SHAPLEY VALUE

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An *n*-person cooperative game in characteristic function form consists of a set of players $N = \{1,\ldots,n\}$ and a superadditive function v defined on subsets of N with $v(E) = 0$, where E denotes the empty set. Normally, $v \geq 0$ and in particular $v(\{i\}) = 0$ for all singleton sets with $i \in N$ since it is assumed that players can't achieve anything on their own. Superadditivity means that $v(S \bigcup T) \geq v(S) + v(T)$ for any disjoint subsets S and T. We call v the characteristic function of the game, and we call the various subsets of N coalitions. Given a coalition $S \subset N$, $v(S)$ is the payoff or utility that the players in S can jointly obtain, without regard to the remaining players. Given this setup, the objective is to determine which coalitions might form and how the payoffs might be split among the cooperating players.

Since the outcome of such a game depends on how the various players bargain with one another, there is no clear-cut result in most cases. Various solution concepts have been defined which attempt to delimit the potential outcomes or to measure the players' relative bargaining powers. One such measure of relative bargaining power is Shapley value which follows from three logical axioms. It defines an "expected" payoff for each player $i \in N$ as

$$s(i) = \sum_{\substack{T \subset N \\ i \in T}} \frac{(t-1)!(n-t)!}{n!} \left[v(T) - v(T - \{i\}) \right]$$ where t is the number of elements of T.

We have $\sum_{i=1}^{n} s(i) = v(N)$ so $s(i)$ measures what share of the grand payoff player i can expect on average, given all the coalitions that might form and that player's marginal value to each one. Since $s(i) \geq v(\{i\})$, player i can always expect to do at least as well in coalition as independently. Shapley value has proved useful in measuring market power in economic games, political power in voting games, and allocated overhead in cost accounting. Our reference is G. Owen, *Game Theory*, 3rd Ed, Academic Press, 2001.