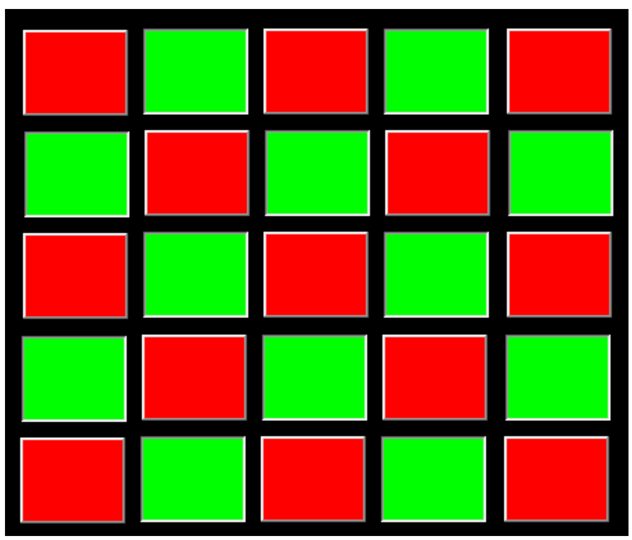
LIGHTS OUT PROBLEM

Applied Mathematics

10/7/2015

ASSIGNMENT # 1



Submitted To : Dr. Désiré Sidibé

Submitted By:

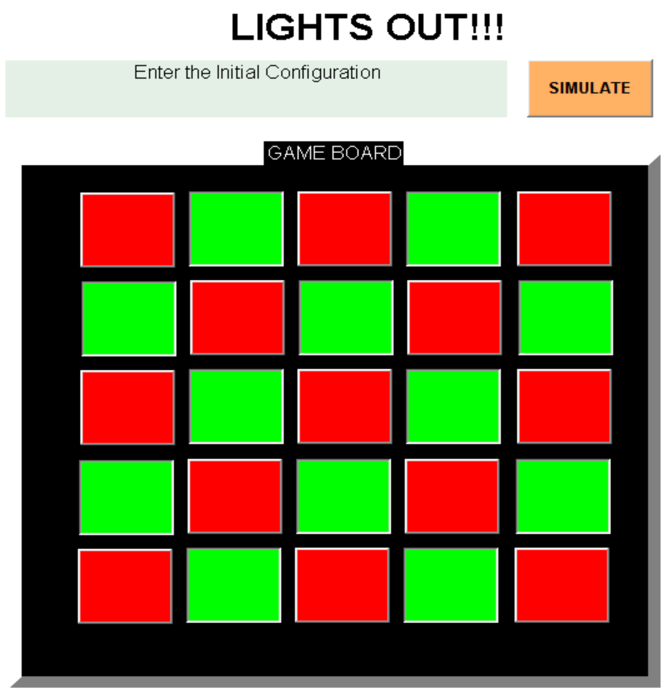
Gourab Ghosh Roy
VIBOT - 10

Raabid Hussain
VIBOT – 10

Problem Statement:-

A typical "Light's Out" game consists of a 5 x 5 electronic logic puzzle board. This game was initially released in 1995 by Tiger Electronics. In this game, the user is provided with a set of initial configuration (some boxes are randomly switched on). The objective of the game is to turn off all the lights in the minimum number of turns. However, the trick is that if a box is pressed (a move is made), the box along with all its adjacent counterparts will also toggle.

"The objective of our project was to simulate a solution, using linear algebra and MATLAB, to the game with the least possible number of toggles, provided any set of random starting configuration by the user."



Methodology:-

The lights out problem is turning all lights out on a preconfigured 5 by 5 board, which can be easily represented in a matrix or vector form with each value being either 1 ('on' state) or 0 ('off' state).

We studied some methods for solving this, but they were not using linear algebra and were not optimal. However, the paper titled "Turning Lights Out with Linear Algebra" by Marlow

Anderson and Todd Feil very clearly describes how the problem can be solved using linear algebra by obtaining a closed form solution. We have based our solution on that paper.

A correct assumption is that if there exists a configuration that can be obtained from an all off state of the board through some button presses, then the same button presses can lead to an all off state from that particular starting configuration.

The solution as described in the paper is basically to solve $Ax=b$, one of our most favorite forms in linear algebra.

Here:-

- *'b'* is the initial configuration vector of size 25 x 1, with values either 1 or 0.
- *'x'* is such that $x_{i,j}=1$ if (i,j) is a button to be pushed or 0 otherwise, reshaped to a 25 x 1 vector.
- *'A'* is created based on how a button press affects the state of the board, basically by figuring out how values in *'x'* add up to one another to give subsequent values in *'b'*.

$$A = \begin{bmatrix} B & I & O & O & O \\ I & B & I & O & O \\ O & I & B & I & O \\ O & O & I & B & I \\ O & O & O & I & B \end{bmatrix}$$

where *'I'* is a 5 x 5 Identity matrix; *'O'* is a 5 x 5 zero matrix; and *'B'* is composed of the following values obtained as per the functioning of each individual buttons:

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Then as learned from the Applied Math class, we apply Gauss Jordan elimination on the matrix '*A'* so that it is transformed to its Row reduced Echelon Form *'E'*:

$$RA=E$$

where '*R'* is the matrix created by multiplication of the individual row reduction operation matrices.

We used MATLAB to perform Gauss Jordan Elimination (did not use the *rref* function because it does not give the matrix *'R'*) and find this matrix '*R*'.

Using results from the paper, we find 'n_1' and 'n_2' which form the basis for *'Null(E)'*, that is null space of '*E'* (since rank of *'E'* is 23 so we need 25-23=2 vectors).

Now from linear algebra in class (and also guided by the paper),

for $Ax=b$ to be solvable,
'b' must lie in *'Col(A)'*, that is column space of *'A'*.

As *'A'* is symmetric, so, *'*$Col(A^T)$*'* is same as *'Col(A)'*, and that is perpendicular to *'Null(A)'* which is *'Null(E)'*.

We use this to decide whether a starting configuration is winnable or not by checking if *'b'* is orthogonal to both 'n_1' and 'n_2', the basis vectors of Null(E). That is equivalent to checking if *'b'* belongs to *'Col(A)'* or not.
This is used in the first part of the code in MATLAB where the program declares if a configuration is not winnable or proceeds to solve it.

A point to mention is that since all values are in 'Z^2', that is either 0 or 1, all arithmetic operations are *modulo 2*.

Now to find the actual solution, we use Theorem 2 in the paper. Rephrasing what we learned from it, we can set the free variables to *zero* (rank=23, two free variables), and then we get as one solution:

$$x=Ex=RAx=Rb$$

where *'b'* is a winnable configuration.

Also since 'n_1' and 'n_2' are in null space of *'A'* that is '$An_1=0$' and '$An_2=0$', *4* possible solutions are:

'Rb'; '$Rb+n_1$'; '$Rb+n_2$'; '$Rb+n_1+n_2$'.

In our code we give that solution out of the 4 which has minimum number of buttons to be pushed.

We have put a MATLAB GUI in front of the Gauss Elimination code to allow a user to interactively configure a 5 by 5 board and then show in the UI how the lights out problem is actually solved through corresponding button presses.

Conclusion:-

So in this homework we solved and successfully implemented a very interesting puzzle using linear algebra with techniques we learned in applied mathematics class and from a very good paper.