Analyzing the Price of Anarchy in Green Routing: A Simulation-Based and Theoretical Approach

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Abstract

We analyze the Price of Anarchy behavior of a selfish routing congestion game, which has the aim of supplying each player's demand through a combination of ICEVs and EVs along different routes, with the goal of minimizing each player's cost, which is a combination of the pollution caused by ICEVs and the delay caused by charging the EVs. We analyze game's behavior as an iterative projected gradient dynamics problem, also known as SIRD, and analyze the PoA analysis by varying the pollution cost. We find that within appropriate ranges, as the value of pollution cost increases, the Price of Anarchy decreases, suggesting that either selfishly or globally, once pollution cost increases too much, the optimal goal would be to completely shift to EVs, after which the problem therein becomes a problem of optimizing the quadratic congestion cost of the EVs.

1. Introduction

The growing demand for sustainable transportation has placed a spotlight on routing optimization problems that minimize environmental impacts. The Green Routing Problem [3] poses a critical challenge in logistic planning, where each player must route a mixed fleet of internal combustion engine vehicles (ICEV) and electric vehicles (EV) through a network of routes. Traditional approaches to routing problems use either distance or travel time when optimizing cost functions [1], disregarding any environmental consideration. In contrast, this Green Routing Problem integrates not only the environmental impact due to the pollution caused by ICEVs, but also the inefficiency of using EVs due to their charging requirement over time. This new set-up becomes highly relevant in the context of rising global emissions and electric vehicle adoption.

The key challenge when addressing a green routing problem lies in the decentralized nature of decision making. Each player operates selfishly, aiming to reduce their cumulative cost - pollution caused by ICEVS and delay caused by charging EVs. This approach deviates our minimized cost from the global minima of the game. This inefficiency is formalized as the term Price of Anarchy (PoA) which quantifies the performance gap between individuals routing selfishly and a central planners optimization.

The primary objective of this paper is to develop a theoretic and simulated framework for the quantification of inefficiencies arising from selfish routing in the game. By extending the classical models to the new constraints, we achieve novel insights on marginal pollution cost, marginal delay cost and inefficiency.

2. Methodology

2.1. Game Setup

The game setup, as outlined in [3] and shown in Figure 1 is as follows:

Players: Each player belongs to the total set of players $\mathcal{N}$=$\{1, ..., N\}$ where N is the number of players. Each player in $i \in \mathcal{N}$ has a demand D_i to fulfill, which it can fulfill via a mixture of ICEVs and EVs along different routes.

Routes: The ICEVs are considered independent of route, assuming that their pollution cost due to traffic far outweighs the congestion cost due to traffic. Thus all ICEVs are assumed to be on a congestion-less path 0.
The EVs, on the other hand, can be distributed along R total routes. Each route r belongs to the set $\mathcal{R}$=$\{1, ..., R\}$

Action profile: Each action profile $a_i \in \mathcal{A}_i$ for each player $i \in \mathcal{N}$ is a column vector in $\mathbf{R}^{R+1}$, as is of the for a_i=$[x_0^i, x_1^i, ..., x_R^i]^T$. where x_0^i is the number of ICEVs player i is using, whereas for $r \in \mathcal{R}$, x_r^i is the number of EVs player i is using on route r.

Congestion: The congestion χ_r of each route $r \in \mathcal{R}$ is said to be the total number of EVs of all players $i \in \mathcal{N}$ on r. To put it mathematically:

$$\chi_r = \sum_i^N x_r^i$$

Cost function: The cost function for each player i, is defined to be as follows:

