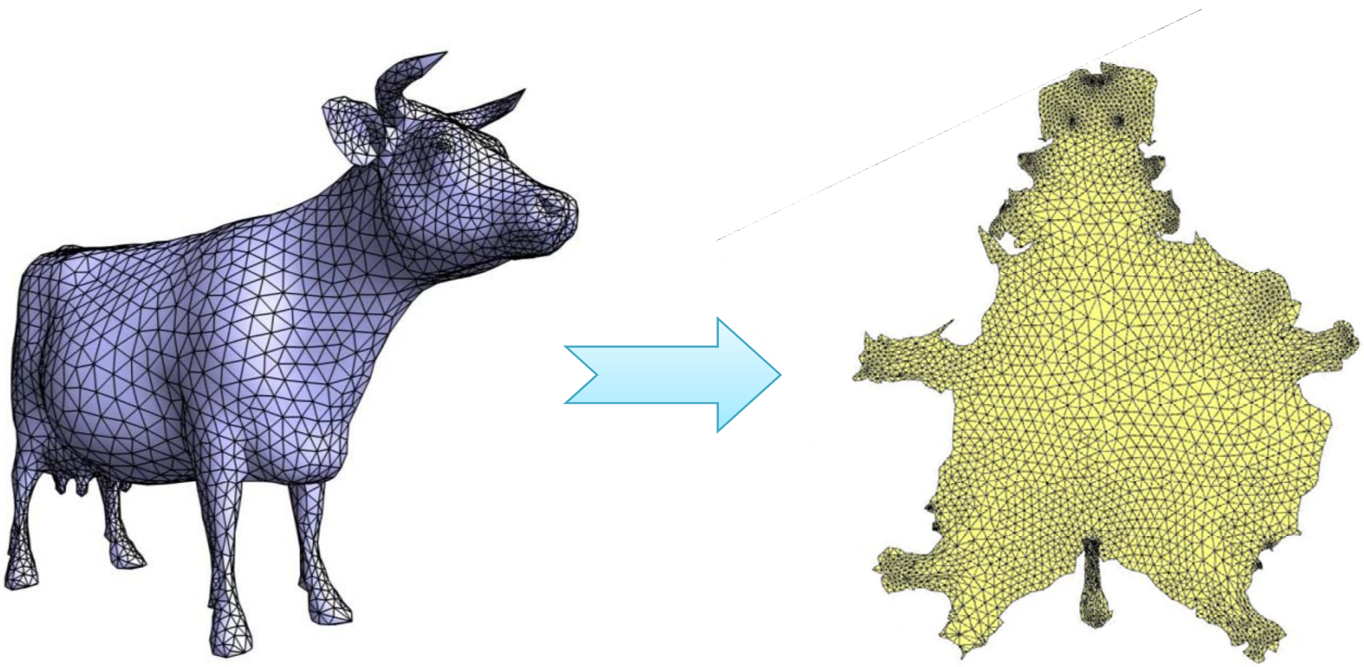
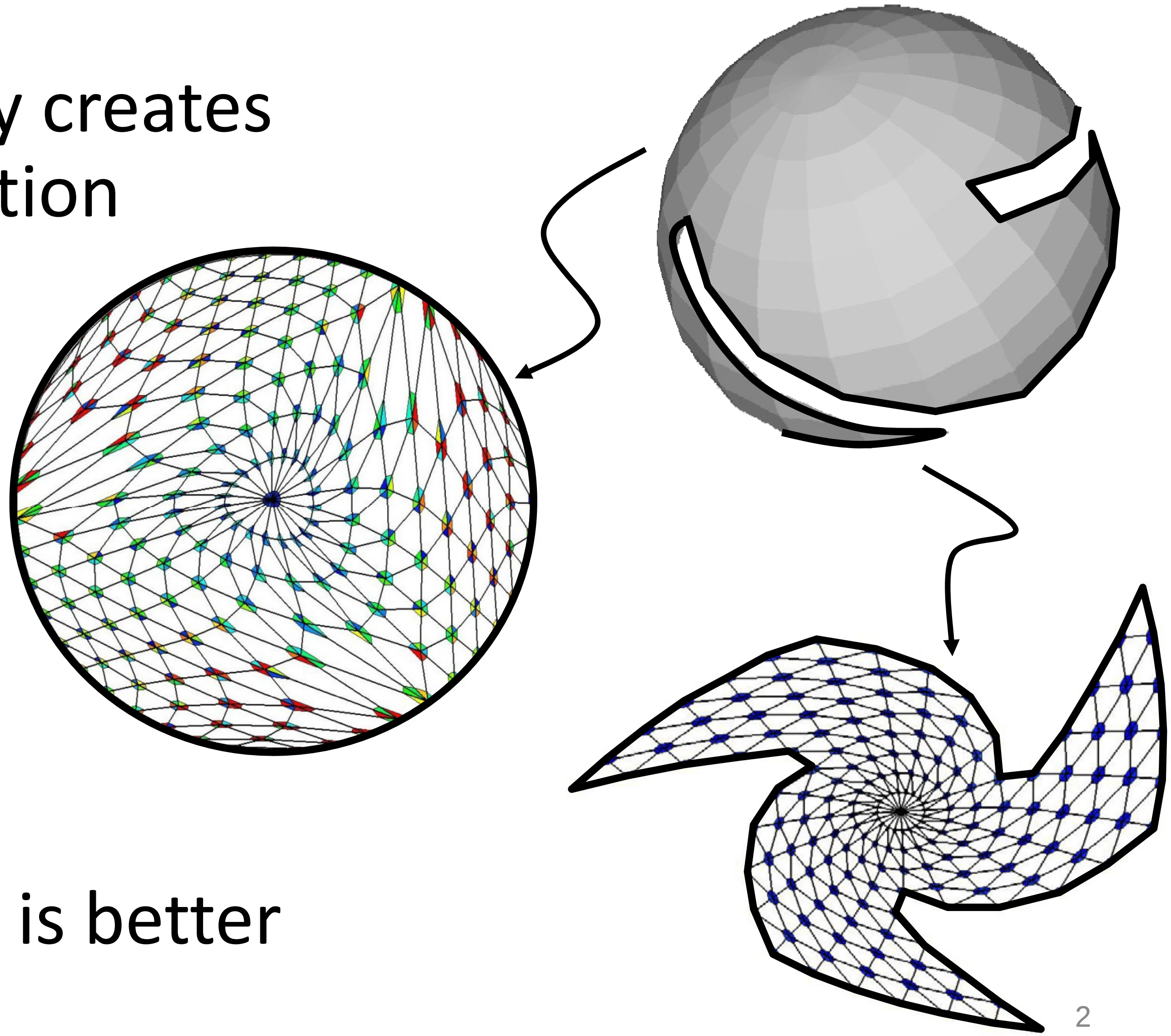
Parameterization II



Some slides from the Mesh Parameterization Course from Siggraph Asia 2008

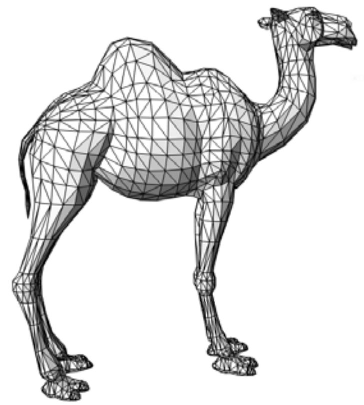
Non-Convex Boundary

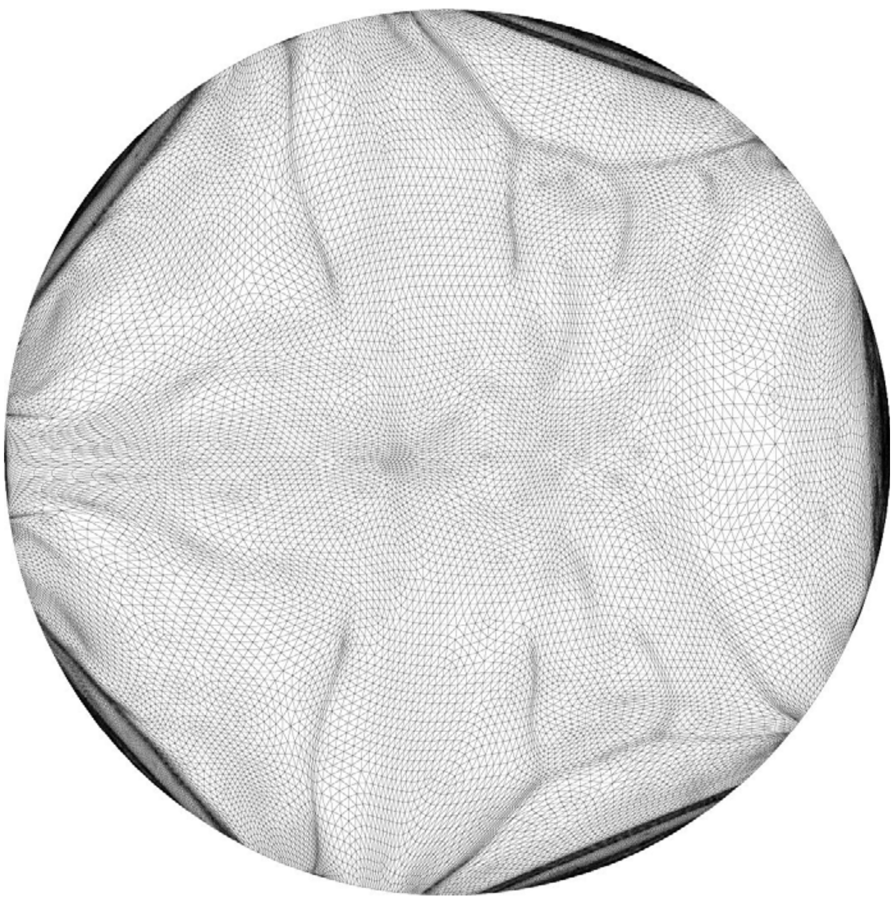
- Convex boundary creates significant distortion

