

**DS & AI
CS & IT**

Linear Algebra

Lecture No. 08



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Recap of previous lecture



Topic

- ① Homogeneous system
- ② Basics of Eigen Values



Topics to be Covered



Topic

PROPERTIES of EIGEN VALUES



Defⁿ! Consider Sq. Mat $A_{n \times n}$ then Non Zero Vector X is called Eigen Vector, corresponding to Eigen value λ (Real/Complex/Zero) if we are able to find a relationship of the type,

$$\boxed{AX = \lambda X}$$

$\swarrow \lambda = \text{Eigen Value}$
 $\searrow X = \text{Eigen Vector.}$

LHS is the Multi of Two Matrices = RHS is the Scalar Multi in a Mat

(Tough)
(Easy)

(*) Here we are considering Homogeneous system as follows

$$AX = \lambda X \Rightarrow AX - \lambda X = 0 \Rightarrow \boxed{(A - \lambda I)X = 0}$$

So it will satisfy all the prop of Homog system.

(*) Consider $AX = \lambda X$

$$(A - \lambda I)X = 0 \quad \text{--- (1)}$$

$$MX = 0$$



Non Zero Eigen Vector

Non Zero solution

$\Rightarrow \infty$ solⁿs.



$$\rho(M) < n \text{ or } |M| = 0$$

$$\Rightarrow \rho(A - \lambda I) < n \text{ or } |A - \lambda I| = 0$$

So Necessary Condition for the existence of Non Zero Eigen Vector is

$$\rho(A - \lambda I) < n \text{ or } |A - \lambda I| = 0$$

Characteristic Equⁿ of A →

Equⁿ (1) is called C.Eq of A & Roots of this equⁿ i.e. values of λ are called E. Values / E. Roots / Char Values / Char Roots / Latent Roots / Sp. Values.



PROPERTIES of E-Values \rightarrow Let $A_{n \times n}$ having Eigen Values $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$



- ① T. Number of E-Values of A = order of A (whether diff or Repeated)
- ② Sum of E-Values = Trace (A) i.e. $\lambda_1 + \lambda_2 + \dots + \lambda_n = \text{Tr}(A)$
- ③ Product of E-Values = Det (A) i.e. $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \dots \cdot \lambda_n = |A|$
- ④ (Zero is an E-Value of A) \iff (A is singular)
- ⑤ Number of Non Zero E-Values of $A \leq \rho(A)$
 eg if $\rho(A_{6 \times 6}) = 4$ then No of Non Zero E-Values ≤ 4
 i.e. we must have at least two eigen values as 0, 0
- ⑥ If sum of all the elements in each Row (or each Column) is unique constant K then that constant K will be one of the E-Value of A .

⑦ Don't use E-operations in a given Mat while calculating E-Values

But we can apply 3rd type of E-operation in it's C-Eqn.

⑧ E-Values of U.T.M, L.T.M, Diag Mat, scalar Mat, Identity Mat are just the diagonal elements.

e.g. $A = \begin{bmatrix} 2 & 4 & -1 & 3 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
 $\lambda = 2, 0, -3, -1$

$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$
 $\lambda = 2, -1, 1$

$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
 $\lambda = 2, -3, 4$ $\lambda = 1, 1, 1$

⑨ If λ is an Eigen Value of A then to find Eigen Value of any algebraic expression formed by A, we can replace A with λ in that expression. i.e. $A \rightarrow \lambda$

→ (*) Equivalent Matrices have same Rank but may be different E-Values

(10) Let $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are the E Values of $A_{n \times n}$ then

(i) E Values of A^T are $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ i.e. A & A^T have Same E Values

(ii) E Values of A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m, \dots, \lambda_n^m$; ($m \in \mathbb{N}$)

(iii) E Values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \dots, \frac{1}{\lambda_n}$ (provided $|A| \neq 0$)

(iv) E-Values of $(\text{adj } A)$ are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}, \dots, \frac{|A|}{\lambda_n}$ (when $|A| \neq 0$)

