Spectral Algorithms II

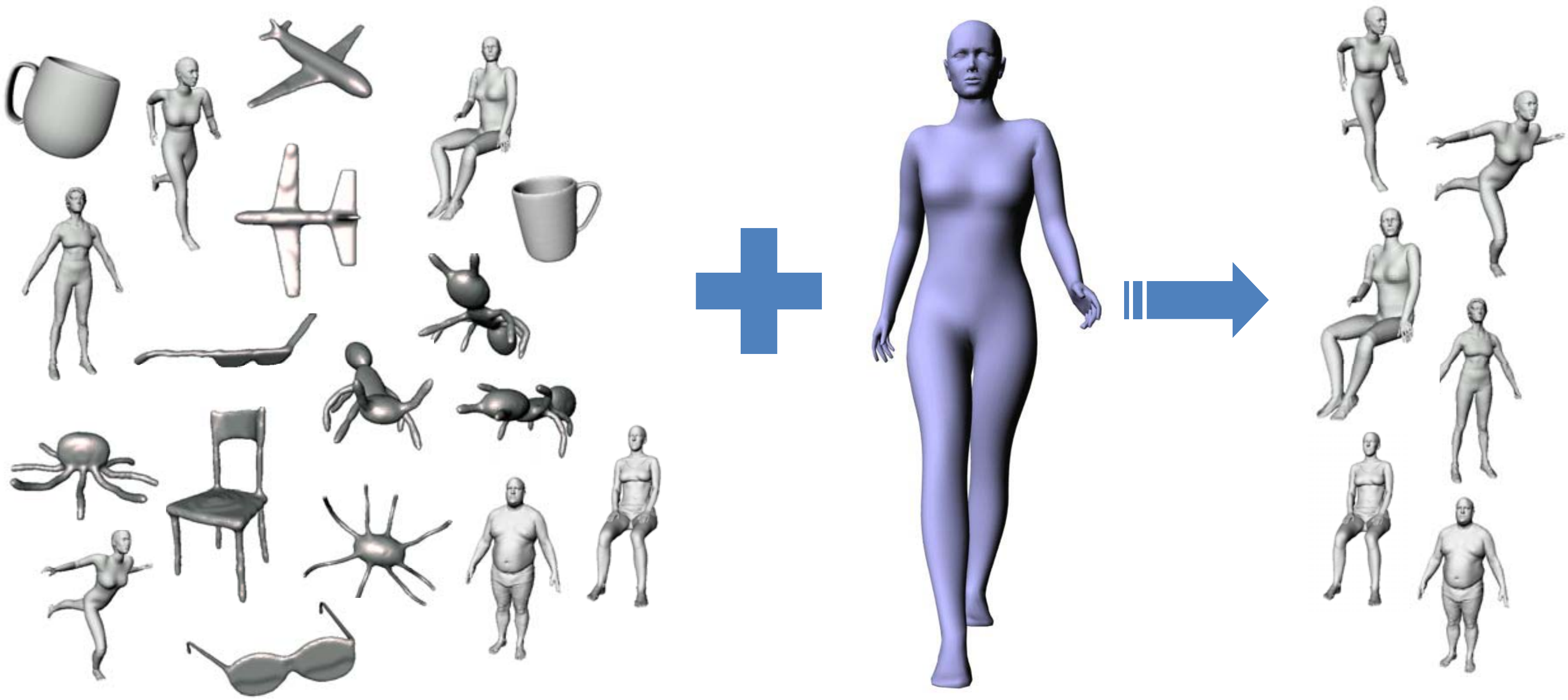
Applications

Slides based on "*Spectral Mesh Processing*" Siggraph 2010 course

Applications

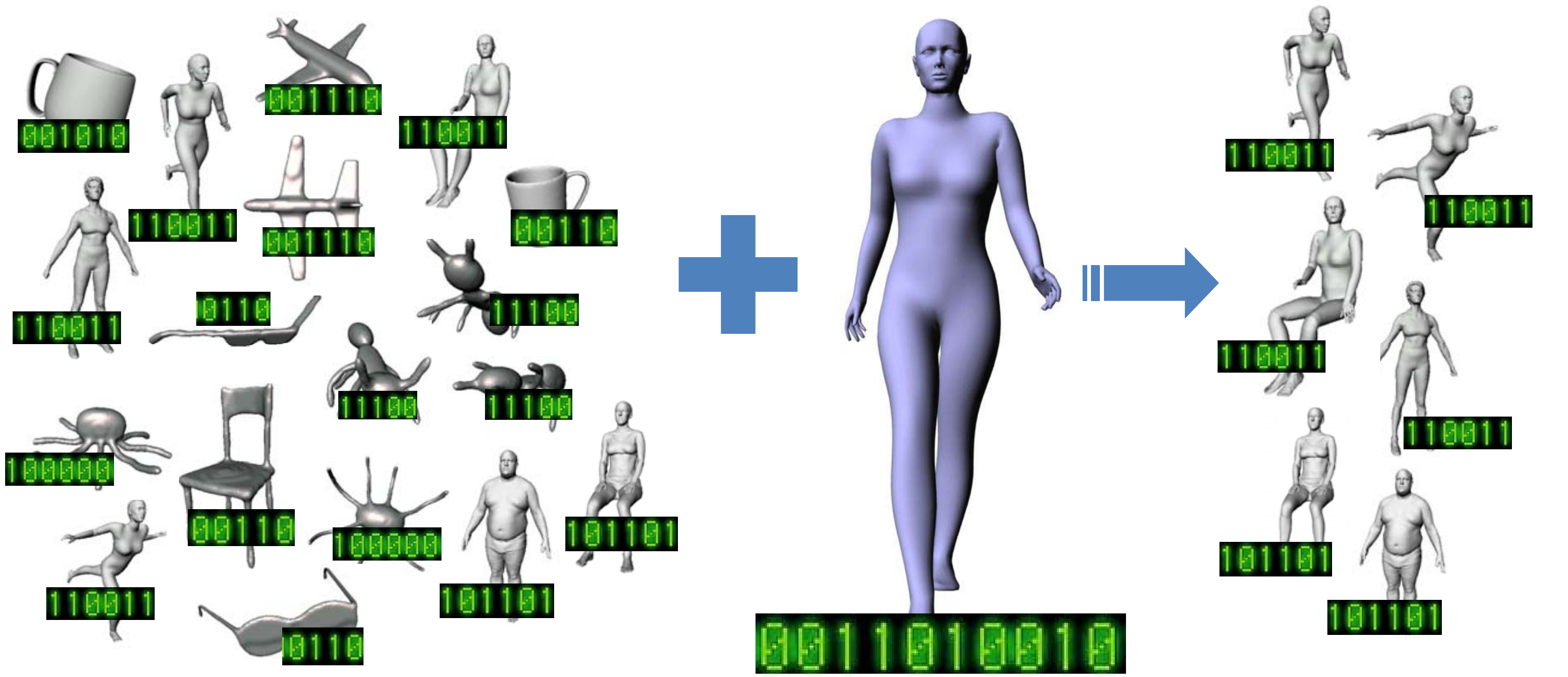
- Shape retrieval
- Parameterization
  - 1D
  - 2D
- Quad meshing

Shape Retrieval



3D Repository　　Query　　Matches

Descriptor based shape retrieval



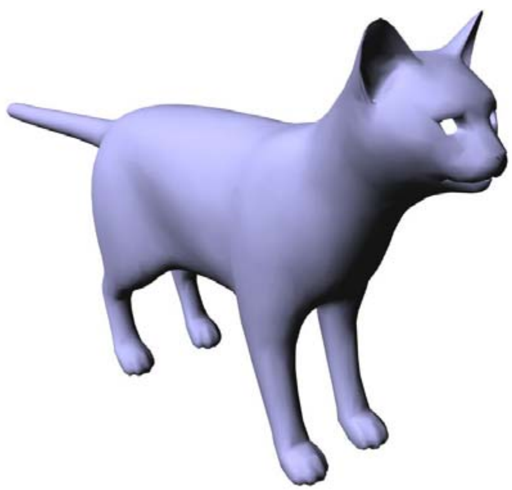
3D Repository descriptors

Query descriptor

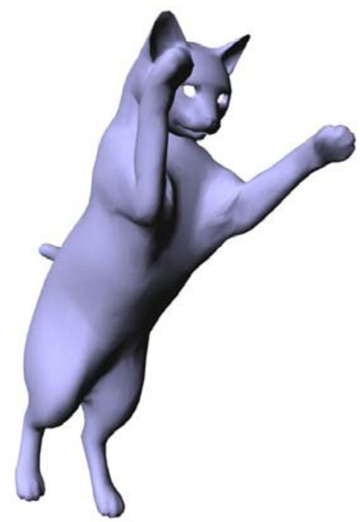
Closest matches

Pose Invariant Shape Descriptor

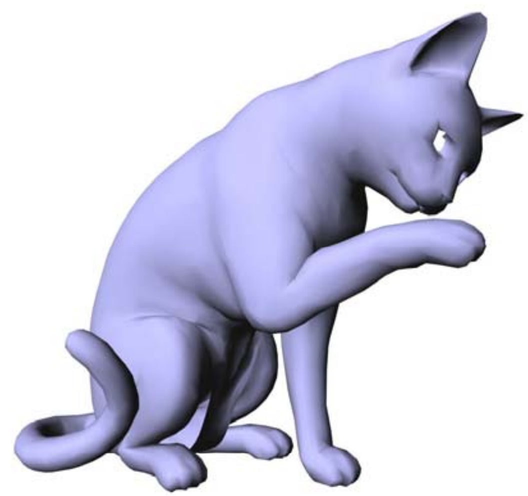
"Similar" descriptors for shape in different poses