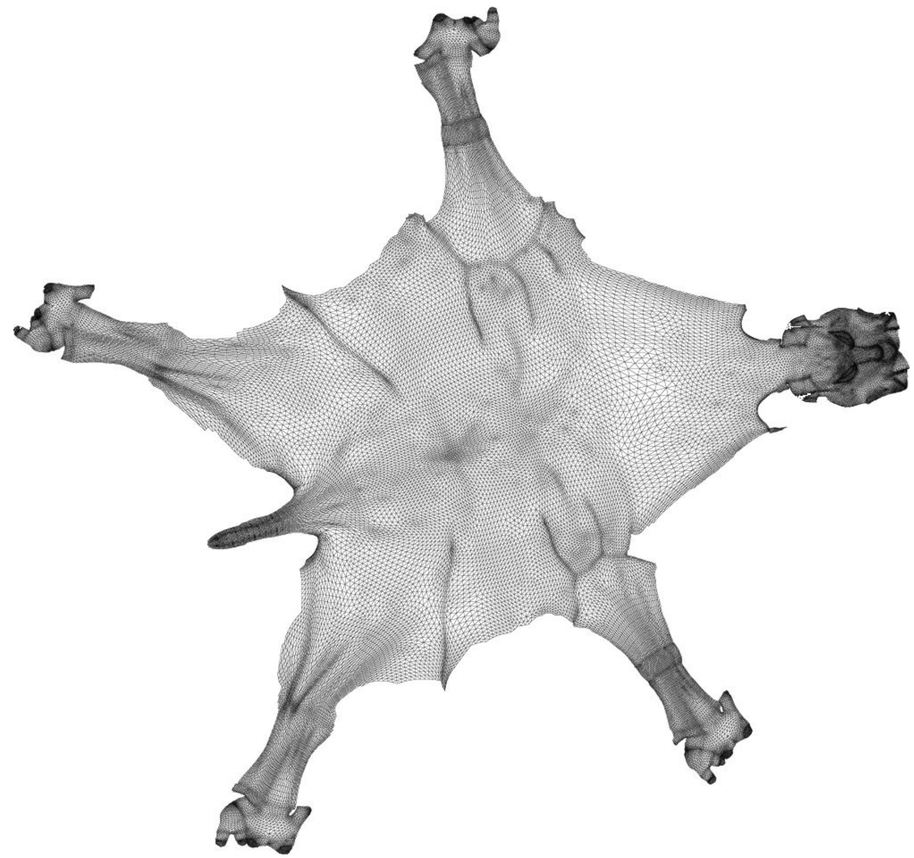


- "Free" boundary is better

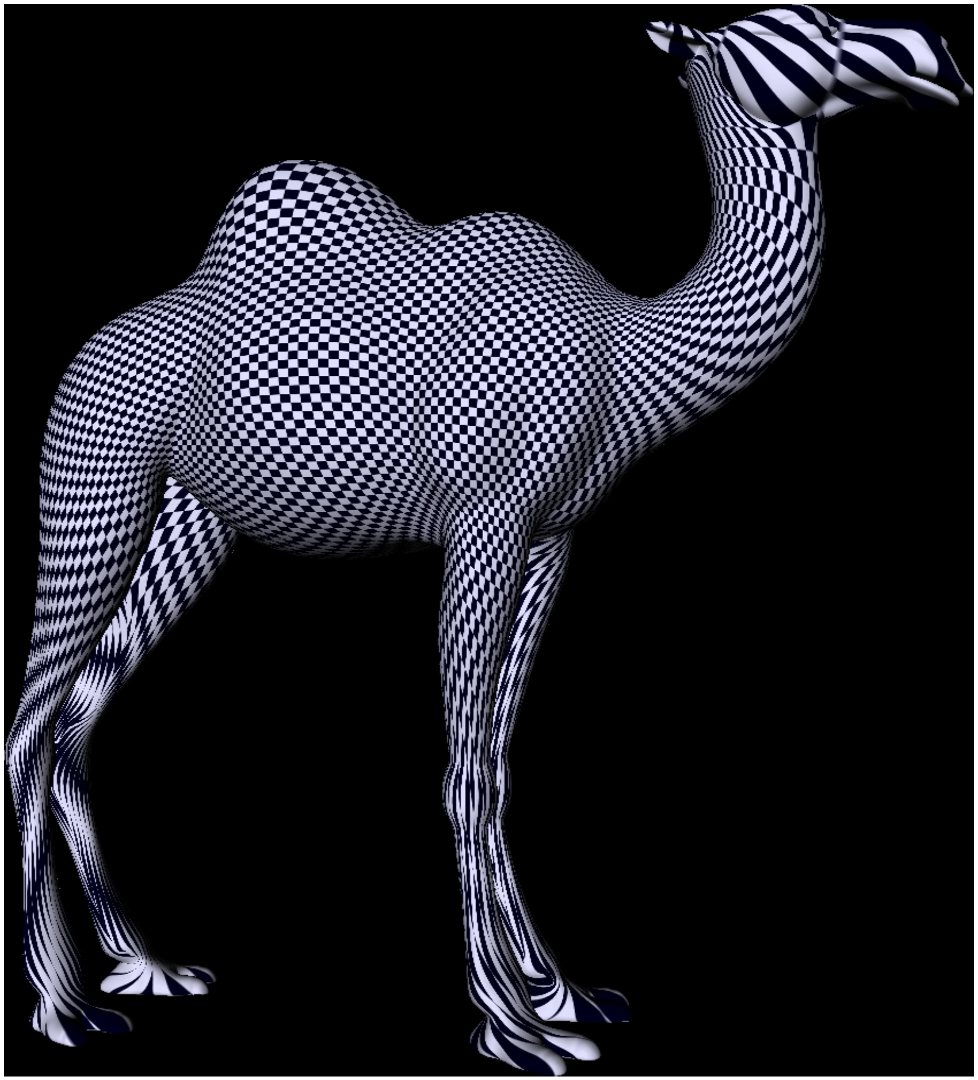
Fixed vs Free Boundary

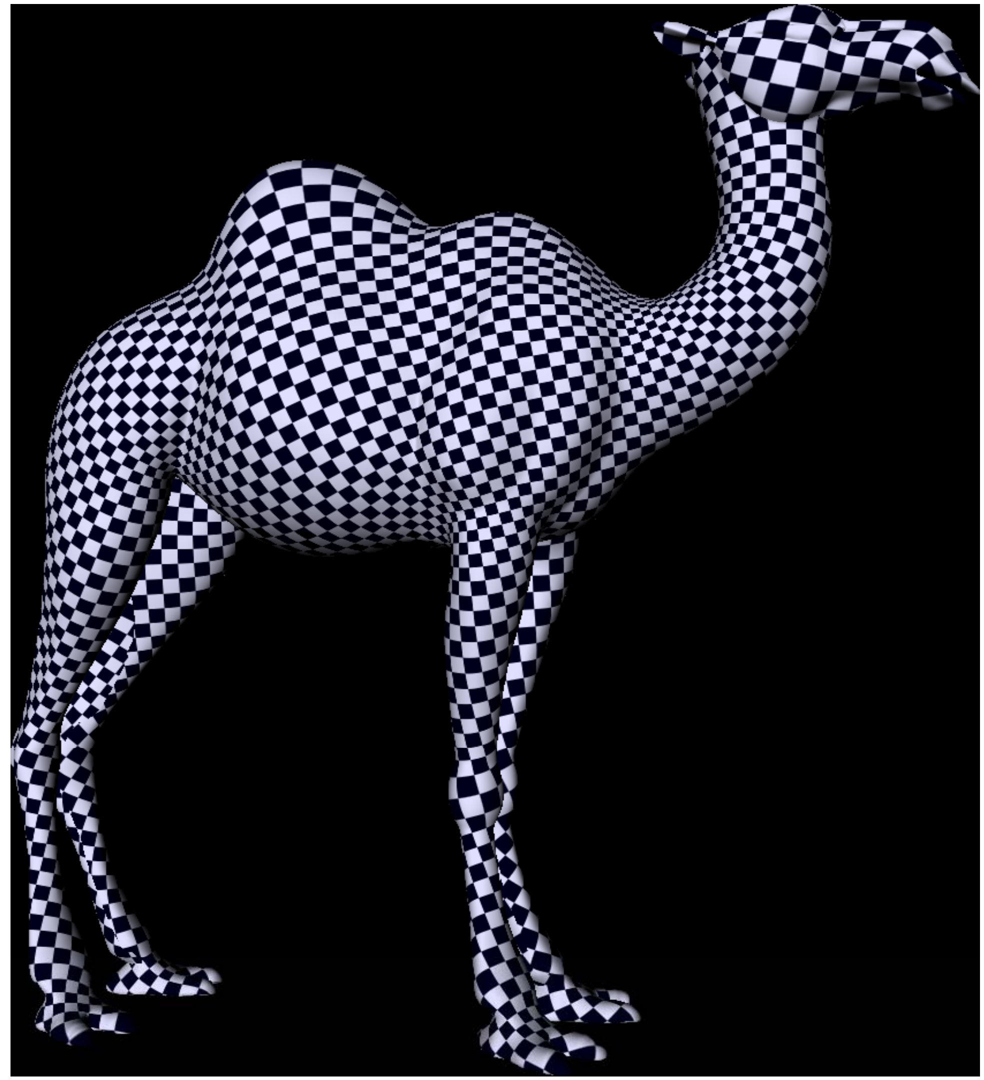


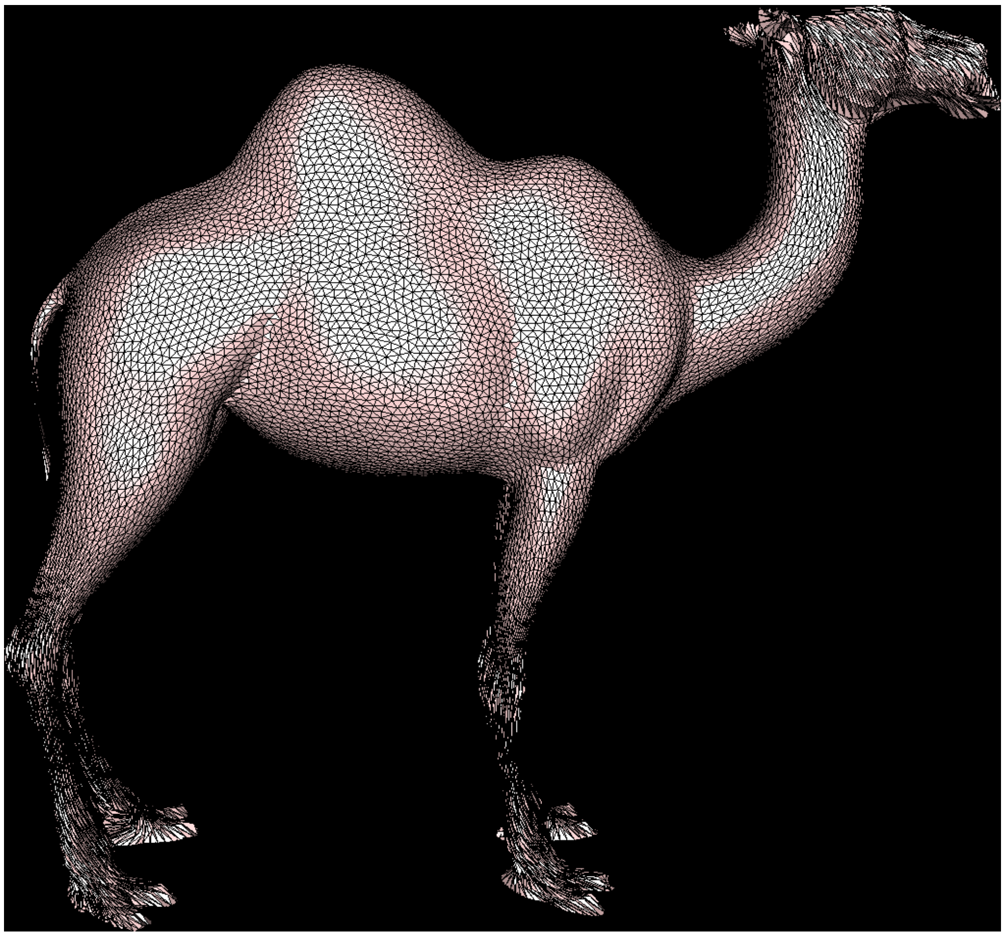


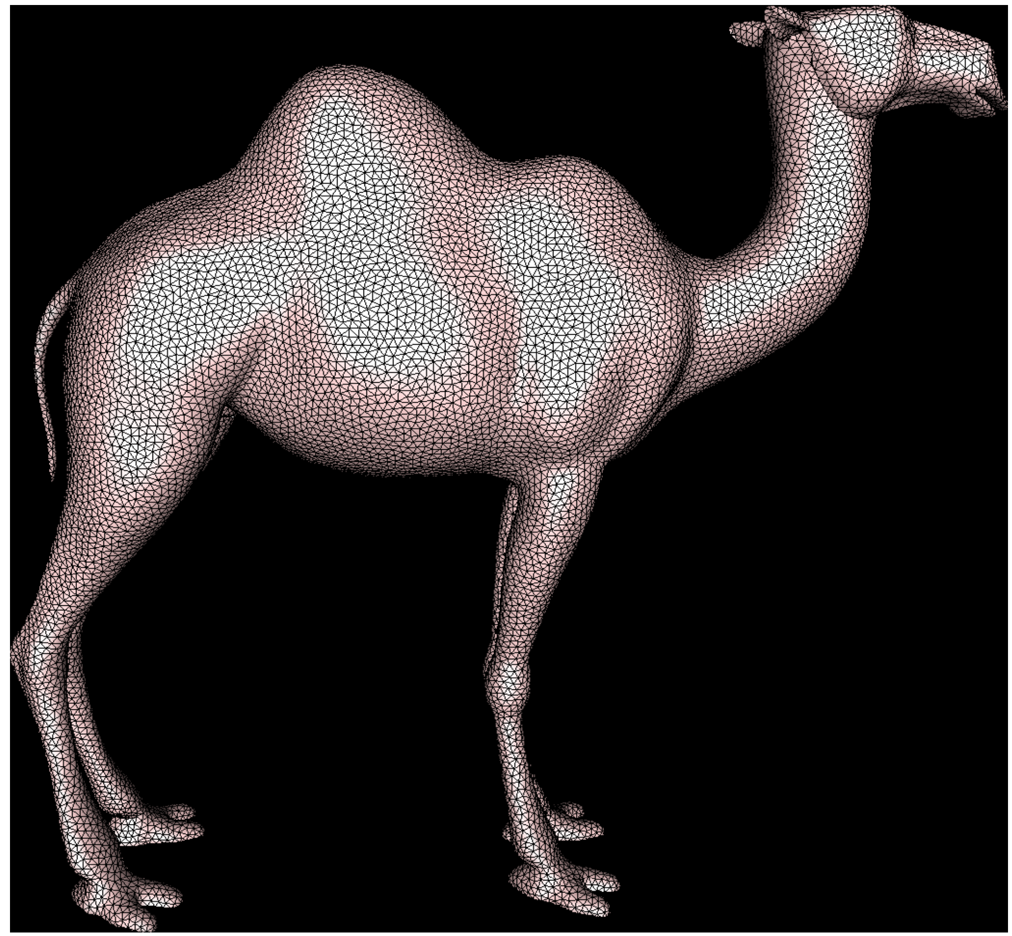


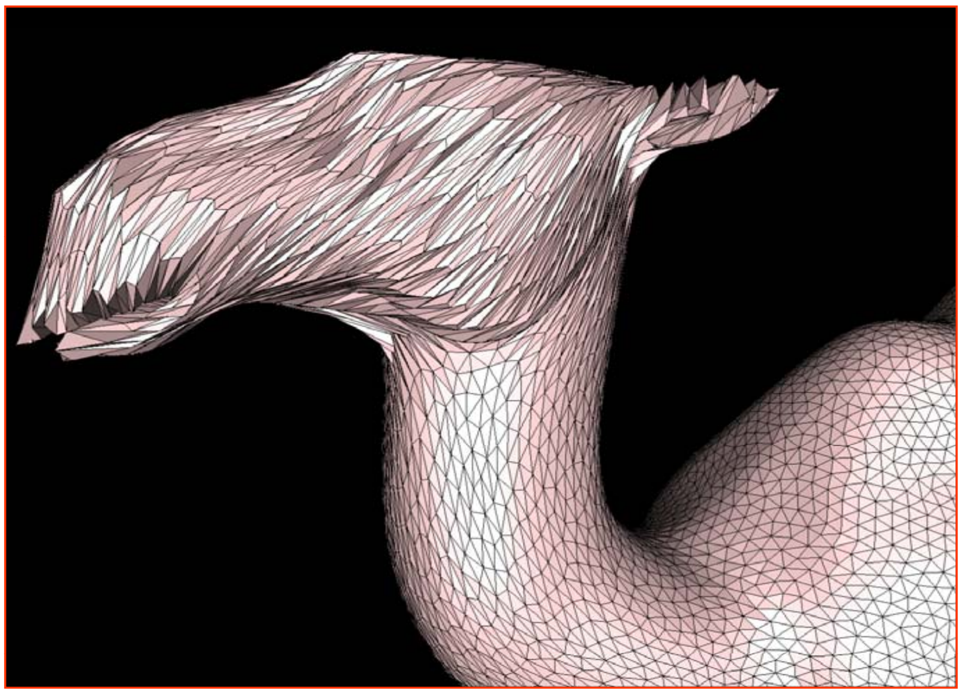
Fixed vs Free Boundary

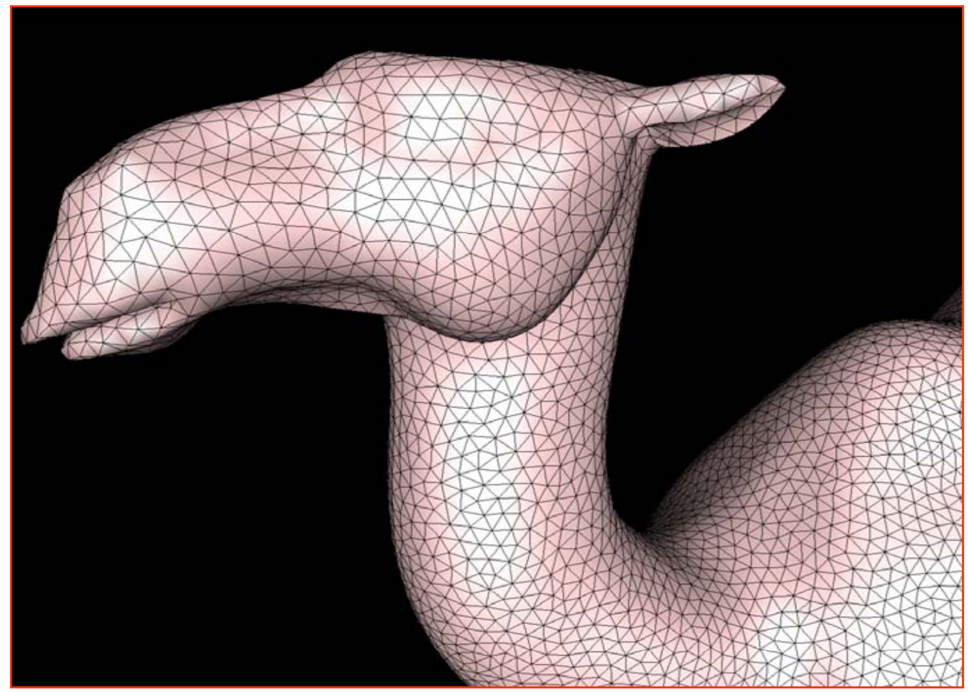


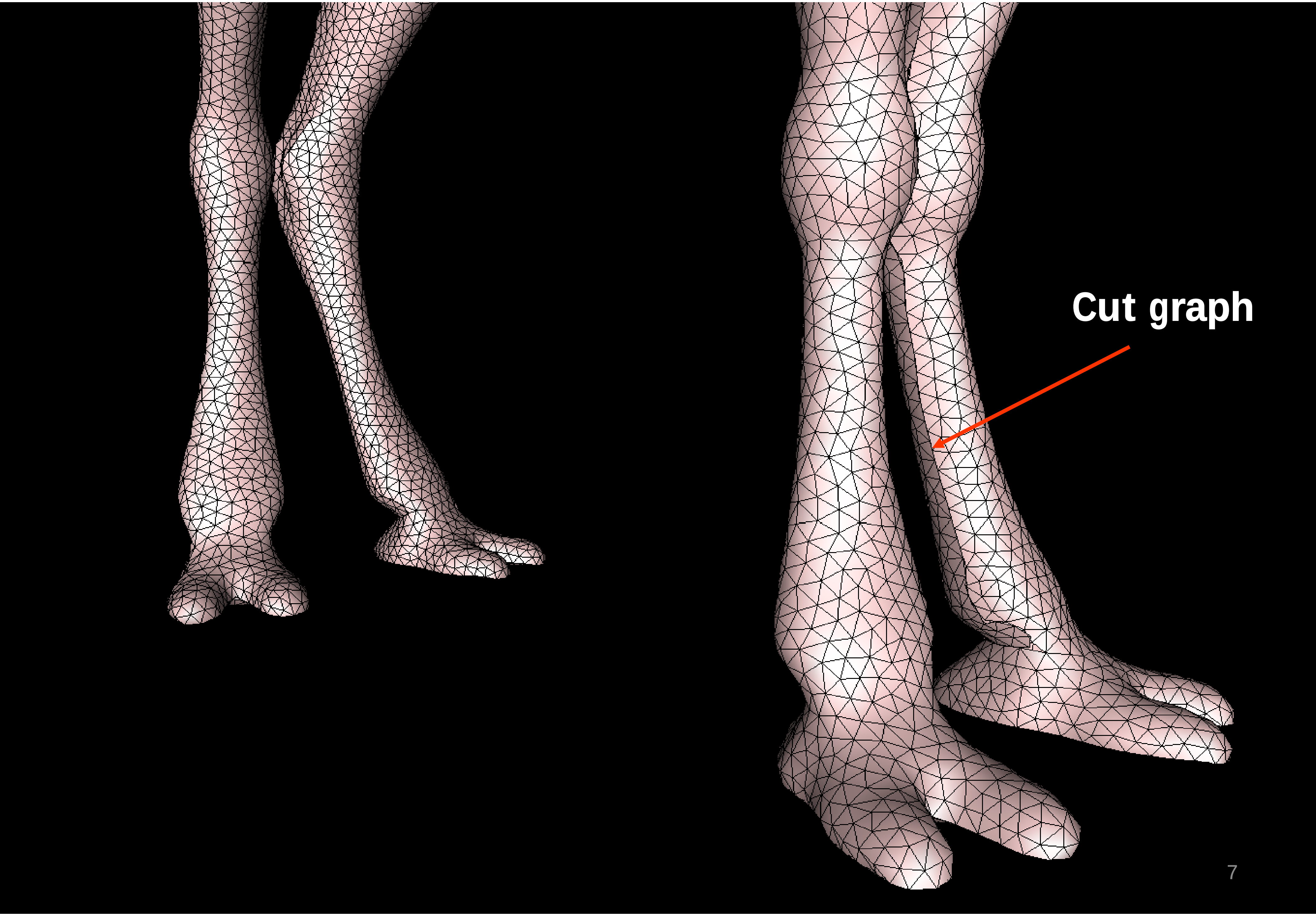


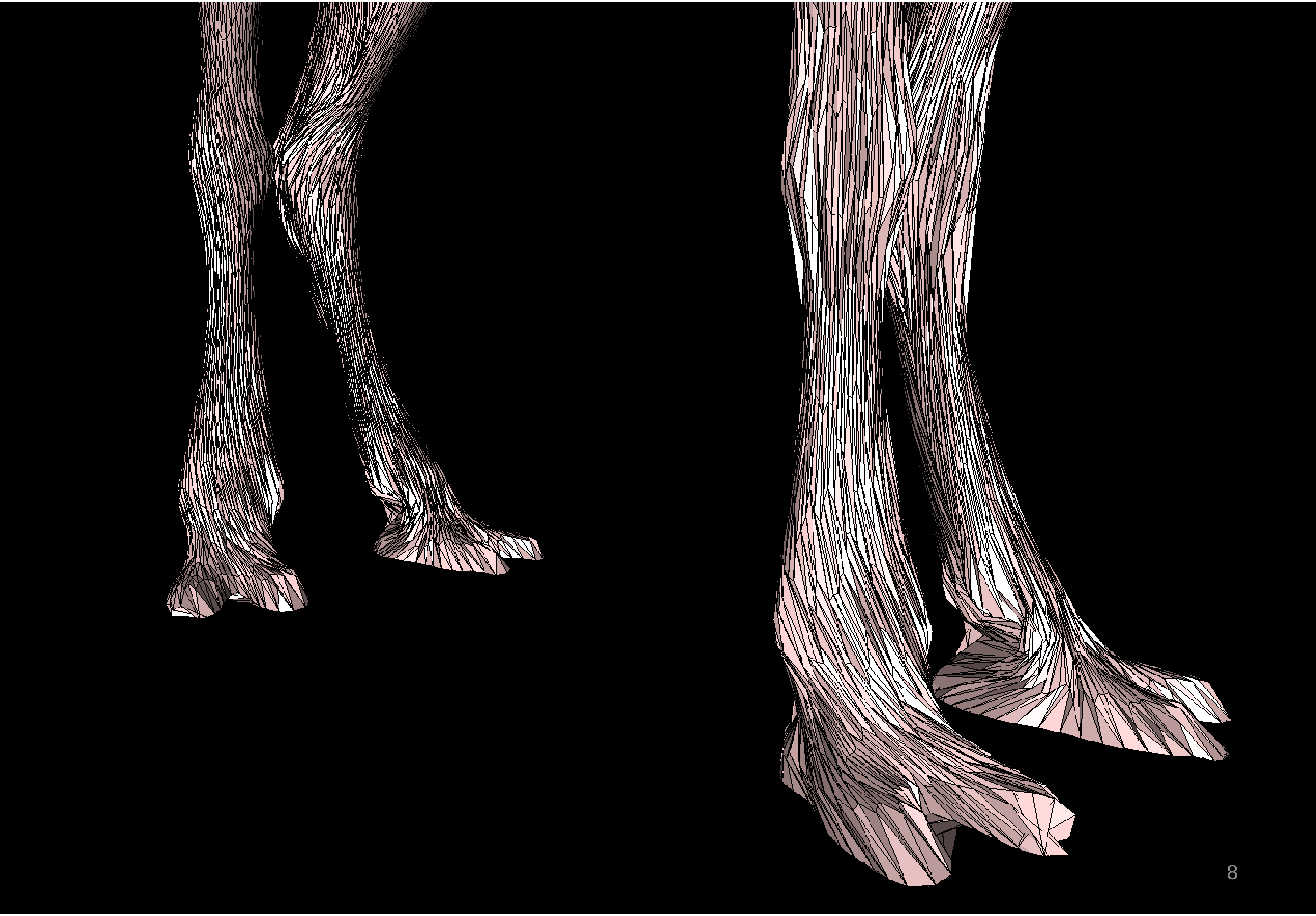









Cut graph



Free Boundary Methods - Tactics

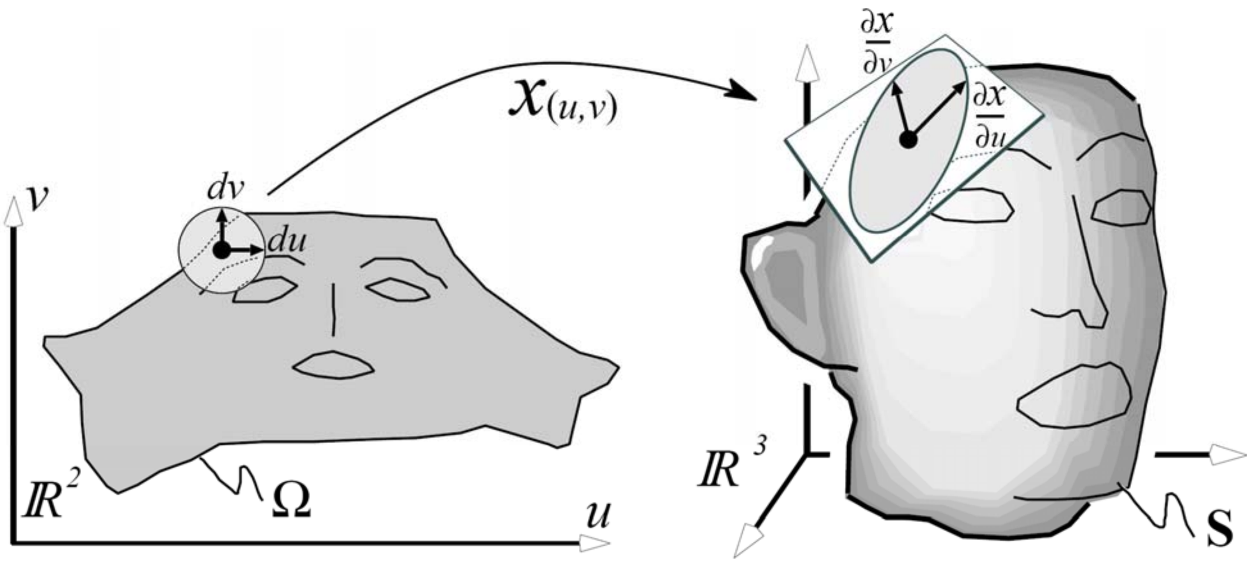
- Solve for the (u,v) coordinates using the discrete first fundamental form
  - MIPS [Hormann et al., 2000]
  - Stretch optimization [Sander et al., 2001]
  - LSCM (conformal, linear) [Levy et al., 2002]
  - DCP (conformal, linear) [Desbrun et al., 2002]

