# GAUSSIAN FORMULAS


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Approximate formulas for service level and average inventory are contained in my 1975 paper *Inequalities with Application in Retail Inventory Analysis* under the assumption of one-at-a-time Poisson demand and periodic review of inventory position (on-hand + on-order) relative to OUTL (order-up-to-level). Here we generalize those formulas to a Gaussian demand process $N(t)$ with $E(N(t)) = \mu t$ and $Var(N(t)) = \sigma^2 t$ over any time interval of length $t$. This serves two purposes. First, Gaussian formulas can provide efficient approximations to Poisson formulas, particularly for high demand items. Second, Gaussian formulas can approximate a more complex demand process like compound Poisson with random demand quantities at Poisson time points *assuming partial fills are allowed*. We utilize four equations from Appendix 4 in *Hadley & Whitin*, *Analysis of Inventory Systems*, *Prentice-Hall, 1963*. Throughout $\phi(r) = \frac{1}{\sqrt{2\pi}} e^{-r^2/2}$ and $\Phi(r) = \int_r^\infty \phi(x)dx$ .

Suppose we have delivery lead time $\nu$ , review period $\omega$ , and OUTL $k$ . Then from my paper, steady-state service level $SVL(k) \cong \frac{E(min(k,N(\nu+\omega))) - E(min(k,N(\nu)))}{\mu\omega}$ where for any $x$ , $min(k,x) = x - max(0,x-k) = k - max(0,k-x)$ . Hence, for any $t$ , $E(min(k,N(t))) = \mu t - E(max(0,N(t)-k))$ and service level $SVL(k) \cong 1 - \frac{E(max(0,N(\nu+\omega)-k)) - E(max(0,N(\nu)-k))}{\mu\omega}$ with good approximation in the high service level domain. For any $t$ , from Equation #25 in Hadley & Whitin, we have $U(k,t) = E\left(max\left(0, N(t)-k\right)\right) = \dfrac{1}{\sigma\sqrt{t}}\int_k^\infty (x-k)\,\phi\left(\dfrac{x-\mu t}{\sigma\sqrt{t}}\right)dx$ =

$$\sigma\sqrt{t}\,\phi\left(\frac{k-\mu t}{\sigma\sqrt{t}}\right)-(k-\mu t)\Phi\left(\frac{k-\mu t}{\sigma\sqrt{t}}\right) = \sigma^2 t\,\phi\left(\frac{k-\mu t}{\sigma\sqrt{t}}\right)/\left(\sigma\sqrt{t}\right)-(k-\mu t)\Phi\left(\frac{k-\mu t}{\sigma\sqrt{t}}\right)$$

for a Gaussian demand process. Thus, $SVL(k) \cong 1 - \dfrac{U(k,\nu+\omega) - U(k,\nu)}{\mu\omega}$ .

Further, from my paper, average inventory $AVI(k) \cong \dfrac{\int_\nu^{\nu+\omega} E\left(max\left(0, k-N(t)\right)\right)dt}{\omega}$ =

$$\frac{\int_\nu^{\nu+\omega}\left(k-\mu t+E\left(max\left(0,N(t)-k\right)\right)\right)dt}{\omega} = k - \mu\left(\nu+\frac{\omega}{2}\right) + \frac{\int_\nu^{\nu+\omega}E\left(max\left(0,N(t)-k\right)\right)dt}{\omega} .$$

For a Gaussian demand process, we have $\int_\nu^{\nu+\omega}E\left(max\left(0,N(t)-k\right)\right)dt$ =

$\int_\nu^{\nu+\omega}\left(\sigma\sqrt{t}\,\phi\left(\dfrac{k-\mu t}{\sigma\sqrt{t}}\right)-(k-\mu t)\Phi\left(\dfrac{k-\mu t}{\sigma\sqrt{t}}\right)\right)dt$ . Splitting this up and using Equation #10 in

Hadley & Whitin, $A = \int_\nu^{\nu+\omega}\sigma\sqrt{t}\,\phi\left(\dfrac{k-\mu t}{\sigma\sqrt{t}}\right)dt = \sigma^2\int_\nu^{\nu+\omega}\dfrac{\sqrt{t}}{\sigma}\phi\left(\dfrac{k-\mu t}{\sigma\sqrt{t}}\right)dt$ =

$\sigma^2\left(W_1(k,\nu+\omega)-W_1(k,\nu)\right)$ where $W_1(k,t) = \dfrac{\sigma^2}{\mu^3}\left(1+\dfrac{\mu k}{\sigma^2}\right)\Phi\left(\dfrac{k-\mu t}{\sigma\sqrt{t}}\right)$ -

$\dfrac{2\sigma^2 t}{\mu^2}\phi\left(\dfrac{k-\mu t}{\sigma\sqrt{t}}\right)/\left(\sigma\sqrt{t}\right) + \dfrac{1}{\mu^2}\left(k-\dfrac{\sigma^2}{\mu}\right)e^{2\mu k/\sigma^2}\,\Phi\left(\dfrac{k+\mu t}{\sigma\sqrt{t}}\right)$ . Using Equation #16 in

