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WHEN ARE SWING OPTIONS BANG-BANG?

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In this paper we investigate a class of swing options with firm constraints in view of the modeling of supply agreements. We show, for a fully general payoff process, that the premium, solution to a stochastic control problem, is concave and piecewise affine as a function of the global constraints of the contract. The existence of bang-bang optimal controls is established for a set of constraints which generates by affinity the whole premium function. When the payoff process is driven by an underlying Markov process, we propose a quantization based recursive backward procedure to price these contracts. A priori error bounds are established, uniformly with respect to the global constraints.

Keywords: Swing option; stochastic control; optimal quantization; energy.

1. Introduction

The deregulation of energy markets has given rise to various families of contracts exchanged by the different energy markets players (banks, utilities...) to mitigate their market risks. Many of them appear as some derivative products whose underlying is some tradable futures or spot contract on gas or electricity (see [14] for an introduction). The class of swing options has been given a special attention in the

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literature, because it includes many of these derivative products. A common feature to all these options is that they introduce some risk sharing between a producer and a trader, of gas or electricity for example. From a probabilistic viewpoint, they appear as some stochastic control problems modeling multiple optimal stopping problems (the control variable is the purchased quantity of energy); see e.g. [10, 11] in a continuous time setting. Gas storage contracts (see [7], [9]) or electricity supply agreements (see [20], [9]) are examples of such swing options. Indeed, energy supply contracts are one simple and important example of such swing options that will be deeply investigated in this paper. It is worth mentioning that this kind of contracts are slightly different from multiple exercise American options as considered in [11] for example. In our setting the volumetric constraints play a key role and thus, the flexibility is not restricted to time decisions, but also has to take into account volume management.

Designing efficient numerical procedures for the pricing of swing option contracts remains a very challenging question as can be expected from a possibly multi-dimensional stochastic control problem subject to various constraints (due to the physical properties of the assets like in storage contracts). Most recent approaches developed in mathematical finance, especially for the pricing of American options, have been adapted and transposed to the swing framework: tree (or "forest") algorithms in the pioneering work [19], least squares regression MC methods (see [7] and references therein), PDE numerical methods (finite elements, see [31]). See also [21,23,30] for other related numerical methods.

The aim of this paper is to elucidate the structure of the optimal control in supply contracts (with firm constraints) and how it impacts the numerical methods of pricing. We will provide in a quite general (and abstract) setting some "natural" (and simple) conditions involving the local and global purchased volume constraints to ensure the existence of a *bang-bang optimal strategy*; that is a strategy such that, at each time step, the control process can only takes as a value one of the bounds of the set of the admissible controls. Such controls usually do not exist, but it is possible to design *a priori* the contract so that its parameters satisfy these conditions. To our knowledge very few theoretical results have been established so far on this problem (see however [7] in a Markovian framework for contracts with penalized constraints and [29], also in a Markovian framework).

This first result of the paper not only enlightens the understanding of the management of a swing contract but it also has some deep repercussions for the numerical methods to price it. As a matter of fact, taking advantage of the existence of a bang-bang optimal strategy, we propose and analyze in detail (when the underlying asset has a Markovian dynamics) a quantized Dynamic Programming procedure to price any swing option whose volume constraints satisfy the "bang-bang" assumption. Furthermore some *a priori* error bounds are established. This procedure turns out to be dramatically efficient, as emphasized in the companion paper [5] where the method is extensively tested with assets having multi-factor Gaussian underlying dynamics and compared to the least squares regression method.

The abstract swing contract with firm constraints. The holder of a supply contract has the right to purchase periodically (daily, monthly, etc) an amount of energy at a given unit price. This amount of energy is subject to some lower and upper "local" constraints. The total amount of energy purchased at the end of the contract is also subject to a "global" constraint. Given the dynamics of the energy price process, the problem is to evaluate the price of such a contract, at time $t = 0$ when it is initiated and during its whole life up to its maturity.

To be precise, the owner of the contract is allowed to purchase at times t_i, $i = 0, \ldots, n-1$ a quantity q_i of energy at a unit *strike price* $K_i := K(t_i)$. At every date t_i, the purchased quantity q_i is subject to the firm "*local*" *constraint*,

$$q_{\min} \le q_i \le q_{\max}, \quad i = 0, \ldots, n-1,$$

($0 \le q_{\min} < q_{\max} < +\infty$) whereas the global purchased quantity $\bar{q}_n := \sum_{i=0}^{n-1} q_i$ is subject to the (firm) global constraint

$$\bar{q}_n \in [Q_{\min}, Q_{\max}] \quad (0 < Q_{\min} \le Q_{\max} < +\infty).$$

For the consistency of the global constraints, it is assumed that

$$nq_{\min} \le Q_{\min} \le Q_{\max} \le nq_{\max},$$

otherwise $Q_{\min}$ or $Q_{\max}$ (or both) has no effect on the contract. The strike price process $(K_i)_{0\le i\le n-1}$ can be either deterministic (even constant) or stochastic, e.g. indexed on a basket of other commodities (oil, etc). Usually, in energy markets the price is known through future contracts $(F_{s,t})_{0\le s\le t}$ where $F_{s,t}$ denotes the price at time s of the forward contract delivered at maturity t. The available data at time 0 are $(F_{0,t})_{0\le t\le T}$ (in real markets this is of course not a continuum).

The underlying asset price process, temporarily denoted $(S_{t_i})_{0\le i\le n-1}$, is often the so-called "day-ahead" contract $F_{t,t+1}$ which is a tradable instrument or the spot price $F_{t,t}$ which is not. All the decisions about the contract need to be adapted to the filtration of (S_{t_i}) i.e. $\mathcal{F}_i := \sigma(S_{t_j}, j = 0, \ldots, i)$, $i = 0, \ldots, n-1$ (with $\mathcal{F}_0 = \{\emptyset, \Omega\}$). This leads to define the price of such a contract at any time t_k, as a function of the *residual global constraints* at time k, namely

$$Q^k := (Q_{\min} - \bar{q}_k, Q_{\max} - \bar{q}_k), \quad \bar{q}_k = \sum_{0\le \ell\le k-1} q_\ell \text{ (cumulative purchased process)},$$

by

$$P_k^n(Q^k) := \operatorname{esssup}\left\{ \mathbb{E}\left(\sum_{j=k}^{n-1} q_j e^{-r(t_j - t_k)}(S_{t_j} - K_j) \,|\, \mathcal{F}_k \right), \right.$$

$$\left. q_j : (\Omega, \mathcal{F}_j) \to [q_{\min}, q_{\max}], j = k, \ldots, n-1, \sum_{j=k}^{n-1} q_j \in [Q^k_{\min}, Q^k_{\max}] \right\}$$

where r denotes the (deterministic) interest rate. This pricing problem clearly appears as a stochastic control problem.

In the pioneering work by [19], this type of contract was computed by using some forests of (multinomial) trees. A natural variant, at least for numerical purpose, is to consider a penalized version of this stochastic control. Thus, in [7], a penalization $Q_{n,\varepsilon}(V_n, \bar{q}_n)$ with $Q_{n,\varepsilon}(V, q) = -(V(q - Q_{\max})^+ + V(q - Q_{\min})^+)/\varepsilon$ is added ($Q_{n,\varepsilon}$ is negative outside $[Q_{\min}, Q_{\max}]$ and zero inside).

Swing options also enable the modeling of physical assets such as underground gas storages or power production assets (see [7]). In those cases, the set of constraints and payoff of the option may become much more complex than for supply agreements.

As concerns the underlying asset dynamics, it is commonly shared in finance to assume that the traded asset has Markovian dynamics (or is a component of a Markov process like with stochastic volatility models). The dynamics of physical assets for many reasons are often modeled using some more deeply non-Markovian models like long memory processes, etc.

All these specific features of energy derivatives suggest to tackle the above pricing problem in a rather general framework, making some Markov assumptions when dealing with numerical aspects. This is what we do in the first part of the paper where the general setting of a swing option defined by a sequence of $\mathcal{F}_k$-adapted payoffs is thoroughly investigated *as a function of its global constraints* $Q = (Q_{\min}, Q_{\max})$ (when the local constraints are normalized i.e. q_i is $[0,1]$-valued for every $i \in \{0, \ldots, n-1\}$). We show that this premium is a concave, piecewise affine, function of the global constraints, affine on triangles of the $(m, M) + \{(u, v), 0 \leq u \leq v \leq 1\}$, $m, M \in \mathbb{N}^2$, $m \leq M \leq n$ and $(m, M) + \{(u, v), 0 \leq v \leq u \leq 1\}$, $m, M \in \mathbb{N}^2$, $m \leq M - 1 \leq n - 1$. We also show that for integer valued global constraints, the optimal controls are always bang-bang i.e. the *a priori* $[0,1]$-valued optimal purchased quantities q_i^* are in fact always equal to 0 or 1. Such a result can be extended in some way to any ordered pair of global constraints when all the payoffs are non-negative.

Then, when V is a Markov functional of a "structure process", we propose an optimal quantization tree approach to efficiently price swing options. This Markov "structure process" can be the underlying traded asset itself or a higher dimensional hidden Markov process like for multi-factor models having some long-memory properties.

Optimal Quantization was first introduced as a numerical method to solve nonlinear problems arising in Mathematical Finance in a series of papers [1–4] devoted to the pricing and hedging of American style multi-asset options. It has also been applied to stochastic control problems, namely portfolio optimization in [25]. The purely numerical aspects of optimal quantization trees, with a special emphasis on the Gaussian distribution, have been investigated in [27, 28]. See [26] for a survey on numerical applications of optimal quantization to Finance. For other applications (to automatic classification, clustering, etc), see [16]. In this paper, we propose a quantized backward dynamic programming algorithm to approximate the premium of a swing contract. We analyze the rate of convergence of this algorithm and provide some *a priori* error bounds in terms of quantization errors. This method, when implemented with optimal quantization grids, can be seen as a "*model-driven*" grid method, optimally "fitting" the marginal distribution by a finitely supported one and automatically taking into account the correlations between assets and/or factors of the model. Its principle asset is to directly approximate the Markov dynamics so that the main part of the procedure is not payoff dependent.

We illustrate the method on a two factor Gaussian model of the future prices of gas. This is a toy model. Some tests are presented (robustness with respect to transition weights estimation, optimal quantization grid size, computation time). Finally the whole graph of the premium as a function of the global constraints is computed by taking advantage of the piecewise affine property of this graph. An extensive study of the pricing of swing options by optimal quantization is carried out from a numerical point of view in [5], including a brief comparison with regression methods "à la Longstaff-Schwartz".

The paper is organized as follows. In Sec. 2 below we define more precisely the stochastic control problem associated to our general swing option problem with firm constraints and the variables of interest (global constraints, local constraints, state variable, etc). We give the dynamic programming formula satisfied by the value function of the problem. We recall the decomposition of such swing options into a swap contract and a normalized swing option with local constraints 0 and 1 respectively. In Sec. 3, we state and prove as our main result that the premium function is piecewise affine and concave and that the optimal purchased quantities satisfy a "0-1" or bang-bang principle (Theorem 3.1). A special attention is paid to the 2-period model for which a more precise result holds and provides an intuitive interpretation of the optimal control. In Sec. 4, after some short background on quantization and its optimization, we propose a quantization based backward dynamic programming formula as a numerical method to solve the swing pricing problem. Then we provide some error bounds for the procedure depending on the quantization error induced by the quantization of the Markov structure process.

Notations.

- The Lipschitz coefficient of a function $f : \mathbb{R}^d \to \mathbb{R}$ is defined by $[f]_{\mathrm{Lip}} := \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|} \leq +\infty$ so that $[f]_{\mathrm{Lip}} < \infty$ if and only if f Lipschitz continuous.
- The canonical Euclidean norm on $\mathbb{R}^d$ will be denoted $|\cdot|$.
- All essential suprema will be taken with respect to the same probability $\mathbb{P}$, so $\mathbb{P}$ will be dropped from now on in the notation esssup.

2. A Model for Swing Options with Firm Constraints

2.1. *Swing option as a stochastic control problem*

One considers a *payoff* sequence $(V_k)_{0\leq k\leq n-1}$ of integrable random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $\mathcal{F}_k^V := \sigma(V_0, V_1, \ldots, V_k)$, $k = 0, \ldots, n-1$ denote its natural filtration. For convenience we introduce a more general discrete time filtration $\mathcal{F} := (\mathcal{F}_k)_{0\le k\le n-1}$ to which $V = (V_k)_{0\le k\le n-1}$ is adapted (i.e. satisfies $\mathcal{F}^V_k \subset \mathcal{F}_k$, $k = 0,\ldots,n-1$). Note that *no Markov or Markov functional assumption is made* on the dynamics of the state process V. Although we are originally motivated by solving the following stochastic control problem (with *maturity* n) defined below at time 0, we will introduce its version at any time $k \in \{0,\ldots,n\}$ for technical matter since the resulting value functionals of these problems will be involved in a dynamic programming formula.

We aim at solving the stochastic control problems

$$(\mathcal{S}_{(V,\mathcal{F})})^n_k \equiv \operatorname{esssup}\left\{\mathbb{E}\left(\sum_{\ell=k}^{n-1} q_\ell V_\ell \,|\, \mathcal{F}_k\right), q_\ell : (\Omega,\mathcal{F}_\ell) \to [q_{\min}, q_{\max}],\right.$$
$$\left. k \le \ell \le n-1, \sum_{\ell=k}^{n-1} q_\ell \in [Q_{\min}, Q_{\max}]\right\} \tag{2.1}$$

where $Q := (Q_{\min}, Q_{\max})$ is an ordered pair of two non-negative $\mathcal{F}_k$-measurable random variables taking values in

$$\mathcal{A}^n_k = \{(Q_1,Q_2) \in \mathbb{R}^2_+ \,|\, (n-k)q_{\min} \le Q_1 \le Q_2 \le (n-k)q_{\max}\}. \tag{2.2}$$

Such an ordered pair will be called an $\mathcal{F}$-*admissible global constraint* at time k (with respect to the local constraints $(q_{\min}, q_{\max})$.