Cat



Same cat



Still the same cat

Spectral Shape Descriptors

- Use pose invariant operators
  - Matrix of geodesic distances
  - Laplace-Beltrami operator
  - Heat kernel

- Derive descriptors from eigen-structure
  - Eigenvalues
  - Distance based descriptors on spectral embedding
  - Heat kernel signature

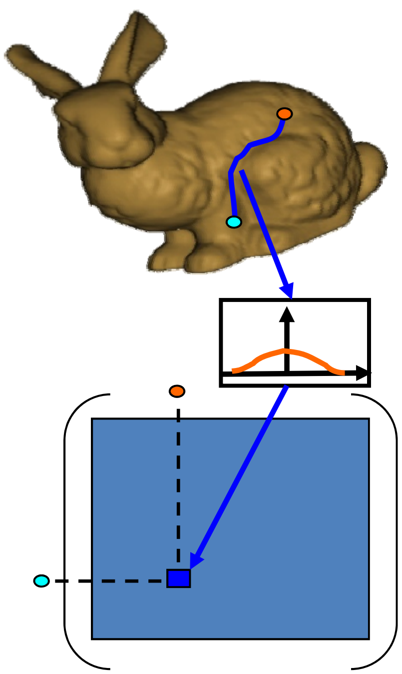
Geodesic Distances Matrix

- Operator: Matrix of Gaussian-filtered pair-wise geodesic distances

$$A_{ij} = e^{-\frac{\left\| p_i - p_j \right\|^2}{2\sigma^2}}$$

- Only take k << n samples

- Descriptor: eigenvalues of matrix



[Jian and Zhang 06]

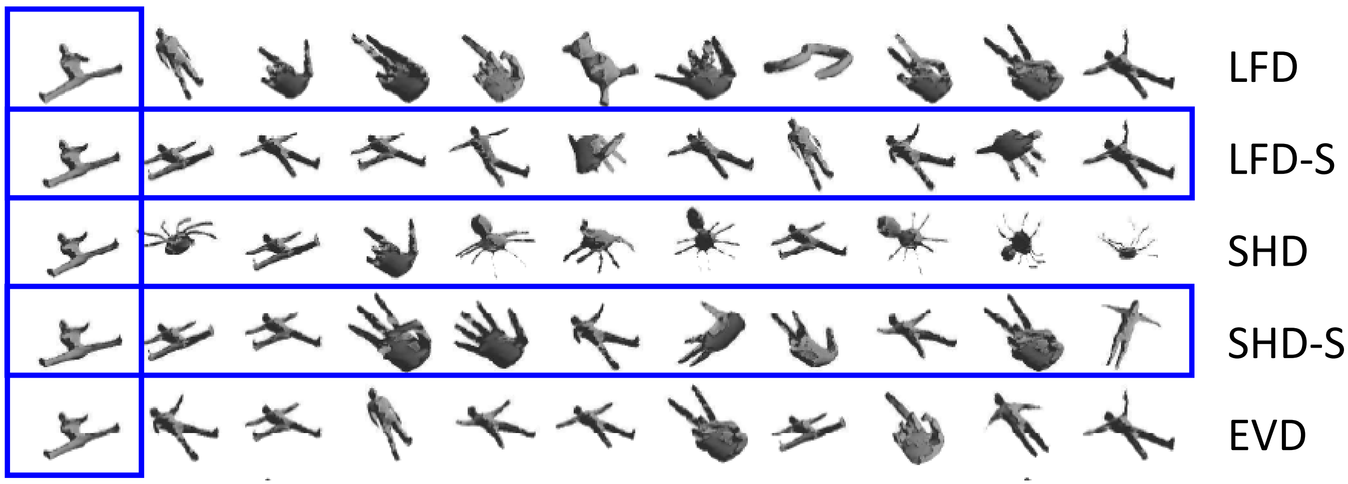
Geodesic Distances Matrix

LFD: Light-field descriptor [Chen et al. 03]

SHD: Spherical Harmonics descriptor [Kazhdan et al. 03]

LFD-S: LFD on spectral embedding

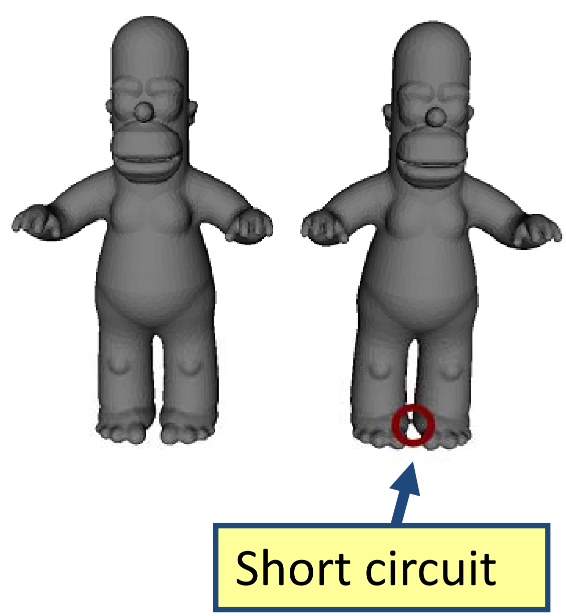
SHD-S: SHD on spectral embedding



Retrieval on McGill Articulated Shape Database

Limitations

- Geodesic distances sensitive to “shortcuts” = small topological holes



Global Point Signatures [Rustamov 07]

Given a point **p** on the surface, define

$$GPS(\mathbf{p}) = \left(\frac{1}{\sqrt{\lambda_1}}\phi_1(\mathbf{p}), \frac{1}{\sqrt{\lambda_2}}\phi_2(\mathbf{p}), \frac{1}{\sqrt{\lambda_3}}\phi_3(\mathbf{p}), \ldots \right)$$

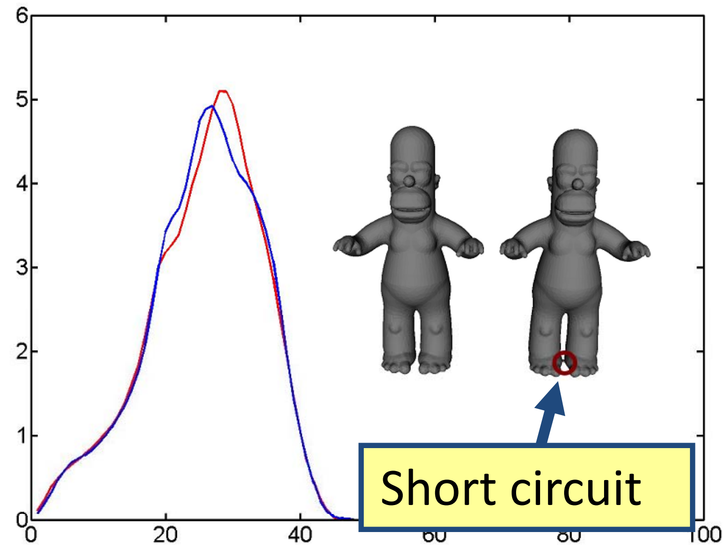
– ϕ_i(p) value of the eigenfunction ϕ_i at the point **p**