Hadley & Whitin $B = \int_\nu^{\nu+\omega}k\,\Phi\left(\dfrac{k-\mu t}{\sigma\sqrt{t}}\right)dt = k\left(V_0(k,\nu+\omega)-V_0(k,\nu)\right)$ where $V_0(k,t)$ =

$$\left(t-\frac{k}{\mu}-\frac{\sigma^2}{2\mu^2}\right)\Phi\left(\frac{k-\mu t}{\sigma\sqrt{t}}\right) + \frac{\sigma^2 t}{\mu}\phi\left(\frac{k-\mu t}{\sigma\sqrt{t}}\right)/\left(\sigma\sqrt{t}\right) + \frac{\sigma^2}{2\mu^2}e^{2\mu k/\sigma^2}\,\Phi\left(\frac{k+\mu t}{\sigma\sqrt{t}}\right) .$$

Using Equation #17 in Hadley & Whitin, $C = \int_{\nu}^{\nu+\omega} \mu t \, \Phi\left( \frac{k - \mu t}{\sigma \sqrt{t}} \right) dt = \mu\left( V_1\left( k, \nu+\omega \right) - V_1\left( k, \nu \right) \right)$

where $V_1\left( k, t \right) = \frac{1}{2}\left( t^2 - \frac{k^2}{\mu^2} - \frac{2\sigma^2 k}{\mu^3} - \frac{3\sigma^4}{2\mu^4} \right)\Phi\left( \frac{k - \mu t}{\sigma \sqrt{t}} \right) +$

$\frac{\sigma^2 t}{2\mu^2}\left( \mu t + \frac{3\sigma^2}{\mu} + k \right)\phi\left( \frac{k - \mu t}{\sigma \sqrt{t}} \right) / \left( \sigma \sqrt{t} \right) - \frac{\sigma^2}{2\mu^3}\left( k - \frac{3\sigma^2}{2\mu} \right) e^{2\mu k/\sigma^2} \, \Phi\left( \frac{k + \mu t}{\sigma \sqrt{t}} \right)$. Finally,

$AVI\left( k \right) \cong k - \mu\left( \nu + \frac{\omega}{2} \right) + \frac{A - B + C}{\omega}$. Now OUTL $k$ can be incremented as indicated in my paper to maximize annual profit, trading off gross margin against inventory carrying cost. This is precisely what is going in in my Excel VBA, R, and Python functions.

What happens when partial fills are not allowed? In that case, we can revert to Poisson formulas with a conservative estimate of "average transaction quantity." Suppose $S = \sum_{i=1}^{N} X_i$ where $N$ is Poisson distributed with mean $E\left( N \right) = \lambda$ and the $X_i$ are independent and identically distributed random variables with $E\left( X_i \right) = \rho$ and $Var\left( X_i \right) = \xi^2$. Then we have $E\left( S \right) = \mu = \lambda\rho$ and $Var\left( S \right) = \sigma^2 = \lambda\left( \rho^2 + \xi^2 \right)$. We can't compute $\lambda, \rho, \xi$ from $\mu, \sigma$ but since $\frac{\sigma^2}{\mu} = \rho + \frac{\xi^2}{\rho}$, we can conservatively estimate $\hat{\rho} = \frac{\hat{\sigma}^2}{\hat{\mu}}$ as average transaction quantity from demand statistics and $\hat{\lambda} = \frac{\hat{\mu}}{\hat{\rho}} = \frac{\hat{\mu}^2}{\hat{\sigma}^2}$ as the estimated Poisson transaction rate. The Poisson calculations then go through with average transaction quantity substituted for the actual unit. In practice, this approximation yields slightly higher OUTL and slightly lower service level than the corresponding Gaussian "partial fills allowed" formulas, as you might expect.

How do we obtain estimates $\hat{\mu}$ and $\hat{\sigma}$ from history? Since we are talking about staple retail items, I would keep it exceedingly simple. For a vector of (say weekly) historical sales $s$ , we have $\hat{\mu} = mean(s)$ and $\hat{\sigma} = sd(s)$ in R. Corresponding functions exist in various Python modules, e.g., $scipy.stats.mean(s)$ and $scipy.stats.tstd(s)$ or $numpy.mean(s)$ and $numpy.std(s)$ or the pandas approach indicated in our estimation script . And of course in Excel we have AVERAGE and STDEV. I would avoid complex procedures like ARIMA or even exponential smoothing. My formulas aren't applicable to items with accentuated seasonality or marked trends. For very high demand items, you could limit estimation to the past year to account for demand drift, but for the great mass of lower demand items my advice is "use what you've got and keep it simple." For Poisson, you only need to estimate the mean demand rate, but you can use the standard deviation estimate to confirm the one-at-a-time Poisson assumption; for Poisson demand, standard deviation should be close to the square root of the mean (variance = mean).