$$J^i = \alpha^i x_0^i + \sum_r^R (\mu \chi_r + v_r - \tau^i)^2$$

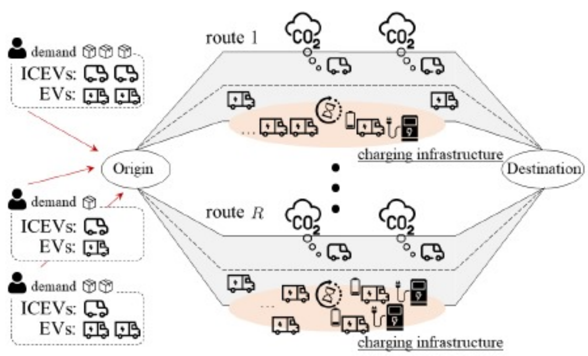

Figure 1. Setup of Green Routing Game

where $\alpha^i x_0^i$ is the linear pollution cost of using ICEVs, scaled by a player-dependent constant α^i, and $(\mu\chi_r + v_r - \tau^i)^2$ is the quadratic congestion cost of using EVs on route r. μ_r is the congestion coefficient of route r, whereas v_r is a constant route cost of route r, and τ^i is player-dependent time delay tolerance threshold, As outlined in [3], rather than being a normal congestion cost, it is a delayed congestion cost, only incurring after χ_r crosses a certain threshold, that is, if $\chi_r \leq \frac{\tau^i - v_r}{\mu_r}$, the congestion cost is zero.

Also shown in [3] is that for the game to have unique Nash Equilirium (NE), the time delay tolerance thresholds must be the same i.e. τ^i=τ, $\forall\ i \in \mathcal{N}$. Thus the cost function then becomes.

$$J^i = \alpha^i x_0^i + \sum_r^R (\mu\chi_r + v_r - \tau)^2$$

2.2. Global Cost Optimization

The global cost is just the sum of all the individual costs i.e.

$$J = \sum_i^N J^i$$

The global optimizing problem is set up as a quadratic programming problem with the aim of minimizing the global. In more formal terms it can be written as:

$$a_{opt} = argmin_{a_{opt}}\ J$$

subject to constraints

$$\begin{gathered} x_0^i + \sum_r^R x_r^i = D_i \quad \forall \text{i} \in \mathcal{N} \\ x_0^i, x_r^i \leq \text{D}_i \quad \forall \text{i} \in \mathcal{N}, \forall \text{r} \in \mathcal{R} \\ x_0^i, x_r^i \geq 0\ \forall \text{i} \in \mathcal{N}, \forall \text{r} \in \mathcal{R} \end{gathered}$$

We are able to solve this problem very easily within MATLAB using the inbuilt quadprog() function. One slight modification we make in this problem is to not consider the delayed cost aspect of the congestion cost and treat it as a simple quadratic cost. We can do that as for the nature of problems we are solving, the quadratic cost will always, at the very least, be convex, and will thus have a global minimum, which also happens to be where the threshold of that route lies as well.

2.3. SIRD Algorithm

Our game-theoretic setup uses an iterative algorithm, as used in [3], hence referred to as Algorithm 1, termed Simultaneous Improving Response Dynamics (SIRD) [4]. This algorithm is a modified version of projected gradient descent, but here, it is done so in a more multi-agent setting. In each iteration, each player tries to minimize their own individual cost by applying gradient descent based on the state in the previous iteration. A more concise and mathematical description of the algorithm is showcased below:

Algorithm 1 Simultaneous Improving Response Dynamics (SIRD)

Require: $x[0]$, γ, ϵ
Ensure: x
$k \leftarrow 0$
repeat
  $k \leftarrow k + 1$
  for $i = 1, \ldots, N$ **do**
    $x^i[k] \leftarrow P_{\mathcal{X}}(x^i[k-1] - \gamma\nabla_i J^i(x[k-1]))$
  end for
until $\|x[k] - x[k-1]\| < \epsilon$
$x \leftarrow x[k]$

Here $P_{\mathcal{X}}$ is the projection of $x^i[k]$ onto the demand polyhedra which satisfies the non-negativity condition for all elements of $x^i[k]$, and the equality constraint that the total distribution of resources across ICEVs and EVs for player i must be equal to D_i. As a result of these constraints, each element of $x^i[k]$ is also upper-bounded by D_i.

from [8], we know that:

$$P_{\mathcal{X}}(x^i) = argmin_{x^*}\ ||x^i - x^*||_2 \quad x^* \in \mathcal{X}^i$$

where $\mathcal{X}^i$ is the feasible polyhedra for x^i

Since the projection minimizes the L2 norm, we can square it and solve it as quadratic optimization problem which would be as follows:

$$x_{proj}^i = argmin_{x_{proj}^i}\ (x^i - x_{proj}^i)^T (x^i - x_{proj}^i)$$

subject to constraints

$$\begin{gathered} x_0^i + \sum_r^R x_r^i = D_i \\ x_0^i, x_r^i \leq \text{D}_i \quad \forall \text{r} \in \mathcal{R} \\ x_0^i, x_r^i \geq 0\ \forall \text{r} \in \mathcal{R} \end{gathered}$$

We can solve this problem very easily too using the MATLAB quadprog() function.

