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Q2)

a) Let G not be connected.
Then G has atleast 2 components.

Let p, q be the number of vertices in two arbitrary components of G

$$n \geq p + q \quad \dots \quad (1)$$

As n is the total number of vertices.

Note that, all vertices in the first component may be connected to at most p-1 other vertices. Or, the degree of any vertex in the first component is at most p-1.

$\therefore \quad \delta(G) \leq p-1$

and $\delta(G) \geq \frac{n-1}{2} \quad \Rightarrow \quad p-1 \geq \frac{n-1}{2}$

By a similar reasoning,

$\delta(G) \leq q-1$

and $\delta(G) \geq \frac{n-1}{2} \quad \Rightarrow \quad q-1 \geq \frac{n-1}{2}$

or, $p-1+q-1 \geq n-1$

or, $p+q \geq n+1$

Combining with (1), $n \geq p+q \geq n+1$

$\Rightarrow \quad n \geq n+1$, which is untrue.

$\therefore$ By proof by contradiction G is connected

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For any two distinct vertices $v_i, v_j \in V(G)$,

If $v_i \sim v_j$ then $d(v_i, v_j) = 1$

If $v_i \nsim v_j$,
Both v_i and v_j must be adjacent to ~~the~~ atleast $\frac{n-1}{2}$ other vertices as the minimum degree $\delta(G) \geq \frac{n-1}{2}$

But there are exactly $n-2$ vertices in $V(G) \setminus \{v_i, v_j\}$
And since $\frac{n-1}{2} + \frac{n-1}{2} > n-2$, by pigeonhole principle, v_i and v_j must be adjacent to atleast one common vertex.

$\therefore \exists\, v_k \in V(G)$ such that $v_i \sim v_k \sim v_j$

$\therefore d(v_i, v_j) = 2$

$\therefore d(v_i, v_j) \neq 1 \Rightarrow d(v_i, v_j) = 2 \quad \forall$ distinct v_i, v_j

$\therefore \operatorname{diam}(G) \leq 2$ $\blacksquare$

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Q2b) G and $\bar{G}$ are both k regular of order n

If $\deg_G v = k$ then $\deg_{\bar{G}} v = n-1-k$

$\therefore \quad k = n-1-k \quad \ldots\ (1)$

By handshaking lemma,

$nk = 2|E|$, an even number

If k is odd then n must be even, or $n-1$ must be odd, or $n-1-k$ is even
Which is not possible as $k = n-1-k$

$\therefore$ k is even $\Rightarrow k = 2t \quad \exists\ t \in \mathbb{Z}$

Substituting in (1) : $2t = n-1-2t$
or $n = 4t+1$ $\blacksquare$

Q2c) G and $\bar{G}$ are both k regular with order n

$\therefore k = n-1-k \Rightarrow k = \frac{n-1}{2}$

$\therefore G$ is $\frac{n-1}{2}$ regular

$\therefore \delta(G) = \frac{n-1}{2} \Rightarrow \operatorname{diam} G \leq 2$ (as proven in Q2a)

$\operatorname{diam}(G) = 0$ iff $G = K_1$, which is not possible as $n > 1$

$\operatorname{diam}(G) = 1$ iff $G = K_n \Rightarrow \bar{G}$ is 0 regular and G is $n-1$ regular. But $n-1 \neq 0 \ \forall\ n > 1$
$\therefore G$ and $\bar{G}$ have different regularity.
or $\operatorname{diam}(G) \neq 1$

$\therefore \operatorname{diam}(G) = 2$ $\blacksquare$

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Q3) Induction Hypotheses:

A^n_{ij} is the number of walks of length n from V_i to V_j $\forall$ $n \geq 1$

Base case: $(n=1)$

$A^1_{ij} = A_{ij} = 1$ iff $V_i \sim V_j$ and therefore A_{ij} denotes number of walks of length one from V_i to V_j

Let Induction hypothesis be true for some k

$$A^{k+1}_{ij} = \sum_{p=1}^{n} A^k_{ip} A_{pj}$$

A^k_{ip} denotes the number of k-length walks from V_i to V_p, and $A_{pj} = 1$ iff $V_p \sim V_j$

$\therefore$ $A^k_{ip} A_{pj}$ is the number of walks of length $k+1$ ~~to~~ from V_i to V_j with second last vertex V_p.

Varying p from 1 to n, we will have counted all walks of length $k+1$ from V_i to V_j

$\therefore$ $A^{k+1}_{ij} = \sum_{p=1}^{n} A^k_{ip} A_{pj}$ = number of $k+1$-length walks from V_i to V_j.