- Solve for the angles of the map (conformal)
  - ABF [Sheffer et al., 2001], ABF++ [Sheffer et al., 2004]
  - LinABF (linear) [Zayer et al., 2007]

Free Boundary Methods - Tactics

- Solve for the edge lengths of the map by prescribing curvature
  - Circle patterns [Kharevych et al., 2006]
  - CPMS (linear) [Ben-Chen et al., 2008]
  - CETM [Springborn et al., 2008]
- Balance area/conformality
  - ARAP [Liu et al., 2008]
- More…

Back to the First Fundamental Form



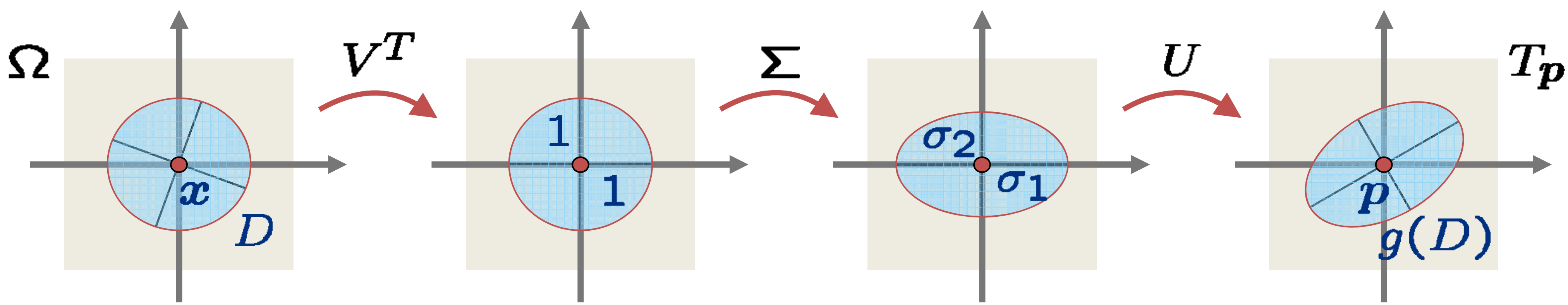
$$\mathbf{X}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} \qquad J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$$

Linear Map Surgery

- **Singular Value Decomposition** (SVD) of J_f

$$J_f = U\,\Sigma\,V^T = U\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\\ 0 & 0\end{pmatrix}V^T$$

