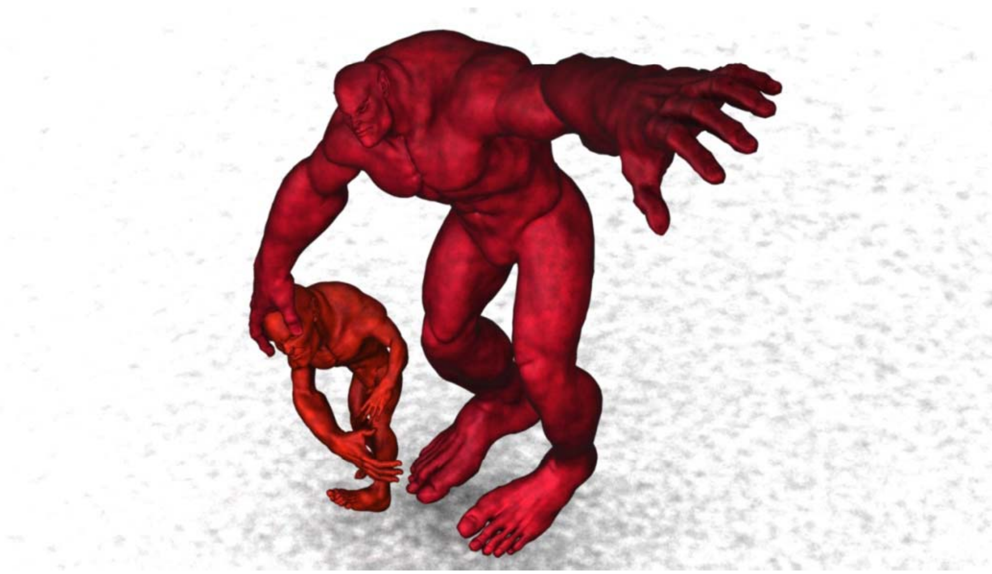
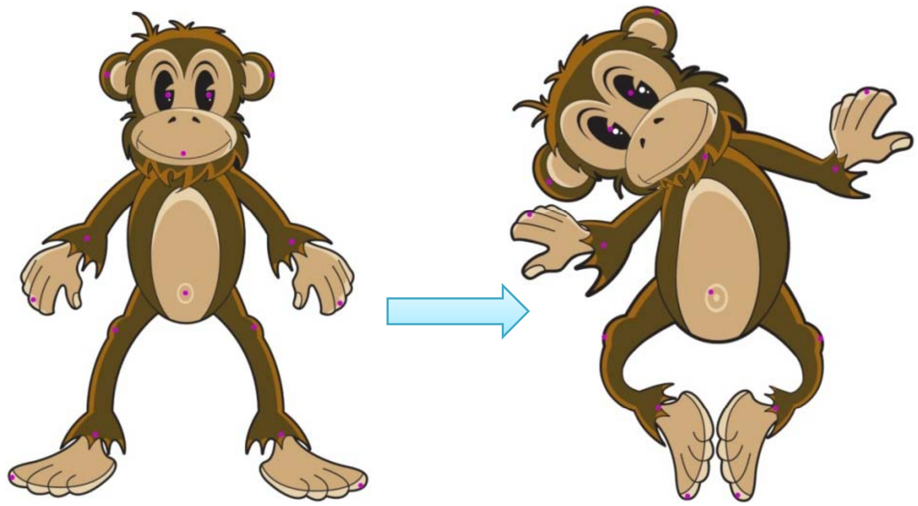
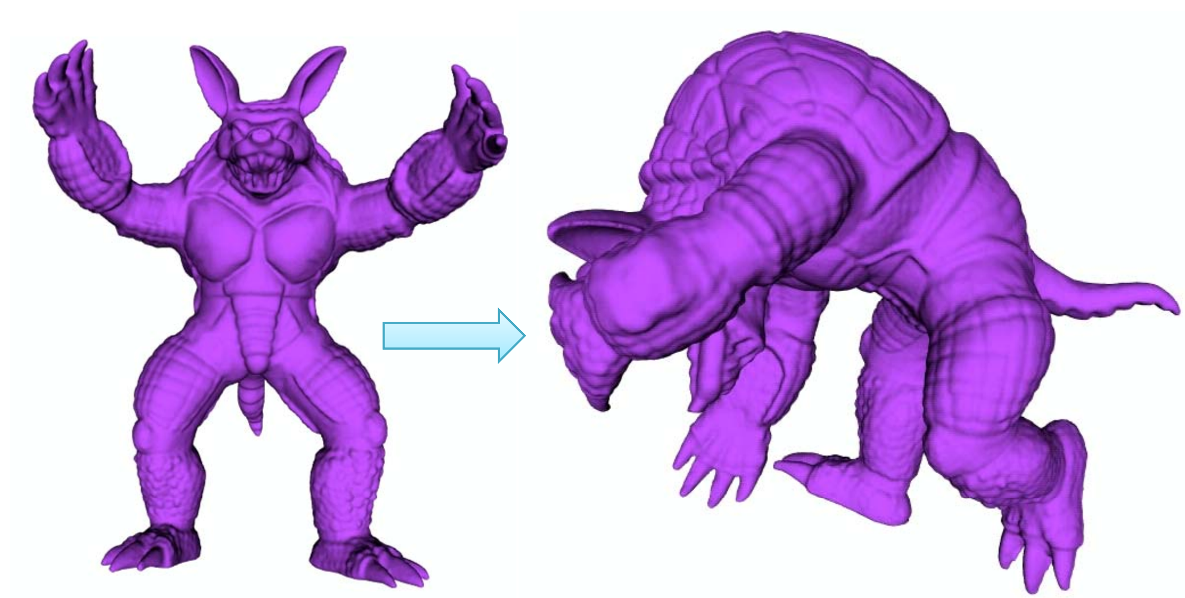
Deformation I