Proof 10(i): $AX = \lambda X$ — (1)

$$A(AX) = A(\lambda X)$$

$$A^2X = \lambda(AX)$$

$$A^2X = \lambda(\lambda X)$$

$$A^2X = \lambda^2 X$$

ie λ^2 is an E Value of A^2
 Similarly λ^3 " " " " A^3

Proof 10(ii): $AX = \lambda X$ — (1)

if $|A| \neq 0 \Rightarrow \bar{A}^{-1}$ exists

$$\bar{A}^{-1}(AX) = \bar{A}^{-1}(\lambda X)$$

$$IX = \lambda(\bar{A}^{-1}X)$$

$$\lambda(\bar{A}^{-1}X) = IX$$

$$\bar{A}^{-1}X = \left(\frac{1}{\lambda}\right)X$$

$\frac{1}{\lambda}$ is an E Value of \bar{A}^{-1}

Proof 10(iv): $AX = \lambda X$ — (1)

Now $\bar{A}^{-1}X = \left(\frac{\text{adj } A}{|A|}\right)X$

$$\left(\frac{1}{\lambda}\right)X = \left(\frac{\text{adj } A}{|A|}\right)X$$

$$\left(\frac{|A|}{\lambda}\right)X = (\text{adj } A)X$$

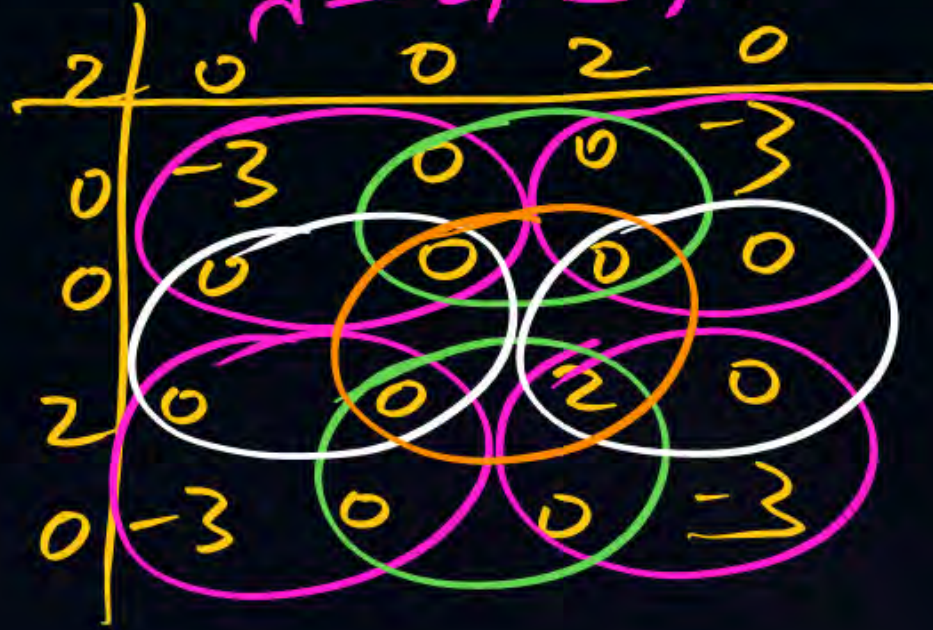
$$\text{or } (\text{adj } A)X = \left(\frac{|A|}{\lambda}\right)X$$

ie $\frac{|A|}{\lambda}$ is an E Value of $(\text{adj } A)$

This Prop holds when A is Non Sing.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda = 2, -3, 0$$



$$(\text{adj } A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

$$\lambda = -6, 0, 0$$

E Value of A are $2, -3, 0$

" " adj A are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}$

$$= \frac{0}{2}, \frac{0}{-3}, \frac{0}{0}$$

Not applicable

eg $A = \begin{bmatrix} -2 & -1 & 3 & 4 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

U.T.M.
 $|A| = -24$

then \rightarrow Eigenvalues of A are $\lambda = -2, 1, 4, 3$

\rightarrow " " A^T are same

\rightarrow " " A^3 are $-8, 1, 64, 27$

\rightarrow " " A^{-1} are $\frac{1}{-2}, 1, \frac{1}{4}, \frac{1}{3}$

\rightarrow " " $\text{adj } A$ are $\frac{-24}{-2}, \frac{-24}{1}, \frac{-24}{4}, \frac{-24}{3}$

eg $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Diag Mat.