In total, we solve the SIRD algorithm iteratively in MATLAB, by applying projected gradient descent for each player to minimize their own cost, till we reach an NE i.e. there is no significant change in action for any player. Once we find the NE, we use it to compute the Price of Anarchy i.e, the ratio of the cost faced in the NE to the global minimum cost.

2.4. Numerical Analysis

We aim to experimentally observe the Price of Anarchy (PoA) for this strategic game before laying down the theoretical basis for it. We initially set our pollution costs to be α={$0.5, 1.5$} and demands D={$100, 150$} for players 1 and 2, respectively. The time delay tolerance is the same for both, i.e. τ=8. The route-specific coefficients are $\mu = \{0.03, 0.05\}$ and v={$5, 6$} for route 1 and 2, respectively.

Intuitively, from observation, we can observe that the pollution cost is significantly higher than the congestion cost, which will most likely produce a solution in which almost all resources will be shifted to EVs. In order to observe the effect of bringing the pollution tradeoff to the congestion tradeoff to the point of indifference, we scale our values of α with multiple values of C_a, $0 \leq C_a \leq 1$ i.e. α_{C_a}={$0.5C_a, 1.5C_a$}, and observe the results of the SIRD algorithm, as well as the PoA results.

2.5. Theoretical Analysis

To develop a theoretical analysis of our game, we will look at the game as very basic form of itself. In a linear Pigou's example, we have two edges: one with $c(x) = x$ and one with $c(x) = 1$, where $c(\cdot)$ is the cost function. In its nonlinear variant, the cost function becomes $c(x) = x^p$ rather than $c(x) = x$.

The PoA of selfish routing can be large or small, but we see that highly nonlinear cost functions can prevent a selfish routing network from having a PoA close to 1. [2] states that highly nonlinear cost functions are the only obstacle to a small PoA—that every selfish routing network with not-too-nonlinear cost functions, no matter how complex, has PoA close to 1. Our analysis helps us reach a PoA of the worst case structure of the game, lower degrees are ignored.

3. Results

The PoAs of the SIRD algorithm obtained from varying the pollution cost, plotted against the disparity between the pollution and the EV congestion cost, mathematically defined as $\frac{max(\alpha)}{min(\mu)}$, is shown below in Figure 2:

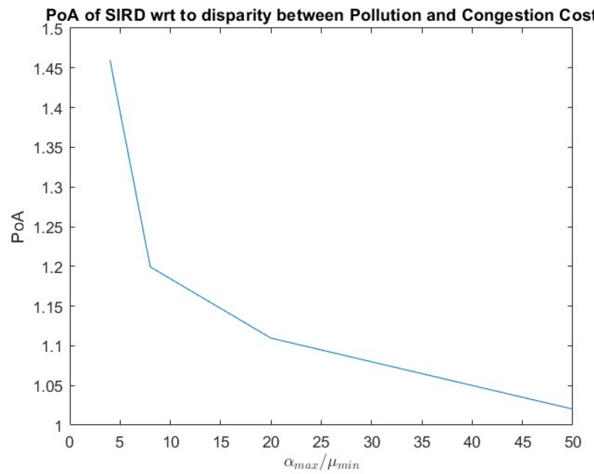

Figure 2. PoA of SIRD wrt disparity between Pollution and Congestion cost.

The definition of the anarchy value defined in [1, Definition 11.2] is simply the worst case inefficiency of selfish routing in Pigou like examples. However, this definition proves to be useless when actually determining the anarchy value of a cost function. we can then further simplify the definition under the assumptions and characteristics of our model. When cost is continuously differentiable and convex, we reach a new formula for our anarchy value with Nash Equilibrium flow as $\theta \cdot c(\theta)$ with θ as the traffic flow on the route, and the optimal flow is given as a fractional combination of the total flow θ:

$$PoA = \frac{\theta \cdot c(\theta)}{\gamma\theta \cdot c(\gamma\theta) + (\theta - \gamma\theta)c(\theta)}$$

$$= \frac{1}{\gamma \cdot \frac{c(\gamma\theta)}{c(\theta)} + (1 - \gamma)}$$

This price of anarchy value becomes the least upperbound, also termed as the supremum. Here we have to determine the expression for γ to be able to compute our bound. For this we look at the generalized Pigou's example with an edge e with the cost function $c_e(x) = \sum_{i=0}^{p} a_i x^i$, the ith edge has a cost of $c_{e,i}(x) = a_i x^i$. We can determine the anarchy value because of the differentiable and convex nature, and the γ value by choice of θ.

