DS & AI
CS & IT

Probability & Statistics

Lecture No. 12

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

① BINOMIAL DISTRIBUTION

② POISSON DISTRIBUTION.

Topics to be Covered

Topic

Continuous Random Variable

(C.R.V)

TAGDAA Question →

Q. wireless sets are manufactured with $\boxed{25}$ soldier joints each. On an Average one joint in 500 is defective. Then find the Number of wireless sets to be free from defective joints in a consignment of 10000 sets?

(a) 488

(b) 9512

(c) 500

(d) 9500

$N = 10000$ sets, For single w-set

$X = \{\text{Number of } \underbrace{\text{Defective joints}}_{\text{success.}} \text{ in one w-set}\}$

$p = P(\text{Def. joint}) = \frac{1}{500}$, $n = 25$ joints, $\lambda = np = 25\left(\frac{1}{500}\right) = \frac{1}{20}$

is Average No. of Def. joints in one set $(\lambda) = \frac{1}{20}$

P(this single w-set is free from Def. joint) $= P(X=0) = \frac{e^{-\lambda}\cdot\lambda^0}{0!} = e^{-\frac{1}{20}} = 0.9512$

i.e. P(" " " " ") $= \frac{0.9512}{1} = \frac{9512}{10000}$

so No. of w-sets to be free from Def. joints = 9512 w-sets.

Note: out of 500 joints, Average No. of Def joints = 1

so " " 1 joint, Av " " " " $= \frac{1}{500}$

& " " 25 joints, Av " " " " $= \frac{1}{500}\times 25 = \frac{1}{20}$ per set.

$= \lambda$

ANALYSIS: Total joints in 10000 W-sets = 10000 × 25 = 250000 joints

Total No. of Def joints in 10000 W-sets $= \frac{250000}{500} = 500$ def. joints

these 500 def joints are Randomly distributed in 10000 − 9512 = 488 W-sets

(a) 488

(b) 9512

(c) 500

(d) 9500

Total W-sets = 10000 sets

No. of W-sets having No. Def. joints = 9512 sets

So No. of W-sets having Def. joints = 10000 − 9512 = 488 sets

(*) Find the Max No. of W-sets containing at least 2 def. joints? = 12

i.e 12 W-sets have exactly 2 Def joints each.

(*) Min No. of W-sets Containing at least 2 Def joints = ? one (& this contains 13 D. joints)

Reccurrence Relation : The Relationship b/n $P(X=r+1)$ & $P(X=r)$ is called R.R.

R.R of Poisson Distn

W.K. that $P(X=r) = \dfrac{e^{-\lambda}\cdot\lambda^{r}}{r!}$ & $P(X=r+1) = \dfrac{e^{-\lambda}\cdot\lambda^{r+1}}{(r+1)!}$

Now, $$\frac{P(X=r+1)}{P(X=r)} = \frac{\dfrac{e^{-\lambda}\cdot\lambda^{r+1}}{(r+1)!}}{\dfrac{e^{-\lambda}\cdot\lambda^{r}}{r!}} = \frac{r!}{(r+1)!}\cdot\frac{\lambda^{r}\cdot\lambda}{\lambda^{r}} = \frac{r!}{(r+1)r!}\cdot\lambda = \frac{\lambda}{r+1}$$

ie $$\boxed{P(X=r+1) = \left(\frac{\lambda}{r+1}\right)P(X=r)}$$

eg $P(X=1) = \lambda\cdot P(X=0)$

$P(X=2) = \dfrac{\lambda}{2}P(X=1)$

$P(X=3) = \dfrac{\lambda}{3}P(X=2)$ & so on . . . . .

If X is a discrete random variable that follows Binomial distribution, then which one of the following ~~response~~ relations is correct?

Reccurrence

(a) $P(r+1) = \frac{n-r}{r+1}P(r)$

(b) $P(r+1) = \frac{p}{q}P(r)$

(c) $P(r+1) = \frac{n+r}{r+1}\frac{p}{q}P(r)$

(d) $P(r+1) = \frac{n-r}{r+1}\frac{p}{q}P(r)$

$$P(X=r) = {}^nC_r\, p^r q^{n-r}$$

$$P(X=r+1) = {}^nC_{r+1}\, p^{r+1} q^{n-(r+1)}$$

Let us Calculate

$$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{\frac{n!}{(r+1)!(n-r-1)!}}{\frac{n!}{r!(n-r)!}}$$

$$= \frac{r!(n-r)!}{(r+1)!(n-r-1)!} = \frac{n-r}{r+1}$$

$$\frac{P(X=r+1)}{P(X=r)} = \frac{{}^{n}C_{r+1}\, p^{r+1} q^{n-r-1}}{{}^{n}C_{r}\, p^{r} q^{n-r}} = \left(\frac{n-r}{r+1}\right)\cdot\frac{p}{q}$$

Hence R.R of B. Dist is

$$\boxed{P(X=r+1) = \left(\frac{n-r}{r+1}\right)\cdot\frac{p}{q}\cdot P(X=r)}$$

BASICS of Integration

Even func^n → If $\boxed{f(-x) = f(x)}$ then $f(x)$ is called an Even function.

& graph of an Even func^n is symmetrical about Y axis i.e Y axis will behave like a mirror.

Odd func^n → if $\boxed{f(-x) = -f(x)}$ then $f(x)$ is called an odd func^n.

& graph of an odd func^n is symmetrical about origin i.e I ⟷ III & II ⟷ IV

Neither Even Nor odd func^n → if $f(-x) \neq f(x) \neq -f(x)$ then $f(x)$ is called NENO func^n

i.e graph is Neither symmetrical about Y axis Nor about origin

Q Check the nature of the following funcⁿ: —

① $y = \boxed{f(x) = x^2}$

Even funcⁿ

$\because f(-x) = (-x)^2$

$= x^2$

$= f(x)$

② $y = \boxed{f(x) = x^3}$

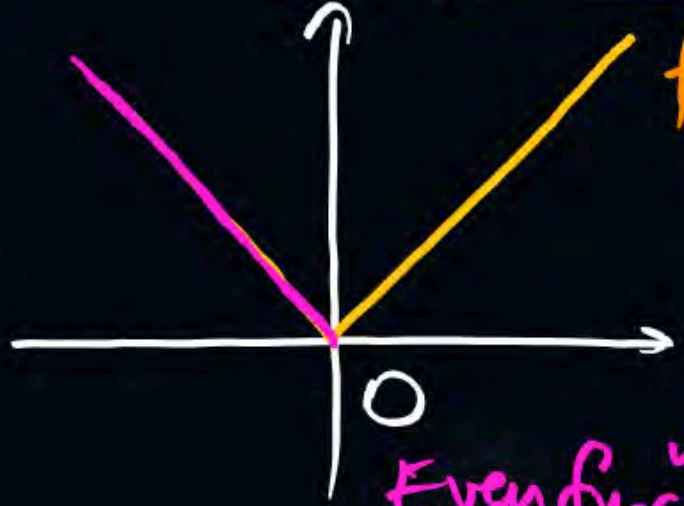
$\because f(-x) = (-x)^3$

$= -x^3$

$= -f(x)$

i.e. odd funⁿ

③ $y = \boxed{f(x) = |x|}$



$f(-x) = |-x|$

$= |x|$

$= f(x)$

Even funcⁿ.

④ $y = \boxed{f(x) = \sin x}$

odd funcⁿ.

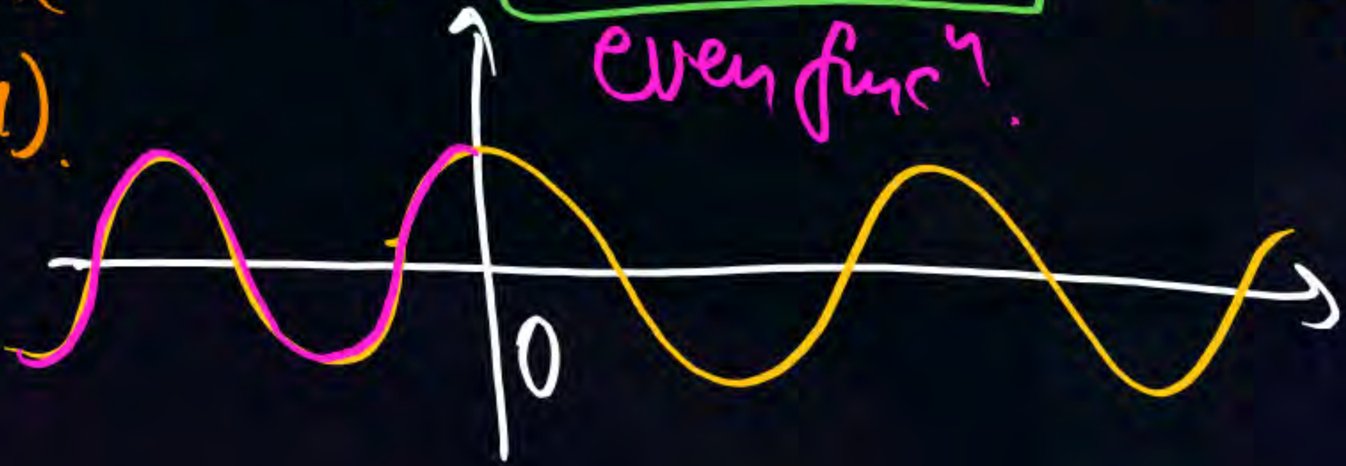
$\because f(-x) = \sin(-x)$

$= -\sin x$

$= -f(x)$.