Eigenvalues of A are $1, -2, 0, 4$

" of A^T are $1, -2, 0, 4$

" of A^2 are $1, 4, 0, 16$

" of A^{-1} are $1, \frac{1}{-2}, \text{ND}, \frac{1}{4}$ (Blunder)

$\therefore A^{-1} = \text{DNE}$ so case of Eigenvalues will not arise

eg $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ 3×3

$\lambda = 3, 1, 1$

$\text{Tr}(A) = 5$

$|A| = 3$

eg $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$
 $= (10) (10) (10) (10)$

ie one E value is $\lambda = 10$
 (Prop 5)

eg $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ 2×2

$\lambda = i, -i$

$\text{Tr} = 0$

$|A| = 1$

eg $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ 3×3

$\lambda = 0, 3, 15$

$\text{Tr} = 18$

$|A| = 0$

$\rho(A) = 2 = \text{Nb. of Non Zero E Values}$
 (Prop 5)

Q If E-Values of $A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$ are 4 & 8 then $x+y = ?$

- (a) -4 (b) 6 (c) 10 (d) 14

Let λ_1 & λ_2 are the E-Values of A

then $\lambda_1 + \lambda_2 = \text{Tr}(A)$ & $\lambda_1 \lambda_2 = |A|$

$$4+8 = 2+y \quad \& \quad (4)(8) = 2y-3x$$

$$y = 10$$

$$32 = 2(10) - 3x$$

$$3x = -12$$

$$x = -4$$

$$x+y = 6 \quad \underline{\underline{A_1}}$$

Q If Trace & Det of $A_{2 \times 2}$ are -2 & -35 resp then $\lambda_1 + \lambda_2 = ?$

- (a) 12 (b) -12 (c) 2 (d) -2

w.k. that $\lambda_1 + \lambda_2 = \text{Tr}(A) = -2$

(ii) Also Find E-Values

w.k. that,

$$\lambda_1 + \lambda_2 = \text{Tr}(A) \quad \& \quad \lambda_1 \cdot \lambda_2 = |A|$$

$$\lambda_1 + \lambda_2 = -2 \quad \& \quad \lambda_1 \cdot \lambda_2 = -35$$

$$\lambda_1 = 5, \lambda_2 = -7$$

Consider a 2×2 square matrix

$$A = \begin{bmatrix} \sigma & x \\ \omega & \sigma \end{bmatrix}$$

where x is unknown. If the eigen values of the matrix A are $(\sigma + j\omega)$ and $(\sigma - j\omega)$, then x is equal to

(a) $+j\omega$

(b) $-j\omega$

(c) $+\omega$

✓ (d) $-\omega$

ATQ, $\lambda_1 = \sigma + j\omega$ & $\lambda_2 = \sigma - j\omega$

$$\lambda_1 + \lambda_2 = \text{Tr}(A)$$

$$\lambda_1 \cdot \lambda_2 = |A|$$

$$2\sigma = 2\sigma$$

(identity)

$$\sigma^2 - j^2 \omega^2 = \sigma^2 - \omega x$$

$$\omega x = j^2 \omega^2$$

$$x = \frac{-\omega^2}{\omega} = \boxed{-\omega}$$

$$j = \sqrt{-1} = j$$

$$j^2 = -1 = j^2$$

Q. If $A_{3 \times 3}$ s.t. $|A - I| = 0$, $\text{Tr}(A) = 13$, $|A| = 32$ then $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = ?$

(a) 12

(b) 13

☒ (c) 81

(d) 80

' $A_{3 \times 3}$ so let it's E values are $\lambda_1, \lambda_2, \lambda_3$

ATQ, $|A - I| = 0$

$|A - \lambda I| = 0$

$\Rightarrow \lambda_1 = 1$

$\text{Tr}(A) = 13$

$\lambda_1 + \lambda_2 + \lambda_3 = 13$

$1 + \lambda_2 + \lambda_3 = 13$

$\lambda_2 + \lambda_3 = 12$

$|A| = 32$

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 32$

$1(\lambda_2)(\lambda_3) = 32$

$\lambda_2 \lambda_3 = 32$

$\lambda_2 = 8, \lambda_3 = 4$

$\therefore \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1^2 + 8^2 + 4^2 = 81$

Q. If P and Q are two sq. Matrices of same order s.t. $PQ = QP = I$ then 0 is an eigen value of ?

$$\because PQ = QP = I \Rightarrow \begin{cases} P^{-1} = Q \\ Q^{-1} = P \end{cases}$$

i.e. Both P & Q are Invertible.

or $|P| \neq 0$ & $|Q| \neq 0$.

i.e. Both are Non Singular

So Zero can't be eigen value of any one Mat.