A $\mathcal{F}$-adapted sequence $(q_k)_{0\le k\le n-1}$ of $[q_{\min}, q_{\max}]$-valued random variable will be a *locally admissible control* (for $(\mathcal{S}_{(V,\mathcal{F})})^n_0$). For any locally admissible control, one defines the cumulative purchase process by

$$\bar{q}_0 := 0, \quad \bar{q}_k := q_0 + \cdots + q_{k-1}, \quad k = 1,\ldots,n,$$

If $\bar{q}_n \in [Q_{\min}, Q_{\max}]$, the sequence $(q_k)_{0\le k\le n-1}$ is called an $(\mathcal{F},Q)$-*admissible control*. It is one natural *state variable* of the problem. So is, by duality, the residual global constraints and we will use both depending on our purpose.

It turns out that the criterion of this stochastic control problem is linear in the control, with a linear state dynamics, the main difficulty coming from the state constraints. Natural adaptations of classical results in a regular Markov framework (like e.g. those established in Chapter 8 in [8]) show that, for every $k \in \{0,\ldots,n\}$ there exists a random function P^n_k defined on $\mathcal{A}^n_k \times \Omega$, $\mathcal{B}or(\mathcal{A}^n_k) \otimes \mathcal{F}_k$-measurable such that for every $\mathcal{F}$-admissible global constraint at time k, say $Q := (Q_{\min}, Q_{\max})$, $P^n_k(Q(\omega),\omega)$ is the solution to the problem $(\mathcal{S}_{(V,\mathcal{F})})^n_k$. Furthermore, $\mathbb{P}(d\omega)$-*a.s.*,

$$Q_{\min} \mapsto P^n_k((Q_{\min}, Q_{\max}),\omega) \text{ is non-increasing and}$$
$$Q_{\max} \mapsto P^n_k((Q_{\min}, Q_{\max}),\omega) \text{ is non-decreasing}$$

(as long as $(Q_{\min}, Q_{\max}) \in \mathcal{A}^n_k$).

A self-contained proof of the existence of the random functions P_k^n and its monotony properties can be found in [6]. From now on, for notational convenience we will refer to this quantity as $P_k^n(Q)$. Furthermore, the value random functions satisfy a backward dynamic programming formula.

Proposition 2.1. (*Local backward dynamic programming formula*) *The random value function* $P_k^n(Q)$ *of Problem* (2.1) *satisfies the following dynamic programming formula*: *for every* $n \geq 1$,

$$P_k^n(Q) = \sup\{qV_k + \mathbb{E}(P_{k+1}^n(Q - q(1,1)) \,|\, \mathcal{F}_0), q \in [q_{\min}, q_{\max}], \\ Q - q(1,1) \in \mathcal{A}_{k+1}^n\} \quad \mathbb{P}\text{-}a.s. \tag{2.3}$$

The optimal control q_k^* *for* (2.3) *is given by*

$$q_k^* = \operatorname{argmax}\{qV_k + \mathbb{E}(P_{k+1}^n(Q - q(1,1)) \,|\, \mathcal{F}_0), q \in [q_{\min}, q_{\max}], \\ Q - q(1,1) \in \mathcal{A}_{k+1}^n\}. \tag{2.4}$$

2.2. *Canonical decomposition, normalized swing option*

As a first step we will reduce the problem to a normalized contract. From practitionners' point of view, this normalization appears as the splitting of the contract into a swap contract and a swing without local minimal constraint (see also [12]). The decomposition follows from the fact that, at every time $k \in \{0, \ldots, n-1\}$, an $\mathcal{F}_k$-measurable random variable q_k is $[q_{\min}, q_{\max}]$-valued if and only if $q_k = q_{\min} + (q_{\max} - q_{\min})q'_k$ for some $[0,1]$-valued $\mathcal{F}_k$-measurable random variable q'_k. Then, under the consistency assumptions on the constraints, for every $k \in \{0, \ldots, n-1\}$,

$$P_k^n(Q) = \underbrace{q_{\min} \sum_{\ell=k}^{n-1} \mathbb{E}(V_\ell \,|\, \mathcal{F}_k)}_{\text{swap contract}} + (q_{\max} - q_{\min}) \underbrace{P_{[0,1],k}^n(\widetilde{Q}_{\min}, \widetilde{Q}_{\max})}_{\text{normalized contract}}$$

where

$$\widetilde{Q}_{\min} = \frac{Q_{\min} - (n-k)q_{\min}}{q_{\max} - q_{\min}} \quad \text{and} \quad \widetilde{Q}_{\max} = \frac{Q_{\max} - (n-k)q_{\min}}{q_{\max} - q_{\min}}$$

and $P_{[0,1],k}^n(\widetilde{Q}_{\min}, \widetilde{Q}_{\max})$ is the value of a *normalized swing contract* with constraints $q_{\min} = 0$ and $q_{\max} = 1$, i.e. the purchased quantity at time k q_k is always $[0,1]$-valued.

As a result, the number of parameters is reduced to 2 (global constraints). This decompositon has no major theoretical consequences although it will simplify some technical aspects of the proofs. From a practical viewpoint, it shows that swing contracts are somewhat over-parametrized so that this reduction makes it possible to compute a large bundle of contracts using a translation-contraction.

2.3. *Objectives of the paper*

We will answer the following questions:

- Existence of an optimal control $q^* = (q_k^*)_{0\le k\le n-1}$ (i.e. an optimal purchasing strategy).
- Regularity of the price at the origin as a function of the global constraints $Q \mapsto P_0^n(Q)$ (piecewise affine and concave).
- Existence of a *bang-bang* optimal control $(q_k^*)_{0\le k\le n-1}$ for certain values of the global constraints Q (namely when Q has integer components). By "bang-bang" we mean that $\mathbb{P}(d\omega)$-*a.s.*
  - all the *local* constraints on the $q_k(\omega)$ are saturated i.e. for every $k \in \{0,\ldots, n-1\}$, $q_k(\omega) \in \{0, 1 \wedge Q_{\max}\}$

    or
  - there exists at most one instant $k_0(\omega)$ such that $q_{k_0(\omega)}(\omega) \in (0, 1 \wedge Q_{\max})$ and one global constraint is saturated.

    Note that if $Q \in \mathbb{N}^2$, then a bang-bang Q-admissible control necessarily satisfies $\mathbb{P}(d\omega)$-*a.s.* $q_k(\omega) \in \{0, 1\}$, $k = 0,\ldots, n-1$. The *existence of bang-bang optimal controls* combined with *the piecewise affinity* of $P_0^n(Q)$ will be the keys, with the canonical reduction below, to devise our quantization based numerical method in Sec. 4.
- When there is an underlying structure Markov process, say $V_k = v_k(Y_k)$, we will show that the optimal control turns out to be a function of Y_k at every time k as well.

3. Affine Value Function with Bang-Bang Optimal Controls

3.1. *The main result*

We define, for every integer $n \ge 1$, the triangular set (simplex) of admissible values for an ordered pair of global constraints by:

$$T^+(n) := \{(u, v) \in \mathbb{R}^2, 0 \le u \le v \le n\}.$$

Then we will define a triangular tiling of $T^+(n)$ as follows: for every ordered pair of integers (i, j), $0 \le i \le j \le n-1$,

$$T_{ij}^+ := \{(u, v) \in [i, i+1] \times [j, j+1], v \ge u + j - i\} \quad \text{and}$$
$$T_{ij}^- := \{(u, v) \in [i, i+1] \times [j, j+1], v \le u + j - i\}.$$

One checks that

$$T^+(n) = \left(\bigcup_{0\le i\le j\le n-1} T_{ij}^+\right) \bigcup \left(\bigcup_{0\le i<j\le n-1} T_{ij}^-\right).$$

Figure 1 displays such a tiling with $n = 5$.

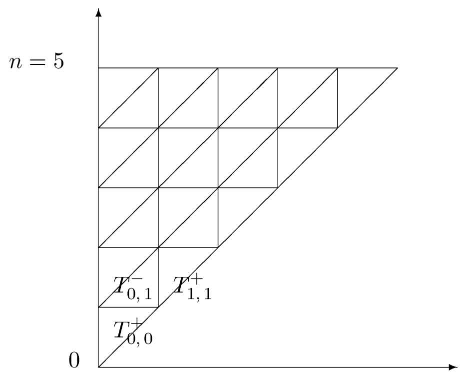

Fig. 1. Triangular tiling of $T^+(5)$.

Theorem 3.1. *The normalized multi-period swing option premium with deterministic global constraints $Q := (Q_{\min}, Q_{\max}) \in T^+(n)$ as defined by (2.1) is always obtained as the result of an optimal strategy.*

(a) *The value function (premium):*

- *the mapping $Q \mapsto P_0^n(Q)$ is a concave, continuous, piecewise affine process, affine on every triangle $T_{ij}^{\pm}$ of the tiling of $T^+(n)$. Furthermore (as long as $Q = (Q_{\min}, Q_{\max}) \in T^+(n)$),*

  $$Q_{\min} \mapsto P_0^n(Q) \text{ is non-increasing} \quad \text{and} \quad Q_{\max} \mapsto P_0^n(Q) \text{ is non-decreasing}$$

  and one has at the vertices of the simplex $T^+(n)$,

  $$P_0^n(0,0) = 0, \quad P_0^n(0,n) = \mathbb{E}\left(\sum_{k=0}^{n-1} V_k^+ \,|\, \mathcal{F}_0\right) \quad \text{and}$$

  $$P_0^n(n,n) = \mathbb{E}\left(\sum_{k=0}^{n-1} V_k \,|\, \mathcal{F}_0\right).$$

- *If $V_i \geq 0$ a.s. for every $i \in \{0,\ldots,n-1\}$, then, for every $Q = (Q_{\min}, Q_{\max}) \in T^+(n)$, $P_0^n(Q) = P_0^n(0, Q_{\max}) = P_0^n(Q_{\max}, Q_{\max})$ a.s. (in particular $Q_{\max} \mapsto P_0^n(Q_{\max}, Q_{\max})$ is a.s. non-decreasing).*

(b) *The optimal control:*

- *If the global constraint $Q = (Q_{\min}, Q_{\max}) \in \mathbb{N}^2 \cap T^+(n)$, then there always exists a bang-bang optimal control $(q_k^*)_{0 \leq k \leq n-1}$ such that, for every $k = 0,\ldots,n-1$, q_k^* is $\{0,1\}$-valued.*
- *If $n = 2$ and $Q_{\max} - Q_{\min} \in \mathbb{N}$, there exists a bang-bang optimal control.*
- *If $V_i \geq 0$ a.s. for every $i \in \{0,\ldots,n-1\}$, then there always exists a bang-bang optimal control $(q_k^*)_{0 \leq k \leq n-1}$ which satisfies $\sum_{0 \leq k \leq n-1} q_k^* = Q_{\max}$.*
- *Otherwise the optimal control is not bang-bang (in a generic setting).*

We will first inspect the case of a two period swing contract. In that setting we obtain a slightly more general result for the existence of a bang-bang control and some explicit forms for the optimal control as well as some intuitive interpretations.

3.2. *The two period option*

When $n = 2$, we obtain a slightly looser condition to establish the existence of a bang-bang control. What is probably more important, we have a clear interpretation in terms of error of prediction to enlighten the situation where "bang-bangness" fails. This is the main reason for the specific investigations that follow.

Proposition 3.1. (a) *The two period* ($n = 2$) *swing option premium with admissible global constraints* $Q = (Q_{\min}, Q_{\max}) \in T^+(2)$ *as defined by* (2.1) *is always obtained as the result of an optimal strategy* (q_0^*, q_1^*) *given by*

$$\begin{aligned} q_0^* &= \operatorname{argmax}_{q \in I_Q^1} \{qV_0 + (1 \wedge (Q_{\max} - q))\mathbb{E}(V_1^+|\mathcal{F}_0) \\ &\quad - (Q_{\min} - q)^+ \mathbb{E}(V_1^-|\mathcal{F}_0)\} \end{aligned} \tag{3.1}$$

$$q_1^* = (1 \wedge (Q_{\max} - q_0^*))\mathbf{1}_{\{V_1 \geq 0\}} + (Q_{\min} - q_0^*)^+ \mathbf{1}_{\{V_1 < 0\}} \tag{3.2}$$

where $I_Q^1 := [(Q_{\min} - 1)^+, Q_{\max} \wedge 1]$ *so that* $P_0^2(Q, \mathcal{F}) = \mathbb{E}(q_0^* V_0 + q_1^* V_1 \mid \mathcal{F}_0)$.

(b) *The optimal control*:

- *If the global constraints* $Q = (Q_{\min}, Q_{\max})$ *satisfy*

$$Q_{\max} - Q_{\min} \in \{0, 1, 2\} \tag{3.3}$$

  there always exists a bang-bang optimal control. If furthermore, they only take integer values (*in* $\{0, 1, 2\}$) *then there always exists a* $\{0, 1\}$*-valued bang-bang optimal control.*
- *If* $V_0, V_1 \geq 0$ *a.s., then any optimal control* (q_0^*, q_1^*) *is a.s. bang-bang and satisfies* $q_0^* + q_1^* = Q_{\max}$ *on* $\{V_i > 0, i = 1, 2\}$.
- *Otherwise the optimal control is not bang-bang* (*generally*).

