L^p-L^q Fourier multipliers on locally compact groups

A thesis presented for the degree of

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by

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Declaration

I certify that the contents of this thesis are my own original work, unless indicated otherwise. Section 1.6, Chapter 5 and Section A.2 are based on my joint work with Erlan Nursultanov and Michael Ruzhansky.

Chapters 1-4 are based on my joint work Michael Ruzhansky.

Section A.1 collects basic facts on linear unbounded operators measurable with respect to semi-finite von Neumann algebras. This section has been written by me, but its contents are based on the existing literature.

SIGNED: .... DATE: ....

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Dedication

This dissertation is dedicated to my loving, always encouraging and supportive wife, Gulnara Akylzhanova (Kulanbaeva) and our curious, exuberant, sweet, kind-hearted little girl, Aylin Akylzhanova.

So I saw that there is nothing better for a person than to enjoy their work, because that is their lot. For who can bring them to see what will happen after them?

Ecclesiastes 3:22

Abstract

We study the $L^p - L^q$ boundedness of both spectral and Fourier multipliers on general locally compact separable unimodular groups G. As a consequence of the established Fourier multiplier theorem we also derive a spectral multiplier theorem on general locally compact separable unimodular groups. We then apply it to obtain embedding theorems as well as time-asymptotics for the $L^p - L^q$ norms of the heat kernels for general positive unbounded invariant operators on G. We illustrate the obtained results for sub-Laplacians on compact Lie groups and on the Heisenberg group, as well as for higher order operators.

With minor modificaitons, our proofs of Paley-type inequalities and $L^p - L^q$ bounds of Fourier multipliers can be adapted to the setting of compact homogeneous manifolds.

Contents

Chapter 1

Introduction

Fourier multipliers arise naturally when formally obtaining the solution to constant coefficient partial differential operator on $\mathbb{R}^n$. A fundamental problem in the study of Fourier multipliers is to relate regularity of the symbol and the boundedness of the operator. My PhD research focuses on Fourier multipliers on locally compact groups. The main insight and the tool is the theory of von Neumann algebras. The key observation is that regularity of the symbol is encoded in the decay of the spectral information of the Fourier multiplier viewed as a linear closed operator affiliated with the group von Neumann algebra. This allows us to establish general unifying results on L^p-L^q-boundedness of Fourier multipliers on topological groups. It is known that on non-compact groups we must have $p \leq q$ and two classical results are available on $\mathbb{R}^n$, namely, Hörmander's multiplier theorem [Hör60] for $1 < p \leq 2 \leq q < \infty$, and Lizorkin's multiplier theorem [Liz67] for $1 < p \leq q < \infty$. There is a philosophical difference between these results: Hörmander's theorem does not require any regularity of the symbol and applies to p and q separated by 2, while Lizorkin theorem applies also for $1 < p \leq q \leq 2$ and $2 \leq p \leq q < \infty$ but imposes certain regularity conditions on the symbol. In this thesis we aim at proving theorems expressing conditions in terms of the sharp decay property of the spectral information associated to the operator, on general locally compact separable unimodular groups based on developing a new approach relying on the analysis in the noncommutative Lorentz spaces on the group von Neumann algebra. This suggested approach seems very effective, implying as special cases known results expressed in terms of symbols, in settings when the symbolic calculus is available. The obtained results are for general Fourier multipliers, in particular also implying new results for spectral multipliers. The crucial difference with the known techniques by Hörmander and Lizorkin is the application of the theory of t-th generalized singular numbers $\mu_t(\cdot)$.

We assume for simplicity that G is unimodular but we do not make assumption that G is either of type I or type II. The assumption for the locally compact group to be separable and unimodular may be viewed as natural allowing one to use basic results of von Neumann-type Fourier analysis, such as, for example, Plancherel formula [Seg50, Mau50]. However, the unimodularity assumption may be in principle avoided, see e.g. [DM76], but the exposition becomes much more technical. For a more detailed discussion of pseudo-differential operators in such settings we refer to [MR15], but we note that compared to the analysis there in this paper we do not need to assume that the group is of type I. The class of groups covered by our analysis is very wide. In particular, it contains abelian, compact, nilpotent groups, exponential, real algebraic or semi-simple Lie groups, solvable groups (not all of which are type I, but we do not need to assume the group to be of type I or II), and many others. As far as we are aware our results are new in all of these non-Euclidean settings.

We focus on the L^p-L^q multipliers as opposed to the L^p-multipliers when theorems of Mihlin-Hörmander or Marcinkiewicz types provide results for both Fourier and spectral multipliers in some settings, based on the regularity of the multiplier. L^p-multipliers have been intensively studied on different kinds of groups, however, mostly L^p spectral multipliers, for which a wealth of results is available: e.g. [MS94, MRS95] on Heisenberg type groups, [CM96] on solvable Lie groups, [MT07] on nilpotent and stratified groups, to mention only very very few. L^p Fourier multiplies have been also studied but to a lesser extent due to lack of symbolic calculus that was not available until recently, e.g. Coifman and Weiss [CW71b, CW71a] on SU_2, [RW13, RW15] and then [Fis16] on compact Lie groups, or [FR14, ?] on graded Lie groups. A characteristic feature of the L^p-L^q multipliers is that less regularity of the symbol is required. Therefore, in this paper we concentrate on the L^p-L^q multiplier theorems, however aiming at obtaining unifying results for general locally compact groups. We give several applications of the obtained results to questions such as embedding theorems and dispersive estimates for evolution PDEs.

The approach to the L^p-Fourier multipliers is different from the technique proposed in this paper allowing us to avoid making an assumption that the group is compact or nilpotent. In this paper we are interested in both Fourier multipliers and spectral multipliers, for the latter some L^p-L^q results being available in some special settings, see e.g. [CGM93], and also [Cow74], as well as [ANR16a] for the case of SU_2, and for the discussion of some relations between those in the group setting we can refer to [RW15] and references therein.

Fourier multipliers in the context of group von Neumann algebras have been studied in [GPJP17]. The authors take a dual point of view. Given an element $m \in L^1(G) \cap L^2(G)$, they construct a certain linear mapping $T_m \colon \mathrm{VN}_L(G) \to \mathrm{VN}_L(G)$, where $\mathrm{VN}_L(G)$ is the left group von Neumann algebra. Fourier multipliers acting on the compact dual of arbitrary discrete groups has been recently investigated in [JMP14b]. The main result is a Hörmander-Mihlin multiplier theorem for finite-dimensional cocycles with optimal smoothness condition. A version of Hörmander-Mihlin theoreom on unimodular locally compact groups has been recently obtained in [JMP14b]. By the combinatorial method it is possible to establish the L^p-L^q estimates for the Poisson-type semigroup $\mathcal{P}_t$ on discrete groups G [JMP14a].

Finally we note that multiplier estimates on noncommutative groups are in general considerably more delicate than those in the commutative case, recall e.g. the asymmetry problem and its resolution in [DGR00]. A link between Fourier multipliers and Lorentz spaces on group von Neumann algebras has been outlined in [AR16].

We now proceed to making a more specific description of the considered problems.

1.1 Hörmander's theorem on locally compact groups

To put this in context, we recall that in [Hör60, Theorem 1.11], Lars Hörmander has shown that for $1 < p \leq 2 \leq q < \infty$, if the symbol $\sigma_A \colon \mathbb{R}^n \to \mathbb{C}$ of a Fourier multiplier A on $\mathbb{R}^n$ satisfies the condition

$$\sup_{s>0} s \left(\int\limits_{\xi \in \mathbb{R}^n \colon |\sigma_A(\xi)| \geq s} d\xi \right)^{\frac{1}{p}-\frac{1}{q}} < +\infty, \tag{1.1}$$

then A is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Here, as usual, the Fourier multiplier A on $\mathbb{R}^n$ acts by multiplication on the Fourier transform side, i.e.

$$\widehat{Af}(\xi) = \sigma_A(\xi)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^n. \tag{1.2}$$

Moreover, it then follows that

$$\|A\|_{L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} \lesssim \sup_{s>0} s \left(\int\limits_{\substack{\xi \in \mathbb{R}^n \\ |\sigma_A(\xi)| \geq s}} d\xi \right)^{\frac{1}{p}-\frac{1}{q}}, \quad 1 < p \leq 2 \leq q < +\infty. \tag{1.3}$$

The L^p-L^q boundedness of Fourier multipliers has been also recently investigated in the context of compact Lie groups, and we now briefly recall the result. Let G be a compact Lie group and $\widehat{G}$ its unitary dual. For $\pi \in \widehat{G}$, we write d_π for the dimension of the (unitary irreducible) representation π. It has been shown in [ANR16b] that, for a Fourier multiplier A acting via

$$\widehat{Af}(\pi) = \sigma_A(\pi)\widehat{f}(\pi)$$

by its global symbol $\sigma_A(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ we have

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \sup_{s>0} s \left(\sum_{\substack{\pi\in\widehat{G} \\ \|\sigma_A(\pi)\|_{\mathrm{op}}\geq s}} d_\pi^2 \right)^{\frac{1}{p}-\frac{1}{q}}, \quad 1 < p \leq 2 \leq q \leq \infty. \tag{1.4}$$

Here for $\pi \in \widehat{G}$, the Fourier coefficients are defined as

$$\widehat{f}(\pi) = \int_G f(x)\pi(x)^* dx,$$

and $\|\sigma_A(\pi)\|_{\mathrm{op}}$ is the operator norm of $\sigma_A(\pi)$ as the linear transformation of the representation space of $\pi \in \widehat{G}$ identified with $\mathbb{C}^{d_\pi}$. For a general development of global symbols and the corresponding global quantization of pseudo-differential operators on compact Lie groups we can refer to [RT13, RT10]. One of the results of this thesis generalises both multiplier theorems (1.3) and (1.4) to the setting of general locally compact separable unimodular groups G.

By a *left Fourier multiplier in the setting of general locally compact unimodular groups we will mean left invariant operators that are measurable with respect to the right group von Neumann algebra* $\mathrm{VN}_R(G)$.

Thus, in Theorem 3.1 we prove the following inequality

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \sup_{s>0} s \left[\int\limits_{t\in\mathbb{R}_+ :\, \mu_t(A)\geq s} dt \right]^{\frac{1}{p}-\frac{1}{q}}, \quad 1 < p \leq 2 \leq q < +\infty, \tag{1.5}$$

where $\mu_t(A)$ are the t-th generalised singular values of A, see Definition A.6 for the precise definition and properties (following [TK86]). An extension of (1.5) to $q = \infty$ will be shown in Theorem 3.6.

The key idea behind the extension (1.5) is that Hörmander's theorem (1.3) can be reformulated as

$$\|A\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \lesssim \sup_{s>0} s \left(\int\limits_{\substack{\xi\in\mathbb{R}^n \\ |\sigma_A(\xi)|\geq s}} d\xi \right)^{\frac{1}{p}-\frac{1}{q}}$$
$$\simeq \|\sigma_A\|_{L^{r,\infty}(\mathbb{R}^n)} \simeq \|A\|_{L^{r,\infty}(\mathrm{VN}(\mathbb{R}^n))}, \quad (1.6)$$

where $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, $\|\sigma_A\|_{L^{r,\infty}(\mathbb{R}^n)}$ is the Lorentz space norm of the symbol σ_A, and $\|A\|_{L^{r,\infty}(\mathrm{VN}(\mathbb{R}^n))}$ is the norm of the operator A in the Lorentz space on the group von Neumann algebra $\mathrm{VN}(\mathbb{R}^n)$ of $\mathbb{R}^n$. In turn, our estimate (1.5) is equivalent to the estimate

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \|A\|_{L^{r,\infty}(\mathrm{VN}_R(G))} \simeq \sup_{s>0} s \left[\int\limits_{t\in\mathbb{R}_+:\ \mu_t(A)\geq s} dt \right]^{\frac{1}{r}}, \quad (1.7)$$

where $\|A\|_{L^{r,\infty}(\mathrm{VN}_R(G))}$ is the norm of the operator A in the noncommutative Lorentz space on the right group von Neumann algebra $\mathrm{VN}_R(G)$ of G. Thus, the noncommutative Lorentz spaces become a key point for the extension of Hörmander's theorem to the setting of general locally compact (unimodular) groups.

The multiplier theorem (1.5) implies Hörmander's theorem (1.3). In Proposition 3.1 we show that the multiplier theorem (1.5) recovers (1.4) in the setting of compact Lie groups. The proof of inequality (1.5) is based on a version of the Hausdorff-Young-Paley inequality on locally compact separable groups that we establish for this purpose. The latter inequality is formulated in Theorem 2.4 in Chapter 2. There are two main new ingredients in our proof of Fourier (and then also spectral)multiplier theorems: Paley/Hausdorff-Young-Paley and Nikolskii inequalities for the Hörmander and Lizorkin versions of multiplier statements, respectively.

1.2 Lizorkin theorem

The classical Lizorkin theorem [Liz67] applies for the range $1 < p \leq q < \infty$. Let A be a Fourier multiplier on $\mathbb{R}$ with the symbol σ_A as in (1.2). Assume that for some $C < \infty$ the symbol $\sigma_A(\xi)$ satisfies the following conditions

$$\sup_{\xi\in\mathbb{R}} |\xi|^{\frac{1}{p}-\frac{1}{q}} |\sigma_A(\xi)| \leq C, \tag{1.8}$$

$$\sup_{\xi\in\mathbb{R}} |\xi|^{\frac{1}{p}-\frac{1}{q}+1} \left| \frac{d}{d\xi}\sigma_A(\xi) \right| \leq C. \tag{1.9}$$

Then $A\colon L^p(\mathbb{R}) \to L^q(\mathbb{R})$ is a bounded linear operator and

$$\|A\|_{L^p(\mathbb{R})\to L^q(\mathbb{R})} \lesssim C. \tag{1.10}$$

An extension to $G = \mathbb{R}^n$ with sharper conditions on the symbol has been obtained in [STT10]. A number of papers [YST16, PST08, PST12] deal with the same problem on $G = \mathbb{T}^n$ and $G = \mathbb{R}^n$.

In order to measure the regularity of the symbol, we introduce (3.34) a new family of difference operators $\widehat{\partial}$ acting on Fourier coefficients and on symbols. These operators are used to formulate and prove a version of the Lizorkin theorem on compact groups. Let G be a compact Lie group of topological dimension n and let $\widehat{G}$ be the unitary dual of G . In Theorem 3.4, we establish followoing version of Lizorkin multiplier on G

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \sup_{\pi\in\widehat{G}} \langle\pi\rangle^{n\left(\frac{1}{p}-\frac{1}{q}\right)} \|\sigma_A(\pi)\|_{\mathtt{op}} + \sup_{\pi\in\widehat{G}} \langle\pi\rangle^{n(\frac{1}{p}-\frac{1}{q}+1)} \|\widehat{\partial}\sigma_A(\pi)\|_{\mathtt{op}}, \tag{1.11}$$

where $\langle\pi\rangle$ are the eigenvalues for the first-order pseudo-differential operator $(I-\Delta_G)^{\frac{1}{2}}$, i.e.

$$(I-\Delta_G)^{\frac{1}{2}}\pi_{ij} = \langle\pi\rangle\pi_{ij}, \quad \text{for all } 1 \leq i,j \leq d_\pi, \tag{1.12}$$

where Δ_G os the Laplace operator on G and π_{ij} are matrix elements of $\pi \in \widehat{G}$.

1.3 Spectral multipliers

Let us illustrate the use of the Fourier multiplier theorem (1.5) in the important case of spectral multipliers on locally compact groups. Later, in Theorem 4.1 we will give a spectral multiplier result on general semifinite von Neumann algebras, however, we now formulate its special case for the case of group von Neumann algebras associated to locally compact groups.

Interestingly, for the range $1 < p \leq 2 \leq q < \infty$ this result asserts that the L^p-L^q norm estimates of spectral multipliers $\varphi(|\mathcal{L}|)$ depend only on the rate of growth of traces of spectral projections of the operator $|\mathcal{L}|$:

Theorem 1.1. Let G be a locally compact separable unimodular group and let $\mathcal{L}$ be a left Fourier multiplier on G. Assume that φ is a monotonically decreasing continuous function on $[0, +\infty)$ such that

$$\varphi(0) = 1,$$
$$\lim_{u \to +\infty} \varphi(u) = 0.$$

Then we have the inequality

$$\|\varphi(|\mathcal{L}|)\|_{L^p(G) \to L^q(G)} \lesssim \sup_{u>0} \varphi(u) \left[\tau(E_{(0,u)}(|\mathcal{L}|))\right]^{\frac{1}{p}-\frac{1}{q}} \tag{1.13}$$

for the range $1 < p \leq 2 \leq q < \infty$.

Here $E_{(0,u)}(|\mathcal{L}|)$ are the spectral projections associated to the operator $|\mathcal{L}|$ to the interval $(0, u)$ We always assume that the spectrum of $|A|$ is separated from zero, i.e.

$$\mathrm{Sp}(|A|) \subset (c, +\infty)$$

for some positive real number $c > 0$. This can always be achieved by shifting the operator of A by a constant. and τ is the canonical trace on the right group von Neumann algebra $\mathrm{VN}_R(G)$.

The estimate (1.13) says that if the supremum on the right hand side is finite then the operator $\varphi(|\mathcal{L}|)$ is bounded from $L^p(G)$ to $L^q(G)$. Moreover, the estimate for the operator norm can be used for deriving asymptotics for propagators for equations on G. For example, we get the following consequences for the L^p-L^q norm for the heat kernel of $\mathcal{L}$, applying Theorem 1.1 with $\varphi(u) = e^{-tu}$, or embedding theorems for $\mathcal{L}$ with $\varphi(u) = \frac{1}{(1+u)^\gamma}$.

Example 1.1. Let $G = \mathbb{H}^n$ and let $\mathfrak{h}^n$ be its Lie algebra. Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_n, T$ be the (usual) basis in $\mathfrak{h}^n$ such that $[X_k, Y_k] = T$ and all other commutators are zero. Let $\mathcal{L} = I + (-1)^N \left(\sum\limits_{k=1}^{n} X_k^{2N} + \sum\limits_{k=1}^{n} Y_k^{2N} \right)$. It can be computed (see Example 4.5) that

$$\tau(E_{(0,s)}(\mathcal{L})) \lesssim s^{\frac{Q}{2N}},$$

where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$ and φ is a monotonically decreasing function as in Theorem 1.1.

Corollary 1.1. Let G be a locally compact unimodular separable group and let $\mathcal{L}$ be a positive left Fourier multiplier such that for some α we have

$$\tau(E_{(0,s)}(\mathcal{L})) \lesssim s^\alpha, s > 0. \tag{1.14}$$

Then for any $1 < p \leq 2 \leq q < \infty$ there is a constant $C = C_{\alpha,p,q} > 0$ such that we have

$$\|e^{-t\mathcal{L}}\|_{L^p(G) \to L^q(G)} \leq C t^{-\alpha\left(\frac{1}{p} - \frac{1}{q}\right)}, \quad t > 0. \tag{1.15}$$

We also have the inequalities

$$\|f\|_{L^q(G)} \leq C \|(1 + \mathcal{L})^\gamma f\|_{L^p(G)}, \tag{1.16}$$

provided that

$$\gamma \geq \alpha \left(\frac{1}{p} - \frac{1}{q} \right), \quad 1 < p \leq 2 \leq q < \infty. \tag{1.17}$$

The number α in (1.14) is determined based on the spectral properties of $\mathcal{L}$ and is often computable. For example, we have

(a) if $\mathcal{L}$ is the sub-Laplacian on a compact Lie group G then $\alpha = \frac{Q}{2}$, where Q is the Hausdorff dimension of G with respect to the control distance associated to $\mathcal{L}$;

(b) if $\mathcal{L}$ is the sub-Laplacian on the Heisenberg group $G = \mathbb{H}^n$ then $\alpha = \frac{Q}{2}$, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$;

(c) More generally, let $X_1, \ldots, X_n, Y_1, \ldots, Y_n, T$ be the (usual) basis in the Lie algebra $\mathfrak{h}^{\mathfrak{n}}$ such that $[X_k, Y_k] = T$ and all other commutators are zero. If $\mathcal{L} = 1 + (-1)^N \left(\sum\limits_{k=1}^{n} X_k^{2N} + \sum\limits_{k=1}^{n} Y_k^{2N} \right)$ then $\alpha = \frac{Q}{2N}$.

(d) Even more generally, if $\mathcal{L}$ is a positive hypoelliptic homogeneous of order ν left-invariant operator on the Heisenberg group $G = \mathbb{H}^n$ then $\alpha = \frac{Q}{\nu}$. Hence, by Theorem 1.1, we get

$$\left\| \varphi \left(1 + (-1)^N \left(\sum_{k=1}^{n} X_k^{2N} + \sum_{k=1}^{n} Y_k^{2N} \right) \right) \right\|_{L^p(\mathbb{H}^n) \to L^q(\mathbb{H}^n)} \leq \sup_{s>0} \varphi(s) s^{\frac{Q}{\nu}\left(\frac{1}{p} - \frac{1}{q}\right)}, \tag{1.18}$$

where φ is a non-increasing function as in Theorem 1.1.

Consequently, in both of the sub-Laplacian cases (a) and (b), Corollary 1.1 implies that for any $1 < p \leq 2 \leq q < \infty$ there is a constant $C = C_{p,q} > 0$ such that we have

$$\|e^{-t\mathcal{L}}\|_{L^p(G) \to L^q(G)} \leq C t^{-\frac{Q}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}, \quad t > 0. \tag{1.19}$$

The embeddings (1.16) under conditions (1.17) show that the *statement* *of Theorem 1.1 is in general sharp.* Taking $\varphi(s) = \frac{1}{(1+s)^{a/2}}$ and applying (1.13) to the sub-Laplacian Δ_{sub} in either of examples (a) or (b) above, we get that the operator $\varphi(1-\Delta_{sub}) = (1-\Delta_{sub})^{-a/2}$ is $L^p(G)$-$L^q(G)$ bounded and the inequality

$$\|f\|_{L^q(G)} \leq C\|(1-\Delta_{sub})^{a/2})f\|_{L^p(G)} \tag{1.20}$$

holds true provided that

$$a \geq Q\left(\frac{1}{p}-\frac{1}{q}\right), \quad 1 < p \leq 2 \leq q < \infty. \tag{1.21}$$

However, this yields the Sobolev embedding theorem which is well-known to be sharp at least in the case (b) of the Heisenberg group ([Fol75]), showing *the sharpness of Theorem 1.1 and hence also of the Fourier multiplier theorem* (1.7). More details are given in Section 4.1.

We also establish spectral multiplier theorem on compact Lie group associated with Lizorkin theorem. For convenience of the following formulation we label the equivalence classes of irreducible unitary representations of G as $\pi_k, k \in \mathbb{N}$ in such a way that the corresponding sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ of $(I-\Delta_G)^{\frac{\dim(G)}{2}}$ is non-decreaasing. More precisely, we have

$$(I-\Delta_G)^{\frac{\dim(G)}{2}}\pi_{ij}^k = \lambda_k \pi_{ij}^k$$

holds for all $1 \leq m, l \leq d_{\pi^k}$ and. From the Weyl asymptotic formula for the eigenvalue counting function we get that

$$\lambda_k \cong \langle \pi^k \rangle^n \cong k, \quad \text{with } n = \dim G. \tag{1.22}$$

Therefore, we can formulate a spectral multipliers corollary of Theorem 3.4.

Corollary 1.2. Let A be a left Fourier multiplier on a compact Lie group G of dimension n. Let $1 < p \leq q < \infty$. Assume that φ is a monotone function on $[0,+\infty)$. Then $\varphi(|A|)\colon L^p(G) \to L^q(G)$ is a bounded linear operator and we have

$$
\begin{aligned}
&\|\varphi(|A|)\|_{L^p(G)\to L^q(G)} \\
&\qquad\qquad\qquad\lesssim \sup_{k\in\mathbb{N}} k^{\frac{1}{p}-\frac{1}{q}} \sup_{t=1,\ldots,d_{\pi^k}} |\varphi(\alpha_{t\,k})| \\
&\qquad\qquad + \sup_{j\in\mathbb{N}} k^{\frac{1}{p}-\frac{1}{q}+1} \sup_{t=1,\ldots,d_{\pi^k}} |\varphi(\alpha_{t\,k}) - \varphi(\alpha_{t+1\,k+1})| \,, \quad (1.23)
\end{aligned}
$$

where $\alpha_{t\,k}$ are the singular numbers of the symbol $\sigma_A(\pi^k)$, $\pi^k \in \widehat{G}$, $t = 1,\ldots,d_{\pi^k}$.

It will be clear from the proof of Corollary 1.2 that the condition that φ is monotone is not essential and is needed only for obtaining a simpler expression under the sum in (4.1). We refer the reader to Remark 4.1 for the analogous statement.

1.4 Hausdorff-Young-Paley inequalities

A fundamental problem in Fourier analysis is that of investigating the relationship between the "size" of a function and the "size" of its Fourier transform.

In [HL27] Hardy and Littlewood proved the following generalisation of the Plancherel's identity on the circle $\mathbb{T}$, namely

$$\sum_{m\in\mathbb{Z}} (1+|m|)^{p-2} |\widehat{f}(m)|^p \leq C_p \|f\|^p_{L^p(\mathbb{T})}, \quad 1 < p \leq 2. \tag{1.24}$$

Hewitt and Ross [HR74] generalised this to the setting of compact abelian groups. Recently, the inequality has been extended [ANR15] to compact homogeneous manifolds. In particular, on a compact Lie group G of topological dimension n, the result can be written as

$$\sum_{\pi\in\widehat{G}} d_\pi^{\,p(\frac{2}{p}-\frac{1}{2})} \langle\pi\rangle^{n(p-2)} \|\widehat{f}(\pi)\|^p_{\mathtt{HS}} \leq C_p \|f\|^p_{L^p(G)}, \tag{1.25}$$

where $\langle\pi\rangle$ are obtained from eigenvalues of the Laplace operator Δ_G on

G by

$$\sqrt{I-\Delta_G}\pi_{ij} = \langle\pi\rangle\pi_{ij}, \quad i,j=1,\dots,d_\pi,$$

where π_{ij} are the matrix entries of $\pi \in \widehat{G}$.

Recall briefly that in [Hör60] Hörmander has shown the following version of the *Paley inequality* on $\mathbb{R}^n$: if a positive function $\varphi \geq 0$ satisfies

$$|\{\xi \in \mathbb{R}^n \colon \varphi(\xi) \geq t\}| \leq \frac{C}{t} \quad \text{for } t > 0, \tag{1.26}$$

then

$$\left(\int_{\mathbb{R}^n} |\widehat{u}(\xi)|^p \, \varphi(\xi)^{2-p} \, d\xi \right)^{\frac{1}{p}} \lesssim \|u\|_{L^p(\mathbb{R}^n)}, \quad 1 < p \leq 2. \tag{1.27}$$

The classical Hardy-Littlewood inequality can be recovered by choosing $\varphi(\xi) = (1+|\xi|)^{-n}$. For functions with monotone Fourier coefficients such results serve as an extension of the Plancherel identity to L^p-spaces: for example, on the circle $\mathbb{T}$, Hardy and Littlewood have shown that for $1 < p < \infty$, if the Fourier coefficients $\widehat{f}(m)$ are monotone, then one has

$$f \in L^p(\mathbb{T}) \quad \text{if and only if} \quad \sum_{m\in\mathbb{Z}} (1+|m|)^{p-2} |\widehat{f}(m)|^p < \infty. \tag{1.28}$$

Hardy-Littlewood inequalities on locally compact groups have been studied e.g. by Kosaki [H.K81], see Theorem 2.1. In Section 2.1 we establish a version of Paley inequality, and consequently of the Hausdorff-Young-Paley inequality on locally compact groups, yielding extensions of the Euclidean version (1.27) as well as of Kosaki's results. The established Hausdorff-Young-Paley inequality (Theorem 2.4) is a crucial ingredient in our proof of Hörmander's version of multiplier theorem in Section 3.1.

1.5 Nikolskii type inequalities

The essential step in our proof of Lizorkin theorem is the Nikolskii inequality which is also referred to as the reverse Hölder inequality in the literature. Originating in Nikolskii's work [Nik51] in 1951 for trigonometric polynomials on the circle, functions on $\mathbb{R}^n$ with compact support of Fourier transform, the *Nikolskii inequality* takes the form (see e.g. [NW78])

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C[\mathrm{vol}[\mathrm{conv}[\mathrm{supp}(\widehat{f})]]^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 \leq p \leq q \leq \infty, \tag{1.29}$$

for every function $f \in L^p(\mathbb{R}^n)$ with Fourier transform $\widehat{f}$ of compact support, where $\mathrm{conv}(E)$ denotes the convex hull of the set E. The Nikolskii inequality plays an important role in many questions of function theory, harmonic analysis, and approximation theory. Its versions on compact Lie groups (and compact homogeneous manifolds) have been established in [NRT15, NRT16], with further applications to Besov spaces and to Fourier multipliers acting in Besov spaces in those settings.

In Section 2.2 we prove a version of the Nikolskii inequality on general locally compact separable unimodular groups. An interesting question in this setting already is how to understand functions with bounded support of the Fourier transform in such generality. In Theorem 2.5 we show that

$$\|f\|_{L^q(G)} \leq \left(\tau(P_{\mathrm{supp}[\widehat{f}]})\right)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^p(G)}, \tag{1.30}$$

provided that $\tau(P_{\mathrm{supp}[\widehat{f}]}) < \infty$ and $1 < p \leq \min(2, q)$, $1 \leq q \leq \infty$. Here $P_{\mathrm{supp}[\widehat{f}]}$ denotes the orthogonal projector associated to the support $\mathrm{supp}[\widehat{f}]$ of the operator-valued Fourier transform $\widehat{f}$ of f. The estimate (1.30) will play a crucial role in proving Lizorkin type multiplier theorems in Section 3.2.

Remark 1.1. The constant in inequality (1.30) depends only on the value of the trace τ on the orthogonal projection $P_{\mathrm{supp}[\widehat{f}]})$ associated to $\mathrm{supp}[\widehat{f}]$. We notice that inequality (1.30) is formulated only for the range $1 < p \leq 2, p \leq q \leq \infty$. Moreover, it can be shown [NW78, p. 10] that inequality (1.30) fails for $p > 2$. Thus, inequality (1.30) cannot be considered as an improvement of (1.29) for the range $2 < p \leq q \leq \infty$.