- (a) P But Not Q
- (b) Q But not P
- (c) Both P & Q
- (d) Neither P Nor Q

Q.8 if one of the E Value of $A = \begin{bmatrix} 40 & -16 & -24 \\ -11 & 30 & -19 \\ 26 & 24 & -50 \end{bmatrix}$ is $\lambda = 0$ then other E Values will be?

(a) $\lambda - 20$

(b) $20 - \lambda$

(c) $20 + \lambda$

(d) 0

So By Prop (6), one E Value is $\lambda = 0$

Now $\lambda_1 + \lambda_2 + \lambda_3 = \text{Tr}(A)$

$0 + \lambda + \lambda_3 = 20$

$\lambda_3 = 20 - \lambda$

M-II $|A| = \begin{vmatrix} 40 & -16 & -24 \\ -11 & 30 & -19 \\ 26 & 24 & -50 \end{vmatrix}$

$C_1 \rightarrow C_1 + C_2 + C_3$

$= \begin{vmatrix} 0 & -16 & -24 \\ 0 & 30 & -19 \\ 0 & 24 & -50 \end{vmatrix}$

$|A| = 0 \Rightarrow \lambda = 0$

Prop (4)

Q one of the E-Value of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$ will be ? (a) 5 (b) 0
 (c) 10 (d) -1



M.I using prop (6), one of the E-Value of A is = 10 Ans

M.II C.Equⁿ of A, $|A - \lambda I| = 0$

$$\begin{vmatrix} (1-\lambda) & 2 & 3 & 4 \\ 2 & (3-\lambda) & 4 & 1 \\ 3 & 4 & (1-\lambda) & 2 \\ 4 & 1 & 3 & (2-\lambda) \end{vmatrix} = 0$$

$$C_1 \rightarrow C_1 + (C_2 + C_3 + C_4)$$

$$\begin{vmatrix} (10-\lambda) & 2 & 3 & 4 \\ (10-\lambda) & (3-\lambda) & 4 & 1 \\ (10-\lambda) & 4 & (1-\lambda) & 2 \\ (10-\lambda) & 1 & 3 & (2-\lambda) \end{vmatrix} = 0$$

$$(10-\lambda) \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & (3-\lambda) & 4 & 1 \\ 1 & 4 & (1-\lambda) & 2 \\ 1 & 1 & 3 & (2-\lambda) \end{vmatrix} = 0 \Rightarrow \lambda = 10$$

Q the E-Values of $A = \begin{bmatrix} | & | & | & | \\ | & | & | & | \\ | & | & | & | \\ | & | & | & | \end{bmatrix}$ are ? $\lambda = 4, ?, ?, ?$
 $= \textcircled{4} \textcircled{4} \textcircled{4} \textcircled{4}$ 4×4



By Prop (6) one E-Value is $\lambda = 4$

$\therefore f(A) = \text{one}$ & Number of Non Zero E-Values of $A \leq \rho(A)$ (Prop 5)

ie No of Non Zero E-Values of $A \leq 1$ ($= \text{one}$)

A has only one Non Zero E-Value which is $\lambda = 4$

\therefore Remaining E-Values are $\lambda = 0, 0, 0$.

overall for A , $\lambda = 4, 0, 0, 0$.