– λ_i's are the Laplace-Beltrami eigenvalues

– Euclidean distance in GPS space =
  commute time distance on the surface

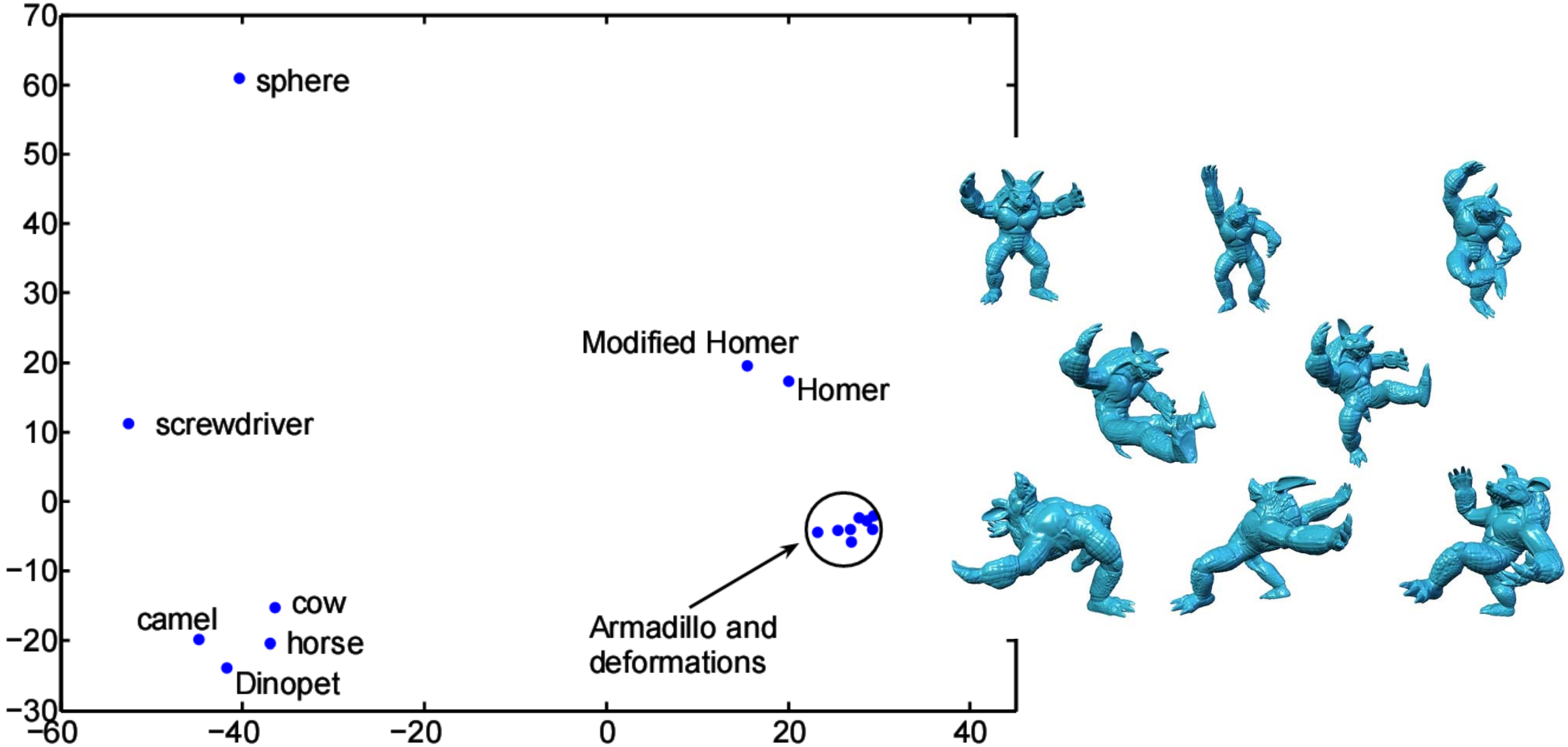
GPS-based shape retrieval

- Use histogram of distances in the GPS embeddings
  - Invariance properties reflected in GPS embeddings
  - Less sensitive to topology changes by using only low-frequency eigenfunctions



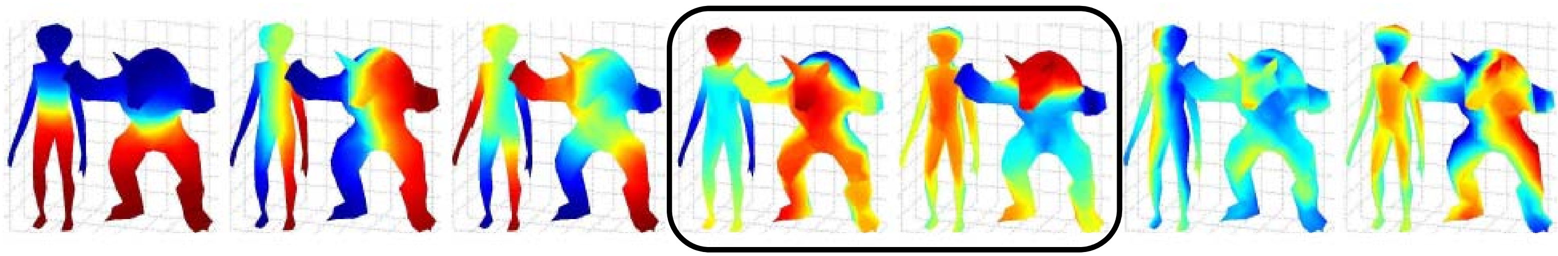
MDS on GPS

2D embedding that "almost" reproduces GPS distances



Use for shape matching?

- Nope. Embedding sensitive to eigenvector "switching"



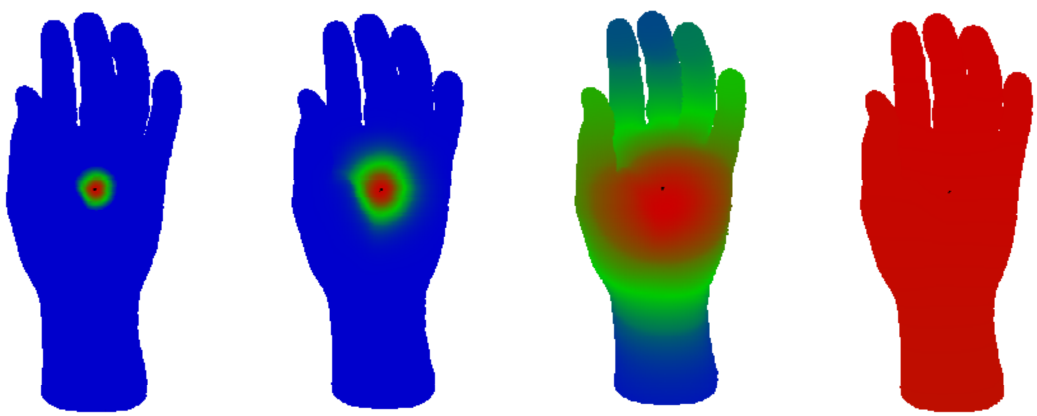
- Eigenvectors are not unique
  - Only defined up to sign
  - If repeating eigenvalues – any vector in subspace is eigenvector

Heat Equation on a Manifold

Heat kernel $k_t(x, y) : \mathbb{R}^+ \times \mathcal{M} \times \mathcal{M} \to \mathbb{R}$

$$f(x,t) = \int_{\mathcal{M}} k_t(x,y) f(y,0) dy$$

$k_t(x, y)$ amount of heat transferred from x to y in time t.



$t = 0.001$ $t = 0.02$ $t = 3$ $t = 40$

Heat Kernel

$$K_t = \exp(-tL) \quad \text{Or} \quad K_t(x,y) = \sum_{i=1}^{n} e^{-t\lambda_i} \varphi_i(x) \varphi_i(y)$$

Eigenvalues of L

Eigenvectors of L

Heat Kernel

K_t = Fundamental solution to
heat diffusion equation

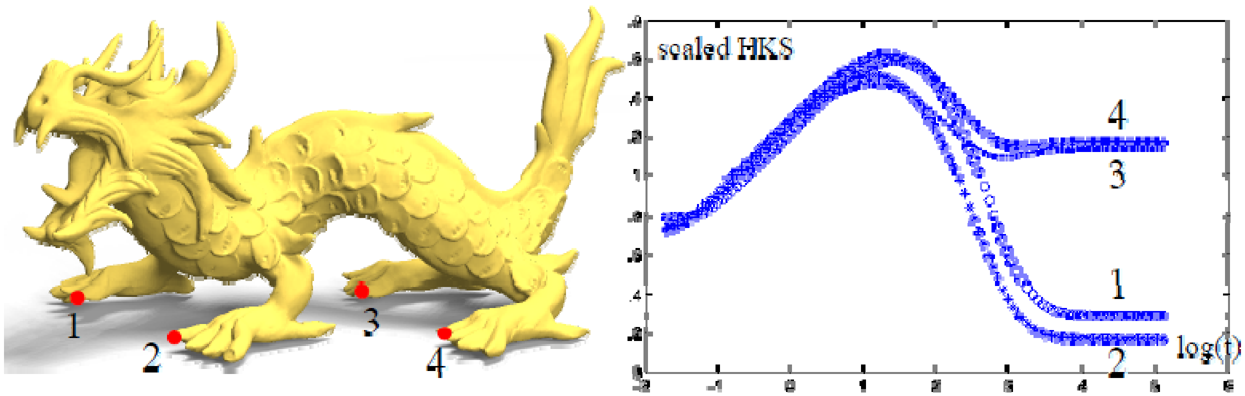
$K_t(x,y) =$ Prob. of reaching y from x after t random steps

$K_t(x,x) =$ Heat Kernel Signature [Sun et al. 09]

Heat Kernel Signature

HKS(p) = $k_t^M(p,p) : \mathbb{R}^+ \to \mathbb{R}^+$: amount of heat left at p at time t.

Signature of a point is a function of one variable.

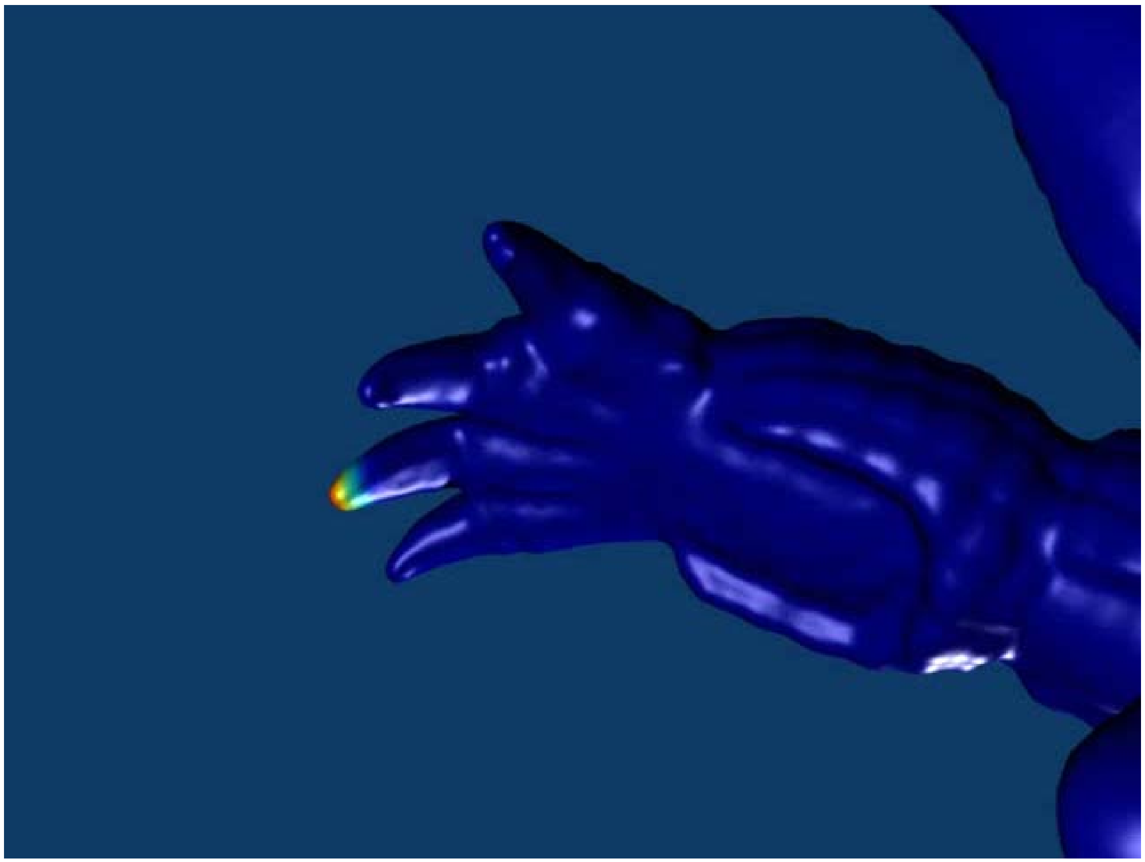


Invariant to isometric deformations. Moreover complete:

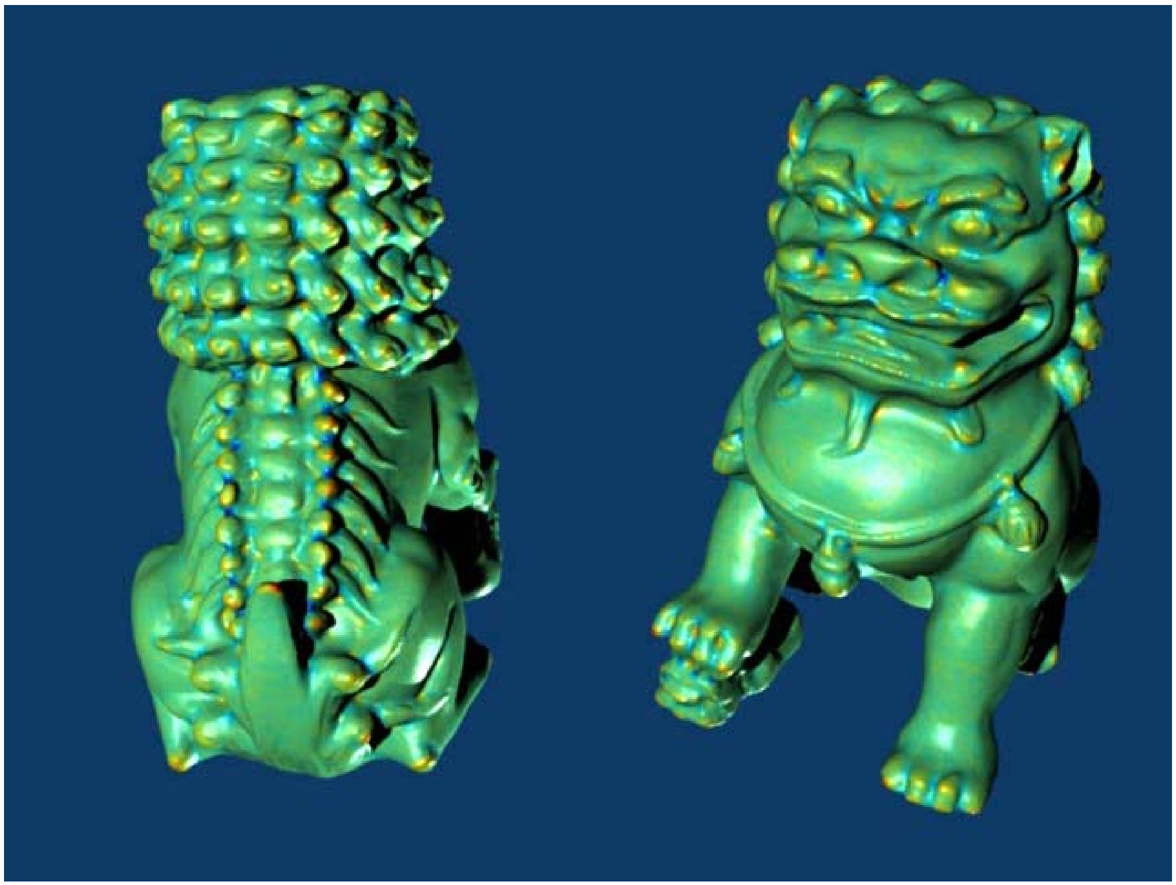
Any continuous map between shapes that preserves HKS must preserve *all* distances.

A Concise and Provably Informative …,
Sun et al., SGP 2009

Heat Kernel Column

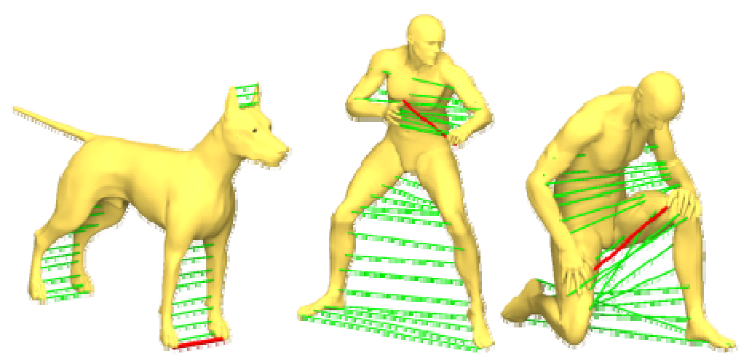


Heat Kernel Signature



Heat Kernel Applied

- Diffusion wavelets
  [Coifman and Maggioni '06]
- Segmentation
  [deGoes et al. '08]
- Heat kernel signature
  [Sun et al. '09]
- Heat kernel matching
  [Ovsjanikov et al. '10]



Applications

- Shape retrieval
- **Parameterization**
  - 1D
  - 2D
- Quad meshing

1D surface parameterization

Graph Laplacian

$a_{i,j} = w_{i,j} > 0$ if (i,j) is an edge

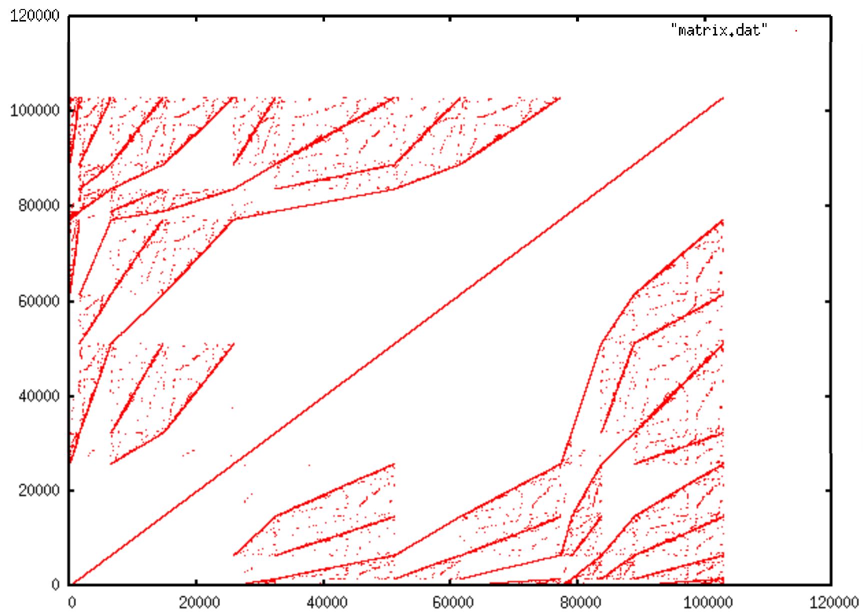
$a_{i,i} = -\sum a_{i,j}$

(1,1 … 1) is an eigenvector assoc. with 0

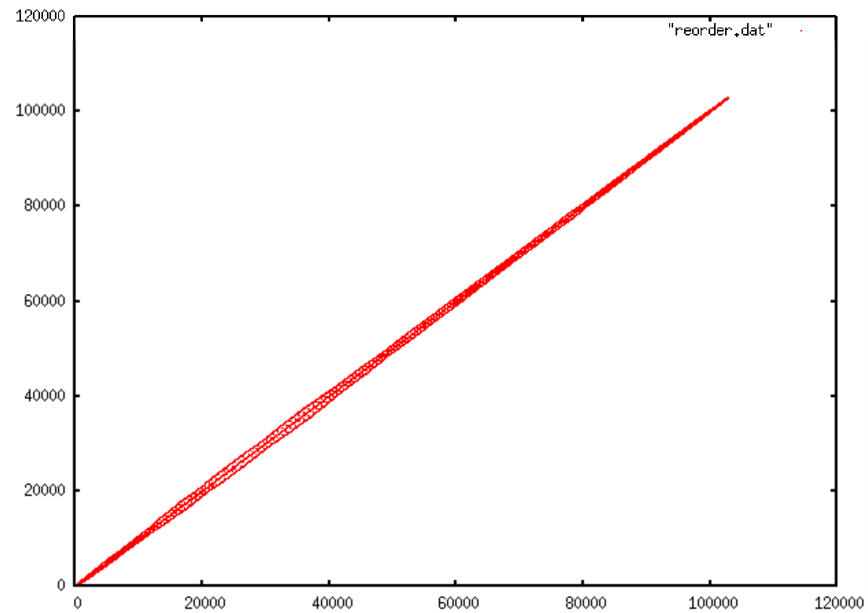
The second eigenvector is interresting
[Fiedler 73, 75]

