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About Gödel

Kurt Friedrich Gödel (April 28, 1906 – January 14, 1978) was a logician, mathematician, and philosopher. Considered along with Aristotle and Gottlob Frege to be one of the most significant logicians in history, Gödel had an immense effect upon scientific and philosophical thinking in the 20th century.

Gödel published his first incompleteness theorem in 1931 when he was 25 years old, one year after finishing his doctorate at the University of Vienna. The first incompleteness theorem states that for any ω-consistent recursive axiomatic system powerful enough to describe the arithmetic of the natural numbers (for example Peano arithmetic), there are true propositions about the natural numbers that can be neither proved nor disproved from the axioms. To prove this, Gödel developed a technique now known as Gödel numbering, which codes formal expressions as natural numbers. The second incompleteness theorem, which follows from the first, states that the system cannot prove its own consistency.

Gödel also showed that neither the axiom of choice nor the continuum hypothesis can be disproved from the accepted Zermelo-Fraenkel set theory, assuming that its axioms are consistent. The former result opened the door for mathematicians to assume the axiom of choice in their proofs. He also made important contributions to proof theory by clarifying the connections between classical logic, intuitionistic logic, and modal logic.

Twin prime conjecture

A twin prime is a prime number that is either 2 less or 2 more than another prime number—for example, either member of the twin prime pair (41, 43). In other words, a twin prime is a prime that has a prime gap of two. Sometimes the term twin prime is used for a pair of twin primes; an alternative name for this is prime twin or prime pair.

Twin primes become increasingly rare as one examines larger ranges, in keeping with the general tendency of gaps between adjacent primes to become larger as the numbers themselves get larger. However, it is unknown whether there are infinitely many twin primes (the so-called twin prime conjecture) or if there is a largest pair. The work of Yitang Zhang in 2013, as well as work by James Maynard, Terence Tao and others, has made substantial progress towards proving that there are infinitely many twin primes, but at present this remains unsolved.

Twin prime conjecture, also known as Polignac's conjecture, in number theory, assertion that there are infinitely many twin primes, or pairs of primes that differ by 2. For example, 3 and 5, 5 and 7, 11 and 13, and 17 and 19 are twin primes. As numbers get larger, primes become less frequent and twin primes rarer still.

The first statement of the twin prime conjecture was given in 1846 by French mathematician Alphonse de Polignac, who wrote that any even number can be expressed in infinite ways as the difference between two consecutive primes. When the even number is 2, this is the twin prime conjecture; that is, $2 = 5 - 3 = 7 - 5 = 13 - 11 = \ldots$. (Although the conjecture is sometimes called Euclid's twin prime conjecture, he gave the oldest known proof that there exist an infinite number of primes but did not conjecture that there are an infinite number of twin primes.) Very little progress was made on this conjecture until 1919, when Norwegian mathematician Viggo Brun showed that the sum of the reciprocals of the twin primes converges to a sum, now known as Brun's constant. (In contrast, the sum of the reciprocals of the primes diverges to infinity.) Brun's constant was calculated in 1976 as approximately 1.90216054 using the twin primes up to 100 billion. In 1994 American mathematician Thomas Nicely was using a personal computer equipped with the then new Pentium chip from the Intel Corporation when he discovered a flaw in the chip that was producing inconsistent results in his calculations of Brun's constant. Negative publicity from the mathematics community led Intel to offer free replacement chips that had been modified to correct the problem. In 2010 Nicely gave a value for Brun's constant of $1.902160583209 \pm 0.000000000781$ based on all twin primes less than 2×1016.

The next big breakthrough occurred in 2003, when American mathematician Daniel Goldston and Turkish mathematician Cem Yildirim published a paper, "Small Gaps Between Primes," that established the existence of an infinite number of prime pairs within a small difference (16, with certain other assumptions, most notably that of the Elliott-Halberstam conjecture). Although their proof was flawed, they corrected it with Hungarian mathematician János Pintz in 2005. American mathematician Yitang Zhang built on their work to show in 2013 that, without any assumptions, there were an infinite number differing by 70 million. This bound was improved to 246 in 2014, and by assuming either the Elliott-Halberstam conjecture or a generalized form of that conjecture, the difference was 12 and 6, respectively. These techniques may enable progress on the

Riemann hypothesis, which is connected to the prime number theorem (a formula that gives an approximation of the number of primes less than any given value).

Game of Life

The Game of Life is a two-dimensional universe in which patterns evolve through time. It is one of the best examples in science of how a few simple rules can result in incredibly complex behaviour. It's also incredibly cool and gorgeous to watch. The Life universe is terrifically simple. A square grid contains cells that are either alive or dead. The behaviour of each cell is dependent only on the state of its eight immediate neighbours, according to the following rules:

Live cells:

1. a live cell with zero or one live neighbors will die.
2. a live cell with two or three live neighbors will remain alive
3. a live cell with four or more live neighbors will die.