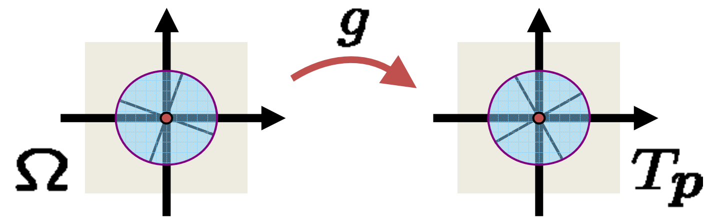
with **rotations** $U \in \mathbb{R}^{3\times 3}$ and $V \in \mathbb{R}^{2\times 2}$
and **scale factors** (singular values) $\sigma_1 \geq \sigma_2 > 0$



Notion of Distortion

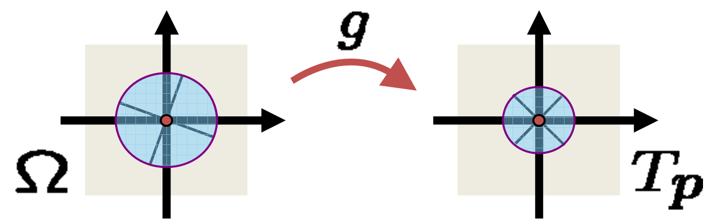
- **isometric** or **length**-preserving

  $$\sigma_1 = \sigma_2 = 1$$



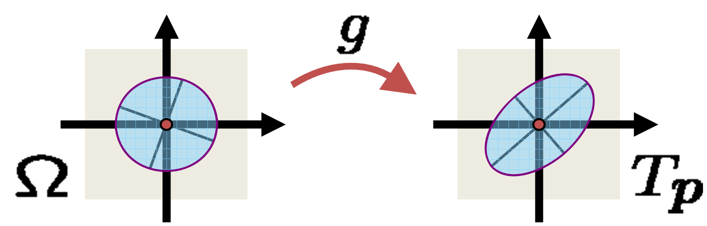
- **conformal** or **angle**-preserving

  $$\sigma_1 = \sigma_2$$



- **equiareal** or **area**-preserving

  $$\sigma_1 \cdot \sigma_2 = 1$$



- everything defined **pointwise** on Ω

Computing the Stretch Factors

- **first fundamental form** $\mathbf{I}_f = {J_f}^T J_f = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$

$$E = {f_u}^T f_u \qquad F = {f_u}^T f_v \qquad G = {f_v}^T f_v$$

- **eigenvalues** of $\mathbf{I}_f$

$$\lambda_{1,2} = \frac{1}{2}\left((E+G) \pm \sqrt{4F^2 + (E-G)^2}\right)$$

- **singular values** of J_f

$$\sigma_1 = \sqrt{\lambda_1} \quad \text{and} \quad \sigma_2 = \sqrt{\lambda_2}$$

Measuring Distortion

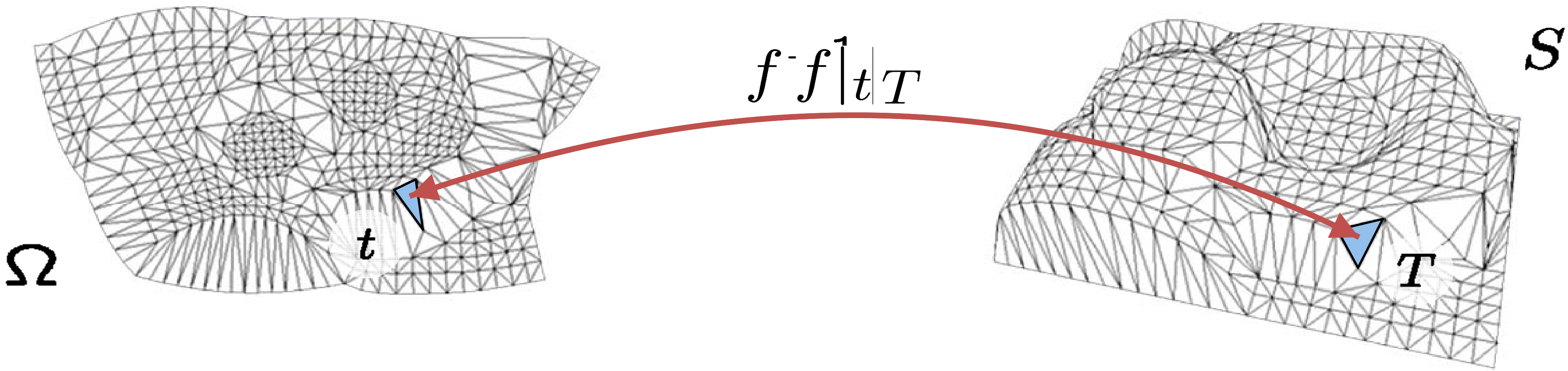
- **local** distortion measure

$$E : (\mathbb{R}_+ \times \mathbb{R}_+) \longrightarrow \mathbb{R}, \qquad (\sigma_1, \sigma_2) \mapsto E(\sigma_1, \sigma_2)$$

- E has **minimum** at
  - $(\sigma_1, \sigma_2) = (1, 1)$ **isometric** measure
  - $(\sigma_1, \sigma_2) = (x, x)$ **conformal** measure
- **overall** distortion

$$E(f) = \int_\Omega E(\sigma_1(u,v), \sigma_2(u,v))\, du\, dv \Big/ \mathrm{Area}(\Omega)$$

Piecewise Linear Parameterizations



- piecewise linear **atomic maps** $f^{-1}|_T : T \rightarrow t$
- distortion **constant** per triangle
- overall distortion $E(f^{-1}) = \sum_{T \in S} E(T)A(T) \Big/ \sum_{T \in S} A(T)$

Distortion Based Methods

- Define energy functional F as a function of J_p, I_p, σ_1,σ_2
- Expand their expression in F in function of the unknown u_i, v_i
- Design an algorithm to find the u_i,v_i's that minimizes F

Linear Methods

- the terms $\sigma_1{}^2 + \sigma_2{}^2$ and $\sigma_1\sigma_2$ are **quadratic** in the parameter points $\boldsymbol{u}_i$
- **Dirichlet** energy

$$E_\mathsf{D} = \tfrac{1}{2}(\sigma_1{}^2 + \sigma_2{}^2) \geq \sigma_1\sigma_2$$

[Pinkall & Polthier 1993]
[Eck et al. 1995]

- **Conformal** energy

$$E_\mathsf{C} = (\sigma_1 - \sigma_2)^2/2$$

[Lévy et al. 2002]
[Desbrun et al. 2002]

- minimization yields **linear** problem

Linear Methods

- both result in **barycentric mappings** with **discrete harmonic** weights for **interior** vertices
- **Dirichlet maps** require to **fix all** boundary vertices
- **Conformal maps** only two
  - result depends on this choice
  - best choice → [Mullen et al. 2008]
- both maps **not** necessarily **bijective**

Non-linear Methods

- **MIPS** energy [Hormann & Greiner 2000]

$$E_{\mathrm{M}} = \kappa_F(J_f) = \|J_f\|_F \|J_f{}^{-1}\|_F = \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}$$

- **Area-preserving** MIPS [Degener et al. 2003]

$$E_\theta = \left(\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}\right)\left(\sigma_1\sigma_2 + \frac{1}{\sigma_1\sigma_2}\right)^\theta$$

Non-linear Methods

- **Green-Lagrange** deformation tensor [Maillot et al. 1993]

$$E_{\mathrm{G}} = \|\mathbf{I}_f - \mathrm{Id}\|_F^2 = ({\sigma_1}^2 - 1)^2 + ({\sigma_2}^2 - 1)^2$$

- **Stretch** energies (L^2, L^∞, and symmetric stretch)

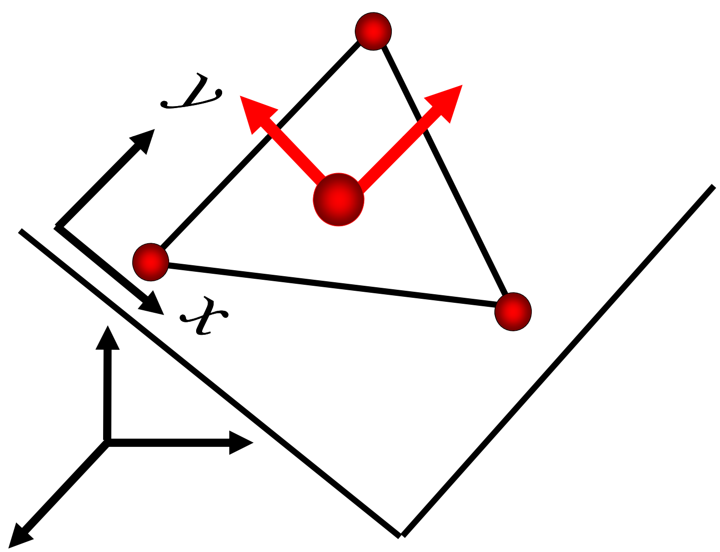
$$E_2 = \frac{1}{\sqrt{2}} \|J_f\|_F = \sqrt{({\sigma_1}^2 + {\sigma_2}^2)/2} = \sqrt{E_{\mathrm{D}}}$$

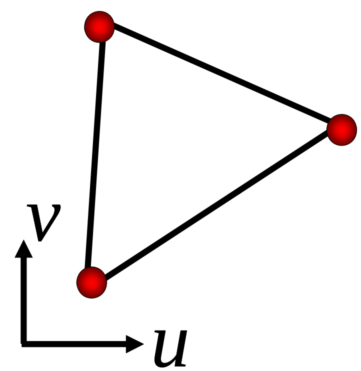
$$E_\infty = \|J_f\|_2 = \sigma_1$$

$$E_{\mathrm{S}} = \max(\sigma_1, \frac{1}{\sigma_2})$$

[Sander et al. 2001]
[Sorkine et al. 2002]

Conformal Parameterization - LSCM





Cauchy Riemann equations:

$$\begin{cases} \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} \end{cases}$$

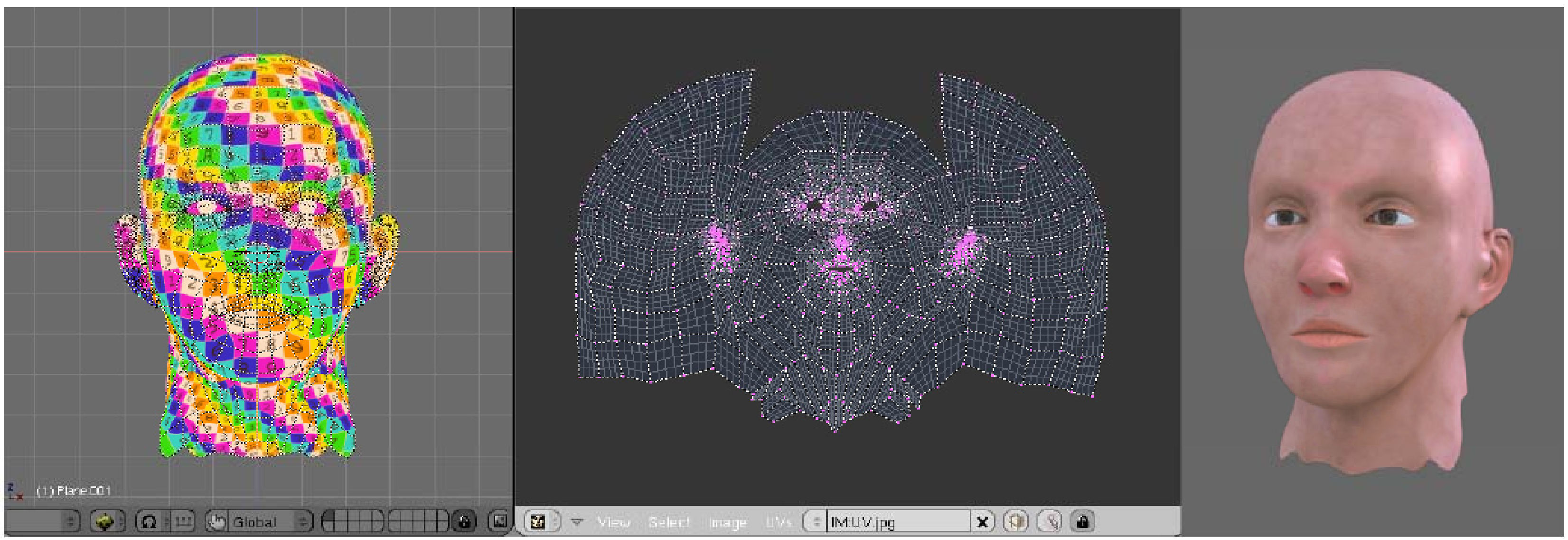
No Piecewise Linear solution in general

Conformal Parameterization – LSCM

[Levy et al., 2002]

Minimize $$\sum_T \left\| \begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix} - \begin{pmatrix} -\frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial x} \end{pmatrix} \right\|^2$$

Fix two vertices to determine rot,transl,scaling

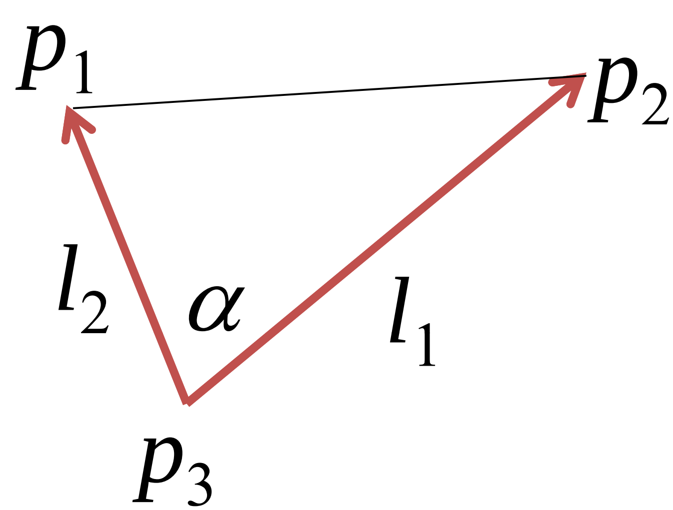


Conformal Parameterization - LSCM

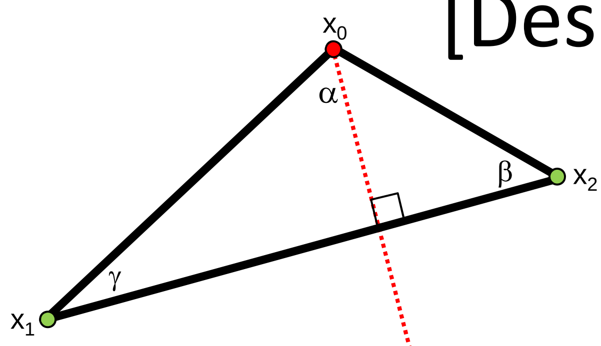
Equivalent formulation, uses only **edge lengths ratios and angles**