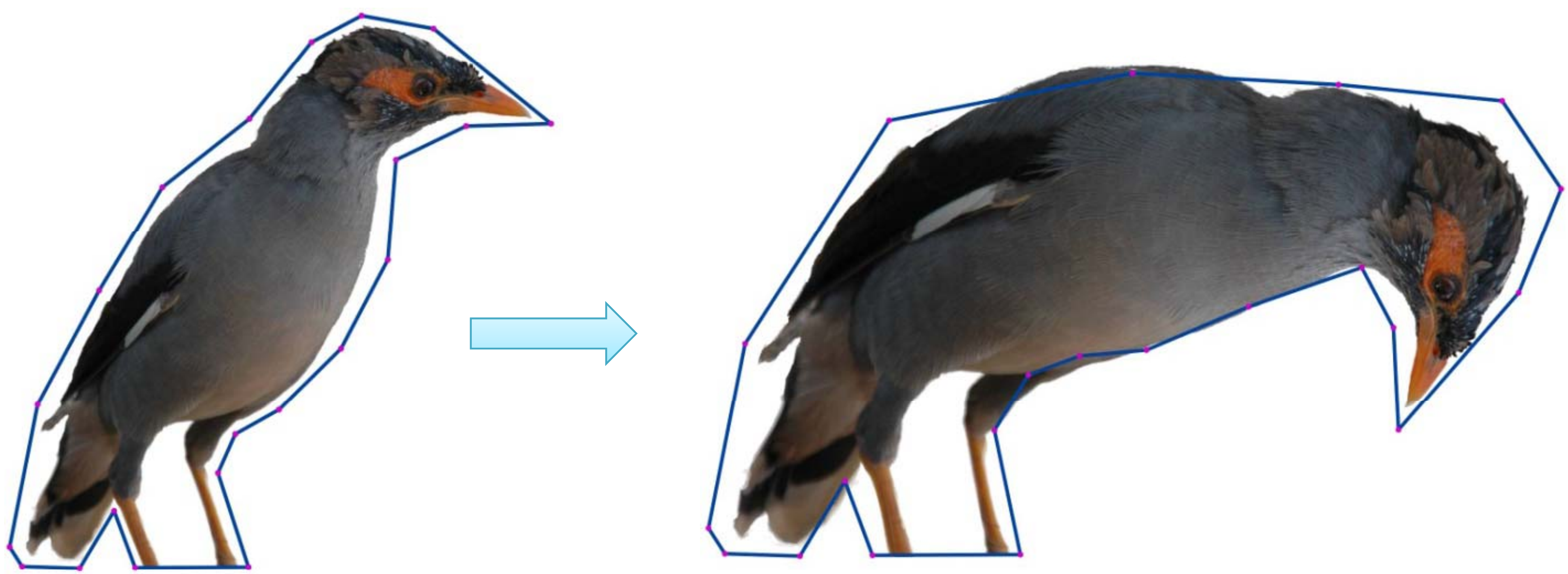
Deformation





Motivation

Easy modeling – generate new shapes by deforming existing ones



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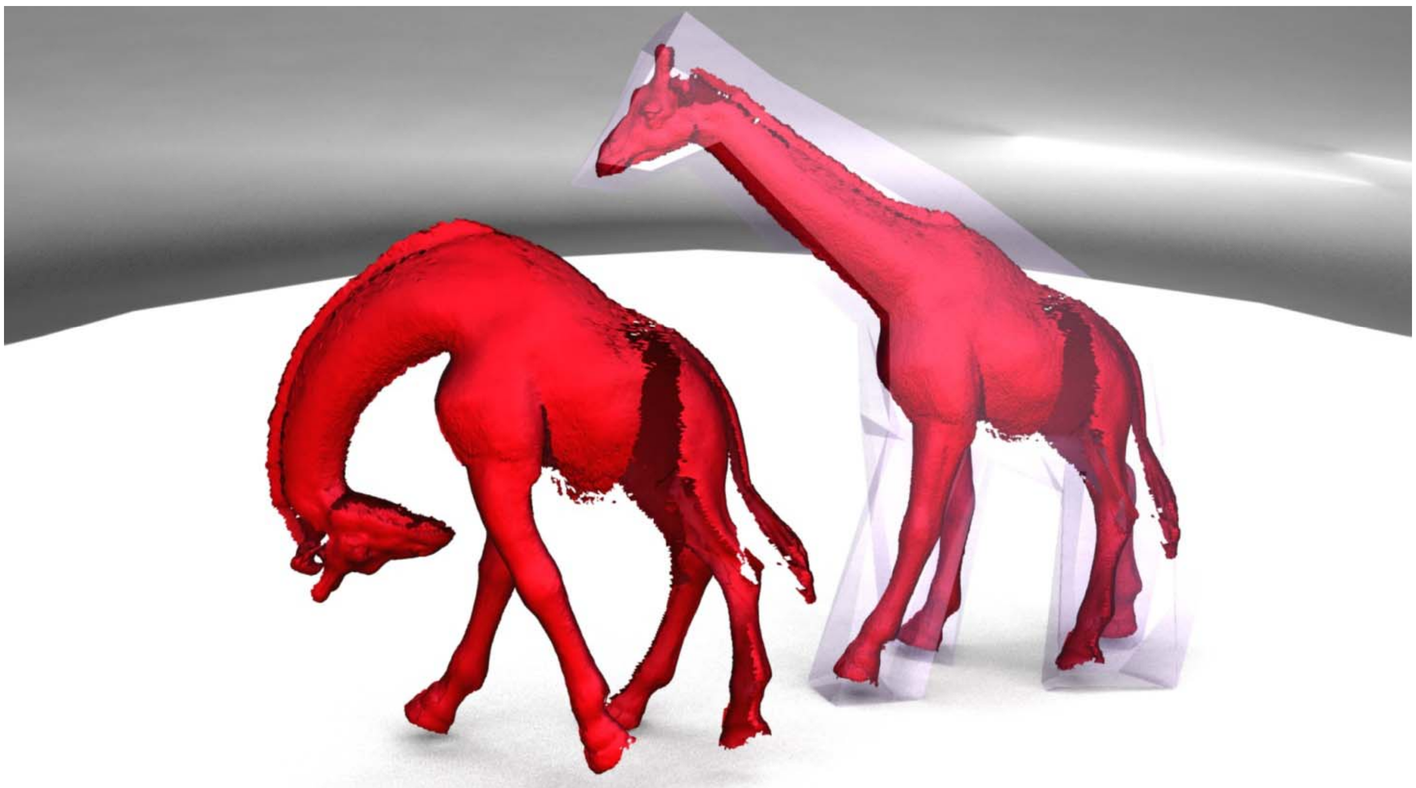
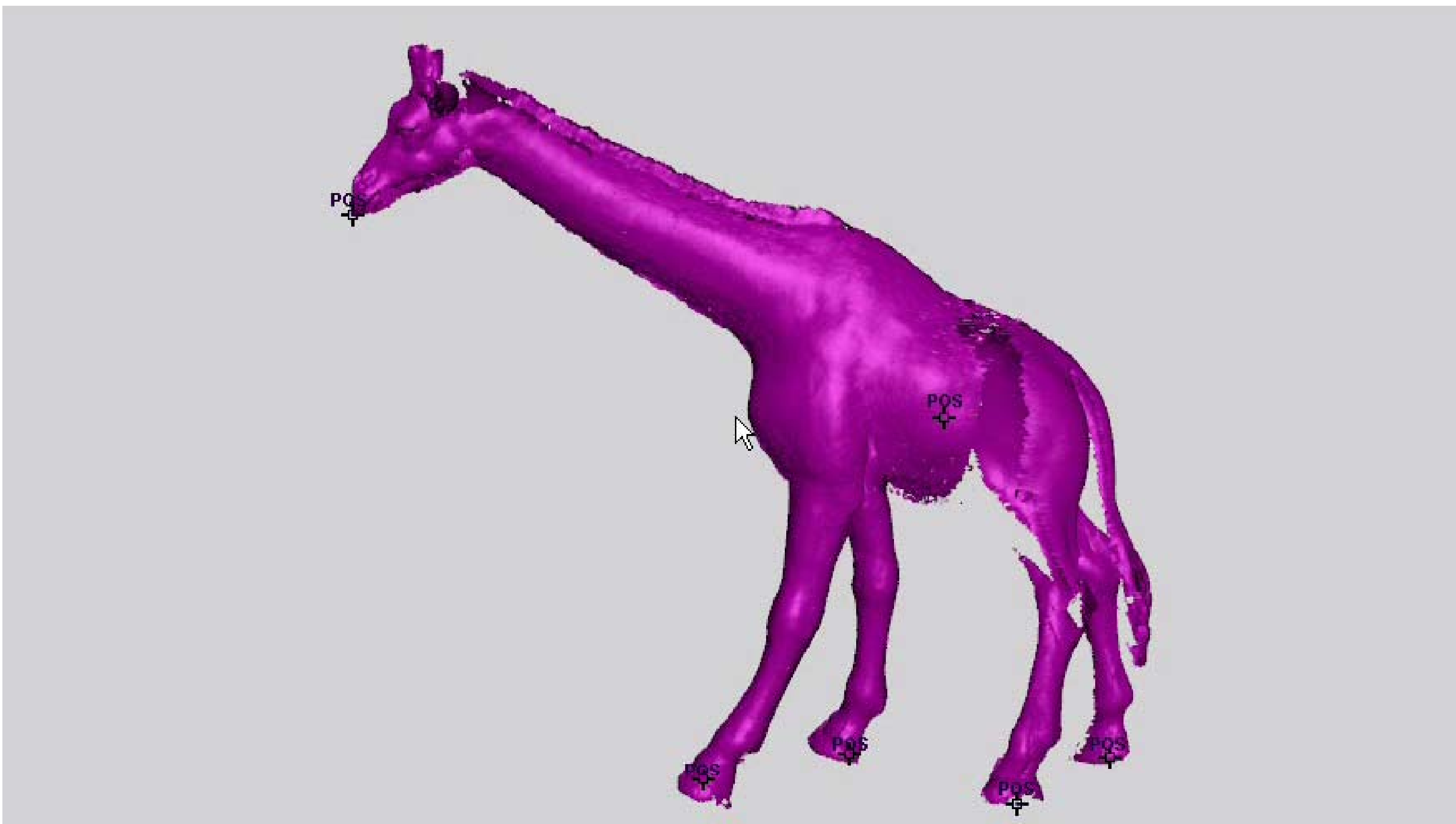