1D surface parameterization

Fiedler vector



FEM matrix,
Non-zero entries

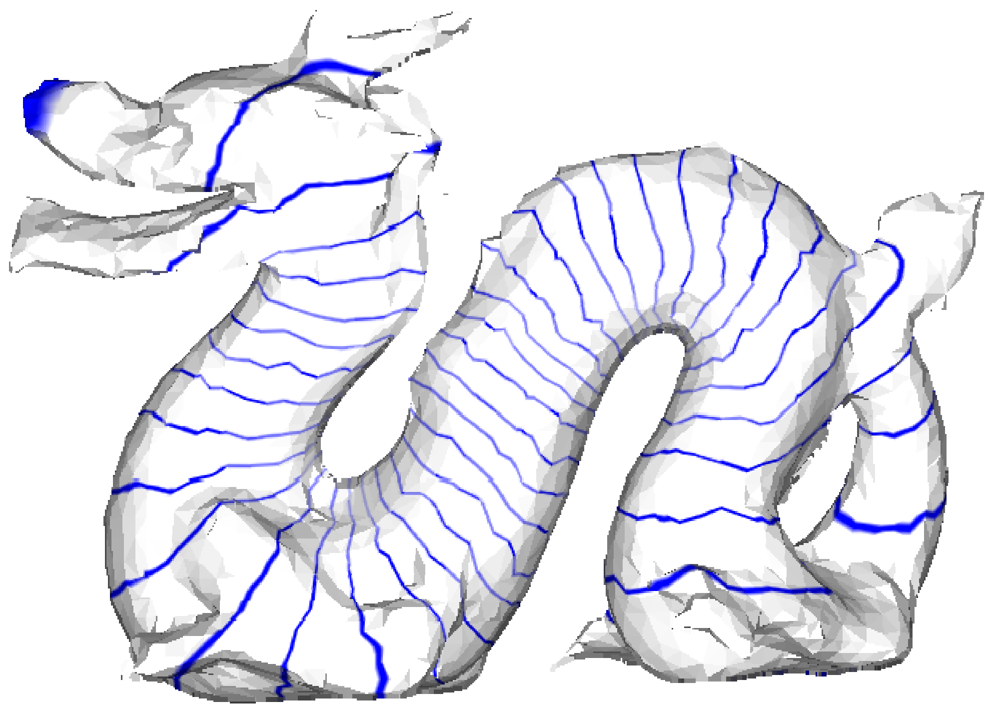


Reorder with
Fiedler vector

1D surface parameterization
Fiedler vector

Streaming meshes
[Isenburg & Lindstrom]

1D surface parameterization
Fiedler vector



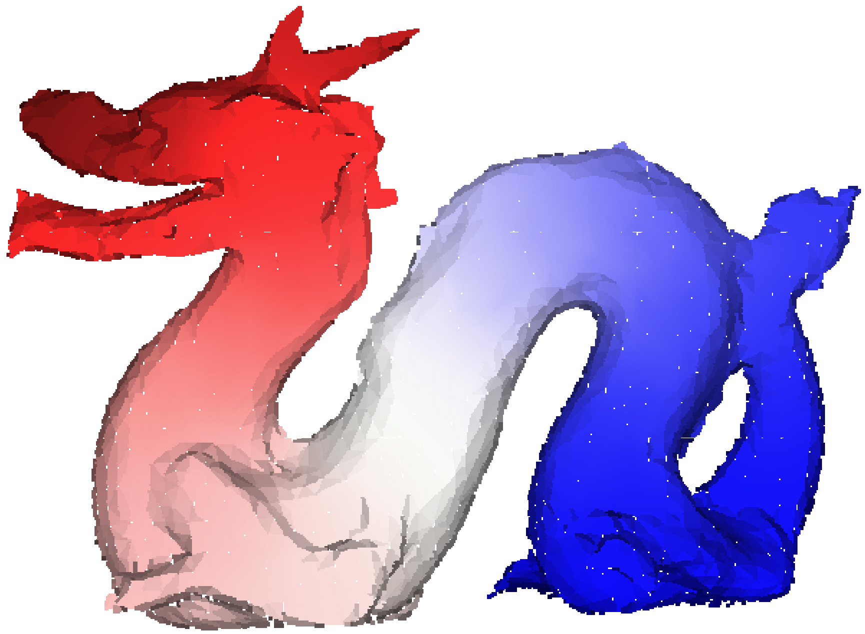
Streaming meshes
[Isenburg & Lindstrom]

1D surface parameterization

Fiedler vector

$$F(u) = \Sigma\, w_{ij}\, (u_i - u_j)^2$$

Minimize $\quad F(u) = \frac{1}{2}\, u^t A\, u$



1D surface parameterization
Fiedler vector

$$F(u) = \Sigma\, w_{ij}\, (u_i - u_j)^2$$

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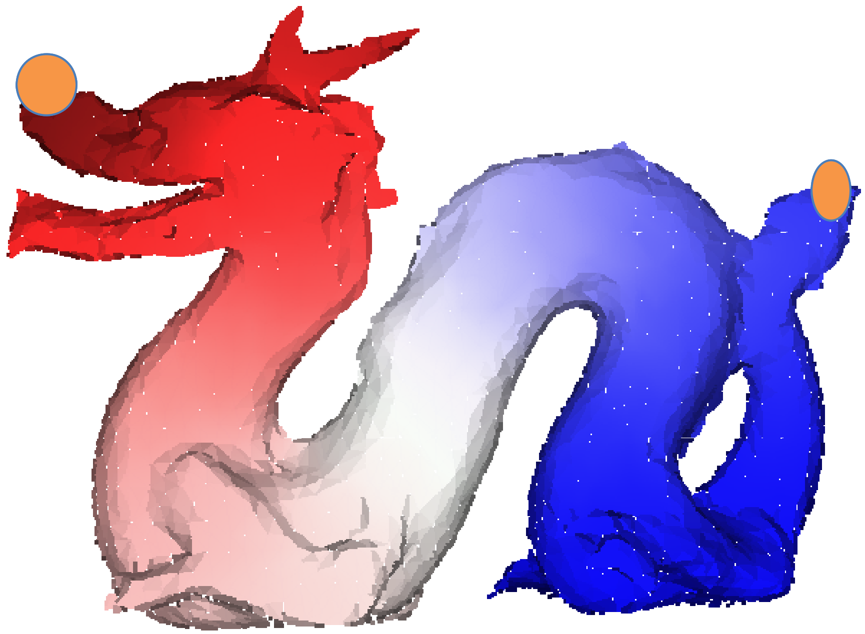


How to avoid trivial solution ?
Constrained vertices ?

1D surface parameterization
Fiedler vector

$$F(u) = \Sigma\, w_{ij}\, (u_i - u_j)^2$$

Minimize $F(u) = \frac{1}{2}\, u^t A\, u$ subject to $\sum u_i = 0$

Global constraints are more elegant !

1D surface parameterization

Fiedler vector

$$F(u) = \Sigma\, w_{ij}\, (u_i - u_j)^2$$

Minimize $F(u) = \frac{1}{2}\, u^t A\, u$ subject to

$$\sum u_i = 0$$

$$\frac{1}{2} \sum u_i^2 = 1$$

Global constraints are more elegant !

We need also to constrain the second mementum

1D surface parameterization
Fiedler vector

$$F(u) = \Sigma\, w_{ij}\, (u_i - u_j)^2$$

Minimize $F(u) = ½\, u^t A\, u$ subject to

$$\sum u_i = 0$$

$$½ \sum u_i^2 = 1$$

$$L(u) = ½\, u^t A\, u - \lambda_1\, u^t\, \mathbf{1} - \lambda_2\, ½\, (u^t u - 1)$$

$$\nabla_u L = A\, u - \lambda_1\, \mathbf{1} - \lambda_2\, u$$

$$\nabla_{\lambda 1} L = u^t\, \mathbf{1}$$

$$\nabla_{\lambda 2} L = ½(u^t\, u - 1)$$

u = eigenvector of A

λ_1 = 0

λ_2 = eigenvalue

1D surface parameterization
Fiedler vector

Rem: Fiedler vector is also a minimizer of the Rayleigh quotient

$$R(A,x) = \frac{x^t A x}{x^t x}$$

The other eigenvectors x_i are the solutions of :

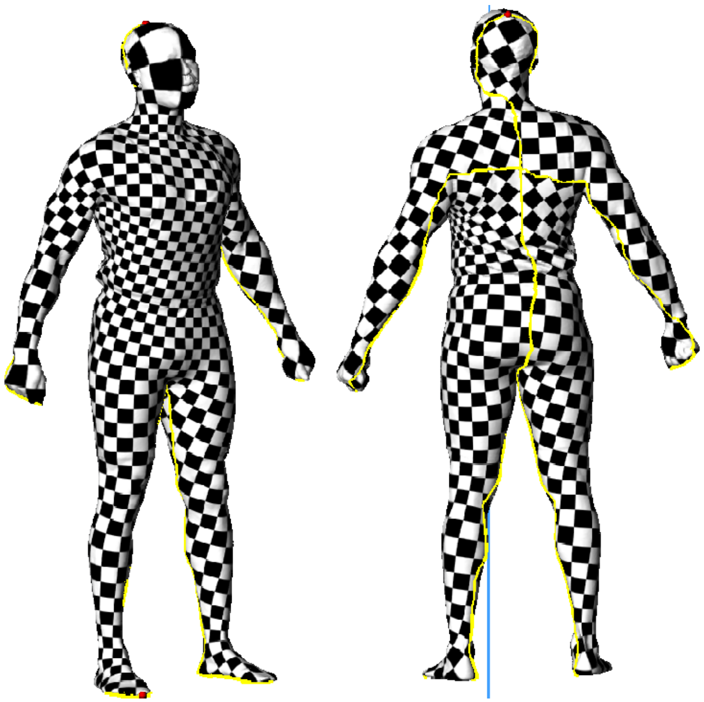
minimize $R(A,x_i)$ subject to $x_i^t x_j = 0$ for $j < i$