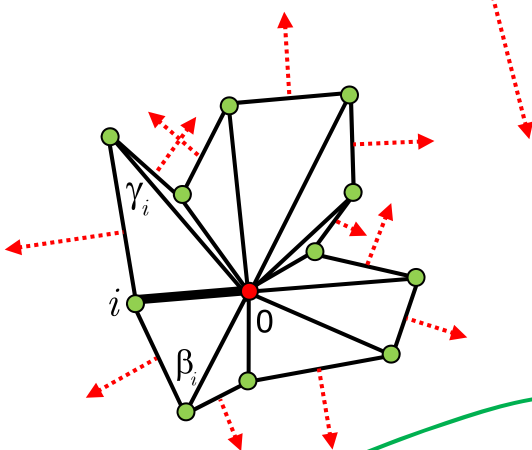
For each triangle:

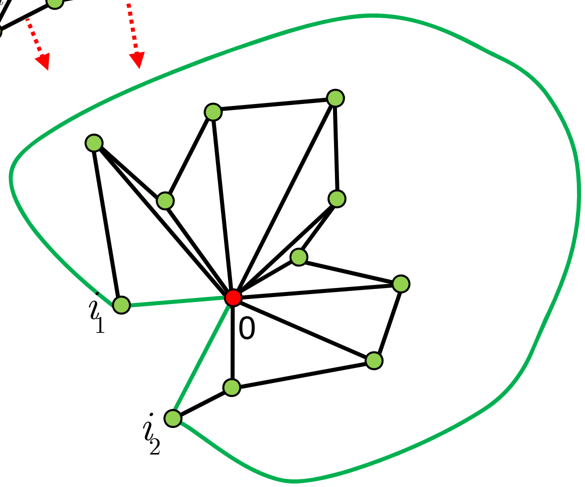
$$(p_1 - p_3) = R^{\alpha}(p_2 - p_3)\frac{l_2}{l_1}$$



Conformal Parameterization – DCP
[Desbrun et al., 2002]







For one triangle:

$$\cot\gamma\,(x_2 - x_0) + \cot\beta\,(x_1 - x_0) = R^{90}(x_1 - x_2)$$

For complete one-ring of triangles (interior vertex):

$$\sum_{i=1}^{n}(\cot\gamma_i + \cot\beta_i)(x_i - x_0) = 0$$

For incomplete one-ring of triangles (boundary vertex):

$$\sum_{i=1}^{n}(\cot\gamma_i + \cot\beta_i)(x_i - x_0) = R^{90}(x_{i_1} - x_{i_2})$$

Isotropic Parameterizations

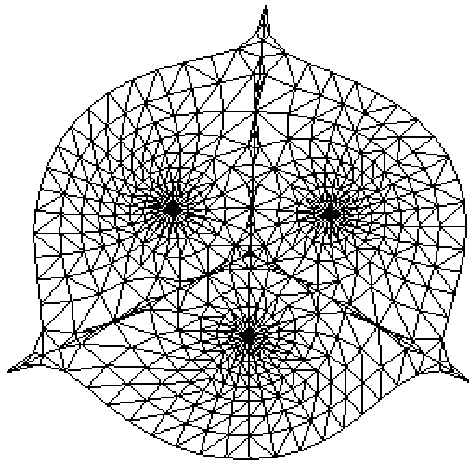
Conformal = Harmonic

$$\underbrace{\int_S \det(\mathbf{J})ds}_{\text{area of the surface}} = \underbrace{\frac{1}{2}\int_S \|f_u\|^2 + \|f_v\|^2 ds}_{\text{Dirichlet's energy}} - \underbrace{\frac{1}{2}\int_S \|f_v - \text{rot}_{90}(f_u X)\|^2}_{\text{conformal energy}}$$

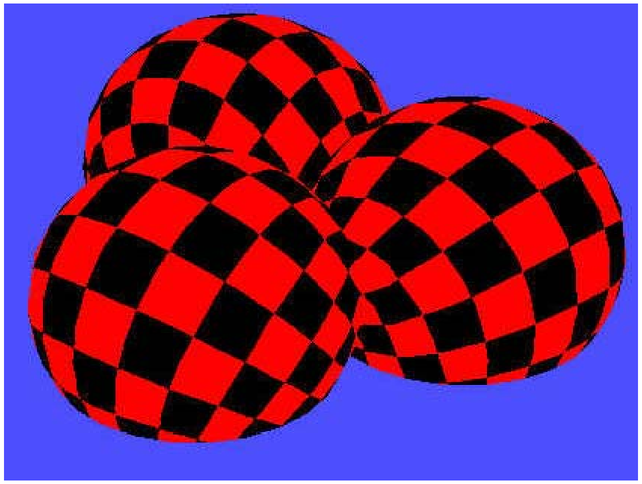
$$\underbrace{\frac{\partial X}{\partial u}\frac{\partial Y}{\partial v} - \frac{\partial X}{\partial v}\frac{\partial Y}{\partial u}}_{\det(\mathbf{J})} = \underbrace{\frac{1}{2}\left(\left\|\begin{pmatrix}\frac{\partial X}{\partial u}\\ \frac{\partial Y}{\partial u}\end{pmatrix}\right\|^2 + \left\|\begin{pmatrix}\frac{\partial X}{\partial v}\\ \frac{\partial Y}{\partial v}\end{pmatrix}\right\|^2\right)}_{\|f_u\|^2+\|f_v\|^2} - \underbrace{\frac{1}{2}\left(\left\|\begin{pmatrix}\frac{\partial X}{\partial v} + \frac{\partial Y}{\partial u}\\ \frac{\partial Y}{\partial v} - \frac{\partial X}{\partial u}\end{pmatrix}\right\|^2\right)}_{\|f_v - \text{rot}_{90}(f_u)\|^2}$$

Angle Based Flattening (ABF)

- Fact: Triangular 2D mesh is defined by its angles (up to similarity)



- Define problem in angle space



- Angle based formulation:
  - Distortion as function of angles
  - Validity - set of angle constraints

Constrained Minimization

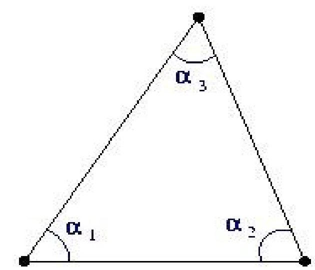
- Notation: β_i are (given) 3D angles. α_i are (unknown) 2D angles.

- Objective: minimize (relative) deviation of angles:

$$\boxed{D(\alpha_i) = \sum_{i=1}^{3T} (\alpha_i - \beta_i)^2}$$

Constraints

- All angles are positive (linear inequalities).

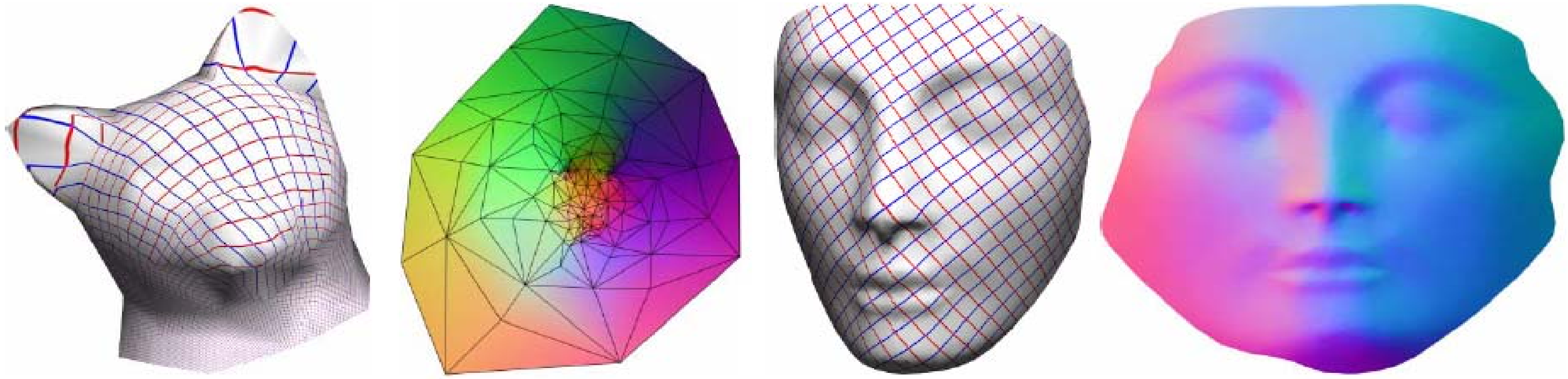


- Sum of angles in each triangle is π (linear equalities).
- Sum of angles around each vertex is 2π (linear equalities).
- All one-rings close properly (non-linear equalities).

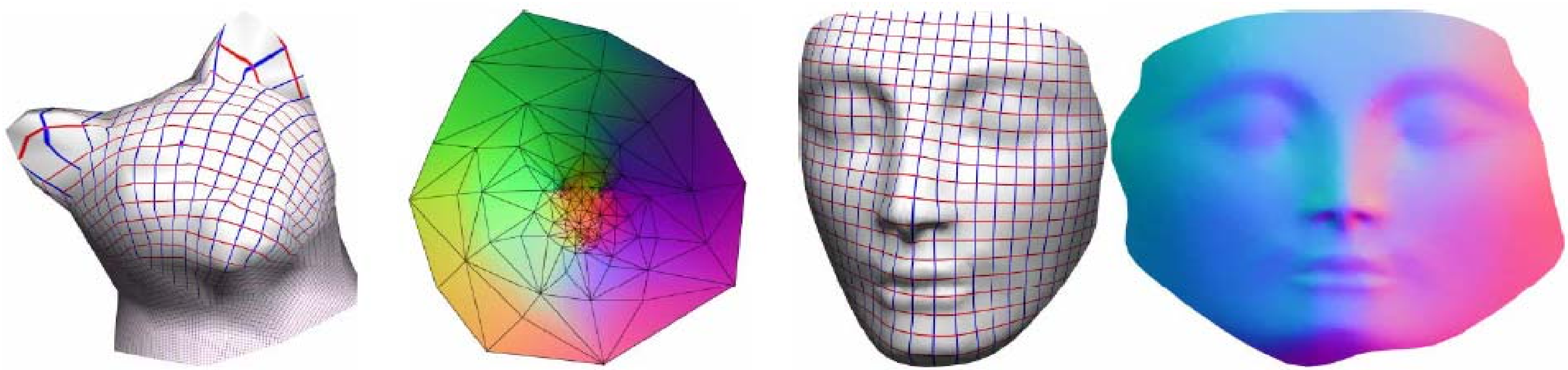
Solving

- Use non-linear solver (Lagrange multipliers, Newton method) to solve for α_i
- Use LSCM to embed in plane based on α_i