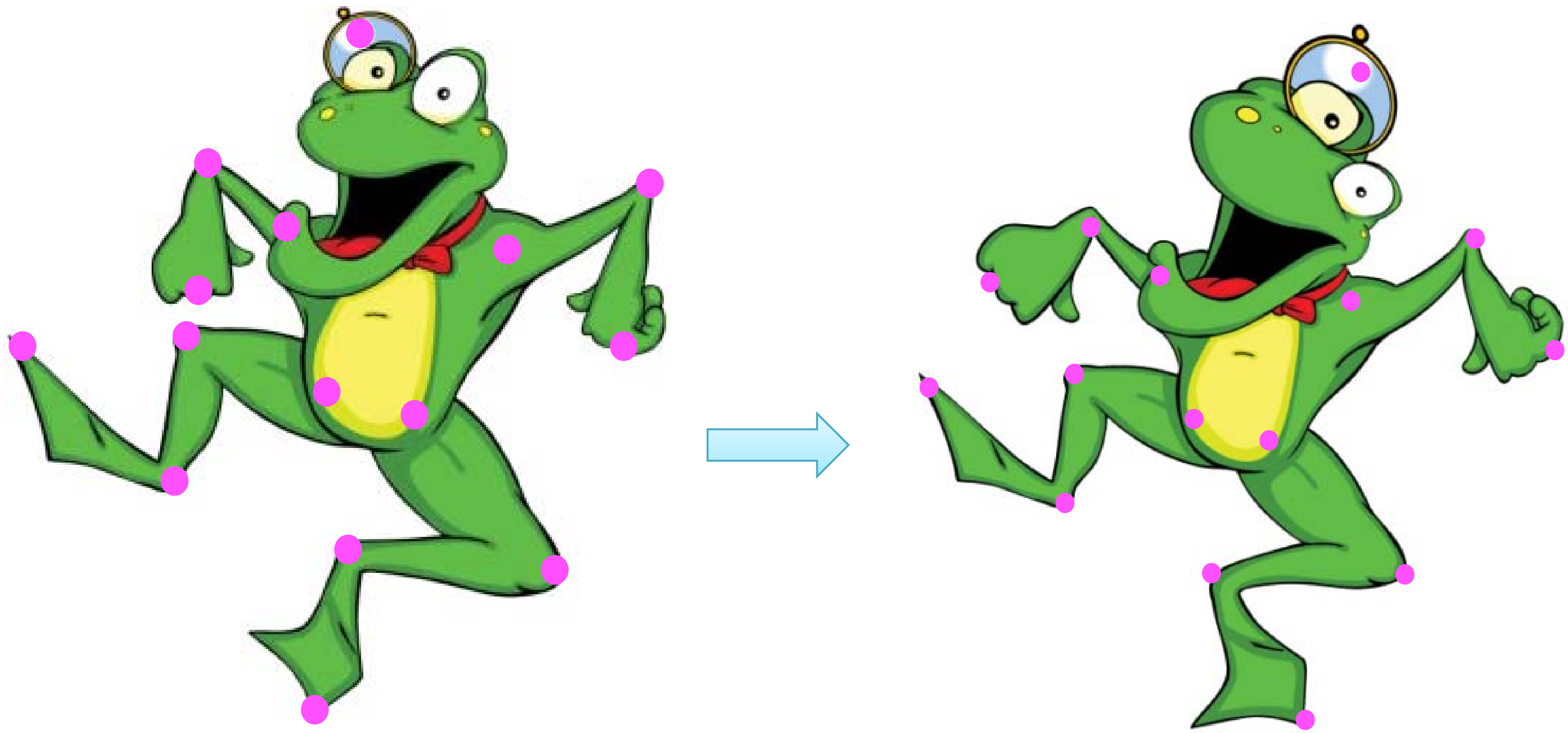
Motivation

Character posing for animation



Challenges

User says as little as possible…

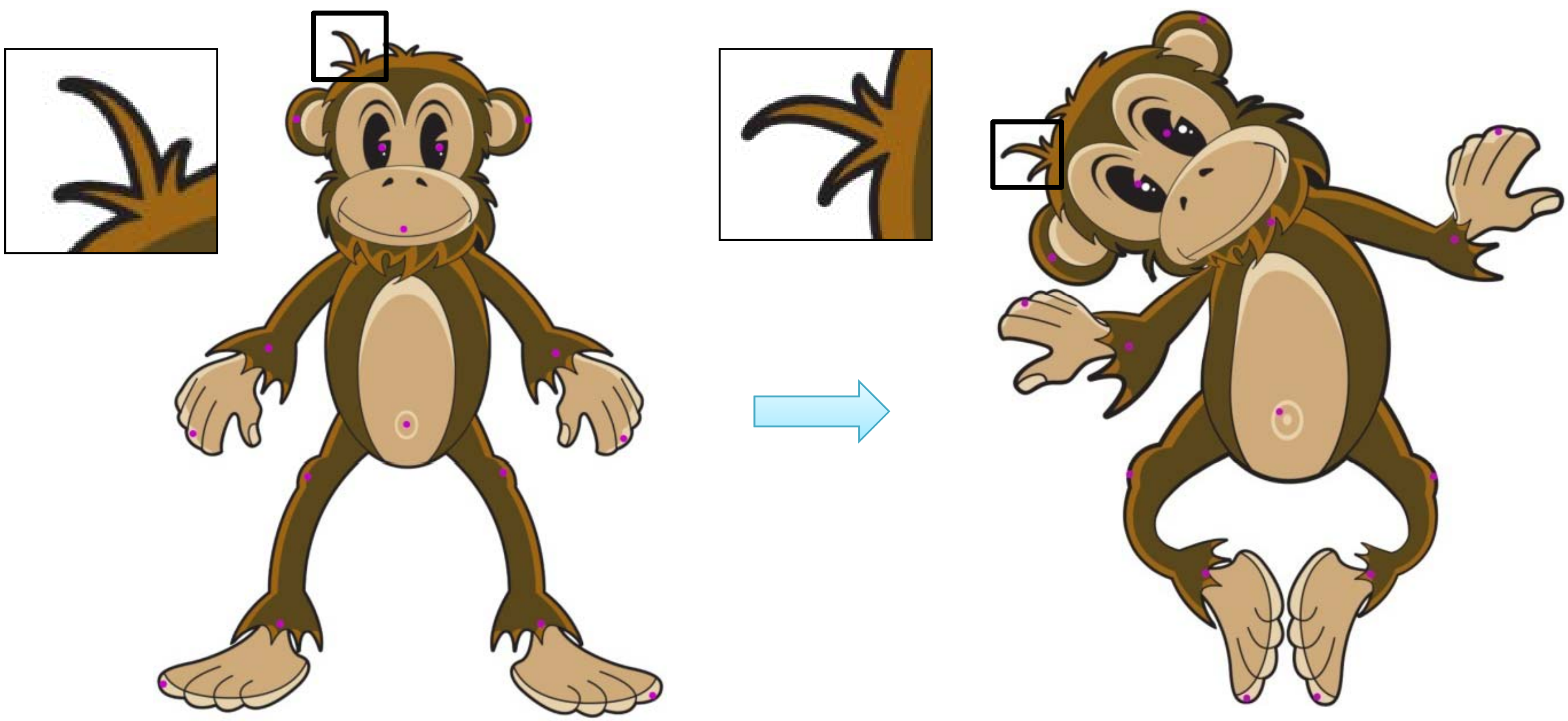


…algorithm deduces the rest

Challenges

"Intuitive deformation"

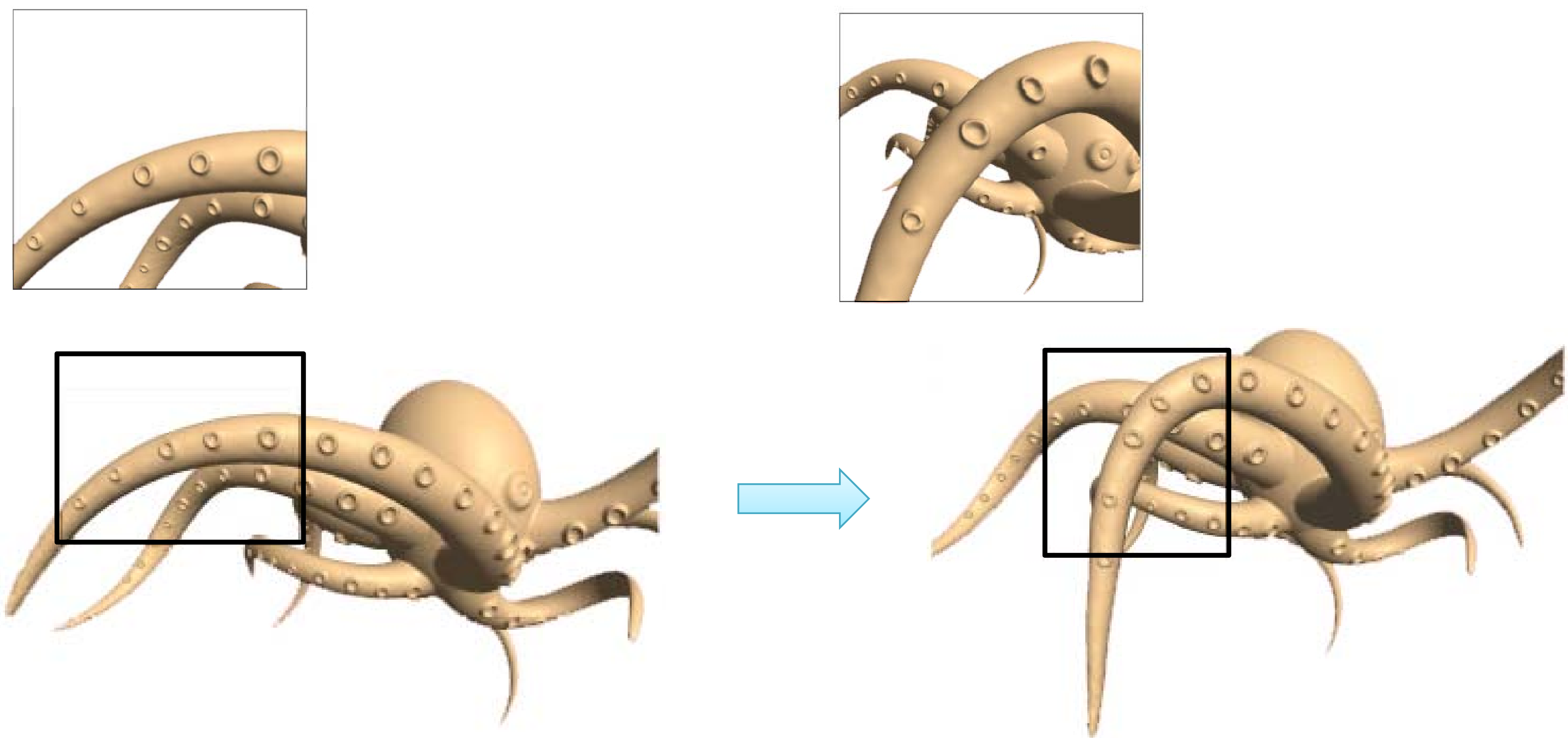
global change + local detail preservation



Challenges

"Intuitive deformation"