1.6 Extension to compact homogeneous manifolds

We study necessary conditions and sufficient conditions for the p-th power integrability of a function on an arbitrary compact homogeneous space G/K by means of its Fourier coefficients. The obtained inequalities provide a noncommutative version of known results of this type on the circle $\mathbb{T}$ and the real line $\mathbb{R}$.

In Chapter 5 we establish the following results, that we now summarise and briefly discuss. In Theorem 5.2 we obtain the Hardy-Littlewood type inequality

$$\sum_{\pi\in\widehat{G}_0} d_\pi k_\pi \langle\pi\rangle^{n(p-2)} \left(\frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}}\right)^p \leq C\|f\|^p_{L^p(G/K)}, \quad 1<p\leq 2, \quad (1.31)$$

on arbitrary compact homogeneous manifolds G/K, interpreting

$$\mu(Q)=\sum_{\pi\in Q} d_\pi k_\pi \qquad (1.32)$$

as the Plancherel measure on the set $\widehat{G}_0$, the 'unitary dual' of the homogeneous manifold G/K, and k_π the maximal rank of Fourier coefficients matrices $\widehat{f}(\pi)$, so that e.g. $\|\widehat{\delta}(\pi)\|_{\mathtt{HS}}=\sqrt{k_\pi}$ for the delta-function δ on G/K and $\pi\in\widehat{G}_0$.

Using the Hilbert-Schmidt norms of Fourier coefficients in (1.31) rather than Schatten norms (leading to a different version of ℓ^p-spaces on the unitary dual) leads to the sharper estimate [ANR15]

The exact form of (1.31) is justified in Section 5.1 by comparing the differential interpretations (5.4) and (5.15) of the classical Hardy-Littlewood inequality (5.1) and of (1.31), respectively. In fact, it is exactly from these differential interpretations is how we arrive at the desired expression in (1.31). Roughly, both are saying that for $1 < p \leq 2$,

$$g \in L^p_{2n(\frac{1}{p}-\frac{1}{2})}(G/K) \Longrightarrow \widehat{g} \in \ell^p(\widehat{G}_0) \tag{1.33}$$

with the corresponding norm estimate $\|\widehat{g}\|_{\ell^p(\widehat{G}_0)} \leq C\|g\|_{L^p_{2n(\frac{1}{p}-\frac{1}{2})}(G/K)}$, where $L^p_{2n(\frac{1}{p}-\frac{1}{2})}$ is the Sobolev space over L^p of order $2n(\frac{1}{p}-\frac{1}{2})$, and $\ell^p(\widehat{G}_0)$ is an appropriately defined Lebesgue space ℓ^p on the unitary dual $\widehat{G}_0$ of representations relevant to G/K, with respect to the corresponding Plancherel measure. In particular, as a special case we have the original Hardy-Littlewood inequality (5.1), which can be reformulated as

$$g \in L^p_{2(\frac{1}{p}-\frac{1}{2})}(\mathbb{T}) \Longrightarrow \widehat{g} \in \ell^p(\mathbb{Z}), \quad 1 < p \leq 2,$$

see (5.4), since $\ell^p(\widehat{\mathbb{T}}_0) \simeq \ell^p(\mathbb{Z})$, and the Plancherel measure is the counting measure on $\mathbb{Z}$ in this case.

By duality, the inequality (1.31) remains true (with the reversed inequality) also for $2 \leq p < \infty$. It has also been shown in [ANR16a] that the inequality (1.31) is sharp on SU_2. We refer to [ANR16a] for more details.

In Theorem 5.4 we show that the Paley inequality (5.3) and the Hausdorff-Young inequalities on G/K imply the Hausdorff-Young-Paley inequality.

This estimate becomes instrumental in obtaining L^p-L^q Fourier multiplier theorems on G/K for indices $1 < p \leq 2 \leq q < 2$.

In Section 5.3 we give such results for Fourier multipliers on G/K: for a Fourier multiplier A acting by $\widehat{Af}(\pi) = \sigma_A(\pi)\widehat{f}(\pi)$ and $1 < p \leq 2 \leq q < 2$ we have

$$\|A\|_{L^p(G/K)\to L^q(G/K)} \lesssim \sup_{s>0} \left\{ s\mu(\pi \in \widehat{G}_0 : \|\sigma_A(\pi)\|_{\mathtt{op}} > s)^{\frac{1}{p}-\frac{1}{q}} \right\},$$

where μ is the Plancherel measure as in (1.32), see Theorem 5.5.

Consequently, in Theorem 3.8 we also give a general $L^p(G)$-$L^q(G)$ boundedness result for general (not necessarily invariant) operators A on a compact Lie group G in terms of their matrix symbols $\sigma_A(x,\xi)$.

Main inequalities in Chapter 5 are established on general compact homogeneous manifolds of the form G/K, where G is a compact Lie group and K is a compact subgroup. Important examples are compact Lie groups themselves when we take the trivial subgroup $K=\{e\}$ in which case $k_\pi = d_\pi$, or spaces like spheres $\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ or complex spheres (projective spaces) $\mathbb{CS}^n = \mathrm{SU}(n+1)/\mathrm{SU}(n)$ in which cases the subgroups are massive and so $k_\pi = 1$ for all $\pi \in \widehat{G}_0$. We briefly describe such spaces and their representation theory in Section 5.1. When we want to show the sharpness of the obtained inequalities, we may restrict to the case of semisimple Lie groups G.

An analogue of the Hardy-Littlewood criterion for integrability of functions in $L^p(\mathrm{SU2})$ in terms of their Fourier coefficients has been obtained in [ANR16a]. This provides the converse to Hardy-Littlewood inequalities on SU_2 previously obtained by the authors in [ANR16a].

We shall use the symbol C to denote various positive constants, and $C_{p,q}$ for constants which may depend only on indices p and q. We shall write $x \lesssim y$ for the relation $|x| \leq C|y|$, and write $x \cong y$ if $x \lesssim y$ and $y \lesssim x$.

1.7 The model example of $\mathbb{S}^3$

We show how the approach developed in this thesis provides us with new results on $L^p - L^q$ bounds of global pseudo-differential operators on the three-dimensional sphere $\mathbb{S}^3$. It is known that $\mathbb{S}^3$ is globally diffeomorphic to SU_2. In other words, there is a global diffeomorphism $\Phi\colon \mathrm{SU}_2 \to \mathbb{S}^3$. Moreover, the mapping Φ is a global diffeomorphism. For more details we refer to [RT10].

We start by recalling some details on representation theory of SU_2. The unitary dual of SU_2 is

$$\widehat{\mathrm{SU}_2} = \{t^l(g), g \in \mathrm{SU}_2\}_{l \in \frac{1}{2}\mathbb{N}_0}, \tag{1.34}$$

where $t^l(g)$ is an irreducible (hence finite-dimensional) continuous unitary representation of SU_2 and its matrix components $t^l_{mn} \in C^\infty(\mathrm{SU}_2)$ can be expressed in terms of Legendre-Jacobi functions, see Vilenkin [Vil68]. It is also usual to vary indices m, n in the range from l to l, equi-spaced with step one.

Fourier series take the form

$$f = \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1) \operatorname{Tr} \widehat{f}(l), \; f \in L^2(\mathrm{SU}_2), \tag{1.35}$$

where

$$\widehat{f}(l) = \int\limits_{\mathrm{SU}_2} f(u) t^l(u)^* \, du. \tag{1.36}$$

The Peter-Weyl theorem on SU_2 implies that the Plancherel identity

$$\|f\|^2_{L^2(\mathrm{SU}_2)} = \sum_{l \in \frac{1}{2}} (2l+1) \|\widehat{f}(l)\|^2_{\mathtt{HS}} \tag{1.37}$$

holds true for all $f \in L^2(\mathrm{SU}_2)$.

The diffeomorphism Φ induces the Fourier analysis on $\mathbb{S}^3$. Hence, with suitable modifications, equations (1.34),(1.35),(1.36) and (1.37) remain valid on $\mathbb{S}^3$.

Let $A \colon C^\infty(\mathbb{S}^3) \to C^\infty(\mathbb{S}^3)$ be a continuous linear operator. Let $\sigma_A(l, u)$ be the full symbol of A defined as follows

$$\sigma_A(x, l) = t^l(x)^* (At^l)(x) \in \mathbb{C}^{(2l+1)\times(2l+1)}.$$

Then we have the representation of A in the form [RT10, Theorem 10.4.4]

$$Af(x) = \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1) \operatorname{Tr} \sigma_A(x, l) \widehat{f}(l) t^l(x),$$

We refer to [RT10, Section 12.5] for the details. Let $\{X_j\}_{j=1}^3$ be a basis for the Lie algebra of SU_2 and let ∂_j be the left-invariant vector fields corresponding to X_j. For $\alpha \in \mathbb{N}_0^3$, let us denote $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$.

Now, by Theorem 5.6, we get

$$\|A\|_{L^p(\mathbb{S}^3)\to L^q(\mathbb{S}^3)} \lesssim \sum_{|\alpha|\le l} \sup_{x\in\mathbb{S}^3} \sup_{s>0} s \left(\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0 \\ \|\partial_x^\alpha \sigma_A(x,l)\|_{\mathtt{op}}\ge s}} (2l+1)^2 \right)^{\frac{1}{p}-\frac{1}{q}}.$$

where $1 < p \le 2 \le q < \infty$.

If $A\colon C^\infty(\mathbb{S}^3) \to C^\infty(\mathbb{S}^3)$ is left-invariant, then we recover the Hörmander theorem multiplier theorem (Theorem 3.1) applied to SU_2.

Now, we illustrate Lizorkin multiplier theorem for left-invariant operators on $\mathbb{S}^3$ (see Theorem 3.4 below). Let A be a left Fourier multiplier on $\mathbb{S}^3$. It can be seen (see calculations in [RT10, p. 633] that

$$\langle t^l \rangle = \sqrt{1 + l(l+1)} \cong (2l+1),\ l \in \frac{1}{2}\mathbb{N}_0. \tag{1.38}$$

Assume that its global symbol $\sigma_A(l)$ satisfies the following

$$\sup_{l\in\frac{1}{2}\mathbb{N}_0} (2l+1)^{3(\frac{1}{p}-\frac{1}{q})} \|\sigma_A(l)\|_{\mathtt{op}} < \infty,$$
$$\sup_{l\in\frac{1}{2}\mathbb{N}_0} (2l+1)^{3(\frac{1}{p}-\frac{1}{q}+1)} \|\widehat{\partial}\sigma_A(l)\|_{\mathtt{op}} < \infty.$$

where $\widehat{\partial}$ is the difference operator acting on the singular numbers (see (3.34)). Then, by Theorem 3.4, $A\colon L^p(\mathbb{S}^3) \to L^q(\mathbb{S}^3)$ is a bounded linear operator and we have

$$\|A\|_{L^p(\mathbb{S}^3)\to L^q(\mathbb{S}^3)} \lesssim \sup_{l\in\frac{1}{2}\mathbb{N}_0} (2l+1)^{3(\frac{1}{p}-\frac{1}{q})} \|\sigma_A(l)\|_{\mathtt{op}} + \sup_{l\in\frac{1}{2}\mathbb{N}_0} (2l+1)^{3(\frac{1}{p}-\frac{1}{q}+1)} \|\widehat{\partial}\sigma_A(l)\|_{\mathtt{op}},$$

where $1 < p \le q < \infty$.

Chapter 2

Hausdorff-Young-Paley and Nikolskii inequalities

Let G be a locally compact unimodular separable group. We denote by $\pi_R(g)$ the right regular representation of G on $L^2(G)$, i.e. the unitary representations given by

$$\pi_R(g)\colon L^2(G) \ni f \mapsto f(\cdot g) \in L^2(G).$$

Let us denote by $\mathrm{VN}_R(G)$ be the right group von Neumann algebra generated by $\{\pi_R(g)$ i.e.

$$\mathrm{VN}_R(G) := \{\pi_R(g)\}_{g\in G}^{!!},$$

where !! is the double commutant.

The group Fourier transform $\mathcal{F}[f]$ is usually defined as the family of operators $\{\widehat{f}(\pi)\}_{\pi\in\widehat{G}}$ indexed by the elements $\pi \in \widehat{G}$. Nevertheless, following Kunze [Kun58] and Terp [T$^+$17], we take the "global view" and consider the Fourier transform $\mathcal{F}[f]$ of $f \in L^1(G)$ to be the operator R_f of right convolution.

For $f \in L^1(G)$, we say that f on G *has a Fourier transform* whenever the convolution operator

$$R_f h(x) := (h * f)(x) = \int\limits_G h(g) f(g^{-1}x)\, dg \tag{2.1}$$

is a τ-measurable operator with respect to $\mathrm{VN}_R(G)$. We shall sometimes interchangeably use notations $\mathcal{F}[f]$ or $\widehat{f}$ to denote the convolution operator R_f

$$\mathcal{F}[f] = \widehat{f} = R_f.$$

The Fourier transformation $\mathcal{F}$ maps $L^1(G)$ into the space $\mathrm{VN}_R(G)$, i.e.

$$\mathcal{F} \colon L^1(G) \ni f \mapsto \widehat{f} \in \mathrm{VN}_R(G).$$

For type I locally compact groups G, it is possible to recover the usual local definition of the Fourier coefficients.

Remark 2.1. Let G be a locally compact group of type I. Let $f \in L^1(G)$.

$$\widehat{f} = \bigoplus_{\widehat{G}_r} \int \widehat{f}(\pi) d\pi, \tag{2.2}$$

where $d\pi$ is the Plancherel measure and $\widehat{G}_r$ is the reduced unitary dual of G, i.e. the set of all unitary irreducible representations π weakly contained in π_R and $\widehat{f}(\pi)$ is the Fourier coefficient of f at $\pi \in \widehat{G}$, i.e.

$$\widehat{f}(\pi) = \int\limits_G f(x)\pi(x)^*\, dx, \tag{2.3}$$

The reduction theory [Dix81, Part II] applied to $\widehat{f}$ immediately yields Remark 2.1.

Example 2.1. Let $G = \mathbb{T}$ and $f \in L^1(\mathbb{T})$. Then the Fourier transformation $\widehat{f}$ is the convolution operator, i.e.

$$L^2(\mathbb{T}) \ni h \mapsto \widehat{f}(h) = f * h \in L^2(\mathbb{T}).$$

By the standard calculations, we get

$$\widehat{f} = \bigoplus_{n\in\mathbb{Z}} \widehat{f}(n),$$

where each Fourier coefficient $\widehat{f}(n) \in \mathbb{C}^{1\times 1}$ is regarded as one-dimensional operator.

By Young convolution inequality, we immediately obtain

$$\|\widehat{f}\|_{B(L^2(G))} \leq \|f\|_{L^1(G)}. \tag{2.4}$$

The Plancherel identity takes ([Seg50, Theorem 3 on page 282]) the form

$$\|\widehat{f}\|_{L^2(\mathrm{VN}_R(G))} = \|f\|_{L^2(G)}. \tag{2.5}$$

In this setting, the Hausdorff-Young inequality has been established in [Kun58]

$$\|\widehat{f}\|_{L^{p'}(\mathrm{VN}_R(G))} \leq \|f\|_{L^p(G)}, \quad 1 < p \leq 2. \tag{2.6}$$

In [H.K81], as an application of the technique of the t-th generalised singular values, the Hardy-Littlewood theorem ([HL27]) has been generalised to an arbitrary locally compact separable unimodular group G:

Theorem 2.1 ([H.K81])**.** Let $1 < p \leq 2$ and $f \in L^p(G)$. Then we have

$$\|\widehat{f}\|_{L^{p'}{}^{p}(\mathrm{VN}_R(G))} \leq \|f\|_{L^p(G)}. \tag{2.7}$$

Remark 2.2. By duality, the inequality (2.7) remains true (with the reversed inequality) also for $2 \leq p < \infty$, i.e.

$$\|f\|_{L^p(G)} \leq \|\widehat{f}\|_{L^{p'}{}^{p}(\mathrm{VN}_R(G))}, \quad 2 \leq p < \infty. \tag{2.8}$$

The Plancherel equality (2.5) by Segal [Seg50] and Kosaki's version [H.K81] of Hardy-Littlewood inequality (2.7) have been originally established for the left convolution $L_f h = f * h$. However, the same line of reasoning yields inequalities (2.7) and (2.5) with the right convolution R_f. We work with the right convolution operators R_f here since it naturally corresponds to left-invariant operators when analysing the Fourier multipliers on groups.

Using the technique of the t-th generalised singular values developed in [TK86], we can formulate both the Hausdorff-Young (2.6) and Hardy-Littlewood (2.7) inequalities in the forms (for $1 < p \leq 2$):

$$\left(\int_0^{+\infty} \mu_t(\widehat{f})^{p'} dt \right)^{\frac{1}{p'}} \equiv \|\widehat{f}\|_{L^{p'}(\mathrm{VN}_R(G))} \leq \|f\|_{L^p(G)}, \tag{2.9}$$

$$\left(\int_0^{+\infty} t^{p-2} \mu_t(\widehat{f})^p dt \right)^{\frac{1}{p}} \equiv \|\widehat{f}\|_{L^{p',p}(\mathrm{VN}_R(G))} \leq \|f\|_{L^p(G)}. \tag{2.10}$$

The Hardy-Littlewood inequalities (2.7) and (2.10) (for the right convolution R_f) will also follow as special cases of a more general Hausdorff-Young-Paley inequality. We prove the latter in Theorem 2.4.

2.1 Paley and Hausdorff-Young-Paley

Our analysis of L^p-L^q multipliers will be based on a general version of the Hausdorff-Young-Paley inequality. It will be obtained by interpolation between the Hausdorff-Young inequality and Paley inequality that we discuss first.

We start first with an inequality that can be regarded as a Paley type inequality.

Theorem 2.2 (Paley inequality). Let G be a locally compact unimodular separable group. Let $1 < p \leq 2$. Suppose that a positive function

$\varphi(t)$ satisfies the condition

$$M_\varphi := \sup_{s>0} s \int\limits_{\substack{t\in\mathbb{R}_+ \\ \varphi(t)\geq s}} dt < +\infty. \tag{2.11}$$

Then for all $f \in L^p(G)$ we have

$$\left(\int\limits_0^{+\infty} \mu_t(\widehat{f})^p \varphi(t)^{2-p}\, dt \right)^{\frac{1}{p}} \leq M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(G)}. \tag{2.12}$$

As usual, the integral over an empty set in (2.11) is assumed to be zero.

We note that taking $\varphi(t) = \frac{1}{t}$ we recover Kosaki's Hardy-Littlewood inequality in Theorem 2.1. In this sense, the Paley inequality can be viewed as an extension of (one of) the Hardy-Littlewood inequalities.

Proof of Theorem 2.2. Let ν be such measure that its Radon-Nikodym derivative at t is given by $\varphi^2(t)$, i.e. for any measurable subset $B \subset \mathbb{R}_+$ we have

$$\nu(B) = \int\limits_B \varphi^2(t)\, dt. \tag{2.13}$$

We define the corresponding space $L^p(\mathbb{R}_+, \nu)$, $1 \leq p < \infty$, as the space of complex (or real) valued functions $f = f(t)$ such that

$$\|f\|_{L^p(\mathbb{R}_+,\nu)} := \left(\int\limits_{\mathbb{R}_+} |f(t)|^p \varphi^2(t)\, dt \right)^{\frac{1}{p}} < \infty. \tag{2.14}$$

We will show that the sub-linear operator

$$T\colon L^p(G) \ni f \mapsto Tf := \mu_t(\widehat{f})/\varphi(t) \in L^p(\mathbb{R}_+, \nu)$$

is well-defined and bounded from $L^p(G)$ to $L^p(\mathbb{R}_+, \nu)$ for $1 < p \leq 2$. In other words, we claim that we have the estimate

$$\|Tf\|_{L^p(\mathbb{R}_+,\nu)} = \left(\int_{\mathbb{R}_+} \left(\frac{\mu_t(\widehat{f}\,)}{\varphi(t)} \right)^p \varphi^2(t)\, dt \right)^{\frac{1}{p}} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(G)}, \quad (2.15)$$

which would give (2.12), and where we set $M_\varphi := \sup_{t>0} t \int\limits_{\substack{t\in\mathbb{R}_+ \\ \varphi(t)\geq s}} dt$. We will show that T is of weak-type (2,2) and of weak-type (1,1). More precisely, with the distribution function ν , we show that

$$\nu_{\mathbb{R}_+}(y; Tf) \leq \left(\frac{M_2\|f\|_{L^2(G)}}{y} \right)^2 \quad \text{with norm } M_2 = 1, \quad (2.16)$$

$$\nu\mathbb{R}_+(y; Tf) \leq \frac{M_1\|f\|_{L^1(G)}}{y} \quad \text{with norm } M_1 = M_\varphi, \quad (2.17)$$

where $\nu_{\mathbb{R}_+}$ is defined as

$$\nu_{\mathbb{R}_+}(y; Tf) = \int\limits_{\substack{t\in\mathbb{R}_+ \\ \frac{\mu_t(\widehat{f}\,)}{\varphi(t)}\geq y}} \varphi^2(t)\, dt. \quad (2.18)$$

Then (2.15) would follow from (2.16) and (2.17) by the Marcinkiewicz interpolation theorem.

By the Plancherel formula, T is of strong type $(2, 2)$, i.e. inequality (2.16) holds true. Therefore, we concentrate on showing that T is of weak-type $(1, 1)$. More precisely, we show that

$$\nu\{t \in \mathbb{R}_+ \colon \frac{\mu_t(\widehat{f}\,)}{\varphi(t)} > y\} \lesssim M_\varphi \frac{\|f\|_{L^1(G)}}{y}. \quad (2.19)$$

The left-hand side here is the integral $\int \varphi^2(t)\, dt$ taken over those $t \in \mathbb{R}_+$ for which $\frac{\mu_t(\widehat{f}\,)}{\varphi(t)} > y$ (see (2.18)).

It can be easily seen that

$$\mu_t(\widehat{f}\,) \leq \|f\|_{L^1(G)}. \quad (2.20)$$

Indeed, from the Definition A.6, we have

$$\mu_t(\widehat{f}) \leq \|\widehat{f}\|_{B(L^2(G))} \leq \|f\|_{L^1(G)},$$

where we used (2.4). This shows (2.20).

Therefore, we have

$$y < \frac{\mu_t(\widehat{f})}{\varphi(t)} \leq \frac{\|f\|_{L^1(G)}}{\varphi(t)}.$$

Using this, we get

$$\left\{t \in \mathbb{R}_+ \colon \frac{\mu_t(\widehat{f})}{\varphi(t)} > y\right\} \subset \left\{t \in \mathbb{R}_+ \colon \frac{\|f\|_{L^1(G)}}{\varphi(t)} > y\right\}$$

for any $y > 0$. Consequently,

$$\nu\left\{t \in \mathbb{R}_+ \colon \frac{\mu_t(\widehat{f})}{\varphi(t)} > y\right\} \leq \nu\left\{t \in \mathbb{R}_+ \colon \frac{\|f\|_{L^1(G)}}{\varphi(t)} > y\right\}.$$

Setting $v := \frac{\|f\|_{L^1(G)}}{y}$, we get

$$\nu\left\{t \in \mathbb{R}_+ \colon \frac{\mu_t(\widehat{f})}{\varphi(t)} > y\right\} \leq \int\limits_{\substack{t\in\mathbb{R}_+ \\ \varphi(t)\leq v}} \varphi^2(t)\, dt. \tag{2.21}$$

We now claim that

$$\int\limits_{\substack{t\in\mathbb{R}_+ \\ \varphi(t)\leq v}} \varphi^2(t)\, dt \lesssim M_\varphi v. \tag{2.22}$$

Indeed, first we notice that we have

$$\int\limits_{\substack{t\in\mathbb{R}_+ \\ \varphi(t)\leq v}} \varphi^2(t)\, dt = \int\limits_{\substack{t\in\mathbb{R}_+ \\ \varphi(t)\leq v}} dt \int\limits_0^{\varphi^2(t)} d\tau.$$

We can interchange the order of integration to get

$$\int\limits_{\substack{t\in\mathbb{R}_+\\ \varphi(t)\leq v}} dt \int\limits_{0}^{\varphi^2(t)} d\tau = \int\limits_{0}^{v^2} d\tau \int\limits_{\substack{t\in\mathbb{R}_+\\ \tau^{\frac{1}{2}}\leq\varphi(t)\leq v}} dt.$$

Further, we make a substitution $\tau = s^2$, yielding

$$\int\limits_{0}^{v^2} d\tau \int\limits_{\substack{t\in\mathbb{R}_+\\ \tau^{\frac{1}{2}}\leq\varphi(t)\leq v}} dt = 2\int\limits_{0}^{v} s\,ds \int\limits_{\substack{s\in\mathbb{R}_+\\ s\leq\varphi(t)\leq v}} dt \leq 2\int\limits_{0}^{v} s\,ds \int\limits_{\substack{t\in\mathbb{R}_+\\ s\leq\varphi(t)}} dt.$$

Since

$$s \int\limits_{\substack{t\in\mathbb{R}_+\\ s\leq\varphi(t)}} dt \leq \sup_{s>0} s \int\limits_{\substack{t\in\mathbb{R}_+\\ s\leq\varphi(t)}} dt = M_\varphi$$

is finite by the assumption that $M_\varphi < \infty$, we have

$$2\int\limits_{0}^{v} s\,ds \int\limits_{\substack{t\in\mathbb{R}_+\\ s\leq\varphi(t)}} dt \lesssim M_\varphi v.$$

This proves (5.31) and hence also (5.29). Thus, we have proved inequalities (2.16) and (2.17). Then by using the Marcinkiewicz interpolation theorem with $p_1 = 1$, $p_2 = 2$ and $\frac{1}{p} = 1 - \theta + \frac{\theta}{2}$ we now obtain

$$\left(\int\limits_{\mathbb{R}_+} \left(\frac{\mu_t(\widehat{f})}{\varphi(t)}\right)^p \varphi^2(t)dt\right)^{\frac{1}{p}} = \|Af\|_{L^p(\mathbb{R}_+,\nu)} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(G)}.$$

This completes the proof of Theorem 2.2. □

Further, we recall a result on the interpolation of weighted spaces from [BL76]: Let $d\mu_0(x) = \omega_0(x)d\mu(x)$ and $d\mu_1(x) = \omega_1(x)d\mu(x)$. We write $L^p(\omega) = L^p(\omega d\mu)$ for the weight ω.

Theorem 2.3 (Interpolation of weighted spaces)**.** Let $0 < p_0, p_1 < \infty$. Then

$$(L^{p_0}(\omega_0), L^{p_1}(\omega_1))_{\theta,p} = L^p(\omega),$$

where $0<\theta<1$, $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$, and $\omega=\omega_0^{p\frac{1-\theta}{p_0}}\omega_1^{p\frac{\theta}{p_1}}$.

From this, interpolating between the Paley-type inequality (2.12) in Theorem 2.2 and Hausdorff-Young inequality (2.9), we readily obtain an inequality that will be crucial for our consequent analysis of L^p-L^q multipliers:

Theorem 2.4 (Hausdorff-Young-Paley inequality). Let G be a locally compact unimodular separable group. Let $1<p\leq b\leq p'<\infty$. If a positive function $\varphi(t)$, $t\in\mathbb{R}_+$, satisfies condition

$$M_\varphi := \sup_{s>0} s \int\limits_{\substack{t\in\mathbb{R}_+\\ \varphi(t)\geq s}} dt<\infty, \tag{2.23}$$

then for all $f\in L^p(G)$ we have

$$\left(\int\limits_{\mathbb{R}_+}\left(\mu_t(\widehat{f})\varphi(t)^{\frac{1}{b}-\frac{1}{p'}}\right)^b dt\right)^{\frac{1}{b}} \lesssim M_\varphi^{\frac{1}{b}-\frac{1}{p'}}\|f\|_{L^p(G)}. \tag{2.24}$$

Naturally, this reduces to the Hausdorff-Young inequality (2.9) when $b=p'$ and to the Paley inequality in (2.12) when $b=p$.

2.2 Nikolskii inequality

In this section we establish the Nikolskii inequality (sometimes called the reverse Hölder inequality) in the setting of locally compact groups. This complements the knowledge on locally compact groups as well as gives an extension of known results on compact and on nilpotent Lie groups.

Let us denote by $\mathrm{supp}(\widehat{f})$ the subspace of $L^2(G)$ orthogonal to the kernel $\mathrm{Ker}(\widehat{f})$ of the Fourier transform $\widehat{f}$, i.e.

$$\mathrm{supp}[\widehat{f}] := \mathrm{Ker}\left(\widehat{f}\right)^{\perp}, \tag{2.25}$$

where $\mathrm{Ker}(\widehat{f})\subset L^2(G)$ is the kernel of the operator $\widehat{f}$.

We note that the classical Nikolskii inequality is an L^p-L^q estimate for norms of the same functions for $p < q$ provided that the Fourier transforms of the functions under consideration have bounded support.

In [NRT16, NRT15] Nikolskii inequality has been established on compact Lie groups and on compact homogeneous manifolds, respectively, for functions with bounded support of the noncommutative Fourier coefficients.

The main question in the setting of locally compact groups is to find an analogue of the condition for bounded support of Fourier transforms since we may not have a canonical operator to use its spectral decomposition for the definition of the bounded spectrum.

Let $P_{\mathrm{supp}[\widehat{f}]}$ be the orthogonal projector associated to the support $\mathrm{supp}[\widehat{f}]$. We say that $f \in L^1(G)$ has *bounded spectrum* if $\tau(P_{\mathrm{supp}[\widehat{f}]}) < +\infty$.

Theorem 2.5. Let G be a locally compact separable unimodular group. Let $1 < q \leq \infty$ and $1 < p \leq \min(2, q)$. Assume that $\tau(P_{\mathrm{supp}[\widehat{f}]}) < \infty$. Then we have

$$\|f\|_{L^q(G)} \leq \left(\tau(P_{\mathrm{supp}[\widehat{f}]})\right)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^p(G)}, \tag{2.26}$$

with the constant in (2.26) independent of f.