Therefore, by Induction, A^n_{ij} is the number of walks of length n from V_i to V_j.

$\blacksquare$

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$\therefore A^4_{ii}$ is the number of walks of length 4 from v_i to v_i

Each cycle of length 4 containing v_i contains 2 walks from v_i to v_i (As walks are ordered) ..... (1)

For non cyclic walks of length 4, let the walk be $(v_i, u_1, u_2, u_3, v_i)$

If $u_1 \neq u_3$

Then $u_2 = v_i$. Else if $u_2 \neq v_i$ then the vertices v_i, u_1, u_2, u_3 are distinct and hence the walk is a cycle.

(Since u_2 can not be u_1 or u_3 as $u_1 \sim u_2 \sim u_3$)

$\therefore$ The walk is $(v_i, u_1, v_i, u_3, v_i)$

Number of ways of choosing u_1 and $u_3 = \binom{\deg v_i}{2}$ as u_1 and u_3 must be adjacent to v_i

$\therefore$ Total number of such walks $= 2\binom{\deg v_i}{2}$ as walks are ordered. ...... (2)

If $u_1 = u_3$

Then the walk is $(v_i, u_1, u_2, u_1, v_i)$.

Total number of such walks containing u_1 is $\deg u_1$ as u_2 may be any vertex adjacent to u_1.

But u_1 may be ~~adja~~ any vertex adjacent to v_i

$\therefore$ total number of such walks $= \sum_{j : v_j \sim v_i} \deg v_j$ ...... (3)

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combining statements (1), (2) and (3)

$$A^4_{ii} = 2C_4(v_i) + \sum_{j:\, v_j \sim v_i} \deg v_j + 2\binom{\deg v_i}{2}$$

$$\Rightarrow \quad C_4(v_i) = \frac{1}{2}A^4_{ii} - \frac{1}{2}\sum_{j:\, v_j\sim v_i} \deg v_j - \binom{\deg v_i}{2}$$

$$\therefore \quad \sum_{i=1}^{n} C_4(v_i) = \frac{1}{2}\sum_{i=1}^{n} A^4_{ii} - \frac{1}{2}\sum_{i}\sum_{\cancel{j:v_j\sim v_i}} \deg v_j - \sum_{i=1}^{n}\binom{\deg v_i}{2}$$

$$\sum_{i=1}^{n} C_4(v_i) = \frac{1}{2}\operatorname{tr} A^4 - \frac{1}{2}\sum_{i=1}^{n}\sum_{j:v_i\sim v_j} \deg v_j - \sum_{i=1}^{n}\frac{(\deg v_i)^2}{2} + \sum_{i=1}^{n}\frac{\deg v_i}{2}$$

But $\frac{1}{2}\sum \deg v_i = |E(G)|$

$$\therefore \quad \sum_{i=1}^{n} C_4(v_i) = \frac{1}{2}\operatorname{tr} A^4 - \frac{1}{2}\sum_{i=1}^{n}\sum_{j:v_i\sim v_j} \deg v_j - \frac{1}{2}\sum_{i=1}^{n}\frac{(\deg v_i)^2}{2} + |E(G)|$$

Note that $\sum_{i=1}^{n}\sum_{j:v_i\sim v_j} \deg v_j$ counts the degree of v_j for every vertex adjacent to it. But there are $\deg v_j$ vertices adjacent to v_j

$$\therefore \quad \sum_{i}\sum_{j:v_i\sim v_j} \deg v_j = \sum_{i=1}^{n} (\deg v_i)^2$$

Also, $\sum_{i=1}^{n} C_4(v_i)$ counts number of cyclic graphs of length 4 four times, once for each vertex.

$$\therefore \quad 4C_4(G) = \frac{1}{2}\operatorname{tr} A^4 - \frac{1}{2}\sum_{i=1}^{n}(\deg v_i)^2 - \frac{1}{2}\sum_{i=1}^{n}(\deg v_i)^2 + |E(G)|$$

or $$c_4(G) = \frac{1}{8} \operatorname{tr} A^4 - \frac{1}{4} \sum_{i=1}^{n} (\deg v_i)^2 + \frac{|E(G)|}{4}$$