(M-II) Also find E. Values by conventional approach? $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$



C. Eqnⁿ is $|A - \lambda I| = 0$

$$\begin{vmatrix} (1-\lambda) & 1 & 1 & 1 \\ 1 & (1-\lambda) & 1 & 1 \\ 1 & 1 & (1-\lambda) & 1 \\ 1 & 1 & 1 & (1-\lambda) \end{vmatrix} = 0$$

$$C_1 \rightarrow C_1 + (C_2 + C_3 + C_4)$$

$$\begin{vmatrix} (4-\lambda) & 1 & 1 & 1 \\ (4-\lambda) & (1-\lambda) & 1 & 1 \\ (4-\lambda) & 1 & (1-\lambda) & 1 \\ (4-\lambda) & 1 & 1 & (1-\lambda) \end{vmatrix} = 0$$

$$\begin{vmatrix} (4-\lambda) & 1 & 1 & 1 \\ 1 & (1-\lambda) & 1 & 1 \\ 1 & 1 & (1-\lambda) & 1 \\ 1 & 1 & 1 & (1-\lambda) \end{vmatrix} = 0$$

$R_2 - R_1, R_3 - R_1, R_4 - R_1$

$$\begin{vmatrix} (4-\lambda) & 1 & 1 & 1 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(-\lambda^3) = 0$$

$$\lambda = 4, 0, 0, 0.$$

M-III (WRONG APP) →

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\lambda = 4, 0, 0, 0$$

$$\begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\lambda = 1, 0, 0, 0$$

ie we have justified that equivalent Matrices have different E Values

Qs

if $A_{2 \times 2}$ s.t $a_{11} = a_{12} = a_{21} = 1$ & $a_{22} = -1$ then

E. values of A^{19} are ?

(a) $\pm \sqrt{2}$

(b) ± 2

(c) ± 1

(d) $\pm 512\sqrt{2}$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

C. Eq is $|A - \lambda I| = 0$

$$\boxed{\lambda^2 - 2 = 0} \Rightarrow \lambda = \pm \sqrt{2}$$

E. values of A are $\sqrt{2}$ & $-\sqrt{2}$
 " " A^{19} are $(\sqrt{2})^{19}$ & $(-\sqrt{2})^{19}$
 $= (\sqrt{2})^{18}(\sqrt{2})$ & $(-\sqrt{2})^{18}(-\sqrt{2})$
 $= 2^9 \sqrt{2}$ & $2^9(-\sqrt{2})$
 $= 512\sqrt{2}$ & $-512\sqrt{2}$

Q. $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & \frac{-1+i\sqrt{3}}{2} & 0 \\ 3 & 4 & \frac{-1-i\sqrt{3}}{2} \end{bmatrix}$ then $\text{Tr}(A^{102}) = ?$

Sol: $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & \omega & 0 \\ 3 & 4 & \omega^2 \end{bmatrix} = L \cdot T \cdot M$

$$\omega = \frac{-1+i\sqrt{3}}{2}, \omega^2 = \frac{-1-i\sqrt{3}}{2}$$

$$\omega^3 = 1, 1 + \omega + \omega^2 = 0, \overline{\omega} = \omega^2$$

$$\frac{1}{\omega} = \omega^2$$

$$\overline{\omega^2} = \omega$$

E Values of A are $1, \omega, \omega^2$

" " A^{102} are $1^{102}, \omega^{102}, (\omega^2)^{102}$

$$= 1, \omega^{102}, \omega^{204}$$

$$= 1, (\omega^3)^{34}, (\omega^3)^{68} = 1, 1, 1$$

E Values of A^{102} are $1, 1, 1$

$$\text{Tr}(A^{102}) = 1 + 1 + 1 = \boxed{3}$$

Let A be a 3×3 matrix with Eigen values $-1, 1, 0$.

Then $|A^{100} + I|$ is ____.

E Values of A are $-1, 1, 0$

" " A^{100} are $(-1)^{100}, (1)^{100}, (0)^{100}$

" " A^{100} are $= 1, 1, 0$

Let $B = A^{100} + I$

Using Prop (9)


$$1 + 1 = 2$$

$$1 + 1 = 2$$

$$0 + 1 = 1$$

E Values of B are $2, 2, 1$

$$\therefore |B| = (2)(2)(1) = 4$$

Q. If $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ & $B = A^2 - 2A + 3I$ then Find the Product of the Eigen values of B ? 

