Research Statement

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Introduction

My research interests are broadly speaking in noncommutative harmonic analysis and its applications to partial differential equations coming from algebraic differential calculi. I am interested in spectral triples and associated pseudo-differential operators on noncommutative spaces. During my PhD, I developed strong interest in Lie group contractions and geometric representation theory. Over the next few years, I plan to work on many unsolved problems and conjectures in noncommutative analysis making use of the recent developments in quantum Riemannian geometry and noncommutative differential geometry. In my graduate thesis I obtained general unifying results on L^p-L^q-boundedness of Fourier multipliers on topological groups via measuring the regularity of the operators in terms of decay of its spectral projections. The decay is measured by computing the underlying trace on the spectral projections. My approach is very effective, implying as special cases known results expressed in terms of symbols, in settings when the symbolic calculus is available.

The spectral nature of the results obtained in my thesis naturally lead me to explore the counterparts on noncommutative spaces. Quantum groups is one of the main source of genuine noncommutative spaces. These objects are quantisations of Poisson Lie groups, and it is natural to expect that every compact quantum group should possess a spectral triple by a quantisation process of some sort. Up to this moment, the notion of 'differential structure' on quantum groups is understood only at an algebraic level and not on the level of 'noncommutative smooth functions'. Alain Connes successfully reformulated notion of compact spin manifold and encoded the classical properties of the geometric Dirac operator by means of a 'spectral triple' that encodes smooth structure. However, such an approach does not work well with the main quantum groups of interest. On the other hand, algebro-geometric spectral triples originating in the works by Beggs and Majid provide natural framework to study quantum groups. Unfortunately, the latter formalism meets the former only at the algebraic level.

In my most recent research paper [Aky18], I investigated possible smooth dense subalgebras on compact quantum groups and constructed the associated Connes spectral triples in terms of Clebsch-Gordan coefficients. As a by-product, I study elements of distribution theory and introduce global class of pseudo-differential operators associateds with regular and summable spectral triples on compact quantum groups. Further, I explored the question of linking the proposed spectral triples to the differential-algebraic notion of differential calculus in the more constructive approach. It turns out the continuous extension of algebraic vector fields with respect to the proposed spectral triple is incompatible with the regularity of the triple (untwisted). In the view of recent papers [MY17, Yun18] by Yuncken and Matassa it seems highly possible that twisted spectral triples will make contact with algebro-geometric spectral triples of Beggs and Majid. As a corollary, we can relate our class of pseudo-differential operators (PDO) in [AMR18] to the Connes-Moscovici PDO. The latter formalism was primarily designed to obtain the local index formula. In the setting of quantum groups we can assign a full symbol to every pseudo-differential operators from the Connes-Moscovici machinery. The development of the classical theory of pseudo-differential operators (PDO) was historically motivated by explicit differential equations arising in mathematical physics. Starting from [AMR18], we can construct a plethora of explicit partial differential operators (PDE) on the algebraic level. We shall then use the philosophy in [CM95] to define a notion of 'smoothness' of the kernel or the symbol and measure its decay using such tools as the Wodzicki residue and algebraic differential calculi over the space.

My work proposal for future research shall start from algebriac PDEs coming from a fixed differential calculus (this can be Connes derivations or algebraic vector fields from bimodule connection) over a fixed involutive algebra. Taking the completion via a fixed Dirac-like operator, we extend our PDEs to act smoothly on regular summable twisted spectral triple. We shall then proceed to study the regularity properties of these PDEs and develop natural symbolic classes relevant to the study of the proposed PDEs. The future research projects I intend to carry out include, but not limited to, derivation of L^p-bounds for pseudo-differential operators arising as asymptotic solutions to our algebro-geometric PDEs, a version of intrinsic hypoellipticity associated with twisted spectral triples. In particular, I intend to obtain time-dependent estimates for the solutions to heat and wave equation associated with a Dirac-like operator. The proposed results will be new even for classical (pseudo)differential operators acting on smooth manifolds.

Finally, it is worth mentioning that I have many other research problems and projects in mind. For example, in a joint paper [AA18] with Arnaudon we study how certain degenerations (contractions) of Lie groups are 'reflected' on the unitary dual. In more detail, we show that under certain geometrical conditions on the coadjoint orbits, every infinite-dimensional unitary representation can be approximated by a sequence of deformed finite-dimensional representations. The main tool in our proof is the geometric quantization of Kirillov, Kostant and Souriau along with the prequantization formula for the canonical transformations on Lie groups [Woo80]. As a by-product, we might obtain an alternative proof of Baum-Connes conjecture for type I solvable Lie groups. The case of semisimple Lie groups has been considered in [Afg16]. An interesting problem suggested by Alain Valette (private communication) is to verify the cocycle conditions on the coadjoint orbits of semisimple Lie groups.

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