$\blacksquare$

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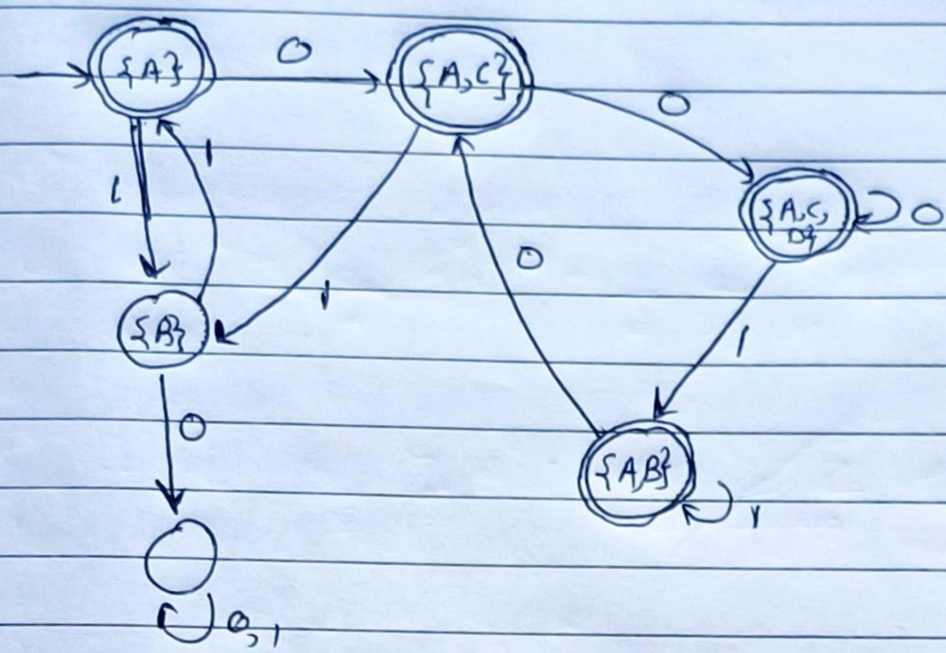
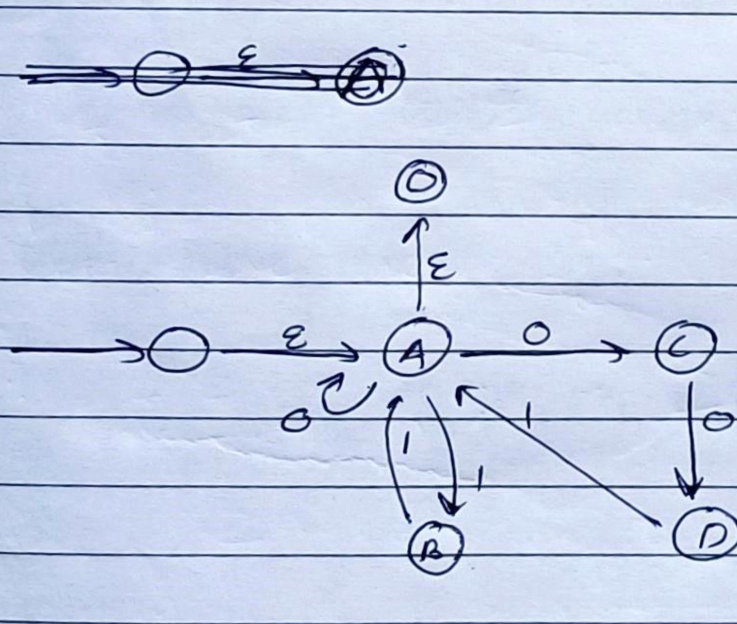
Q1)

a) 1 is not accepted by M

b)



c) Converting given NFA so that there is no transition into the start state and no transition out of the accept state:



Eliminating C :