(c) *The value function* (*premium*):

- *the mapping* $Q \mapsto P_0^2(Q)$ *is concave, affine on each triangle* $T_{i,j}^{\pm}$ *that tiles* $T^+(2)$.
- *Furthermore, when* V_0 *and* V_1 *are a.s. non-negative,*

$$P_0^2(Q) = (1 \wedge (Q_{\max} - 1)^+)(V_0 \wedge \mathbb{E}(V_1|\mathcal{F}_0)) + (Q_{\max} \wedge 1)(V_0 \vee \mathbb{E}(V_1|\mathcal{F}_0)).$$

Proof. We will rely on the elementary remark that a piecewise affine function defined on $[0, 1]$ reaches its extrema either at monotony break points or at the endpoinst 0 and 1.

(a) Let (q_0, q_1) be an admissible control: $q_0 + q_1 \in [Q_{\min}, Q_{\max}]$ and q_i are $[0,1]$-valued $\mathcal{F}_i$-measurable, $i = 0, 1$. Consequently q_0 is I_Q^1-valued and q_1 is $[(Q_{\min} - q_0)^+, (Q_{\max} - q_0) \wedge 1]$-valued. Hence

$$q_0 V_0 + q_1 V_1 \leq q_0 V_0 + (1 \wedge (Q_{\max} - q_0))V_1^+ - (Q_{\min} - q_0)^+ V_1^-. \tag{3.4}$$

Moreover

$$\begin{aligned}&\mathbb{E}(q_0 V_0 + (1 \wedge (Q_{\max} - q_0))V_1^+ - (Q_{\min} - q_0)^+ V_1^- | \mathcal{F}_0)\\ &\quad = q_0 V_0 + (1 \wedge (Q_{\max} - q_0))\mathbb{E}(V_1^+ | \mathcal{F}_0) - (Q_{\min} - q_0)^+ \mathbb{E}(V_1^- | \mathcal{F}_0).\end{aligned}$$

The mapping

$$q \mapsto qV_0 + (1 \wedge (Q_{\max} - q))\mathbb{E}(V_1^+ | \mathcal{F}_0) - (Q_{\min} - q)^+ \mathbb{E}(V_1^- | \mathcal{F}_0)$$

(called the *objective variable* from now on) defined on the interval I_Q^1 is piecewise affine with $\mathcal{F}_0$-measurable coefficients so the above definition of q_0^* defines an $\mathcal{F}_0$-measurable I_Q^1-valued random variable. Now, combining the above inequalities yields

$$\begin{aligned}\mathbb{E}(q_0 V_0 + q_1 V_1 \,|\, \mathcal{F}_0) &\leq q_0 V_0 + (1 \wedge (Q_{\max} - q_0))\mathbb{E}(V_1^+ \,|\, \mathcal{F}_0)\\ &\quad - (Q_{\min} - q_0)^+ \mathbb{E}(V_1^- | \mathcal{F}_0)\\ &\leq \sup_{q_0 \in I_Q^1} \{q_0 V_0 + (1 \wedge (Q_{\max} - q_0))\mathbb{E}(V_1^+ \,|\, \mathcal{F}_0)\\ &\quad - (Q_{\min} - q_0)^+ \mathbb{E}(V_1^- | \mathcal{F}_0)\}\\ &= \mathbb{E}(q_0^* V_0 + q_1^* V_1 \,|\, \mathcal{F}_0).\end{aligned}$$

(b) The objective variable being piecewise affine on I_Q^1, q_0^* is equal either to one of its monotony breaks or to the endpoints of I_Q^1. Consequently, a careful inspection of all possible situations for the global constraints yields the complete set of explicit optimal rules for the optimal exercise of the swing option involving the values V_0 and $\mathbb{E}(V_1^{\pm} | \mathcal{F}_0)$ (expected gain or loss) at time 0 and V_1 at time 1.

$\triangleright$ $Q \in T_{00}^+$ i.e. $Q_{\min} \leq Q_{\max} \leq 1$: $I_Q^1 = [0, Q_{\max}]$ and the objective variable reads

$$q_0 V_0 + (Q_{\max} - q_0)\mathbb{E}(V_1^+ | \mathcal{F}_0) - (Q_{\min} - q_0)^+ \mathbb{E}(V_1^- | \mathcal{F}_0)$$

with one monotony break at $Q_{\min}$. One checks that

$$\begin{aligned}&q_0^* = Q_{\max}, \quad q_1^* = 0 && \text{on } \{V_0 \geq \mathbb{E}(V_1^+ | \mathcal{F}_0)\},\\ &q_0^* = Q_{\min}, \quad q_1^* = (Q_{\max} - Q_{\min})\mathbf{1}_{\{V_1 \geq 0\}} && \text{on } \{\mathbb{E}(V_1 | \mathcal{F}_0) \leq V_0 < \mathbb{E}(V_1^+ | \mathcal{F}_0)\},\\ &q_0^* = 0, \qquad\;\; q_1^* = Q_{\max}\mathbf{1}_{\{V_1 \geq 0\}} + Q_{\min}\mathbf{1}_{\{V_1 < 0\}} && \text{on } \{V_0 < \mathbb{E}(V_1 | \mathcal{F}_0)\}.\end{aligned}$$

Note that

$$q_0^* = Q_{\min}, \quad q_1^* = Q_{\max} - Q_{\min} \quad \text{on } \{\mathbb{E}(V_1 | \mathcal{F}_0) \leq V_0 < \mathbb{E}(V_1^+ | \mathcal{F}_0)\} \cap \{V_1 \geq 0\}$$

so that the above optimal control is not bang-bang on this event except if $Q_{\min} \in \{0, 1\}$ or $Q_{\max} = Q_{\min}$.

$\triangleright$ $Q \in T_{01}^{+}$ i.e. $Q_{\min} \le 1 \le Q_{\max} \le 2$, $Q_{\min} \le Q_{\max} - 1$: $I_Q^1 = [0, 1]$ and the objective variable reads

$$q_0 V_0 + (1 \wedge (Q_{\max} - q_0))\mathbb{E}(V_1^+|\mathcal{F}_0) - (Q_{\min} - q_0)^+\mathbb{E}(V_1^-|\mathcal{F}_0)$$

with monotony breaks at $Q_{\min}$, $Q_{\max} - 1$. One checks that

$$\begin{array}{lll}
q_0^* = 1, & q_1^* = (Q_{\max} - 1)\mathbf{1}_{\{V_1 \ge 0\}} & \text{on } \{V_0 \ge \mathbb{E}(V_1^+|\mathcal{F}_0)\}, \\
q_0^* = Q_{\max} - 1, & q_1^* = \mathbf{1}_{\{V_1 \ge 0\}} & \text{on } \{0 \le V_0 < \mathbb{E}(V_1^+|\mathcal{F}_0)\}, \\
q_0^* = Q_{\min}, & q_1^* = \mathbf{1}_{\{V_1 \ge 0\}} & \text{on } \{-\mathbb{E}(V_1^-|\mathcal{F}_0) \le V_0 < 0\}, \\
q_0^* = 0, & q_1^* = \mathbf{1}_{\{V_1 \ge 0\}} + Q_{\min}\mathbf{1}_{\{V_1 < 0\}} & \text{on } \{V_0 < -\mathbb{E}(V_1^-|\mathcal{F}_0)\}.
\end{array}$$

Note that

$$q_0^* + q_1^* = Q_{\max} - 1 \quad \text{on } \{0 \le V_0 < \mathbb{E}(V_1^+|\mathcal{F}_0)\} \cap \{V_1 < 0\}$$

so that the control is not bang-bang on this event, except if $Q_{\max} \in \{1, 2\}$ or $Q_{\max} = 1 + Q_{\min}$, since the local control q_0^* and the global constraint are not saturated. Likewise

$$q_0^* + q_1^* = 1 + Q_{\min} \quad \text{on } \{-\mathbb{E}(V_1^-|\mathcal{F}_0) \le V_0 < 0\} \cap \{V_1 > 0\}$$

and the optimal control is not bang-bang on this event, except when $Q_{\min} \in \{0, 1\}$ or $Q_{\max} = 1 + Q_{\min}$.

Note that *both events correspond to prediction errors*: V_1 has not the predicted sign. Moreover, these events are *a.s.* empty when $V_i \ge 0$ *a.s.*, $i = 1, 2$. On all other events the optimal control is bang-bang.

$\triangleright$ $Q \in T_{01}^{-}$ i.e. $Q_{\min} \le 1 \le Q_{\max} \le 2$, $Q_{\min} \ge Q_{\max} - 1$: Then the monotony breaks of the objective process (with the same expression as in the former case) still are $Q_{\min}$, $Q_{\max} - 1$. A careful inspection of the four possible cases leads to

$$\begin{array}{lll}
q_0^* = 1, & q_1^* = (Q_{\max} - 1)\mathbf{1}_{\{V_1 \ge 0\}} & \text{on } \{V_0 \ge \mathbb{E}(V_1^+|\mathcal{F}_0)\}, \\
q_0^* = Q_{\min}, & q_1^* = (Q_{\max} - Q_{\min})\mathbf{1}_{\{V_1 \ge 0\}} & \text{on } \{\mathbb{E}(V_1|\mathcal{F}_0) \le V_0 < \mathbb{E}(V_1^+|\mathcal{F}_0)\}, \\
q_0^* = Q_{\max} - 1, & \multicolumn{2}{l}{q_1^* = \mathbf{1}_{\{V_1 \ge 0\}} + (Q_{\min} - Q_{\max} + 1)\mathbf{1}_{\{V_1 < 0\}}} \\
& \multicolumn{2}{l}{\qquad \text{on } \{-\mathbb{E}(V_1^-|\mathcal{F}_0) \le V_0 < \mathbb{E}(V_1|\mathcal{F}_0)\},} \\
q_0^* = 0, & q_1^* = \mathbf{1}_{\{V_1 \ge 0\}} + Q_{\min}\mathbf{1}_{\{V_1 < 0\}} & \text{on } \{V_0 < -\mathbb{E}(V_1^-|\mathcal{F}_0)\}.
\end{array}$$

Note that on the event

$$\{-\mathbb{E}(V_1^-|\mathcal{F}_0) \le V_0 < \mathbb{E}(V_1|\mathcal{F}_0)\} \cap \{V_1 < 0\}$$

the optimal control is not bang-bang (both q_0^* and q_1^* are $(0,1)$-valued), except if $Q_{\max} \in \{1,2\}$ ($q_0^*, q_1^* \in \{0,1\}$) or $Q_{\max} = Q_{\min}$ ($q_0^* = Q_{\max} - 1$, $q_1^* = 1$) or $Q_{\max} = Q_{\min} + 1$ ($q_0^* = Q_{\min}$, $q_1^* = 0$); on the event

$$\{\mathbb{E}(V_1|\mathcal{F}_0) \le V_0 < \mathbb{E}(V_1^+|\mathcal{F}_0)\} \cap \{V_1 > 0\}$$

the optimal control is not bang-bang either (except if $Q_{\min} \in \{0,1\}$ or $Q_{\max} = Q_{\min}$ or $Q_{\max} = Q_{\min}+1$) by similar arguments. Note that these events do not correspond to a prediction error but *to an a priori compromise* made at time 0 between two possible antagonist situations. Otherwise, the optimal control is bang-bang.

$\triangleright$ $Q \in T_{11}^+$ i.e. $1 < Q_{\min} \le Q_{\max} \le 2$: The objective variable is defined on $I_Q^1 = [Q_{\min} - 1, 1]$ by

$$q_0 V_0 + (1 \wedge (Q_{\max} - q_0))\mathbb{E}(V_1^+|\mathcal{F}_0) - (Q_{\min} - q_0)\mathbb{E}(V_1^-|\mathcal{F}_0)$$

with only one breakpoint at $Q_{\max} - 1$. One checks that

$$\begin{aligned}
q_0^* &= 1, & q_1^* &= (Q_{\max} - 1)\mathbf{1}_{\{V_1 \ge 0\}} + (Q_{\min} - 1)\mathbf{1}_{\{V_1 < 0\}} \\
& & &\qquad\qquad\qquad \text{on } \{\mathbb{E}(V_1|\mathcal{F}_0) \le V_0\}, \\
q_0^* &= Q_{\max} - 1, & q_1^* &= \mathbf{1}_{\{V_1 \ge 0\}} + (Q_{\min} - Q_{\max} + 1)\mathbf{1}_{\{V_1 < 0\}} \\
& & &\qquad \text{on } \{-\mathbb{E}(V_1^-|\mathcal{F}_0) \le V_0 < \mathbb{E}(V_1|\mathcal{F}_0)\}, \\
q_0^* &= Q_{\min} - 1, & q_1^* &= 1 \quad \text{on } \{V_0 < -\mathbb{E}(V_1^-|\mathcal{F}_0)\}.
\end{aligned}$$

Once again on the event

$$\{-\mathbb{E}(V_1^-|\mathcal{F}_0) \le V_0 < \mathbb{E}(V_1|\mathcal{F}_0)\} \cap \{V_1 < 0\}$$

the optimal control is not bang-bang, except if $Q_{\max} \in \{1,2\}$ or $Q_{\max} = Q_{\min}$ or $Q_{\max} = Q_{\min} + 1$.