In [NRT16, NRT15] Nikolskii inequality has been established on compact Lie groups and on compact homogeneous manifolds, respectively. We prove Theorem 2.5 along the lines of the proof in [NRT16] adapting the latter to the setting of locally compact groups.

Proof of Theorem 2.5. We will give the proof of (2.26) in three steps.

Step 1. The case $p = 2$ and $q = \infty$. We have (by e.g. [Hay14, Proposition A.1.2. p. 216]) that

$$\left|\mathrm{Tr}(\widehat{f}(\pi)\pi(x))\right| \leq \mathrm{Tr}\left|\widehat{f}(\pi)\right|, \quad x \in G. \tag{2.27}$$

We notice that

$$\mathcal{F}[f]P_{\mathrm{supp}[\widehat{f}]} = \mathcal{F}[f].$$

Then by [TK86, Lemma 2.6, p. 277], we have

$$\mu_s(\mathcal{F}[f]) = 0, \quad s \geq \tau(P_{\mathrm{supp}[\widehat{f}]}). \tag{2.28}$$

From now on we shall denote $t := \tau(P_{\mathrm{supp}[\widehat{f}]})$ throughout the proof. Further, the application of [TK86, Proposition 2.7, p.277] yields

$$\tau(|\mathcal{F}[f]|) = \int\limits_0^\infty \mu_s(\mathcal{F}[f])\, ds = \int\limits_0^t \mu_s(\mathcal{F}[f])\, ds, \tag{2.29}$$

where we used (2.28) in the last equality. Combining (2.29) and (2.27), we obtain

$$\begin{aligned} \|f\|_{L^\infty(G)} &\leq \int\limits_{\widehat{G}} \mathrm{Tr}\left|\widehat{f}(\pi)\right| d\pi = \tau(|\mathcal{F}[f]|) = \int\limits_0^t \mu_s(\mathcal{F}[f]) ds \\ &\leq \left(\int\limits_0^t ds\right)^{\frac{1}{2}} \left(\int\limits_0^t \mu_s^2(\mathcal{F}[f]) ds\right)^{\frac{1}{2}} = \sqrt{\tau(P_{\mathrm{supp}[\widehat{f}]})}\|f\|_{L^2(G)}, \end{aligned} \tag{2.30}$$

where in the last inequality we used the Plancherel identity. Step 2. The case $p = 2$ and $2 < q \leq \infty$. We take $1 \leq q' < 2$ so that $\frac{1}{q} + \frac{1}{q'} = 1$. We set $r := \frac{2}{q'}$ so that its dual index r' satisfies $\frac{1}{r'} = 1 - \frac{q'}{2}$. By the Hausdorff-

Young inequality in (2.9), and by Hölder's inequality, we obtain

$$
\begin{aligned}
\|f\|_{L^q(G)} \le \|\mathcal{F}[f]\|_{L^{q'}(\mathrm{VN}_R(G))} &= \left(\int_0^t \mu_s^{q'}(\mathcal{F}[f])ds\right)^{\frac{1}{q'}} \\
&\le \left(\int_0^t ds\right)^{\frac{1}{q'r'}} \left(\int_0^t \mu_s^{q'r}(\mathcal{F}[f])\, ds\right)^{\frac{1}{q'r}} \\
&= \left(\int_0^t ds\right)^{\frac{1}{q'}-\frac{1}{2}} \left(\int_0^t \mu_s^{2}(\mathcal{F}[f])\, ds\right)^{\frac{1}{2}} \\
&\le \left(\int_0^t ds\right)^{\frac{1}{q'}-\frac{1}{2}} \left(\int_0^\infty \mu_s^{2}(\mathcal{F}[f])\, ds\right)^{\frac{1}{2}} \\
&= \tau(P_{\mathrm{supp}[\widehat{f}]})^{\frac{1}{2}-\frac{1}{q}}\|f\|_{L^2(G)},
\end{aligned}
$$

where we have used that $\frac{q'r}{2}=1$.

<u>Step 3.</u> If $p=\min(2,q)$ and $p\neq 2$, then $p=q$ and there is nothing to prove. For $1<p<\min(2,q)$, we claim to have

$$\|f\|_{L^q(G)} \le \tau(P_{\mathrm{supp}[\widehat{f}]})^{(1/p-1/q)}\|f\|_{L^p(G)}.$$

Indeed, if $q=\infty$, for $f\not\equiv 0$, we get

$$
\begin{aligned}
\|f\|_{L^2} &= \||f|^{1-p/2}|f|^{p/2}\|_{L^2} \le \||f|^{1-p/2}\|_{L^\infty}\||f|^{p/2}\|_{L^2} \\
&= \|f\|_{L^\infty}^{1-p/2}\||f|^{p/2}\|_{L^2} = \|f\|_{L^\infty}\|f\|_{L^\infty}^{-p/2}\||f|^{p/2}\|_{L^2} \\
&= \|f\|_{L^\infty}\|f\|_{L^\infty}^{-p/2}\|f\|_{L^p}^{p/2} \\
&\le \tau(P_{\mathrm{supp}[\widehat{f}]})^{1/2}\|f\|_{L^2}\|f\|_{L^\infty}^{-p/2}\|f\|_{L^p}^{p/2},
\end{aligned}
\tag{2.31}
$$

where we have used (2.30) in the last line. Therefore, using that $f\not\equiv 0$, we have

$$\|f\|_{L^\infty} \le \tau(P_{\mathrm{supp}[\widehat{f}]})^{1/p}\|f\|_{L^p}. \tag{2.32}$$

For $p < q < \infty$ we obtain

$$\begin{aligned}\|f\|_{L^q} &= \||f|^{1-p/q}|f|^{p/q}\|_{L^q} \leq \|f\|_{L^\infty}^{1-p/q}\|f\|_{L^p}^{p/q} \\ &\leq \tau(P_{\mathrm{supp}[\widehat{f}]})^{1/p(1-p/q)}\|f\|_{L^p}^{1-p/q}\|f\|_{L^p}^{p/q} = \tau(P_{\mathrm{supp}[\widehat{f}]})^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^p},\end{aligned} \tag{2.33}$$

where we have used (2.32). □

2.2.1 Compact case

We shall formulate and prove a slight variation (Proposition 2.1 and Proposition 2.2) of results in [NRT16] . These statements are used later in the proof of Theorem 3.4. Let G be a compact Lie group of dimension n. We label the equivalence classes of irreducible unitary representations of G as π^k with $k \in \mathbb{N}$ in such a way that the corresponding sequence of eigenvalues λ_k of $(I - \Delta_G)$ is non-decreasing. More precisely, we have

$$(I - \Delta_G)^{\frac{\dim(G)}{2}} \pi_{ij}^k = \lambda_k \pi_{ij}^k \tag{2.34}$$

holds for all $1 \leq i, j \leq d_{\pi^k}$. From the Weyl asymptotic formula for the eigenvalue counting function we get that

$$\lambda_k \cong \langle \pi^k \rangle^n \cong k, \qquad \text{with } n = \dim G. \tag{2.35}$$

Proposition 2.1. Let $1 \leq p \leq q \leq \infty$ and let G be a compact Lie group. Let $f_N = \sum\limits_{k=1}^{N} d_{\pi^k} \operatorname{Tr} \widehat{f}(\pi^k)\pi^k$, $N \in \mathbb{N}$. Then we have

$$\|f_N\|_{L^q(G)} \lesssim \lambda_N^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^p(G)}. \tag{2.36}$$

Proof of Proposition 2.1. It has been shown in [NRT16] that for every trigonometric polynomial

$$f_L = \sum_{\substack{\xi \in \widehat{G} \\ \langle \xi \rangle \leq L}} d_\pi \operatorname{Tr}(\widehat{f}(\xi)\xi) \tag{2.37}$$

a version of the Nikolskii-Bernshtein inequality can be written as

$$\|f_L\|_{L^q(G)} \leq N(\rho L)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^p(G)}, \quad 1 < p < q \leq \infty$$

where

$$N(\rho L) := \sum_{\substack{\xi \in \widehat{G} \\ \langle \xi \rangle \leq \rho L}} d_\xi^2, \quad \rho = \min(1, [p/2]),$$

where $[p/2]$ is the integer part of $p/2$. The application of Weyl's asymptotic law to the counting function $N(\rho L)$ of the n-th order elliptic pseudo-differential operator $(I - \mathcal{L}_G)^{\frac{1}{2}}$ with $L = \langle \pi \rangle$ yields

$$N(\rho \langle \pi \rangle) \cong \rho^n \langle \pi \rangle^n, \quad n = \dim(G).$$

Hence, we immediately obtain

$$\|f_{\langle \pi \rangle}\|_{L^q(G)} \lesssim \langle \pi \rangle^{n\left(\frac{1}{p}-\frac{1}{q}\right)} \|f_{\langle \pi \rangle}\|_{L^p(G)}.$$

Using notation in (2.35) and (2.34), we immediately get

$$\|f_N\|_{L^q(G)} \leq \lambda_N^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^p(G)}, \quad 1 < p < q \leq \infty. \tag{2.38}$$

This completes the proof of (2.36). □

Proposition 2.2. Let $1 \leq p < q \leq \infty$ and let G be a compact Lie group. Then we have

$$\sum_{k \in \mathbb{N}} \frac{\|f_k\|_{L^q(G)}}{\lambda_k^{\frac{1}{p}-\frac{1}{q}}} \frac{1}{\lambda_k} \leq \|f\|_{L^p(G)}, \tag{2.39}$$

where $\{\lambda_k\}_{k \in \mathbb{N}}$ are the eigenvalues of $(I - \Delta_G)^{\frac{\dim G}{2}}$.

Proof of Proposition 2.2. Let $p_0 < p < p_1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $0 < \theta < 1$. Let us take

$$\alpha = (1 - \theta) \left(\frac{1}{p_0} - \frac{1}{p_1} \right) \tag{2.40}$$

and

$$\beta = \frac{1}{p_0} - \frac{1}{q}. \tag{2.41}$$

Standart calculation yields

$$\beta - \alpha = \frac{1}{p} - \frac{1}{q}. \tag{2.42}$$

Let us denote by $\overline{f}(t)$ the quantity given by

$$\overline{f}(t) = \sup_{\lambda_n \geq t} \frac{\|f_n\|_{L^q(G)}}{\lambda_n^{\beta}}, \quad t \in \mathbb{R}_+. \tag{2.43}$$

Using (2.42),(2.43) and the fact that the function $t \to \overline{f}(t)$ is non-increasing function of $t > 0$, we get

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\|f_k\|_{L^q(G)}}{\lambda_k^{\frac{1}{p}-\frac{1}{q}}} \frac{1}{\lambda_k} &= \sum_{k=1}^{\infty} \lambda_k^{\alpha} \frac{\|f_k\|_{L^q(G)}}{\lambda_k^{\beta}} \frac{1}{\lambda_k} \leq \sum_{k=1}^{\infty} \lambda_k^{\alpha} \overline{f}(\lambda_k) \frac{1}{\lambda_k} \\
&= \sum_{s \in \mathbb{Z}} \sum_{k \in \mathbb{N}:\, 2^s \leq \lambda_k \leq 2^{s+1}} \lambda_k^{\alpha} \overline{f}(\lambda_k) \frac{1}{\lambda_k} \\
\leq \sum_{s \in \mathbb{Z}} 2^{\alpha(s+1)} \overline{f}(2^s) \sum_{k \in \mathbb{N}:\, 2^s \leq \lambda_k \leq 2^{s+1}} \frac{1}{\lambda_k} &= 2^{2\alpha} \sum_{s \in \mathbb{Z}} 2^{\alpha(s-1)} \overline{f}(2^s) \\
\leq 2^{2\alpha} \sum_{s \in \mathbb{Z}} \int_{2^{s-1}}^{2^s} t^{\alpha} \overline{f}(t) \int_{2^{s-1}}^{2^s} \frac{dt}{t} &= 2^{2\alpha} \int_0^{\infty} t^{\alpha} \overline{f}(t) \frac{dt}{t} \\
&= 2^{2\alpha} \int_0^{\infty} t^{(1-\theta)\alpha_1} \overline{f}(t) \frac{dt}{t},
\end{aligned} \tag{2.44}$$

where in the last line we denote

$$\alpha_1 = \frac{1}{p_0} - \frac{1}{p_1}. \tag{2.45}$$

Hence, making substitution $t = v^{\frac{1}{\alpha_1}}$, we get from (2.44)

$$\sum_{k=1}^{\infty} \frac{\|f_k\|_{L^q(G)}}{\lambda_k^{\frac{1}{p}-\frac{1}{q}}} \frac{1}{\lambda_k} \leq \int_0^{\infty} t^{-\theta\alpha_1} t^{\alpha_1} \overline{f}(t) \frac{dt}{t}$$
$$\cong \int_0^{\infty} v^{-\theta} v \overline{f}(v^{\frac{1}{\alpha_1}}) \frac{dv}{v} \leq \int_0^{\infty} v^{-\theta} \left(\sup_{u \leq v^{\frac{1}{\alpha_1}}} u^{\alpha_1} \overline{f}(u) \right) \frac{dv}{v}. \tag{2.46}$$

Let $f = f^0 + f^1 \in L^{p_0} + L^{p_1}$ be an arbitrary decomposition. We shall show that

$$\left(\sup_{u \leq v^{\frac{1}{\alpha_1}}} u^{\alpha_1} \overline{f}(u) \right) \leq \|f^0\|_{L^{p_0}} + v\|f^1\|_{L^{p_1}}. \tag{2.47}$$

It is sufficient to establish

$$\overline{f}(\lambda_k) \leq \|f\|_{L^{p_0}(G)}, \tag{2.48}$$

$$\lambda_k^{\alpha_1} \overline{f}(\lambda_k) \leq \|f\|_{L^{p_1}(G)}, \tag{2.49}$$

where we used By Nikolskii-type inequality (2.36), we have

$$\|f_k\|_{L^q(G)} \leq \lambda_k^{\frac{1}{p_0}-\frac{1}{q}} \|f\|_{L^{p_0}(G)}, \tag{2.50}$$

$$\|f_k\|_{L^q(G)} \leq \lambda_k^{\frac{1}{p_1}-\frac{1}{q}} \|f\|_{L^{p_1}(G)}. \tag{2.51}$$

Rescalling in the second inequality in (2.50), we get

$$\|f_k\|_{L^q(G)} \leq \lambda_k^{\frac{1}{p_0}-\frac{1}{q}} \|f\|_{L^{p_0}(G)}, \tag{2.52}$$

$$\|f_k\|_{L^q(G)} \leq \lambda_k^{\frac{1}{p_1}-\frac{1}{p_0}} \lambda_k^{\frac{1}{p_0}-\frac{1}{q}} \|f\|_{L^{p_1}(G)}. \tag{2.53}$$

Thus, taking $\alpha_1 = \frac{1}{p_0} - \frac{1}{p_1}$ and using (2.43), we rewrite inequalities (2.52) and (2.53) as follows

$$\frac{\|f_k\|_{L^q(G)}}{\lambda_k^{\frac{1}{p_0}-\frac{1}{q}}} \leq \|f\|_{L^{p_0}(G)}, \tag{2.54}$$

$$\lambda_k^{\alpha_1} \frac{\|f_k\|_{L^q(G)}}{\lambda_k^{\frac{1}{p_0}-\frac{1}{q}}} \leq \|f\|_{L^{p_1}(G)}. \tag{2.55}$$

Recalling notation (2.40), (2.41) and (2.43), we get from (2.54) and (2.55)

$$\begin{aligned}\overline{f}(\lambda_k) &\leq \|f\|_{L^{p_0}(G)},\\ \lambda_k^{\alpha_1}\overline{f}(\lambda_k) &\leq \|f\|_{L^{p_1}(G)}.\end{aligned}$$

Thus, we have established inequality (2.48) and inequality (2.49). Now, we shall use these estimates to obtain (2.47). Let $u > 0$. Then there is $k \in \mathbb{N}$ such that

$$\lambda_k \leq u \leq \lambda_{k+1}. \tag{2.56}$$

Since $\overline{f}(u)$ is non-increasing, we get

$$\overline{f}(u) \leq \overline{f}(\lambda_k) \leq \|f\|_{L^{p_0}(G)}, \tag{2.57}$$

where we used estimate (2.48) in the second inequality.

Using (2.35), we get

$$\frac{\lambda_{k+1}}{\lambda_k} \leq C,$$

and hence

$$u^{\alpha_1}\overline{f}(u) \leq \lambda_{k+1}^{\alpha_1}\overline{f}(\lambda_k) \lesssim \lambda_k^{\alpha_1}\overline{f}(\lambda_k) \leq \|f\|_{L^{p_1}(G)}, \tag{2.58}$$

where we used estimate (2.49) in the last inequality.

Let $f = f^0 + f^1$ be an arbitrary decomposition. From (2.57) and (2.58) we obtain

$$\begin{aligned}\sup_{u \leq v^{\frac{1}{\alpha_1}}} u^{\alpha_1}\overline{f}(u) &\leq \sup_{u \leq v^{\frac{1}{\alpha_1}}} u^{\alpha_1}\overline{f^0}(u) + \sup_{u \leq v^{\frac{1}{\alpha_1}}} u^{\alpha_1}\overline{f^1}(u)\\ \leq \sup_{u \leq v^{\frac{1}{\alpha_1}}} u^{\alpha_1}\overline{f^1}(u) &+ v\sup_{u>0}\overline{f^0}(u) \leq \|f^1\|_{L^{p_1}(G)} + v\|f^0\|_{L^{p_0}(G)}.\end{aligned}$$

Thus, we have just shown the validity of (2.47). Since the decomposition $f = f^0 + f^1$ is arbitrary, we take the infimum and get

$$\sup_{u \leq v^{\frac{1}{\alpha_1}}} u^{\alpha_1}\overline{f}(u) \leq K(t, f; L^{p_1}(G), L^{p_0}(G)), \tag{2.59}$$

where the functional $K(t, f)$ is given by

$$K(v, f; L^{p_1}(G), L^{p_0}(G)), = \inf_{f=f^0+f^1} \left\{ \|f^1\|_{L^{p_1}(G)} + v\|f^0\|_{L^{p_0}(G)} \right\}. \quad (2.60)$$

Composing (2.44) and (2.59), we finally obtain

$$\sum_{k=1}^{\infty} \frac{\|f_k\|_{L^q(G)}}{\lambda_k^{\frac{1}{p}-\frac{1}{q}}} \frac{1}{\lambda_k} \leq \|f\|_{L^p(G)}, \quad (2.61)$$

where in the last equality we used that $L^p(G)$ are the interpolation spaces and the embedding of the Lorentz spaces. This completes the proof. $\square$

Chapter 3

Fourier multipliers

3.1 Hörmander's multiplier theorem

In the following statements, to unite the formulations, we adopt the convention that the sum or the integral over an empty set is zero, and that $0^0 = 0$.

Theorem 3.1. Let $1 < p \leq 2 \leq q < +\infty$ and suppose that A is a Fourier multiplier on a locally compact separable unimodular group G. Then we have

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \sup_{s>0} s \left[\int\limits_{t\in\mathbb{R}_+ :\, \mu_t(A)\geq s} dt \right]^{\frac{1}{p}-\frac{1}{q}}. \tag{3.1}$$

For $p = q = 2$ inequality (3.1) is sharp, i.e.

$$\|A\|_{L^2(G)\to L^2(G)} = \sup_{t\in\mathbb{R}_+} \mu_t(A). \tag{3.2}$$

Using the noncommutative Lorentz spaces $L^{r,\infty}$ with $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, $p \neq q$, we can also write (3.1) as

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \|A\|_{L^{r,\infty}(VN_R(G))}. \tag{3.3}$$

Proof of Theorem 3.1. Since the algebra $\mathrm{Aff}(\mathrm{VN}_R(G))$ of left Fourier multipliers A is closed under taking the adjoint $\mathrm{Aff}(\mathrm{VN}_R(G)) \ni A \mapsto A^* \in \mathrm{Aff}(\mathrm{VN}_R(G))$ (see [Seg53, Theorem 4, p. 412] or [Ter81, Theorem 28 on p. 4]), and

$$\|A\|_{L^p(G)\to L^q(G)} = \|A^*\|_{L^{q'}(G)\to L^{p'}(G)}, \tag{3.4}$$

we may assume that $p \leq q'$, for otherwise we have $q' \leq (p')' = p$ and use (A.13) ensuring that $\mu_t(A^*) = \mu_t(A)$. When $f \in L^p(G)$, dualising the Hausdorff-Young inequality (2.9) gives, since $q' \leq 2$,

$$\|Af\|_{L^q(G)} \leq \left(\int_0^{+\infty} [\mu_t(R_{Af})]^{q'} dt \right)^{\frac{1}{q'}}. \tag{3.5}$$

By the left-invariance of A (e.g. [Ter80, Proposition 3.1 on page 31]) we have

$$R_{Af} = AR_f, \quad f \in L^2(G).$$

By our assumptions, A and R_f are measurable with respect to $\mathrm{VN}_R(G)$. This makes it possible to apply Lemma A.2 to obtain the estimate

$$\mu_t(R_{Af}) = \mu_t(AR_f) \leq \mu_{\frac{t}{2}}(A)\mu_{\frac{t}{2}}(R_f). \tag{3.6}$$

Thus, we obtain

$$\|Af\|_{L^q(G)} \lesssim \left(\int_0^{+\infty} [\mu_t(A)\mu_t(R_f)]^{q'} dt \right)^{\frac{1}{q'}}. \tag{3.7}$$

Now, we are in a position to apply the Hausdorff-Young-Paley inequality in Theorem 2.4. With $\varphi(t) = \mu_t(A)^r$ for $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, the assumptions of Theorem 2.4 are then satisfied, and since $\frac{1}{q'} - \frac{1}{p'} = \frac{1}{p} - \frac{1}{q} = \frac{1}{r}$, we obtain

$$\left(\int_0^{+\infty} [\mu_t(R_f)\mu_t(A)]^{q'} dt \right)^{\frac{1}{q'}} \leq \sup_{s>0} \left[s \int\limits_{\substack{t\in\mathbb{R}_+ \\ \mu_t(A)^r \geq s}} dt \right]^{\frac{1}{r}} \|f\|_{L^p(G)}. \tag{3.8}$$

Further, it can be easily checked that

$$\left(\sup_{s>0} s \int\limits_{\substack{t\in\mathbb{R}_+ \\ \mu_t(A)^r\geq s}} dt\right)^{\frac{1}{r}} = \left(\sup_{s>0} s^r \int\limits_{\substack{t\in\mathbb{R}_+ \\ \mu_t(A)\geq s}} dt\right)^{\frac{1}{r}} = \sup_{s>0} s \left(\int\limits_{\substack{t\in\mathbb{R}_+ \\ \mu_t(A)\geq s}} dt\right)^{\frac{1}{r}}. \tag{3.9}$$

Thus, we have established inequality (3.1). This completes the proof. □

3.1.1 The case of compact Lie groups

We compare Theorem 3.1 with known results in the case of G being a compact Lie group. The global symbolic calculus for operators A acting on compact Lie groups has been introduced and consistently developed in [RT13, RT10], to which we refer to further details on global matrix symbols on compact Lie groups. Here we also note that with this matrix global symbol, the Fourier multiplier A must act by multiplication on the Fourier transform side

$$\widehat{Af}(\xi) = \sigma_A(\xi)\widehat{f}(\xi),\ \xi \in \widehat{G},$$

where $\widehat{f}(\xi) = \int_G f(x)\xi(x)^* dx$ is the Fourier coefficient of f at the representation $\xi \in \widehat{G}$, where for simplicity we may identify ξ with its equivalence class. As we have mentioned in (1.4), the L^p-L^q boundedness of Fourier multipliers on compact Lie groups can be controlled by its symbol $\sigma_A(\xi)$. However, Theorem 3.1 gives a better result than the known estimate (1.4); for completeness we recall the exact statement:

Theorem 3.2 ([ANR16b])**.** Let $1 < p \leq 2 \leq q < \infty$ and suppose that A is a Fourier multiplier on the compact Lie group G. Then we have

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \sup_{s\geq 0} s \left(\sum_{\xi\in\widehat{G}:\ \|\sigma_A(\xi)\|_{\mathrm{op}}\geq s} d_\xi^2\right)^{\frac{1}{p}-\frac{1}{q}}, \tag{3.10}$$

where $\sigma_A(\xi) = \xi^*(g)A\xi(g)\big|_{g=e} \in \mathbb{C}^{d_\xi\times d_\xi}$ is the matrix symbol of A.

The fact that Theorem 3.1 implies Theorem 3.2 follows from the following result relating the noncommutative Lorentz norm to the global symbol of invariant operators in the context of compact Lie groups:

Proposition 3.1. Let $1 < p \leq 2 \leq q < \infty$ and let $p \neq q$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Suppose G is a compact group and A is a Fourier multiplier on G. Then we have

$$\|A\|_{L^{r,\infty}(\mathrm{VN}_R(G))} \leq \sup_{s \geq 0} s \left(\sum_{\substack{\xi \in \widehat{G} \\ \|\sigma_A(\xi)\|_{\mathrm{op}} \geq s}} d_\xi^2 \right)^{\frac{1}{p} - \frac{1}{q}}, \tag{3.11}$$

where $\sigma_A(\xi) = \xi^*(g) A \xi(g)\big|_{g=e} \in \mathbb{C}^{d_\xi \times d_\xi}$ is the matrix symbol of A.

If G is a compact Lie group, the sufficient condition (3.10) on the Fourier multiplier A implies τ-measurability of A with respect to the group von Neumann algebra $\mathrm{VN}_R(G)$, so we do not need to assume it explicitly in the setting of compact Lie groups. Indeed, the condition of τ-measurability does not arise in the setting of compact Lie groups due to the fact [Ter81, Proposition 21, p. 16] that

$$A \text{ is } \tau\text{-measurable with respect to } M$$

if and only if

$$\lim_{\lambda \to +\infty} d_A(\lambda) = 0. \tag{3.12}$$

For more detailed discussion and the proof we refer to [AR17].

Proof of Proposition 3.1. We first compute the norm $\|A\|_{L^{r,\infty}(VN_R(G))}$ with $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, $p \neq q$. By definition, we have

$$\|A\|_{L^{r,\infty}(VN_R(G))} = \sup_{t>0} t^{\frac{1}{p} - \frac{1}{q}} \mu_t(A). \tag{3.13}$$

The application of the property (A.19) from Proposition A.1 yields

$$\sup_{t>0} t^{\frac{1}{r}} \mu_t(A) = \sup_{s>0} s [d_A(s)]^{\frac{1}{p} - \frac{1}{q}}.$$

Therefore, it is sufficient to show that

$$\sup_{s>0} s[d_A(s)]^{\frac{1}{p}-\frac{1}{q}} \leq \sup_{s>0} s \left(\sum_{\substack{\xi\in\widehat{G} \\ \|\sigma_A(\xi)\|\geq s}} d_\xi^2 \right)^{\frac{1}{p}-\frac{1}{q}}. \tag{3.14}$$

If A is a left Fourier multiplier, then its modulus $|A|$ is affiliated with $\mathrm{VN}_R(G)$ (see Lemma [MVN36, p. 33, Lemma 4.4.1]),

To proceed, we will use the following property:

Claim 3.1. Let $A \in \mathrm{Aff}(\mathrm{VN}_R(G))$ and let $E_{[s,+\infty)}(|A|)$ be the spectral measure of $|A|$ corresponding to the interval $[s, +\infty)$. Then we have

$$d_A(s) = \sum_{\xi\in\widehat{G}} d_\xi \sum_{\substack{n=1,\ldots,d_\xi \\ s_{n,\xi}\geq s}} 1, \tag{3.15}$$

where for fixed $n = 1, \ldots, d_\xi$, the number $s_{n,\xi}$ is the joint eigenvalue for the eigenfunctions ξ_{kn}, $k = 1, \ldots, d_\xi$, of $|A|$. These functions $\xi_{kn}, k = 1, \ldots, d_\xi$, generate the subspace $\mathcal{H}^{n,\xi} = \mathrm{span}\{\xi_{kn}\}_{k=1}^{d_\xi}$.

The operators A and $|A|$ leave the spaces $\mathcal{H}^{n,\xi}$ invariant since these operators are Fourier multipliers. Assuming Claim 3.1 for the moment, the proof proceeds as follows. Without loss of generality, we can reorder, for each $\xi \in \widehat{G}$, the numbers $s_{n,\xi}$ putting them in a decreasing order with respect to $n = 1, \ldots, d_\xi$ (thus, also reordering the corresponding eigenfunctions). Then we can estimate

$$d_A(s) = \sum_{\xi\in\widehat{G}} d_\xi \sum_{\substack{n=1,\ldots,d_\xi \\ s_{n,\xi}\geq s}} 1 \leq \sum_{\xi\in\widehat{G}} d_\xi \sum_{\substack{n=1,\ldots,d_\xi \\ s_{1,\xi}\geq s}} 1 = \sum_{\substack{\xi\in\widehat{G} \\ s_{1,\xi}\geq s}} d_\xi^2,$$

where in the first inequality we used the inclusion

$$\{\xi \in \widehat{G}, n = 1, \ldots, d_\xi \colon s_{n,\xi} \geq s\} \subset \{\xi \in \widehat{G}, n = 1, \ldots, d_\xi \colon s_{1,\xi} \geq s\} \tag{3.16}$$

since for fixed $\xi \in \widehat{G}$ the sequence $\{s_{n,\xi}\}_{n=1}^{d_\xi}$ monotonically decreases. We notice that

$$s_{1,\xi} = \|\sigma_A(\xi)\|_{\mathtt{op}}.$$

Thus, we obtain

$$d_A(s) \leq \sum_{\substack{\xi\in\widehat{G} \\ \|\sigma_A(\xi)\|_{\mathtt{op}}\geq s}} d_\xi^2.$$

From this, we get

$$s[d_A(s)]^{\frac{1}{p}-\frac{1}{q}} \leq s\left(\sum_{\substack{\xi\in\widehat{G} \\ \|\sigma_A(\xi)\|_{\mathtt{op}}\geq s}} d_\xi^2\right)^{\frac{1}{p}-\frac{1}{q}}. \tag{3.17}$$

Taking supremum in the right-hand side of (3.17), we get

$$s[d_A(s)]^{\frac{1}{p}-\frac{1}{q}} \leq \sup_{s>0} s\left(\sum_{\substack{\xi\in\widehat{G} \\ \|\sigma_A(\xi)\|_{\mathtt{op}}\geq s}} d_\xi^2\right)^{\frac{1}{p}-\frac{1}{q}}. \tag{3.18}$$

Then taking again the supremum in the left-hand side of (3.18), we finally obtain

$$\sup_{s>0} s[d_A(s)]^{\frac{1}{p}-\frac{1}{q}} \leq \sup_{s>0} s\left(\sum_{\substack{\xi\in\widehat{G} \\ \|\sigma_A(\xi)\|_{\mathtt{op}}\geq s}} d_\xi^2\right)^{\frac{1}{p}-\frac{1}{q}}.$$

This proves (3.14). Now, it remains to justify (5.21) in Claim 3.1.

