)^T$ and similarly $(x-b)(x-b)^T$, we get

$$(x-a)(x-a)^T = x^2 - xa^T - ax^T + a^2 = x^2 - 2a^T x + a^2$$
$$x^2 - 2a^T x + a^2 \leq x^2 - 2b^T x + b^2$$
$$a^2 - 2a^T x \leq b^2 - 2b^T x \tag{1}$$

Rewriting the inequality 1 as, $2(b^T - a^T)x \leq b^2 - a^2$. We get the expression we were looking for, where $c^T = 2(b^T - a^T)$ and $d = b^2 - a^2$.



6 Convex Illumination Problem (3 pts)

Show that the solution $p^* = (p_1^*, p_2^*, ..., p_n^*)^T \in \mathbb{R}^n$ of the non-convex illumination problem from the lecture

$$\text{minimize} \quad \max_{k=1...m} \mid logI_k - logI_{des} \mid \tag{2a}$$
$$\text{subject to} \quad 0 \leq p_j \leq p_{max}, j = 1...n \tag{2b}$$

with $I_k = \sum_{j=1}^n a_{kj} p_j$ for geometric constants $a_{jk} \in \mathbb{R}$, a constant desired ilumination $i_{des} \in \mathbb{R}$ and a upper bound $p_{max} \in \mathbb{R}$ on the lamp power, is identical to the solution of the following equivalent (convex) problem

$$\begin{aligned} &\text{minimize} && \max_{k=1\ldots m} h(I_k/I_{des}) && (3a)\\ &\text{subject to} && 0 \leq p_j \leq p_{max}, j = 1\ldots n && (3b)\end{aligned}$$

with $h(u) = \max\{u, 1/u\}$.

Proof. To prove that solving the optimization problem (2a) is equal to solving (3a), we are going to transform the maximum expression (3a) into the maximum expression (2a).
Taking $h(u) = \max\{u, 1/u\}$, and $u = I_k/I_{des}$, it is possible to rewrite the expression (3a) as

$$h(I_k/I_{des}) = \max\{I_k/I_{des}, I_{des}/I_k\}$$

Knowing that the maximum value of the convex function $h(I_k/I_{des})$ is not going to change if we take its logarithm. And, that $\log(a/b) = \log(a) - \log(b)$. We obtain

$$\begin{aligned}\log(\max_{k=1,\ldots,m} h(I_k/I_{des})) &= \log(\max_{k=1,\ldots,m} \max\{I_k/I_{des}, I_{des}/I_k\})\\ &= \max_{k=1,\ldots,m} \max\{\log(I_k/I_{des}), \log(I_{des}/I_k)\}\\ &= \max_{k=1,\ldots,m} \{\log(I_k) - \log(I_{des}), \log(I_{des}) - \log(I_k)\} \qquad (4)\end{aligned}$$

If we call the $\log(I_k) - \log(I_{des})$ as A in the expression 4. We can clearly see that has the shape of $\max\{A, -A\}$, what we can rewrite as $\max \mid A \mid$. Our expression will be

$$\max_{k=1,\ldots,m} \mid \log(I_k) - \log(I_{des}) \mid$$

Being demonstrated that solving the non-convex ilumination problem, is identical to the solution of equivalent convex problem. □

References

[1] L. Vandenberghe S. Boyd. *Convex Optimization*. Cambridge University Press, 2004.