Finally, note that when $V_0, V_1 \ge 0$, the events on which the optimal controls are not bang-bang are empty. □

3.3. *Proof of Theorem 3.1*

Step 1 (Concavity and monotony)**.** The set $T^+(n)$ of admissible global constraints is clearly convex. Now let Q and Q' be two ordered pairs of global constraints. Note that if $q = (q_k)_{0 \le k \le n-1}$ and $q' = (q'_k)_{0 \le k \le n-1}$ are locally admissible controls then for every $[0,1]$-valued, $\mathcal{F}_0$-measurable random variable λ, $\lambda q + (1-\lambda)q' := (\lambda q_k + (1-\lambda)q'_k)_{0 \le k \le n-1}$ is still locally admissible. If q and q' satisfy the global constraints induced by Q and Q' respectively, then $\lambda q + (1-\lambda)q'$ always satisfies those induced by $\lambda Q + (1-\lambda)Q'$. Consequently,

$$\begin{aligned}
&P_0^n(\lambda Q + (1-\lambda)Q') \\
&\quad \geq \operatorname{esssup}\left\{\mathbb{E}\left(\sum_{k=0}^{n-1}(\lambda q_k + (1-\lambda)q'_k)V_k \,|\, \mathcal{F}_0\right), q, q' \text{ locally admissible,}\right. \\
&\qquad\qquad \left.\bar{q}_n \in [Q_{\min}, Q_{\max}], \bar{q}'_n \in [Q'_{\min}, Q'_{\max}]\right\} \\
&\quad = \lambda \operatorname{esssup}\left\{\mathbb{E}\left(\sum_{k=0}^{n-1} q_k V_k \,|\, \mathcal{F}_0\right), q \text{ locally admissible}, \bar{q}_n \in [Q_{\min}, Q_{\max}]\right\} \\
&\qquad + (1-\lambda)\operatorname{esssup}\left\{\mathbb{E}\left(\sum_{k=0}^{n-1} q'_k V_k \,|\, \mathcal{F}_0\right), q' \text{ locally admissible,}\right. \\
&\qquad\qquad \left.\bar{q}'_n \in [Q'_{\min}, Q'_{\max}]\right\} \\
&\quad = \lambda P_0^n(Q) + (1-\lambda)P_0^n(Q').
\end{aligned}$$

As concerns the values of $P_0^n(Q)$ when Q is a vertex of $T^+(n)$, this follows for $Q=(0,0)$ and $Q=(n,n)$ from the fact that the only admissible purchasing process is $q_k \equiv 0$ and $q_k \equiv 1$ respectively. When $Q=(0,n)$, the global constraint induces no further constraint to the local ones so that the locally optimal strategy $q_k = \mathbf{1}_{\{V_k>0\}}$ is admissible, hence optimal.

Finally, the monotony property is obvious from the very definition of $P_0^n(Q)$ as the value functional of the problem $(S_{(V,\mathcal{F})})_0^n$.

Step 2 (Proof of the main results, except those about non-negative V_i)**.** First, in view of our objective, we rewrite the dynamic programming in terms of global constraints. Set $P_n^n \equiv 0$. Taking into account the "normalized" form of the constraints, the local backward dynamical programming formula (2.3) can be rewritten in a slightly more computational form as follows: for every $k \in \{0,\ldots,n-1\}$ and every ordered pair $Q=(Q_{\min}, Q_{\max})$ of deterministic admissible global constraints at time k,

$$P_k^n(Q) = \sup\{qV_k + \mathbb{E}(P_{k+1}^n(\chi^{n-k-1}(Q,q)) \,|\, \mathcal{F}_k), q \in I_Q^{n-1-k}\} \tag{3.5}$$

where $\chi^M(Q,q) = ((Q_{\min}-q)^+, (Q_{\max}-q)\wedge M)$ and $I_Q^M := [(Q_{\min}-M)^+ \wedge 1, Q_{\max}\wedge 1]$. The optimal control q_0^* (at the origin) is then solution to

$$q_0^* = \underset{q \in I_Q^{n-1}}{\operatorname{argmax}}\{qV_0 + \mathbb{E}(P_1^n(\chi^{n-1}(Q,q)) \,|\, \mathcal{F}_0)\}. \tag{3.6}$$

(a) We proceed by induction on n. When $n=1$, the result is trivial since $T^+(1) = T_{00}^+$ and $P_0^1(Q) = Q_{\max}\mathbf{1}_{\{V_0\geq 0\}} + Q_{\min}\mathbf{1}_{\{V_0<0\}}$. When $n=2$, this follows from Proposition 3.1.

Now, we pass from n to $n+1$. Using (3.5) with $k=0$ yields

$$P_0^{n+1}(Q) = \sup\{qV_0 + \mathbb{E}(P_1^{n+1}(\chi^n(Q,q)) \,|\, \mathcal{F}_0), q \in I_Q^n\}. \tag{3.7}$$

At this stage, we make the following key remark: let $k \in \{1,\ldots,n\}$. It is clear that the two problems $(S_{(V,\mathcal{F})})_{k+1}^{n+1}$ related to the payoff dynamics $(V_\ell,\mathcal{F}_\ell)_{0\le\ell\le n}$ and $(S_{((V_{1+\ell}),(\mathcal{F}_{1+\ell}))})_k^n$ related to the payoff dynamics $(V_{1+\ell},\mathcal{F}_{1+\ell})_{0\le\ell\le n-1}$ are identical so that the random value function P_k^n satisfies a kind of homogeneity property that could be written with (temporary) obvious notations

$$P_1^{n+1}(Q,(V_\ell,\mathcal{F}_\ell)_{0\le\ell\le n}) = P_0^n(Q,(V_{1+\ell},\mathcal{F}_{1+\ell})_{0\le\ell\le n-1}) \tag{3.8}$$

where Q is an admissible global constraint for both problems. Consequently, the induction assumption implies that the mapping $Q \mapsto P_1^{n+1}(Q)$ is concave, piecewise affine, affine on every tile of the tiling of $T^+(n)$.

We inspect successively all the triangles of the tiling of $T^+(n+1)$ as follows: the upper and lower triangles which lie strictly inside the tiling, then the triangles which lie on the boundary of the tiling.

$\triangleright$ $Q \in T_{ij}^+, 1 \le i \le j \le n-1$: Then, $\chi^n(Q,q) = Q - q(1,1)$ and $I_Q^n = [0,1]$. One checks that $\chi^n(Q,q) \in T_{ij}^+$ if $q \in [0, Q_{\min} - i]$, $\chi^n(Q,q) \in T_{i-1,j}^-$ if $q \in [Q_{\min} - i,$ $Q_{\max} - j]$ and $\chi^n(Q,q) \in T_{i-1j-1}^+$ if $q \in [Q_{\max} - j, 1]$ (see Fig. 2). These three triangles T_{ij}^+, T_{i-1j}^- and T_{i-1j-1}^+ are included in $T^+(n)$. It follows from the induction assumption that $(u,v) \mapsto P_1^{n+1}(u,v)$ is *a.s.* affine on them. Hence there exists three triplets of $\mathcal{F}_1$-measurable random variables (A^m,B^m,C^m), $m=1,2,3$, such that, for every $Q \in T_{ij}^+$,

$$P_1^{n+1}(\chi^n(Q,q)) = \sum_{m=1}^{3} \mathbf{1}_{\tilde{I}_Q^m}(A^m(Q_{\min} - q) + B^m(Q_{\max} - x) + C^m)$$

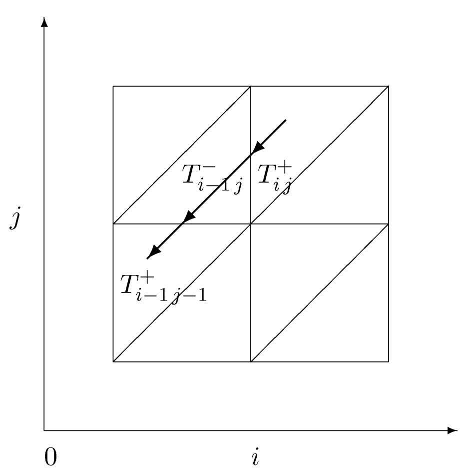

Fig. 2. $q \mapsto \chi^n(q)$ for $Q \in T_{ij}^+, 1 \le i \le j \le n-1$.

where $\tilde{I}_Q^1 = [0, Q_{\min} - i]$, $\tilde{I}_Q^2 = [Q_{\min} - i, Q_{\max} - j]$ and $\tilde{I}_Q^3 = [Q_{\max} - j, 1]$. Note that these random coefficients satisfy some compatibility constraints to ensure concavity (and continuity).

Consequently

$$\begin{aligned} &qV_0 + \mathbb{E}(P_1^{n+1}(\chi^n(Q,q)) \,|\, \mathcal{F}_0) \\ &\qquad = \sum_{m=1}^{3} \mathbf{1}_{\tilde{I}_Q^m}(qV_0 + A_0^m(Q_{\min} - q) + B_0^m(Q_{\max} - q) + C_0^m) \end{aligned}$$

where $A_0^m = \mathbb{E}(A^m \,|\, \mathcal{F}_0)$, etc. A piecewise affine function reaches its maximum on a compact interval either at its endpoint or at its monotony breakpoints $q_1 = 0$, $q_2 = Q_{\min} - i$, $q_3 = Q_{\max} - j$, $q_4 = 1$. Hence,

$$\begin{aligned} &\sup_{q \in I_Q^n} (qV_0 + \mathbb{E}(P_1^{n+1}(\chi^n(Q,q)) \,|\, \mathcal{F}_0)) \\ &\qquad = \max\{q_\ell V_0 + A_0^m(Q_{\min} - q_\ell) + B_0^m(Q_{\max} - q_\ell) + C_0^m, \\ &\qquad\qquad (\ell, m) = (1,1), (2,1), (3,2), (4,3)\}. \end{aligned}$$

It is clear that the right hand side of the previous equality stands as the maximum of four affine functions of Q. One derives that $Q \mapsto P_0^{n+1}(Q)$ is a convex function on T_{ij}^+ as the maximum of affine functions. Hence it is affine since we know that it is also concave.

- ▷ $Q \in T_{ij}^-, 1 \le i < j \le n-1$: This case can be treated likewise.
- ▷ $Q \in T_{0j}^{\pm}, 1 \le j \le n-1$: In that case $I_Q^n = [0,1]$, $\chi^n(Q,q) = ((Q_{\min} - q)^+, Q_{\max} - q), q \in I_Q^n$.
  - If $Q \in T_{0j}^+$, $\chi^n(Q,q) = Q - q(1,1) \in T_{0j}^+$, $q \in [0, Q_{\min}]$, $\chi^n(Q,q) = (0, Q_{\max} - q) \in T_{0j}^+$, $q \in [Q_{\min}, Q_{\max} - j]$, $\chi^n(Q,q) = (0, Q_{\max} - q) \in T_{0j-1}^+$, $q \in [Q_{\max} - j, 1]$ (see Fig. 3, Left). The induction assumption implies that

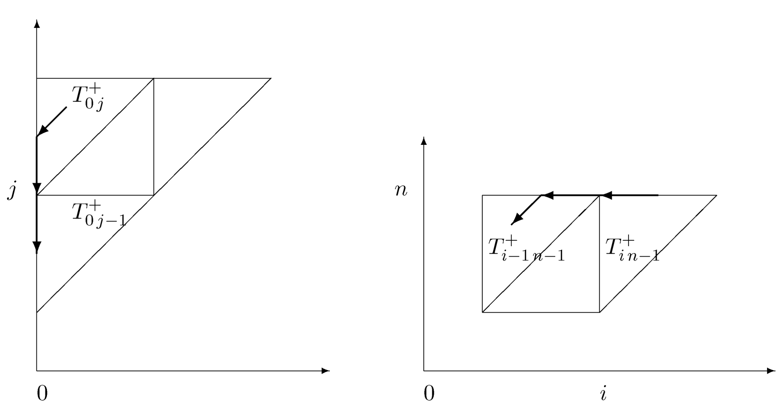

Fig. 3. Left: $q \mapsto \chi^n(q)$ for $Q \in T_{0j}^+, 1 \le j \le n-1$. Right: $q \mapsto \chi^n(q)$ for $Q \in T_{in}^+, 1 \le i \le n-1$.

$q \mapsto P_1^{n+1}(\chi^n(Q,q))$ is piecewise affine with monotony breaks at $Q_{\min}$ and $Q_{\max} - j$.

– If $Q \in T_{0j}^-$, $\chi^n(Q,q)$ crosses the upper (horizontal) edge of T_{0j-1}^+ at $q = Q_{\max} - j$ and the left (vertical) edge of $T^+(n)$ at $q = Q_{\min}$. Hence $x \mapsto P_1^{n+1}(\chi^n(Q,q))$ is again piecewise affine with monotony breaks at $Q_{\min}$ and $Q_{\max} - j$.

In both cases one concludes as above.

▷ $Q \in T_{00}^\pm$: One proceeds like with T_{0j}^+ except that $I_Q^n = [0, Q_{\max}]$ which yields only one monotony break at $Q_{\min}$.

▷ $Q \in T_{in}^\pm, 1 \leq i \leq n-1$: Assume first $Q \in T_{in}^+$. $I_Q^n = [0,1]$ and $\chi^n(Q,q) = (Q_{\min} - q, n)$ if $q \in [0, Q_{\max} - n]$. Otherwise $\chi^n(Q,q) = (Q_{\min} - q, Q_{\max} - q)$. It follows (see Fig. 3, Right) that $\chi^n(Q,q) \in T_{i,n-1}^+$ if $q \in [0, Q_{\min} - i]$ and $\chi^n(Q,q) \in T_{i-1,n-1}^+$ if $q \in [Q_{\min} - i, 1]$. Both T_{in-1}^+ and T_{i-1n-1}^+ are included in $T^+(n)$. Hence $(u,v) \mapsto P_1^{n+1}(u,v)$ being affine on both triangles still owing to (3.8), one derives that

$$\begin{aligned} P_1^{n+1}(\chi^n(Q,q)) = {} & \mathbf{1}_{q\in[0,Q_{\max}-n]}(A^1(Q_{\min} - q) + B^1 n + C^1) \\ & + \mathbf{1}_{q\in[Q_{\max}-n,1]}(A^2(Q_{\min} - q) + B^2(Q_{\max} - q) + C^2), \end{aligned}$$

where $A^m, B^m, C^m, m = 1, 2$ are $\mathcal{F}_1$-measurable r.v. Then one concludes just as in the first case.