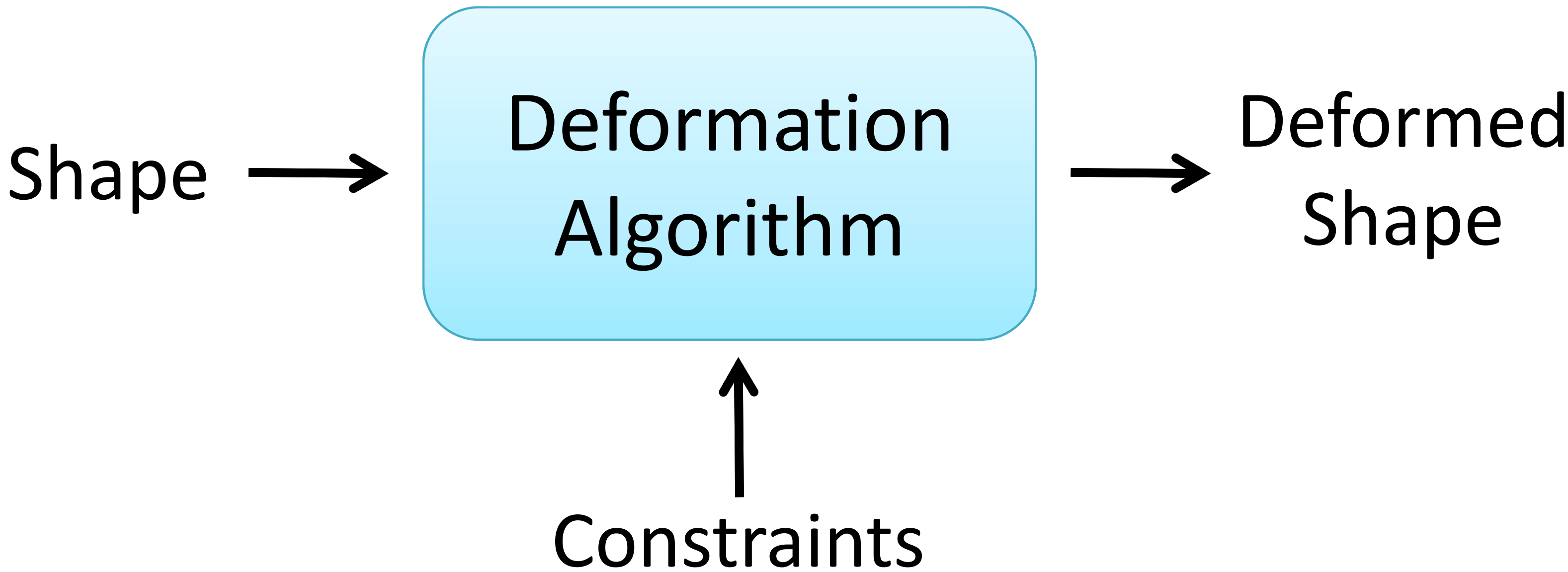
global change + local detail preservation



Challenges

Efficient!

Rules of the Game



Position:
"This point goes there"

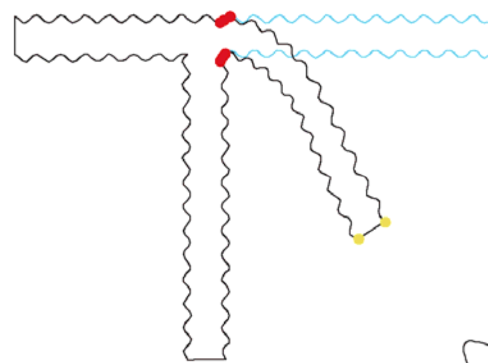
Orientation/Scale:
"The environment of this point should rotate/scale"

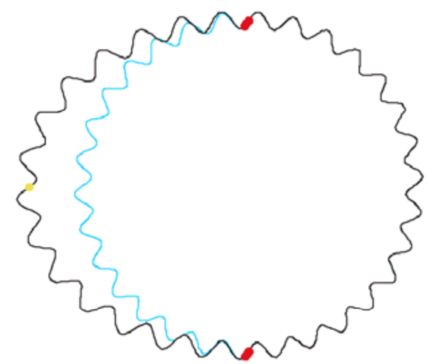
Other shape property:
Curvature, perimeter,…

[Parameterization is also "deformation": constraints = curvature 0 everywhere]

Approaches

- Surface deformation
  - Shape is empty shell
    - Curve for 2D deformation
    - Surface for 3D deformation

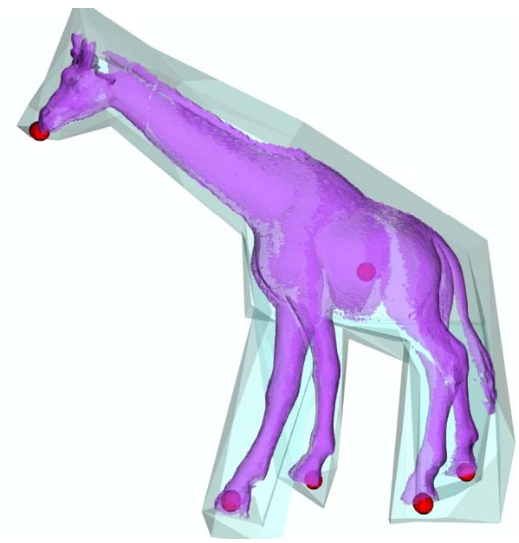




  - Deformation only defined **on** shape
  - Deformation coupled with shape representation

Approaches

- Space deformation
  - Shape is volumetric
    - Planar domain in 2D
    - Polyhedral domain in 3D



  - Deformation defined in neighborhood of shape
  - Can be applied to any shape representation

Approaches

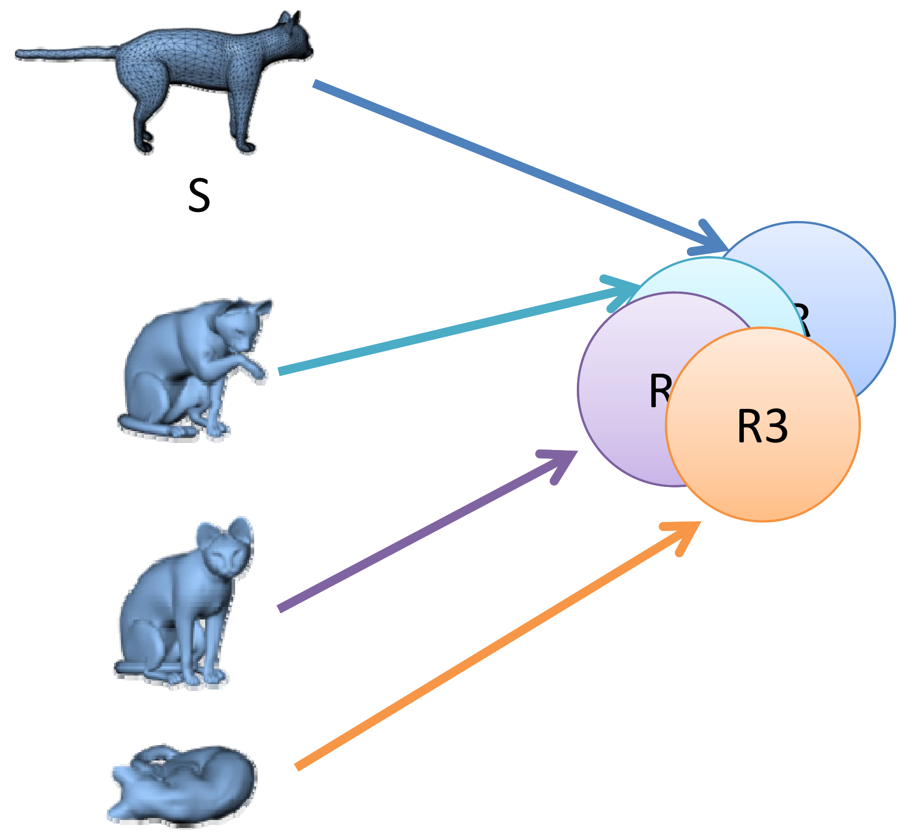
- Surface deformation
  - Find alternative representation which is "deformation invariant"
- Space deformation
  - Find a space map which has "nice properties"

Surface Deformation

Setup:

- Choose alternative representation *f*(S)

- Given S find S' such that
  - Constraints(S') are true
  - *f*(S') = *f*(S)
    (or close)
  - An optimization problem



Shape Representation

How good is the representation?

- Representation should **always** be invariant to:
  - Global translation
  - Global rotation
  - Global scale? Depends on application

- Shapes we want “reachable” should have similar representations
  - Almost isometric deformation
    local translation + rotation
  - Almost “conformal” deformation
    local translation + rotation + scale

Shape Representation

Robustness

- How hard is it to solve the optimization problem?
- Can we find the global minimum?
- Small change in constraints → similar shape?

Efficiency

- Can it be solved at interactive rates?

Shape Representations

Rule of thumb:

If representation is a **linear** function of the coordinates, deformation is:

Robust

Fast

But representation is **not rotation invariant**!

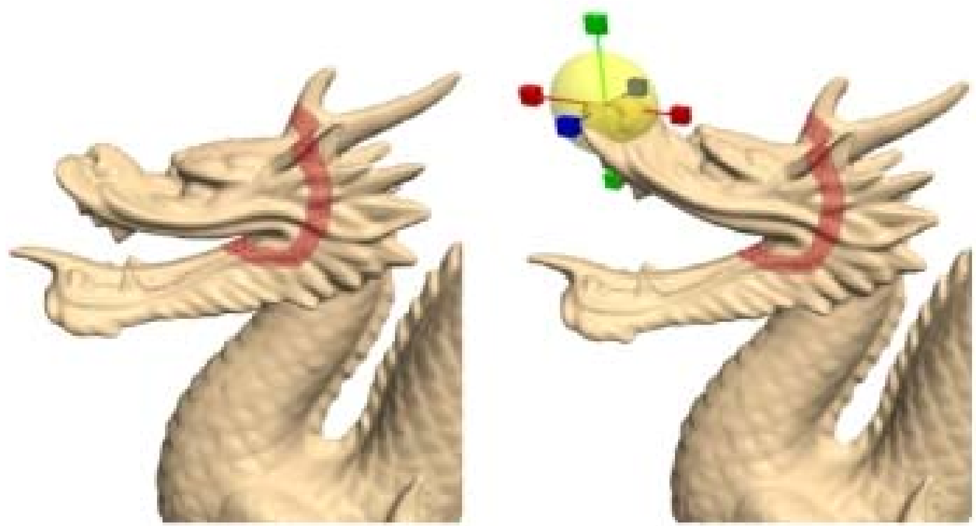
(for large rotations)

Surface Representations

- Laplacian coordinates
- Edge lengths + dihedral angles
- Pyramid coordinates
- Local frames
- ….