~~(a) 81~~ (M-I) $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ $\lambda_A = 6$ (Do yourself/HW)
 $\lambda_A = 2$

(b) 30

(c) 12

(d) 144

using Prop (9):

$$B = (A^2 - 2A + 3I) \rightarrow \lambda_B = (6)^2 - 2(6) + 3(1) = 27$$

$$\lambda_B = (2)^2 - 2(2) + 3(1) = 3$$

So Req. Product = $27 \times 3 = 81$

Verification:- $B = A^2 - 2A + 3I = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - 2 \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 12 & 15 \end{bmatrix}$ $\begin{matrix} 27 \\ 27 \end{matrix}$

So E-Values of B are 27 & 3 & Product = $|B| = 81$

Let A be a 4×4 matrix with real entries such that $-1, 1, 2, -2$ are its Eigen values. If $B = A^4 - 5A^2 + 5I$ then trace of $A + B$ is _____.

$$\begin{array}{l}
 A \rightarrow \begin{cases} \lambda_A = -1 \\ \lambda_A = 1 \\ \lambda_A = -2 \\ \lambda_A = 2 \end{cases} \\
 B = A^4 - 5A^2 + 5I
 \end{array}$$

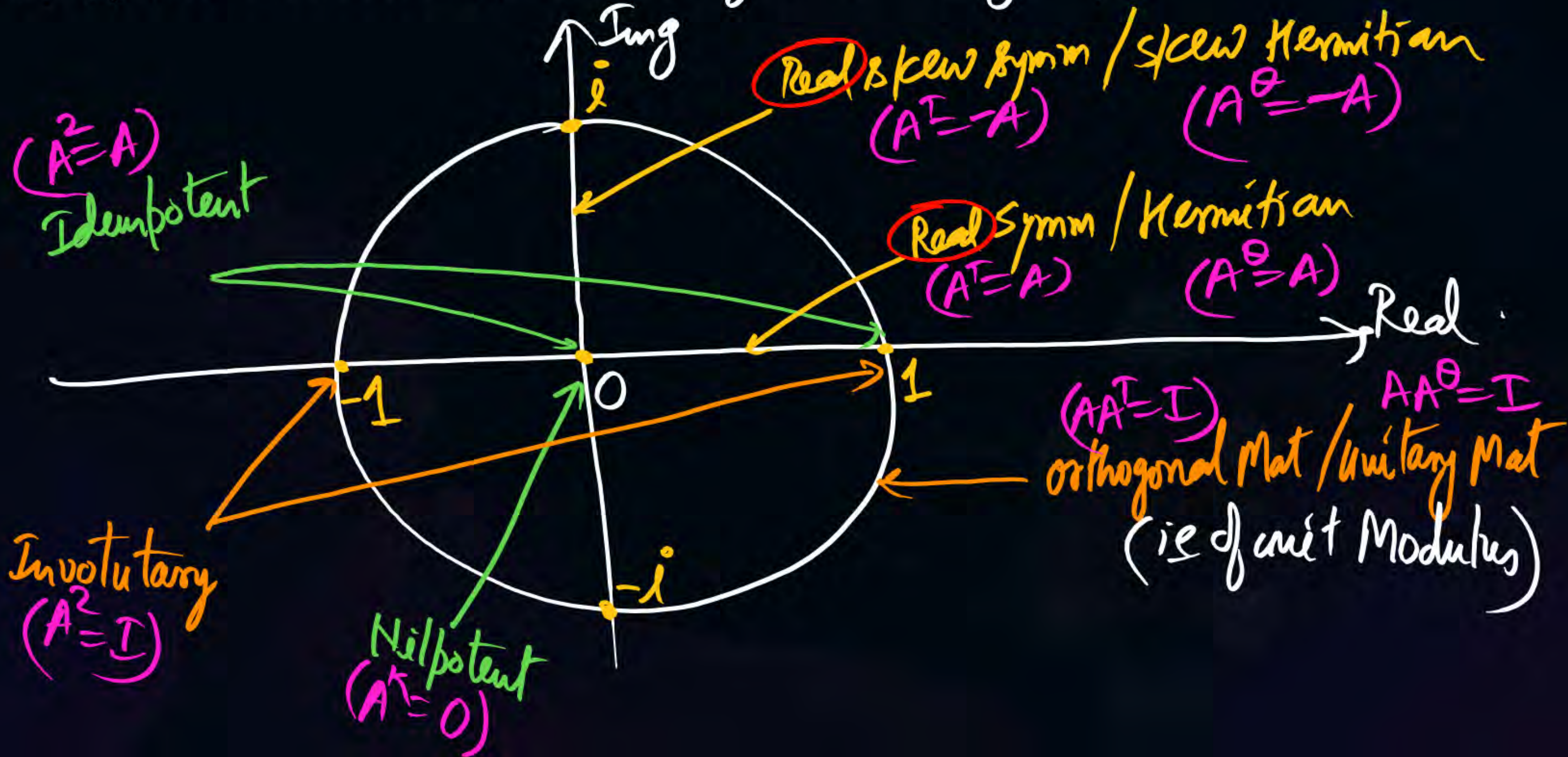
$$\begin{array}{l}
 (-1)^4 - 5(-1)^2 + 5(1) = 1 - 5 + 5 = 1 \\
 (1)^4 - 5(1)^2 + 5(1) = 1 - 5 + 5 = 1 \\
 (-2)^4 - 5(-2)^2 + 5(1) = 16 - 20 + 5 = 1 \\
 (2)^4 - 5(2)^2 + 5(1) = 16 - 20 + 5 = 1
 \end{array}$$