If $Q \in T_{in}^-$, one proceeds likewise except that the two "visited" triangles of $T^+(n)$ are $T_{i-1,n-1}^\pm$.

▷ $Q \in T_{0n}^\pm$: $I_Q^n = [0,1]$ and

– $\chi^n(Q,q) = ((Q_{\min} - q)^+, Q_{\max} - q), q \in [0, Q_{\max} - n]$, $\chi^n(Q,q) = ((Q_{\min} - q)^+, n)$, $q \in [Q_{\max} - n, 1]$ if $Q \in T_{0n}^+$,
– $\chi^n(Q,q) = (Q_{\min} - q, n), q \in [0, Q_{\max} - n]$, $\chi^n(Q,q) = (Q_{\min} - q, Q_{\max} - q), q \in [Q_{\max} - n, Q_{\min}]$ if $Q \in T_{0n}^-$ and $\chi^n(Q,q) = (0, Q_{\max} - q)$, $q \in (Q_{\min}, 1]$.

In both cases the only "visited" triangle is $T_{0n-1}^+ \subset T^+(n)$ and one concludes as usual.

▷ $Q \in T_{nn}^+$: $I_Q^n = [Q_{\min} - n, 1]$ and $\chi^n(Q,q) = (Q_{\min} - q, n)$ if $q \in [Q_{\min} - n, Q_{\max} - n]$, $\chi^n(Q,q) = (Q_{\min} - q, Q_{\max} - q)$ otherwise. Hence $\chi^n(Q,q)$ takes its values in T_{n-1n-1}^+ on which $(u,v) \mapsto P_1^{n+1}(u,v)$ is affine. The conclusion follows.

The inspection of all these cases completes the induction.

(b) We deal successively with the two announced settings.

– **Global Constraints in $\mathbb{N}^2$:** Let $n \geq 1$. We rely on the characterization (3.6) of q_0^*. We know from item (a) that $(u,v) \mapsto P_1^n(u,v)$ is affine on every tile $T_{ij}^\pm$ of $T^+(n)$ owing to the pseudo-homogenity property established in (3.8).

If $Q \in T^+(n) \cap \mathbb{N}^2$ then one checks that $q \mapsto \chi^{n-1}(Q,q)$, $q \in I_Q^{n-1}$, is always affine with I_Q^{n-1} having 0 and/or 1 as endpoints. To be precise

– if $Q = (i,j)$, $1 \le i \le j \le n-1$, $\chi^{n-1}(Q,q) = (i-q, j-q) \in \partial T^+_{i-1j-1} \cap \partial T^-_{i-1j-1}$, $I_Q^{n-1} = [0,1]$,
– if $Q = (0,j)$, $1 \le j \le n-1$, $\chi^{n-1}(Q,q) = (0, j-x) \in \partial T^+_{0j-1}$, $I_Q^{n-1} = [0,1]$,
– if $Q = (0,0)$, $\chi^{n-1}(Q,q) = (0,0)$, $I_Q^{n-1} = \{0\}$,
– if $Q = (i,n)$, $1 \le i \le n-1$, $\chi^{n-1}(Q,q) = (i-q, n-1) \in \partial T^-_{i-1n-1}$, $I_Q^{n-1} = [0,1]$,
– if $Q = (0,n)$, $\chi^{n-1}(Q,q) = (0, n-1)$, $I_Q^{n-1} = [0,1]$,
– if $Q = (n,n)$, $\chi^{n-1}(Q,q) = (n-1, n-1)$, $I_Q^{n-1} = \{1\}$.

As a consequence, affinity being stable by composition, $q \mapsto P_0^{n-1}(\chi^{n-1}(Q,q))$ is affine on $I_Q^{n-1} \in \{[0,1], \{0\}, \{1\}\}$. The pseudo-homogenity property (3.8) implies that $q \mapsto P_1^n(\chi^{n-1}(Q,q))$ is affine on I_Q^{n-1} as well. As a consequence, the function $q \mapsto qV_0 + \mathbb{E}(P_1^n(\chi^{n-1}(Q,q)) \,|\, \mathcal{F}_0)$ is affine and reaches its maximum at some endpoint of I_Q^{n-1} i.e. $q_0^* = 0$ or at $q_0^* = 1$. Then, inspecting the above cases shows that $Q - q_0^*(1,1) \in T^+(n-1) \cap \mathbb{N}^2$. Using (3.5) and (3.6), one shows by induction on k that q_k^* is always $\{0,1\}$-valued.

Step 3 (Non-Negative V_i)**.** This step is again divided into two steps.

Step 3.1. *Global constraint on $Q_{\max}$ can be saturated.* Let $n \ge 1$. Let $q^* = (q_k^*)_{0 \le k \le n-1}$ be an optimal Q-admissible control. We introduce the $\mathcal{F}$-stopping time

$$\tau(q^*) := \min\{k \,|\, q_0^* + \cdots + q_k^* < Q_{\max} - (n-1) + k\}$$

with the convention $\min \emptyset = +\infty$.

On $\tau(q^*) = +\infty$, $q_0^* + \cdots + q_k^* \ge Q_{\max} - (n-1) + k$ for every $k = 0, \ldots, n-1$. In particular the global constraint is saturated at time $n-1$, i.e.

$$q_0^* + \cdots + q_{n-1}^* = Q_{\max}.$$

Set

$$\tilde{q}_k = q_k^* \mathbf{1}_{\{k \ne \tau(q^*)\}} + (Q_{\max} - (n-1) + k - (q_0^* + \cdots + q_{k-1}^*))\mathbf{1}_{\{k = \tau(q^*)\}}.$$

One checks that $\tilde{q}$ is a Q-admissible control since $\tau(q^*)$ is an $\mathcal{F}$-stopping time and $q^*_{\tau(q^*)}$ is $[0,1]$-valued on $\{\tau(q^*) < +\infty\}$ (note that $0 \le q^*_{\tau(q^*)} < \tilde{q}_{\tau(q^*)} := Q_{\max} - (n-1) + \tau(q^*) - (q_0^* + \cdots + q^*_{\tau(q^*)-1}) \le 1$ on $\{\tau(q^*) < +\infty\}$). Likewise one shows that $\tilde{q}_k \ge q_k^*$ for every $k = 0, \ldots, n-1$.

Now note that if $Q_{\max}$ is an integer and q^* is $\{0,1\}$-valued then $\tilde{q}$ is still $\{0,1\}$-valued.

The random variables V_k being non negative

$$\sum_{k=0}^{n-1} \tilde{q}_k V_k \ge \sum_{k=0}^{n-1} q_k^* V_k$$

hence $\tilde{q}$ is still an optimal control. Furthermore $\tilde{q}_0 + \cdots + \tilde{q}_k \geq Q_{\max} - (n-1) + k$ on $\{k \geq \tau(q^*)\}$ so that the stopping time $\tau(\tilde{q})$ satisfies by construction

$$\tau(\tilde{q}) \leq \tau(q^*) - 1 \text{ on } \{1 \leq \tau(q^*) < +\infty\} \quad \text{and}$$
$$\tau(\tilde{q}) = +\infty \text{ on } \{\tau(q^*) = 0\} \cup \{\tau(q^*) = +\infty\}.$$

One defines by induction a sequence $q^{(i)}$ of optimal Q-admissible controls such that $q^{(0)} = q^*$, $\tau(q^{(i+1)}) \leq \tau(q^{(i)}) - 1$ on the event $\{\tau(q^{(i)}) < +\infty\}$ and $\tau(q^{(i+1)}) = +\infty$ on $\{\tau(q^{(i)}) = 0, +\infty\}$. Hence in at most n steps one obtains an optimal Q-admissible control q^{opt} such that $\tau(q^{\text{opt}}) = +\infty$ *a.s.* Such a control q^{opt} saturates the global constraint.

As a consequence, this shows that $P_0^n(Q) = P_0^n(0, Q_{\max}) = P_0^n(Q_{\max}, Q_{\max})$ *a.s.* so that $Q_{\max} \mapsto P_0^n(Q_{\max}, Q_{\max})$ is *a.s.* non-decreasing and concave.

Step 3.2. *Local constraints.* Since there is an optimal control q^* which saturates the global constraint, one may assume without loss of generality that $Q_{\min} = Q_{\max}$. We proceed again by induction on n based on the dynamic programming formula (2.3). When $n = 1$ the result is obvious (and true when $n = 2$ as well).

Assume now the announced result is true for $n \geq 1$.

Let $j \in \{0, \ldots, n-1\}$ and $Q_{\max} \in [j, j+1]$. Then, $I_Q^n = [0, Q_{\max} \wedge 1]$ and $\chi^n((Q_{\max}, Q_{\max}), q) = (Q_{\max} - q, Q_{\max} - q)$, $q \in I_Q^n$. Hence $\chi^n((Q_{\max}, Q_{\max}), q) \in T_{jj}^+$, $q \in [0, Q_{\max} - j]$ and $\chi^n((Q_{\max}, Q_{\max}), q) \in T_{j-1,j-1}^+$, $q \in [Q_{\max} - j, Q_{\max} \wedge 1]$. Now $v \mapsto P_1^{n+1}((v, v))$ is *a.s.* concave, non-decreasing, affine on $[j-1, j]$ and on $[j, j+1]$ owing to (3.8). Consequently, there exists B^m, C^m, $m = 1, 2$, $\mathcal{F}_1$-measurable random variables satisfying

$$P_1^{n+1}(\chi^n((Q_{\max}, Q_{\max}), q)) = B^1(Q_{\max} - q) + C^1, \quad q \in [0, Q_{\max} - j],$$
$$P_1^{n+1}(\chi^n((Q_{\max}, Q_{\max}), q)) = B^2(Q_{\max} - q) + C^2, \quad q \in [Q_{\max} - j, Q_{\max} \wedge 1].$$

with $0 \leq B^1 \leq B^2$ and $B^2 j + C^2 = B^1 j + C^1$ *a.s.* Set temporarily

$$\Psi(q) := qV_0 + \mathbb{E}(P_1^{n+1}(\chi^n((Q_{\max}, Q_{\max}), q)) \,|\, \mathcal{F}_0).$$

Hence,

$$\sup_{q \in I_Q^n} \Psi(q) = \max(\Psi(0), \Psi(Q_{\max} - j), \Psi(Q_{\max} \wedge 1)).$$

Set $B_0^m := \mathbb{E}(B^m \,|\, \mathcal{F}_0)$ and $C_0^m := \mathbb{E}(C^m \,|\, \mathcal{F}_0)$ and note that $B_0^1 \leq B_0^2$ and $B_0^2 j + C_0^2 = B_0^1 j + C_0^1$ *a.s.* Elementary computations show that:

- $\Psi(0) \leq \Psi(Q_{\max} - j)$ on $\{V_0 \geq B_0^1\}$ and $\Psi(0) \geq \Psi(Q_{\max} - j)$ on $\{V_0 \leq B_0^1\}$,
- $\Psi(Q_{\max} - j) \leq \Psi(Q_{\max} \wedge 1)$ on $\{V_0 \geq B_0^2\}$ and $\Psi(Q_{\max} - j) \geq \Psi(Q_{\max} \wedge 1)$ on $\{V_0 \leq B_0^2\}$.

Consequently q_0^* can be chosen $\{0, Q_{\max} \wedge 1\}$-valued on $E_0 := \{V_0 \notin (B_0^1, B_0^2)\} \in \mathcal{F}_0$ and equal to $Q_{\max} - j$ on ${}^cE_0 := \{V_0 \in (B_0^1, B_0^2)\} \in \mathcal{F}_0$.

On $E_0^1 = E_0 \cap \{q_0^* = Q_{\max} \wedge 1\} \in \mathcal{F}_0$, one has with obvious (temporary) notations

$$P_0^{n+1}((Q_{\max}, Q_{\max}), \mathcal{F}) = P_0^{n+1}((Q_{\max}, Q_{\max}), \mathcal{F} \cap E_0^1)$$

where $\mathcal{F} \cap E_0^1$ denotes the trace filtration of $\mathcal{F}$ on E_0^1.

Then, the dynamic programming formula (2.3) shows that the other components $(q_k^*)_{1 \leq k \leq n}$ of the optimal control on E_0 can be obtained as the optimal control of the pricing problem $P_1^{n+1}(((Q_{\max} - 1)^+, (Q_{\max} - 1)^+), \mathcal{F} \cap E_0^1)$. One derives from the induction assumption at time n that $(q_k^*)_{1 \leq k \leq n}$ can be chosen bang-bang and $((Q_{\max} - 1)^+, (Q_{\max} - 1)^+)$-admissible which implies that q is Q-admissible and bang-bang since $q_0^* = Q_{\max} \wedge 1$ (on E_0^1). A similar proof holds on $E_0^0 = E_0 \cap \{q_0^* = 0\}$.