Examples

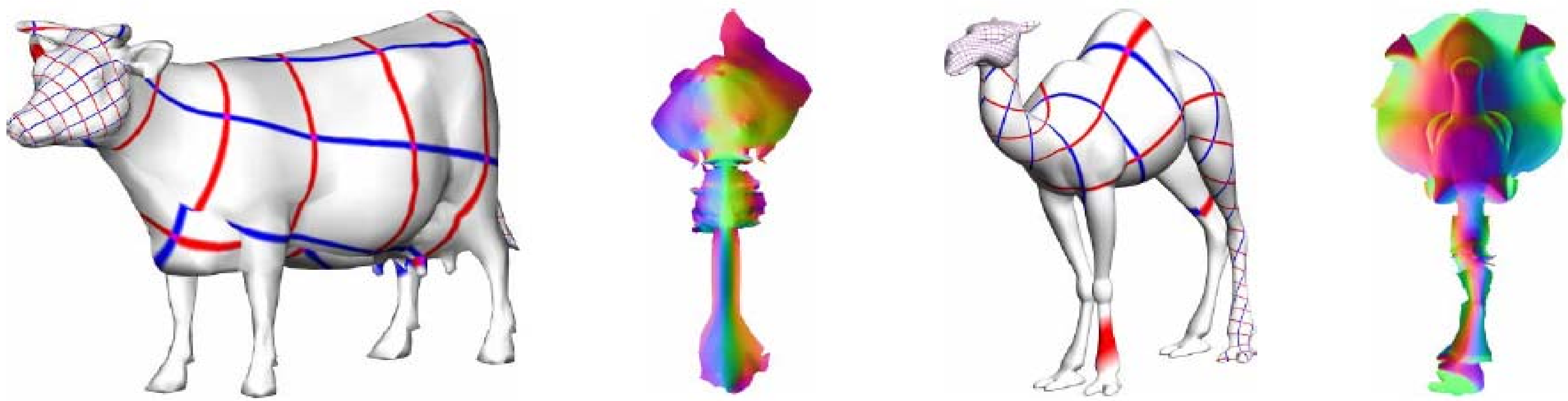


LSCM

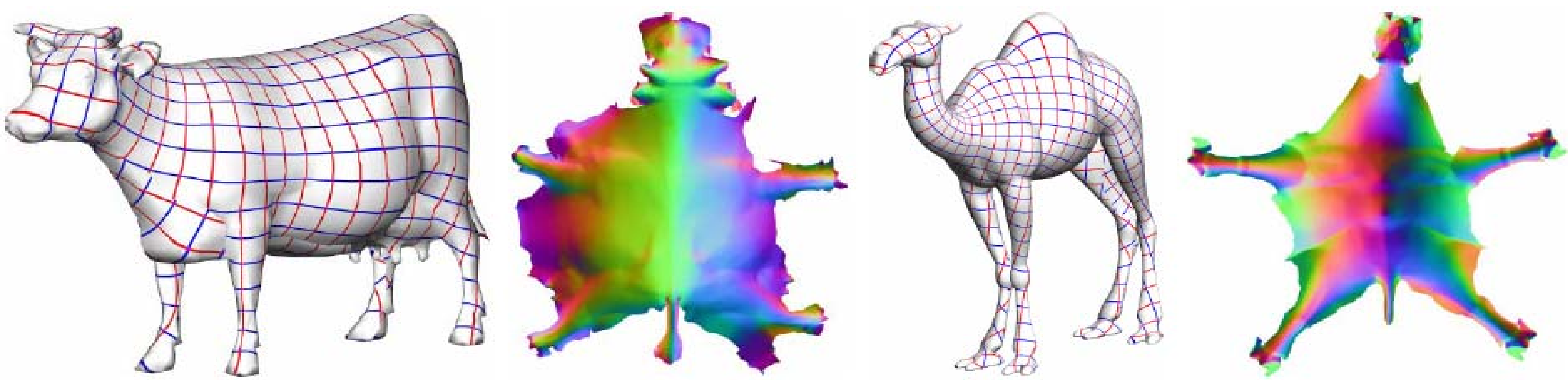


ABF

Examples

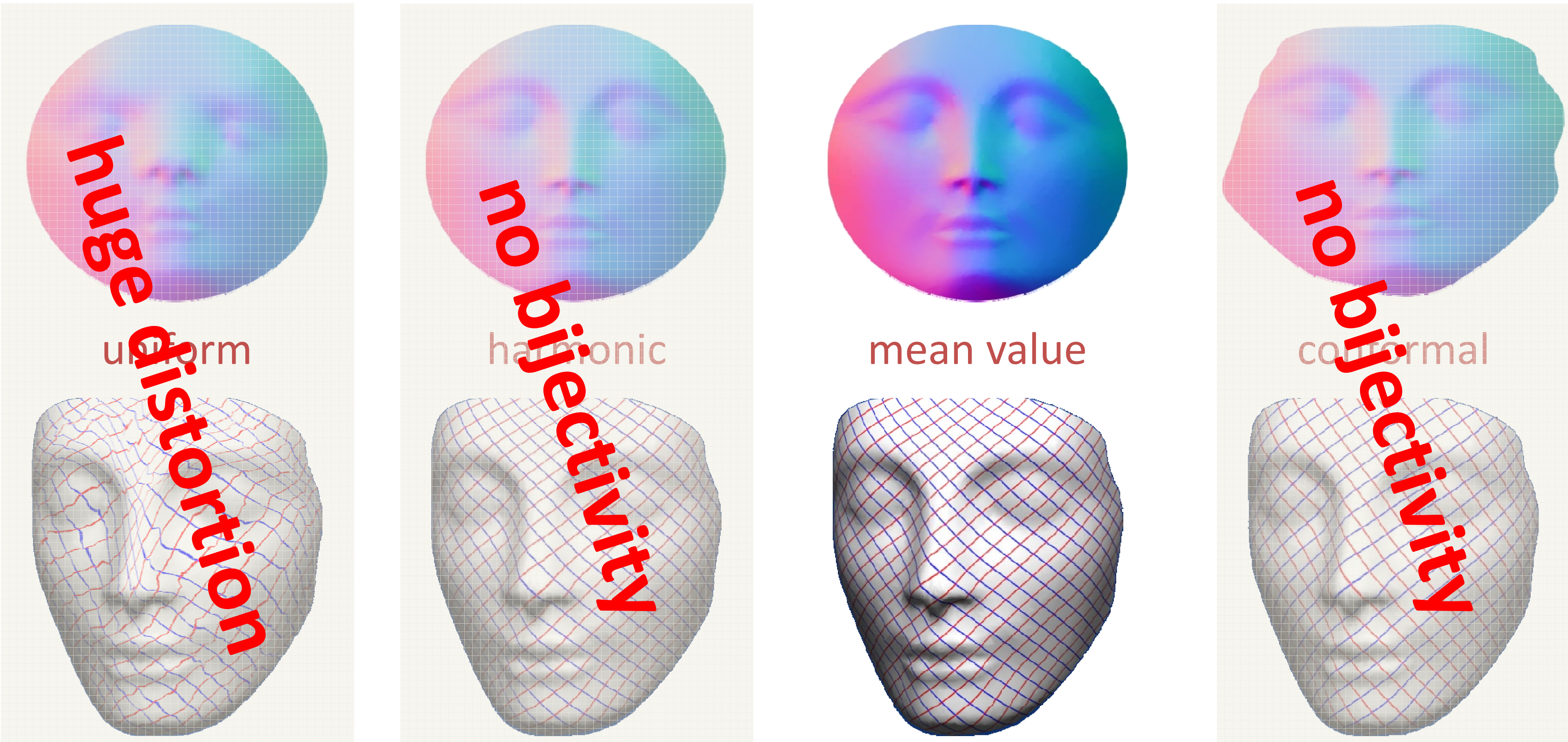


LSCM

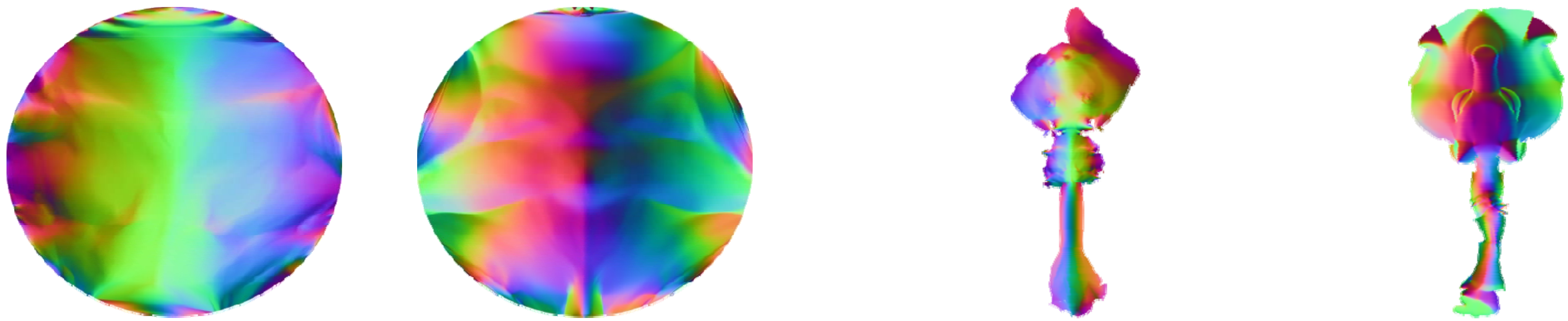


ABF

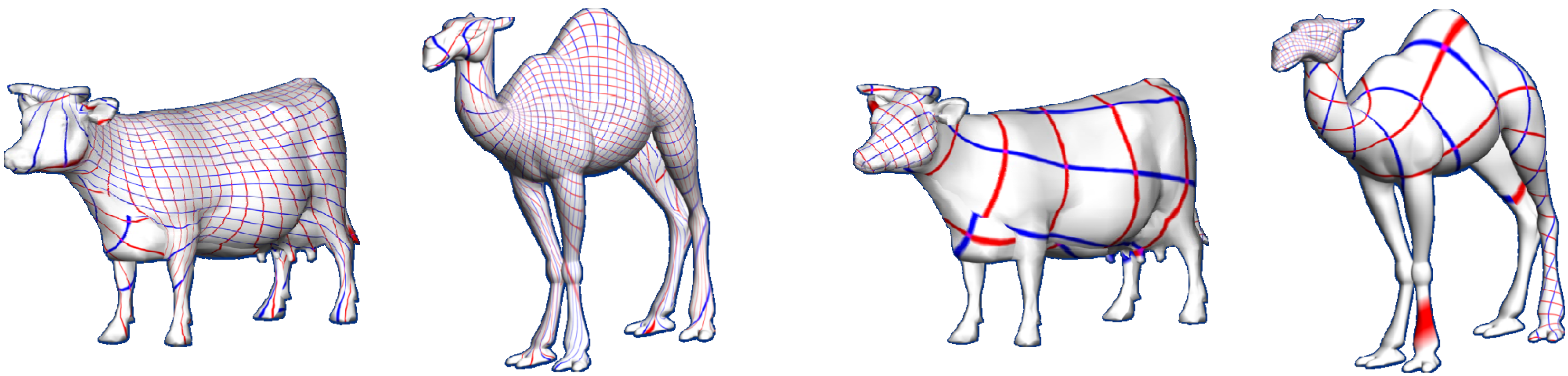
Linear Methods



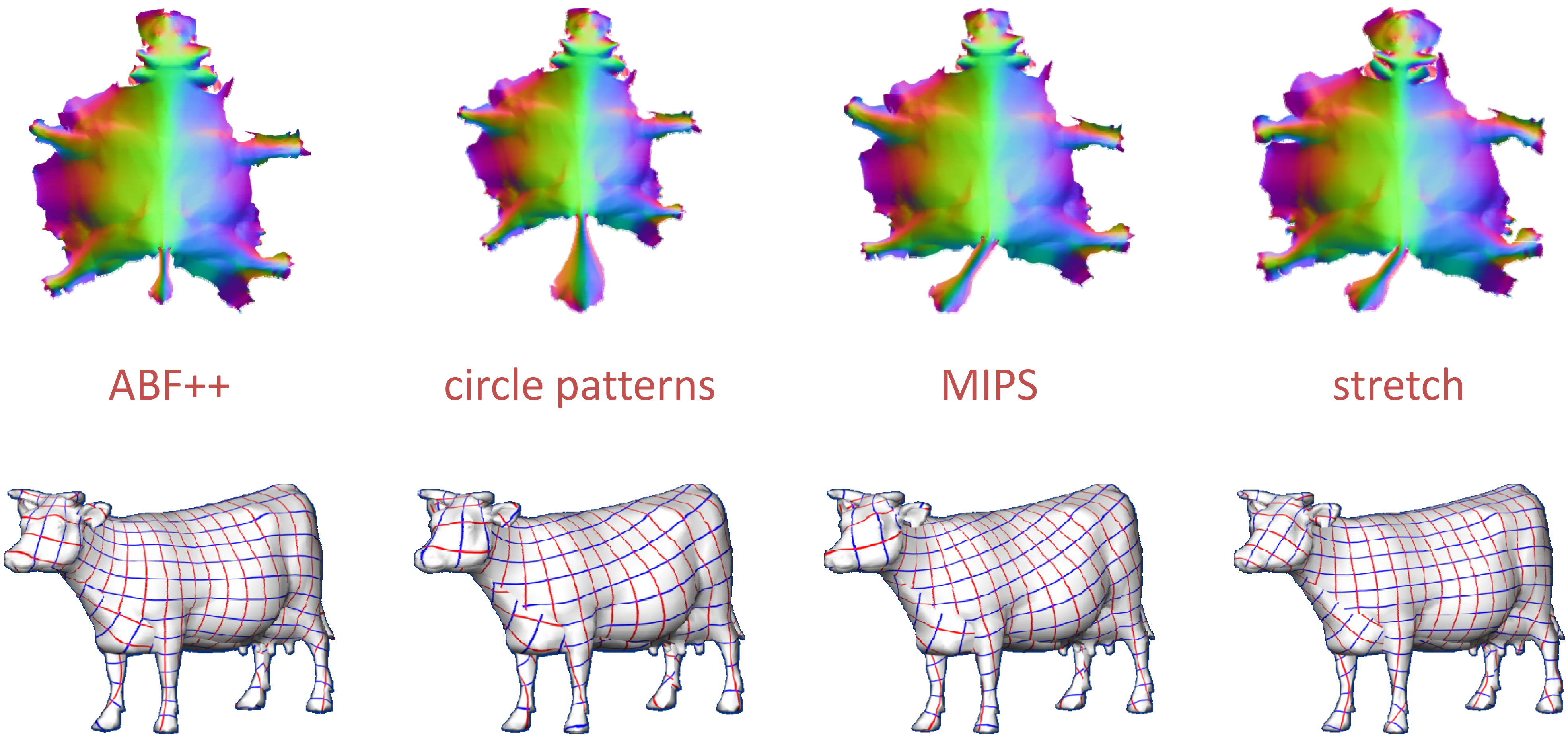
Linear Methods



mean value conformal

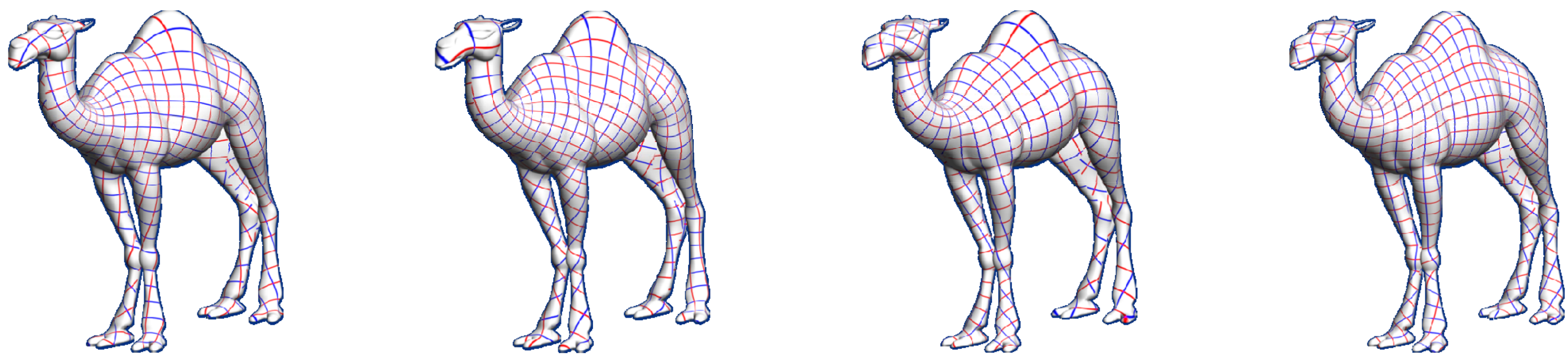


Non-Linear Methods



Non-Linear Methods

ABF++ circle patterns MIPS stretch



Curvature Prescription

LSCM [2002]

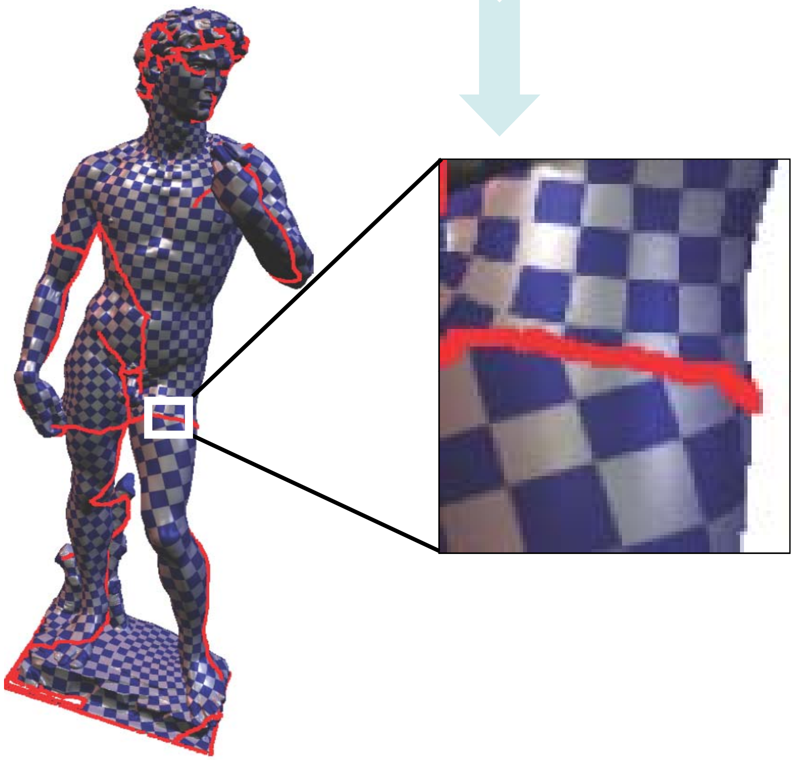
ABF++ [2005]

LinABF [2007]

(and many more...)

