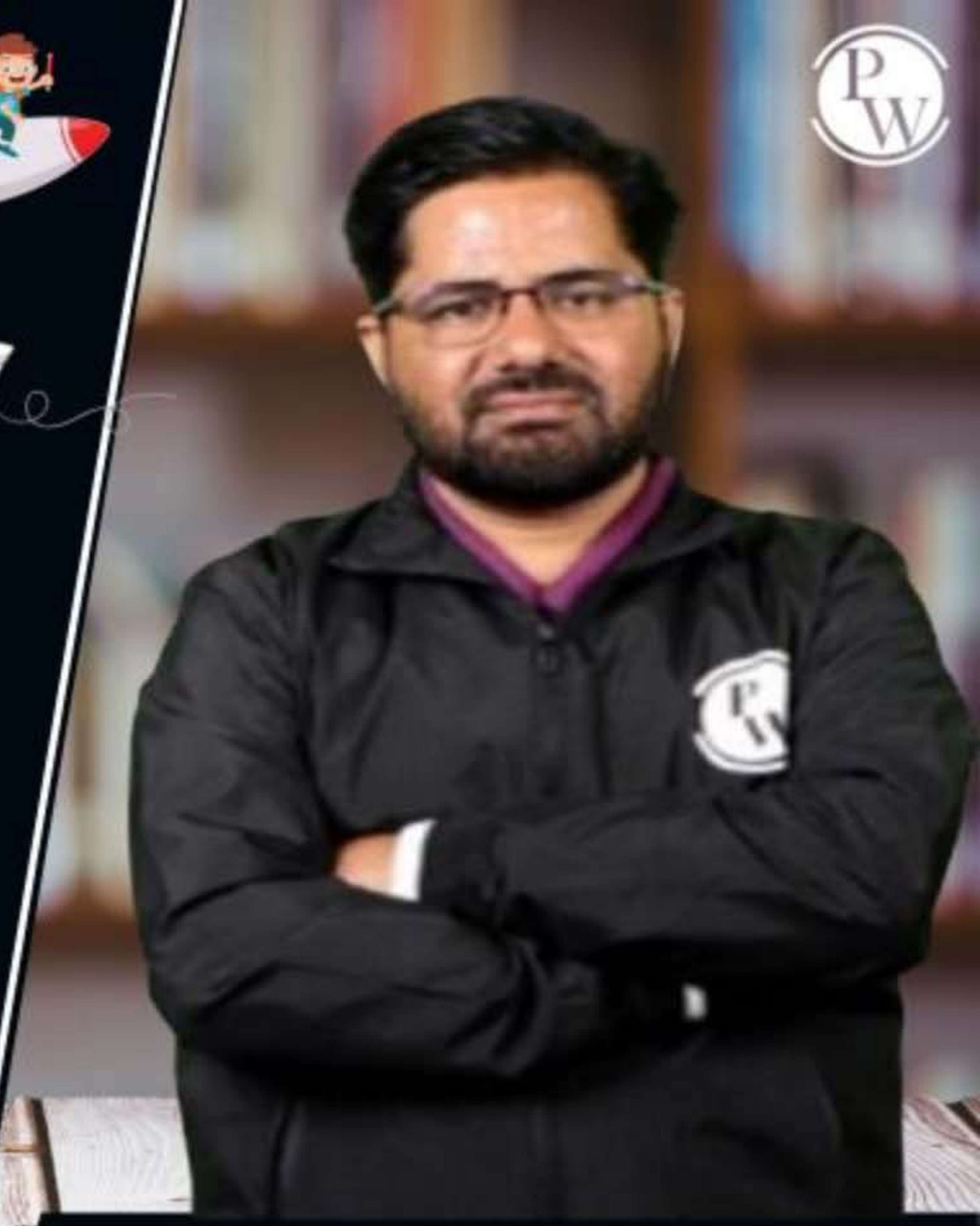
DS & AI
CS & IT

Linear Algebra

Lecture No. 06



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Recap of previous lecture

Topic

→ RANK of MATRIX

→ VECTORS & their PROPERTIES

Topics to be Covered

Topic

System of Equations

(Non Homogeneous)