Further, we determine the singular values of A, or equivalently we will look for the eigenvalues of $|A|$.

Indeed, we recall that $|A|\big|_{\bigoplus_{k=1}^{d_\xi}\mathcal{H}^{k,\xi}} = \sigma_{|A|}(\xi)$ and use the fact that $s_{1,\xi} = \|\sigma_{|A|}(\xi)\|_{\mathtt{op}}$. It is convenient to enumerate the singular values $s_{k,\xi}$ by two elements (k,ξ), $k = 1,\ldots,d_\xi$, in view of the decomposition into the closed subspaces invariant under the group action. Since G is compact, its von Neumann algebra $\mathcal{M} = \mathrm{VN}_R(G)$ is a type I factor. Thus, by (A.9), we get

$$\operatorname{Tr}(|A|) = \sum_{\pi\in\widehat{G}} d_\xi \sum_{n=1}^{d_\xi} s_{n,\xi}, \tag{3.19}$$

where we write $s_{n,\xi}$ for the eigenvalue of the restriction $|A|\big|_{\bigoplus_{n=1}^{d_\xi}\mathcal{H}^{n,\xi}}$ of $|A|$ to the subspaces $\mathcal{H}^{k,\xi}$ which are spanned by the eigenfunctions ξ_{kn}, $n = 1, \ldots, d_\xi$, corresponding to $s_{k,\xi}$. In other words, the multiplicity of $s_{k,\xi}$ is d_ξ. From this place, we write π rather than ξ to emphasize our choice of an element ξ from the equivalence class $[\pi]$. Each element $\pi \in \widehat{G}$ can be realised as a finite-dimensional matrix via some choice of a basis in the representation space. Denote by π_{kn} the matrix elements of π, i.e.

$$\pi\colon G \ni g \mapsto \pi(g) = [\pi_{kn}(g)]_{k,n=1}^{d_\pi} \times \mathbb{C}^{d_\pi\times d_\pi}. \tag{3.20}$$

By the Peter-Weyl theorem (see e.g. [RT10, Theorem 7.5.14]), we have the decomposition

$$L^2(G) = \bigoplus_{\pi\in\widehat{G}} \bigoplus_{n=1}^{d_\pi} \operatorname{span}\{\pi_{kn}\}_{k=1}^{d_\pi}. \tag{3.21}$$

In other words, we can write

$$L^2(G) \ni f = \sum_{\pi\in\widehat{G}} d_\pi \sum_{n=1}^{d_\pi}\sum_{k=1}^{d_\pi} (f, \pi_{kn})_{L^2(G)}\pi_{kn} \in \bigoplus_{\pi\in\widehat{G}} \bigoplus_{n=1}^{d_\pi} \operatorname{span}\{\pi_{kn}\}_{k=1}^{d_\pi}. \tag{3.22}$$

The action of A can be written in the form

$$Af = \sum_{\pi\in\widehat{G}} d_\pi \sum_{n=1}^{d_\pi}\sum_{k=1}^{d_\pi}\sum_{s=1}^{d_\pi} \sigma_A(\pi)_{ns}(f, \pi_{ks})_{L^2(G)}\pi_{kn}, \tag{3.23}$$

This implies

$$A = \bigoplus_{\pi\in\widehat{G}} \bigoplus_{n=1}^{d_\pi} \sigma_A(\pi), \tag{3.24}$$

where $\sigma_A(\pi)$ is the global matrix symbol of A (cf. [RT13, RT10]). Then for the modulus $|A| = \sqrt{AA^*}$ we get

$$|A| = \bigoplus_{\pi \in \widehat{G}} \bigoplus_{n=1}^{d_\pi} |\sigma_A(\pi)|. \tag{3.25}$$

Choosing a representative $\xi \in [\pi]$ from the equivalence class $[\pi]$, we can diagonalise the matrix $|\sigma_A(\pi)|$ as

$$\sigma_{|A|}(\xi) = \begin{pmatrix} s_{1,\xi} & 0 \dots & 0 \\ 0 & s_{2,\xi} \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & s_{d_\xi,\xi} \end{pmatrix}. \tag{3.26}$$

Thus, we obtain

$$|A| f = \sum_{\xi \in \widehat{G}} d_\xi \sum_{n=1}^{d_\xi} s_{k,\xi} \cdot \sum_{k=1}^{d_\xi} (f, \xi_{kn})_{L^2(G)} \xi_{kn}. \tag{3.27}$$

Each $s_{n,\xi}$ is a joint eigenvalue of $|A|$ with the eigenfunctions ξ_{kn}

$$|A| \xi_{k,n} = s_{k,\xi} \xi_{k,n}, \quad n = 1, \dots, d_\xi. \tag{3.28}$$

Since each singular value $s_{k,\xi}$, $k = 1, \dots, d_\xi$, has the multiplicity d_ξ, we obtain

$$E_{[t,+\infty)}(|A|) = \bigoplus_{\xi \in \widehat{G}} \bigoplus_{\substack{k=1,\dots,d_\xi \\ s_{k,\xi} \geq t}} E^{n,\xi}, \tag{3.29}$$

where $E^{n,\xi}$ is the projection to the left-invariant subspace $\operatorname{span}\{\xi_{kn}\}_{k=1}^{d_\xi}$. Consequently, we have

$$\operatorname{Tr}(E_{[t,+\infty)}(|A|)) = \sum_{\xi \in \widehat{G}} d_\xi \sum_{\substack{k=1,\dots,d_\xi \\ s_{k,\xi} \geq t}} 1. \tag{3.30}$$

The proof is now complete. $\square$

3.2 Lizorkin theorem

In this section we prove an analogue of the Lizorkin theorem for the L^p-L^q boundedness of Fourier multipliers on compact Lie groups for the range of indices $1 < p \leq q < \infty$. We recall the classical Lizorkin theorem on the real line $\mathbb{R}$:

Theorem 3.3 ([Liz67]). Let $1 < p \leq q < \infty$ and let A be a Fourier multiplier on $\mathbb{R}$ with the symbol σ_A, i.e.

$$Af(x) = \int_{\mathbb{R}} e^{2\pi i x\xi} \sigma_A(\xi) \widehat{f}(\xi) d\xi.$$

Assume that the symbol $\sigma_A(\xi)$ satisfies the following conditions

$$\sup_{\xi \in \mathbb{R}} |\xi|^{\frac{1}{p}-\frac{1}{q}} |\sigma_A(\xi)| \leq C < \infty, \tag{3.31}$$

$$\sup_{\xi \in \mathbb{R}} |\xi|^{\frac{1}{p}-\frac{1}{q}+1} \left| \frac{d}{d\xi} \sigma_A(\xi) \right| \leq C. \tag{3.32}$$

Then $A\colon L^p(\mathbb{R}) \to L^q(\mathbb{R})$ is a bounded linear operator and

$$\|A\|_{L^p(\mathbb{R}) \to L^q(\mathbb{R})} \lesssim C. \tag{3.33}$$

There have been recent works extending this statement to higher dimension, as well as to the operators on the torus, see e.g. [PST12, PST08, STT10, YST16] deal with the same problem on $G = \mathbb{T}^n$ and $G = \mathbb{R}^n$. We prove an analogue of Theorem 3.3 on compact Lie groups.

The crucial information about operators (Fourier multipliers) is the spectral information measured in terms of the generalised t-singular numbers discussed in Appendix A.

In order to measure the regularity of the symbol, we introduce a new family of difference operators $\widehat{\partial}$ acting on Fourier coefficients and on symbols. These operators are used to formulate and prove a version of the Lizorkin theorem on compact groups.

Let $\sigma = \{\sigma(\pi)\}_{\pi \in \widehat{G}}$ be a field of operators and define

$$\widehat{\partial}_\pi \sigma(\pi^j) := \\ U_\pi \begin{pmatrix} \mu_1(\sigma(\pi^k)) - \mu_1(\sigma_A(\pi^{k+1})) & 0 & \ldots & 0 \\ 0 & \mu_2(\sigma(\pi^k)) - \mu_2(\sigma_A(\pi^{k+1})) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \mu_{d_{\pi^j}}(\sigma(\pi^k)) \end{pmatrix}, \quad (3.34)$$

where U_π is a partial isometry matrix in the polar decomposition $\sigma(\pi) = U_\pi |\sigma(\pi)|$. Here $\mu_k(\sigma(\pi))$ are the eigenvalues of $|\sigma(\pi)|$. The latter acts on the left invariant subspace $\mathcal{H}^{k,\pi}$ spanned by the eigenfunctions π_{kn}.

Now, using the direct sum decomposition $\mathrm{Op}(\sigma) = \bigoplus\limits_{\pi \in \widehat{G}} d_\pi \sigma(\pi)$, we can also lift the difference operators ∂_π to $\mathrm{Op}(\sigma)$ by

$$\widehat{\partial}\, \mathrm{Op}(\sigma) = \bigoplus_{\pi \in \widehat{G}} d_\pi \, \widehat{\partial}_\pi \sigma_A(\pi). \quad (3.35)$$

Theorem 3.4. Let $1 < p \leq q < \infty$ and let A be a left Fourier multiplier on a compact Lie group G of dimension n. Assume that the symbol $\sigma_A(\pi)$ satisfies the following

$$C_1 = \sup_{\pi \in \widehat{G}} \langle \pi \rangle^{n\left(\frac{1}{p} - \frac{1}{q}\right)} \|\sigma_A(\pi)\|_{\mathtt{op}} < \infty, \quad (3.36)$$

$$C_2 = \sup_{\pi \in \widehat{G}} \langle \pi \rangle^{n\left(\frac{1}{p} - \frac{1}{q} + 1\right)} \|\widehat{\partial}\sigma_A(\pi)\|_{\mathtt{op}} < \infty, \quad (3.37)$$

Then $A \colon L^p(G) \to L^q(G)$ is a bounded linear operator and we have

$$\|A\|_{L^p(G) \to L^q(G)} \lesssim C_1 + C_2. \quad (3.38)$$

We also have

$$\|A\|_{L^p(G) \to L^q(G)} \lesssim \sup_{\pi \in \widehat{G}} \langle \pi \rangle^{n\left(\frac{1}{p} - \frac{1}{q}\right)} \|\sigma_A(\pi)\|_{\mathtt{op}} + \sum_{\pi \in \widehat{G}} \langle \pi \rangle^{n\left(\frac{1}{p} - \frac{1}{q}\right)} \|\widehat{\partial}\sigma_A(\pi)\|_{\mathtt{op}}. \quad (3.39)$$

Proof of Theorem 3.4. By the density, it is sufficient to establish inequality (3.38) and inequality (3.39) for trigonometric polynomials.

By the Plancherel identity, we get

$$\left|(Af,g)_{L^2(G)}\right| = \left|\sum_{k=1}^{N} d_k \operatorname{Tr} \sigma_A(\pi^k)\widehat{f}(\pi^k)\widehat{g}(\pi^k)^*\right|$$
$$\leq \sum_{k=1}^{N} d_k \left|\operatorname{Tr} \sigma_A(\pi^k)\widehat{f}(\pi^k)\widehat{g}(\pi^k)^*\right|, = \sum_{k=1}^{N}\sum_{t=1}^{d_\pi} d_k \mu_t \left[\sigma_A(\pi^k)\widehat{f}(\pi^k)\widehat{g}(\pi^k)^*\right], \tag{3.40}$$

where

$$f = \sum_{k=1}^{N} d_k \operatorname{Tr} \widehat{f}(\pi^k)\pi^k,$$

and we write $d_k = d_{\pi^k}, k \in \mathbb{N}$.

Splitting the last sum in (3.40) into even and odd terms, we obtain

$$\begin{aligned}
&\sum_{k=1}^{N}\sum_{t=1}^{[\frac{d_k}{2}]} d_\pi \mu_{2t} \left[\sigma_A(\pi^k)\widehat{f}(\pi^k)\widehat{g}(\pi^k)^*\right] + \sum_{k=1}^{N}\sum_{t=1}^{[\frac{d_k+1}{2}]} d_k \mu_{2t-1} \left[\sigma_A(\pi^k)\widehat{f}(\pi^k)\widehat{g}(\pi^k)^*\right] \\
&\leq \\
&\sum_{k=1}^{N}\sum_{t=1}^{[\frac{d_k}{2}]} d_k \mu_{2t} \left[\sigma_A(\pi^k)\widehat{f}(\pi^k)\widehat{g}(\pi^k)^*\right] + \sum_{k=1}^{N}\sum_{t=1}^{[\frac{d_k+1}{2}]} d_k \mu_{2(t-1)} \left[\sigma_A(\pi^k)\widehat{f}(\pi^k)\widehat{g}(\pi^k)^*\right] \\
&\leq \sum_{k=1}^{N}\sum_{t=1}^{[\frac{d_k}{2}]} d_\pi \mu_t \left[\sigma_A(\pi^k)]\mu_t[\widehat{f}(\pi^k)\widehat{g}(\pi^k)^*\right] \\
&+ \sum_{k=1}^{N}\sum_{u=1}^{[\frac{d_k+1}{2}]} d_k \mu_{u-1} \left[\sigma_A(\pi^k)\right] \mu_{u-1} \left[\widehat{f}(\pi^k)\widehat{g}(\pi^k)^*\right] \\
&\lesssim \sum_{k=1}^{N}\sum_{t=1}^{d_k} d_\pi \mu_t \left[\sigma_A(\pi^k)\right] \mu_t \left[\widehat{f}(\pi^k)\widehat{g}(\pi^k)^*\right] \qquad ,
\end{aligned} \tag{3.41}$$

where in the second inequality used the sub-multiplicativity $\mu_{2t}(\cdot) \leq \mu_t(\cdot) \cdot \mu_t(\cdot)$ of the singular values μ_t. Composing (3.40) with (3.41), we get

$$\left|(Af,g)_{L^2(G)}\right| \lesssim \sum_{k=1}^{N}\sum_{t=1}^{d_k} d_\pi \mu_t \left[\sigma_A(\pi^k)\right] \mu_t \left[\widehat{f}(\pi^k)\widehat{g}(\pi^k)^*\right].$$

We define all the singular numbers μ_t to be zero for $t > d_k$, $k \in \mathbb{N}$, i.e.

$$\mu_t(\widehat{f}(\pi^k)) = 0, \quad \mu_t(\widehat{g}(\pi^k)) = 0, \quad \mu_t(\sigma_A(\pi^k)) = 0, \quad \text{for} \quad t > d_k,\ k \in \mathbb{N}.$$

We have thus shown that

$$\left|(Af, g)_{L^2(G)}\right| \lesssim \sum_{k=1}^{N} \sum_{t=1}^{\infty} d_k \mu_t \left[\sigma_A(\pi^k)\right] \mu_t \left[\widehat{f}(\pi^k)\widehat{g}(\pi^k)^*\right] \tag{3.42}$$

for all $N \in \mathbb{N}$ and $g \in L^{q'}(G)$.

Changing the order of summation in (3.42) and using the convention above, we get

$$\left|(Af, g)_{L^2(G)}\right| \leq \sum_{t=1}^{\infty} \sum_{k=1}^{N} d_k \mu_t \left[\sigma_A(\pi^k)\right] \mu_t \left[\widehat{f}(\pi^k)\widehat{g}(\pi^k)^*\right]. \tag{3.43}$$

Now, we shall write

$$\alpha_{t,k} = \mu_t[\sigma_A(\pi^k)], \quad \beta_{t,k} = \mu_t[\widehat{f}(\pi^k)\widehat{g}(\pi^k)^*]. \tag{3.44}$$

Let us apply the Abel transform with respect to k in the right hand side of (3.43):

$$\sum_{k=1}^{N} \alpha_{t,k} d_k \beta_{t,k} = \alpha_{t,N} \sum_{w=1}^{N} d_w \beta_{t,w} + \sum_{k=1}^{N-1} (\Delta_k \alpha_{t,k}) \sum_{w=1}^{k} d_w \beta_{t,w},$$

where

$$\Delta_k \alpha_{t,k} = \alpha_{t,k} - \alpha_{t+1,k}.$$

Combining this with (3.42), we get

$$
\begin{aligned}
&\left|(Af,g)_{L^2(G)}\right| \\
&\qquad \le \sum_{t=1}^{\infty} \alpha_{t,N} \sum_{w=1}^{N} d_w \beta_{t,w} + \sum_{t=1}^{\infty} \sum_{k=1}^{N-1} \Delta_k \alpha_{t,k} \sum_{w=1}^{k} d_w \beta_{t,w} \\
&\qquad \le \alpha_{1,N} \sum_{t=1}^{\infty} \sum_{w=1}^{N} d_w \beta_{t,w} \\
&\qquad\qquad + \sum_{k=1}^{N-1} \left(\sup_{t\in\mathbb{N}} \Delta_k \alpha_{t,k} \right) \sum_{t=1}^{\infty} \sum_{w=1}^{k} d_w \beta_{t,w}.
\end{aligned}
$$

Interchanging the order of summation and applying the Plancherel formula, we get

$$
\begin{aligned}
\sum_{t=1}^{\infty} \sum_{w=1}^{k} d_w\, \mu_t \left[\widehat{f}(\pi^w) \widehat{g}(\pi^w)^* \right] &= \sum_{w=1}^{k} \sum_{t=1}^{\infty} d_w\, \mu_t \left[\widehat{f}(\pi^w) \widehat{g}(\pi^w)^* \right] \\
= \sum_{w=1}^{k} d_w \sum_{t=1}^{d_w} \mu_t \left[\widehat{f}(\pi^w) \widehat{g}(\pi^w)^* \right] &= \sum_{w=1}^{k} d_w \operatorname{Tr}[\widehat{f}(\pi^w) \widehat{g}(\pi^w)^*] \\
&= (f_k, g)_{L^2(G)} \le \|f_k\|_{L^q(G)} \|g\|_{L^{q'}(G)},
\end{aligned}
$$

where we write

$$
f_k = \sum_{w=1}^{k} d_w \operatorname{Tr}(\widehat{f}(\pi^w)\pi^w). \tag{3.45}
$$

Collecting these estimates, we obtain

$$
\begin{aligned}
&\left|(Af,g)_{L^2(G)}\right| \\
&\qquad \le \left(\alpha_{1\,N} \|f\|_{L^q(G)} + \sum_{k=1}^{N-1} \sup_{t\in\mathbb{N}} \Delta_k \alpha_{t,k} \|f_k\|_{L^q(G)} \right) \|g\|_{L^{q'}(G)}.
\end{aligned}
$$

By the duality of L^p-spaces we immediately get

$$
\begin{aligned}
\|Af\|_{L^q(G)} &\lesssim \alpha_{1,N} \|f\|_{L^q(G)} + \sum_{k=1}^{N-1} \sup_{t\in\mathbb{N}} \Delta_k \alpha_{t,k} \|f_k\|_{L^q(G)} \\
&\lesssim \alpha_{1,N} \lambda_N^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^p(G)} + \sum_{k=1}^{N-1} \sup_{t\in\mathbb{N}} \Delta_k \alpha_{t,k} \|f_k\|_{L^q(G)},
\end{aligned} \tag{3.46}
$$

where in the last inequality we used Proposition 2.1.

The application of Proposition 2.2 yields

$$\begin{aligned}
\sum_{k=1}^{N-1} \lambda_k^{\frac{1}{p}-\frac{1}{q}+1} \left(\sup_{t\in\mathbb{N}} \Delta_k \alpha_{t,k} \right) \frac{\|f_k\|_{L^q(G)}}{\lambda_k^{\frac{1}{p}-\frac{1}{q}}} \frac{1}{\lambda_k} \\
&\leq \sum_{k=1}^{\infty} \lambda_k^{\frac{1}{p}-\frac{1}{q}+1} \left(\sup_{t\in\mathbb{N}} \Delta_k \alpha_{t,k} \right) \frac{\|f_k\|_{L^q(G)}}{\lambda_k^{\frac{1}{p}-\frac{1}{q}}} \frac{1}{\lambda_k} \\
&\leq \sup_{k\in\mathbb{N}} \lambda_k^{\frac{1}{p}-\frac{1}{q}+1} \left(\sup_{t\in\mathbb{N}} \Delta_k \alpha_{t,k} \right) \sum_{k=1}^{\infty} \frac{\|f_k\|_{L^q(G)}}{\lambda_k^{\frac{1}{p}-\frac{1}{q}}} \frac{1}{\lambda_k} \\
&\leq \sup_{k\in\mathbb{N}} \lambda_k^{\frac{1}{p}-\frac{1}{q}+1} \left(\sup_{t\in\mathbb{N}} \Delta_k \alpha_{t,k} \right) \|f\|_{L^p(G)}, \quad (3.47)
\end{aligned}$$

where we divide and multiply by $\lambda_k^{\frac{1}{p}-\frac{1}{q}+1}$ in the first line and apply inequality (2.39) in the last estimate. Thus, collecting (3.46) and (3.47), we finally obtain

$$\|Af\|_{L^q(G)} \leq \left(\sup_{k\in\mathbb{N}} \lambda_k^{\frac{1}{p}-\frac{1}{q}} \alpha_{1,k} + \sup_{t\in\mathbb{N}} \lambda_k^{\frac{1}{p}-\frac{1}{q}+1} \sup_{t\in\mathbb{N}} \Delta_k \alpha_{t,k} \right) \|f\|_{L^p(G)}. \quad (3.48)$$

Returning to 'the unitary dual notation' in (3.48) and in (3.46), we immediately get (3.38) and (3.39) respectively. This completes the proof. □

Example 3.1. Let $G = \mathbb{T}$. It can be shown that $\widehat{\mathbb{T}} = \{e^{2\pi kx}\}_{k\in\mathbb{Z}}$. The direct calculation yields

$$\langle e^{2\pi kx} \rangle \cong (1 + |k|).$$

By Theorem 3.4, we get

$$\|A\|_{L^p(\mathbb{T})\to L^q(\mathbb{T})} \leq \sup_{k\in\mathbb{Z}} |k|^{\frac{1}{p}-\frac{1}{q}} |\sigma_A(k)| + \sup_{k\in\mathbb{Z}} |k|^{\frac{1}{p}-\frac{1}{q}+1} |\sigma_A(k) - \sigma_A(k+1)| . \quad (3.49)$$

3.3 The case of non-invariant operators

Our main results on the $L^p - L^q$ boundedness of Fourier multipliers can be extended to to non-invariant operators. This extension relies on two ingridients: Schwartz kernel theorem and Sobolev embedding theorem.

For our purposes, it is convenient to use the Schwartz-Bruhat spaces $\mathcal{D}(G)$ that have been developed by Bruhat [Bru61] as a way of doing distribution theory on locally compact groups. The strong dual of $\mathcal{D}'G)$ is denoted by $\mathcal{D}'(G)$. It contains $\mathcal{D}(G)$ and its elements are called distributions. The topological vector space is barelled and bornological. The space $\mathcal{D}(G)$ is densely contained in the space $C_0(G)$ of continuous functions with compact support. If G is a Lie group, then $\mathcal{D}(G) = C_0^\infty(G)$. The crucial property of the space $\mathcal{D}(G)$ is nuclearity. We refer to [Bru61] for further details. The theory of topological vector spaces has been significantly developed [Gro55] by Alexander Grothendieck. The modern exposition of the theory of topological vector spaces and their tensor products can be found in [Trè67]. It turns out that the property of being nuclear is crucial and these spaces are 'closest' to finite-dimensional spaces. The nuclearity is the necessary and sufficient condition for the existence of abstract Schwartz kernels.

Theorem 3.5. Let G be a locally compact group and $\mathcal{D}(G)$ is the Schwartz-Bruhat space on G. Let $A\colon \mathcal{D}(G) \mapsto \mathcal{D}'(G)$ be a linear continuous operator. Then there is distribution $K_A \in \mathcal{D}'(G)\widehat{\otimes}\mathcal{D}'(G)$ such that

$$Af(\varphi) = K_A(f \otimes \varphi), \tag{3.50}$$

where $\widehat{\otimes}$ denotes the topological tensor product. The proof of Theorem 3.5 follows from the application of Theorem [Trè67, Proposition 50,5, p.522]. We refer to [Trè67, Part III] for more details.

Let us assume that, in addition, G is unimodular. Then we have

$$Af(x) = \int\limits_G K_A(x,t)f(t)\,dt = \int\limits_G f(t)R_A(x,t^{-1}x)\,dt, \tag{3.51}$$

in the standard distributional interpretation. where

$$R_A(x,y) = K_A(x,xy^{-1}).$$

We refer to [RT10, Proposition 10.4.1, p.550] for more details and proofs. By varying the first argument in the kernel R_A, we define a family of operators

$$A_u f(x) := \int\limits_G f(t) R_A(u, t^{-1}x)\, dt.. \tag{3.52}$$

Theorem 3.6. Let G be a locally compact unimodular separable group and let A be a left Fourier multipler on G. Let Let $1 < \beta \leq 2$. Then we have

$$\|A\|_{L^\beta(G)\to L^\infty(G)} \leq \|A\|_{L^\beta(\mathrm{VN}_R(G))}. \tag{3.53}$$

Proof of Theorem 3.6. Since A is a left invariant operator, the kernel R_A in (3.51) does not depend on the first argument, i.e.

$$Af(x) = \int\limits_G f(t) R_A(e, t^{-1}x)\, dt,$$

where e is the identity element in G. We refer to [RW15, pp. 623-624] for more details. By Hölder inequality, we get

$$|Af(x)| \leq \|K_A\|_{L^{\beta'}(G)} \|f\|_{L^\beta(G)}, \quad \frac{1}{\beta'} + \frac{1}{\beta} = 1. \tag{3.54}$$

The application of Hardy-Littlewood inequality (2.8) yields

$$\|K_A\|_{L^{\beta'}(G)} \leq \|A\|_{L^\beta(\mathrm{VN}_R(G))}, \quad 1 < \beta \leq 2. \tag{3.55}$$

Composing (3.54) and (3.55), we get

$$\|Af(x)\| \leq \|A\|_{L^\beta(\mathrm{VN}_R(G))} \|f\|_{L^\beta(G)}. \tag{3.56}$$

Taking the supremum in the left hand-side of (3.56), we obtain (3.53). This completes the proof. $\square$

Theorem 3.7. Let G be a locally compact unimodular separable group. Let $\mathcal{X}$ be a closed densely defined operator affiliated with $\mathrm{VN}_R(G)$ such that its inverse $\mathcal{X}^{-1}$ is measurable with respect to $\mathrm{VN}_R(G)$ and such that for some $1 < \beta \leq 2$ we have

$$\|\mathcal{X}^{-1}\|_{L^{\beta}(\mathrm{VN}_R(G))} < +\infty. \tag{3.57}$$

Then we have

$$\|f\|_{L^{\infty}(G)} \leq \|\mathcal{X}^{-1}\|_{L^{\beta}(\mathrm{VN}_R(G))} \|\mathcal{X} f\|_{L^{\beta}(\mathrm{VN}_R(G))}. \tag{3.58}$$

Proof of Theorem 3.7 . The application of Theorem 3.6 with $A = \mathcal{X}^{-1}$ yields (3.58). This completes the proof. $\square$

One can build Sobolev spaces associated with $\mathcal{X}$. However, we shall not investigate this direction here.

Theorem 3.8. Let G be a locally compact unimodular separable group. Let $\mathcal{X}$ be a closed densely defined operator affiliated with $\mathrm{VN}_R(G)$ such that its inverse $\mathcal{X}^{-1}$ is measurable with respect to $\mathrm{VN}_R(G)$ and such that for some $1 < \beta \leq 2$ we have

$$\|\mathcal{X}^{-1}\|_{L^{\beta}(\mathrm{VN}_R(G))} < +\infty. \tag{3.59}$$

Let A be a linear continuous operator on the Schwartz-Bruhat space $\mathcal{D}(G)$. Then for any $1 < p \leq 2 \leq q < \infty$ we have

$$\|A\|_{L^p(G) \to L^q(G)} \lesssim \left(\int\limits_G \left(\|\mathcal{X} \circ A_u\|_{L^{r,\infty}(\mathrm{VN}_R(G))} \right)^{\beta} du \right)^{\frac{1}{\beta}}, \tag{3.60}$$

where $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ and A_u is defined in (3.52).

But first we observe that choosing various $\mathcal{X}$, we get different inequalities in (3.57). Thus, before proving Theorem 3.8, we illustrate it in a few examples. We use a "minimal" version (Theorem 3.7 below) of Sobolev embedding inequalities. Sobolev inequalities in diverse setting from smooth manifolds to finitely generated groups are discussed in [SC09]. We refer an interested reader to [SC09] and to references therein.

Example 3.2. Let G be a compact Lie group of dimension n and let Δ_G be the Laplace operator on G. Let us take $\mathcal{X} = (I - \Delta_G)^{\frac{n}{2}}$. By the Weyl's asymptotic law, we get $\lambda_k \cong k$, where λ_k are the eigenvalues of $\mathcal{X}$. Then, up to constant, we obtain

$$\|\mathcal{X}^{-1}\|_{L^\beta(\mathrm{VN}_R(G))} \simeq \sum_{k=1}^{\infty} \frac{1}{k^\beta} < +\infty,$$

for any $\beta > 1$. Thus, condition (3.57) is satisfied.