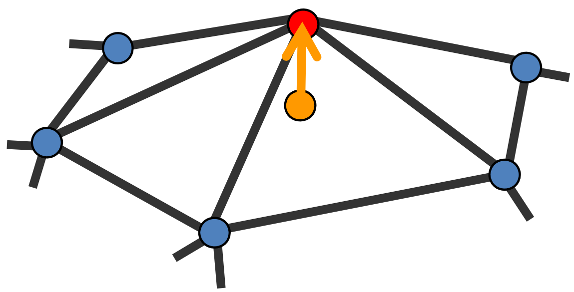
Laplacian Coordinates [Sorkine et al. 04]

- Control mechanism
  – Handles (vertices) moved by user
  – Region of influence (ROI)



Movie

Laplacian Coordinates



$$\boldsymbol{\delta}_i = \mathbf{v}_i - \sum_{j \in N(i)} \frac{1}{d_i} \mathbf{v}_j = \sum_{j \in N(i)} \frac{1}{d_i} \left(\mathbf{v}_i - \mathbf{v}_j \right)$$

$$\boldsymbol{\delta} = \boldsymbol{L}\boldsymbol{V} = (\boldsymbol{I}\text{-}\boldsymbol{D}^{-1}\boldsymbol{A})\boldsymbol{V}$$

$\boldsymbol{I}$ = Identity matrix

$\boldsymbol{D}$ = Diagonal matrix $[d_{ii} = deg(\boldsymbol{v}_i)]$

$\boldsymbol{A}$ = Adjacency matrix

$\boldsymbol{V}$ = Vertices in mesh

Approximation to normals - unique up to translation

Reconstruct by solving $\boxed{\boldsymbol{L}\boldsymbol{V} = \delta}$ for $\boldsymbol{V}$, with one constraint

Poisson equation

Deformation

- Pose modeling constraints for vertices $C \subset V$
  - $v'_i = u_i \quad i \in C$

- No exact solution, minimize error

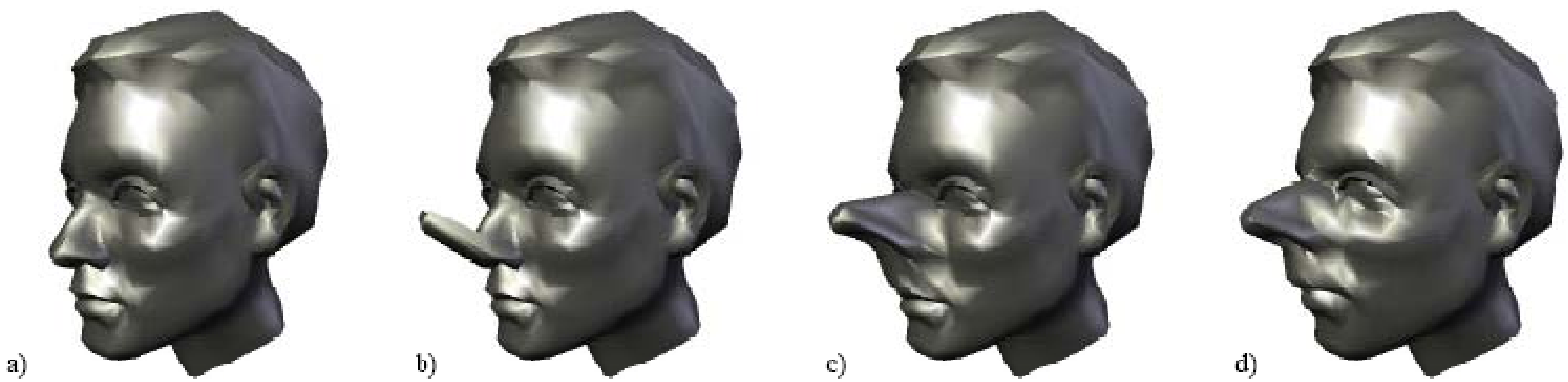
$$\mathbf{V}' = \arg\min_{\mathbf{V}'} \sum_{i=1}^{n} \left\| \delta_i - L(\mathbf{v}'_i) \right\|^2 + \sum_{i \in C} \left\| \mathbf{v}'_i - \mathbf{u}_i \right\|^2$$

Laplacian coordinates of original mesh

Laplacian coordinates of deformed mesh

User Constraints

Deformation



$$\mathbf{V}' = \arg\min_{\mathbf{V}'} \sum_{i=1}^{n} \left\| \delta_i - L(\mathbf{v}'_i) \right\|^2 + \sum_{i \in C} \left\| \mathbf{v}'_i - \mathbf{u}_i \right\|^2$$

Laplacian coordinates of original mesh

Laplacian coordinates of deformed mesh

User Constraints

Linear Least Squares

- $\boldsymbol{Ax} = \boldsymbol{b}$ with *m* equations, *n* unknowns

- Normal equations: $(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})\boldsymbol{x} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$
- Solution by pseudo inverse:

$$\boxed{\boldsymbol{x} = \boldsymbol{A}^{+}\boldsymbol{b} = [(\boldsymbol{A}^{T}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}]\boldsymbol{b}}$$

If system under determined: $\boldsymbol{x} = \boldsymbol{arg\ min}\ \{\ ||\boldsymbol{x}||\ \boldsymbol{:}\ \boldsymbol{Ax} = \boldsymbol{b}\ \}$

If system over determined: $\boldsymbol{x} = \boldsymbol{arg\ min}\ \{\ ||\boldsymbol{Ax\text{-}b}||^{2}\ \}$

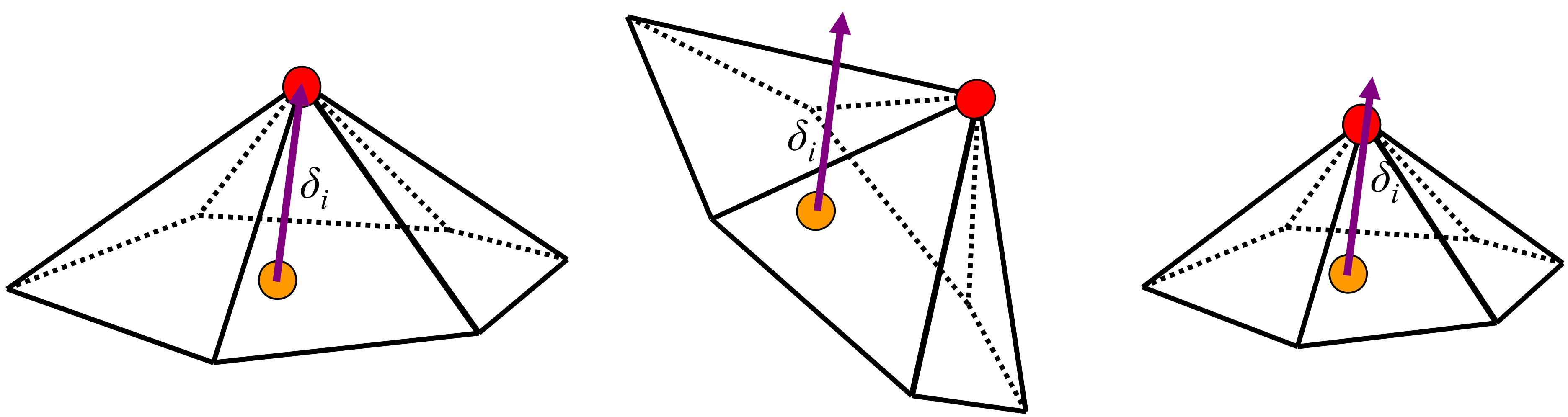
Laplacian Coordinates

Sanity Check

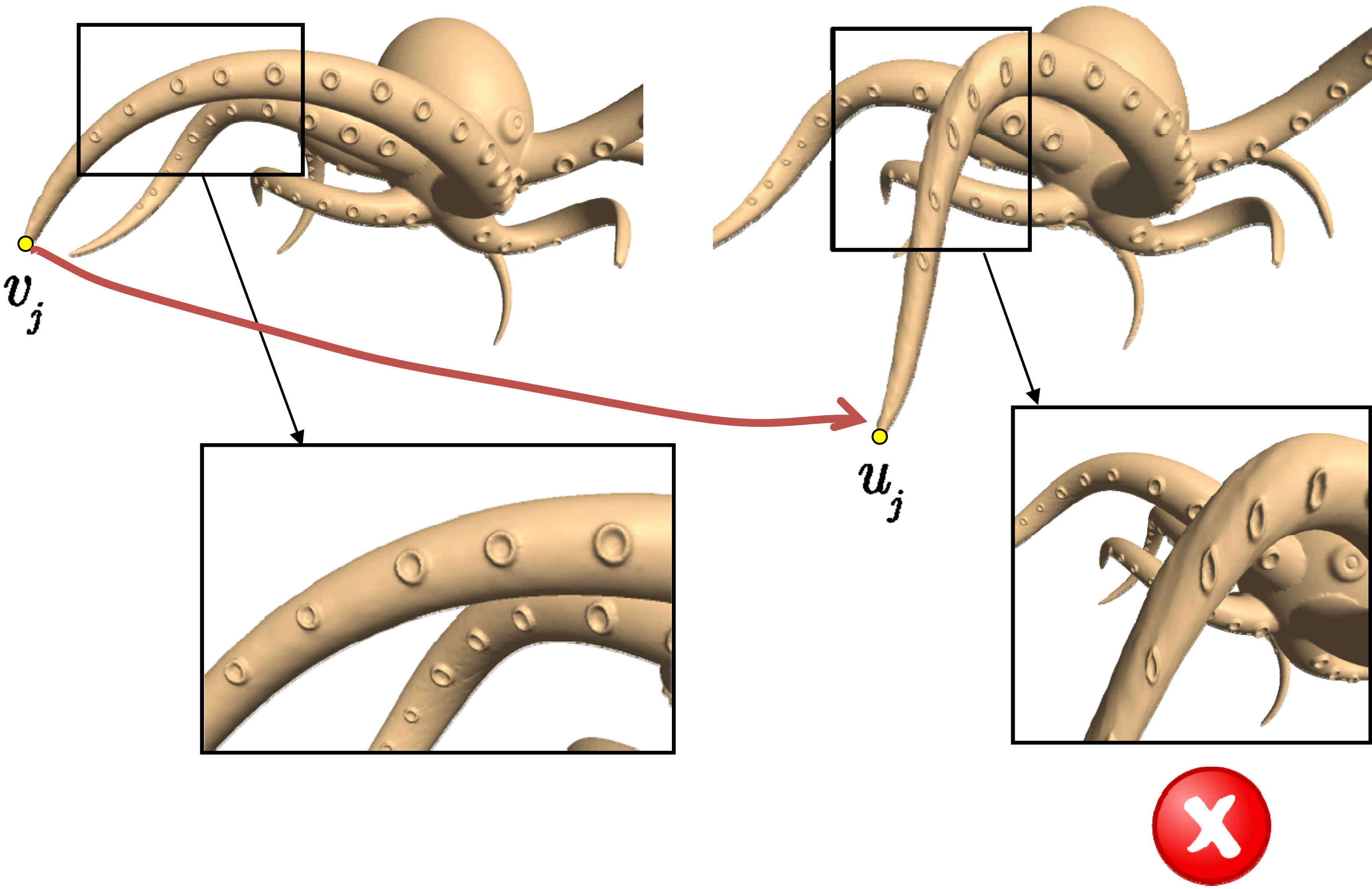
- Translation invariant?