Surface parameterization

Minimize $$\sum_{T} \left\| \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} - \begin{bmatrix} -\frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial x} \end{bmatrix} \right\|^2$$

Discrete conformal mapping:

[L, Petitjean, Ray, Maillot 2002]
[Desbrun, Alliez 2002]



Surface parameterization

Minimize $$\sum_{T} \left\| \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} - \begin{bmatrix} -\frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial x} \end{bmatrix} \right\|^2$$

Discrete conformal mapping:

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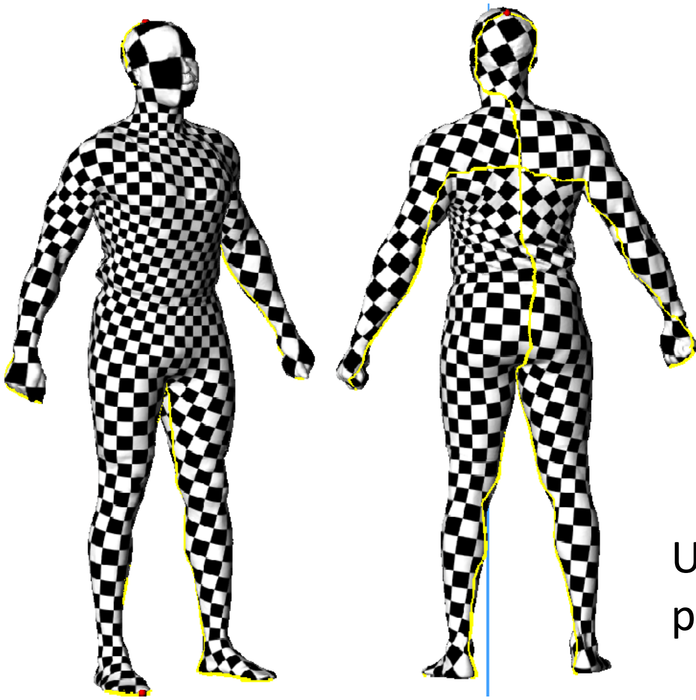
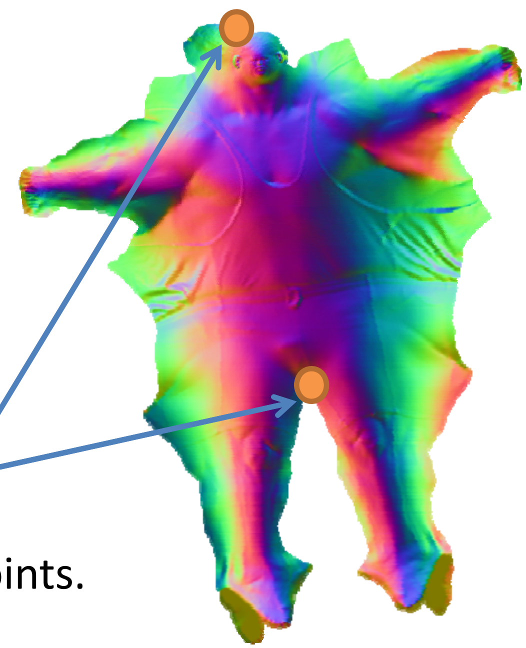
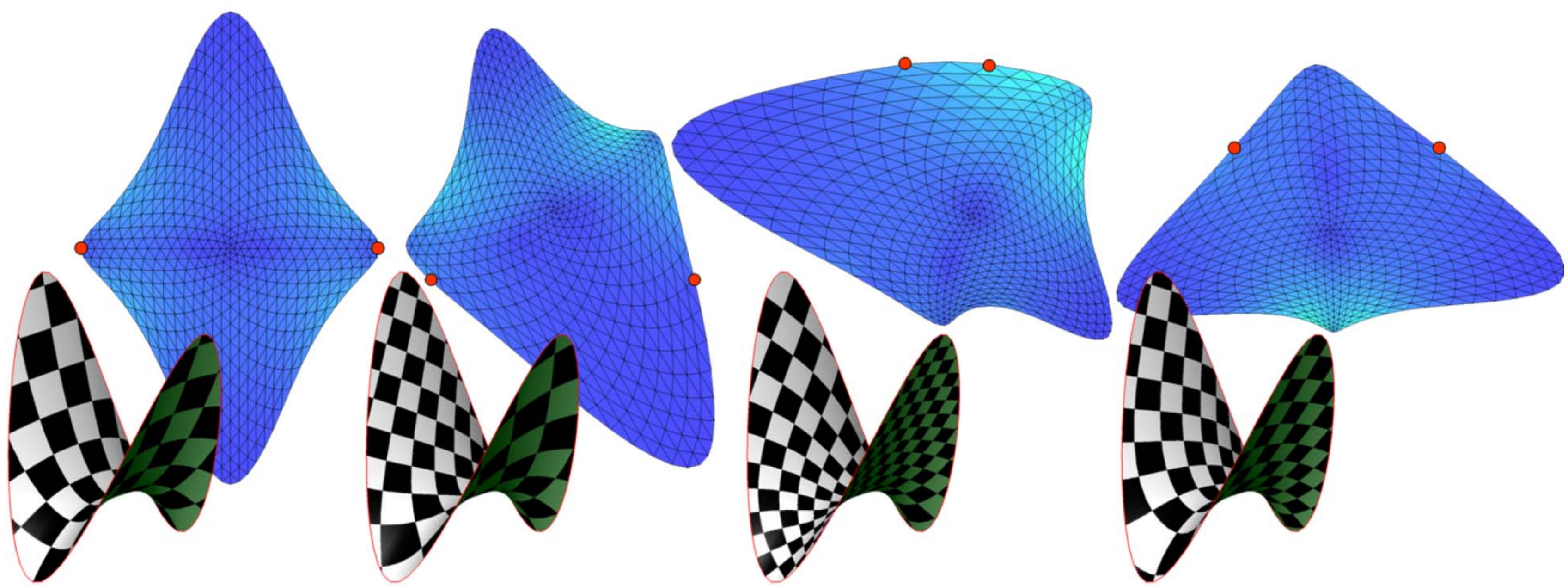


Uses pinned points.



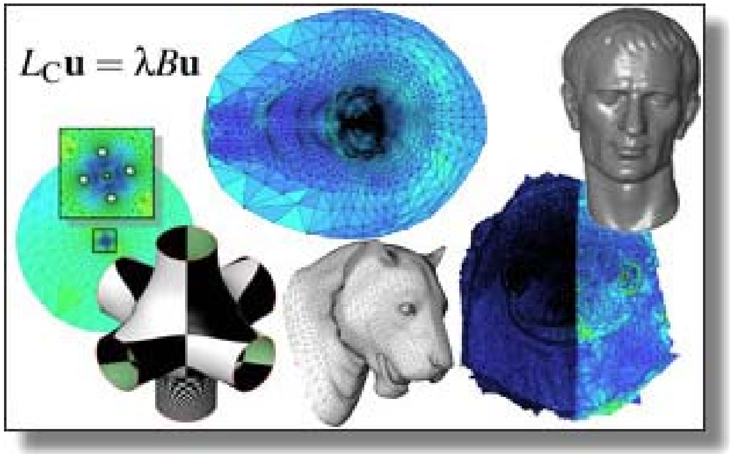
Sensitive to Pinned Vertices



Surface parameterization

[Muellen, Tong, Alliez, Desbrun 2008]

Use Fiedler vector,
i.e. the minimizer of $R(A,x) = x^t A x / x^t x$
that is orthogonal to the trivial constant solution



Implementation:

(1) assemble the matrix of the discrete conformal parameterization
(2) compute its eigenvector associated with the first non-zero eigenvalue

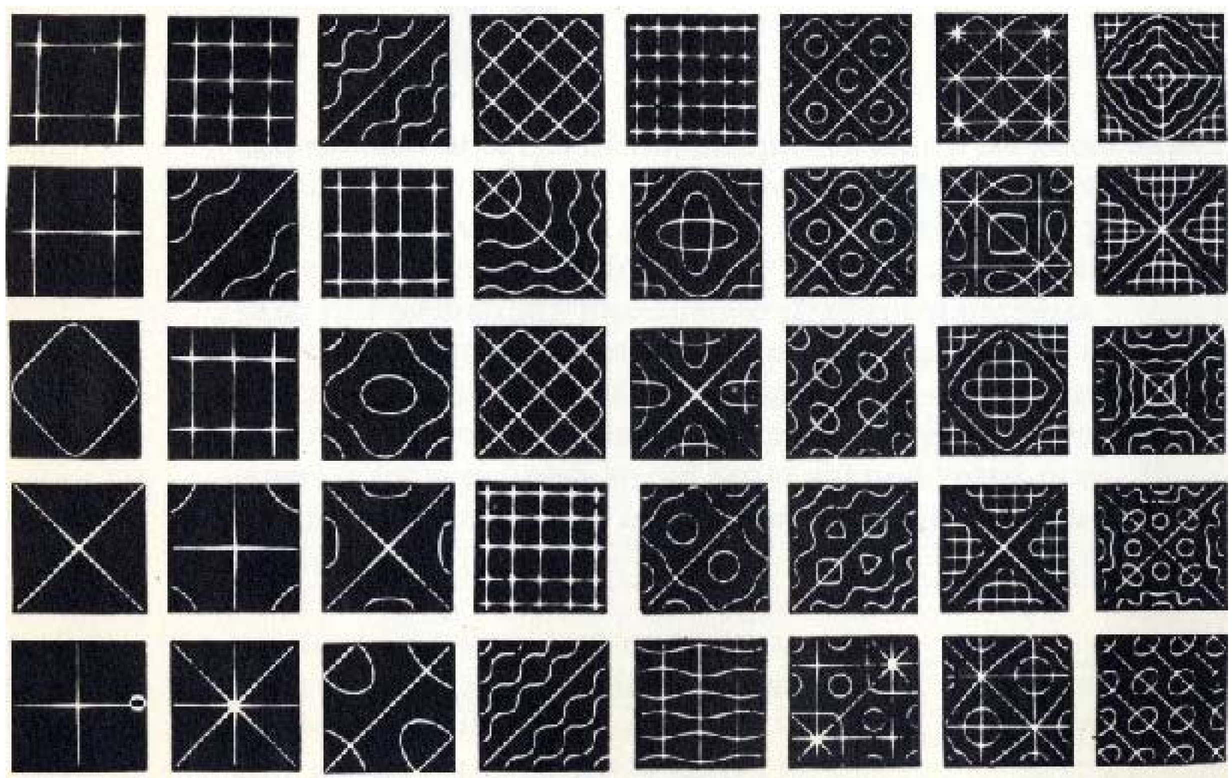
See http://alice.loria.fr/WIKI/ Graphite tutorials – Manifold Harmonics