Dead cells:

1. a dead cell with exactly three live neighbors becomes alive
2. in all other cases a dead cell will stay dead.

We start with a pattern on the grid (generation 0) and we apply the rules simultaneously on all cells. This action results in a new pattern (generation 1). We then apply the rules again on all the cells, which creates another pattern (generation 2), and so on, and so on.

That's it. There are no other rules. The British mathematician John Conway, who is currently at Princeton University, invented the Game of Life in the late 1960s. He chose rules that produced the most unpredictable behavior. One of the most interesting early shapes was the R-pentomino. All patterns made up of up to five live cells die out or become stable after ten generations. But the R-pentomino is active for more than a thousand. The five-cell pattern that moves diagonally across the screen from the 69th generation is called a glider, and is probably the most famous pattern in Life.

It moves one cell vertically and one horizontally every four generations, giving the impression that it is crawling across the screen. Another important pattern is the eater, so called because it eats gliders and other spaceships. One of Conway's early interests was whether or not the Game of Life could emulate the internal workings of a computer. To do this, Life patterns had to be designed that could emulate the behavior of the three basic logic gates: the NOT, AND and OR gates: It is remarkable to think that with all these patterns each cell is only listening to its immediate neighbor.

Set Theory

An important exchange of letters with Richard Dedekind, mathematician at the Brunswick Technical Institute, who was his lifelong friend and colleague, marked the beginning of Cantor's ideas on the theory of sets. Both agreed that a set, whether finite or infinite, is a collection of objects (e.g., the integers, {0, ±1, ±2, …}) that share a particular property while each object retains its own individuality. But when Cantor applied the device of the one-to-one correspondence (e.g., {a, b, c} to {1, 2, 3}) to study the characteristics of sets, he quickly saw that they differed in the extent of their membership, even among infinite sets. (A set is infinite if one of its parts, or subsets, has as many objects as itself.) His method soon produced surprising results.

In 1873 Cantor demonstrated that the rational numbers, though infinite, are countable (or denumerable) because they may be placed in a one-to-one correspondence with the natural numbers (i.e., the integers, as 1, 2, 3…). He showed that the set (or aggregate) of real numbers (composed of irrational and rational numbers) was infinite and uncountable. Even more paradoxically, he proved that the set of all algebraic numbers contains as many components as the set of all integers and that transcendental numbers (those that are not algebraic, as π), which are a subset of the irrationals, are uncountable and are therefore more numerous than integers, which must be conceived as infinite.

Cantor's theory became a whole new subject of research concerning the mathematics of the infinite (e.g., an endless series, as 1, 2, 3…, and even more complicated sets), and his theory was heavily dependent on the device of the one-to-one correspondence. In thus developing new ways of asking questions concerning continuity and infinity, Cantor quickly became controversial. When he argued that infinite numbers had an actual existence, he drew on ancient and medieval philosophy concerning the "actual" and "potential" infinite and also on the early religious training given him by his parents.

Transfinite Numbers: -

In 1895–97 Cantor fully propounded his view of continuity and the infinite, including infinite ordinals and cardinals, in his best-known work, *Beiträge zur Begründung der transfiniten Mengenlehre.*

This work contains his conception of transfinite numbers, to which he was led by his demonstration that an infinite set may be placed in a one-to-one correspondence with one of its subsets. By the smallest transfinite cardinal number, he meant the cardinal number of any set that can be placed in one-to-one correspondence with the positive integers. This transfinite number he referred to as aleph-null. Larger transfinite cardinal numbers were denoted by aleph-one, aleph-two…. He then developed an arithmetic of transfinite numbers that was analogous to finite arithmetic. Thus, he further enriched the concept of infinity. The opposition he faced and the length of time before his ideas were fully assimilated represented in part the difficulties of mathematicians in reassessing the ancient question: "What is a number?" Cantor demonstrated that the set of points on a line possessed a higher cardinal number than aleph-null. This led to the famous problem of the continuum hypothesis, namely, that there are no cardinal numbers between aleph-null and the cardinal number of the points on a line. This problem was of great interest to the mathematical world and was studied by many subsequent mathematicians, including the Czech-Austrian-American Kurt Gödel and the American Paul Cohen.

Cantor Diagonalization Proof

Cantor shocked the world by showing that the real numbers are not countable… there are "more" of them than the integers! His proof was an ingenious use of a proof by contradiction. In fact, he could show that there exist infinities of many different "sizes"!

Cantor's diagonalization argument, which goes as follows. If the reals were countable, it can be put in 1-1 correspondence with the natural numbers, so we can list them in the order given by those natural numbers. Now use this list to construct a real number X that differs from every number in our list in at least one decimal place, by letting X differ from the N-th digit in the N-th decimal place. (A little care must be exercised to ensure that X does not contain an infinite string of 9's.) This gives a contradiction, because the list was supposed to contain all real numbers, including X. Therefore, hence a 1-1 correspondence of the reals with the natural numbers must not be possible.

