EXERCISE 2: CONVEX FUNCTIONS

31099/61099, APPLIED OPTIMIZATION

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1 Convex Functions (5 pt)

1.1 Simple functions (3 pt)

Are the following functions convex?
In order to know if the following functions are convex, we need to see if the $dom f$ is a convex set, and then study the first and second order derivatives.

1. $f(x) = x^2, x \in \mathbb{R}$

   $dom f(x) = \mathbb{R} \rightarrow$ convex set
   $f'(x) = 2x$; $f''(x) = 2 \geq 0 \in \mathbb{R}$
   As $f''(x) \geq 0$ and $dom f$ is a convex set, we can say that $f(x)$ is a convex function.

2. $f(x) = e^{x^2}, x \in \mathbb{R}$

   $dom f(x) = \mathbb{R} \rightarrow$ convex set
   $f'(x) = 2xe^{x^2}$; $f''(x) = \underbrace{2e^{x^2}}_{\geq 0} + \underbrace{\underbrace{4x^2}_{\geq 0}\underbrace{2e^{x^2}}_{\geq 0}}_{\geq 0}$
   As $f''(x) \geq 0$ and $dom f$ is a convex set, we can say that $f(x)$ is a convex function.

3. $f(x) = x^2 + 3xy + 2y^2, x \in \mathbb{R}$

   $dom f(x) = \mathbb{R}^2 \rightarrow$ convex set

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f(x,y)}{\partial x} \\ \frac{\partial f(x,y)}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x + 3y, & 3x + 4y \end{bmatrix}$$

$$\nabla^2 f(x,y) = \begin{bmatrix} \frac{\partial f(x,y)}{\partial x^2} & \frac{\partial f(x,y)}{\partial xy} \\ \frac{\partial f(x,y)}{\partial yx} & \frac{\partial f(x,y)}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

To know if $\nabla^2 f(x,y)$ is positive semidefinite, necessary condition to be convex, it is necessary to calculate the eigenvalues

$$\det A = 0, \text{ with } A = \nabla^2 f(x,y) \text{ and } I \text{ as the identity matrix 2x2}$$
$$(2-\lambda)(4-\lambda) - 9 = 0; \lambda^2 - 6\lambda - 1 = 0$$

If we solve the equation we get the following solution for the eigenvalues λ

$$\lambda = 3 + \sqrt{10} \text{ and } \lambda = 3 - \sqrt{10}$$

How one of the eigenvalues is < 0 we can conclude that the function $f(x,y)$ is not a convex function.

1.2 Log-sum-exp (2 pt)

Show that the function is convex on R^n?

$$f(x) = log(e^{x_1} + ... + e^{x_n})$$

We use the convexity definition to check whether Log-sum-exp is convex or not:

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$
$$\forall x, y \in dom f$$
$$\theta \in [0,1]$$
$$f(x) = log(e^{x_1} + ... + e^{x_n}) = \log(\sum_{i=1}^{n} e^{x_i})$$

Hölder inequality theorem:

$$\sum_{i=1}^{n} |x_i y_i| \leq (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{n} |y_i|^q)^{\frac{1}{q}}$$
$$\frac{1}{p} + \frac{1}{q} = 1$$

for all $(x_1, ..., x_n)$ and $(y_1, ..., y_n) \in R^n$

Now we can substitute $f(x) = log(e^{x_1} + ... + e^{x_n})$ in Hölder inequality:

$$\log(\sum_{i=1}^{n} e^{\theta x_i} e^{(1-\theta)y_i}) \leq \log[(\sum_{i=1}^{n} e^{\theta x_i \frac{1}{\theta}})^{\theta} (\sum_{i=1}^{n} e^{(1-\theta)y_i \frac{1}{(1-\theta)}})^{(1-\theta)}]$$
$$\log(\sum_{i=1}^{n} e^{\theta x_i} e^{(1-\theta)y_i}) \leq \log(\sum_{i=1}^{n} e^{\theta x_i \frac{1}{\theta}})^{\theta} + \log(\sum_{i=1}^{n} e^{(1-\theta)y_i \frac{1}{(1-\theta)}})^{(1-\theta)}$$
$$\log(\sum_{i=1}^{n} e^{\theta x_i + (1-\theta)y_i}) \leq \theta \log(\sum_{i=1}^{n} e^{x_i}) + (1-\theta) \log(\sum_{i=1}^{n} e^{y_i})$$

It's the same as the convexity definition. So, we can conclude that f(x) is convex.

1.3 Geometric mean (Bonus (2 pt))

Show that the geometric mean is concave on R^n_{++}?

Mahler's inequality theory:

$$\prod_{i=1}^{n}(x_i + y_i)^{\frac{1}{n}} \geq \prod_{i=1}^{n}(x_i)^{\frac{1}{n}} + \prod_{i=1}^{n}(y_i)^{\frac{1}{n}}$$

$x_i, y_i > 0$

Now we can substitute $f(x) = \prod_{i=1}^{n}(x_i)^{\frac{1}{n}}$ in Mahler's inequality:

$$\prod_{i=1}^{n}(\theta x_i + (1-\theta)y_i)^{\frac{1}{n}} \geq \prod_{i=1}^{n}(\theta)^{\frac{1}{n}}(x_i)^{\frac{1}{n}} + \prod_{i=1}^{n}(1-\theta)^{\frac{1}{n}}(y_i)^{\frac{1}{n}}$$

$$\prod_{i=1}^{n}(\theta x_i + (1-\theta)y_i)^{\frac{1}{n}} \geq ((\theta)^{\frac{1}{n}})^n \prod_{i=1}^{n}(x_i)^{\frac{1}{n}} + ((1-\theta)^{\frac{1}{n}})^n \prod_{i=1}^{n}(y_i)^{\frac{1}{n}}$$

$$f(\theta x + (1-\theta)y) \geq \theta f(x) + (1-\theta)f(y)$$

So, it's concave.