On ${}^cE_0$, one has likewise $P_0^{n+1}((Q_{\max}, Q_{\max}), \mathcal{F}) = P_0^{n+1}((Q_{\max}, Q_{\max}), \mathcal{F} \cap {}^cE_0)$. Then, the dynamic programming formula (2.3) shows that the other components $(q_k^*)_{1 \leq k \leq n}$ of the optimal control on ${}^cE_0$ can be obtained as the optimal control of the pricing problem $P_1^{n+1}((j, j), \mathcal{F} \cap {}^cE_0)$. Since $(j, j) \in \mathbb{N}^2$, there exists a (j, j)-admissible bang-bang optimal control $(q_1^*)_{1 \leq k \leq n}$ (with respect to $\mathcal{F} \cap {}^cE_0$ on ${}^cE_0$. Then q_k^* is $\{0, 1\}$-valued for every $k = 1, \ldots, n$ (in fact identically 0 if $j = 0$). At this stage one can recursively modify $(q_k^*)_{1 \leq k \leq n}$ using the procedure described in Step 1 to saturate the upper global constraint. Finally one may assume that $\sum_{0 \leq k \leq n-1} q_k^* = j$ which in turn implies that $(q_k^*)_{0 \leq k \leq n}$ is a bang-bang $(Q_{\max}, Q_{\max})$-admissible optimal control.

In Proposition 3.1 ($n = 2$), we showed that optimal controls are bang-bang under the assumption $Q_{\max} - Q_{\min} \in \mathbb{N}$. But this is specific to the two period model and becomes false when $n \geq 3$. This comes from the fact that at integer valued global constraints the bang-bang optimal control *may saturate none of the global constraints*. Indeed, so is the case at $Q = (0, 2)$ when $n = 2$. This phenomenon also explains why the premium is not affine but only piecewise affine on elementary triangles of the simplex $T^+(n)$. □

Application. When a global constraint Q belongs to the interior of a triangle $T_{ij}^{\pm}$, *one only needs to compute the value of* P_0^n *at the vertices of this triangle to derive the value of the premium at every* $Q \in T_{ij}^{\pm}$. When Q is itself an integer valued ordered pair, at most six further points allow to compute the premium in a neighborhood of Q. We will use this result extensively when designing our quantization based numerical procedure in Sec. 4.

An additional result. Using the same approach as developed in the proof of Theorem 3.1, one can show the following result, whose details of proof are left to the reader.

Corollary 3.1. *Suppose the assumptions of Theorem* 3.1 *hold. If an ordered pair of admissible constraints* $(Q_{\min}, Q_{\max}) \in T^+(n)$ *satisfies*

$$Q_{\max} - Q_{\min} \in \{0, \ldots, n\},$$

then there exists a quasi-bang-bang control *in the following sense*: $\mathbb{P}$-*a.s.*, q_k^* *is* $\{0, 1\}$-*valued except for at most one local constraint* $q_{k_0}^*$.

3.4. *The Markov setting*

By Markov setting we simply mean that the payoffs V_k are functions of an $\mathbb{R}^d$-valued underlying $\mathcal{F}$-Markov structure process $(Y_k)_{0\le k\le n-1}$ i.e.

$$V_k = v_k(Y_k), \quad k = 0, \dots, n-1, \tag{3.9}$$

where $v_k : (\mathbb{R}^d, \mathcal{B}or(\mathbb{R}^d)) \to \mathbb{R}$, $k = 0, \dots, n-1$ are Borel functions and $\sum_{k=0}^{n-1} |v_k(Y_k)| \in L^1$. The Markovian dynamics of Y reads on Borel functions $g : \mathbb{R}^d \to \mathbb{R}$

$$\mathbb{E}(g(Y_{k+1}) \,|\, \mathcal{F}_k) = \mathbb{E}(g(Y_{k+1}) \,|\, Y_k) = \Theta_k(g)(Y_k)$$

where $(\Theta_k)_{0\le k\le n-1}$ is a sequence of Borel probability transitions on $(\mathbb{R}^d, \mathcal{B}or(\mathbb{R}^d))$.

Since we intend to take advantage of this Markovian setting to devise a numerical scheme for the computation of the premium of swing options, it turns out to be more convenient and natural to switch back to the local control as the state variable of the stochastic control problem. This induces a slight change of notation: in this section we will denote by $P_k^n(Q, \bar{q}_k, \omega)$ the price at time $k \in \{0, \dots, n\}$ of a swing contract with *global constraints at time* 0 given by $Q = (Q_{\min}, Q_{\max}) \in \mathcal{A}_0^n$ and cumulative purchased quantity until time $k-1$ is $\bar{q}_k$. This notation, which is classical when there is an underlying Markov dynamics, amounts to writing $P_k^n(Q, \bar{q}_k(\omega), \omega)$ instead of $P_k^n(Q - \bar{q}_k(\omega)(1,1), \omega)$ as we did in the previous sections.

One derives from the expression of the set of admissible global constraints $\mathcal{A}_k^n$ at time $k \in \{0, \dots, n-1\}$, that the cumulated purchased quantity $\bar{q}_k$ may vary (see Fig. 4) in

$$J_k^n = J_k^n(Q_{\min}, Q_{\max}) = [(Q_{\min} - (n-k))_+, \min(Q_{\max}, k)]. \tag{3.10}$$

Then, one straightforwardly shows that the value function (swing premium) at time k becomes a function of $(\bar{q}_k, Y_k)$. To be precise, for every (deterministic) global constraints at time 0, there exist Borel functions $p_k^n : \mathcal{A}_0^n \times J_k^n \times \mathbb{R}^d \to \mathbb{R}$ such that $p_n^n \equiv 0$ and for every $k = 0, \dots, n-1$,

$$\mathbb{P}(d\omega)\text{-}a.s. \quad P_k^n(Q, \bar{q}, \omega) = p_k^n(Q, \bar{q}, Y_k(\omega)), \quad \bar{q} \in J_k^n, \;\; Q \in \mathcal{A}_0^n.$$

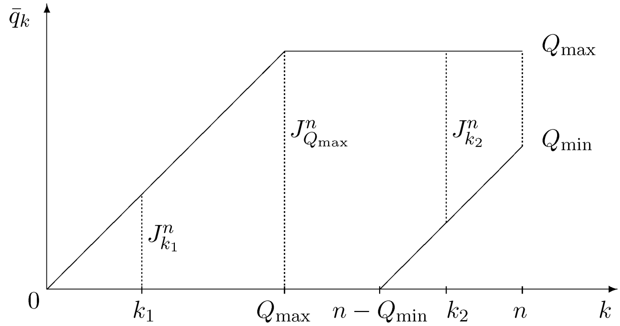

Fig. 4. Set of admissible local constraints: examples of intervals J_k^n (vertical doted lines).

Standard arguments (see e.g. [6] for details) show that this identity extends to possibly $\mathcal{F}_0$-measurable random constraints by simply setting $P_k^n(Q, \bar{q}, \omega) = p_k^n(Q(\omega), \bar{q}, Y_k(\omega))$.

Then the backward dynamic programming principle (2.3) can be rewritten as follows (with $\bar{q}$ as state variable, the global constraint vector Q at time 0 being fixed): for every $k = 0, \ldots, n-1$ and every $(\bar{q}, y) \in J_k^n \times \mathbb{R}^d$,

$$p_k^n(Q, \bar{q}, y) = \sup\{qv_k(y) + \Theta_k(p_{k+1}^n(Q, \bar{q}+q, .))(y), q \in [0,1], \bar{q}+q \in J_{k+1}^n\}. \tag{3.11}$$

Pointwise estimation of $P_0^n(Q)$, $Q \in T^+(n)$. As established in Theorem 3.1, one only needs to compute the value function $P_0^n(Q)$ at global constraints $Q = (Q_{\min}, Q_{\max}) \in T^+(n) \cap \mathbb{N}^2$ i.e. with integer components. Moreover, for these constraints, the local optimal control q_k^* is always bang-bang i.e. $q_k^* \in \{0,1\}$.

Consequently, taking full advantage of this bang-bang feature yields the following form for the backward dynamic programming formula:

$$p_k^n(Q, \bar{q}, y) = \max\{qv_k(y) + \Theta_k(p_{k+1}^n(Q, \bar{q}+q, .))(y), q \in \{0,1\}, \bar{q}+q \in J_{k+1}^n \cap \mathbb{N}\}, \tag{3.12}$$

for every $\bar{q} \in J_k^n$, $y \in \mathbb{R}^d$, $k = 0, \ldots, n-1$ (see Fig. 4). At this stage, no numerical computation is possible yet since the state space discretization has not been achieved. This is the aim of the next section where we will approximate the above dynamic programming principle by (optimal) quantization of the state process Y.

4. Computing Swing Contracts Using an (Optimal) Quantization Tree

Throughout this section we assume that we are in the Markov setting as described in Sec. 3.4. We aim at pricing a swing option with an abstract (Markov) payoff process $V_k = v_k(Y_k)$ as defined in (3.9) where (Y_k) is an $\mathbb{R}^d$-valued discrete time Markov process (possibly not homogeneous).

4.1. *The quantization tree approach*

Vector quantization. In this section, we propose a quantization based approach to compute the premium of the swing contracts with firm constraints. Quantization has been originally introduced and developed in the early 1950s in signal processing (see [15]). Nowadays it is widely used in image compression. The starting idea is simply to replace every random vector $Y : (\Omega, \mathcal{A}) \to \mathbb{R}^d$ by a random vector $\widehat{Y} = g(Y)$ taking finitely many values in a *grid* (or *codebook*) $\Gamma := \{y^1, \ldots, y^N\}$ (of size N). The grid Γ is also called an N-quantizer of Y. When the Borel function g satisfies

$$|Y - \widehat{Y}| = d(Y, \Gamma) = \min_{1 \le i \le N} |Y - y^i|, \tag{4.1}$$

$\widehat{Y}$ is called a *Voronoi quantization* of Y (and g as well) and is often denoted $\widehat{Y}^{\Gamma}$. One easily checks that g is necessarily a nearest neighbor projection on Γ i.e. satisfies

$$\forall\, i \in \{1,\ldots,N\}, \quad \{g = y^i\} \subset \left\{ u \in \mathbb{R}^d : |u - y^i| = \min_{1\le j\le N} |u - y^j|]\right\}.$$

The so-called *Voronoi cells* $\{g = y^i\}, 1 \le i \le N$, make up a (Borel) *Voronoi tessellation* or partition of $\mathbb{R}^d$ (induced by Γ). Note that when the distribution $\mathbb{P}_{Y}$ of Y is *strongly continuous* that is any hyperplane is $\mathbb{P}_{Y}$-negligible, the boundaries of all the Voronoi tessellation of Γ are also $\mathbb{P}_{Y}$-negligible so that the $\mathbb{P}_{Y}$-weights of the Voronoi cells entirely characterize the distribution of $\widehat{Y}^{\Gamma}$.

When $p \in [1,\infty)$, the L^p-mean error induced by replacing Y by $\widehat{Y}$, namely

$$\|Y - \widehat{Y}\|_{p} = \left(\mathbb{E} \min_{1\le i\le N} |Y - y^i|^p\right)^{\frac{1}{p}}$$

is called the L^p*-mean quantization error* induced by Γ and its pth power is known as the L^p-distortion. We will see in the next section that the codebook Γ can be optimized so as to minimize the above L^p-quantization error with respect to the distribution of Y.

Our aim in this section is to design an algorithm based on the quantization of the Markov chain (Y_k) at every time k to approximate the premium of the swing contract with firm constraints and to provide some *a priori* error estimates in terms of quantization errors.

A quantization tree for pricing swing options. Our strategy is to use optimal quantization to first devise a procedure for $T^+(n) \cap \mathbb{N}^2$-valued global constraints (at time 0) to take full advantage of the existence of bang-bang optimal controls and, as a second step, to use the fact that the premium is affine on each elementary "tile" of $T^+(n)$ to compute values of the premium for other global constraints by a simple barycentric interpolation of the premium at the vertices of the tile.