Methods of Checking the Nature of Vectors →

Consider the given vectors are $x_1, x_2, x_3, \ldots\ldots, x_r$

then Construct a Matrix A as follows; $A = [x_1 \, x_2 \, x_3 \ldots\ldots x_r]$ ≠ Row Mat

(M-I) General Method (always applicable) →

(i) if $\rho(A) =$ No. of vectors $\Rightarrow$ Vectors are LI

(ii) if $\rho(A) <$ " " " $\Rightarrow$ " " LD

(M-II) Tricky Method (applicable only when A is Sq Mat) →

(i) if $|A| \neq 0 \Rightarrow$ Vectors are LI

(ii) if $|A| = 0 \Rightarrow$ " " LD

linear System of Equations

(1) Non Homogeneous System

$(AX = B)$

(2) Homogeneous System

$(AX = 0)$

eg

$$\left.\begin{aligned} 2x - y + 4z &= 0 \\ x - 2 + 3z &= 0 \\ -2x + 2y - z &= 0 \\ 4x - 3y + 2z &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 2x - y + 4z &= 0 \\ x + 0y + 3z &= 2 \\ -2x + 2y - z &= 0 \\ 4x - 3y + 2z &= 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 2 & -1 & 4 \\ 1 & 0 & 3 \\ -2 & 2 & -1 \\ 4 & -3 & 2 \end{bmatrix}_{4\times 3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}_{4\times 1}$$

$$A_{4\times 3} X_{3\times 1} = B_{4\times 1}$$

Here $[A \vdots B] = \left[\begin{array}{ccc:c} 2 & -1 & 4 & 0 \\ 1 & 0 & 3 & 2 \\ -2 & 2 & -1 & 0 \\ 4 & -3 & 2 & 0 \end{array}\right]_{4\times 4}$ = Augmented Mat.

1

Coefficient Mat

Constant Mat

$$\boxed{A_{m\times n} X_{n\times 1} = B_{m\times 1}}, \quad [A \vdots B]_{m\times(n+1)} = \text{Augumented Mat}$$

No. of equ^{n}s

No of Variables

Variable Mat

Solution of system

unknown vector

where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$

2. if $m > n$ then system is overdetermined (tough)

if $m = n$ then " " equally determined

if $m < n$ then " " underdetermined. (Easy)

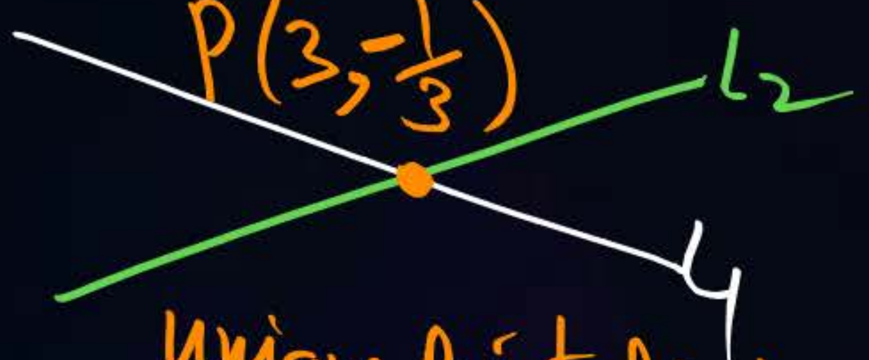
(3) Nature of solution →

eg

$$2x+3y=5$$
$$x-3y=4$$

$$X=\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 3 \\ -1/3 \end{bmatrix}= \text{unique V.}$$

$\simeq$ unique sol.



unique point of intersection

eg

$$2x+3y=5$$
$$4x+6y=10$$

$$X=\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix},\begin{bmatrix} 2 \\ 1/3 \end{bmatrix},\begin{bmatrix} 1/2 \\ 4/3 \end{bmatrix},\ldots\ldots$$

= Multiple vectors exist

= ∞ soln exist.

overlapping lines

eg

$$2x+3y=5$$
$$4x+6y=9$$

$$X=\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} ? \\ ? \end{bmatrix}=\text{No sol.}$$

$\because$ Both the equn contradict each other.