Then by Theorem 3.8, we get

$$\begin{aligned}
\|A\|_{L^p(G)\to L^q(G)} &\lesssim \left(\int\limits_G \left(\|\mathcal{X} \circ A_u\|_{L^{r,\infty}(\mathrm{VN}_R(G))} \right)^\beta du \right)^{\frac{1}{\beta}}, \\
&\lesssim \left(\left(\int\limits_G \sup_{s>0} s \left(\sum_{\substack{\pi \in \widehat{G} \\ \|(I-\Delta_G)^{\frac{n}{2}} \sigma_A(u,\pi)\|_{\mathrm{op}} \geq s}} d_\pi^2 \right)^{\frac{1}{r}} \right)^\beta du \right)^{\frac{1}{\beta}},
\end{aligned}$$

where we used Proposition 3.1 in the second inequality. Here, for every fixed $u \in G$ the restriction of A_u onto the subspace $\mathcal{H}^\pi$ is the global symbol $\sigma_A(u,\pi)$.

Example 3.3. Let us take $\mathcal{X} = (I + \mathcal{L})^{\frac{Q}{2}}$, where $\mathcal{L}$ is the positive sub-Laplacian on $\mathbb{H}^n$. It can be computed (see (4.42)) that

$$\mu_t(\mathcal{X}^{-1}) \cong \frac{1}{(1 + t^{\frac{2}{Q}})^{\frac{Q}{2}}}.$$

From this we obtain

$$\|\mathcal{X}^{-1}\|^\beta_{L^\beta(\mathrm{VN}_R(\mathbb{H}^n))} \cong \int\limits_0^{+\infty} \frac{1}{(1 + t^{\frac{2}{Q}})^{\frac{Q}{2}\beta}}.dt. \tag{3.61}$$

The integral in (3.61) is convergent for $\beta > 1$. Thus, by Theorem 3.7, we get

$$\|f\|_{L^\infty(\mathbb{H}^n)} \leq C \|(I + \mathcal{L})^{\frac{Q}{2}} f\|_{L^\beta(\mathrm{VN}_R(\mathbb{H}^n))}, \quad 1 < \beta < 2. \tag{3.62}$$

By different methods, inequality (3.62) has been established in [FR16, Chapter 4] for $1 < \beta < \infty$.

Example 3.4. Let G be a compact Lie group of dimension n and let $\mathcal{X}$ be as in Example 3.2. Let $A\colon C^\infty(G) \to C^\infty(G)$ be a continuous linear operator. Then, by Theorem 3.8, we get

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \left(\int\limits_G \|(I-\Delta_G)^{\frac{n}{2}} A_u\|^\beta_{L^\beta(\mathrm{VN}_R(G))}\, du \right)^\beta . \tag{3.63}$$

Since G is compact, we obtain from (3.63)

$$\begin{aligned}\|A\|_{L^p(G)\to L^q(G)} &\leq \sup_{u\in G} \|(I-\Delta_G)^{\frac{n}{2}} A_u\|^\beta_{L^\beta(\mathrm{VN}_R(G))} \\ &\leq \sup_{u\in G}\sup_{s>0} s \left(\sum_{\substack{\pi\in\widehat{G} \\ \|(I-\Delta_G)^{\frac{n}{2}}\sigma_A(\pi,u)\|_{\mathrm{op}}\geq s}} d_\pi^2 \right)^{\frac{1}{p}-\frac{1}{q}} ,\end{aligned} \tag{3.64}$$

where we used Proposition 3.1 in the second inequality. Hence, we recover a version of Theorem 5.6 for $(I-\Delta_G)^{\frac{n}{2}}$.

Here $\sigma_A(\pi,u)$ is the global symbol of A_u, $u\in G$,

Example 3.5. Let us take G to be the Heisenberg group $\mathbb{H}^n$ with the homogeneous dimension $Q = 2n+2$, and let $\mathcal{X}$ be as in Example 3.3. Let $A\colon C_0^\infty(\mathbb{H}^n) \to C_0^\infty(\mathbb{H}^n)$ be a continuous linear operator. Then, by Theorem 3.8, we get

$$\|A\|_{L^p(\mathbb{H}^n)\to L^q(\mathbb{H}^n)} \leq \left(\int\limits_{\mathbb{H}^n} \|(1+\mathcal{L}_u)^{\frac{Q}{2}} A_u\|_{L^{r,\infty}(\mathrm{VN}_R(\mathbb{H}^n))}\, du \right)^{\frac{1}{\beta}} . \tag{3.65}$$

Proof of Theorem 3.8. Let us define

$$A_u f(x) := \int\limits_G f(t) R_A(u, t^{-1}x) dt,$$

so that $A_x f(x) = Af(x)$. For each fixed $u\in G$ the operator A_u is affiliated with $\mathrm{VN}_R(G)$. Then

$$\|Af\|_{L^q(G)} = \left(\int\limits_G |Af(g)|^q \, dg \right)^{\frac{1}{q}} \leq \left(\int\limits_G \sup_{u \in G} |A_u f(g)|^q \, dg \right)^{\frac{1}{q}}. \quad (3.66)$$

By Theorem 3.7, we get

$$\sup_{u \in G} |A_u f(g)| \leq \|\mathcal{X}^{-1}\|_{L^\beta(\mathrm{VN}_R(G))} \|\mathcal{X} A_u f\|_{L^\beta(G)}, \quad (3.67)$$

where the operator $\mathcal{X}$ acts on $A_u f(g)$ as a function of $u \in G$ variable.

Therefore, using the Minkowski integral inequality to change the order of integration, we obtain

$$\|Af\|_{L^q(G)} \lesssim \left(\int\limits_G \left(\int\limits_G |\mathcal{X} \, A_u f(g)|^\beta \, du \right)^{\frac{q}{\beta}} dg \right)^{\frac{1}{q}} =$$

$$\left[\left\| \int\limits_G |\mathcal{X} \, A_u f(g)|^\beta \, du \right\|_{L^{\frac{q}{\beta}}(G)} \right]^{\frac{1}{\beta}} \leq \left[\int\limits_G \left\| |\mathcal{X} \, A_u f(g)|^\beta \right\|_{L^{\frac{q}{\beta}}(G)} du \right]^{\frac{1}{\beta}} =$$

$$\left(\int\limits_G \left(\int\limits_G |\mathcal{X} \, A_u f(g)|^q \, dg \right)^{\frac{\beta}{q}} du \right)^{\frac{1}{\beta}} \leq$$

$$\left(\int\limits_G \left(\|\mathcal{X} \, A_u\|_{L^{r,\infty}(\mathrm{VN}_R(G))} \right)^\beta du \right)^{\frac{1}{\beta}} \|f\|_{L^p(G)},$$

where the last inequality holds due to Theorem 3.1. This also completes the proof of Theorem 3.8. □

Finally we note that as a corollary of Theorem 3.4 on compact Lie groups we get the boundedness result also for non-invariant operators. Indeed, a rather standard argument (see e.g. the proof of Theorem 3.8) immediately yields:

Theorem 3.9. Let G be a compact Lie group of dimension n. Let $1 < p \leq q < \infty$. Let A be a continuous linear operator on $C_0^\infty(G)$. Then we have

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \sup_{u\in G}\sup_{\pi\in\widehat{G}} \langle\pi\rangle^{n(\frac{1}{p}-\frac{1}{q})}\|(I-\Delta_G)_u^{\frac{n}{2}}\sigma_A(u,\pi)\|_{\mathtt{op}}$$
$$+\sup_{u\in G}\langle\pi\rangle^{n(\frac{1}{p}-\frac{1}{q}+1)}\|(I-\Delta_G)_u^{\frac{n}{2}}\widehat{\partial}\sigma_A(u,\pi)\|_{\mathtt{op}}. \quad (3.68)$$

Proof of Theorem 3.9. Let us define

$$A_u f(x) := \int\limits_G f(t)R_A(u,t^{-1}x)dt, \quad (3.69)$$

so that $A_x f(x) = Af(x)$. For each fixed $u \in G$ the operator A_u is affiliated with $\mathrm{VN}_R(G)$. Then

$$\|Af\|_{L^q(G)} = \left(\int\limits_G |Af(x)|^q\, dg\right)^{\frac{1}{q}} \leq \left(\int\limits_G \sup_{u\in G}|A_u f(x)|^q\, dx\right)^{\frac{1}{q}}. \quad (3.70)$$

By Theorem 3.7, we get

$$\sup_{u\in G}|A_u f(x)| \leq \|\mathcal{X}^{-1}\|_{L^\beta(\mathrm{VN}_R(G))}\|\mathcal{X}_u A_u f(x)\|_{L^\beta_u(G)}, \quad (3.71)$$

where take

$$\mathcal{X} = (I-\Delta_G)^{\frac{n}{2}}$$

and choose $1 < \beta \leq \min(2,q)$. Here the operator $\mathcal{X}$ acts on u-variable. It can be seen from (3.69) that

$$(\mathcal{X}_u \circ A_u)f(x) = \int\limits_G f(t)\mathcal{X}_u R_A(u,t^{-1}x)dt. \quad (3.72)$$

Hence, we get

$$\sigma_{\mathcal{X}_u\circ A_u}(\pi) = \mathcal{X}_u\sigma_A(\pi,u), \quad (3.73)$$

where we used (3.72) and the fact that the global symbol $\sigma_A(\pi,u)$ is Fourier transform of the right-convolutional kernel $R_A(u,t)$ in t-variable, i.e.

$$\sigma_A(\pi,u)=\int\limits_G R_A(u,t)\pi^*(t)\,dt.$$

We refer to [RT10, Section 10.4.1] for more details and proofs.

Further, composing (3.70) and (3.71), we get

$$\begin{gathered}
\|Af\|_{L^q(G)}\lesssim\left(\int\limits_G\left(\int\limits_G|\mathcal{X}\,A_uf(x)|^\beta\,du\right)^{\frac{q}{\beta}}dx\right)^{\frac{1}{q}}= \\
\left[\left\|\int\limits_G|\mathcal{X}\,A_uf(\cdot)|^\beta\,du\right\|_{L^{\frac{q}{\beta}}(G)}\right]^{\frac{1}{\beta}}\le\left[\int\limits_G\left\||\mathcal{X}\,A_uf(\cdot)|^\beta\right\|_{L^{\frac{q}{\beta}}(G)}du\right]^{\frac{1}{\beta}}= \\
\left(\int\limits_G\left(\int\limits_G|\mathcal{X}\,A_uf(x)|^q\,dx\right)^{\frac{\beta}{q}}du\right)^{\frac{1}{\beta}}\le\sup_{u\in G}\left(\int\limits_G|\mathcal{X}\,A_uf(x)|^q\,dx\right)^{\frac{1}{q}},
\end{gathered}\tag{3.74}$$

where we used the Minkowski integral inequality to change the order of integration and we used the compactness of G in the last inequality.

Now, fixing $u\in G$, we apply Theorem 3.4 to each A_u to get

$$\begin{aligned}
&\left(\int\limits_G|\mathcal{X}\,A_uf(g)|^q\,dg\right)^{\frac{1}{q}} \\
&\le \\
&\left(\sup_{\pi\in\widehat{G}}\langle\pi^n\rangle^{\frac{1}{p}-\frac{1}{q}}\|\sigma_{\mathcal{X}_uA_u}(\pi)\|_{\mathsf{op}}+\sup_{\pi\in\widehat{G}}\langle\pi^n\rangle^{\frac{1}{p}-\frac{1}{q}+1}\|\widehat{\partial}\sigma_{\mathcal{X}A}(\pi)\|_{\mathsf{op}}\right)\|f\|_{L^p(G)} \\
&= \\
&\left(\sup_{\pi\in\widehat{G}}\langle\pi^n\rangle^{\frac{1}{p}-\frac{1}{q}}\|\mathcal{X}_u\sigma_A(\pi,u)\|_{\mathsf{op}}+\sup_{\pi\in\widehat{G}}\langle\pi^n\rangle^{\frac{1}{p}-\frac{1}{q}+1}\|\widehat{\partial}\mathcal{X}_u\sigma_A(\pi,u)\|_{\mathsf{op}}\right)\|f\|_{L^p(G)}
\end{aligned}\tag{3.75}$$

where the last inequality holds due to Theorem 3.1 and in the equality above we used (3.73). Finally, composing (3.74) and (3.75), we obtain

$$\|Af\|_{L^q(G)}$$

$$\leq$$

$$\sup_{u\in G}\left(\sup_{\pi\in\widehat{G}}\langle\pi^n\rangle^{(\frac{1}{p}-\frac{1}{q})}\|\mathcal{X}_u\sigma_A(\pi,u)\|_{\mathtt{op}}+\sup_{\pi\in\widehat{G}}\langle\pi^n\rangle^{(\frac{1}{p}-\frac{1}{q}+1)}\|\|\mathcal{X}_u\widehat{\partial}\sigma_A(\pi,u)\|_{\mathtt{op}}\right)\|f\|_{L^p(G)}.$$

This completes the proof. $\square$

Chapter 4

Spectral multipliers

Let G be a compact Lie group and let Δ_G be the Laplace operator on G. For convenience of the following formulation we label the equivalence classes of irreducible unitary representations of G as $\pi_k, k \in \mathbb{N}$ in such a way that the corresponding sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ of $(I - \Delta_G)^{\frac{\dim(G)}{2}}$ is non-decreaasing. More precisely, we have

$$(I - \Delta_G)^{\frac{\dim(G)}{2}} \pi^k_{ij} = \lambda_k \pi^k_{ij}$$

holds for all $1 \leq m, l \leq d_{\pi^k}$ and.

Let A be a left Fourier multiplier. We can now give an illustration of Theorem 3.4 related to spectral multipliers $\varphi(A)$ for a monotone continuous function φ on $[0, +\infty)$.

Corollary 4.1. Let A be a left Fourier multiplier on a compact Lie group G of dimension n. Let $1 < p \leq q < \infty$. Assume that φ is a monotone function on $[0, +\infty)$. Then $\varphi(|A|)\colon L^p(G) \to L^q(G)$ is a bounded linear operator and we have

$$\begin{aligned} \|\varphi(|A|)\|_{L^p(G)\to L^q(G)} & \\ \lesssim \sup_{j\in\mathbb{N}} j^{\frac{1}{p}-\frac{1}{q}} & \sup_{t=1,\ldots,d_{\pi^j}} |\varphi(\alpha_{t\,j})| \\ + \sup_{j\in\mathbb{N}} j^{\frac{1}{p}-\frac{1}{q}+1} & \sup_{t=1,\ldots,d_{\pi^j}} |\varphi(\alpha_{t\,j}) - \varphi(\alpha_{t\,j+1})|, \end{aligned} \quad (4.1)$$

where $\alpha_{t\,j}$ are the singular numbers of the symbol $\sigma_A(\pi^j)$, $\pi^j \in \widehat{G}$, $t = 1, \ldots, d_{\pi^j}$.

It will be clear from the proof of Corollary 4.1 that the condition that φ is monotone is not essential and is needed only for obtaining a simpler expression under the sum in (4.1).

Remark 4.1. Let A and p, q be as in Corollary 4.1. Assume φ is a continuous function. Then $\varphi(|A|)\colon L^p(G) \to L^q(G)$ is a bounded linear operator and we have

$$\begin{aligned} \|\varphi(|A|)\|_{L^p(G)\to L^q(G)} &\leq \sup_{j\in\mathbb{N}} j^{\frac{1}{p}-\frac{1}{q}} \sup_{t=1,\ldots,d_{\pi^j}} (\varphi(\alpha_{t\,k}))^*_t \\ &+ \sup_{j\in\mathbb{N}} j^{\frac{1}{p}-\frac{1}{q}+1} \sup_{t=1,\ldots,d_{\pi^j}} \left[(\varphi(\alpha_{t\,k}))^*_t - (\varphi(\alpha_{t\,k}))^*_{t+1}\right], \end{aligned} \tag{4.2}$$

where $(\varphi(\alpha_{t\,k}))^*_t$ is the non-increasing rearrangement of the sequence $\{\varphi(\alpha_{t\,k})\}_{t=1}^{d_{\pi^j}}$ and $\alpha_{t\,k}$ are the singular values of the symbol $\sigma_A(\pi^j)$, $\pi^j \in \widehat{G}$, $t = 1, \ldots, d_{\pi^j}$.

Remark 4.2. Let G be a compact Lie group of dimension n and let A and φ be as in Corollary 4.1. Let us also assume that φ is boundedly differentiable, i.e. $\|\varphi'\|_{L^\infty(0,+\infty)} < +\infty$. Then $\varphi(|A|)$ is L^p-L^q bounded.

Proof of Corollary 4.1. The functional calculus for linear operators affiliated with semi-finite von Neumann algebras has been developed in [DNSZ16, Section 4]. In particular, by [DNSZ16, Proposition 4.2], we get

$$\varphi(|A|) = \bigoplus_{j=1}^{\infty} d_{\pi^j} \varphi(\sigma_A(\pi^j)), \tag{4.3}$$

where we used the identity

$$\sigma_{\varphi(|A|)}(\pi^j) = \varphi(\sigma_{|A|}(\pi^j)).$$

Let us denote by α_{t,π^j} the singular values of $\sigma_A(\pi^j)$. From (4.3), we get that the singular numbers β_{t,π^j} of $\varphi(\sigma_A(\pi^j))$ are given by

$$\{\beta_{t,\pi^j}\}_{t=1}^{d_{\pi^j}} = \{\varphi(\alpha_{1,\pi^j}), \varphi(\alpha_{2,\pi^j}), \ldots, \varphi(\alpha_{d_{\pi^j},\pi^j})\}$$

if φ is increasing, and by

$$\{\beta_{t,\pi^j}\}_{t=1}^{d_{\pi^j}} = \{\varphi(\alpha_{d_\pi,\pi}), \varphi(\alpha_{d_\pi-1,\pi}), \ldots, \varphi(\alpha_{1,\pi})\}$$

otherwise. By definition (3.34), we get

$$\widehat{\partial}_\pi \sigma_{\varphi(|A|)}(\pi^j) = \begin{pmatrix} \beta_{1,j}-\beta_{1,j+1} & 0 & \cdots & 0 \\ 0 & \beta_{2,j}-\beta_{2,j+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \beta_{t,j}-\beta_{t,j+1} & \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \beta_{d_j-1,j}-\beta_{d_j,j+1} \\ 0 & 0 & 0 & \beta_{d_j,j} \end{pmatrix}, \quad (4.4)$$

By Theorem 3.4 we have

$$\|\varphi(A)\|_{L^p(G)\to L^q(G)} \lesssim \sup_{\pi\in\widehat{G}} \langle\pi\rangle^{\frac{n}{r}} \|\varphi(\sigma_A(\pi))\|_{\mathtt{op}} + \sum_{\pi\in\widehat{G}} \langle\pi\rangle^{\frac{n}{r}} \|\widehat{\partial}\varphi(\sigma_A(\pi))\|_{\mathtt{op}}. \quad (4.5)$$

Combining (4.5) and (4.4) we obtain

$$\begin{aligned}&\|\varphi(A)\|_{L^p(G)\to L^q(G)} \\ &\lesssim \sup_{\pi\in\widehat{G}} \langle\pi\rangle^{\frac{n}{r}} \sup_{t=1,\ldots,d_\pi} |\varphi(\alpha_{t\,\pi})| + \sum_{\pi\in\widehat{G}} \langle\pi\rangle^{\frac{n}{r}} \sup_{t=1,\ldots,d_\pi} |\varphi(\alpha_{t\,\pi}) - \varphi(\alpha_{t+1\,\pi})|\,.\end{aligned}$$

By using (1.22) and the numbering of the representations as explained before Corollary 4.1, it establishes (4.1) and completes the proof. □

In this and next section we will give an application of Theorem 3.1 to spectral multipliers.

The classical Laplace operator $\Delta_{\mathbb{R}^n}$ is affiliated with the von Neumann algebra $\mathrm{VN}(\mathbb{R}^n) = \mathrm{VN}_L(\mathbb{R}^n) = \mathrm{VN}_R(\mathbb{R}^n)$ of all convolution operators, but is not measurable on $\mathrm{VN}(\mathbb{R}^n)$. However, the Bessel potential $(I-\Delta_{\mathbb{R}^n})^{-\frac{s}{2}}$ is measurable with respect to $\mathrm{VN}(\mathbb{R}^n)$. Therefore, one of the aims of spectral multiplier theorems is to "renormalise" operators in Hilbert space $\mathcal{H}$ making them not only measurable but also bounded. In the next theorem we first describe such a relation for general semifinite von Neumann algebras, and then in Corollary 4.2 give its application to spectral multipliers.

Theorem 4.1. Let $\mathcal{L}$ be a closed unbouned operator affiliated with a semifinite von Neumann algebra $\mathcal{M} \subset B(\mathcal{H})$. Assume that φ is a monotonically decreasing continuous function on $[0, +\infty)$ such that

$$\varphi(0) = 1, \tag{4.6}$$

$$\lim_{u \to +\infty} \varphi(u) = 0. \tag{4.7}$$

Then for every $1 \leq r < \infty$ we have the equality

$$\|\varphi(|\mathcal{L}|)\|_{L^{r,\infty}(\mathcal{M})} = \sup_{u>0} \left(\tau(E_{(0,u)}(|\mathcal{L}|))\right)^{\frac{1}{r}} \varphi(u) < +\infty. \tag{4.8}$$

Let $\mathcal{L}$ be an arbitrary unbounded linear operator affiliated with $(\mathcal{M}, \tau)$. Then Theorem 4.1 says that the function $\varphi(|\mathcal{L}|)$ is necessarily affiliated with $(\mathcal{M}, \tau)$ and $\varphi(|\mathcal{L}|) \in (\mathcal{M}, \tau)$ if and only if the r-th power φ^r of φ grows at infitiy not faster than $\frac{1}{\tau(E_{(0,u)}(|\mathcal{L}|))}$, i.e. if we have the estimate

$$\varphi(u)^r \lesssim \frac{1}{\tau(E_{(0,u)}(|\mathcal{L}|))}, \quad u > 0. \tag{4.9}$$

We now give a corollary of Theorem 4.1 for $\mathcal{M} = \mathrm{VN}_R(G)$ being the right von Neumann algebra of a locally compact unimodular group. This is formulated in Theorem 1.1 but we recall it here for readers' convenience. From now on, we assume that the spectrum of $|\mathcal{L}|$ is separated from zero, i.e.

$$\mathrm{Sp}(|\mathcal{L}|) \subset (c, +\infty) \tag{4.10}$$

for some real positive $c \in \mathbb{R}_+$.

Corollary 4.2. Let G be a locally compact unimodular separable group and let $\mathcal{L}$ be a left Fourier multiplier on G. Let φ be as in Theorem 4.1 Then we have the inequality

$$\|\varphi(|\mathcal{L}|)\|_{L^p(G)\to L^q(G)} \lesssim \sup_{s>0} \varphi(u) \left[\tau(E_{(0,s)}(|\mathcal{L}|))\right]^{\frac{1}{p}-\frac{1}{q}}, \tag{4.11}$$

where $1 < p \leq 2 \leq q < \infty$.

This corollary follows immediately from combining Theorem 3.1 and Theorem 4.1 with $M = \mathrm{VN}_R(G)$, also proving Theorem 1.1.

The Borel functional calculus allows to make sense of $\varphi(\mathcal{L})$ for arbitrary Borel function φ on $\mathrm{Sp}(|\mathcal{L}|)$. However, it is not clear how to establish time dependent estimate of $\|\varphi(\mathcal{L})\|_{L^{r\infty}(\mathrm{VN}_R(G))}$.

Proof of Theorem 4.1. By defintion

$$\|\varphi(|\mathcal{L}|)\|_{L^{r,\infty}(M)} = \sup_{t>0} t^{\frac{1}{p}-\frac{1}{q}} \mu_t(\varphi(|\mathcal{L}|)), \quad \frac{1}{r} = \frac{1}{p} - \frac{1}{q}.$$

Using Property (A.19) from Proposition A.1, we get

$$\sup_{t>0} t^{\frac{1}{p}-\frac{1}{q}} \mu_t(\varphi(|\mathcal{L}|)) = \sup_{u>0} u[\tau(E_{(u,\infty)})(\varphi(|\mathcal{L}|))]^{\frac{1}{p}-\frac{1}{q}}.$$

Hence, we have

$$\|\varphi(|\mathcal{L}|)\|_{L^{r,\infty}(M)} = \sup_{u>0} u[\tau(E_{(u,+\infty)})(\varphi(|\mathcal{L}|))]^{\frac{1}{p}-\frac{1}{q}}. \tag{4.12}$$

Since $\mathcal{L}$ is affiliated with M the spectral projections $E_\Omega(|\mathcal{L}|)$ belong to M. Let φ be a Borel measurable function on the spectrum $\mathrm{Sp}(|\mathcal{L}|)$. Then by Borel functional calculus [Arv06, Section 2.6] it is possible to construct the operator $\varphi(|\mathcal{L}|)$. This operator is a strong limit of the spectral projections $E_\Omega(|\mathcal{L}|) \in M$. Therefore $\varphi(|\mathcal{L}|)$ is affiliated with M. The distribution function of the operator $\varphi(|\mathcal{L}|)$ is given by

$$d_u(\varphi(|\mathcal{L}|)) = \tau(E_{(u,\infty)}(\varphi(|\mathcal{L}|)). \tag{4.13}$$

This proves Corollary 4.2.

The monotonic decrease of φ and the spectral mapping theorem ([KR97, Theorem 4.1.6] yield

$$\tau(E_{(u,\infty)}(\varphi(|\mathcal{L}|))) = \tau(E_{(0,\varphi^{-1}(u))}(|\mathcal{L}|). \tag{4.14}$$

From the hypothesis (4.7) imposed on φ and using the properies of the spectral measure $\{E_\Omega\}_{\Omega\subset \mathrm{Sp}(|\mathcal{L}|)}$, we get

$$\lim_{u\to+\infty}\tau(E_{(u,\infty)}(\varphi(|\mathcal{L}|))) = \lim_{u\to+\infty}\tau(E_{(0,\varphi^{-1}(u))}(|\mathcal{L}|) = 0. \tag{4.15}$$

Hence, the operator $\varphi(|\mathcal{L}|)$ is τ-measurable with respect to $\mathrm{VN}_R(G)$. Combining (4.12) and (4.14), we finally obtain

$$\begin{aligned}\|\varphi(|\mathcal{L}|)\|_{L^{r,\infty}(M)} &= \sup_{t>0} t^{\frac{1}{p}-\frac{1}{q}}\mu_t(\varphi(|\mathcal{L}|)) = \sup_{u>0} u[\tau(E_{(u,\infty)})(\varphi(|\mathcal{L}|))]^{\frac{1}{p}-\frac{1}{q}}\\ &= \sup_{u>0} u[\tau(E_{(0,\varphi^{-1}(u))}(|\mathcal{L}|)]^{\frac{1}{p}-\frac{1}{q}} = \sup_{s>0}\varphi(s)[\tau(E_{(0,s)}(|\mathcal{L}|)]^{\frac{1}{p}-\frac{1}{q}},\end{aligned}$$

where in the last equality we used the monotonicity of $u=\varphi(s)$. This completes the proof of Theorem 4.1. $\square$

4.1 Heat kernels and embedding theorems

In this section we show that the spectral multipliers estimate (4.2) may be also used to relate spectral properties of the operators with the time decay rates for propagators for the corresponding evolution equations. We illustrate this in the case of the heat equation, when the the functional calculus and the application of Theorem 4.1 to a family of functions $\{e^{-ts}\}_{t>0}$ yield the time decay rate for the solution $u=u(t,x)$ to the heat equation

$$\partial_t u + \mathcal{L}u = 0, \quad u(0)=u_0.$$

For each $t>0$, we apply Borel functional calculus [Arv06, Section 2.6] to get

$$u(t,x) = e^{-t\mathcal{L}}u_0. \tag{4.16}$$

One can check that $u(t,x)$ satisfies equation (4.16) and the initial condition. Then by Theorem 3.1, we get

$$\|u(t,\cdot)\|_{L^q(G)} \le \|e^{-t\mathcal{L}}\|_{L^{r,\infty}(\mathrm{VN}_R(G))}\|u_0\|_{L^p(G)}, \tag{4.17}$$

reducing the L^p-L^q properties of the propagator to the time asymptotics of its noncommutative Lorentz space norm.

Corollary 4.3 (The $\mathcal{L}$-heat equation)**.** Let G be a locally compact unimodular separable group and let $\mathcal{L}$ be an unbounded positive operator affiliated with $\mathrm{VN}_R(G)$. Assume that

$$\tau(E_{(0,s)}(\mathcal{L})) \lesssim s^{\alpha}, s > 0, \tag{4.18}$$

for some α. Then for any $1 < p \leq 2 \leq q < \infty$ we have

$$\|e^{-t\mathcal{L}}\|_{L^p(G)\to L^q(G)} \leq C_{\alpha,p,q} t^{-\alpha\left(\frac{1}{p}-\frac{1}{q}\right)}, \quad t > 0. \tag{4.19}$$

Proof of Theorem 4.3. The application of Theorem 4.1 yields

$$\|e^{-t\mathcal{L}}\|_{L^{r,\infty}(\mathrm{VN}_R(G))} = \sup_{s>a}[\tau(E_{(0,s)}(|\mathcal{L}|)]^{\frac{1}{r}} e^{-ts}.$$

Now, using this and hypothesis (4.18), we get

$$\|e^{-t\mathcal{L}}\|_{L^{r,\infty}(\mathrm{VN}_R(G))} \lesssim \sup_{s>0} s^{\frac{\alpha}{r}} e^{-ts}.$$

The standart theorems of mathematical analysis yield that

$$\sup_{s>0} s^{\frac{\alpha}{r}} e^{-ts} = \left(\frac{\alpha}{tr}\right)^{\frac{\alpha}{r}} e^{-\frac{\alpha}{r}}. \tag{4.20}$$

Indeed, let us consider a function

$$\varphi(s) = s^{\frac{\alpha}{r}} e^{-ts}.$$

We compute its derivative

$$\varphi'(s) = s^{\frac{\alpha}{r}-1} e^{-ts} \left(\frac{\alpha}{r} - st\right).$$

The only zero is $s_0 = \frac{\alpha}{rt}$ and the derivative $\varphi'(s)$ changes its sign from positive to negative at s_0. Thus, the point s_0 is a point of maximum.