$\triangleright$ **The quantized transitions.** As a first step we consider at every time k a grid $\Gamma_k := \{y_k^1,\ldots,y_k^{N_k}\}$ of size $|\Gamma_k| := N_k$. Then, we design the quantization tree algorithm to price swing contracts by simply mimicking the original dynamic programming formula (3.12). This means in particular that we force in some way the Markov property on $(\widehat{Y}_k)_{0\le k\le n-1}$ by considering the quantized transition operator defined by

$$\widehat{\Theta}_k(g)(y_k^i) = \sum_{j=1}^{N_{k+1}} g(y_{k+1}^j)\pi_k^{ij}, \quad \pi_k^{ij} := \mathbb{P}(\widehat{Y}_{k+1} = y_{k+1}^j \,|\, \widehat{Y}_k = y_k^i)$$

which obviously satisfies

$$\widehat{\Theta}_k(g)(\widehat{Y}_k) = \mathbb{E}(g(\widehat{Y}_{k+1}) \,|\, \widehat{Y}_k), \quad k = 0,\ldots,n-1.$$

$\triangleright$ **The quantized bang-bang dynamic programming.** Let $Q \in T^+(n) \cap \mathbb{N}^2$ be an ordered pair of (deterministic) global constraints (at time 0). The *quantized bang-bang dynamic programming principle* is defined as follows

$$\widehat{P}_k^n(Q, \bar{q}, \omega) = \widehat{p}_k^n(Q, \bar{q}, \widehat{Y}_k(\omega)), \quad k = 0, \ldots, n, \;\; \bar{q} \in J_k^n \cap \mathbb{N},$$

where

$$\begin{aligned}
&\widehat{p}_n^n(Q, \widehat{\bar{q}}_n, \widehat{Y}_n) := 0, \quad \widehat{\bar{q}}_n \in J_n^n \cap \mathbb{N}, \\
&\widehat{p}_k^n(Q, \widehat{\bar{q}}_k, \widehat{Y}_k) := \max(q v_k(\widehat{Y}_k) + \mathbb{E}(\widehat{p}_{k+1}^n(Q, \widehat{\bar{q}}_k + q, \widehat{Y}_{k+1}) \,|\, \widehat{Y}_k), q \in \{0, 1\}, \\
&\qquad\qquad\qquad \widehat{\bar{q}}_k + q \in J_{k+1}^n), \quad \widehat{\bar{q}}_k \in J_k^n \cap \mathbb{N}, \;\; k = 0, \ldots, n-1,
\end{aligned} \tag{4.2}$$

where J_k^n is still given by (3.10). An easy induction yields the true quantized backward algorithm

$$\begin{aligned}
&\widehat{p}_n^n(Q, \widehat{\bar{q}}_n, y_n^i) = 0, \quad i = 1, \ldots, N_n, \\
&\widehat{p}_k^n(Q, \widehat{\bar{q}}_k, y_k^i) = \max\{q v_k(y_k^i) + \widehat{\Theta}_k(\widehat{p}_{k+1}^n(Q, \widehat{\bar{q}}_k + q, .))(y_k^i), x \in \{0, 1\}, \\
&\qquad\qquad \widehat{\bar{q}}_k + q \in J_{k+1}^n\} i = 1, \ldots, N_k, \quad \widehat{\bar{q}}_k \in J_k^n \cap \mathbb{N}, \;\; k = 0, \ldots, n-1.
\end{aligned} \tag{4.3}$$

$\triangleright$ **Interpolation by affinity on the elementary tiles.** When $Q \in T^+(n) \setminus \mathbb{N}^2$, one defines $\widehat{P}_0^n(Q, \mathcal{F})$ (and $\widehat{p}_0^n(Q, .)$) by affinity on each elementary triangle $T_{ij}^{\pm}$ that tiles $T^+(n)$: if $\tau_{ij}^{\pm,\ell}$, $\ell = 1, 2, 3$, denote the vertices of the tile $T_{ij}^{\pm}$, then for every convex combination $\sum_{\ell=1}^{3} \alpha_\ell \tau_{ij}^{\pm,\ell}$ of these vertices

$$\widehat{p}_0^n \left(\sum_{\ell=1}^{3} \tau_{ij}^{\pm,\ell}, . \right) := \sum_{\ell=1}^{3} \alpha_\ell \widehat{p}_0^n \left(\sum_{\ell=1}^{3} \alpha_\ell \tau_{ij}^{\pm,\ell}, . \right). \tag{4.4}$$

Transition weight estimation. The second ingredient to make the quantized backward dynamic programming formula work is the estimation of the transition weights π_k^{ij} on the quantization tree. This can be done by computing all the empirical frequencies when letting some simulated paths of the process Y pass through the grids. Such an approach can be improved in a Gaussian framework (see the annex of [5]). In all cases, this phase relies on repeated nearest neighbour searches (among the grids at each time step). For that purpose, we use some fast procedures like the K-d-tree introduced in [13] which reduce the complexity of this phase (see [5] for details).

At this stage, let us emphasize again that, once the grids and the transition weights have been computed, it becomes possible to compute any (reasonable) swing option price using the quantized dynamic programming formula.

Complexity of the dynamic programming formula. Let us briefly discuss the complexity of this quantized backward dynamic procedure. Let $k \in \{0, \ldots, n-1\}$. At every "node" y_k^i and for every cumulated purchased quantity $\widehat{\bar{q}}_k$, the computation of $\widehat{\Theta}_k(\widehat{p}^n_{k+1}(Q,\widehat{\overline{q}}_k+q,.))(y^i_k)$, requires N_{k+1} products (up to a constant), hence, one checks that $\widehat{\overline{q}}_k$ takes (at most) $\mathcal{Q}^n_k=\mathrm{card}(J^n_k)$ values with

$$\mathcal{Q}^n_k := (Q_{\max}\wedge k)+1-(Q_{\min}-(n-k))^+ \le (Q_{\max}\wedge k)+1\le k+1.$$

One derives the complexity of the computation of $\widehat{p}^n_0(\widehat{Y}_0,0)$ is proportional to

$$\sum_{k=0}^{n-1}\mathcal{Q}^n_k N_k N_{k+1}. \tag{4.5}$$

Note that replacing $\mathcal{Q}^n_k$ by $k+1$ in (4.5) yields the complexity of the quantization tree algorithm devised to solve the pricing of swing options subject to some *penalized global volume constraints* (see the companion paper [5]).

A priori error bounds for the quantized procedure

Theorem 4.1. *Assume that the Markov process $(Y_k)_{0\le k\le n-1}$ is Lipschitz Feller in the following sense: for every bounded Lipschitz continuous function $g:\mathbb{R}^d\to\mathbb{R}$ and every $k\in\{0,\dots,n-1\}$, $\Theta_k(g)$ is a Lipschitz function satisfying $[\Theta_k(g)]_{\mathrm{Lip}}\le[\Theta_k]_{\mathrm{Lip}}[g]_{\mathrm{Lip}}$. Assume that every function $v_k:\mathbb{R}^d\to\mathbb{R}$ is Lipschitz continuous with Lipschitz coefficient $[v_k]_{\mathrm{Lip}}$. Let $p\in[1,\infty)$ such that $\max_{0\le k\le n-1}|Y_k|\in L^p(\mathbb{P})$. Then, there exists a real constant $C_p>0$ such that*

$$\Big\|\sup_{Q\in T^+(n)}|\widehat{P}^n_0(Q)-P^n_0(Q)|\Big\|_p\le C_p\sum_{k=0}^{n-1}\|Y_k-\widehat{Y}_k\|_p. \tag{4.6}$$

Remark. In most situations $\mathcal{F}_0=\{\emptyset,\Omega\}$ so that the error term $|\widehat{P}^n_0(Q)-P^n_0(Q)|$ is deterministic. When $\mathcal{F}_0$ is not trivial, it is straightforward from (4.6) (with $p=1$) that

$$\sup_{Q\in T^+(n)}|\mathbb{E}(\widehat{P}^n_0(Q))-\mathbb{E}(P^n_0(Q))|\le C_1\sum_{k=0}^{n-1}\|Y_k-\widehat{Y}_k\|_1.$$

We first need a lemma about the Lipschitz regularity of the p^n_k functions.

Lemma 4.1. *For every $k\in\{0,\dots,n-1\}$ and every $Q\in T^+(n)\cap\mathbb{N}^2,\bar{q}\in J^n_k$, the function $y\mapsto p^n_k(Q,\bar{q},y)$ is Lipschitz on $\mathbb{R}^d$, uniformly with respect to $(Q,\bar{q})$ and its Lipschitz coefficient $[p^n_k]_{\mathrm{Lip},y}:=\sup_{Q\in T^+(n),\bar{q}\in J^n_k}[p^n_k(Q,\bar{q},.)]_{\mathrm{Lip}}$ satisfies for every $k\in\{0,\dots,n-1\}$,*

$$[p^n_{n-1}]_{\mathrm{Lip},y}\le[v_{n-1}]_{\mathrm{Lip}},\quad [p^n_k]_{\mathrm{Lip},y}\le[v_k]_{\mathrm{Lip}}+[\Theta_k]_{\mathrm{Lip}}[p^n_{k+1}]_{\mathrm{Lip},y}.$$

Proof. This follows easily by a backward induction on k, based on the dynamic programming formula (3.11) and the elementary inequality $|\sup_{i\in I}a_i-\sup_{i\in I}b_i|\le\sup_{i\in I}|a_i-b_i|$ for any indexed families of real numbers $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$. $\square$

Proof of Theorem 4.1. First we set $T^+_{\mathbb{N}}(n) := T^+(n) \cap \mathbb{N}^2$ for convenience. By piecewise affinity of $\widehat{P}^n_0$ and P^n_0, we have

$$\begin{aligned}\sup_{Q\in T^+(n)} |\widehat{P}^n_0(Q) - P^n_0(Q)| &= \sup_{Q\in T^+_{\mathbb{N}}(n)} |\widehat{P}^n_0(Q) - P^n_0(Q)| \\ &= \sup_{Q\in T^+_{\mathbb{N}}(n)} |\widehat{p}^n_0(Q,0,Y_0) - p^n_0(Q,0,\widehat{Y}_0)|.\end{aligned}$$

Let $k \in \{0,\dots,n-1\}$. Let $Q \in T^+_{\mathbb{N}}(n)$ and $\bar{q} \in J^n_k$.

$$\begin{aligned}&|p^n_k(Q,\bar{q},Y_k) - \widehat{p}^n_k(Q,\bar{q},\widehat{Y}_k)| \le |v_k(Y_k) - v_k(\widehat{Y}_k)| \\ &\quad + \max_{q=0,1,\bar{q}+q\in J^n_{k+1}} |\mathbb{E}(p^n_{k+1}(Q,\bar{q}+q,Y_{k+1})|\mathcal{F}_k) - \mathbb{E}(\widehat{p}^n_{k+1}(Q,\bar{q}+q,\widehat{Y}_{k+1})\,|\,\widehat{Y}_k)|.\end{aligned} \tag{4.7}$$

Now, using that Θ_k is a Markov transition and that $\sigma(\widehat{Y}_k) \subset \sigma(Y_k)$, one gets

$$\begin{aligned}&\mathbb{E}(p^n_{k+1}(Q,\bar{q}+q,Y_{k+1})|\mathcal{F}_k) - \mathbb{E}(\widehat{p}^n_{k+1}(Q,\bar{q}+q,\widehat{Y}_{k+1})\,|\,\widehat{Y}_k) \\ &\quad = \Theta_k(p^n_{k+1}Q,\bar{q}+q,,.))(Y_k) - \mathbb{E}(\Theta_k(p^n_{k+1}(Q,\bar{q}+q,,.)(Y_k)\,|\,\widehat{Y}_k) \\ &\qquad + \mathbb{E}(p^n_{k+1}(Q,\bar{q}+q,Y_{k+1}) - \widehat{p}^n_{k+1}(Q,\bar{q}+q,,\widehat{Y}_{k+1})\,|\,\widehat{Y}_k) \\ &\quad = \Theta_k(p^n_{k+1}(Q,\bar{q}+q,.))(Y_k) - \Theta_k(p^n_{k+1}(Q,\bar{q}+q,.)(\widehat{Y}_k) \\ &\qquad + \mathbb{E}(\Theta_k(p^n_{k+1}(Q,\bar{q}+q,.)(\widehat{Y}_k) - \Theta_k(p^n_{k+1}(Q,\bar{q}+q,.)(Y_k)\,|\,\widehat{Y}_k) \\ &\qquad + \mathbb{E}(p^n_{k+1}(Q,\bar{q}+q,Y_{k+1}) - \widehat{p}^n_{k+1}(Q,\bar{q}+q,\widehat{Y}_{k+1})\,|\,\widehat{Y}_k).\end{aligned}$$

Consequently, still using the above elementary triangular inequality for the supremum, one has

$$\begin{aligned}&\sup_{Q\in T^+_{\mathbb{N}}(n),\bar{q}\in J^n_k} |p^n_k(Q,\bar{q},Y_k) - \widehat{p}^n_k(Q,\bar{q},\widehat{Y}_k)| \le |v_k(Y_k) - v_k(\widehat{Y}_k)| \\ &\quad + \sup_{Q\in T^+_{\mathbb{N}}(n),\bar{q}'\in J^n_{k+1}} |\Theta_k(p^n_{k+1}(Q,\bar{q}',.))(Y_k) - \Theta_k(p^n_{k+1}Q,\bar{q}',.)(\widehat{Y}_k)| \\ &\quad + \mathbb{E}\left(\sup_{Q\in T^+_{\mathbb{N}}(n),\bar{q}'\in J^n_{k+1}} |\Theta_k(p^n_{k+1}(Q,\bar{q}',.)(\widehat{Y}_k) - \Theta_k(p^n_{k+1}(Q,\bar{q}',.)(Y_k)|\,|\,\widehat{Y}_k\right) \\ &\quad + \mathbb{E}\left(\sup_{Q\in T^+_{\mathbb{N}}(n),\bar{q}'\in J^n_{k+1}} |p^n_{k+1}(Q,\bar{q}',Y_{k+1}) - \widehat{p}^n_{k+1}(Q,\bar{q}',\widehat{Y}_{k+1})|\,|\,\widehat{Y}_k\right).\end{aligned}$$

Temporarily set for convenience, $\Delta^{n,p}_k := \|\sup_{Q\in T^+_{\mathbb{N}}(n),\bar{q}\in J^n_k} |p^n_k(Q,\bar{q},Y_k) - \widehat{p}^n_k(Q,\bar{q},\widehat{Y}_k)|\|_p$. One derives that for every $k = 0,\dots,n-1$,

$$\Delta^{n,p}_k \le ([v_k]_{\mathrm{Lip}} + 2[\Theta_k]_{\mathrm{Lip}}[p^n_{k+1}]_{\mathrm{Lip},y})\|Y_k - \widehat{Y}_k\|_p + \Delta^{n,p}_{k+1}.$$

Furthermore, $\Delta^{n,p}_{n-1} \le [v_{n-1}]_{\mathrm{Lip}}\|Y_{n-1} - \widehat{Y}_{n-1}\|_p$. The result follows by induction. $\square$

Remark. Inside a tile, the error is bounded by the maximum of the errors at the three vertices of the tile. This follows from (4.4).

4.2. *Optimal quantization*

In this section, we provide a few basic elements about optimal quantization in order to give some error bounds for the premium of the swing option. We refer to [17] for more details about theoretical aspects and to [27, 28] for the algorithmic aspects and numerical applications.