$\parallel^r$ lines $\simeq$ Never intersect

Methods of Solving Non Homog System → Consider $\boxed{A_{m\times n} X_{n\times 1} = B_{m\times 1}}$

RANK Method (always applicable)

(i.e. for $m > n$, $m = n$, $m < n$)

Matrix Method

(applicable only when $m = n$)

① if $\boxed{\rho(A) = \rho(A:B)}$ = No. of Variables ⇒ unique soln exist ⇐ if $|A| \neq 0$

② if $\boxed{\rho(A) = \rho(A:B)}$ < No. of Variables ⇒ ∞ soln. exist ⇐ if $|A| = 0$ & $(\operatorname{adj} A)B = 0$

③ if $\rho(A) \neq \rho(A:B)$ ⟹ No soln exist ⇐ if $|A| = 0$ & $(\operatorname{adj} A)B \neq 0$

(*) **Consistent system** → System is called consistent if $\exists$ solution.
(whether unique or ∞ sol.)

Inconsistent system → System is called Inconsistent if we have No sol.

(*) Necessary condition for a system $\boxed{AX = B}$ to be consistent is ?

$$\boxed{\rho(A) = \rho(A:B)}$$

Gate → Write the condition for the existence of $\underbrace{\text{at least one solution}}_{\text{Consistent}}$ of an $\underbrace{\text{unknown vector}}_{Y}$ in the system $\boxed{PY = Q}$

$$\rho(P) = \rho(P:Q)$$

(*) **Another form of N. Condn for consistency** :→ B must be L.D on columns of A.

(*) $\boxed{\rho(A) \not> \rho(A:B)}$ i.e. Rank of Coeff Mat can never exceed Rank of Augumented Mat.

eg $[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 5 & 14 \end{array}\right]$, $[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 10 \end{array}\right]$, $[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$= \quad c_1 \quad c_2 \quad c_3 \quad B$

$\rho(A) = 3$, $\rho(A:B) = 3$ (unique sol)

$\rho(A) = 2$, $\rho(A:B) = 3$ (No sol.)

$\rho(A) = 2 = \rho(A:B)$ (∞ sol.)

$\Downarrow$

$\rho(A:B) = 2 <$ No of vectors

i.e. c_1, c_2, c_3, B are L.D

i.e. for consistency B must be L.D on columns of A.

$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 5 & 0 \end{array}\right]$

$\rho(A) = 3$, $\rho(A:B) = 3$ (unique sol)

Q. For the system $Ax = B$ which of the following can be taken as condition for no soln.

(a) $\rho(A) = \rho(A:B)$

(b) $\rho(A) < \rho(A:B)$

(c) $\rho(A) > \rho(A:B)$

(d) Both (b) & (c)

$$\rho(A) \neq \rho(A:B) \Rightarrow \begin{cases} \rho(A) < \rho(A:B) & ✓ \\ \rho(A) > \rho(A:B) & ✗ \end{cases}$$

(x) underdetermined system can not have unique solution (Learn)

Note: underdetermined, Non Homog system if Consistent
or $\rho(A) = \rho(A:B)$
always consist Infinite sol.

The system of equations :

$$\begin{cases} 2x + y = 5 \\ x - 3y = -1 \\ 3x + 4y = k \end{cases}$$

is consistent when k is ______

(a) 1 (b) 2

(c) 5 (d) 10

$$[A:B] = \left[\begin{array}{cc:c} 2 & 1 & 5 \\ 1 & -3 & -1 \\ 3 & 4 & k \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc:c} 1 & -3 & -1 \\ 2 & 1 & 5 \\ 3 & 4 & k \end{array}\right]$$

$$\xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{cc:c} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 13 & k+3 \end{array}\right]$$

$$\xrightarrow{R_2/7} \left[\begin{array}{cc:c} 1 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 13 & k+3 \end{array}\right] \longrightarrow \left[\begin{array}{cc:c} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & k-10 \end{array}\right]$$

$\rho(A) = 2$ so for consistency

$\rho(A) = \rho(A:B) = 2 \Rightarrow k = 10$

{ Here we have, unique sol.