$$\gamma = (i + 1)^{-1/i}$$

$$\frac{c(\gamma\theta)}{c(\theta)} = \frac{a(\gamma\theta)^i}{a(\theta)^i} = \gamma^i = (i + 1)^{-1}$$

These simplifications allow us to determine the bound on the PoA with merely the degree of the cost function, i=2 for the Green Rounting Problem.

$$
\begin{aligned}
PoA_{bound} &= \frac{1}{(i+1)^{-1/i} \cdot (i+1)^{-1} + 1 - (i+1)^{-1/i}} \\
&= \frac{1}{1 + (i+1)^{-1/i} \cdot [(i+1)^{-1} - 1]} \\
&= \frac{1}{1 + (i+1)^{-1/i} \cdot [-i(i+1)^{-1}]} \\
&= \frac{1}{1 - i(i+1)^{-(1/i)-1}} \\
&= 1.62575 for i = 2
\end{aligned}
$$

this is the least upperbound on our models PoA.

4. Discussion

The results show a very clear decreasing correlation with the disparity between the pollution and the congestion costs, and the PoA. As mentioned previously, this makes intuitive sense. As the pollution cost increases, the players shift more towards the less costly EV routes; after a certain threshold, they will always choose completely EVs, and the problem boils down to optimizing just the quadratic EV cost. Here only the iterative projections play a part in stopping the solution before the global minimum.

$$J^i = \alpha^i x_0^i + \sum_r^R (\mu \chi_r + v_r - \tau)^2$$

$$J^1 = \sum_r^R (\mu_r \chi_r + v_r - \tau)^2$$

$$J^2 = \sum_r^R (\mu_r \chi_r + v_r - \tau)^2$$

On the other hand, as we decrease the pollution costs, the PoA goes up because of the increasing indifference between pollution cost and delay cost. Additionally, the pollution cost is a linear and monotonically increasing function that is convex in the demand polyhedra i.e. has a global minimum. Because of this, in the line search, the ICEV variable x_0^i is always decreasing for each player, which is only dependent on the scalar factor α^i. As we approach the point of indifference and preference for ICEVs, it causes us to deviate from the global optimum as that would place more preference on EVs, thus having a higher PoA.

We determined the least upper bound of the PoA by the worst-case game, which came out to be 1.62575 for i=2. However, it is important to note here that this analysis is done on a generalized Pigou example that follows the nature of our problem but does not capture the intricacies of it. [2] proposes that the price of anarchy of a game is very small unless the cost functions are extremely steep. something that we have tried to capture in our analysis of disparity between the two kinds of costs being incurred to the players. when it i

5. Conclusion

We simulated the Price of Anarchy behavior of a selfish routing congestion game, which aimed to supply each player's demand through a combination of ICEVs and EVs along different routes, with the goal of minimizing each player's cost, We analyzed the game's behavior as an iterative projected gradient dynamics problem, also known as SIRD, and analyze the PoA analysis by varying the pollution cost. We find that within appropriate ranges, as the value of pollution cost increases, the Price of Anarchy decreases, suggesting that either selfishly or globally, once pollution cost increases too much, the optimal goal would be to completely shift to EVs, after which the problem therein becomes a problem of optimizing the quadratic congestion cost of the EVs.

In the future, we aim to explore extrapolating even more extreme values to scale the pollution cost is to. We aim to implement even more algorithms on this game setup to see how they fare, most notably the Cycling Best Response Dynamics Algorithm (CBRD)[14]. We also aim to see how fine-tuning other hyperparameters affects the centralized and multi-agent cost.

6. Contributions

- **Raafay:** MATLAB Simulation, Report Writing
- **Fatima:** Theoretical Analysis, Report Writing

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