⑤ $y = \boxed{f(x) = \cos x}$

even funcⁿ.

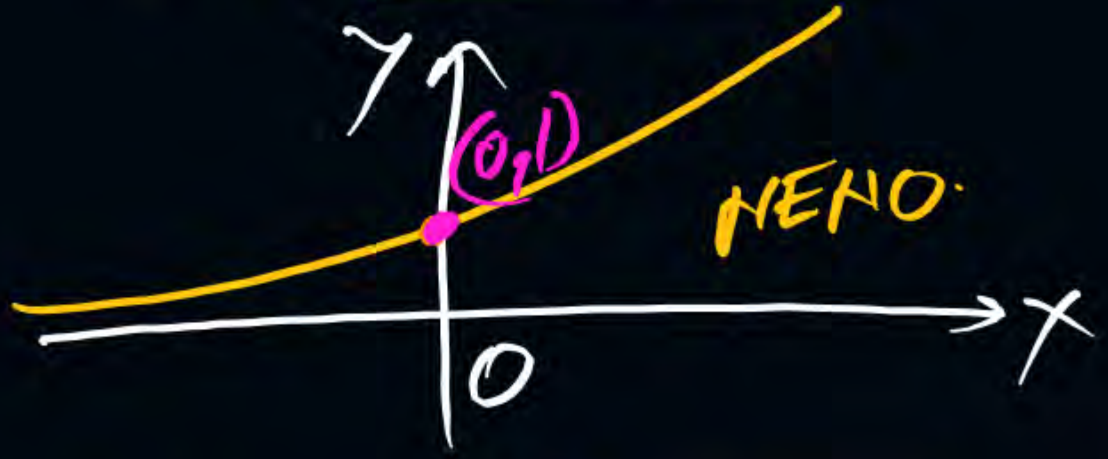


$\because f(-x) = \cos(-x)$

$= \cos(x)$

$= f(x)$

⑥ $y = \boxed{f(x) = e^{x}}$



$\because f(-x) = e^{-x} = \frac{1}{e^{x}} = \frac{1}{f(x)}$

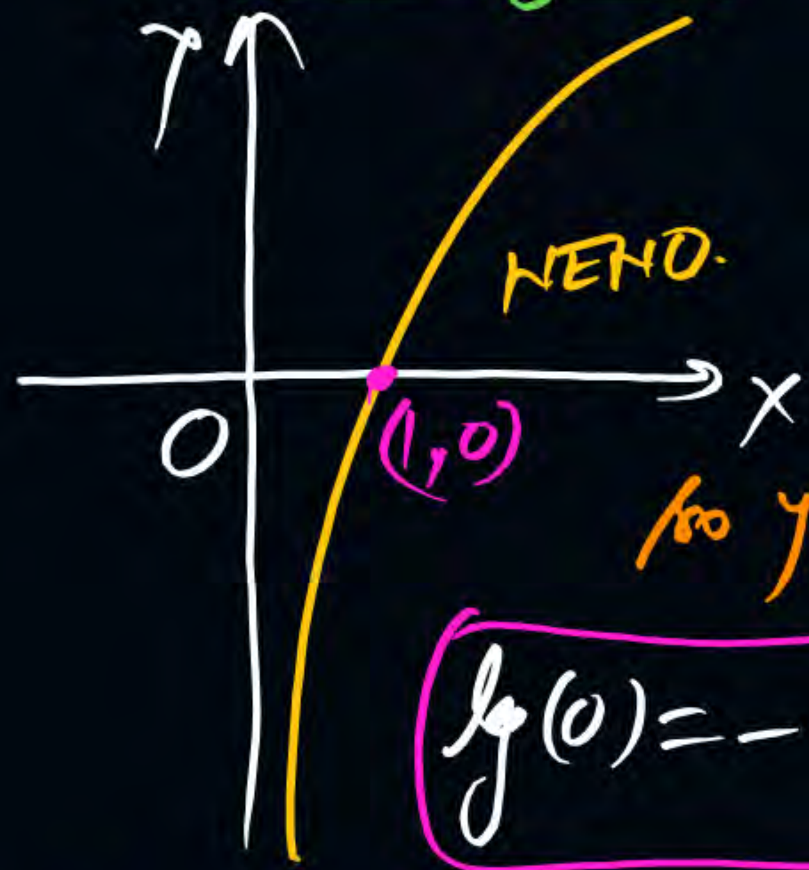
$\because f(-x) \neq f(x)$

$\neq -f(x)$

so $f(x)$ is NENO func^n

$$\boxed{e^{-\infty} = 0, \quad e^{0} = 1, \quad e^{\infty} = \infty}$$

⑦ $y = \boxed{f(x) = \log x}$



$\because f(-x) = \log(-x)$

$= N.D$

$\because f(-x) \neq f(x)$

$\neq -f(x)$

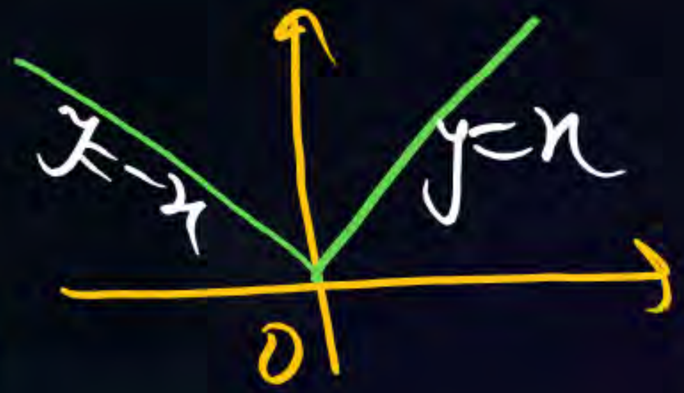
so $y = \log x$ is NENO func^n

$$\boxed{\log(0) = -\infty, \quad \log(1) = 0, \quad \log(\infty) = +\infty}$$

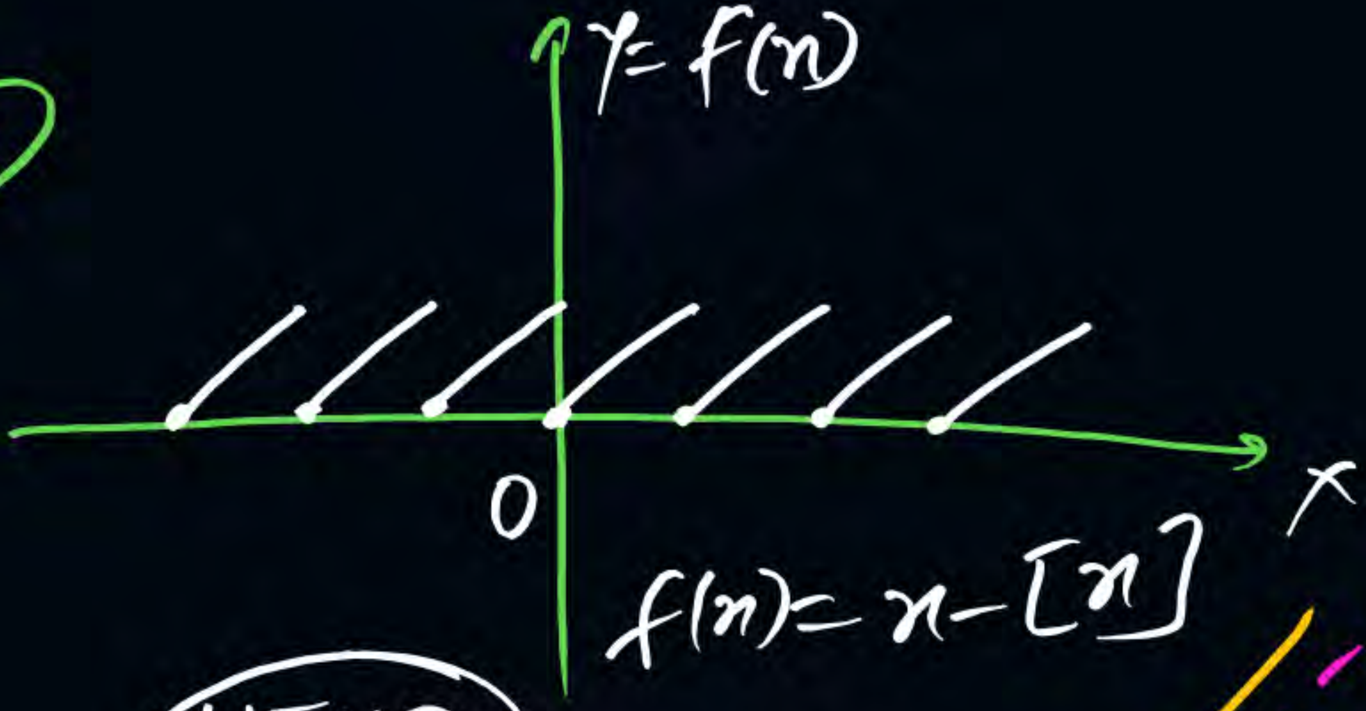
⑧ <u>Mod func^n :-</u>

$$y = |x| = \begin{cases} -x, & x < 0 \\ +x, & x \geq 0 \end{cases}$$

$|-3| = -(-3) = +3$

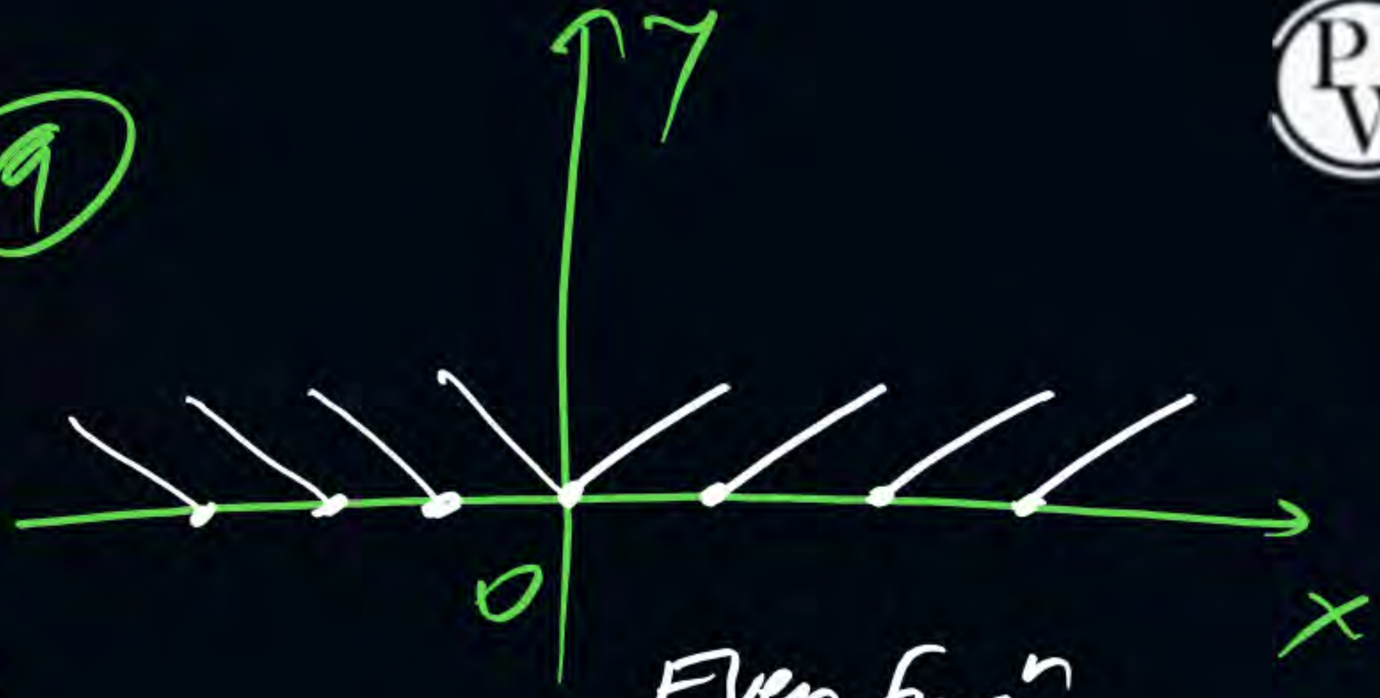


(8)



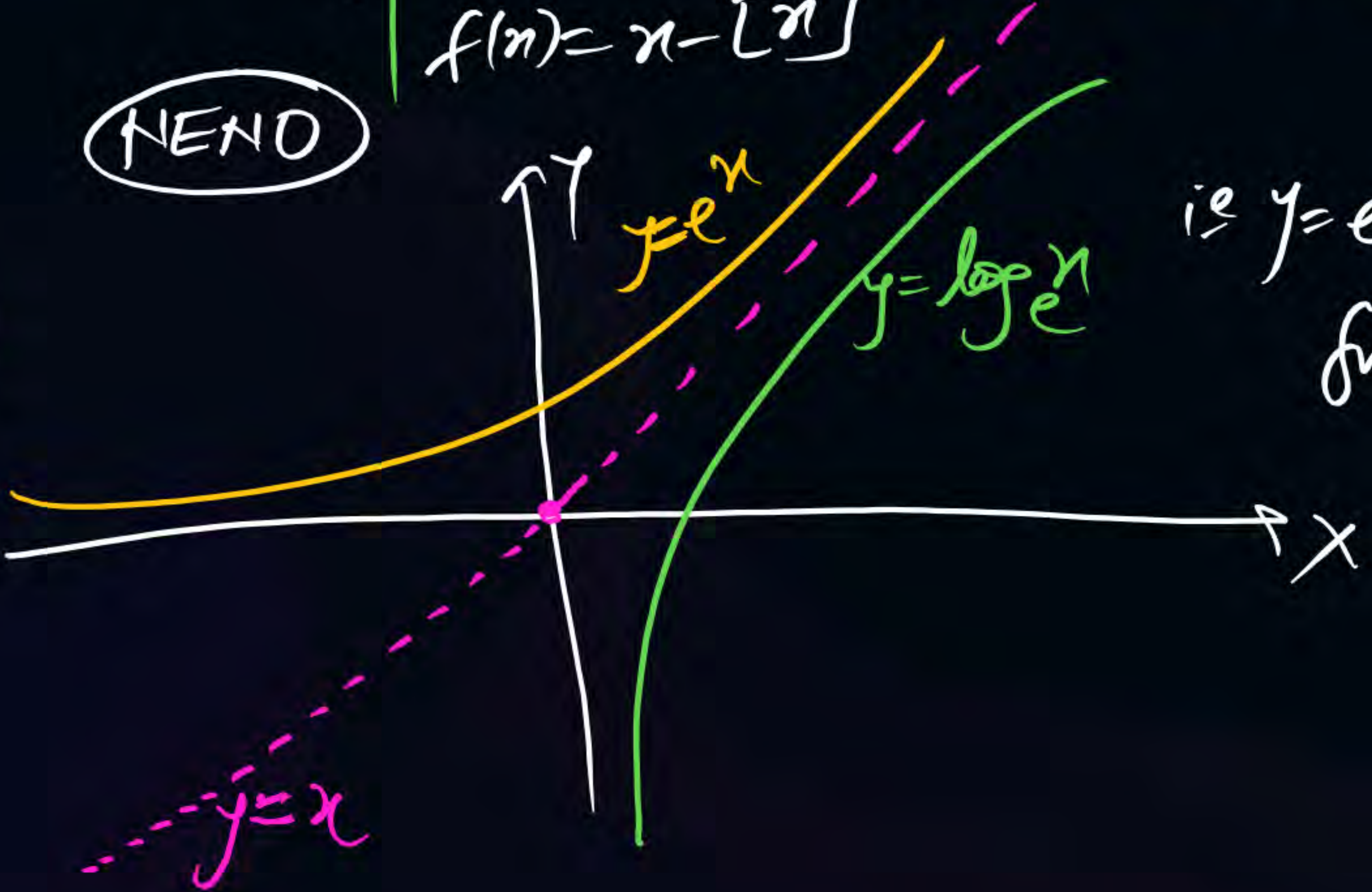