This shows (4.20) and completes the proof. □

Let us now show an application of Theorem 4.1 in the case of $\varphi(s) = \frac{1}{(1+s)^\gamma}$, $s \geq 0$. It shows that for the range $1 < p \leq 2 \leq q < \infty$, the Sobolev type embedding theorems for an operator $\mathcal{L}$ depend only on the spectral behaviour of $\mathcal{L}$.

Corollary 4.4 (Embedding theorems)**.** Let G be a locally compact unimodular separable group and let $\mathcal{L}$ be an unbounded positive operator affiliated with $\mathrm{VN}_R(G)$. Assume that

$$\tau(E_{(a,s)}(\mathcal{L})) \lesssim s^\alpha, \quad s > a, \tag{4.21}$$

for some real positive $\alpha > 0$. Then for any $1 < p \leq 2 \leq q < \infty$ we have

$$\|f\|_{L^q(G)} \leq C\|(1+\mathcal{L})^\gamma f\|_{L^p(G)}, \tag{4.22}$$

provided that

$$\gamma \geq \alpha\left(\frac{1}{p} - \frac{1}{q}\right), \quad 1 < p \leq 2 \leq q < \infty. \tag{4.23}$$

Proof. By Theorem 4.1 with $\varphi(s) = \frac{1}{(1+s)^\gamma}$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ we have

$$\|(1+\mathcal{L})^{-\gamma}\|_{L^p(G)\to L^q(G)} \lesssim \|(1+\mathcal{L})^{-\gamma}\|_{L^{r,\infty}(\mathrm{VN}_R(G))} \lesssim \sup_{s>0} s^{\frac{\alpha}{r}}(1+s)^{-\gamma}.$$

This supremum is finite for $\gamma \geq \frac{\alpha}{r}$, giving the condition (4.23). □

Now, we illustrate Theorem 4.1 and Corollary 4.3 on a number of further examples, showing that the spectral estimate (4.18) required for the L^p-L^q estimate can be readily obtained in different situations.

4.1.1 Sub-Riemannian structures on compact Lie groups

First we consider the example of sub-Laplacians on compact Lie groups in which case the number α in (4.18) can be related to the Hausdorff dimension generated by the control distance of the sub-Laplacian. Moreover, we illustrate Theorem 4.1 with examples of other functions φ than in Corollary 4.3, for example $\varphi(s) = \frac{1}{(1+s)^{\alpha/2}}$, leading to the Sobolev embedding theorems.

Example 4.1. Let $\mathcal{L}$ be a positive sub-Laplacian on a compact Lie group G, with discrete spectrum λ_k. Then by [HK16] the trace of the spectral projections $E_{(0,s)}(\mathcal{L})$ has the following asymptotics

$$\tau(E_{(0,s)}(\mathcal{L})) \lesssim s^{\frac{Q}{2}}, \quad s > 0, \tag{4.24}$$

where Q is the Hausdorff dimension of G relative to the control distance generated by the sub-Laplacian. Let $u(t)$ be the solution to Δ_{sub}-heat equation

$$\begin{gathered}\frac{\partial}{\partial t}u(t,x) + (I - \Delta_{sub})u(t,x) = 0, \quad t > 0,\\ u(0,x) = u_0(x), \quad u_0 \in L^p(G), \quad 1 < p \leq 2.\end{gathered}$$

Then by Corollary 4.3, we obtain

$$\|u(t,\cdot)\|_{L^q(G)} \leq C_{n,p,q} t^{-\frac{Q}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|u_0\|_{L^p(G)}, \quad 1 < p \leq 2 \leq q < +\infty. \tag{4.25}$$

Let us now take $\varphi(s) = \frac{1}{(1+s)^{a/2}}$, $s \geq 0$. Then by Theorem 4.1 the operator $\varphi(-\Delta_{sub}) = (I - \Delta_{sub})^{-a/2}$ is $L^p(G)$-$L^q(G)$ bounded and the inequality

$$\|f\|_{L^q(G)} \leq C\|(1 - \Delta_{sub})^{a/2})f\|_{L^p(G)} \tag{4.26}$$

holds true provided that

$$a \geq Q\left(\frac{1}{p}-\frac{1}{q}\right), \quad 1<p\leq 2\leq q<\infty. \tag{4.27}$$

Here the constant C in (4.26) is given by

$$C := \|(I-\Delta_{sub})^{-a/2}\|_{L^{r,\infty}(\mathrm{VN}_R(G))}.$$

One can always associate with Δ_{sub} a version of Sobolev spaces. Let us define

$$\|f\|_{W^{a,p}_{\Delta_{sub}}(G)} := \|(I-\Delta_{sub})^{a/2}f\|_{L^p(G)}. \tag{4.28}$$

Then the Borel functional calculus (see e.g. [Arv06]) together with (4.26)-(4.27) immediately yield

$$\|f\|_{W^{b,q}_{\Delta_{sub}}(G)} \leq C\|f\|_{W^{a,p}_{\Delta_{sub}}(G)}, \quad a-b\geq Q\left(\frac{1}{p}-\frac{1}{q}\right). \tag{4.29}$$

Each sub-Riemannian structure yields a sub-Laplacian Δ_{sub} on G . If we fix a group von Neumann algebra $\mathrm{VN}_R(G)$, then the constant in inequality (4.29) depends only on the values of the trace τ on the algebra $\mathrm{VN}_R(G)$ and not on a particular choice of a sub-Laplacian Δ_{sub}. Similarly, the Sobolev spaces $W^{a,p}_{\Delta_{sub}}(G)$ do not depend on a particular choice of a sub-Laplacian.

4.1.2 Sub-Laplacian on the Heisenberg group

Here we look at the example of the Heisenberg group determining the value of α in (4.18) for the sub-Laplacian. The interesting point here is that while the spectrum of the sub-Laplacian is continuous, Theorem 4.1 can be effectively used in this situation as well.

Corollary 4.5. Let $\mathcal{L}$ be a positive sub-Laplacian on the Heisenberg group $\mathbb{H}^n$ and let $Q=2n+2$ be the homogeneous dimension of $\mathbb{H}^n$. We claim that

$$\tau(E_{(0,s)}(\mathcal{L})) \lesssim s^{Q/2}, \quad s>0. \tag{4.30}$$

Thus, under conditions of Theorem 4.1 on φ, the spectral multiplier $\varphi(\mathcal{L})$ is τ-measurable with respect to $\mathrm{VN}_R(\mathbb{H}^n)$ and

$$\|\varphi(\mathcal{L})\|_{L^{r,\infty}(\mathrm{VN}_R(\mathbb{H}^n))} \simeq \sup_{u>0} u^{\frac{Q}{2r}}\varphi(u), \quad \frac{1}{r} = \frac{1}{p} - \frac{1}{q}. \tag{4.31}$$

For example, by choosing $\varphi(u) = \frac{1}{(1+u)^{a/2}}$, $\alpha > 0$, we recover the Sobolev embedding inequalities

$$\|(I+\mathcal{L})^{b/2} f\|_{L^q(\mathbb{H}^n)} \leq C\|(I+\mathcal{L})^{a/2} f\|_{L^p(\mathbb{H}^n)}, \tag{4.32}$$

provided

$$a - b \geq Q\left(\frac{1}{p} - \frac{1}{q}\right). \tag{4.33}$$

Inequality (4.32) has been established by Folland [Fol75], and it can be extended further for Rockland operators on general graded Lie groups [FR16].

Proof of Corollary 4.5. By Theorem 3.1, we get

$$\|\varphi(\mathcal{L})\|_{L^p(\mathbb{H}^n)\to L^q(\mathbb{H}^n)} \lesssim \|\varphi(\mathcal{L})\|_{L^{r,\infty}(\mathrm{VN}_R(\mathbb{H}^n))}. \tag{4.34}$$

Hence it is sufficient to find the conditions on φ so that the right-hand side in (4.34) is finite. By Theorem 4.1 we have

$$\|\varphi(\mathcal{L})\|_{L^{r,\infty}(\mathrm{VN}_R(G))} = \sup_{u>0}[\tau(E_{(0,u)}(|\mathcal{L}))]^{\frac{1}{r}}\varphi(u). \tag{4.35}$$

We shall now show (4.30). Since $\Delta_{sub}^{\mathbb{H}^n}$ is affiliated with $\mathrm{VN}_R(\mathbb{H}^n)$ it can be decomposed (see e.g. [DNSZ16, Proposition 4.4.])

$$\Delta_{sub}^{\mathbb{H}^n} = \bigoplus_{\widehat{\mathbb{H}^n}} \int \Delta_{sub}^{\mathbb{H}^n}(\lambda) d\nu(\lambda) \tag{4.36}$$

with respect to the center

$$C = \mathrm{VN}_R(\mathbb{H}^n) \cap \mathrm{VN}_R(\mathbb{H}^n)^!$$

of the group von Neumann algebra $\mathrm{VN}_R(\mathbb{H}^n)$ (see (A.23)).

Here the collection $\{\Delta_{sub}^{\mathbb{H}^n}(\lambda)\}_{\lambda\in\widehat{\mathbb{H}^n}}$ of the (densely defined) operators

$$\Delta_{sub}^{\mathbb{H}^n}(\lambda)\colon L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$$

can be interpreted as the global symbol of the operator $\Delta_{sub}^{\mathbb{H}^n}$, as developed in [FR16].

Hence, the spectral projections $E_{(0,s)}(\Delta_{sub}^{\mathbb{H}^n})$ can be decomposed

$$E_{(0,s)}(\Delta_{sub}^{\mathbb{H}^n}) = \bigoplus_{\widehat{\mathbb{H}^n}}\int E_{(0,s)}(\Delta_{sub}^{\mathbb{H}^n}(\lambda))|\lambda|^n d\lambda. \tag{4.37}$$

As a consequence [Dix81, Theorem 1 on page 225], we get

$$\tau(E_{(0,s)}(\Delta_{sub}^{\mathbb{H}^n}) = \int_{\mathbb{H}^n}\tau(E_{(0,s)}[\Delta_{sub}^{\mathbb{H}^n}(\lambda)])|\lambda|^n d\lambda. \tag{4.38}$$

The global symbol $\Delta_{sub}^{\mathbb{H}^n}(\lambda)\colon \mathcal{S}(\mathbb{R}^n)\subset L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ of the sub-Laplacian $\Delta_{sub}^{\mathbb{H}^n}(\lambda)$ can be found in [FR16, Lemma 6.2.1]

$$\Delta_{sub}^{\mathbb{H}^n}(\lambda)f(u) = -|\lambda|(\Delta_{\mathbb{R}^n}f(u)-|u|^2f(u)),\quad f\in\mathcal{S}(\mathbb{R}^n),\quad u\in\mathbb{R}^n, \tag{4.39}$$

and is a rescaled harmonic oscillator on $\mathbb{R}^n$, see also Folland [Fol16b]. It is known that for each $\lambda\in\mathbb{R}\setminus\{0\}$ the operator $\mathcal{L}_\lambda$ has discrete spectrum

$$\mathrm{Sp}(\Delta_{sub}^{\mathbb{H}^n}(\lambda)) = \{s_{1,\lambda}\le s_{2,\lambda}\le\ldots\le s_{m,\lambda}\le\ldots\}.$$

Since $\mathbb{H}^n$ is of type I, we have

$$\tau(E_{(0,s)}[\Delta_{sub}^{\mathbb{H}^n}(\lambda)]) = \sum_{\substack{k\in\mathbb{N}^n\\ s_{k,\lambda}<s}} 1. \tag{4.40}$$

The eigenvalues $s_{k,\lambda}$ are well-known and are given by

$$s_{k,\lambda} = \lambda\prod_{j=1}^{n}(2k_j+1), \tag{4.41}$$

see e.g. [NR10]. Thus, collecting (4.38), (4.40) and (4.41), we finally obtain

$$\tau(E_{(0,s)}(\Delta_{sub}^{\mathbb{H}^n})) = \int\limits_{\widehat{\mathbb{H}^n}} \sum_{\substack{k\in\mathbb{N}^n \\ s_{k,\lambda}<s}} 1|\lambda|^n d\lambda \;=$$

$$\int\limits_{\widehat{\mathbb{H}^n}} \sum_{\substack{k\in\mathbb{N}^n \\ |\lambda|\prod_{j=1}^n(2k_j+1)<s}} 1|\lambda|^n d\lambda \;=$$

$$\sum_{k\in\mathbb{N}^n} \int\limits_{\substack{\widehat{\mathbb{H}^n} \\ |\lambda|\leq \frac{s}{\prod_{j=1}^n (2k_j+1)}}} |\lambda|^n d\lambda \;=\; \frac{s^{n+1}}{n+1}\prod_{j=1}^n \sum_{k_j\in\mathbb{N}} \frac{1}{(2k_j+1)^{n+1}}.$$

Summarising, we have

$$\tau(E_{(0,s)}(\Delta_{sub}^{\mathbb{H}^n})) = C_n s^{\frac{Q}{2}},$$

where we used the fact the homogeneous dimension Q of the Heiseneberg group $\mathbb{H}^n$ equals $2n+2$, i.e.

$$Q = 2n+2.$$

Now, we notice that

$$\tau(E_{(0,s)}(I+\Delta_{sub}^{\mathbb{H}^n})) = \tau(E_{(0,s-1)}(\Delta_{sub}^{\mathbb{H}^n})) = C_n(s-1)^{\frac{Q}{2}} \lesssim s^{\frac{Q}{2}}. \tag{4.42}$$

Finally, using (4.35) it can be seen that $\|\varphi(\mathcal{L})\|_{L^{r,\infty}(\mathrm{VN}_R(\mathbb{H}^n))}$ is finite if and only if condition (4.33) holds. $\square$

The same argument extends to general Rockland operators, at least on the Heisenberg group. Indeed, if $\mathcal{R}$ is a positive hypoelliptic homogeneous of degree ν left-invariant differential operator on $\mathbb{H}^n$, it follows from [tER97, Theorem 4.1]that the Weyl eigenvalue counting function of $\pi_1(\mathcal{R})$ satisfies $N(\mu) \sim \mu^{\frac{2n}{\nu}}$, and hence its eigenvalues are $\mu_k \sim k^{\frac{\nu}{2n}}$. Moreover, we have $\pi_\lambda(\mathcal{R}) = |\lambda|^{\frac{\nu}{2}}\pi_1(\mathcal{R})$ for the Schrödinger representations π_λ. Consequently, the same argument as above shows that

$$\tau(E_{(0,s)}(I+\mathcal{R})) \cong s^{\frac{Q}{\nu}}, \tag{4.43}$$

determining the value of $\alpha = \frac{Q}{\nu}$ in (4.18). Hence also

$$\|\varphi(I+\mathcal{R})\|_{L^p(\mathbb{H}^n)\to L^q(\mathbb{H}^n)} \lesssim \|\varphi(I+\mathcal{R})\|_{L^{r,\infty}(\mathrm{VN}_R(\mathbb{H}^n))} = \sup_{u>0} s^{\frac{Q}{\nu}\left(\frac{1}{p}-\frac{1}{q}\right)}\varphi(u), \tag{4.44}$$

for any monotonically decreasing function φ satisfying conditions of Theorem 4.1.

Chapter 5

Hausdorff-Young-Paley inequalities and Fourier multipliers on compact homogeneous manifolds

In Section 5.1 we fix the notation for the representation theory of compact Lie groups and formulate estimates relating functions to the behaviour of their Fourier coefficients: the version of the Hardy–Littlewood inequalities on arbitrary compact homogeneous manifold G/K and further extensions. We develop Paley and Hausdorff-Young-Paley inequalities in Section 5.2 In Section 5.3 we obtain L^p-L^q Fourier multiplier theorem on G/K and the L^p-L^q boundedness theorem for general operators on G.

5.1 Notation and Hardy-Littlewood inequalities

Hardy and Littlewood proved the following generalisation of
the Plancherel's identity on the circle $\mathbb{T}$.

Theorem 5.1 (Hardy–Littlewood [HL27])**.** The following holds.

1. Let $1 < p \leq 2$. If $f \in L^p(\mathbb{T})$, then

$$\sum_{m\in\mathbb{Z}} (1+|m|)^{p-2}|\widehat{f}(m)|^p \leq C_p \|f\|^p_{L^p(\mathbb{T})}, \tag{5.1}$$

where C_p is a constant which depends only on p.

2. Let $2 \leq p < \infty$. If $\{\widehat{f}(m)\}_{m\in\mathbb{Z}}$ is a sequence of complex numbers such that

$$\sum_{m\in\mathbb{Z}} (1+|m|)^{p-2}|\widehat{f}(m)|^p < \infty, \tag{5.2}$$

then there is a function $f \in L^p(\mathbb{T})$ with Fourier coefficients given by $\widehat{f}(m)$, and

$$\|f\|^p_{L^p(\mathbb{T})} \leq C'_p \sum_{m\in\mathbb{Z}} (1+|m|)^{p-2}|\widehat{f}(m)|^p.$$

Hewitt and Ross [HR74] generalised this theorem to the setting of compact abelian groups. We note that if $\Delta = \partial_x^2$ is the Laplacian on $\mathbb{T}$, and $\mathcal{F}_{\mathbb{T}}$ is the Fourier transform on $\mathbb{T}$, the Hardy-Littlewood inequality (5.1) can be reformulated as

$$\|\mathcal{F}_{\mathbb{T}}\Big((1-\Delta)^{\frac{p-2}{2p}} f\Big)\|_{\ell^p(\mathbb{Z})} \leq C_p \|f\|_{L^p(\mathbb{T})}. \tag{5.3}$$

Denoting $(1-\Delta)^{\frac{p-2}{2p}} f$ by f again, this becomes also equivalent to the estimate

$$\|\widehat{f}\|_{\ell^p(\mathbb{Z})} \leq C_p \|(1-\Delta)^{-\frac{p-2}{2p}} f\|_{L^p(\mathbb{T})} \equiv C_p \|(1-\Delta)^{\frac{1}{p}-\frac{1}{2}} f\|_{L^p(\mathbb{T})},\ 1 < p \leq 2. \tag{5.4}$$

The first purpose of this section is to argue what could be a noncommutative version of these estimates and then to establish an analogue of Theorem 5.1 in the setting of compact homogeneous manifolds. To motivate the formulation, we start with a compact Lie group G. Identifying a representation π with its equivalence class and choosing some bases in the representation spaces, we can think of $\pi \in \widehat{G}$ as a unitary matrix-valued mapping $\pi : G \to \mathbb{C}^{d_\pi \times d_\pi}$. For $f \in L^1(G)$, we define its

Fourier transform at $\pi \in \widehat{G}$ by

$$(\mathcal{F}_G f)(\pi) \equiv \widehat{f}(\pi) := \int_G f(u)\pi(u)^* du,$$

where du is the normalised Haar measure on G. This definition can be extended to distributions $f \in \mathcal{D}'(G)$, and the Fourier series takes the form

$$f(u) = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\left(\pi(u)\widehat{f}(\pi)\right). \tag{5.5}$$

The Plancherel identity on G is given by

$$\|f\|^2_{L^2(G)} = \sum_{\pi \in \widehat{G}} d_\pi \|\widehat{f}(\pi)\|^2_{\mathtt{HS}} =: \|\widehat{f}\|^2_{\ell^2(\widehat{G})}, \tag{5.6}$$

yielding the Hilbert space $\ell^2(\widehat{G})$. Thus, Fourier coefficients of functions and distributions on G take values in the space

$$\Sigma = \left\{\sigma = (\sigma(\pi))_{\pi \in \widehat{G}} : \sigma(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}\right\}. \tag{5.7}$$

For $1 < p < \infty$, we define the space $\ell^p(\widehat{G})$ by the norm

$$\|\sigma\|_{\ell^p(\widehat{G})} := \left(\sum_{\pi \in \widehat{G}} d_\pi^{p\left(\frac{2}{p}-\frac{1}{2}\right)} \|\sigma(\pi)\|^p_{\mathtt{HS}}\right)^{1/p}, \ \sigma \in \Sigma, \ 1 < p < \infty, \tag{5.8}$$

where $\|\cdot\|_{\mathtt{HS}}$ denotes the Hilbert-Schmidt matrix norm i.e.

$$\|\sigma(\pi)\|_{\mathtt{HS}} := \left(\operatorname{Tr}(\sigma(\pi)\sigma(\pi)^*)\right)^{\frac{1}{2}}.$$

The power of d_π in (5.8) can be naturally interpreted if we rewrite it in the form

$$\|\sigma\|_{\ell^p(\widehat{G})} := \left(\sum_{\pi \in \widehat{G}} d_\pi^2 \left(\frac{\|\sigma(\pi)\|_{\mathtt{HS}}}{\sqrt{d_\pi}}\right)^p\right)^{1/p}, \ \sigma \in \Sigma, \ 1 < p < \infty, \tag{5.9}$$

and think of $\mu(Q) = \sum_{\pi \in Q} d_\pi^2$ as the Plancherel measure on $\widehat{G}$, and of $\sqrt{d_\pi}$ as the normalisation for matrices $\sigma(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$, in view of $\|I_{d_\pi}\|_{\mathtt{HS}} = \sqrt{d_\pi}$ for the identity matrix $I_{d_\pi} \in \mathbb{C}^{d_\pi \times d_\pi}$.

It was shown in [RT10, Section 10.3] that, among other things, these are interpolation spaces, and that the Fourier transform $\mathcal{F}_G$ and its inverse $\mathcal{F}_G^{-1}$ satisfy the Hausdorff-Young inequalities in these spaces. These can be written as follows

$$\left(\sum_{\pi \in \widehat{G}} d_\pi^{2-\frac{p}{2}} \|\widehat{f}(\pi)\|_{\mathtt{HS}}^p \right)^{\frac{1}{p}} = \|\widehat{f}\|_{\ell^{p'}_{sch}(\widehat{G})} \leq \|f\|_{L^p(G)} \text{ for } 1 < p \leq 2.$$

We now describe the setting of Fourier coefficients on a compact homogeneous manifold M following [DR14] or [NRT16], and referring for further details with proofs to Vilenkin [Vil68] or to Vilenkin and Klimyk [VK91].

Let G be a compact group acting transitively on M and let K be the stationary subgroup of some point. Alternatively, we can start with a compact Lie group G with a closed subgroup K, and identify $M = G/K$ as an analytic manifold in a canonical way. We normalise measures so that the measure on K is a probability one. Typical examples are the spheres $\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ or complex spheres $\mathbb{CS}^n = \mathrm{SU}(n+1)/\mathrm{SU}(n)$.

Let us denote by $\widehat{G}_0$ the subset of $\widehat{G}$ of representations that are class I with respect to the subgroup K. This means that $\pi \in \widehat{G}_0$ if π has at least one non-zero invariant vector a with respect to K, i.e. that

$$\pi(h)a = a \text{ for all } h \in K.$$

Let $\mathcal{B}_\pi$ denote the space of these invariant vectors and let

$$k_\pi := \dim \mathcal{B}_\pi.$$

Let us fix an orthonormal basis in the representation space of π so that its first k_π vectors are the basis of B_π. The matrix elements $\pi(x)_{ij}$, $1 \leq j \leq k_\pi$, are invariant under the right shifts by K.

We note that if $K = \{e\}$ so that $M = G/K = G$ is the Lie group, we have $\widehat{G} = \widehat{G}_0$ and $k_\pi = d_\pi$ for all π. As the other extreme, if K is a massive subgroup of G, i.e., if for every such π there is precisely one invariant vector with respect to K, we have $k_\pi = 1$ for all $\pi \in \widehat{G}_0$. This is, for example, the case for the spheres $M = \mathbb{S}^n$. Other examples can be found in Vilenkin [Vil68] .

We can now identify functions on $M = G/K$ with functions on G which are constant on left cosets with respect to K. Then, for a function $f \in C^\infty(M)$ we can recover it by the Fourier series of its canonical lifting $\widetilde{f}(g) := f(gK)$ to G, $\widetilde{f} \in C^\infty(G)$, and the Fourier coefficients satisfy $\widehat{\widetilde{f}}(\pi) = 0$ for all representations with $\pi \notin \widehat{G}_0$. Also, for class I representations $\pi \in \widehat{G}_0$ we have $\widehat{\widetilde{f}}(\pi)_{ij} = 0$ for $i > k_\pi$.

With this, we can write the Fourier series of f (or of $\widetilde{f}$, but we identify these) in terms of the spherical functions π_{ij} of the representations $\pi \in \widehat{G}_0$, with respect to the subgroup K. Namely, the Fourier series (5.5) becomes

$$f(x) = \sum_{\pi \in \widehat{G}_0} d_\pi \sum_{i=1}^{d_\pi} \sum_{j=1}^{k_\pi} \widehat{f}(\pi)_{ji} \pi(x)_{ij} = \sum_{\pi \in \widehat{G}_0} d_\pi \operatorname{Tr}(\widehat{f}(\pi)\pi(x)), \tag{5.10}$$

where, in order to have the last equality, we adopt the convention of setting $\pi(x)_{ij} := 0$ for all $j > k_\pi$, for all $\pi \in \widehat{G}_0$. With this convention the matrix $\pi(x)\pi(x)^*$ is diagonal with the first k_π diagonal entries equal to one and others equal to zero, so that we have

$$\|\pi(x)\|_{\mathtt{HS}} = \sqrt{k_\pi} \text{ for all } \pi \in \widehat{G}_0,\ x \in G/K. \tag{5.11}$$

Following [DR14], we will say that *the collection of Fourier coefficients* $\{\widehat{f}(\pi)_{ij} : \pi \in \widehat{G}, 1 \leq i, j \leq d_\pi\}$ *is of class I with respect to* K if $\widehat{f}(\pi)_{ij} = 0$ whenever $\pi \notin \widehat{G}_0$ or $i > k_\pi$. By the above discussion, if the collection of Fourier coefficients is of class I with respect to K, then the expressions

(5.5) and (5.10) coincide and yield a function f such that $f(xh) = f(h)$ for all $h \in K$, so that this function becomes a function on the homogeneous space G/K. For the space of Fourier coefficients of class I we define the analogue of the set Σ in (5.7) by

$$\Sigma(G/K) := \{\sigma : \pi \mapsto \sigma(\pi) : \ \pi \in \widehat{G}_0\}, \tag{5.12}$$

where we assume that $\sigma(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$, $\sigma(\pi)_{ij} = 0$ for $i > k_\pi$. In analogy to (5.8), we can define the Lebesgue spaces $\ell^p(\widehat{G}_0)$ by the following norms which we will apply to Fourier coefficients $\widehat{f} \in \Sigma(G/K)$ of $f \in \mathcal{D}'(G/K)$. Thus, for $\sigma \in \Sigma(G/K)$ we set

$$\|\sigma\|_{\ell^p(\widehat{G}_0)} := \left(\sum_{\pi \in \widehat{G}_0} d_\pi k_\pi^{p(\frac{1}{p}-\frac{1}{2})} \|\sigma(\pi)\|_{\mathtt{HS}}^p \right)^{1/p}, \ 1 \leq p < \infty. \tag{5.13}$$

In the case $K = \{e\}$, so that $G/K = G$, these spaces coincide with those defined by (5.8) since $k_\pi = d_\pi$ in this case. Again, by the same argument as that in [RT10], these spaces are interpolation spaces and the Hausdorff-Young inequality holds for them. We refer to [NRT16] for some more details on these spaces.

Similarly to (5.9), the power of k_π in (5.13) can be naturally interpreted if we rewrite it in the form

$$\|\sigma\|_{\ell^p(\widehat{G}_0)} := \left(\sum_{\pi \in \widehat{G}_0} d_\pi k_\pi \left(\frac{\|\sigma(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}} \right)^p \right)^{1/p}, \ \sigma \in \Sigma(G/K), \ 1 \leq p < \infty, \tag{5.14}$$

and think of $\mu(Q) = \sum_{\pi \in Q} d_\pi k_\pi$ as the Plancherel measure on $\widehat{G}_0$, and of $\sqrt{k_\pi}$ as the normalisation for matrices $\sigma(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ under the adopted convention on their zeros in (5.12).

Let $\Delta_{G/K}$ be the differential operator on G/K obtained by the Laplacian Δ_G on G acting on functions that are constant on right cosets of G, i.e., such that $\widetilde{\Delta_{G/K} f} = \Delta_G \widetilde{f}$ for $f \in C^\infty(G/K)$.

Recalling, that the Hardy-Littlewood inequality can be formulated as

(5.3) or (5.4), we will show that the analogue of (5.4) on a compact homogeneous manifold G/K becomes

$$\|\widehat{f}\|_{\ell^p(\widehat{G}_0)} \leq C_p \|(1-\Delta_{G/K})^{n(\frac{1}{p}-\frac{1}{2})} f\|_{L^p(G/K)}, \quad 1 < p \leq 2, \tag{5.15}$$

where $n = \dim G/K$. This yields sharper results compared to using the Schatten-based space $\ell^p_{sch}(\widehat{G}_0)$ in view of the inequality

$$\|\widehat{f}\|_{\ell^p_{sch}(\widehat{G}_0)} \leq \|\widehat{f}\|_{\ell^p(\widehat{G}_0)}.$$

For more extensive analysis and description of Laplace operators on compact Lie groups and on compact homogeneous manifolds we refer to e.g. [Ste70] and [Pes08], respectively. We note that every representation $\pi(x) = (\pi_{ij}(x))_{i,j=1}^{d_\pi} \in \widehat{G}_0$ is invariant under the right shift by K. Therefore, $\pi(x)_{ij}$ for all $1 \leq i,j \leq d_\pi$ are eigenfunctions of $\Delta_{G/K}$ with the same eigenvalue, and we denote by $\langle\pi\rangle$ the corresponding eigenvalue for the first order pseudo-differential operator $(1-\Delta_{G/K})^{1/2}$, so that we have

$$(1-\Delta_{G/K})^{1/2}\pi(x)_{ij} = \langle\pi\rangle\pi(x)_{ij} \text{ for all } 1 \leq i,j \leq d_\pi.$$

We now formulate the analogue of the Hardy-Littlewood Theorem 5.1 on a compact homogeneous manifolds G/K as the inequality (5.15) and its dual:

Theorem 5.2 (Hardy-Littlewood inequalities)**.** Let G/K be a compact homogeneous manifold of dimension n. Then the following holds.

