$\boxed{\text{PROBLEM 1}}$

$$m(v) = g^T v + \frac{1}{2} v^T B v \; ; \quad B = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \; ; \; g = \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \quad \Delta = 1/2$$

$\boxed{\text{Case 1}}$ and $\boxed{\text{case 2}}$

$$g^T B g = (1, 1) \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1, 1) \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6 > 0$$

$\hookrightarrow$ Case 2 : $t = \min \{ \|g\|^3 / (\Delta g^T B g) , 1 \}$

$$\|g\|^3 = (\sqrt{1+1})^3 = 2\sqrt{2} \quad \Bigg| \quad \|g\|^3 / \Delta g^T B g = \frac{2\sqrt{2}}{3} \simeq 0.94$$

$$\Delta g^T B g = 0.5 \cdot 6 = 3$$

$$v^c = - t \Delta \frac{g}{\|g\|} = - \frac{2\sqrt{2}}{3} \cdot \frac{1}{2} \cdot \frac{g}{\sqrt{2}} = - \frac{1}{3} g$$

PROBLEM 2

$$m(v) = g^T v + \frac{1}{2} v^T B v \quad ; \quad B = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \quad g = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Delta = 1/2$$

1) Calculate v^G and v^N

$v^G \rightarrow$ minimizer of $m(v)$ along $-g$ $\qquad v^* = \lambda g$

$$m(v^*) = g^T \lambda g + \frac{1}{2} \lambda g^T B \lambda g = \lambda g^T g + \frac{1}{2} \lambda^2 g^T B g$$

$$\nabla_\lambda m(v^*) = g^T g + \lambda g^T B g \overset{!}{=} 0 \Rightarrow \lambda = \frac{-g^T g}{g^T B g} \Rightarrow v^G = \left(-\frac{g^T g}{g^T B g} \right) g$$

$v^N \rightarrow$ optimal solution to the quadratic model

$$\nabla_v m(v) = g + B v \overset{!}{=} 0 \Rightarrow v^N = -B^{-1} g$$

2) Dogleg Path $v^{dog}(z)$ with $z \in [0,2]$

$$v^{dog}(z) = \begin{cases} z v^G & 0 \leq z \leq 1 \\ v^G + (z-1)(v^N - v^G) & 1 \leq z \leq 2 \end{cases}$$

$\boxed{2.1}$ $v^* = z v^G$

$$v^* = z v^G = z \left(\frac{-g^T g}{g^T B g} \right) g$$

$$m(v^*) = g^T z \left(\frac{-g^T g}{g^T B g} \right) g + \frac{1}{2} \left[z \left(\frac{-g^T g}{g^T B g} \right) g \right]^T B \left[z \left(\frac{-g^T g}{g^T B g} \right) g \right]$$

$$\nabla_z m(v^*) = g^T \left(-\frac{g^T g}{g^T B g} \right) g + z \left(\frac{g^T g}{g^T B g} \right)^2 g^T B g =$$

$$= -\frac{(g^T g)^2}{g^T B g} + z \left(\frac{g^T g}{g^T B g} \right)^2 g^T B g \overset{!}{=} 0 \Rightarrow \underline{z = 1}$$

$\boxed{2.2}$ $v^* = v^G + (z-1)(v^N - v^G)$

$$v^* = \left(\frac{-g^T g}{g^T B g} \right) g + (z-1) \left(-B^{-1} g + \left(\frac{g^T g}{g^T B g} \right) g \right) =$$

$$= -\frac{1}{3} g + (z-1) \left(\begin{pmatrix} -1/4 \\ -1/2 \end{pmatrix} + \frac{1}{3} g \right) = -\frac{1}{3} g + (z-1) \begin{pmatrix} 1/12 \\ -1/6 \end{pmatrix} =$$

$$= z \begin{pmatrix} 1/12 \\ -1/6 \end{pmatrix} + \begin{pmatrix} -5/12 \\ -1/6 \end{pmatrix}$$

2.1

$$m(v^*) = g^T\left(z\begin{pmatrix}1/12\\-1/6\end{pmatrix} - \begin{pmatrix}5/12\\1/6\end{pmatrix}\right) + \frac{1}{2}\left(z\begin{pmatrix}1/12\\-1/6\end{pmatrix} - \begin{pmatrix}5/12\\1/6\end{pmatrix}\right)^T B\left(z\begin{pmatrix}1/12\\-1/6\end{pmatrix} - \begin{pmatrix}5/12\\1/6\end{pmatrix}\right) =$$

$$= \cancel{2}\,\frac{-1}{12}z - \frac{7}{12} + \frac{1}{2}\left(z\begin{pmatrix}1/12\\-1/6\end{pmatrix} - \begin{pmatrix}5/12\\1/6\end{pmatrix}\right)^T\left(z\begin{pmatrix}1/3\\-1/3\end{pmatrix} - \begin{pmatrix}5/3\\1/3\end{pmatrix}\right) =$$

$$= \frac{-1}{12}z - \frac{7}{12} + \frac{1}{2}\left(\frac{+1}{12}z^2 + \frac{1}{12}z - \frac{1}{12}z + 3/4\right) =$$

$$= +\frac{1}{2\cdot 12}z^2 - \frac{1}{6}z - \frac{5}{24}$$

$$\nabla_z m(v^*) = +\frac{1}{12}z - \frac{1}{6} \overset{!}{=} 0 \Rightarrow \underline{z = +2}$$