NENO

(9)



Even funcⁿ

(10)



i.e. $y = e^x$ & $y = \log x$ are inverse funcⁿs of each other.

Special formula of Integration :-

$$\int_{-\alpha}^{\alpha} f(x)\,dx = \begin{cases} 2\int_{0}^{\alpha} f(x)\,dx & , \text{ if } f(x) \text{ is an Even func}^n \\ 0 & , \text{ if } f(x) \text{ is an odd func}^n. \end{cases}$$

eg $\int_{-\pi/3}^{\pi/3} \sin^3 x\,dx = ? = 0$ $\left(\because f(x) = \sin^3 x \text{ is an odd func}^n\right)$

$\because f(-x) = \sin^3(-x) = (-\sin x)^3 = -\sin^3 x = -f(x)$

eg $\int_{-\pi/4}^{\pi/4} \log\left(\frac{2+\sin x}{2-\sin x}\right) dx = ? = 0$ $\because f(x) = \log\left(\frac{2+\sin x}{2-\sin x}\right)$ is also an odd func^n

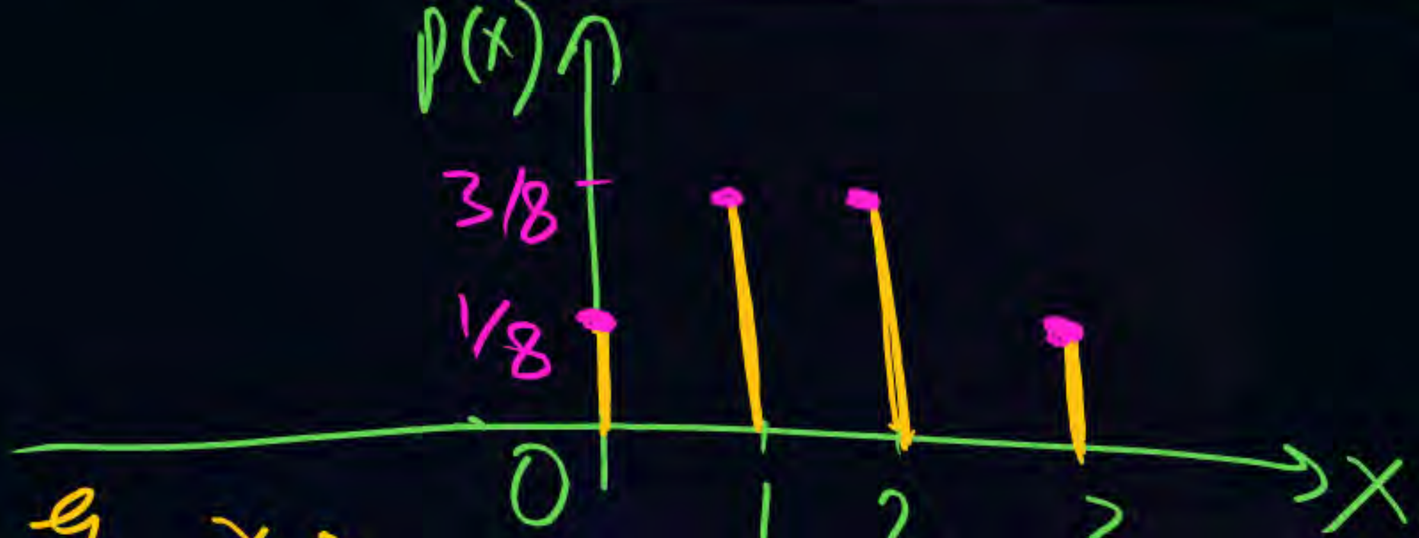
$\because f(-x) = \log\left(\frac{2+\sin(-x)}{2-\sin(-x)}\right) = \log\left(\frac{2-\sin x}{2+\sin x}\right) = \log\left(\frac{2+\sin x}{2-\sin x}\right)^{-1} = -f(x)$

Rough work

$X =$ D.R.V.