1. Let $1 < p \leq 2$. If $f \in L^p(G/K)$, then

$$\|\mathcal{F}_{G/K}\Big((1-\Delta_{G/K})^{n(\frac{1}{2}-\frac{1}{p})} f\Big)\|_{\ell^p(\widehat{G}_0)} \leq C_p \|f\|_{L^p(G/K)}. \tag{5.16}$$

Equivalently, we can rewrite this estimate as

$$\sum_{\pi\in\widehat{G}_0} d_\pi k_\pi^{p(\frac{1}{p}-\frac{1}{2})} \langle\pi\rangle^{n(p-2)} \|\widehat{f}(\pi)\|^p_{\mathtt{HS}} \leq C_p \|f\|^p_{L^p(G/K)}. \tag{5.17}$$

2. Let $2 \leq p < \infty$. If $\{\sigma(\pi)\}_{\pi\in\widehat{G}_0} \in \Sigma(G/K)$ is a sequence of complex matrices such that $\langle\pi\rangle^{n(p-2)}\sigma(\pi)$ is in $\ell^p(\widehat{G}_0)$, then there is a function $f \in L^p(G/K)$ with Fourier coefficients given by $\widehat{f}(\pi) = \sigma(\pi)$, and

$$\|f\|_{L^p(G/K)} \leq C_p' \|\langle\pi\rangle^{\frac{n(p-2)}{p}} \widehat{f}(\pi)\|_{\ell^p(\widehat{G}_0)}. \tag{5.18}$$

Using the definition of the norm on the right hand side we can write (5.18) as

$$\|f\|^p_{L^p(G/K)} \leq C_p' \sum_{\pi\in\widehat{G}_0} d_\pi k_\pi^{p(\frac{1}{p}-\frac{1}{2})} \langle\pi\rangle^{n(p-2)} \|\widehat{f}(\pi)\|^p_{\mathtt{HS}}. \tag{5.19}$$

For $p = 2$, both of these statements reduce to the Plancherel identity (5.6). We note that the formulations in terms of the Hilbert-Schmidt L^p-spaces $\ell^p(\widehat{G}_0)$ are sharper (see [ANR15] for more details).

Proof of Theorem 5.2. The second part of Theorem 5.2 follows from the first by duality, so we will concentrate on proving the first part.

Denote by $N(L)$ the eigenvalue counting function of eigenvalues (counted with multiplicities) of the first order elliptic pseudo-differential operator $(I - \Delta_{G/K})^{\frac{1}{2}}$ on the compact manifold G/K, i.e.

$$N(L) := \sum_{\substack{\pi\in\widehat{G}_0 \\ \langle\pi\rangle\leq L}} d_\pi k_\pi. \tag{5.20}$$

Using the eigenvalue counting function $N(L)$, we can reformulate condition (5.23) for $\varphi(\pi) = \langle\pi\rangle^{-n}$ in the following form

$$\sup_{0<u<+\infty} uN(u^{-\frac{1}{n}}) < \infty. \tag{5.21}$$

Since $N(L)$ is a right-continuous monotone function, the set of discontinuity points on $(0, +\infty)$ is at most countable. Therefore, without loss of generality, we can assume that $\psi(u) = uN\left(\left(\frac{1}{u}\right)^{\frac{1}{n}}\right)$ is a continuous function on $(0, +\infty)$. It is clear that $\lim_{u\to+\infty} \psi(u) = 0$. Further, we use the asymptotic of the Weyl eigenvalue counting function $N(L)$ for the first order elliptic pseudo-differential operator $(1-\Delta_{G/K})^{1/2}$ on the compact manifold G/K, to get that the eigenvalue counting function $N(L)$ (see e.g. Shubin [Shu87]) satisfies

$$N(L) = \sum_{\substack{\pi\in\widehat{G}_0 \\ \langle\pi\rangle\leq L}} d_\pi k_\pi \cong L^n \quad \text{for large } L. \tag{5.22}$$

With $L = \left(\frac{1}{u}\right)^{\frac{1}{n}}$ and $n = \dim G/K$, this implies

$$\lim_{u\to 0} \psi(u) = \lim uN\left(\left(\frac{1}{u}\right)^{\frac{1}{n}}\right) = \lim_{u\to 0} u\left(\frac{1}{u^{\frac{1}{n}}}\right)^n = \lim_{u\to 0} 1 = 1.$$

Thus, we showed that $\psi(u)$ is a bounded function on $(0,+\infty)$, or equivalently, we established (5.21). Then, it is clear that $\varphi(\pi) = \langle\pi\rangle^{-n}$ satisfies condition (5.23). The application of the Paley inequality from Theorem 5.3 yields the Hardy-Littlewood inequality. This completes the proof. $\square$

5.2 Paley and Hausdorff-Young-Paley inequalities

In [Hör60], Lars Hoŗmander proved a Paley-type inequality for the Euclidean Fourier transform on $\mathbb{R}^n$. Here we give an analogue of this inequality on compact homogeneous manifolds.

Theorem 5.3 (Paley-type inequality). Let G/K be a compact homogeneous manifold. Let $1 < p \leq 2$. If $\varphi(\pi)$ is a positive sequence over $\widehat{G}_0$ such that

$$M_\varphi := \sup_{t>0} t \sum_{\substack{\pi\in\widehat{G}_0 \\ \varphi(\pi)\geq t}} d_\pi k_\pi < \infty \tag{5.23}$$

is finite, then we have

$$\left(\sum_{\pi\in\widehat{G}_0} d_\pi k_\pi^{p(\frac{1}{p}-\frac{1}{2})} \|\widehat{f}(\pi)\|_{\mathtt{HS}}^p \,\varphi(\pi)^{2-p}\right)^{\frac{1}{p}} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(G/K)}. \tag{5.24}$$

As usual, the sum over an empty set in (5.23) is assumed to be zero.

With $\varphi(\pi) = \langle\pi\rangle^{-n}$, where $n = \dim G/K$, using the asymptotic formula (5.22) for the Weyl eigenvalue counting function, we recover the first part of Theorem 5.2 (see the proof of Theorem 5.2). In this sense, the Paley inequality is an extension of one of the Hardy-Littlewood inequalities.

Proof of Theorem 5.3. Let ν give measure $\varphi^2(\pi)d_\pi k_\pi$ to the set consisting of the single point $\{\pi\}, \pi \in \widehat{G}_0$, i.e.

$$\nu(\{\pi\}) := \varphi^2(\pi)d_\pi k_\pi.$$

We define the corresponding space $L^p(\widehat{G}_0, \nu)$, $1 \le p < \infty$, as the space of complex (or real) sequences $a = \{a_\pi\}_{\pi\in\widehat{G}_0}$ such that

$$\|a\|_{L^p(\widehat{G}_0,\nu)} := \left(\sum_{\pi\in\widehat{G}_0} |a_\pi|^p \varphi^2(\pi) d_\pi k_\pi\right)^{\frac{1}{p}} < \infty. \tag{5.25}$$

We will show that the sub-linear operator

$$A\colon L^p(G/K) \ni f \mapsto Af = \left\{\frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)}\right\}_{\pi\in\widehat{G}_0} \in L^p(\widehat{G}_0, \nu)$$

is well-defined and bounded from $L^p(G/K)$ to $L^p(\widehat{G}_0, \nu)$ for $1 < p \le 2$. In other words, we claim that we have the estimate

$$\|Af\|_{L^p(\widehat{G}_0,\nu)} = \left(\sum_{\pi\in\widehat{G}_0} \left(\frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)}\right)^p \varphi^2(\pi) d_\pi k_\pi\right)^{\frac{1}{p}} \lesssim N_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(G/K)}, \tag{5.26}$$

which would give (5.26) and where we set $N_\varphi := \sup_{t>0} t \sum\limits_{\substack{\pi\in\widehat{G}_0 \\ \varphi(\pi)\geq t}} d_\pi k_\pi$. We will show that A is of weak type (2,2) and of weak-type (1,1). For definition and discussions we refer to Section A.2 where we give definitions of weak-type, formulate and prove Marcinkiewicz-type interpolation Theorem A.5 to be used in the present setting. More precisely, with the distribution function ν as in Theorem A.5, we show that

$$\begin{aligned} \nu_{\widehat{G}_0}(y; Af) &\leq \left(\frac{M_2\|f\|_{L^2(G/K)}}{y}\right)^2 \quad \text{with norm } M_2 = 1, \quad (5.27) \\ \nu_{\widehat{G}_0}(y; Af) &\leq \frac{M_1\|f\|_{L^1(G/K)}}{y} \quad \text{with norm } M_1 = M_\varphi, \quad (5.28) \end{aligned}$$

where $\nu_{\widehat{G}_0}$ is defined in Section A.2. Then (5.26) would follow by Marcinkiewicz interpolation theorem (Theorem A.5 from Section A.2) with $\Gamma = \widehat{G}_0$ and $\delta_\pi = d_\pi, \kappa_\pi = k_\pi$.

Now, to show (5.27), using Plancherel's identity (5.6), we get

$$\begin{aligned} y^2\nu_{\widehat{G}_0}(y; Af) \leq \|Af\|^2_{L^p(\widehat{G}_0,\nu)} &= \sum_{\pi\in\widehat{G}_0} d_\pi k_\pi \left(\frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)}\right)^2 \varphi^2(\pi) \\ &= \sum_{\pi\in\widehat{G}_0} d_\pi \|\widehat{f}(\pi)\|^2_{\mathtt{HS}} = \|\widehat{f}\|^2_{\ell^2(\widehat{G}_0)} = \|f\|^2_{L^2(G/K)}. \end{aligned}$$

Thus, A is of type (2,2) with norm $M_2 \leq 1$. Further, we show that A is of weak-type (1,1) with norm $M_1 = M_\varphi$; more precisely, we show that

$$\nu_{\widehat{G}_0}\{\pi \in \widehat{G}_0 \colon \frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y\} \lesssim M_\varphi \frac{\|f\|_{L^1(G/K)}}{y}. \quad (5.29)$$

The left-hand side here is the weighted sum $\sum \varphi^2(\pi) d_\pi k_\pi$ taken over those $\pi \in \widehat{G}_0$ for which $\frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y$. From the definition of the Fourier transform it follows that

$$\|\widehat{f}(\pi)\|_{\mathtt{HS}} \leq \sqrt{k_\pi}\|f\|_{L^1(G/K)}.$$

Therefore, we have

$$y < \frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)} \leq \frac{\|f\|_{L^1(G/K)}}{\varphi(\pi)}.$$

Using this, we get

$$\left\{\pi \in \widehat{G}_0 \colon \frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y\right\} \subset \left\{\pi \in \widehat{G}_0 \colon \frac{\|f\|_{L^1(G/K)}}{\varphi(\pi)} > y\right\}$$

for any $y > 0$. Consequently,

$$\nu\left\{\pi \in \widehat{G}_0 \colon \frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y\right\} \leq \nu\left\{\pi \in \widehat{G}_0 \colon \frac{\|f\|_{L^1(G/K)}}{\varphi(\pi)} > y\right\}.$$

Setting $v := \frac{\|f\|_{L^1(G/K)}}{y}$, we get

$$\nu\left\{\pi \in \widehat{G}_0 \colon \frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y\right\} \leq \sum_{\substack{\pi\in\widehat{G}_0 \\ \varphi(\pi)\leq v}} \varphi^2(\pi) d_\pi k_\pi. \tag{5.30}$$

We claim that

$$\sum_{\substack{\pi\in\widehat{G}_0 \\ \varphi(\pi)\leq v}} \varphi^2(\pi) d_\pi k_\pi \lesssim M_\varphi v. \tag{5.31}$$

In fact, we have

$$\sum_{\substack{\pi\in\widehat{G}_0 \\ \varphi(\pi)\leq v}} \varphi^2(\pi) d_\pi k_\pi = \sum_{\substack{\pi\in\widehat{G}_0 \\ \varphi(\pi)\leq v}} d_\pi k_\pi \int\limits_0^{\varphi^2(\pi)} d\tau.$$

We can interchange sum and integration to get

$$\sum_{\substack{\pi\in\widehat{G}_0 \\ \varphi(\pi)\leq v}} d_\pi k_\pi \int\limits_0^{\varphi^2(\pi)} d\tau = \int\limits_0^{v^2} d\tau \sum_{\substack{\pi\in\widehat{G}_0 \\ \tau^{\frac{1}{2}}\leq\varphi(\pi)\leq v}} d_\pi k_\pi.$$

Further, we make a substitution $\tau = t^2$, yielding

$$\int\limits_0^{v^2} d\tau \sum_{\substack{\pi\in\widehat{G}_0 \\ \tau^{\frac{1}{2}}\leq\varphi(\pi)\leq v}} d_\pi k_\pi = 2\int\limits_0^{v} t\,dt \sum_{\substack{\pi\in\widehat{G}_0 \\ t\leq\varphi(\pi)\leq v}} d_\pi k_\pi \leq 2\int\limits_0^{v} t\,dt \sum_{\substack{\pi\in\widehat{G}_0 \\ t\leq\varphi(\pi)}} d_\pi k_\pi.$$

Since

$$t \sum_{\substack{\pi\in\widehat{G}_0 \\ t\leq\varphi(\pi)}} d_\pi k_\pi \leq \sup_{t>0} t \sum_{\substack{\pi\in\widehat{G}_0 \\ t\leq\varphi(\pi)}} d_\pi k_\pi = M_\varphi$$

is finite by the assumption that $M_\varphi < \infty$, we have

$$2\int\limits_0^{v} t\,dt \sum_{\substack{\pi\in\widehat{G}_0 \\ t\leq\varphi(\pi)}} d_\pi k_\pi \lesssim M_\varphi v.$$

This proves (5.31). Thus, we have proved inequalities (5.27), (5.28). Then by using the Marcinkiewicz interpolation theorem (Theorem A.5 from Section A.2) with $p_1 = 1, p_2 = 2$ and $\frac{1}{p} = 1 - \theta + \frac{\theta}{2}$ we now obtain

$$\left(\sum_{\pi\in\widehat{G}_0} \left(\frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\varphi(\pi)}\right)^p \varphi^2(\pi) d_\pi k_\pi\right)^{\frac{1}{p}} = \|Af\|_{L^p(\widehat{G}_0,\mu)} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(G/K)}.$$

This completes the proof. $\square$

Now we recall the Hausdorff-Young inequality:

$$\left(\sum_{\pi\in\widehat{G}_0} d_\pi k_\pi^{p'(\frac{1}{p'}-\frac{1}{2})} \|\widehat{f}(\pi)\|_{\mathtt{HS}}^{p'}\right)^{\frac{1}{p'}}$$
$$\equiv \|\widehat{f}\|_{\ell^{p'}(\widehat{G}_0)} \lesssim \|f\|_{L^p(G/K)},\ 1 < p \leq 2, \quad (5.32)$$

where, as usual, $\frac{1}{p} + \frac{1}{p'} = 1$. The inequality (5.32) was argued in [NRT16] in analogy to [RT10, Section 10.3], so we refer there for its justification. Interpolating (see Theorem 2.3) between the Paley-type inequality (5.24) in Theorem 5.3 and Hausdorff-Young inequality (5.32), we obtain:

Theorem 5.4 (Hausdorff-Young-Paley inequality). Let G/K be a compact homogeneous manifold. Let $1 < p \leq b \leq p' < \infty$. If a positive sequence $\varphi(\pi)$, $\pi \in \widehat{G}_0$, satisfies condition

$$M_\varphi := \sup_{t>0} t \sum_{\substack{\pi \in \widehat{G}_0 \\ \varphi(\pi) \geq t}} d_\pi k_\pi < \infty, \tag{5.33}$$

then we have

$$\left(\sum_{\pi \in \widehat{G}_0} d_\pi k_\pi^{b(\frac{1}{b}-\frac{1}{2})} \left(\|\widehat{f}(\pi)\|_{\mathtt{HS}}\, \varphi(\pi)^{\frac{1}{b}-\frac{1}{p'}} \right)^b \right)^{\frac{1}{b}} \lesssim M_\varphi^{\frac{1}{b}-\frac{1}{p'}} \|f\|_{L^p(G/K)}. \tag{5.34}$$

This reduces to the Hausdorff-Young inequality (5.32) when $b = p'$ and to the Paley inequality in (5.24) when $b = p$.

Proof. We consider a sub-linear operator A which takes a function f to its Fourier transform $\widehat{f}(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ divided by $\sqrt{k_\pi}$, i.e.

$$L^p(G/K) \ni f \mapsto Af = \left\{ \frac{\widehat{f}(\pi)}{\sqrt{k_\pi}} \right\}_{\pi \in \widehat{G}_0} \in \ell^p(\widehat{G}_0, \omega),$$

where the spaces $\ell^p(\widehat{G}_0, \omega)$ is defined by the norm

$$\|\sigma(\pi)\|_{\ell^p(\widehat{G}_0,\omega)} := \left(\sum_{\pi \in \widehat{G}_0} \|\sigma(\pi)\|_{\mathtt{HS}}^p\, \omega(\pi) \right)^{\frac{1}{p}},$$

and $\omega(\pi)$ is a positive scalar sequence over $\widehat{G}_0$ to be determined. Then the statement follows from Theorem 2.3 if we regard the left-hand sides of inequalities (5.24) and (5.32) as $\|Af\|_{\ell^p(\widehat{G}_0,\omega)}$-norms in weighted sequence spaces over $\widehat{G}_0$, with the weights given by $\omega_0(\pi) = d_\pi k_\pi \varphi(\pi)^{2-p}$ and $\omega_1(\pi) = d_\pi k_\pi$, $\pi \in \widehat{G}_0$, respectively. $\square$

5.3 L^p-L^q boundedness of operators

In this section we use the Hausdorff-Young-Paley inequality in Theorem 5.4 to give a sufficient condition for the L^p-L^q boundedness of Fourier multipliers on compact homogeneous spaces. It extends the condition that was obtained by a different method in [NT00] on the circle $\mathbb{T}$. In the case of compact Lie groups, we extend the criterion for Fourier multipliers in a rather standard way, to derive a condition for the L^p-L^q boundednes of general operators, all for the range of indices $1 < p \leq 2 \leq q < \infty$.

In the case of a compact Lie group G, the Fourier multipliers correspond to left-invariant operators, and these can be characterised by the condition that their symbols do not depend on the space variable. Thus, we can write such operators A in the form

$$\widehat{Af}(\pi) = \sigma_A(\pi)\widehat{f}(\pi), \tag{5.35}$$

with the symbol $\sigma_A(\pi)$ depending only on $\pi \in \widehat{G}$. The Hörmander-Mihlin type multiplier theorem for such operators to be bounded on $L^p(G)$ for $1 < p < \infty$ was obtained in [RW13].

Now, in the context of compact homogeneous spaces G/K we still want to keep the formula (5.35) as the definition of Fourier multipliers, now for all $\pi \in \widehat{G}_0$. Indeed, due to properties of zeros of the Fourier coefficients, we have that both sides of (5.35) are zero for $\pi \notin \widehat{G}_0$. Also, for $\pi \in \widehat{G}_0$, we have $\widehat{f}(\pi) \in \Sigma(G/K)$ with the set $\Sigma(G/K)$ defined in (5.12), which means that

$$\widehat{f}(\pi)_{ij} = \widehat{Af}(\pi)_{ij} = 0 \text{ for } i > k_\pi.$$

Therefore, we can assume that the symbol σ_A of a Fourier multiplier A on G/K satisfies

$$\sigma_A(\pi) = 0 \text{ for } \pi \notin \widehat{G}_0; \text{ and } \sigma_A(\pi)_{ij} = 0 \text{ for } \pi \in \widehat{G}_0, \text{ if } i > k_\pi \text{ or } j > k_\pi. \tag{5.36}$$

Therefore, only the upper-left block in $\sigma_A(\pi)$ of the size $k_\pi \times k_\pi$ may be non-zero. Thus, we will say that A is a Fourier multiplier on G/K if conditions (5.35) and (5.36) are satisfied.

Theorem 5.5. Let $1 < p \leq 2 \leq q < \infty$ and suppose that A is a Fourier multiplier on the compact homogeneous space G/K. Then we have

$$\|A\|_{L^p(G/K)\to L^q(G/K)} \lesssim \sup_{s>0} s \left(\sum_{\substack{\pi\in\widehat{G}_0 \\ \|\sigma_A(\pi)\|_{\mathrm{op}}>s}} d_\pi k_\pi \right)^{\frac{1}{p}-\frac{1}{q}} . \tag{5.37}$$

We note that if $\mu(Q) = \sum_{\pi\in Q} d_\pi k_\pi$ denotes the Plancherel measure on $\widehat{G}_0$, then (5.37) can be rewritten as

$$\|A\|_{L^p(G/K)\to L^q(G/K)} \lesssim \sup_{s>0} \left\{ s\mu(\pi\in\widehat{G}_0 : \|\sigma_A(\pi)\|_{\mathrm{op}} > s)^{\frac{1}{p}-\frac{1}{q}} \right\}.$$

Remark 5.1. Inequality (5.37) is sharp for $p = q = 2$.

Proof of Remark 5.1. First, we have the estimate

$$\|A\|_{L^2(G/K)\to L^2(G/K)} \leq \sup_{\pi\in\widehat{G}_0} \|\sigma_A(\pi)\|_{\mathrm{op}}.$$

Since the set

$$\{\pi\in\widehat{G}_0 \colon \|\sigma_A(\pi)\|_{\mathrm{op}} \geq s\}$$

is empty for $s > \|A\|_{L^2(G/K)\to L^2(G/K)}$ and a sum over the empty set is set to be zero, we have by (5.37)

$$\begin{aligned}
\|A\|_{L^2(G/K)\to L^2(G/K)} &\leq \sup_{s>0} s \left(\sum_{\substack{\pi\in\widehat{G}_0 \\ \|\sigma_A(\pi)\|_{\mathrm{op}}\geq s}} d_\pi k_\pi \right)^{0} \\
&= \sup_{0<s\leq\|A\|_{L^2(G/K)\to L^2(G/K)}} s\cdot 1 = \|A\|_{L^2(G/K)\to L^2(G/K)}.
\end{aligned}$$

Thus, for $p = q = 2$ we attain equality in (5.37). $\square$

Proof of Theorem 5.5. Recall that A is a Fourier multiplier on G/K, i.e.

$$\widehat{Af}(\pi) = \sigma_A(\pi)\widehat{f}(\pi),$$

with σ_A satisfying (5.36). Since the application of [ANR16a, p. 17, Theorem 4.2] with $X = G/K$ and $\mu = \{\text{Haar measure on } G\}$ yields

$$\|A\|_{L^p(G/K)\to L^q(G/K)} = \|A^*\|_{L^{q'}(G/K)\to L^{p'}(G/K)}, \tag{5.38}$$

we may assume that $p \leq q'$, for otherwise we have $q' \leq (p')' = p$ and $\|\sigma_{A^*}(\pi)\|_{\mathtt{op}} = \|\sigma_A(\pi)\|_{\mathtt{op}}$. When $f \in C^\infty(G/K)$ the Hausdorff-Young inequality gives, since $q' \leq 2$,

$$\|Af\|_{L^q(G/K)} \lesssim \|\widehat{Af}\|_{\ell^{q'}(\widehat{G}_0)} = \|\sigma_A\widehat{f}\|_{\ell^{q'}(\widehat{G}_0)}.$$

We set $\sigma(\pi) := \|\sigma_A(\pi)\|_{\mathtt{op}}^r I_{d_\pi}$. It is obvious that

$$\|\sigma(\pi)\|_{\mathtt{op}} = \|\sigma_A(\pi)\|_{\mathtt{op}}^r. \tag{5.39}$$

Now, we are in a position to apply the Hausdorff-Young-Paley inequality in Theorem 5.4. With $\sigma(\pi) = \|\sigma_A\|^r I_{d_\pi}$ and $b = q'$, the assumption of Theorem 5.4 are then satisfied and since $\frac{1}{q'} - \frac{1}{p'} = \frac{1}{p} - \frac{1}{q} = \frac{1}{r}$, we obtain

$$\|\sigma_A\widehat{f}\|_{\ell^{q'}(\widehat{G}_0)} \lesssim \left(\sup_{s>0} s \sum_{\substack{\pi\in\widehat{G}_0 \\ \|\sigma(\pi)\|_{\mathtt{op}}\geq s}} d_\pi k_\pi \right)^{\frac{1}{r}} \|f\|_{L^p(G/K)}, \quad f \in L^p(G/K).$$

Further, it can be easily checked that

$$\left(\sup_{s>0} s \sum_{\substack{\pi\in\widehat{G}_0 \\ \|\sigma(\pi)\|_{\mathrm{op}}>s}} d_\pi k_\pi\right)^{\frac{1}{r}} = \left(\sup_{s>0} s \sum_{\substack{\pi\in\widehat{G}_0 \\ \|\sigma_A(\pi)\|_{\mathrm{op}}^r>s}} d_\pi k_\pi\right)^{\frac{1}{r}}$$

$$= \left(\sup_{s>0} s^r \sum_{\substack{\pi\in\widehat{G} \\ \|\sigma_A(\pi)\|_{\mathrm{op}}>s}} d_\pi k_\pi\right)^{\frac{1}{r}} = \sup_{s>0} s \left(\sum_{\substack{\pi\in\widehat{G} \\ \|\sigma_A(\pi)\|_{\mathrm{op}}>s}} d_\pi k_\pi\right)^{\frac{1}{r}}.$$

This completes the proof. $\square$

A standard addition to the proof of the preceding theorem extends Theorem 5.5 to the non-invariant case. For the simplicity in the formulation and in the understanding a variant of (5.35) in the non-invariant case, the following result is given in the context of general compact Lie groups. To fix the notation, we note that according to [RT10, Theorem 10.4.4] any linear continuous operator A on $C^\infty(G)$ can be written in the form

$$Af(g) = \sum_{\pi\in\widehat{G}} d_\pi \operatorname{Tr}\left(\pi(g)\sigma_A(g,\pi)\widehat{f}(\pi)\right)$$

for a symbol σ_A that is well-defined on $G\times\widehat{G}$ with values $\sigma_A(g,\pi)\in\mathbb{C}^{d_\pi\times d_\pi}$. Let $\{X_j\}_{j=1}^{\dim(G)}$ be a basis for the Lie algebra of G and let ∂_1 be the left-invariant vector fields corresponding to X_j. For $\alpha\in\mathbb{N}_0^n$, let us denote $\partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}\, n = \dim(G)$.

Theorem 5.6. Let $1 < p \le 2 \le q < \infty$. Suppose that $l > \frac{p}{\dim(G)}$ is an integer. Let A be a linear continuous operator on $C^\infty(G)$. Then we have

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \sum_{|\alpha|\le l} \sup_{u\in G}\sup_{s>0} s \left(\sum_{\substack{\pi\in\widehat{G} \\ \|\partial_u^\alpha\sigma_A(u,\pi)\|_{\mathrm{op}}\ge s}} d_\pi k_\pi\right)^{\frac{1}{p}-\frac{1}{q}}. \tag{5.40}$$

Proof of Theorem 5.6. Let us define

$$A_u f(g) := \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\left(\pi(g)\sigma_A(u,\pi)\widehat{f}(\pi)\right)$$

so that $A_g f(g) = Af$. Then

$$\|Af\|_{L^q(G)} = \left(\int_G |Af(g)|^q \, dg\right)^{\frac{1}{q}} \leq \left(\int_G \sup_{u\in G} |A_u f(g)|^q \, dg\right)^{\frac{1}{q}}. \tag{5.41}$$

By an application of the Sobolev embedding theorem we get

$$\sup_{u\in G} |A_u f(g)|^q \leq C \sum_{|\alpha|\leq l} \int_G |\partial_u^\alpha A_u f(g)|^q \; dy.$$

Therefore, using the Fubini theorem to change the order of integration, we obtain

$$\begin{aligned}
\|Af\|^q_{L^q(G)} &\leq C \sum_{|\alpha|\leq l} \int_G \int_G |\partial_u^\alpha A_u f(g)|^q \, dg \, du \\
&\leq C \sum_{|\alpha|\leq l} \sup_{u\in G} \int_G |\partial_u^\alpha A_y f(g)|^p \, dg \\
&= C \sum_{|\alpha|\leq l} \sup_{u\in G} \|\partial_u^\alpha A_u f\|^q_{L^q(G)} \\
&\leq C \sum_{|\alpha|\leq l} \sup_{u\in G} \|f \mapsto \operatorname{Op}(\partial_u^\alpha \sigma_A) f\|^q_{\mathcal{L}(L^p(G)\to L^q(G))} \|f\|^q_{L^p(G)} \\
&\lesssim \left[\sum_{|\alpha|\leq l} \sup_{u\in G} \sup_{s>0} s \left(\sum_{\substack{\pi\in\widehat{G} \\ \|\partial_u^\alpha \sigma_A(u,\pi)\|_{\mathrm{op}}\geq s}} d_\pi^2\right)^{\frac{1}{p}-\frac{1}{q}}\right]^q \|f\|^q_{L^p(G)},
\end{aligned}$$

where the last inequality holds due to Theorem 5.5. This completes the proof. □

Appendix A

Background

A.1 Elements of noncommutative integration

If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.

John von Neumann

We recall basic facts on spectral theory and von Neumann algebras. Main references are [KR97] and [Dix81]. Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ be the set of continuous linear operators acting in $\mathcal{H}$. This set $B(\mathcal{H})$ can be endowed with the structure of an algebra over the field of complex numbers. Let $\mathcal{M}$ be any subset of $B(\mathcal{H})$. The commutant $\mathcal{M}^!$ of $\mathcal{M}$ is the set of all elements $A \in B(\mathcal{H})$ commuting with all the elements of M, i.e.

$$\mathcal{M}^! = \{B \in B(\mathcal{H})\colon BA = AB, \quad A \in \mathcal{M}\}.$$

The bicommutant $\mathcal{M}^{!!}$ of $\mathcal{M}$ is the commutant of $\mathcal{M}^!$

$$\mathcal{M}^{!!} = [\mathcal{M}^!]^!.$$

It is clear that for any subset $\mathcal{M}$ of $B(\mathcal{H})$ we have

$$\mathcal{M} \subset \mathcal{M}^{!!} \tag{A.1}$$

It turns out that the algebraic condition (A.1) yields purely topological description.