$$\delta_i = L(\mathbf{v}_i) = L(\mathbf{v}_i + \mathbf{t}) \quad \forall \mathbf{t} \in \mathbb{R}^3$$

- Rotation/scale invariant?

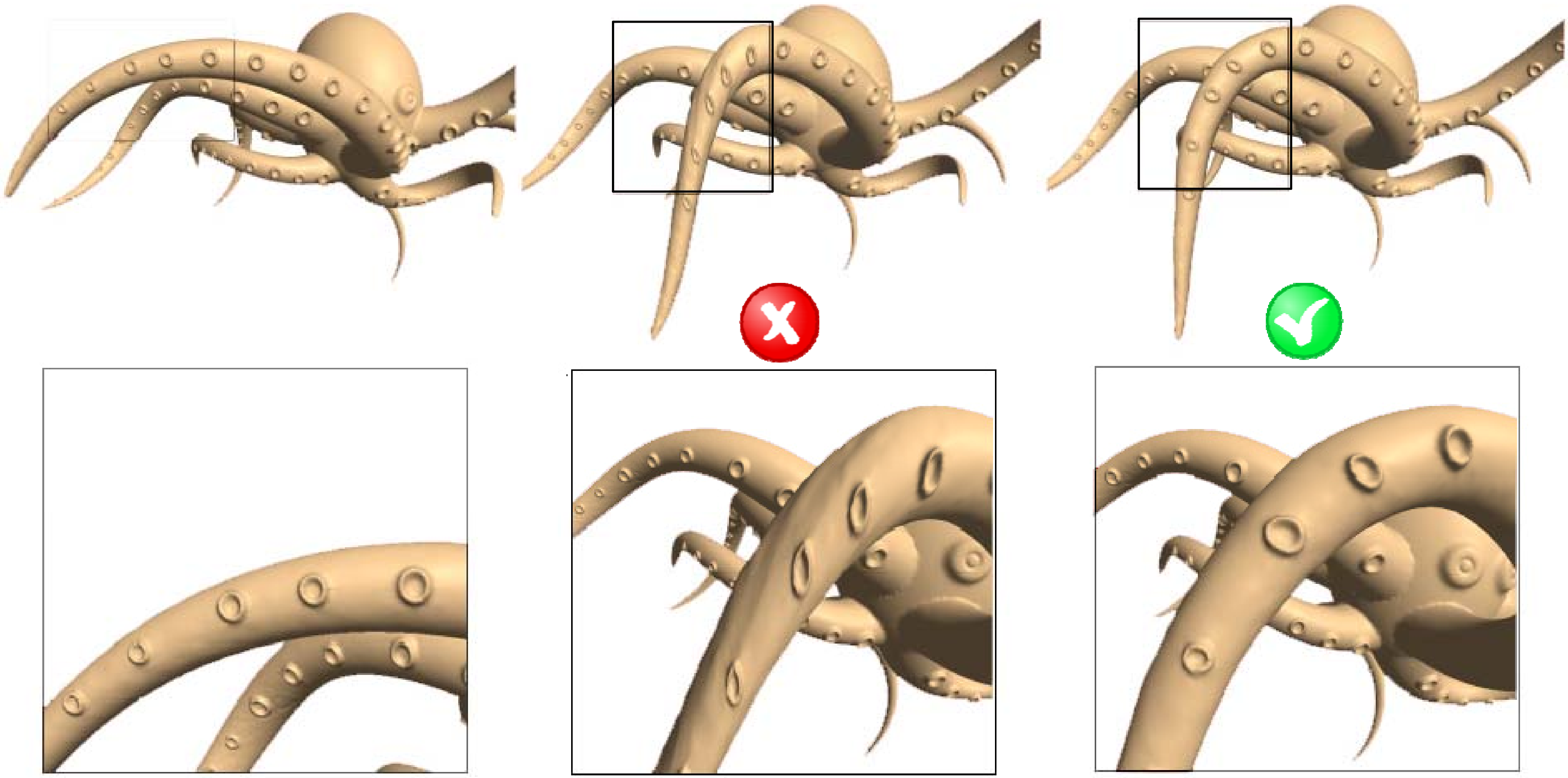


Problem



$$Lv' = \delta$$

$$v'_j = u_j$$



input

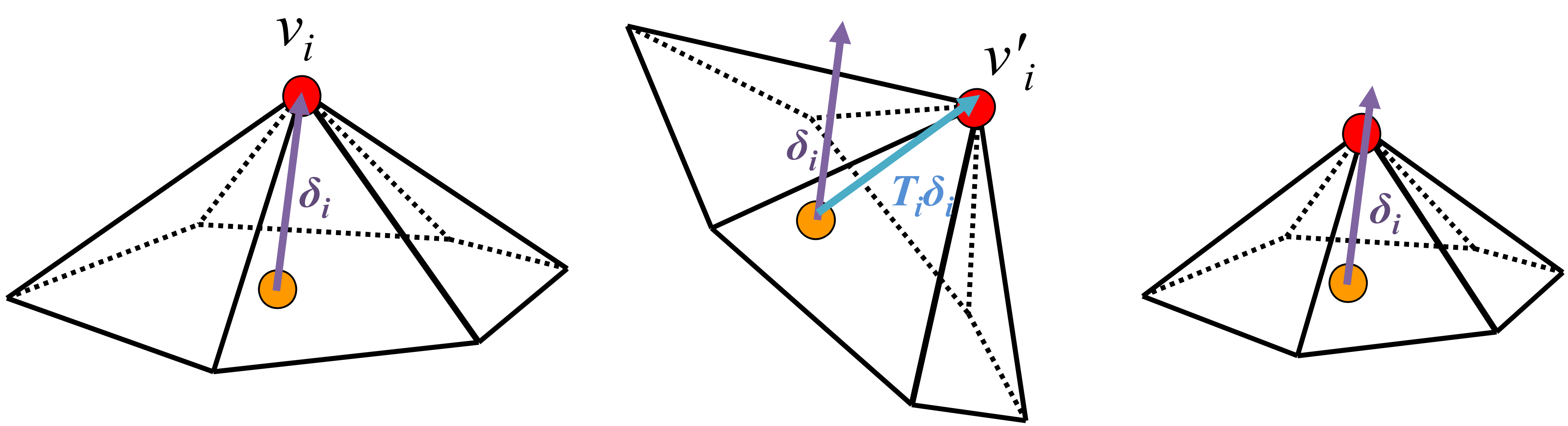
Laplacian coords

“Rotation invariant” coords

“Rotation Invariant” Coords

The representation should take into account local rotations + scale

$$\boldsymbol{\delta_i} = \boldsymbol{L}(v_i) \qquad \boldsymbol{T_i \delta_i} = \boldsymbol{L}(v'_i)$$



“Rotation Invariant” Coords

The representation should take into account local rotations + scale

$$\boldsymbol{\delta}_{\boldsymbol{i}} = \boldsymbol{L}(v_i) \qquad \boldsymbol{T}_{\boldsymbol{i}}\boldsymbol{\delta}_{\boldsymbol{i}} = \boldsymbol{L}(v'_i)$$

Problem: $\boldsymbol{T}_{\boldsymbol{i}}$ depends on **deformed** position v'_i

Solution: Implicit Transformations

Idea: solve for local transformation and deformed surface simultaneously

$$V' = \arg\min_{V'} \left(\sum_{i=1}^{n} \left\| L(\mathbf{v}'_i) - T_i(\boldsymbol{\delta}_i) \right\|^2 + \sum_{j \in C} \left\| \mathbf{v}'_j - \mathbf{u}_j \right\|^2 \right)$$

Transformation of the local frame

Similarities

Restrict T_i to "good" transformations = rotation + scale → similarity transformation

$$V' = \arg\min_{V'} \left(\sum_{i=1}^{n} \left\| L(\mathbf{v}'_i) - T_i(\boldsymbol{\delta}_i) \right\|^2 + \sum_{j \in C} \left\| \mathbf{v}'_j - \mathbf{u}_j \right\|^2 \right)$$

Similarity Transformation

Similarities

- Conditions on T_i to be a similarity matrix?