1. Cut mesh to disk
2. **Compute new angles/lengths**
3. Embed in plane

→ Discontinuities in scale



Circle Patterns [2006]

Ricci Flow [2007]

CPMS, CETM [2008]

Curvature Prescription

LSCM [2002]

ABF++ [2005]

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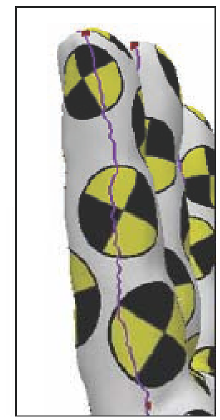
(and many more...)



Circle Patterns [2006]

Ricci Flow [2007]

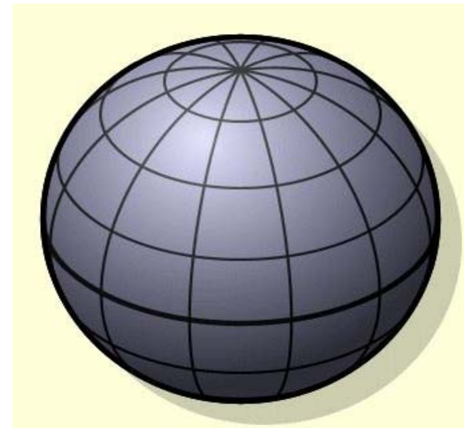
CPMS, CETM [2008]

1. Compute new angles/lengths
2. Cut Mesh to disk
3. Embed in plane

→ No discontinuities in scale

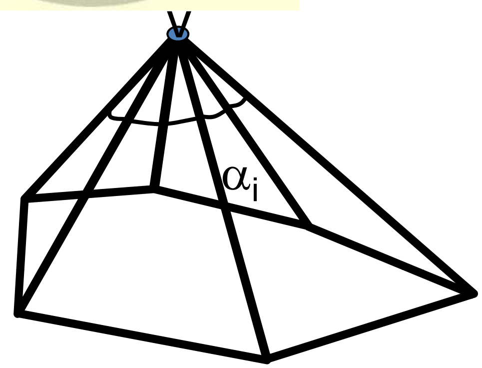
Gaussian Curvature

Continuous definition
Scale dependent



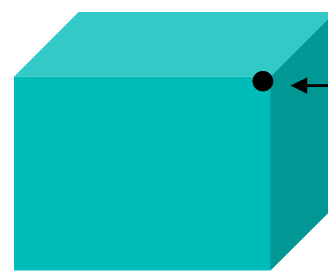
$\kappa \equiv 1/r^2$

Gauss-Bonnet theorem: $\int \kappa = 2\pi\chi$



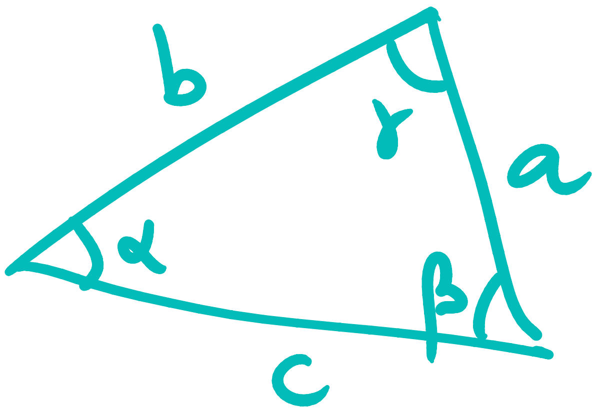
Discrete definition: $\kappa(v) = 2\pi - \Sigma\alpha_i$
Scale independent !

Gauss-Bonnet theorem: $\sum \kappa = 2\pi\chi$



$\kappa = \pi/2$

Discrete Metric



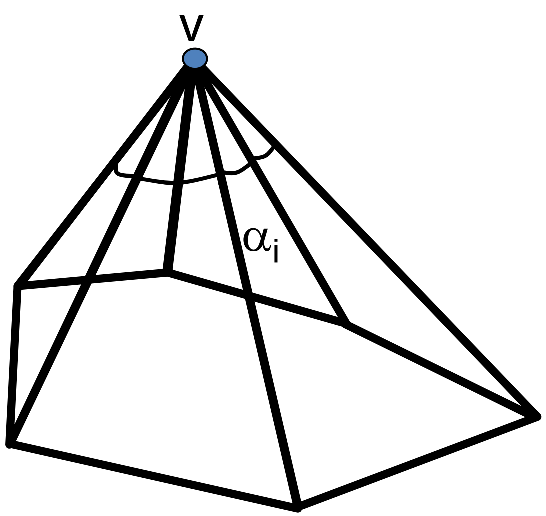
Triangle edge lengths (a,b,c)
are the discrete metric

The metric defines the angles by the cosine law

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

The angles define the Gaussian curvature

$$k = 2\pi - \sum \alpha_i$$



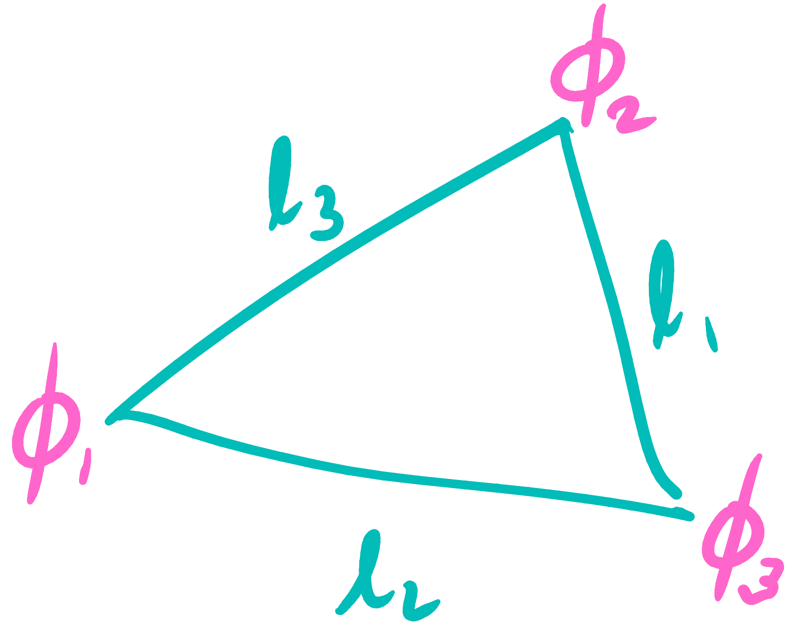
CPMS [2008]

Original edge lengths (metric)

Conformal factor (per vertex)

Extend ϕ to edge

$$\phi(t) = \phi_2 t + \phi_3 (1 - t)$$



Integrate e^{ϕ} on edge

Scale to get new edge lengths

$$\tilde{l}_1 = l_1 \int e^{\phi(t)}$$

What Happens to the Curvature?

- Continuous conformal map

$$e^{2\phi}\tilde{\kappa} = \kappa - \nabla^2 \phi$$

Curvature after map

Curvature before map

Conformal factor

What Happens to the Curvature?

- Continuous conformal map

$$e^{2\phi}\tilde{\kappa} = \kappa - \nabla^2\phi$$

- Discrete conformal map (for small changes)

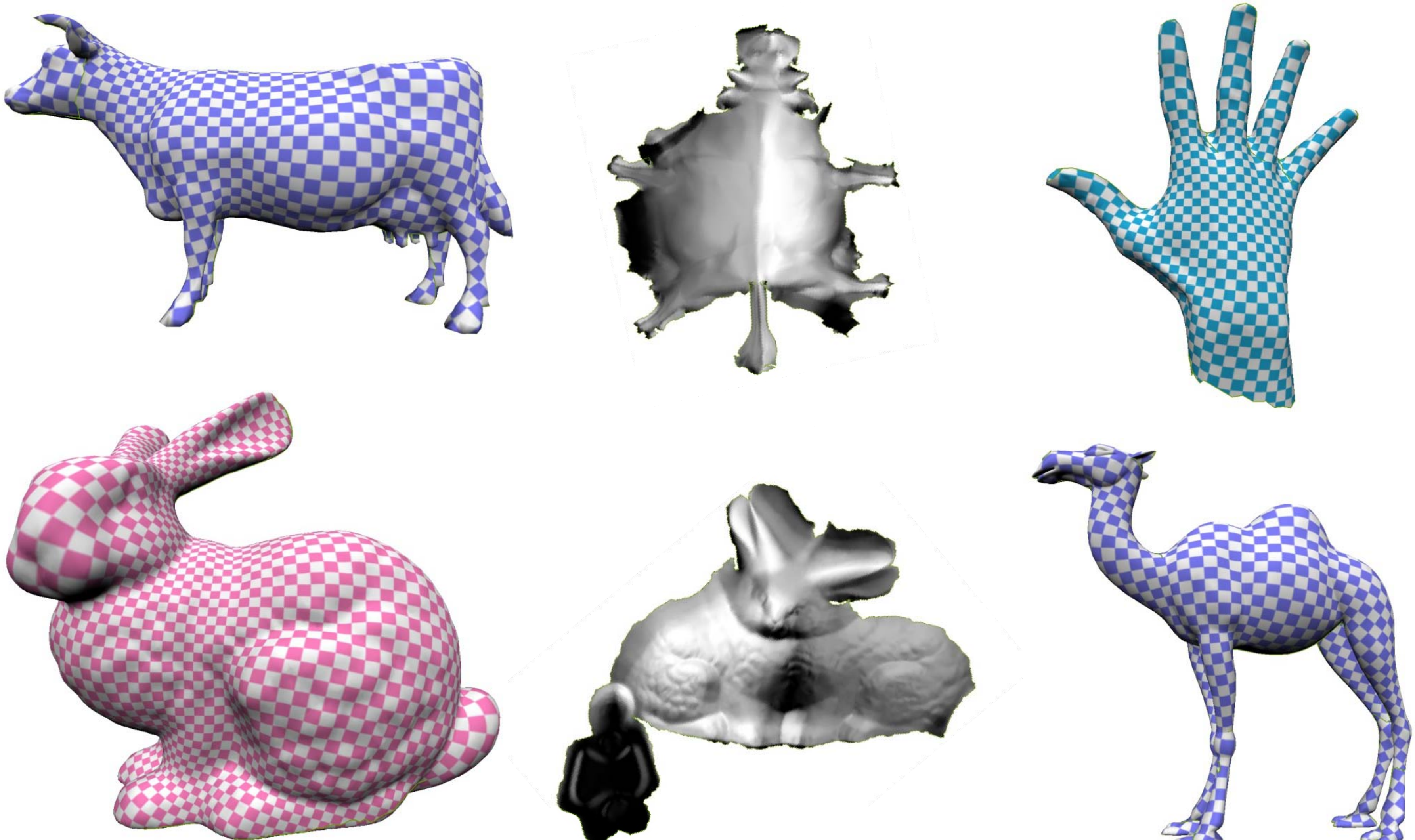
$$\tilde{\kappa} \approx \kappa - \nabla^2\phi$$

- Where did the exponent go?
  - Discrete curvature doesn't scale!