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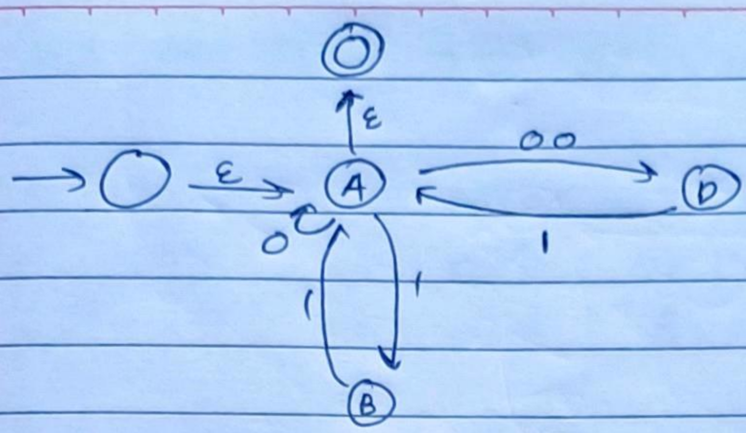
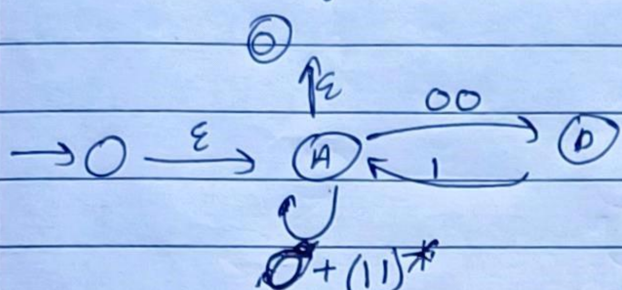
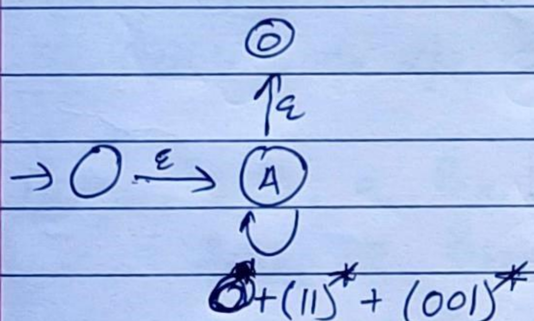


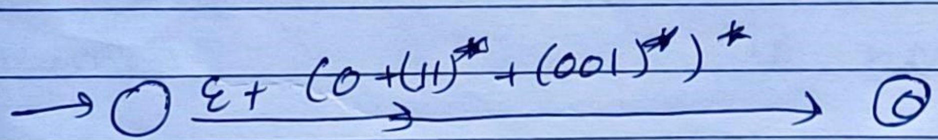
Elimaniting B



Eliminating D



Eliminating A:



$\therefore$ Regular Expression = $\boxed{\varepsilon + (0 + (11)^* + (001)^*)^*}$

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Let AB be an adjacency matrix.

Q4)

a) Let G and H have a common edge
or $v_p \sim_G v_k$ and $v_k \sim_H v_p$
or $A_{pk} = 1$ and $B_{kp} = 1$

Then $AB_{pp} = \sum_{j=1}^{n} A_{pj} B_{jp} \neq 0$ as $A_{pk} B_{kp} = 1$ and all entries of A and B are non negetive.

Which is not possible as AB is an adjacency matrix, hence diagonal elements must be 0.

$\therefore$ AB is an adjacency matrix $\Rightarrow$ G and H have no common edges. ∎

b) Let there exist atleast 2 vertices $v_{k_1}, v_{k_2} \in V(G)$ such that
$v_i \sim_G v_{k_1} \sim_H v_j$ and $v_i \sim_G v_{k_2} \sim_H v_j$

Then $A_{ik_1} = B_{k_1 j} = 1$ and
$A_{ik_2} = B_{k_2 j} = 1$

$\therefore AB_{ij} = \sum_{p=1}^{n} A_{ip} B_{pj} > 1$

as $A_{ik_1} B_{k_1 j} + A_{ik_2} B_{k_2 j} = 2$, and all elements in A and B are non negetive.

This is not possible as AB is an adjacency matrix whose elements may be at most 1.

$\therefore$ AB is an adjacency matrix $\Rightarrow \exists$ at most one v_k such that $v_i \sim_G v_k \sim_H v_j$

∎

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c) Let $v_i \sim_G v_k \sim_H v_j$

Then $A_{ik} = A_{ki} = 1$ and
$B_{kj} = B_{jk} = 1$

$\therefore \; AB_{ij} = \sum_r A_{ir} B_{rj} = 1$, as $A_{ik} B_{kj} = 1$

and all elements of AB are non negetive, and at masst one.

But AB is symmetric $\Rightarrow AB_{ij} = AB_{ji}$

$\therefore \; AB_{ji} = 1 = \sum_r A_{jr} B_{ri}$

$\therefore \; \exists \; p$ such that $A_{jp} = A_{pj} = 1$ and
$B_{pi} = B_{ip} = 1$

$\therefore \; v_i \sim_H v_p \sim_G v_j$

To prove the converse,
Let AB be a matrix that satisfies the three conditions.

Then :

(i) diagonal elements of AB are 0 (by part(a))

(ii) All elements are at most 1 (by part(b))

(iii) All elements are non-negetive (As elements of A and be are non negetive)

(iv) AB is symmetric (by part (c))

$\therefore$ AB is an adjacency matrix

∎