- Linear in 2D:

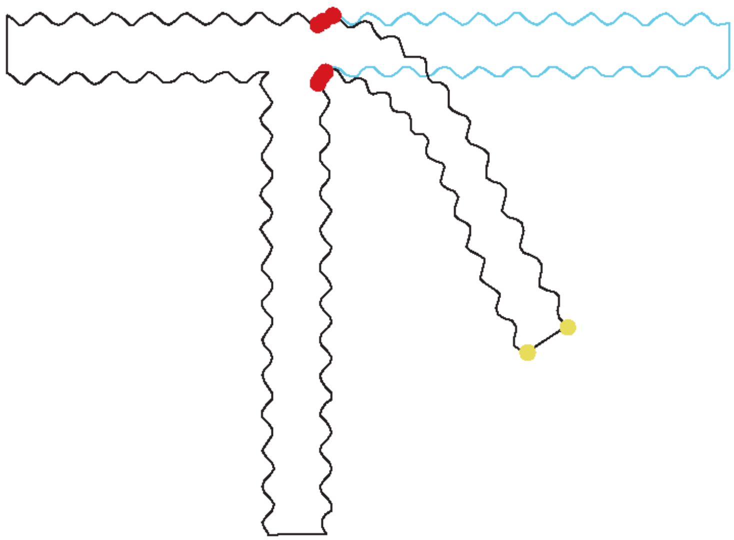
Auxiliary variables

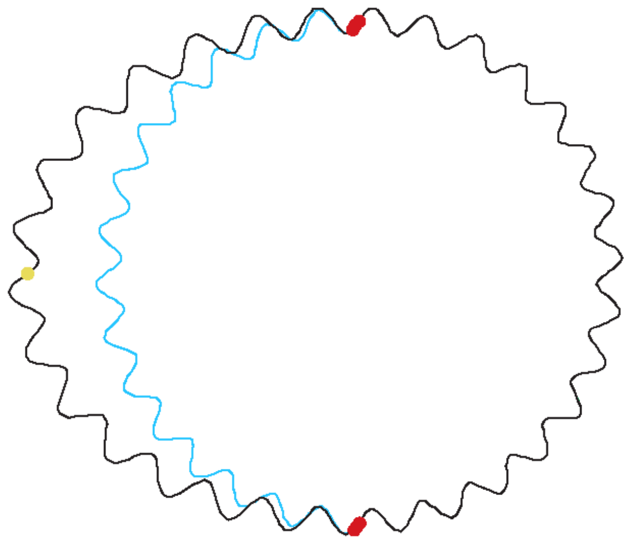
$$T_i = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & d_x \\ -\sin\theta & \cos\theta & d_y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} w & a & t_x \\ -a & w & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

Uniform scale

Rotation + translation

Similarities 2D





Similarities – 3D case

- Not linear in 3D:

$$\begin{pmatrix} \text{rotation} + \\ \text{uniform scale} \end{pmatrix} = s \, \exp H = s\left(\alpha I + \beta H + \mathbf{h}^T \mathbf{h}\right)$$

H is 3×3 skew-symmetric, $H\mathbf{x} = \mathbf{h} \times \mathbf{x}$

- Linearize by dropping the quadratic term
  - Effectively: only **small** rotations are handled

Laplacian Coordinates

Realtime?

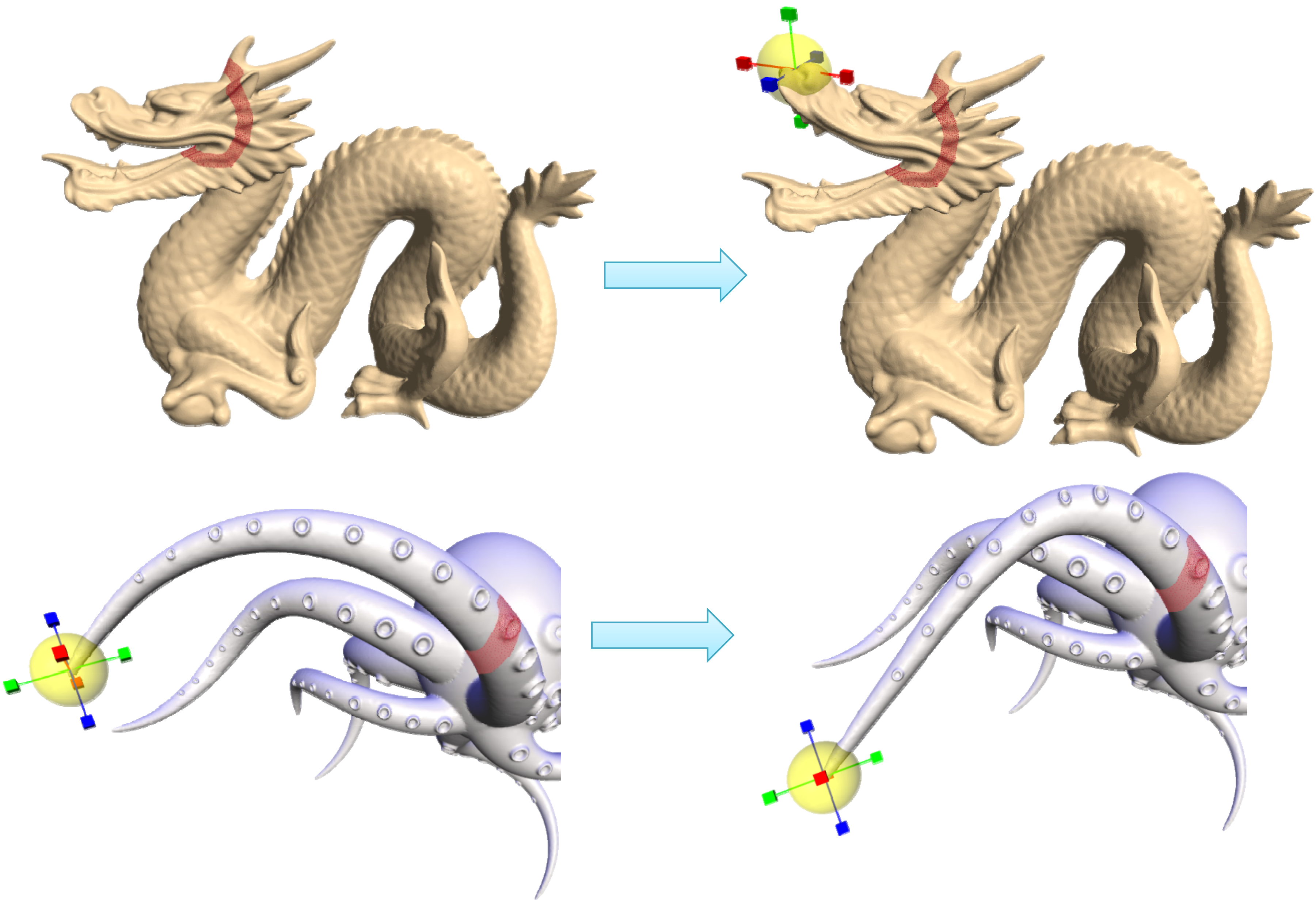
- Need to solve a linear system each frame