3) Approximate dogleg solution

$$\|v^N\| = \|-B^{-1}g\| = \left\|-\begin{pmatrix}1/4 & 0\\0 & 1/2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\| = \sqrt{\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{2}\right)^2} = 0.56$$

$\boxed{\text{Case 1}}$

$\|v^N\| > \Delta \Rightarrow$ DON'T APPLY

$\boxed{\text{Case 2}}$ $$\|v^G\| = \left\|\left(\frac{-g^Tg}{g^TBg}\right)g\right\| = \left\|-\frac{1}{3}g\right\| = \sqrt{\frac{1}{9} + \frac{1}{9}} = 0.47$$

$\|v^G\| < \Delta \Rightarrow$ DON'T APPLY

From 2.2

$\boxed{\text{Case 3}}$ $$\left\|z\begin{pmatrix}1/12\\-1/6\end{pmatrix} + \begin{pmatrix}-5/12\\-1/6\end{pmatrix}\right\|^2 =$$

$$= \left(\frac{1}{12}z - \frac{5}{12}\right)^2 + \left(\frac{-1}{6}z + \frac{1}{6}\right)^2 = \left(\frac{1}{12}\right)^2 z^2 + \frac{2\cdot 5}{12\cdot 12}z + \left(\frac{5}{12}\right)^2 + \frac{1}{36}z^2 - \frac{1}{36}z + \frac{1}{36} =$$

$$= \frac{5}{144}z^2 + \frac{1}{24}z + \frac{29}{144} = 0.5^2$$

$$\frac{5}{144}z^2 + \frac{1}{24}z + \frac{7}{144} = 0 \quad \begin{cases}\hat{z} = 0.73\\ z = \cancel{-1.93} \rightarrow z \geq 0\end{cases}$$

$$\hat{v} = -\frac{1}{3}g + (\hat{z} - 1)\left(-B^{-1}g + \frac{1}{3}g\right) = \begin{pmatrix}-0.36\\-0.29\end{pmatrix}$$

$$\boxed{\hat{v} = (-0.36, -0.29)^T}$$

Problem 3:

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \quad g = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Delta = 1 \quad v = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathcal{L}(v^*, \lambda^*) = g^T v^* + \frac{1}{2} v^{*T} B v^* + \frac{\lambda}{2}(v^{*T} v^* - \Delta^2)$$

$\| v^* \| \leq 1$

$\lambda^* \geq 0$

$\lambda^* (v^{*T} v^* - \Delta^2) = 0 \Rightarrow \lambda^* (v^{*T} v^* - 1) = 0$

$(B + \lambda^* I) v^* = -g \Rightarrow \begin{cases} (4 + \lambda^*) x = -1 \\ (2 + \lambda^*) y = -1 \end{cases}$

$\lambda^* (x^2 + y^2 - 1) = 0 \Rightarrow \lambda^* = 0 \Rightarrow x = \frac{-1}{4}, \; y = \frac{-1}{2} \Rightarrow v^* = \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{2} \end{pmatrix}$

$\|v^*\|^2 = 0.31 \not\leq \frac{1}{4} \Rightarrow \lambda^* \neq 0$

$x^2 + y^2 - 1 = 0 \Rightarrow x^2 + y^2 = 1$

$\Rightarrow \frac{+1}{(4 + \lambda^*)^2} + \frac{1}{(2 + \lambda^*)^2} = 1 \Rightarrow \lambda^* = -0.94 \not\geq 0$

$\lambda^* = -5.05 \not\geq 0$

$\Rightarrow v^*$ not defined

problem 4:

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \qquad g = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \Delta = \frac{1}{2} \qquad v^* = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathcal{L}(v^*, \lambda^*) = g^T v^* + \frac{1}{2} v^{*T} B v^* + \frac{\lambda}{2}(v^{*T} v^* - \Delta^2)$$

$$\|v^*\|^2 \leq \Delta^2 \Rightarrow \|v^*\|^2 \leq \frac{1}{4}$$

$$\lambda^* \geq 0$$

$$\lambda^*(v^{*T} v^* - \Delta^2) = 0 \Rightarrow \lambda^*(x^2 + y^2 - \frac{1}{4}) = 0$$

$$(B + \lambda^* I) v^* = -g \Rightarrow \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} \lambda^* & 0 \\ 0 & \lambda^* \end{pmatrix} v^* = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4 + \lambda^* & 0 \\ 0 & 2 + \lambda^* \end{pmatrix} v^* = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\Rightarrow (4 + \lambda^*) x = -1$$
$$(2 + \lambda^*) y = -1$$

$$\lambda^*(x^2 + y^2 - \frac{1}{4}) = 0 \Rightarrow \lambda^* = 0 \Rightarrow x = \frac{-1}{4}, \; y = \frac{-1}{2} \Rightarrow v = \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{2} \end{pmatrix}$$

$$\|v^*\|^2 = 0.31 \not< \frac{1}{4} \Rightarrow \lambda^* \neq 0$$

$$\Rightarrow x^2 + y^2 = \frac{1}{4} \Rightarrow \left(\frac{-1}{4 + \lambda^*}\right)^2 + \left(\frac{-1}{2 + \lambda^*}\right)^2 = \frac{1}{4}$$

$$\Rightarrow \lambda^* = 0.26 > 0$$
$$\lambda^* = -6.26 < 0 \; \times$$

$$\Rightarrow x = -0.23 \quad , y = -0.44 \Rightarrow v^* = \begin{pmatrix} -0.23 \\ -0.44 \end{pmatrix}$$

$$\|v^*\|^2 = 0.24 < \frac{1}{4}$$

Problem 5:

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \quad g = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \Delta = \frac{1}{2} \quad v^* = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathcal{L}(v^*, \lambda^*) = g^T v^* + \frac{1}{2} v^{*T} B v^* + \frac{\lambda}{2}(v^{*T} v^* - \Delta^2)$$

$$\|v^*\|^2 \leq \frac{1}{4}$$

$$\lambda^* \geq 0$$

$$\lambda^* (x^2 + y^2 - \frac{1}{4}) = 0$$

$$(B + \lambda^* I) v^* = -g \Rightarrow (4 + \lambda^*) x = -2$$
$$(2 + \lambda^*) y = 0$$

$$\lambda^* = 0 \Rightarrow x = -\frac{1}{2}, \; y = 0 \Rightarrow v^* = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} \quad \|v^*\|^2 = \frac{1}{4} \leq \frac{1}{4}$$

Problem 6:

$$B = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix} \quad g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Delta = 1 \quad v^* = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathcal{L}(v^*, \lambda^*) = g^T v^* + \frac{1}{2} v^{*T} B v^* + \frac{\lambda}{2}(v^{*T} v^* - \Delta^2)$$

$$\|v^*\|^2 \leq 1$$

$$\lambda^* \geq 0$$

$$(B + \lambda^* I) v^* = -g \Rightarrow (4 + \lambda^*) x = -1$$
$$(-2 + \lambda^*) y = 0$$

$$\lambda^* (x^2 + y^2 - 1) = 0 \rightarrow \lambda^* = 0 \Rightarrow x = -\frac{1}{4}, \; y = 0 \Rightarrow v^* = \begin{pmatrix} -\frac{1}{4} \\ 0 \end{pmatrix} \quad \|v^*\|^2 < \frac{1}{4}$$

$$\hookrightarrow x^2 + y^2 - 1 = 0$$

$$\lambda^* = 2 \Rightarrow x = -\frac{1}{6}, \; y = \pm 0.98 \Rightarrow v^* = \begin{pmatrix} -\frac{1}{6} \\ 0.98 \end{pmatrix}$$

$$\|v^*\|^2 = 0.98 \not< \frac{1}{4}$$

$$\Rightarrow v^* = \begin{pmatrix} -\frac{1}{4} \\ 0 \end{pmatrix}$$

Problem 7:

$$B = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix} \qquad g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Delta = 1$$

$$\mathcal{L}(v^*, \lambda^*) = g^T v + \frac{1}{2} v^{*T} B v^* + \frac{\lambda}{2} (v^{*T} v^* - \Delta^2)$$

$$\|v^*\|^2 \leqslant 1$$

$$\lambda^* \geqslant 0$$

$$(B + \lambda^* I) v^* = -g \implies \begin{array}{l} (4 + \lambda^*) x = -1 \\ (-2 + \lambda^*) y = 0 \end{array}$$

$$\lambda^* (x^2 + y^2 - 1) = 0 \implies \lambda^* = 0 \implies x = \frac{-1}{4}, \; y = 0$$

$$\implies x^2 + y^2 = 1, \; \lambda^* = 2 \implies x = \frac{-1}{6} \implies y = \frac{\pm\sqrt{35}}{6} = \pm 0.98$$

$$\implies v^* = \begin{pmatrix} -\frac{1}{4} \\ 0 \end{pmatrix} \qquad \|v^*\| \leqslant \frac{1}{4}$$