M-II Solving (1) & (2), $x = 2,\ y = 1$

So By (3), $3x + 4y = k$

$3(2) + 4(1) = k$

$k = 10$

Q3 The solution of the system:

$$\begin{cases} x+y+z=6 \\ x+2y+3z=10 \\ x+2y+5z=14 \end{cases}$$ will be.

(a) $\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$ (c) ✓ $\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix}$

M-I (Using option ellimination App)

$\longrightarrow$ (c) is correct.

M-II $\rightarrow$ Solve it simultaneously as discussed in class 8th...... Do yourself

M-III Using Gauss Ellimination Method $\rightarrow$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & 5 & 14 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 4 & 8 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{array}\right], \quad \rho(A)=3, \quad \rho(A:B)=3$$

$\because \rho(A)=\rho(A:B)=$ No. of Variables (3)

$\Rightarrow$ unique sol. exist.

Process of finding Sol. $\rightarrow$ See Next Slide.

Process of Finding Sol :-

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 14 \end{bmatrix}$$

- - - - - - - - -

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} (x+y+z) \\ (y+2z) \\ 2z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}$$

$$x+y+z = 6 \Rightarrow x+0+2 = 6 \Rightarrow x = 4$$

$$y+2z = 4 \Rightarrow y+2(2) = 4 \Rightarrow y = 0$$

$$2z = 4 \Rightarrow z = 2$$

ie Solⁿ is $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$ ie unique solⁿ exist.

Note- this method of solving Linear system of Eqⁿ is called ECHELON FORM method.
or Gauss-Ellimination method.
or Backward substitution Method.
or RANK method.

Qs Find the values of λ & μ for which following system has

① ∞ sol. ② No sol ③ unique sol.

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

$$[A:B]=\left[\begin{array}{ccc|c}1&1&1&6\\1&2&3&10\\1&2&\lambda&\mu\end{array}\right] \dashrightarrow \left[\begin{array}{ccc|c}1&1&1&6\\0&1&2&4\\0&0&\boxed{(\lambda-3)}&(\mu-10)\end{array}\right]_{3\times 4}$$

3×3

For ∞ sol.: $\rho(A)=\rho(A:B)<3$

Let $\underbrace{\rho(A)=2}_{\lambda=3}=\underbrace{\rho(A:B)}_{\mu=10}$ Ans

For No sol: $\rho(A)\neq\rho(A:B) \Longrightarrow \rho(A)<\rho(A:B)$

Let $\rho(A)=2$ & $\rho(A:B)=3$

$\Downarrow$

$\lambda=3$ & $\mu\neq 10$ Ans

$\because x\neq y \begin{cases} x>y \\ x<y \end{cases}$, $\rho(A)\neq\rho(A:B) \begin{cases} \cancel{\rho(A)>\rho(A:B)} \\ \rho(A)<\rho(A:B) \checkmark \end{cases}$

(3) For unique sol →

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \boxed{(\lambda-3)} & (\mu-10) \end{array}\right]$$

for unique sol $\Rightarrow \rho(A) = \rho(A:B) = 3$

$\rho(A) = 3$ & $\rho(A:B) = 3$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$\lambda \neq 3$ $\qquad$ μ can take any value.

i.e. $\lambda \neq 3$ & $\mu \in R$ Ans

Note: for $\lambda = 5$, $\mu = 14$, sol of above system is unique

& for $\lambda = 11$, $\mu = 10$, sol of " " is unique

Q. Solve $3x+3y+2z=1$

$$x+2y=4$$

$$10y+3z=-2$$

$$2x-3y-z=5$$

unique sol

Q. Find the Nature of soln of

(MSQ) $x+y+z=1$

$2x+y+4z=1$

$4x+y+10z=1$

(a) No soln.