$$(A^{\mathrm{T}}A)\boldsymbol{x} = A^{\mathrm{T}}\boldsymbol{b}$$

- Precompute sparse Cholesky factorization

- Only back substitution per frame

Some Results



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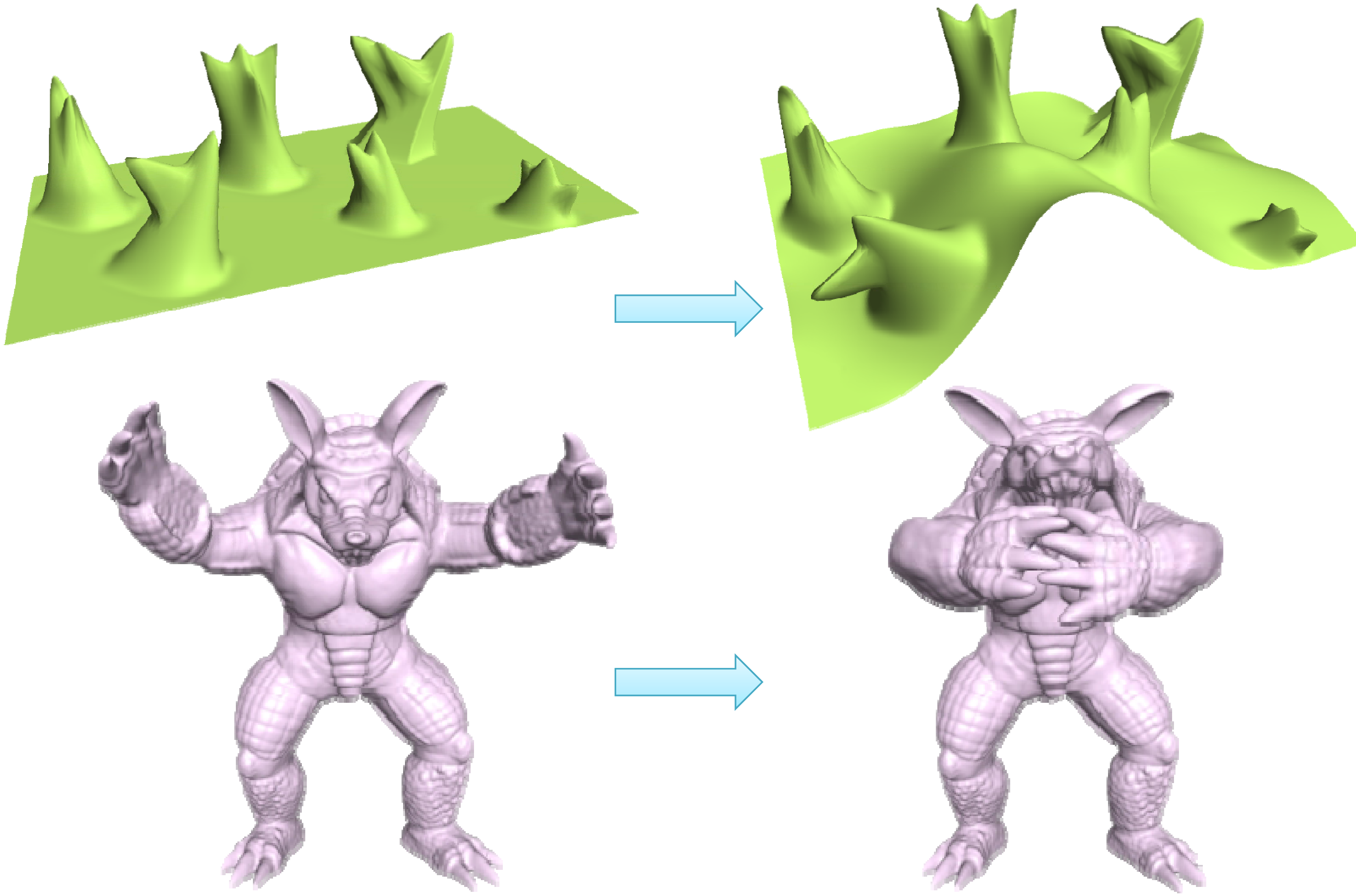
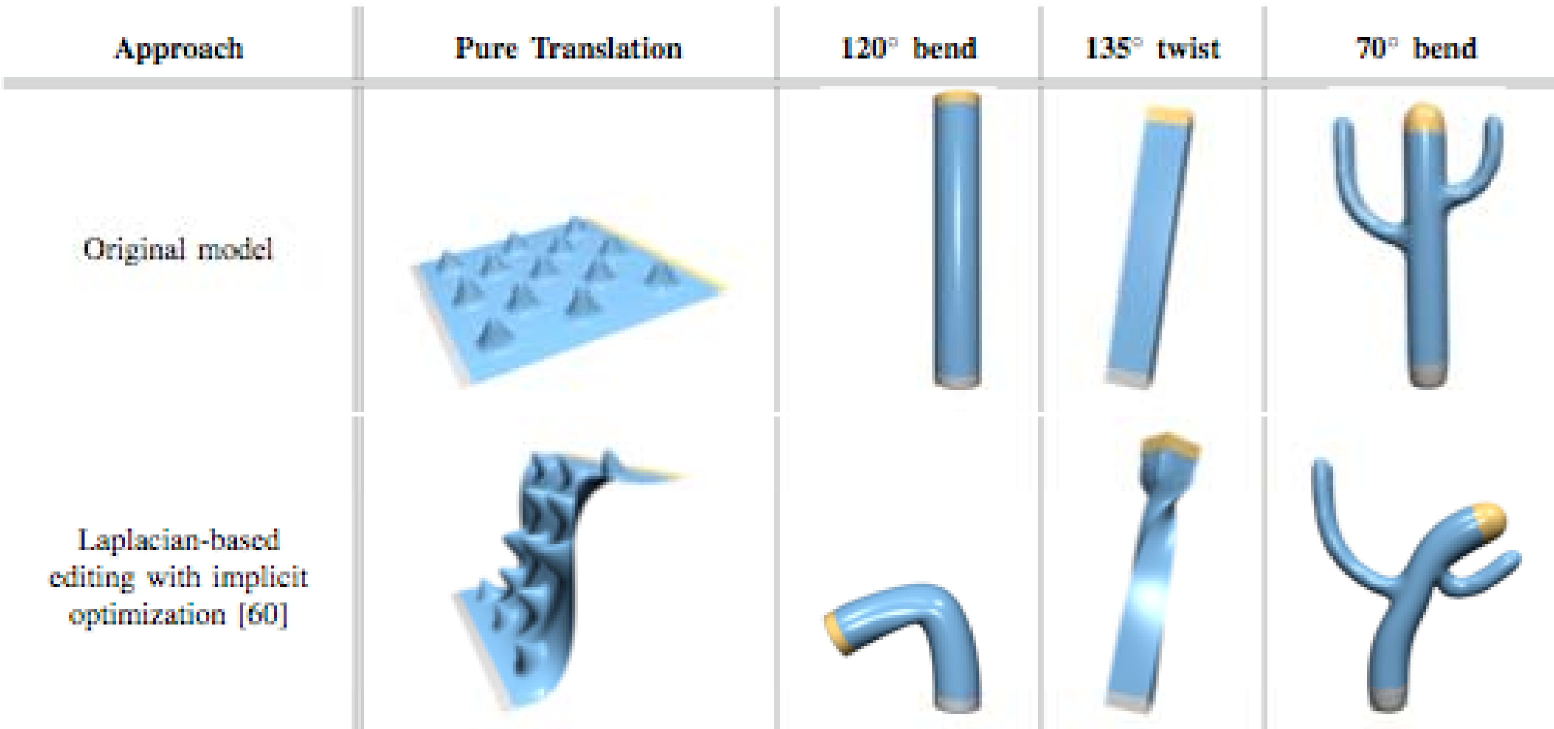


Limitations: Large Rotations



How to Find the Rotations?

- Laplacian coordinates – solve for them
  - Problem: not linear

- Another approach: propagate rotations from handles

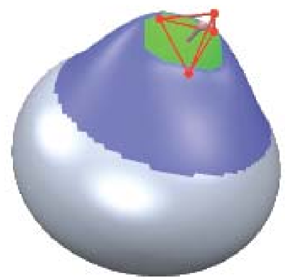
Rotation Propagation

- Compute handle's "deformation gradient"
- Extract rotation and scale/shear components
- Propagate damped rotations over ROI

Deformation Gradient

- Handle has been transformed affinely

$$\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$



- Deformation gradient is:

$$\nabla \mathbf{T}(\mathbf{x}) = \mathbf{A}$$

- Extract rotation $\mathbf{R}$ and scale/shear $\mathbf{S}$

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \quad \Rightarrow \quad \mathbf{R} = \mathbf{U}\mathbf{V}^T,\ \mathbf{S} = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^T$$

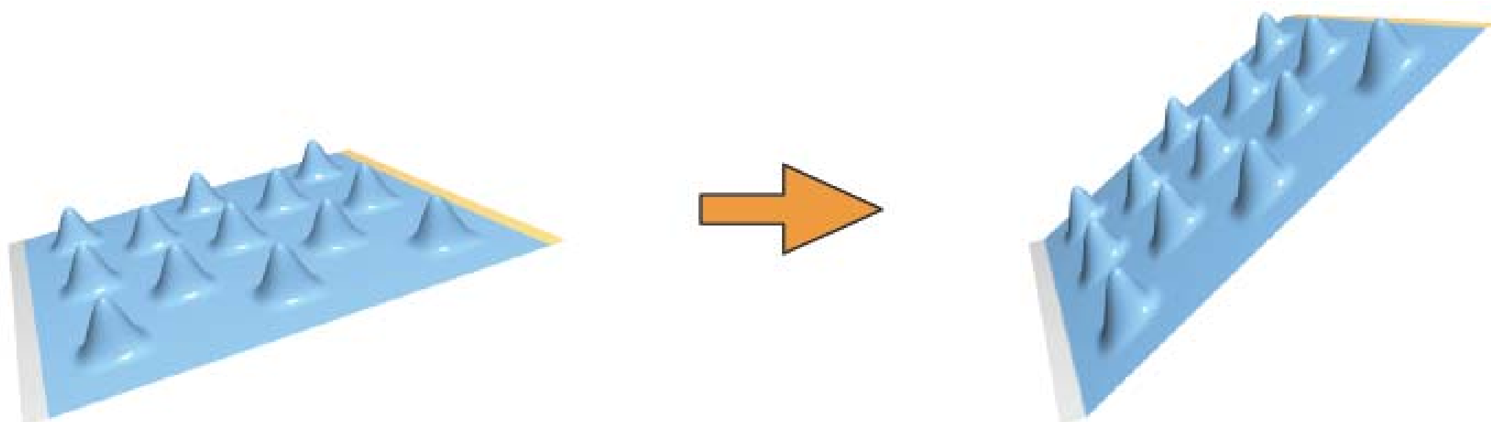
Smooth Propagation

- Construct smooth scalar field [0,1]
  - $\alpha(\mathbf{x})=1$ Full deformation (handle)
  - $\alpha(\mathbf{x})=0$ No deformation (fixed part)
  - $\alpha(\mathbf{x})\in[0,1]$ Damp transformation (in between)

- Linearly damp scale/shear:
  $\mathbf{S}(\mathbf{x})= \alpha(\mathbf{x})\mathbf{S}(\mathit{handle})$
- Log scale damp rotation:
  $\mathbf{R}(\mathbf{x}) = \exp(\alpha(\mathbf{x})\log(\mathbf{R}(\mathit{handle})))$

Limitations

- Works well for rotations
- Translations don’t change deformation gradient
  - “Translation insensitivity”



The Curse of Rotations

- Can’t solve for them directly using a linear system
- Can’t propagate if the handles don’t rotate

- Some linear methods work for rotations
- Some work for translations
- None work for both

The Curse of Rotations

- Non linear methods work for both large rotations and translation only

- No free lunch: much more expensive