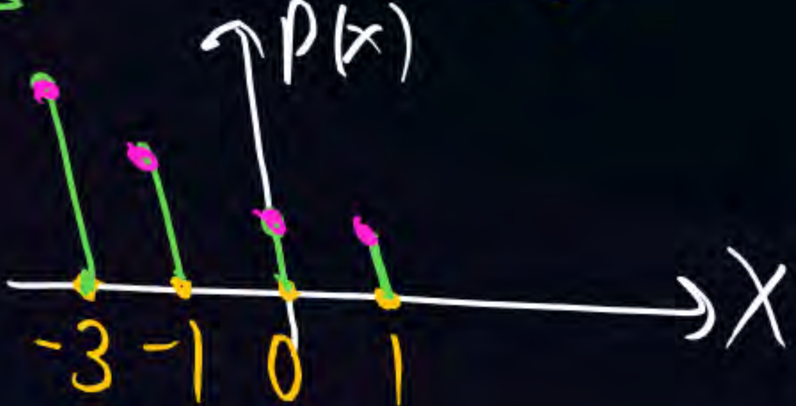
eg

X:	0	1	2	3
$P(X)$:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$



eg

X:	1	0	-1	-3
$P(X)$:	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{25}{216}$	$\frac{125}{216}$



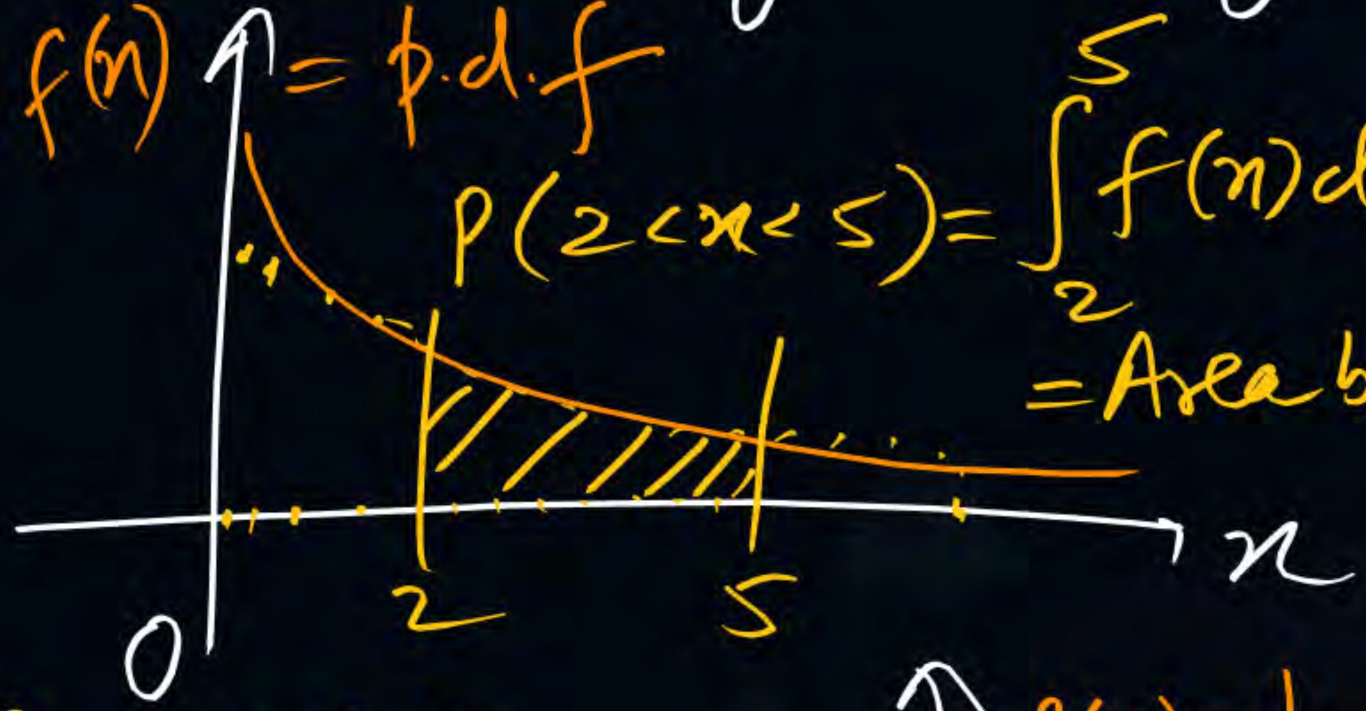
$x =$ C.R.V.

eg $x =$ waiting time

eg $f(x)$ = p.d.f

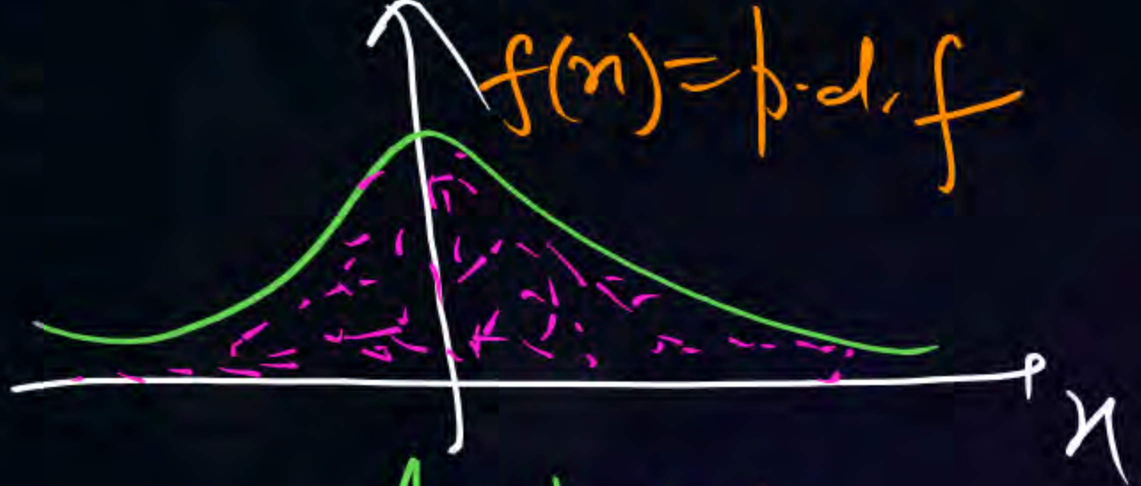
$P(2 < x < 5) = \int_2^5 f(x)\,dx$

= Area b/w 2 & 5



eg $x =$ weight

$f(x)$ = p.d.f



Av wt = 65 kg

Rough work.

- If $X = $ DRV then $P(-\infty < X < \infty) = \sum p_i = 1$.
- if $x = $ C.R.V then $P(-\infty < x < \infty) = \int_{-\infty}^{\infty} f(x)\,dx = 1$.

i.e. Total area under $f(x) = 1$ unit.

- so $P(a < x < b) = ? = \int_a^b f(x)\,dx = $ Area under $f(x)$ b/w a & b.
- In DRV: $E(X) = \sum p_i x_i$, $E(x^2) = \sum p_i x_i^2$

In C.R.V: $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$, $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$ & so on ......

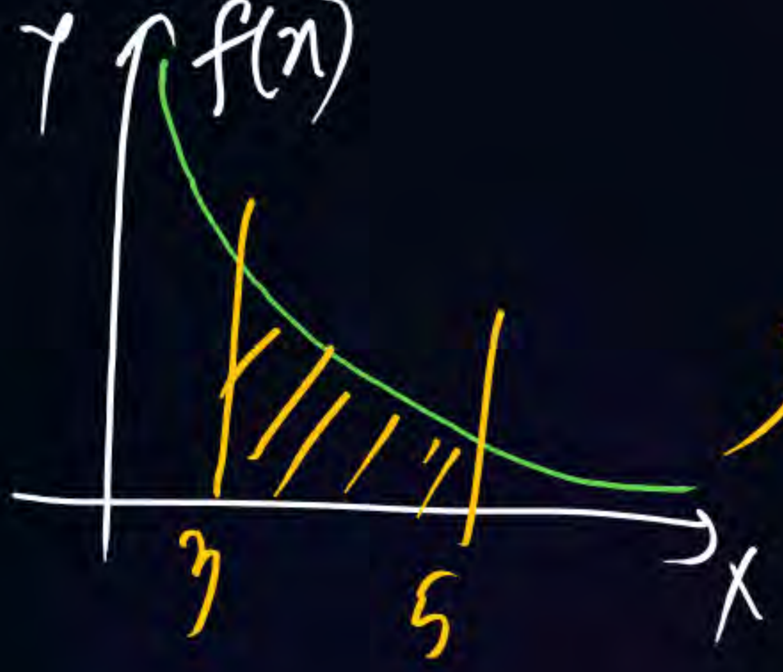
Continuous Random Variable (C.R.V = x).

