Assignment 01

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Solution 1.

Let $S = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$. Compute SP.

$$SP = S \cdot P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -3 \end{bmatrix}$$

The transpose of S is $S^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. Then we can compute SPS^T:

$$SPS^T = SP \cdot S^T = \begin{bmatrix} 1 & 3 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

Obviously, SPS^T is a diagonal matrix. So SPS^T is a symmetric matrix.

Solution 2.

Let

$$S = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

(a) First we prove that S is orthogonal.

$$S^T = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Then we calculate $S^T \cdot S$:

$$S^T \cdot S = \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & 0 \\ 0 & \cos^2\alpha + \sin^2\alpha \end{bmatrix} = I$$

Thus S is orthogonal.
(b)The proof is as follows:
Considering $B = SAS^T$, we have

$$B = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Expanding the above expression, we get

$$B = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} a_{11}\cos\alpha + a_{12}\sin\alpha & -a_{11}\sin\alpha + a_{12}\cos\alpha \\ a_{21}\cos\alpha + a_{22}\sin\alpha & -a_{21}\sin\alpha + a_{22}\cos\alpha \end{bmatrix}$$

We obtain

$$B = \begin{bmatrix} a_{11}\cos^2\alpha + 2a_{12}\cos\alpha\sin\alpha + a_{22}\sin^2\alpha & (a_{22} - a_{11})\cos\alpha\sin\alpha + a_{12}(\cos^2\alpha - \sin^2\alpha) \\ (a_{22} - a_{11})\cos\alpha\sin\alpha + a_{12}(\cos^2\alpha - \sin^2\alpha) & a_{11}\sin^2\alpha - 2a_{12}\cos\alpha\sin\alpha + a_{22}\cos^2\alpha \end{bmatrix}$$

Since B is diagonal, the off-diagonal elements must be zero. Therefore, we have

$$(a_{22} - a_{11})\cos\alpha\sin\alpha + a_{12}(\cos^2\alpha - \sin^2\alpha) = 0$$

$$(a_{22} - a_{11}) + a_{12}\tan(2\alpha) = 0$$

$$\tan(2\alpha) = \frac{2a_{12}}{a_{11} - a_{22}}$$

Thus, we have proved that if $\tan(2\alpha) = \frac{2a_{12}}{a_{11}-a_{22}}$, then $B = SAS^T$ is diagonal.
(c) Verify that Tr[B] = Tr[A].

$$Tr[B] = Tr[SAS^T] = Tr[ASS^T] = Tr[AI] = Tr[A]$$

Thus, we have proved that Tr[B] = Tr[A].

Solution 3.
Let F be the event that the coin picked is fair, and H be the event that heads shows both times. We want to find $P(F|H)$.

$$P(F|H) = \frac{P(F \cap H)}{P(H)}$$

$$P(F \cap H) = P(F) \cdot P(H|F) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

$$P(H) = P(F) \cdot P(H|F) + P(F^c) \cdot P(H|F^c) = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot 1 = \frac{5}{8}$$

Thus, we have

$$P(F|H) = \frac{P(F \cap H)}{P(H)} = \frac{1}{5}$$

SOLUTION 4.
Let D_A be the event that a bulb is defective from Box A.
Let D_B be the event that a bulb is defective from Box B.
(a)The probability that both bulbs are defective can be calculated as:

$$P(\text{both defective}) = P(\text{both defective from A}) + P(\text{both defective from B})$$

$$= P(D_A \cap D_A) + P(D_B \cap D_B) = P(D_A)^2 + P(D_B)^2$$
$$= (0.1)^2 + (0.05)^2 = 0.01 + 0.0025 = 0.0125$$

Therefore, the probability that both bulbs are defective is 0.0125.
(b) Using Bayes' theorem:

$$P(\text{Box A — both defective}) = \frac{P(\text{both defective — Box A}) * P(\text{Box A})}{P(\text{both defective})}$$

$$= \frac{0.01 * 0.5}{0.0125} = \frac{2}{5}$$

If both are defective, the probability that they come from box A is 0.4.