(b) unique soln.

(c) ∞ solns

(d) Finite No. of Ind soln.

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 1 \\ 4 & 1 & 10 & 1 \end{array}\right] \xrightarrow[R_3-4R_1]{R_2-2R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & -3 & +6 & -3 \end{array}\right]$$

$$\xrightarrow{R_3-3R_2} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & -1 & +2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A)=2=\rho(A:B)$

$\Rightarrow \infty$ solns exist.

$$AX=B \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\cdots\cdots \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & +2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} (x+y+z) \\ (-y+2z) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$x+y+z=1$
$-y+2z=-1$

let $z=k$, $-y+2k=-1 \Rightarrow y=2k+1$

& $x+y+z=1 \Rightarrow x+2k+1+k=1 \Rightarrow x=-3k$

Sol is $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -3k \\ 2k+1 \\ k \end{bmatrix}=\infty$ sol. $=\begin{bmatrix} 0-3k \\ 1+2k \\ 0+k \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}+\begin{bmatrix} -3k \\ 2k \\ k \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}+k\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

($\because k=$ arbitrary const.)

$=\alpha+k\beta$.

$X=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}=X_1$, $\begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}=X_2$, $\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}=X_3$, $\begin{bmatrix} -6 \\ 5 \\ 2 \end{bmatrix}=X_4$, - - - - - - ∞ sol. exist.

$(k=0)$ $(k=1)$ $k=-1$ $k=2$

$X_3=2X_1-X_2$ $X_4=-X_1+2X_2$ & so on - - - -

ie only X_1 & X_2 are LI & Rest are LD on them.

Eg the Nature of the soln of following system:

(1) $x_1 - 2x_2 + 4x_3 = 5$

$2x_1 - 4x_2 + 8x_3 = 7$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & -2 & 4 & 5 \\ 2 & -4 & 8 & 7 \end{array}\right]_{2\times 4}$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 4 & 5 \\ 0 & 0 & 0 & -3 \end{array}\right]_{2\times 4}$$

$\rho(A) = 1, \rho(A:B) = 2$

No soln

(2) $x_1 - 2x_2 + 4x_3 = 5$

$2x_1 - 4x_2 + 8x_3 = 10$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & -2 & 4 & 5 \\ 2 & -4 & 8 & 10 \end{array}\right]_{2\times 4}$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]_{2\times 4}$$

$\rho(A) = 1 = \rho(A:B) <$ No of variables (3)

∞ solns.

Given a system of equations :

$$x + 2y + 2z = b_1$$
$$5x + y + 3z = b_2$$

underdetermined

Which of the following is true regarding its solution ?

(a) The system has a unique solution for any given b_1 and b_2 X

(b) The system will have infinitely many solutions for any ~~given~~ b_1 and b_2

(c) Whether or not a solution exists depends on the given b_1 and b_2 X

(d) The system would have no solution for any values of b_1 and b_2 X

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 2 & 2 & b_1 \\ 5 & 1 & 3 & b_2 \end{array}\right]_{2\times 4}$$

2×2

$\rho(A) = 2 = \rho(A:B)$

Consistent $\Rightarrow \infty$ soln

Not depends upon the values of b_1 & b_2

Q. if $A_{n\times n}$ s.t $A^2 = I$ then $AX = B$ has ________ sol.

(a) unique sol.

(b) ∞ sol.

(c) No sol.

(d) More than one but finite No of sol.

Sol: $A^2 = I$

$|A^2| = |I|$

$|A|^2 = 1$

$|A| = \pm 1$

i.e $|A| \neq 0$

So By Matrix method unique sol exist

M-II (w/o using property) →

$AX = B$

$A(AX) = AB$

$A^2X = A_{n\times n} B_{n\times 1}$

$IX = (AB)_{n\times 1}$

$X = \begin{bmatrix} - \\ - \\ \vdots \\ - \end{bmatrix}_{n\times 1}$ = unique vector exist.

So unique sol exist.

THANK - YOU

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