Theorem A.1 (The von Neumann bicommutant theorem)**.** Let $\mathcal{M}$ be a sub $*$-algebra of $B(\mathcal{H})$. Then

$$\mathcal{M}^{!!} = \mathcal{M} \tag{A.2}$$

if and only if

$$\mathcal{M} \text{ is closed in the strong operator topology.} \tag{A.3}$$

Definition A.1. A sub *-algebra $\mathcal{M}$ of $B(\mathcal{H})$ shall be called von Neumann algebra if $\mathcal{M}$ is closed with respect to the strong operator topology.

Definition A.2 (Affiliated operators)**.** A linear closed operator A (possibly unbounded in $\mathcal{H}$) is said to be *affiliated with* $\mathcal{M}$, symbolically $A\nu\mathcal{M}$, if it commutes with the elements of the commutant $\mathcal{M}^{!}$ of $\mathcal{M}$, i.e.

$$AU = UA, \qquad \text{for all } \; U \in \mathcal{M}^{!}. \tag{A.4}$$

Here (A.4) includes the statement that $\mathrm{Dom}(AU) = \mathrm{Dom}(A)$. The set of all linear operators affiliated with $\mathcal{M}$ will be denoted by $\mathrm{Aff}(\mathcal{M})$.

It is common to define the affiliation as the commutativity with unitary elements in $\mathcal{M}$. It can be shown [KR97, Theorem 4.1.7] that every bounded linear operator $U \in \mathcal{M}^{!}$ is a linear combination of four unitary operators. Thus, we get the equivalent definition.

This relation ν is a natural relaxation of the relation $\in$: if A is a bounded operator affiliated with $\mathcal{M}$, then by the double commutant theorem $A \in \mathcal{M}$. One of the original motivations [MVN36, MvN37] of John von Neumann was to build a mathematical foundation for quantum mechanics. In this framework, the observables with unbounded spectrum correspond to closed densely defined unbounded operators. Although the algebra $\mathcal{M}$ consists primarily of bounded operators, the technique of projections makes it possible to approximate unbounded operators.

The polar decomposition for arbitrary closed densely defined possibly unbounded operators A acting on a Hilbert space $\mathcal{H}$ has been established in [vN32]. Thus, we apply [vN32, page 307, Theorem 7] to get

$$A = W|A|, \tag{A.5}$$

where W is a partial isometry. This means that the operators W^*W and WW^* are projections in $\mathcal{H}$.

Lemma A.1 ([MVN36, p. 33, Lemma 4.4.1])**.** Let M be a von Neumann algebra. Suppose A is affiliated with M. Then $|A|$ is affiliated with M as well and $W \in M$.

The following definition is taken from [Dix81, Definition I.6.1, p.93]:

Definition A.3. Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{M}_+$ be its positive part, i.e.

$$\mathcal{M}_+ = \{A \in \mathcal{M} \colon A^* = A > 0\}.$$

A trace $\tau \colon \mathcal{M}_+ \mapsto [0, +\infty]$ a map taking non-negative, possibly infinite, real values, possessing the following properties:

- If $A \in \mathcal{M}_+$ and $B \in \mathcal{M}_+$, we have $\tau(A + B) = \tau(A) + \tau(B)$;
- If $A \in \mathcal{M}_+$ and $\lambda \in \mathbb{R}_+$, we have $\tau(\lambda A) = \lambda\tau(A)$ (with the convention that $0 \cdot +\infty = 0$);
- If $A \in \mathcal{M}_+$ and if U is a unitary operator of $\mathcal{M}$, then $\tau(UAU^{-1}) = \tau(A)$.

We say that a trace τ is

- faithful (or exact) if the condition $A \in \mathcal{M}_+$, $\tau(A) = 0$, implies that $A = 0$;

- finite if $\tau(A) < +\infty$ for all $A \in \mathcal{M}_+$;

- semi-finite if, for each $A \in \mathcal{M}_+$, $\tau(A)$ is the supremum of the numbers $\tau(B)$ over those $B \in \mathcal{M}_+$ such that $B \leq A$ and $\tau(B) < +\infty$;

- normal if, for each increasing filtering set $\mathcal{S} \subset \mathcal{M}_+$ with supremum $S \in \mathcal{M}_+$, $\tau(S)$ is the supremum of $\{\tau(B)\}_{B \in \mathcal{S}}$;

Definition A.4. A von Neumann algebra $\mathcal{M}$ is said to be *semifinite* if there exists a semifinite faithful normal trace τ on $\mathcal{M}_+$.

The theory of integration on von Neumann algebras has long history. It started with [MVN36, MvN43]. The foundation of the theory has bee laid down by Segal [Seg53]. He introduced a "noncommutative version" of the $*$-algebra of measurable complex-valued functions. It was based on the notion of measurability of an unbounded affiliated operator with respect to a trace faithful normal semi-finite trace τ. An alternative approach based on the Grotendieck's [Gro95] rearrangement of operators was developed by Yeadon [Yea81, TK86].

Definition A.5 (τ-topology)**.** Let $\mathcal{M}$ be a semi-finite von Neumann algebra with a trace τ. We define τ-topology on $\mathrm{Aff}(\mathcal{M})$ by saying that the sets

$$N(\varepsilon, s) = \{A \in \mathrm{Aff}(\mathcal{M})\colon \tau(E_{(u,+\infty)}(|A|)) < \varepsilon\} \tag{A.6}$$

are basis of open neighbourhood of zero.

We shall denote by $\mathrm{Meas}(\mathcal{M})$ the set $\mathrm{Aff}(\mathcal{M})$ of affiliated operators endowed with τ-topology.

We note that the notion of τ-measurability does not appear in the classical theory of Schatten classes since for $\mathcal{M} = B(\mathcal{H})$ we have $\mathrm{Aff}(B(\mathcal{H})) = B(\mathcal{H})$. The structure of commutative von Neumann algebras allows nice description. Every abelian algebra $\mathcal{C}$ is isometrically isomorphic [Dix81, Theorem 1, p.132] to $L^{\infty}(X,\nu)$, where X is a locally compact space with a positive measure ν on X. This justifies thinking of von Neumann algebras as a noncommutative measure space.

Example A.1. Let $\mathcal{M} = \{M_\varphi\}_\varphi$ be the abelian von Neumann algebra, where each element M_φ is the multiplication operator acting in $L^2(X,\mu)$ as follows

$$L^2(X,\mu) \ni f \mapsto M_\varphi f = \varphi f \in L^2(X,\mu)\},$$

where $\varphi \in L^{\infty}(X,\mu)$. Let us take $\tau(M_\varphi) := \int\limits_X \varphi d\mu$, where (X,μ) is a measure space. Then an operator M_φ is τ-measurable if and only if φ is measurable is a μ-almost everywhere finite function.

The $*$-algebra Meas($\mathcal{M}$) is a basic constructon for the noncommutative integration. Let $A = U|A|$ be the polar decomposition. The spectral theorem yields that

$$|A| = \int\limits_{\mathrm{Sp}(|A|)} \lambda dE_\lambda(|A|), \tag{A.7}$$

where $E_\lambda(|A|) = I - E_{(\lambda,+\infty)}$. Since A is affiliated with $\mathcal{M}$, the projections satisfy $E_\lambda(|A|) \in \mathcal{M}$. Composed with the spectral measure $\{E_\lambda(|A|)\}$, the trace τ induces a positive measure on the spectrum $\mathrm{Sp}(|A|) \subset \mathbb{R}_+$ of A. We shall denote it by $\tau(E_\lambda(|A|))$. The spectral decomposition implies that we have

$$\tau(\varphi(|A|)) = \int\limits_{\mathrm{sp}(|A|)} \varphi(\lambda) d\tau(E_\lambda(|A|)) \tag{A.8}$$

for every Borel function $\varphi \in L^{\infty}(\mathrm{sp}(|A|))$.

The trace τ takes integer values on the projections of type I von Neumann algebras $\mathcal{M}$. We denote τ by Tr in this case.

For each value of λ the projection E_λ is the sum of minimal mutually orthogonal projection operators, hence the value $\tau(E_\lambda)$ can increase only in jumps [Naj72, page 478] and its points of growth s_n are the singular values of the operator A. Thus, we get

$$\mathrm{Tr}(A) = \sum_{n\in\mathbb{N}} m_n s_n, \tag{A.9}$$

where m_n is the corresponding jump of the function $\tau(E_\lambda)$.

Now, we are ready to 'measure the speed of decay' of the operator A.

Definition A.6 (Generalised t-th singular numbers)**.** For an operator $A \in \mathrm{Meas}(\mathcal{M})$, define the distribution function $d_\lambda(A)$ by

$$d_A(\lambda) := \tau(E_{(\lambda,+\infty)}(|A|)), \quad \lambda \geq 0, \tag{A.10}$$

where $E_{(\lambda,+\infty)}(|A|)$ is the spectral projection of $|A|$ corresponding to the interval $(\lambda, +\infty)$. For any $t > 0$, we define the generalised t-th singular numbers by

$$\mu_t(A) := \inf\{\lambda \geq 0 \colon d_A(\lambda) \leq t\}. \tag{A.11}$$

Example A.2. For the operator M_φ in Example A.1, from Defintion A.6 we can show its generalised t-th singular numbers to be

$$\mu_t(M_\varphi) = \varphi^*(t),$$

where $\varphi^*(t)$ is the classical function rearrangement (see e.g. [BS88]).

Here we formulate some properties of μ_t that we use in the proof of Theorem 3.1.

Lemma A.2 ([TK86, Lemma 2.5, p. 275])**.** Let $A, B \in \mathrm{Meas}(\mathcal{M})$. Then the following properties hold true.

1. The map $(0, +\infty) \ni t \mapsto \mu_t(A)$ is non-increasing and continuous from the right. Moreover,

$$\lim_{t\to 0} \mu_t(A) = \|A\| \in [0, +\infty]. \tag{A.12}$$

2.
$$\mu_t(A) = \mu_t(A^*). \tag{A.13}$$

3.
$$\mu_{t+s}(AB) \leq \mu_t(A)\mu_s(B). \tag{A.14}$$

4.
$$\mu_t(ACB) \leq \|A\|\|B\|\mu_t(C), \quad \text{for any } C \in \mathrm{Meas}(\mathcal{M}). \tag{A.15}$$

5. For any continuous increasing function f on $[0, +\infty)$ we have
$$\mu_t(f(|A|)) = f(\mu_t(|A|)). \tag{A.16}$$

In Lemma A.2, we formulate only the properties we use, whereas in [TK86, Lemma 2.5, p. 275] the reader can find more details. For the sake of the exposition clarity we now formulate some properties of the distribution function d_A which we will be using in the proofs.

Proposition A.1. Let $A \in \mathrm{Meas}(\mathcal{M})$. Then we have

$$d_A(\mu_t(A)) \leq t; \tag{A.17}$$
$$\mu_t(A) > s \quad \text{if and only if} \quad t < d_A(s); \tag{A.18}$$
$$\sup_{t>0} t^\alpha \mu_t(A) = \sup_{s>0} s[d_A(s)]^\alpha \quad \text{for } 0 < \alpha < \infty. \tag{A.19}$$

The proof of this proposition is almost verbatim to the proof of [Gra08, Proposition 1.4.5 on page 46]. The word ‘almost’ stands for the right-continuity of the non-commutative distribution function $d_A(s)$ which is discussed after [TK86, Definition 1.3 on page 272]. Therefore, in the following proof we shall use the right-continuity of $d_A(s)$ without any justification.

Proof of Proposition A.1. Let $s_n \in \{s > 0\colon d_A(s) \leq t\}$ be such that $s_n \searrow \mu_t(A)$. Then $d_A(s_n) \leq t$ and the right-continuity of d_A implies that $d_A(\mu_t(A)) \leq t$. This proves (A.17). Now, we apply this property to derive (A.18). If $s < \mu_t(A) = \inf\{s > 0\colon d_A(s) \leq t\}$, then s does not belong to the set $\{s > 0\colon d_A(s) \leq t\} \implies d_A(s) > t$. Conversely, if for some t and s, we had $\mu_t(A) < s$, then the application of d_A and property (A.18) would yield the contradiciton $d_A(s) \leq d_A(\mu_t(A)) \leq t$. Property (A.18) is established. Finally, we show (A.19). Given $s > 0$, pick ε satisfying $0 < \varepsilon < s$. Property (A.18) yields $\mu_{d_A(s)-\varepsilon}(A) > s$ which implies that

$$\sup_{t>0} t^\alpha \mu_t(A) \geq (d_A(s) - \varepsilon)^\alpha \mu_{d_A(s)-\varepsilon}(A) > (d_A(s) - \varepsilon)^\alpha s. \tag{A.20}$$

We first let $\varepsilon \to 0$ and then take the supremum over all $s > 0$ to obtain one direction. Conversely, given $t > 0$, pick $0 < \varepsilon < \mu_A(t)$. Property (A.18) yields that $d_A(\mu_A(t)-\varepsilon) > t$. This implies that $\sup_{s>0} s(d_A(s))^\alpha \geq (\mu_A(t) - \varepsilon)(d_A(\mu_A(t) - \varepsilon))^\alpha > (\mu_A(t) - \varepsilon)t^\alpha$. We first let $\varepsilon \to 0$ and then take the supremum over all $t > 0$ to obtain the opposite direction of (A.19). $\square$

As a noncommutative extension [H.K81] of the classical Lorentz spaces, we define Lorentz spaces $L^{p,q}(\mathcal{M})$ associated with a semifinite von Neumann algebra $\mathcal{M}$ as follows:

Definition A.7 (Noncommutative Lorentz spaces)**.** For $1 \leq p < \infty$, $1 \leq q < \infty$, denote by $L^{p,q}(\mathcal{M})$ the set of all operators $A \in \mathrm{Aff}(\mathcal{M})$ satisfying

$$\|A\|_{L^{p,q}(\mathcal{M})} := \left(\int_0^{+\infty} \left(t^{\frac{1}{p}} \mu_t(A) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < +\infty. \tag{A.21}$$

For $q = \infty$, we define $L^{p,\infty}(\mathcal{M})$ as the space of all operators $A \in \mathrm{Aff}(\mathcal{M})$ satisfying

$$\|A\|_{L^{p,\infty}(\mathcal{M})} := \sup_{t>0} t^{\frac{1}{p}} \mu_t(A). \tag{A.22}$$

With this, for $1 \le p < \infty$, we can also define L^p-spaces on $\mathcal{M}$ by

$$\|A\|_{L^p(\mathcal{M})} := \|A\|_{L^{p,p}(M)} = \left(\int_0^{+\infty} \mu_t(A)^p \; dt \right)^{\frac{1}{p}} .$$

The classical Lorentz spaces $L^{p,q}(X,\mu)$ correspond to the case of commutative von Neumann algebra. Modulus technical details [Dix81, p. 132, Theorem 1], an arbitrary abelian von Neumann algebra in a Hilbert space $\mathcal{H}$ is isometrically isomorphic to the algebra $\{M_\varphi\}_{\varphi \in L^\infty(X,\mu)}$ from Example A.1. Then noncommutative Lorentz spaces coincide with the classical ones:

Example A.3 (Classical Lorentz spaces)**.** Let $\mathcal{M}$ be the abelian algebra $\{M_\varphi\}$ from Example A.1 consisting of all the multiplication operators $M_\varphi \colon L^2(X,\mu) \ni f \mapsto M_\varphi f = \varphi f \in L^2(X,\mu)$. By Example A.2, we have

$$\mu_t(M_\varphi) = \varphi^*(t).$$

Thus, the Lorentz space $L^{p,q}(M)$ consists of all operators M_φ such that

$$\int_0^{+\infty} [t^{\frac{1}{p}} \varphi^*(t)]^q \frac{dt}{t} < +\infty,$$

which gives the classical Lorentz space.

A.1.1 The reduction theory

The reduction theory is way of 'breaking down' von Neumann algebras into a direct integrals (a generalisation of direct sums) of von Neumann algebras. The center $\mathcal{C}$ of a von Neumann algebra $\mathcal{M}$ is the intersection of $\mathcal{M}$ and its commutant $\mathcal{M}^!$, i.e.

$$\mathcal{C} = \mathcal{M} \cap \mathcal{M}^!.$$

A factor is a von Neumann algebra $\mathcal{M}$ whose center $\mathcal{C}$ only contains the scalar operators, i.e.

$$\mathcal{C} = \lambda I.$$

Murray and von Neumann classified factors into three types [MVN36, vN40]. Major achievements have been further made by [Pow67] and [Con76]. We shall not go into details here but refer to [Sun87] for a brief and concise introduction to the classification of factors. Let us just mention that von Neumann algebras have found many applications in the diverse range of fields: knot theory, statistical mechanics, quantum field theory, free probability.

Let X be a Borel space, ν a positive measure on X.

A measurable field of Hilbert spaces is a function $\lambda \mapsto \mathcal{H}^\lambda$, where each $\mathcal{H}^\lambda$ is a Hilbert space, together with a set S of functions $\lambda \mapsto x(\lambda) \in \mathcal{H}^\lambda$ that are said to be measurable and that satisfy

- the function $\lambda \mapsto (x(\lambda), y(\lambda))_{\mathcal{H}^\lambda}$ is measurable for all $x, y \in S$ and
- if v is a vector field and the function $\lambda \mapsto (x(\lambda), v(\lambda)$ is measurable for each $x \in S$, then $v \in S$.

and let $\lambda \mapsto \mathcal{H}_\lambda$ be a measurable field of Hilbert spaces $\mathcal{H}_\lambda$. For every $\lambda \in X$, let A_λ be an element of $B(\mathcal{H}_\lambda)$, i.e. a linear bounded operator on $\mathcal{H}_\lambda$. The mapping $\lambda \mapsto A_\lambda$ is called a field of bounded linear operators over X. A field $\lambda \mapsto A_\lambda$ of bounded operators is said to be measurable if for every measurable vector field $x(\lambda)$ the field $\lambda \mapsto A(\lambda)x(\lambda)$ is measurable. We refer to [Dix81, II.1.1-II.1.5] for more details. Measurable fields of operators associated to group von Neumann algebras have been discussed in detail in [FR16, Appendix B].

Definition A.8. A measurable field $\{A_\lambda\}_{\lambda \in X}$ is said to be essentially bounded if the essential supremum of the function $\lambda \mapsto \|A_\lambda\|_{B(\mathcal{H}_\lambda)}$ is finite. A linear operator $A \colon \bigoplus \int \mathcal{H}_\lambda \to \bigoplus \int \mathcal{H}_\lambda$ is said to be decomposable if it is defined as an essentially bounded measurable field $\{A_\lambda\}_{\lambda \in X}$.

We then write

$$A = \bigoplus \int A_\lambda d\nu(\lambda),$$

where $Af(\lambda) = A_\lambda x(\lambda)$.

For every $\lambda \in X$, let $\mathcal{M}_\lambda$ be a von Neumann algebra in $\mathcal{H}_\lambda$. The mapping $\lambda \mapsto \mathcal{M}^\lambda$ is called a field of von Neumann algebras over X.

Definition A.9. A von Neumann algebra $\mathcal{M} \subset B(\mathcal{H})$ is said to be decomposable if it is defined by a measurable field $\lambda \mapsto \mathcal{M}^\lambda$ of von Neumann algebras. We then write

$$\mathcal{M} = \bigoplus \int \mathcal{M}^\lambda d\nu(\lambda),$$

The importance of abelian subalgebras is motivated by the following

Theorem A.2 ([vN49, Theorem 7, p. 460])**.** Let $\mathcal{M}$ be a semi-finite von Neumann algebra. Then $\mathcal{M} = \bigoplus\limits_X \int \mathcal{M}_\lambda d\nu(\lambda)$, where each M_λ is a factor.

Here X is the measure space assigned to the center $\mathcal{C} = \mathcal{M} \cap \mathcal{M}^!$. The following example is a brief summary of [Dix77, Chapter 18, Part II].

Example A.4 ([Dix77, II.18])**.** Let G be a locally compact unimodular group and $\mathrm{VN}_R(G)$ its right group von Neumann algebra. Now, we apply Theorem A.2 with $\mathcal{M} = \mathrm{VN}_R(G)$ and $\mathcal{C} = \mathrm{VN}_R(G) \cap (\mathrm{VN}_R(G))^!$. The unitary dual $\widehat{G}$ of G can be endowed with the Mackey-Borel structure in a canonical way [Dix77, II.18.5.3]. There exists a standard measure μ on $X = \widehat{G}$ and a measurable field $\{\mathcal{H}^\pi\}_{\pi \in \widehat{G}}$ of Hilbert spaces and a measurable field of factor von Neumann algebras VN_R^π such that

$$\mathrm{VN}_R(G) = \bigoplus_{\widehat{G}} \int \mathrm{VN}_R^\pi(G) d\nu(\pi), \tag{A.23}$$

where $\mathrm{VN}_R^\pi(G)$ acts in $\mathcal{H}^\pi$

For more details we refer to [Dix77] or [Fol16a, Theorem 7.37, p. 227].

We recall the basic result on central decomposition of traces of von Neumann algebras.

Theorem A.3 ([Dix81, Theorem II.5.2]). Let $\mathcal{M} = \bigoplus \mathcal{M}_\lambda d\nu(\lambda)$ be a semifinite decomposable von Neumann algebra. Suppose that ν is standard. Then we have

1. The $\mathcal{M}_\lambda$'s are semifinite almost everywhere.

2. Let τ be a semifinite faithful normal trace on $\mathcal{M}_+$. Then there exists a measurable field $\lambda \mapsto \tau_\lambda$ of semifinite faithful normal traces on the $(\mathcal{M}_\lambda)_+$'s such that

$$\tau = \int \tau_\lambda d\nu(\lambda). \tag{A.24}$$

A.2 Marcinkiewicz interpolation theorem

In this section we formulate the Marcinkiewicz interpolation theorem on arbitrary σ-finite measure spaces. Then we show how to use this theorem for linear mappings between $C^\infty(G)$ and the space Σ of finite matrices on the discrete unitary dual $\widehat{G}$ or on the discrete set $\widehat{G}_0$ of class I representations with different measures on $\widehat{G}$ and $\widehat{G}_0$.

This approach will be instrumental in the proof of the Hardy-Littlewood Theorem 5.2 and of the Paley inequality in Theorem 5.3.

We now formulate the Marcinkiewicz theorem for linear mappings between functions on arbitrary σ-finite measure spaces (X, μ_X) and (Γ, ν_Γ).

Let $PC(X)$ denote the space of step functions on (X, μ_X). We say that a linear operator A is of strong type (p, q), if for every $f \in L^p(X, \mu_X) \cap PC(X)$, we have $Af \in L^q(\Gamma, \nu_\Gamma)$ and

$$\|Af\|_{L^q(\Gamma,\nu_\Gamma)} \leq C\|f\|_{L^p(X,\mu_X)},$$

where C is independent of f, and the space $\ell^q(\Gamma, \nu_\Gamma)$ defined by the norm

$$\|h\|_{L^q(\Gamma,\nu_\Gamma)} := \left(\int\limits_\Gamma |h(\pi)|^p \nu(\pi) \right)^{\frac{1}{q}}. \tag{A.25}$$

The least C for which this is satisfied is taken to be the strong (p, q)-norm of the operator A.

Denote the distribution functions of f and h by $\mu_X(x; f)$ and $\nu_\Gamma(y; h)$, respectively, i.e.

$$\begin{aligned} \mu_X(x; f) &:= \int\limits_{\substack{t\in X \\ |f(t)|\geq x}} d\mu(t), \quad x > 0, \\ \nu_\Gamma(y; h) &:= \int\limits_{\substack{\pi\in\Gamma \\ |h(\pi)|\geq y}} d\nu(\pi), \quad y > 0. \end{aligned} \tag{A.26}$$

Then

$$\begin{aligned} \|f\|^p_{L^p(X,\mu_X)} &= \int\limits_X |f(t)|^p \, d\mu(t) = p \int\limits_0^{+\infty} x^{p-1} \mu_X(x; f) \, dx, \\ \|h\|^q_{L^q(\Gamma,\nu_\Gamma)} &= \int\limits_{\pi\in\Gamma} |h(\pi)|^q \nu(\pi) = q \int\limits_0^{+\infty} y^{q-1} \nu_\Gamma(y; h) \, dy. \end{aligned}$$

A linear operator $A\colon \mathcal{P}C(X) \to L^q(\Gamma, \nu_\Gamma)$ satisfying

$$\nu_\Gamma(y; Af) \leq \left(\frac{M}{y} \|f\|_{L^p(X,\mu_X)} \right)^q, \quad \text{for any } y > 0. \tag{A.27}$$

is said to be of *weak type* (p, q); the least value of M in (A.27) is called the weak (p, q) norm of A.

Every operation of strong type (p, q) is also of weak type (p, q), since

$$y \left(\nu_\Gamma(y; Af)\right)^{\frac{1}{q}} \leq \|Af\|_{L^q(\Gamma)} \leq M\|f\|_{L^p(X)}.$$

Theorem A.4. Let $1 \leq p_1 < p < p_2 < \infty$. Suppose that a linear operator A from $\mathcal{P}C(X)$ to $L^q(\Gamma, \nu_\Gamma)$ is simultaneously of *weak types* (p_1, p_1) and (p_2, p_2), with norms M_1 and M_2, respectively, i.e.

$$\begin{aligned} \nu_\Gamma(y; Af) &\leq \left(\frac{M_1}{y}\|f\|_{L^{p_1}(X,\mu_X)}\right)^{p_1}, \\ \nu_\Gamma(y; Af) &\leq \left(\frac{M_2}{y}\|f\|_{L^{p_2}(X,\mu_X)}\right)^{p_2} \qquad \text{hold for any } y > 0. \end{aligned}$$

Then for any $p \in (p_1, p_2)$ the operator A is of strong type (p, p) and we have

$$\|Af\|_{L^p(\Gamma,\nu_\Gamma)} \lesssim M_1^{1-\theta} M_2^{\theta} \|f\|_{L^p(X,\mu_X)}, \quad 0 < \theta < 1,$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

The proof is given in e.g. Folland [Fol99]. Now, we adapt this theorem to the setting of matrix-valued mappings.

Suppose Γ is a discrete set. Integral over Γ is defined as sum over Γ, i.e.

$$\int_\Gamma \nu_\Gamma(\pi) := \sum_{\pi\in\Gamma} \nu(\pi). \tag{A.28}$$

In this case, to define a measure on Γ means to define a real-valued positive sequence $\nu = \{\nu_\pi\}_{\pi\in\Gamma}$, i.e.

$$\Gamma \ni \pi \mapsto \nu_\pi \in \mathbb{R}_+.$$

We turn Γ into a σ-finite measure space by introducing a measure

$$\nu_\Gamma(Q) := \sum_{\pi\in Q} \nu_\pi,$$

where Q is arbitrary subset of Γ.

We consider two sequences $\delta = \{\delta_\pi\}_{\pi\in\Gamma}$ and $\kappa = \{\kappa_\pi\}_{\pi\in\Gamma}$, i.e.

$$\Gamma \ni \pi \mapsto \delta_\pi \in \mathbb{N},$$
$$\Gamma \ni \pi \mapsto \kappa_\pi \in \mathbb{N}.$$

We denote by Σ the space of matrix-valued sequences on Γ that will be realised via

$$\Sigma := \left\{ h = \{h(\pi)\}_{\pi\in\Gamma}, h(\pi) \in \mathbb{C}^{\kappa_\pi \times \delta_\pi} \right\}.$$

The ℓ^p spaces on Σ can be defined, for example, motivated by the Fourier analysis on compact homogeneous spaces, in the form

$$\|h\|_{\ell^p(\Gamma,\Sigma)} := \left(\sum_{\pi\in\Gamma} \left(\frac{\|h(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}} \right)^p \nu_\pi \right)^{\frac{1}{p}}, \quad h \in \Sigma.$$

If we put $X = G$, where G is a compact Lie group and let $\Gamma = \widehat{G}$, then Fourier transform can be regarded as an operator mapping a function $f \in L^p(G)$ to the matrix-valued sequence $\widehat{f} = \{\widehat{f}(\pi)\}_{\pi\in\widehat{G}}$ of the Fourier coefficients, with $\delta_\pi = \kappa_\pi = d_\pi$. For $\Gamma = \widehat{G}_0$ we put $\delta_\pi = d_\pi$ and $\kappa_\pi = k_\pi$, these spaces thus coincide with the $\ell^p(\widehat{G}_0)$ spaces introduced in [RT10]. In Section 5, choosing different measures $\{\nu_\pi\}_{\pi\in\Gamma}$ on the unitary dual $\widehat{G}$ or on the set $\widehat{G}_0$, we use this to prove the Paley inequality and Hausdorff-Young-Paley inequalitites. Let us denote by $|h|$ the sequence consisting of $\{\frac{\|h(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}}\}$, i.e.

$$|h| = \left\{ \frac{\|h(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}} \right\}_{\pi\in\Gamma}.$$

Then, we have

$$\|h\|_{\ell^q(\Gamma,\Sigma)} = \||h|\|_{L^q(\Gamma,\nu_\Gamma)}.$$

Thus, we obtain

Theorem A.5. Let $1 \le p_1 < p < p_2 < \infty$. Suppose that a linear operator A from $\mathcal{PC}(X)$ to Σ is simultaneously of *weak types* (p_1, p_1)

and (p_2, p_2), with norms M_1 and M_2, respectively, i.e.

$$\begin{aligned}
\nu_\Gamma(y; Af) &\leq \left(\frac{M_1}{y}\|f\|_{L^{p_1}(X)}\right)^{p_1}, && \text{(A.29)}\\
\nu_\Gamma(y; Af) &\leq \left(\frac{M_2}{y}\|f\|_{L^{p_2}(X)}\right)^{p_2} \quad \text{hold for any } y > 0. && \text{(A.30)}
\end{aligned}$$

Then for any $p \in (p_1, p_2)$ the operator A is of strong type (p, p) and we have

$$\|Af\|_{\ell^p(\Gamma,\Sigma)} \leq M_1^{1-\theta} M_2^{\theta} \|f\|_{L^p(X)}, \quad 0 < \theta < 1, \tag{A.31}$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

A version of Marcinkiewicz theorem in the form with general (possibly noncommutative) Lorentz norms can be found in [DDdP92, Theorem 4.1].

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