Applications

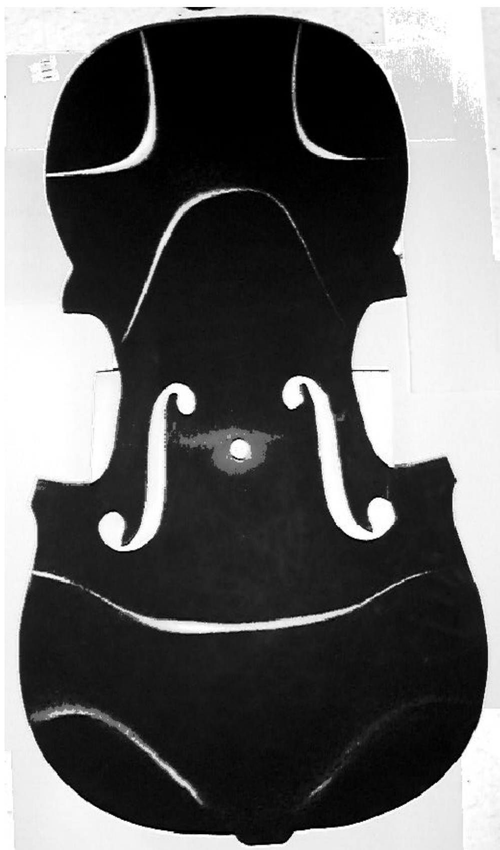
- Shape retrieval
- Parameterization
  - 1D
  - 2D
- **Quad meshing**

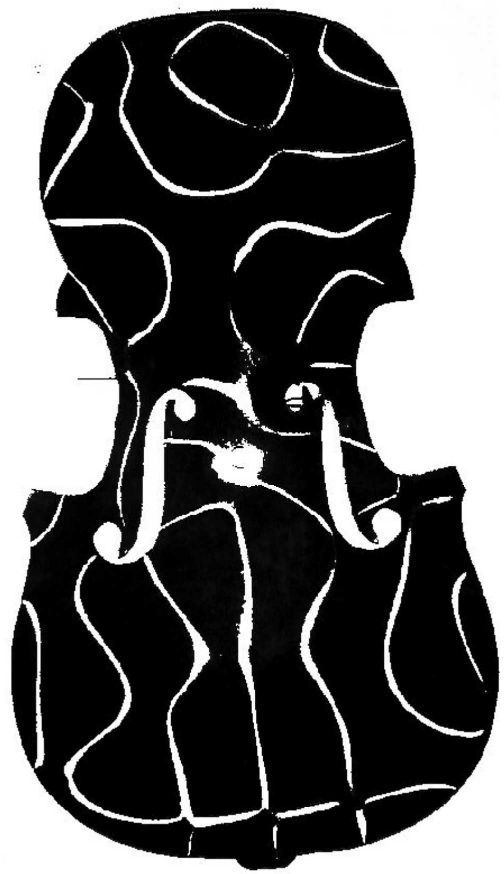
Chladni Patterns

Nodal sets of eigenfunctions of Laplacian



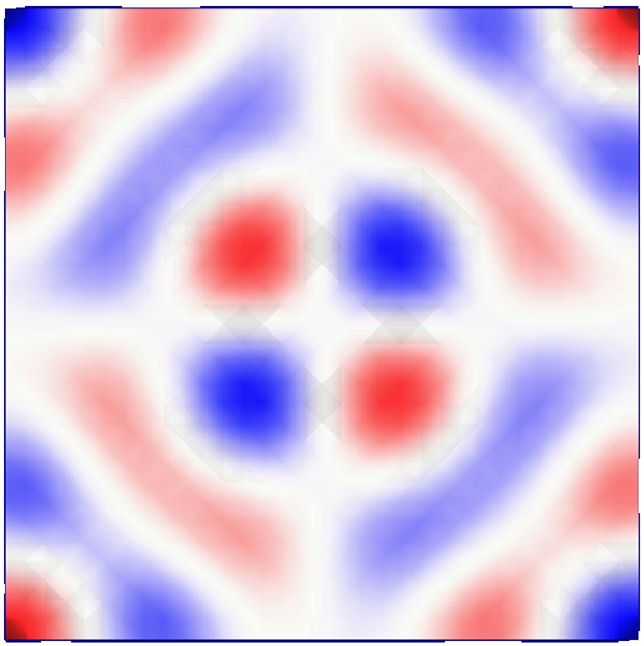
Chladni Patterns

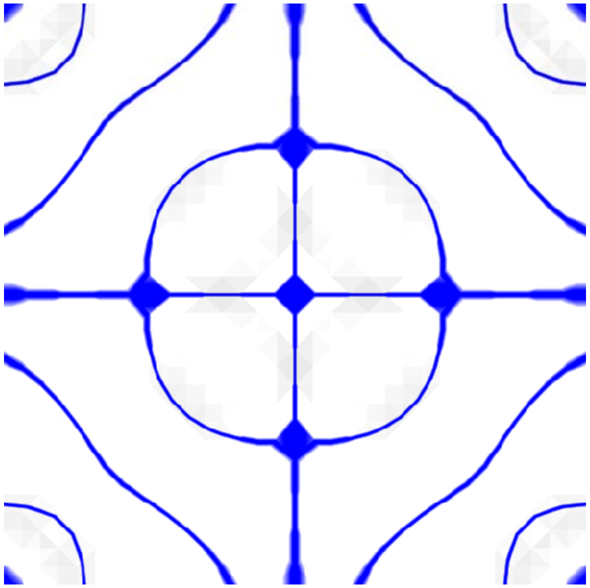




Quad Remeshing

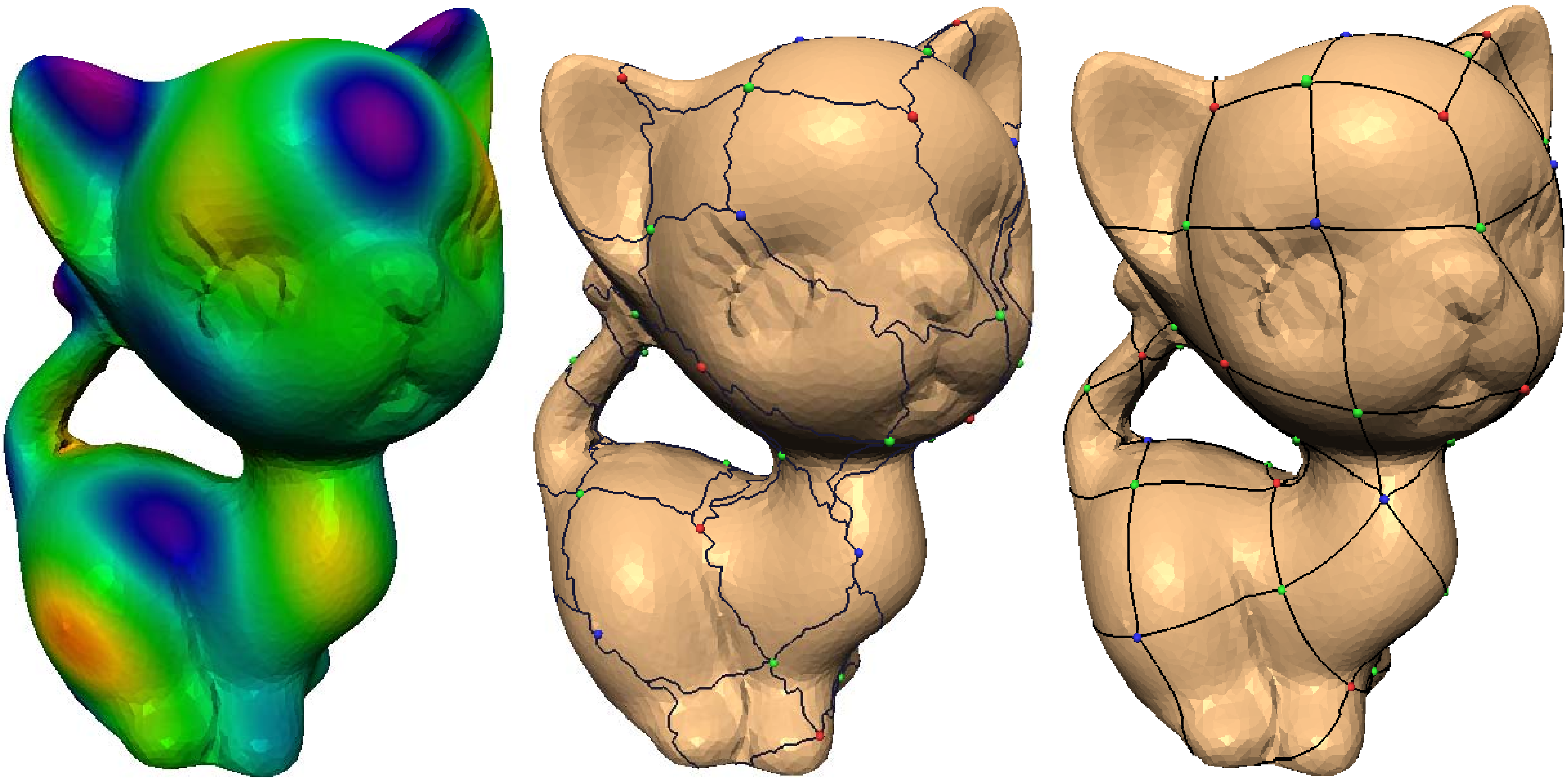
Nodal sets are sets of curves intersecting at constant angles