If a Random Variable x has ∞ possibilities in a given Range then it is Called C.R.V.

for eg Height, weight, time etc.

Let x is C.R.V and it's prob Density func^n is $f(x)$ then

for eg, $P(3<x<5) = ? = \int_3^5 f(x)\,dx$ = shaded area



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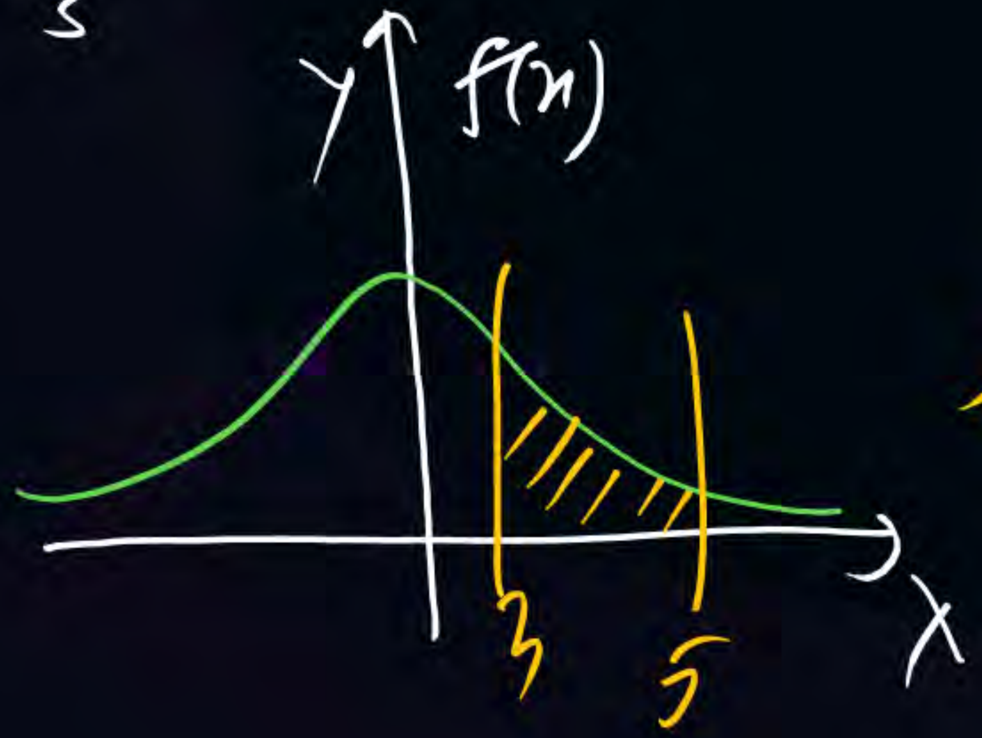

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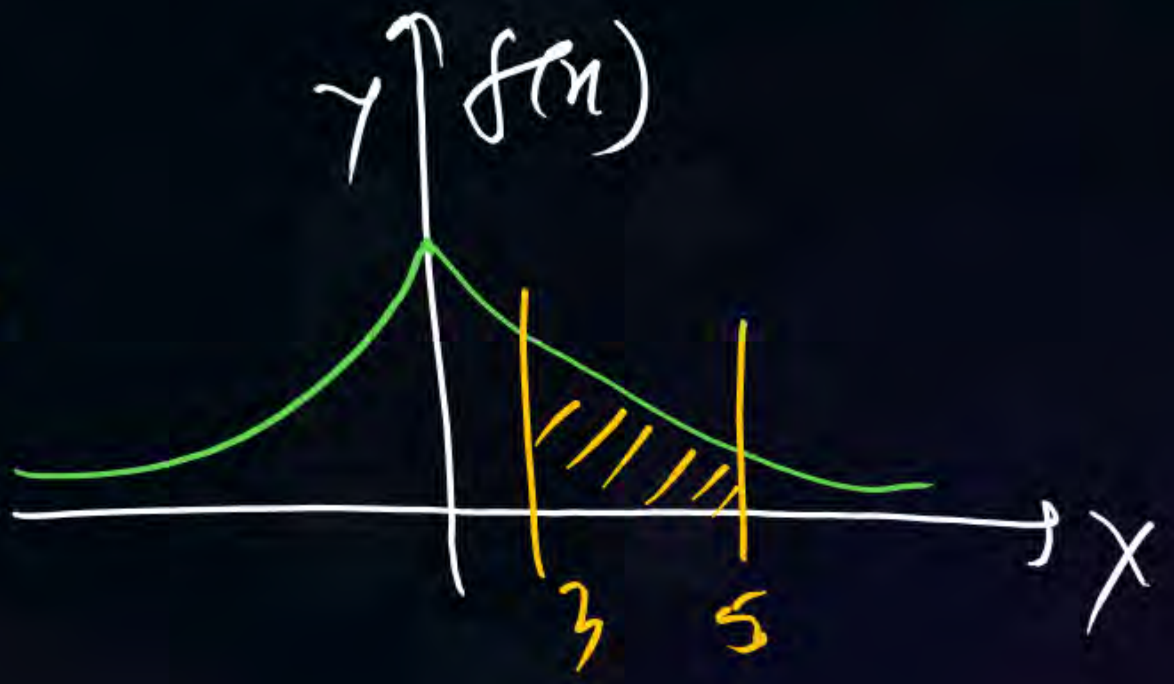


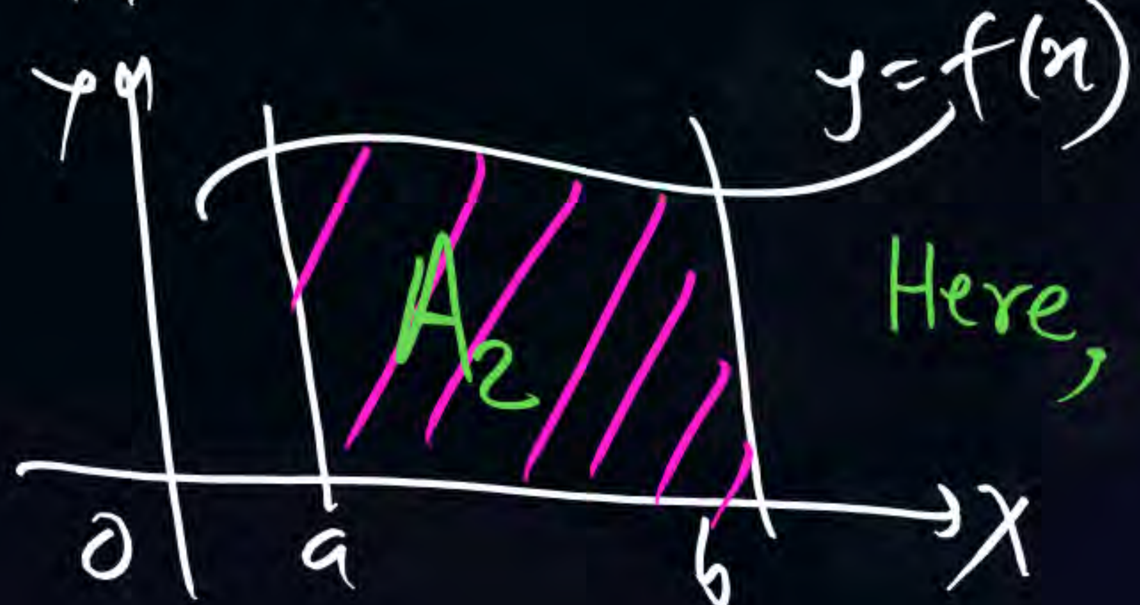
INFO Let x is C.R.V, & $f(x)$ is its p.d.f then we have following Results;

① $\boxed{f(x) \geq 0}$ i.e Graph of p.d.f always lies above x axis.