Let $p \in [1, +\infty)$. Let $Y \in L^p(\Omega, \mathcal{A}, \mathbb{P})$ be an $\mathbb{R}^d$-valued random vector and let $N \geq 1$ be a given integer. The best L^p-approximation of Y by a random vector taking its values in a given grid Γ of size (at most) N is given by a Voronoi quantizer $\widehat{Y}^\Gamma$ obtained as a (Borel) projection following the nearest neighbour rule on Γ. Such quantizers induce an $L^p(\mathbb{P})$- mean quantization error

$$e_{N,p}(Y, \Gamma) = \|Y - \widehat{Y}^\Gamma\|_p = \left(\mathbb{E} \min_{y \in \Gamma} |Y - y^i|^p\right)^{\frac{1}{p}}.$$

When the grids Γ run over all the subsets of $\mathbb{R}^d$ of size at most N, it has been shown that $e_{N,p}(Y, \Gamma)$ reaches a minimum denoted $e_{N,p}(Y)$ (see e.g. [17] or [24]) at some grid $\Gamma^{(N,*)}$:

$$e_{N,p}(Y) = \min\{e_{N,p}(Y, \Gamma), \Gamma \subset \mathbb{R}^d,\ \mathrm{card}(\Gamma) \leq N\}$$

Several algorithms have been designed to compute some optimal or close to optimality quantizers, especially in the quadratic case ($p = 2$). They all rely on the stationarity property satisfied by optimal quantizers. In the quadratic case, a grid Γ is stationary if

$$\widehat{Y}^\Gamma = \mathbb{E}(Y \,|\, \widehat{Y}^\Gamma)$$

This follows from some differentiability property of the L^2-distortion on the grids of size N (in the non quadratic case, we refer to [18]). In 1-dimension, a regular Newton-Raphson zero search procedure is dramatically efficient if properly initialized. In higher dimension (at least when $d \geq 3$ or 4) only stochastic procedures can be implemented like the $CLVQ$ (a stochastic gradient descent, see [24, 28]) or the Lloyd's I procedure (a randomized fixed point procedure, see [15, 28]). One must keep in mind that both procedures rely on repeated nearest neighbour searches.

As a result of these methods, some optimized grids of the (centered) normal distribution $\mathcal{N}(0; I_d)$ are available on line at the URL: www.quantize.maths-fi.com for dimensions $d = 1, \ldots, 10$ and sizes from $N = 2$ up to 5 000.

It is clear by considering a sequence of grids $\Gamma^{(N)} := \{r^1, \ldots, r^N\}$ where $(r^n)_{n \geq 1}$ is an everywhere dense sequence in $\mathbb{R}^d$ that $e_{N,p}(Y)$ decreases to 0 as $N \to \infty$. The rate of convergence of this sequence is ruled by the so-called Zador Theorem (see [32] for a first statement of the result, until the first rigorous proof in [17]).

Theorem 4.2 (Non asymptotic Zador Theorem, see [22]). *Let $p \geq 1$, $\eta > 0$. There exists a real constant $C_{d,p,\eta} > 0$ and an integer $N_{d,p,\eta} \geq 1$ such that for any*

$\mathbb{R}^d$-valued random vector Y, for every $N \geq N_{d,p,\eta}$,

$$e_{N,p}(Y) \leq C_{d,p,\eta} \|Y\|_{p+\eta} N^{-\frac{1}{d}}.$$

Theoretical rate of convergence of the quantization pricing method. Now we are in position to apply the above results to provide an error bound for the pricing of swing options by optimal quantization: assume there is a real exponent $p \in [1, +\infty)$ such that the (d-dimensional) Markov structure process $(Y_k)_{0 \leq k \leq n-1}$ satisfies

$$\max_{0 \leq k \leq n-1} |Y_k| \in L^{p+\eta}(\mathbb{P}), \quad \eta > 0$$

At each time $k \in \{0, \ldots, n-1\}$, we implement a (quadratic) optimal quantization grid $\Gamma^{\bar{N}}$ of Y_k with constant size $\bar{N}$. Then the general error bound result (4.6) combined with Theorem 4.2 says that, if $\bar{N} \geq N_{d,p,\eta}$,

$$\left\| \sup_{Q \in T^+(n)} |P_0^n(Q) - \widehat{p}_0^n(Q)| \right\|_p \leq C \frac{n}{\bar{N}^{\frac{1}{d}}}$$

as the complexity of the procedure is bounded by $n(n+1)\bar{N}^2$ (up to a constant).

In fact this error bound turns out to be conservative and several numerical experiments, as those presented below, suggest that in fact the true rate (for a fixed number n of purchase instants) behaves like $O(\bar{N}^{-\frac{2}{d}})$.

Another approach could be to minimize the complexity of the procedure by considering (optimal) grids with variable sizes N_k satisfying $\sum_{k=0}^{n-1} N_k = n\,\bar{N}$. We refer to [5] for further results in that direction. However, numerical experiments were carried out with constant size grids for both programming convenience and memory saving.

4.3. *A numerical illustration*

We considered a two factor continuous model for the price of future contracts which leads to the following dynamics for the spot price

$$S_t = F_{0,t} \exp\left(\sigma_1 \int_0^t e^{-\alpha_1(t-s)} dW_s^1 + \sigma_2 \int_0^t e^{-\alpha_2(t-s)} dW_s^2 - \frac{1}{2}\Lambda_t\right), \quad t \in [0, T],$$

where W^1 and W^2 are two standard Brownian motions with correlation coefficient ρ and

$$\Lambda_t = \frac{\sigma_1^2}{2\alpha_1}(1 - e^{-2\alpha_1 t}) + \frac{\sigma_2^2}{2\alpha_2}(1 - e^{-2\alpha_2 t}) + \frac{2\rho\sigma_1\sigma_2}{\alpha_1 + \alpha_2}(1 - e^{-(\alpha_1+\alpha_2)t}).$$

The (deterministic) process $(F_{0,t})_{t \in [0,T]}$ denotes the price at time 0 of the future contracts on gas of maturity t. Then, we consider a (daily) discretization of the Gaussian process $\log(S_t/F_{0,t})$ at times $\frac{kT}{n}$ (e.g. $T = 1/12$ and $n = 30$ or $T = 1$ and $n = 365$). The sequence $(\log(S_{t_k}/F_{0,t_k}))_{0 \leq k \leq n-1}$ is clearly not Markov since the stochastic convolution introduces some memory in the dynamics of the process.

However, it is possible to consider a higher dimensional homogenous Markov process $Y = (Y_k)_{0 \le k \le n-1}$ for which $\log(S_{t_k}/F_{0,t_k}) = AY_k$ for some matrix A. The process Y is an autoregressive process taking values in $\mathbb{R}^2$ in the above example. This calls upon classical methods from time series analysis (see further developments in the annex of the companion paper [5]). As a by-product, a fast method has been devised that makes possible a parallel implementation of the transition weights of the quantization tree of Y. For further details we refer to [5].

The model parameters have been settled at the following values

$$\alpha_1 = 0.21, \quad \alpha_2 = 5.4, \quad \sigma_1 = 36\%, \quad \sigma_2 = 111\%, \quad \rho = -0.11.$$

Firstly, the future curve has been assumed to be flat: $F_{0,t_k} = 20$, $k \ge 0$. The contract is a regular swing contract with parameters (before normalization)

$$n = 30, \quad q_{\min} = 0, \quad q_{\max} = 6, \quad Q_{\min} = 78, \quad Q_{\max} = 144, \quad K_k = K = 10.$$

First we tried to evaluate the rate of convergence of the method in this 2-dimensional setting as a function of the (common) size $\bar{N}$ of the quantization grids Γ_k used at time t_k (this choice was motivated by some implementation considerations). The grids were designed from some pre-computed optimal $\bar{N}$-quantizers of the bi-variate normal distribution (available at www.quantize.maths-fi.com). We computed the transition weights π_k^{ij} with $M = 3.10^6$ sample paths to get rid of the Monte Carlo error (instead of 100 000 which is the regular size for this phase). The reference price is computed as an average of prices computed from large grids, once they are stabilized.

Making the hypothesis that the error behaves like $\bar{N}^{-\vartheta}$, we made an estimation by regression of the best fitting exponent ϑ. We obtained $\vartheta = 1.26$ which suggests a much faster rate of convergence than expected by theoretical results ($1/d = 1/2$).

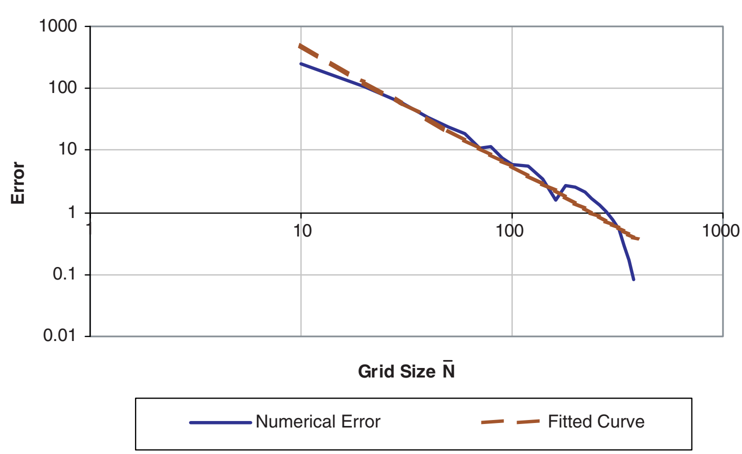

Fig. 5. Numerical rate of convergence as a function of the size $\bar{N}$ of the quantization grid. $n = 30$, $F_{0,t_k} = 20$, $K_k = K = 10$, $Q_{\min} = 78$, $Q_{\max} = 144$.

Further tests with different values of the strike K are reported in [5] always providing some exponents $\vartheta \approx 1 = \frac{2}{d}$. A similar phenomenon had been observed with the pricing of American style options (see [4]). This leads us to settle the size of the grid at $\bar{N} = 200$.

Then we tested the robustness of the approach with respect to the size of the Monte Carlo weight estimation. We came to the conclusion that a Monte Carlo simulation of size $M_{MC} = 100\,000$ is sufficient. This holds true for various (reasonable) values of the strike K and of the constraints.

Based on this simulation parameter, the transition weights estimation takes approximately 6 *seconds* and the swing option pricing itself (by the quantized dynamic programming) 1/3 *second* (Processor Celeron, CPU 2,4 GHz, 1,5 Go RAM). In case of a non-Gaussian framework, the grid computation time should be added to the transition weights one (unless it is carried out *off line* as is often the case). In daily operating use, many contracts with various parameters requires to be priced (at least for negotiation purpose) so this ability to compute almost instantly is quite valuable.

Our last illustration (see Fig. 6) is a complete graph of the premium function $Q := (Q_{\min}, Q_{\max}) \mapsto P_0^n(Q)$ when Q runs over the whole set $T^+(n)$ of admissible global constraints $T^+(n)$. The function is computed at integer valued global constraints and interpolated inside the tiling triangles. The model parameters are the same as above, except that the implemented future curve is made up with real data (future price contracts of 30 daily maturities quoted on 17/01/2003). The contract parameters are $q_{\min} = 0$, $q_{\max} = 1$ and $K_k = K = 18$.

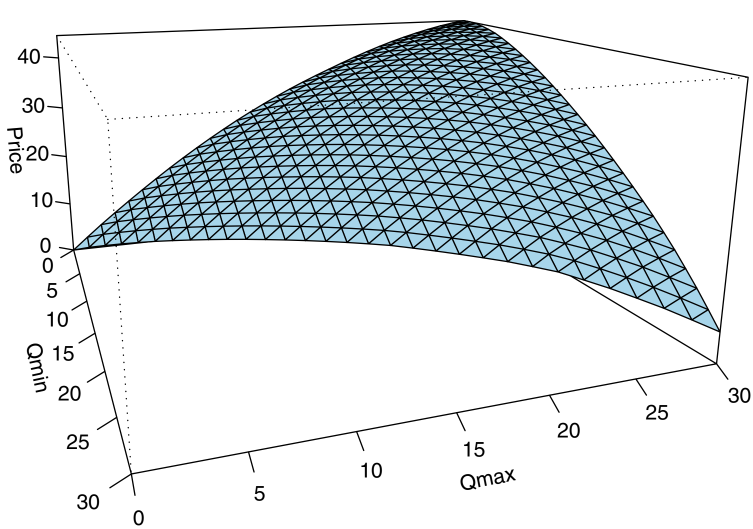

Fig. 6. The mapping $Q \mapsto \widehat{p}_0^{\,n}(Q)$ affinely interpolated from integer-valued global constraints. $n = 30$, $K_k = K = 18$, $q_{\min} = 0$, $q_{\max} = 1$, $\bar{N} = 200$, $M_{MC} = 100\,000$.

In [5], the performances of the optimal quantization tree method described above have been extensively tested from a numerical viewpoint (rates of convergence, needed memory, swapping effect, etc). Its performances have been compared to those of the least squares regression approach introduced for swing option pricing in [7]. This comparison emphasizes the accuracy and the efficiency of our approach, even if only one contract is to be computed and the computation of the transition weights is included in the computation time of the quantization method. Furthermore, it seems that quantization needs significantly less memory capacity than regression methods (for a given accuracy).

As a conclusion, the quantization tree method directly approximates the dynamics of the underlying structure process. It is in some way "model-driven": this means that very few choices are to be made prior to the implementation by the user (mainly the size of the quantization grids) whereas in regression methods the set of regression functions has to be specified which needs much care and some *a priori* knowledge on the dynamics. Conversely it seems clear that one could take advantage of the above theoretical results to improve these purely Monte Carlo approaches as well.

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