The Math Behind the Fact:
The theory of countable and uncountable sets came as a big surprise to the mathematical community in the late 1800's.
By the way, a similar "diagonalization" argument can be used to show that any set S and the set of all S's subsets (called the *power set* of S's) cannot be placed in one-to-one correspondence. The idea goes like this: if such a correspondence were possible, then every element A of S has a subset K(A) that corresponds to it. Now construct a new subset of S, call it J, such that an element A is in J if and only if A is NOT in K(A). This new set, by construction, can never be K(A) for any A, because it differs from every K(A) in the "A-th element". So, K(A) must not run through the entire power set of A, hence the 1-1 correspondence asserted above must not be possible.
This proof shows that there are infinite sets of many different "sizes" by considering the natural numbers and its successive power sets! The "size" of a set is called is cardinality.

David Hilbert's 3 questions

Is math complete, Is math consistent and Is math decidable

At the International Congress of Mathematicians held in Bologna, Italy , in 1928 mathematician and physicist David Hilbert returned to the second of the twenty-three problems posed in his 1900 paper Mathematische Probleme, asking "**is mathematics complete, is it consistent, and is it decidable**?"

Three years later, the first two of these questions were answered in the negative by Kurt Gödel. Working independently, Alonzo Church, Alan Turing, and Emil Post published answers to the third question in 1936**.**

Gödel Incompleteness Theorem

Mathematicians of the era sought a solid foundation for mathematics: a set of basic mathematical facts, or axioms, that was both consistent — never leading to contradictions — and complete, serving as the building blocks of all mathematical truths.

But Gödel's shocking incompleteness theorems, published when he was just 25, crushed that dream. He proved that any set of axioms you could posit as a possible foundation for math will inevitably be incomplete; there will always be true facts about numbers that cannot be proved by those axioms. He also showed that no candidate set of axioms can ever prove its own consistency. His incompleteness theorems meant there can be no mathematical theory of everything, no unification of what's provable and what's true. What mathematicians can prove depends on their starting assumptions, not on any fundamental ground truth from which all answers spring.

For example, Gödel himself helped establish that the continuum hypothesis, which concerns the sizes of infinity, is undecidable, as is the halting problem, which asks whether a computer program fed with a random input will run forever or eventually halt. Undecidable questions have even arisen in physics, suggesting that Godelian incompleteness afflicts not just math, but — in some ill-understood way — reality.

First Incompleteness Theorem:

"Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of F which can neither be proved nor disproved in F."

The unprovable statement G_F referred to by the theorem is often referred to as "the Gödel sentence" for the system F. The proof constructs a particular Gödel sentence for the system F, but there are infinitely many statements in the language of the system that share the same properties, such as the conjunction of the Gödel sentence and any logically valid sentence.

The first incompleteness theorem shows that the Gödel sentence G_F of an appropriate formal theory F is unprovable in F. Because, when interpreted as a statement about arithmetic, this unprovability is exactly what the sentence (indirectly) asserts, the Gödel sentence is, in fact, true. For this reason, the sentence G_F is often said to be "true but unprovable." However, since the Gödel sentence cannot itself formally specify its intended interpretation, the truth of the sentence G_F may only be arrived at via a meta-analysis from outside the system. In general, this meta-analysis can be carried out within the weak formal system known as primitive recursive arithmetic, which proves the implication $\mathrm{Con}(F)\rightarrow G_F$, where $\mathrm{Con}(F)$ is a canonical sentence asserting the consistency of F

Second Incompleteness Theorem:

"Assume *F* is a consistent formalized system which contains elementary arithmetic. Then ."

$$F \nvdash \mathrm{Cons}(F)$$

This theorem is stronger than the first incompleteness theorem because the statement constructed in the first incompleteness theorem does not directly express the consistency of the system. The proof of the second incompleteness theorem is obtained by formalizing the proof of the first incompleteness theorem within the system F itself.

The standard proof of the second incompleteness theorem assumes that the provability predicate $\mathrm{Prov}_A(P)$ satisfies the Hilbert–Bernays provability conditions. Letting #(P) represent the Gödel number of a formula P, the provability conditions say:

1. If F proves P, then F proves $\mathrm{Prov}_A(\#(P))$.
2. F proves 1.; that is, F proves $\mathrm{Prov}_A(\#(P)) \rightarrow \mathrm{Prov}_A(\#(\mathrm{Prov}_A(\#(P))))$.
3. F proves $\mathrm{Prov}_A(\#(P \rightarrow Q)) \wedge \mathrm{Prov}_A(\#(P)) \rightarrow \mathrm{Prov}_A(\#(Q))$