② $P(-\infty < x < \infty) = \boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$ = Total area under $f(x)$ with x axis

③ $\boxed{P(a < x < b) = \int_{a}^{b} f(x)\,dx}$ = Area under $f(x)$ b/w $x=a$ & $x=b$

or $P(a \leq x \leq b) = \int_{a}^{b} f(x)\,dx =$ " " " " " "



Here, $A_1 = A_2$

INFO

(5) Mean(x) = $\boxed{E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx}$

2nd Moment $= E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

3rd Moment $= E(x^3) = \int_{-\infty}^{\infty} x^3 f(x)\,dx$

& similarly $E(|x|) = \int_{-\infty}^{\infty} |x| f(x)\,dx$

In all these Results, $f(x)$ is the p.d.f of x only

Now in general,
If $f(x)$ is the p.d.f of x and $g(x)$ be another func^n of x then

$$\boxed{E\{g(x)\} = \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx}$$

(6) $\boxed{Var(x) = E(x^2) - E^2(x)}$

(7) S.D $(\sigma) = +\sqrt{Var(x)}$

$\underline{\text{Q}}$ if $\boxed{f(x) = Ke^{-\alpha|x|}}$, $\alpha \in R^+$ is $\underbrace{\text{density func}^n}_{\text{p.d.f}}$ of x then find $K = ?$

$= \text{Even fun}^c.$

$\underline{\text{Sol}}$: $\because f(x)$ is p.d.f of x so

Total area under $f(x) = 1$

ie $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$2\int_0^{\infty} f(x)\,dx = 1$$

$$2\int_0^{\infty} Ke^{-\alpha|x|}\,dx = 1$$

$$2K\int_0^{\infty} e^{-\alpha(+x)}\,dx = 1$$

$$2K\left[\frac{e^{-\alpha x}}{-\alpha}\right]_0^{\infty} = 1$$

$$-\frac{2K}{\alpha}\left[e^{-\infty} - e^{0}\right] = 1$$

$$-\frac{2K}{\alpha}[0-1] = 1 \Rightarrow K = \frac{\alpha}{2}$$

Q. If $\alpha = 2$ in previous Question then $P(-1 \le x \le 1) = ?$

Sol: $f(x) = k e^{-\alpha|x|} = \frac{\alpha}{2} e^{-\alpha|x|} = \frac{2}{2} e^{-2|x|} = e^{-2|x|}$ = Even func^n

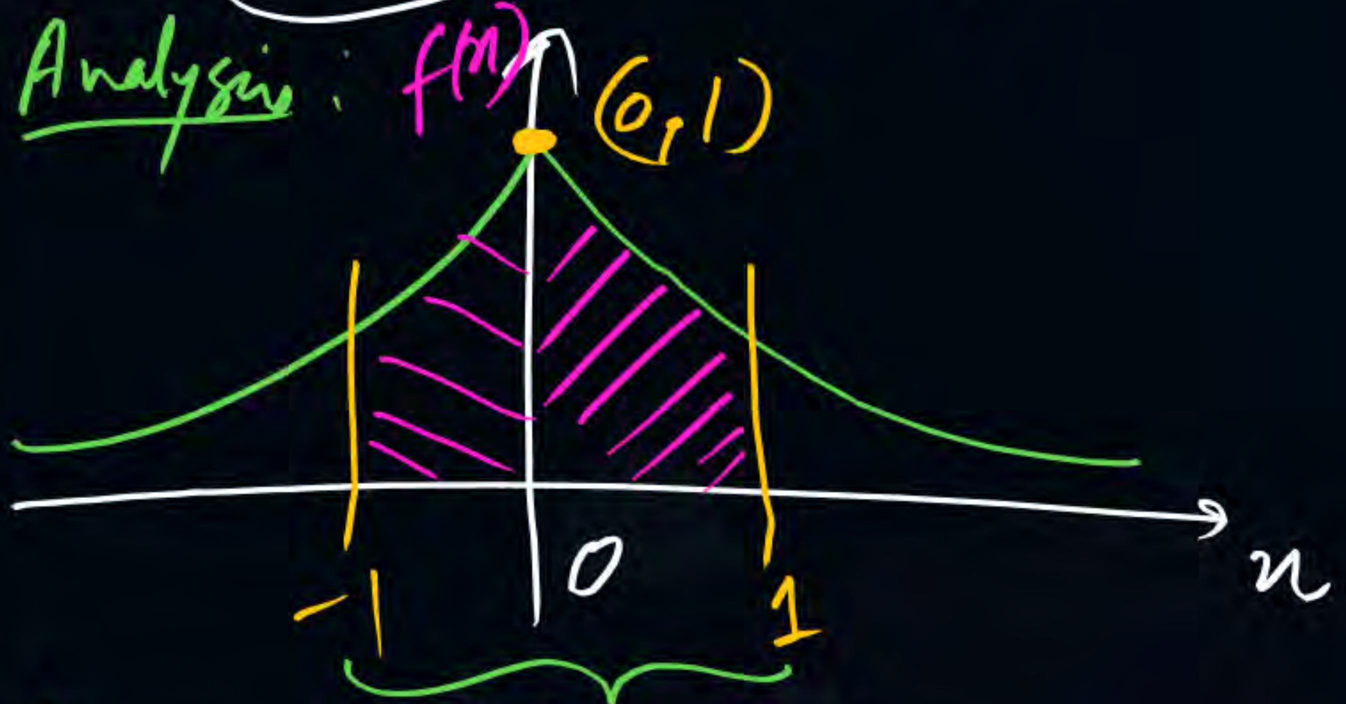
$$P(-1 \le x \le 1) = \int_{-1}^{1} f(x)\,dx = 2\int_{0}^{1} f(x)\,dx$$

$$= 2\int_{0}^{1} e^{-2|x|}\,dx = 2\int_{0}^{1} e^{-2(x)}\,dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)\Bigg|_{0}^{1} = -\left[e^{-2} - e^{0}\right]$$

$$= 1 - \frac{1}{e^2} = 0.867 = \frac{86.7}{100}$$

Analysis:

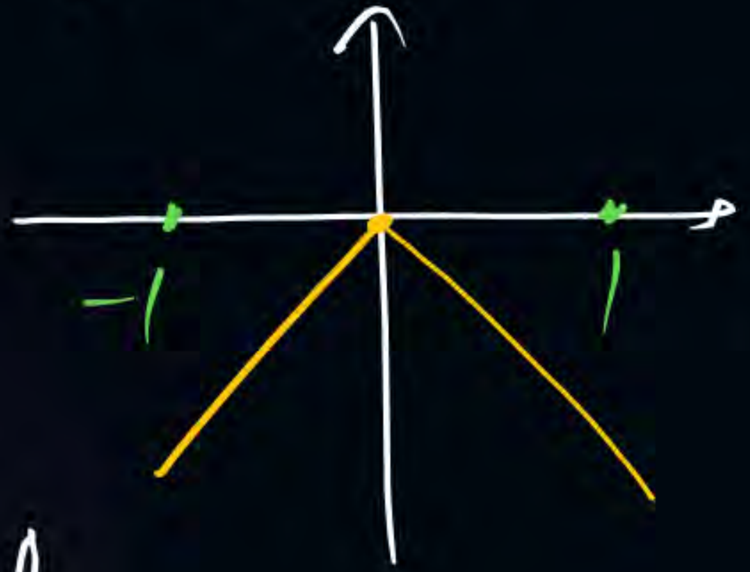


86.7% area lies in b/n $x = -1$ & $x = 1$

Q. Draw the Graph of $y = 1-|x|$, $-1 \le x \le 1$

Sol: $y = |x|$

$y = -|x|$



$y = -|x| + 1$

Verify that $f(x)$ is p.d.f ?

$\because$ Graph of $f(x)$ lies above X axis so $\boxed{f(x) \ge 0}$

$\boxed{f(x) = 1-|x|}$, $-1 \le x \le 1$

$\because$ Total area under $f(x) = \frac{1}{2} \times \text{Base} \times \text{Height} = \frac{1}{2} \times 2 \times 1 = 1$ Hence Verified

Qs (2012) If x is C.R.V s.t its p.d.f is $f(x) = \begin{cases} 1+x, & -1 \le x \le 0 \\ 1-x & 0 \le x \le 1 \end{cases}$ then find Mean, 2^{nd} Moment, Variance & S.D of x?

(a) $SD = 0$

(b) $SD = 1/6$

(c) $SD = 1/\sqrt{6}$

(d) $SD = 2$

$$f(x) = \begin{cases} 1+x, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le 1 \end{cases} = \boxed{1-|x|}, \quad -1 \le x \le 1$$

= Even func^n.

$$\text{Mean}(x) = E(x) = \int_{-\infty}^{\infty} x.f(x)\,dx = \int_{-1}^{1} \underbrace{x.f(x)}_{\text{odd}}\,dx = 0$$

$$2^{nd}\text{ Moment} = E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-1}^{1} \underbrace{x^2 f(x)}_{\text{Even}}\,dx = 2\int_0^1 x^2 f(x)\,dx$$

$$= 2\int_0^1 x^2(1-x)\,dx = \dots\dots = \frac{1}{6}$$

So $\text{Var}(x) = E(x^2) - E^2(x) = \frac{1}{6} - (0)^2 = \frac{1}{6}$

Q. If $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is p.d.f of x & $g(x) = e^{\frac{3x}{4}}$ then find Mean of $g(x)$?