The N-th eigenfunction has at most N eigendomains

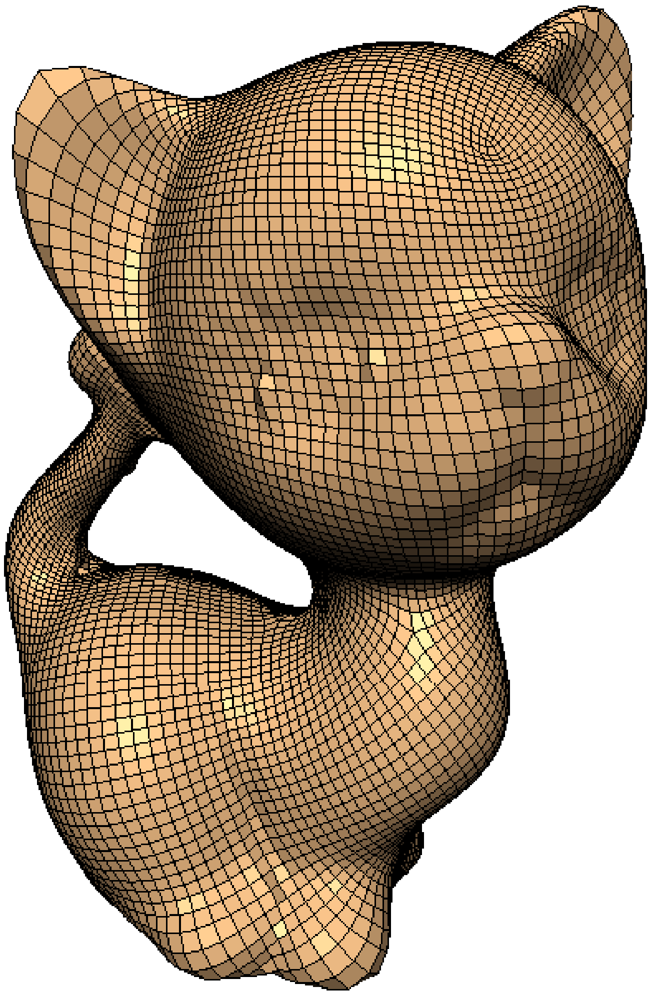
Surface quadrangulation

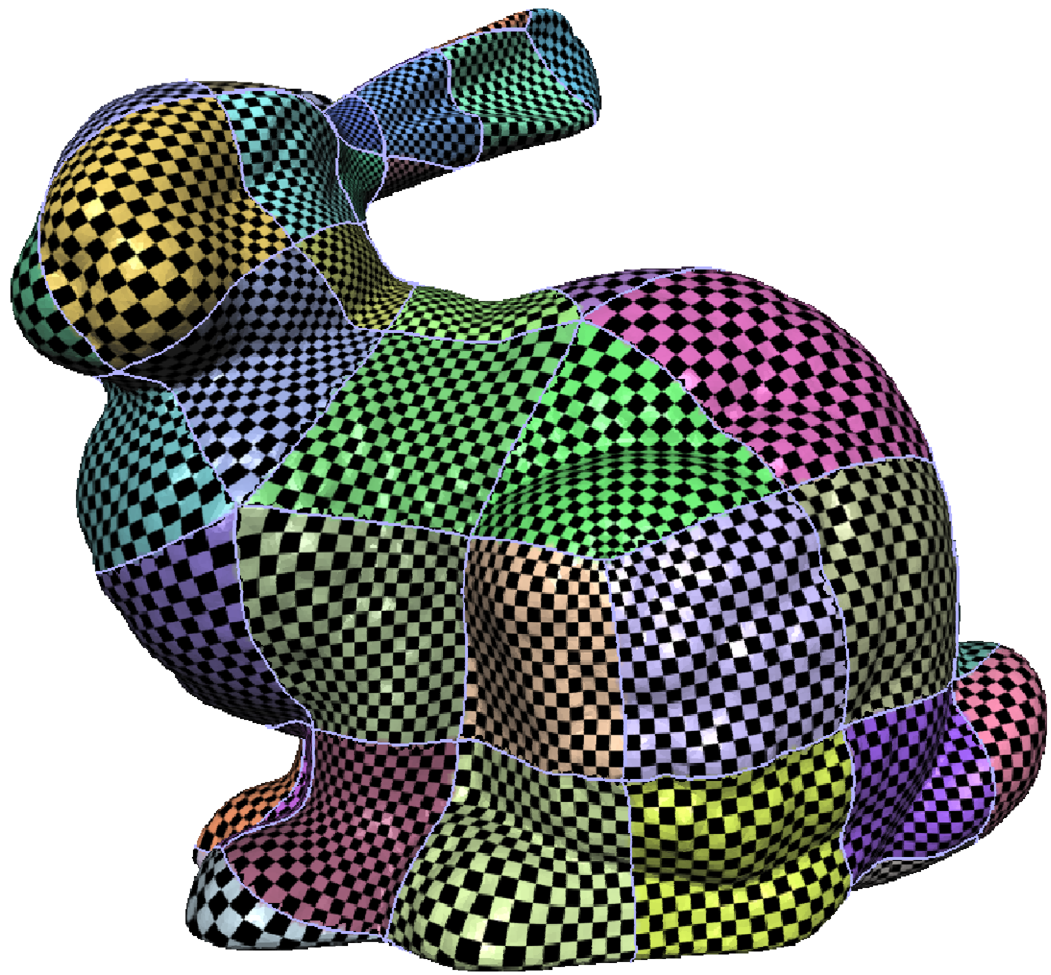


One eigenfunction   Morse complex   Filtered morse complex

[Dong and Garland 2006]

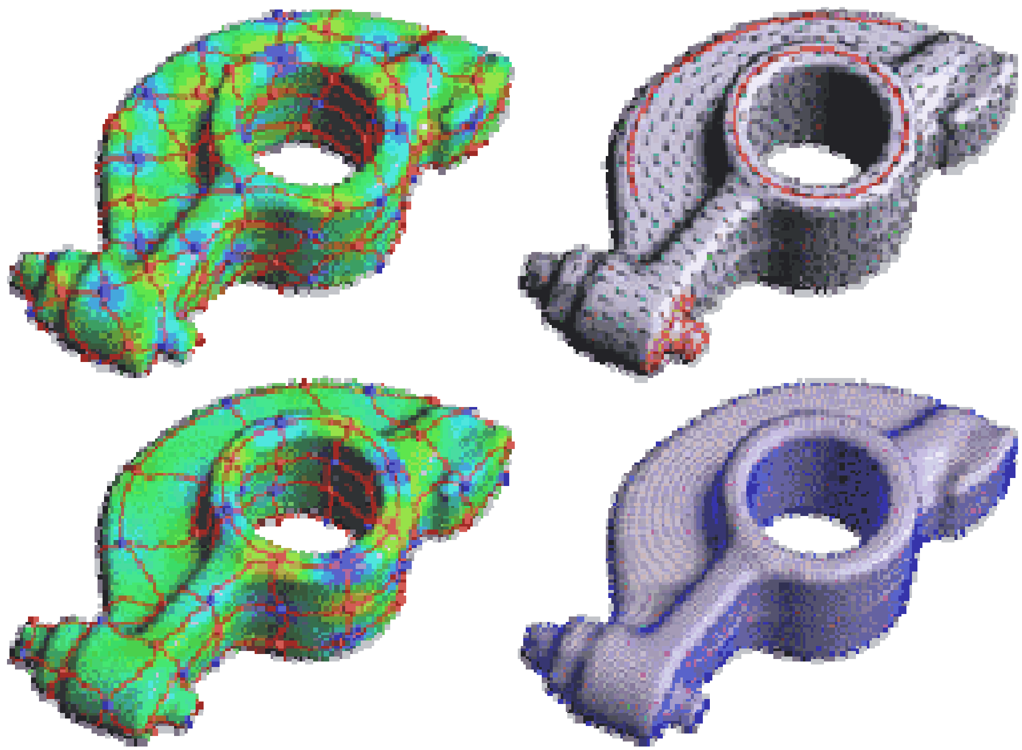
Surface quadrangulation





Reparameterization of the quads

Surface quadrangulation



Improvement in [Huang, Zhang, Ma, Liu, Kobbelt and Bao 2008], takes a guidance vector field into account.