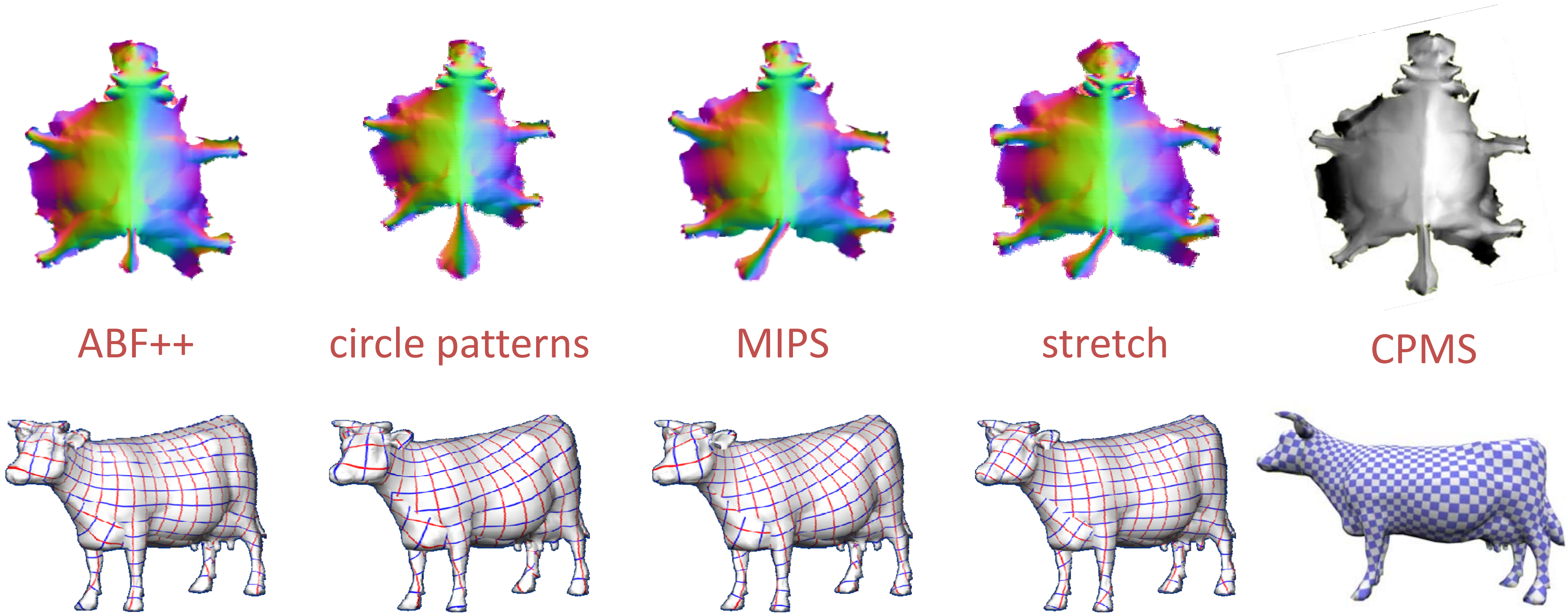
The Flattening Algorithm for a Topological Disk

- Set target curvature to 0 on interior vertices
- Set ϕ to 0 on the boundary
- Find ϕ on interior vertices by solving $\nabla^2 \phi = \tilde{\kappa} - \kappa$
- Scale edge lengths to get conformal metric
- Embed new edge lengths using LSCM

Some Results

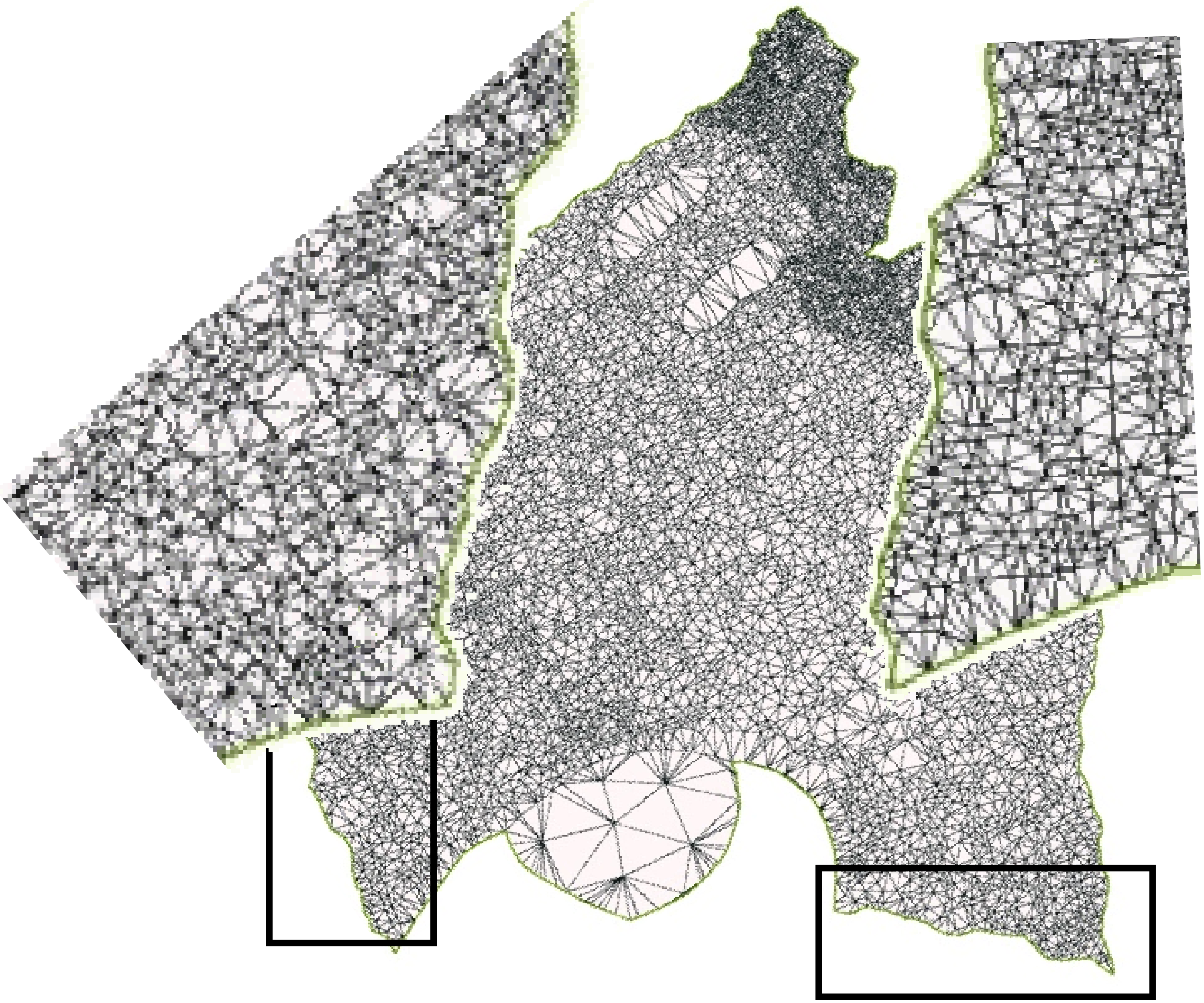


Comparison with Non-Linear Methods



Reducing Area Distortion

- Gather Gaussian curvature into “cone points”
  - Target curvature not 0 everywhere
- First compute new edge lengths, then cut through cone points
- No discontinuities in scale



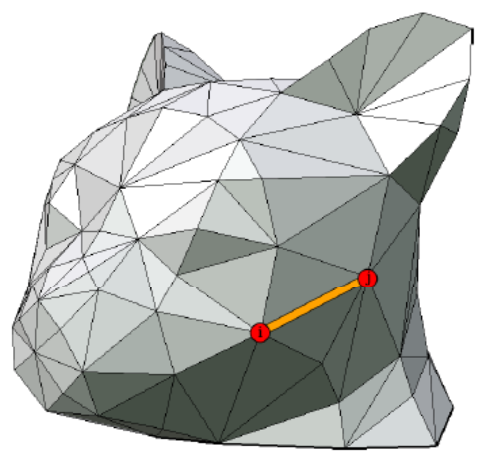
Conformal Equivalence of Triangle Meshes [2008]

Definition [Luo 2004]

Two discrete metrics ℓ, $\tilde{\ell}$ on M are *(discretely) conformally equivalent* if

$$\tilde{\ell}_{ij} = e^{\frac{1}{2}(u_i+u_j)}\ell_{ij}$$

for some function $u : V \to \mathbb{R}$



- use $\lambda_{ij} = 2\log \ell_{ij}$

  so $\ell_{ij} = e^{\lambda_{ij}/2}$

  and $\tilde{\lambda}_{ij} = \lambda_{ij} + u_i + u_j$

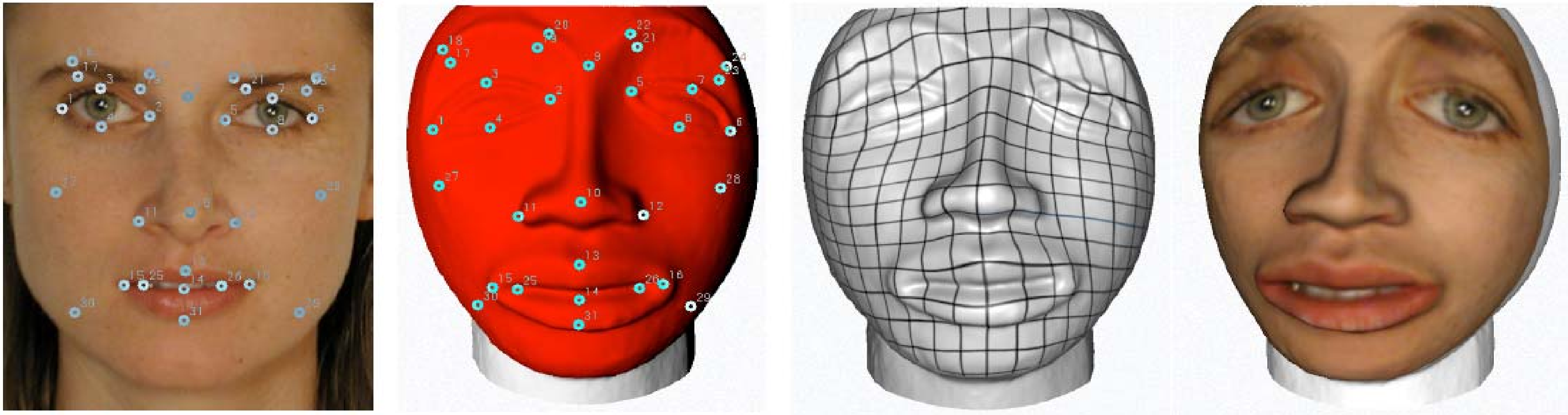
Conformal Equivalence of Triangle Meshes [2008]

- Given input mesh and target curvatures, find u s.t.
  - New mesh is conformally equivalent to source mesh
  - Target curvatures are achieved
- Minimize convex energy → global minimum
- First optimization step equivalent to CPMS

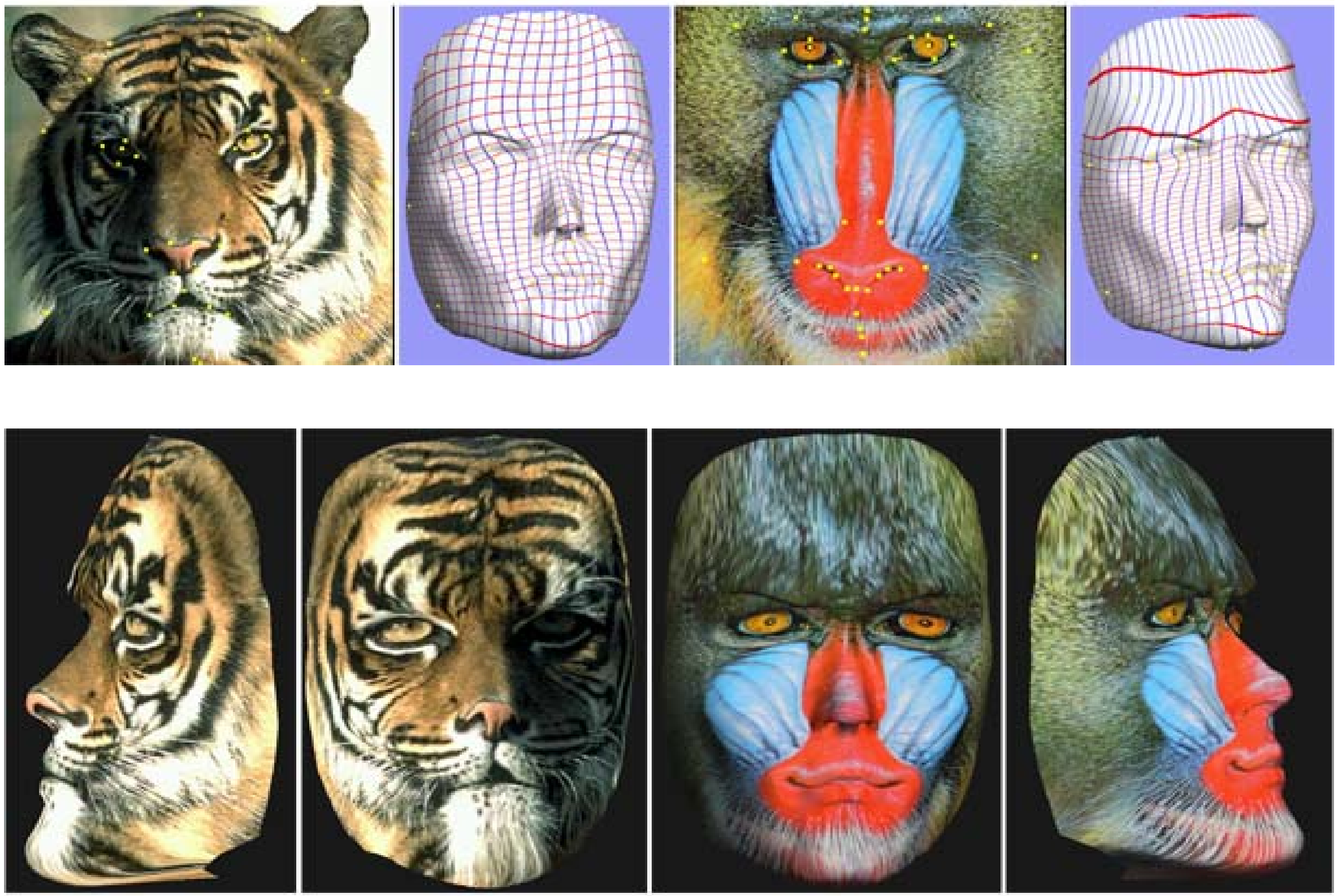
Conformal Equivalence of Triangle Meshes [2008]

Constrained Parameterizations

- Construct well-behaved mapping while satisfying constraints



Constrained Parameterizations



Constrained Parameterizations

- Objective function

points in domain

constrained positions

$$C(f) = \sum_{\substack{j \in M \\ \text{set of constraints}}} \underbrace{(f(\mathbf{u}_j) - \mathbf{v}_j)^2}_{\text{fitting constraints}} + \epsilon \underbrace{\int_S \gamma(\mathbf{u}) d\mathbf{u}}_{\text{distortion}}$$

Parameterization - Conclusions

- Many MANY methods out there
  - Fixed / free boundary
  - Bijective / non bijective
  - Conformal / area preserving
  - Linear / non linear
  - First cut then embed / cone points

- Best method depends on mesh
  - Close to disk → linear methods can work well
  - Require large distortion → prefer non-linear methods