(a) 1

(b) 2

(c) 1/4

(d) ~~4~~

w.k.that $\boxed{E\{g(x)\} = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx} = \int_{-\infty}^{0} g(x)(0)\,dx + \int_{0}^{\infty} g(x)(e^{-x})\,dx$

$= \int_{0}^{\infty} e^{\frac{3x}{4}} \cdot e^{-x}\, dx = \int_{0}^{\infty} e^{-\frac{x}{4}}\, dx$

$= - - - - - = 4$

Q. The variance of R.V x for which p.d.f is $f(x) = \frac{1}{2}|x|e^{-|x|}$ will be ?

HW

$= \text{Even func}^n.$

(a) 0

(b) 2

(c) 6

(d) $\sqrt{6}$

$$Var(x) = E(x^2) - E^2(x)$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\int_{-\infty}^{\infty} \underbrace{x f(x)}_{\text{odd.}}\,dx\right)^2$$

$$= 2\int_0^{\infty} x^2\left(\frac{1}{2}|x|e^{-|x|}\right)dx - (0)^2$$

$$= \int_0^{\infty} x^3 e^{-x}\,dx = ?$$

M-I

M-II

HW

Q→ if $f(x) = \begin{cases} kx+1, & 0 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$ is p.d.f for x then $k = ?$

(a) $-3/8$ (b) $8/3$ (c) $\frac{16}{3}$ (d) f(x) can't be a p.d.f

Probability density function is given as

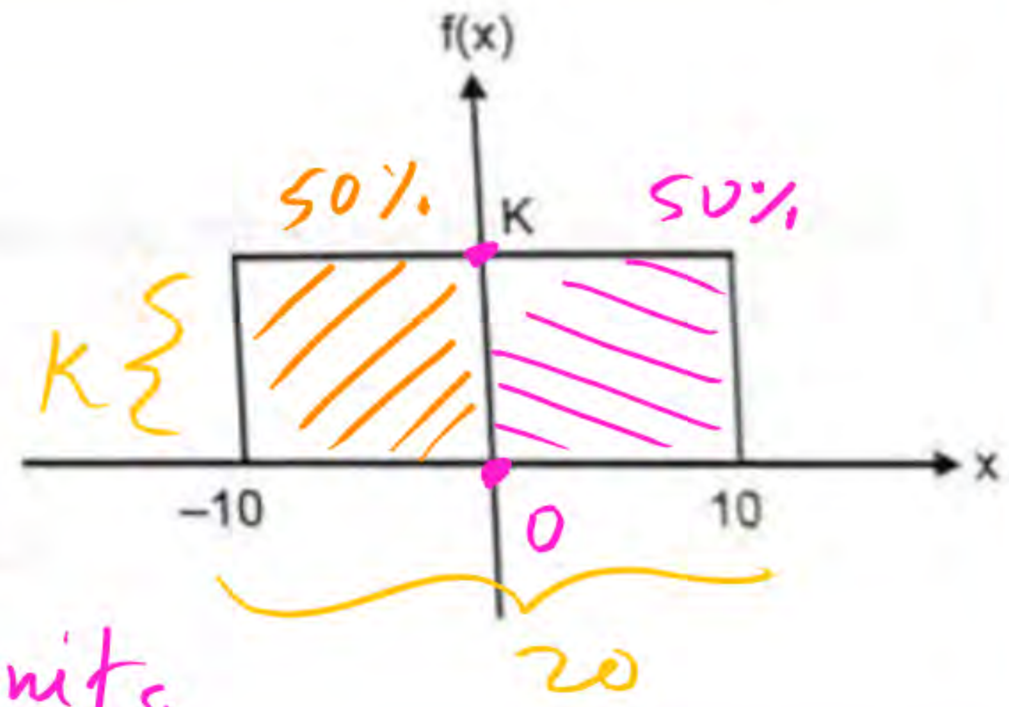
$$f(x) = \begin{cases} k & \text{for } -10 < x < 10 \\ 0, & \text{otherwise} \end{cases}$$

Calculate :

(a) k

(b) $P[-10 < x < 0]$ = left side area = 0.5

(c) $P[x > 0]$ = Right side Area = 0.5 units



w.k.that, Total area under $f(x) = 1$

Length × Height = 1

$20 \times K = 1 \Rightarrow K = \frac{1}{20}$

50% area ≈ $\frac{1}{2}$ units

so $P(-10 \le x \le 0) = \frac{1}{2}$

& $P(0 < x < 10) = \frac{1}{2}$

(iii) $P(x>0) = \int_0^{\infty} f(x)\,dx = \int_0^{10} f(x)\,dx = \int_0^{10} (k)\,dx = \frac{1}{20}(10-0) = \frac{1}{2}$

If probability density function of a random variable X is $f(x) = x^2$ for $-1 \le x \le 1$ and $f(x) = 0$ for any other values of x, then the percentage probability

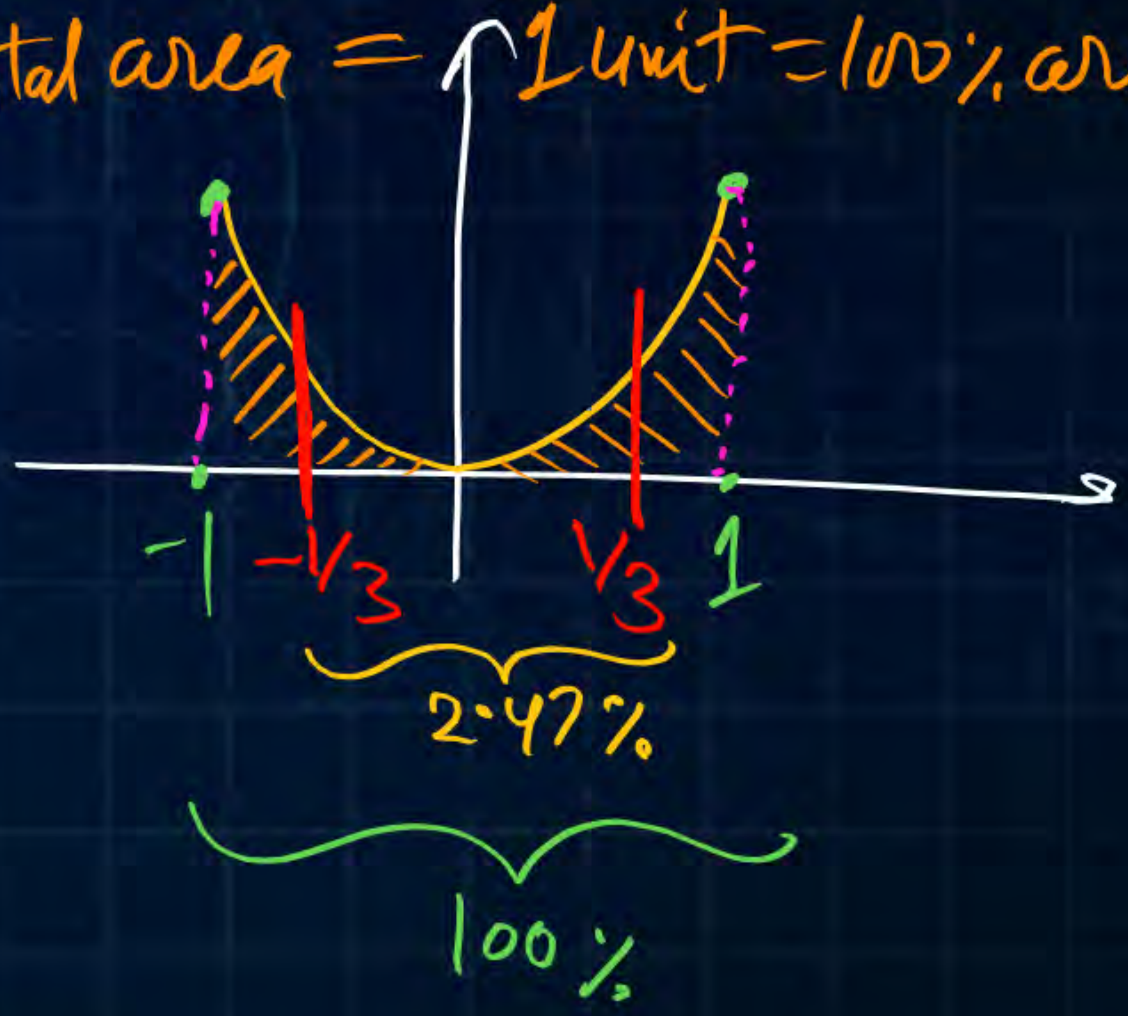
$P\left(-\frac{1}{3} \le x \le \frac{1}{3}\right)$ is

(a) 0.247 (b) 2.47

(c) 24.7 (d) 247

$$y = f(x) = \begin{cases} x^2, & -1 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Total area = 1 unit = 100% area.



$$P\left(-\frac{1}{3} \le x \le \frac{1}{3}\right) = \int_{-1/3}^{1/3} f(x)\,dx = \int_{-1/3}^{1/3} (x^2)\,dx$$

$$= 2\int_0^{1/3} (x^2)\,dx = 2\left(\frac{x^3}{3}\right)_0^{1/3} = \frac{2}{3}\left(\frac{1}{27} - 0\right) = \frac{2}{81} = 0.0247 = 2.47\%$$

Rough work

$p.d.f = f(x)$, $C.D.f = F(x)$

Cummulative Density func^n (c.d.f)

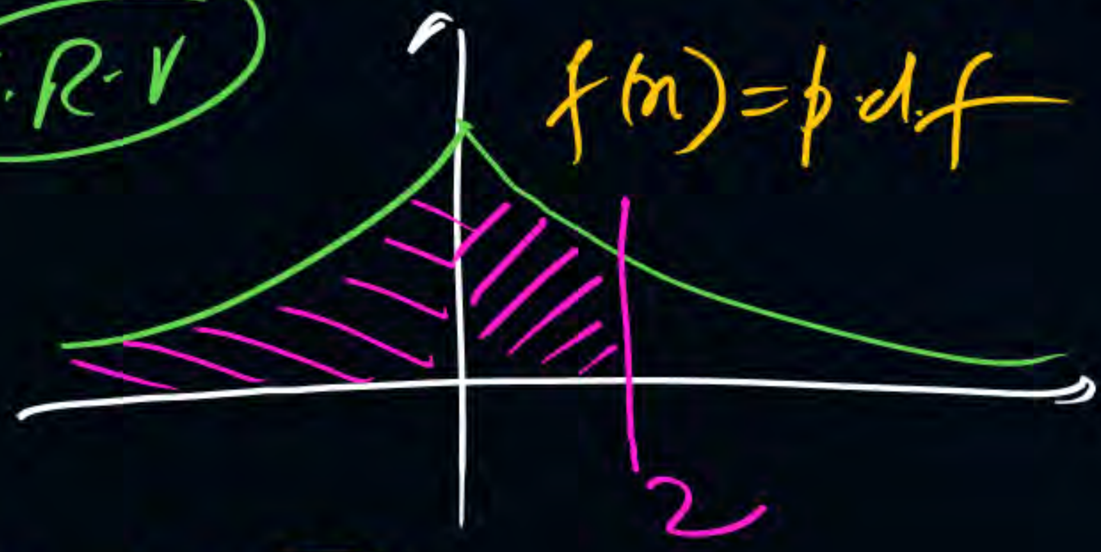
D.R.V

x :	0	1	2	3
$p(x)$:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

C.D.f at $x=2$ $= p_0 + p_1 + p_2$

C.D.f at $x=1$ $= p_0 + p_1$

C.R.V



C.D.f at $x=2$

i.e $F(2) = \int_{-\infty}^{2} f(x)\,dx$

Similarly C.D.f at $x=5$

$$F(5) = \int_{-\infty}^{5} f(x)\,dx$$

eg

Rough

p.d.f at x : $= \boxed{f(x) = F(x) - F(x^-)} = \int_{x^-}^{x} f(x)\,dx$

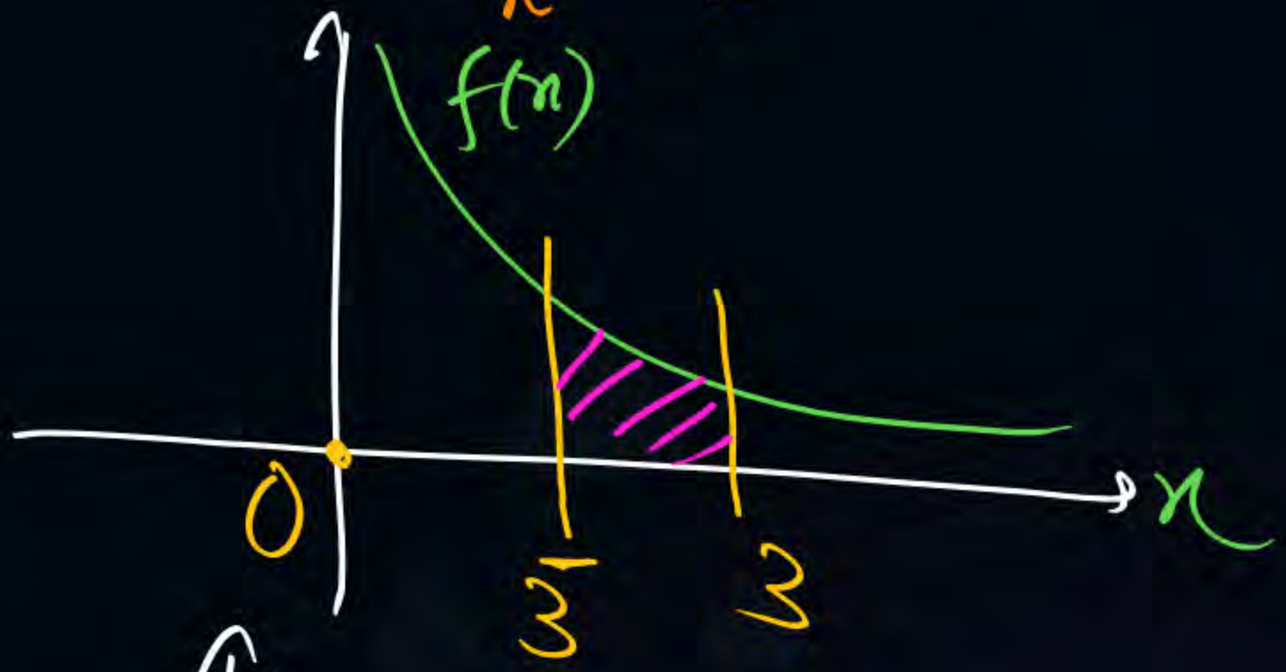
= Shaded Area

eg

p.d.f = $f(3) = ? = F(3) - F(3^-)$

$= \int_0^3 f(x)\,dx - \int_0^{3^-} f(x)\,dx$

= shaded area

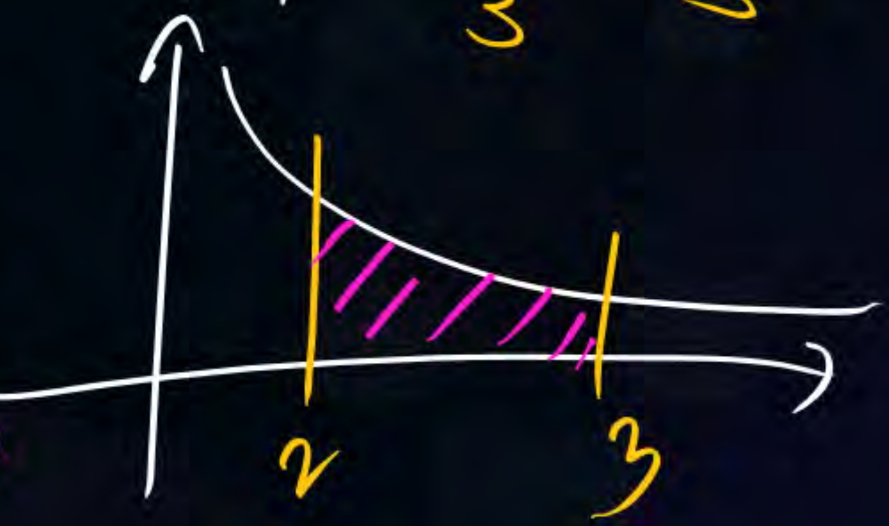


eg

$P(2 \le x \le 3)$

$= F(3) - F(2)$

$= \int_2^3 f(x)\,dx$ = shaded area



Cummulative Density func^n (c.d.f) / Distribution func^n.

Let x is C.R.V and $f(x)$ is it's p.d.f then it's c.d.f is denoted by $F(x)$ and is defined as

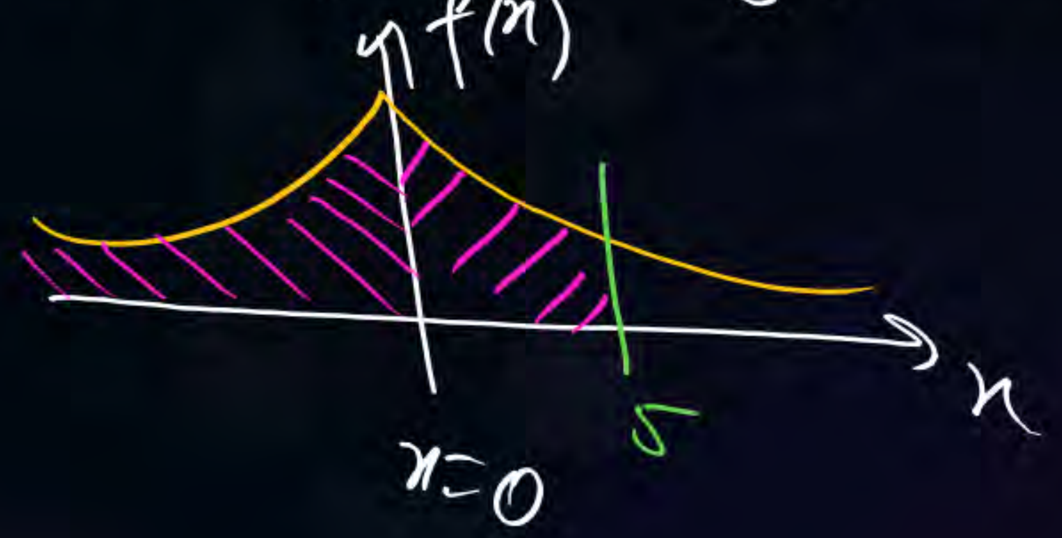
$$\boxed{F(x) = \int_{-\infty}^{x} f(x)\,dx}$$

$$F(-\infty) = \int_{-\infty}^{-\infty} f(x)\,dx = 0$$

$$F(\infty) = \int_{-\infty}^{\infty} f(x)\,dx = 1$$

for eg, $\underset{\text{(c.d.f at 5)}}{F(5)} = \int_{-\infty}^{5} f(x)\,dx$ = Sum of all the probabilities from starting point upto $x = 5$

= Shaded area.



ASLI Conclusion :-

① $f(x) = $ p.d.f of $x \neq$ Probability. & $\boxed{0 \leq f(x) < \infty}$

But $\int_{-\infty}^{\infty} f(x)\,dx = $ Total area $= 1$ unit.

② $F(x) = $ C.D.F at $x = $ Probability. & $\boxed{0 \leq F(x) \leq 1}$

$$\boxed{f(x) = F(x) - F(\bar{x})}$$

Note ① Graph of p.d.f as well as c.d.f can not lies below X axis.

② Graph of p.d.f can go from $y=0$ to $y=\infty$

③ Graph of c.d.f can go only in b/w $y=0$ & $y=1$

$$f(2) = \int_{1.999}^{2} f(x)\,dx \text{ or } F(2) - F(1.999)$$

$$P(a < x < b) = ?$$

M-I $= \int_a^b f(x)\,dx$

M-II $= \boxed{F(b) - F(a)}$

$\underset{(C.D.F)}{F(x)} = \int_{-\infty}^{x} f(x)\,dx$ is By Integrating p.d.f we can find c.d.f

& $\underset{(p.d.f)}{f(x)} = \frac{d}{dx} F(x)$ is By Differentiating c.d.f we can find p.d.f.

Thank

YOU