$$\text{Tr}(A+B) = \text{Tr}(A) + \text{Tr}(B)$$

$$= (-1 + 1 - 2 + 2) + (1 + 1 + 1 + 1) = 4$$

(1) Short cut Method to Learn Various Th based on E Values →

Consider a unit circle centered at origin i.e. $x^2 + y^2 = 1$



$$g \quad A = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 5 & 1 \\ 4 & 1 & 0 \end{bmatrix}$$

= Real Symm.

$\lambda \in \mathbb{R}$

$$, A = \begin{bmatrix} 2 & -3 & 4+i \\ -3 & 5 & 1 \\ 4+i & 1 & 0 \end{bmatrix}$$

= Complex Symm.

it is not Necessary that λ is Real.

$$g \quad A = \begin{bmatrix} 0 & -3 & 4 \\ 3 & 0 & 1 \\ -4 & -1 & 0 \end{bmatrix}$$

= Real Skew Symm

$\lambda \in$ Purely Imag Number

$$, A = \begin{bmatrix} 0 & -3 & 4i \\ 3 & 0 & 2-i \\ -4i & -2+i & 0 \end{bmatrix}$$

= Complex Skew Symm.

it is not Necessary that λ is Purely Imag

Q If E Values of $A = \begin{bmatrix} 2 & 5+i & -3 \\ x & -1 & 4 \\ -3 & 4 & 3 \end{bmatrix}$ are all Real Nos then $x = ?$

- (a) $5+i$
- ☒ (b) $5-i$
- (c) 0
- (d) Both a & b

Real E-Values \Rightarrow Either A is Real Symm. \times ($\because a_{12} = \text{Complex No}$)
 or A is Hermitian \checkmark

\Downarrow
 Corresponding elements are Conjugates of each other.

\Downarrow
 $x = 5-i$

Q. If $A = \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}_{4 \times 4}$ is an O-Mat then $(AA^T)^{-1} = ? = (\mathbf{I})^{-1} = \mathbf{I} = I_{4 \times 4}$

(ii) which of the following Can not be the E Value of A ?

(a) $-\frac{1}{2} + i\frac{\sqrt{3}}{2} \Rightarrow |\lambda| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$ ✓

(b) $-1 \Rightarrow |\lambda| = 1$ ✓

(c) $\frac{\sqrt{3}}{2} + \frac{i}{2} \Rightarrow |\lambda| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1$ ✓

~~(d)~~ $\frac{1}{4} + i\frac{\sqrt{3}}{4} \Rightarrow |\lambda| = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2} = \sqrt{\frac{1}{16} + \frac{3}{16}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$ (Not)

(e) None

* If $\{\lambda_i, x_i\}$ is an E-pair of $A_{n \times n}$ then which is false?

- (a) If Non Zero E vector of A if $\rho(A - \lambda I) < n$ **T** (\because it is not condition)
- (b) If $A^D = A$ then $\lambda_i \in \mathbb{R} \forall i$ **(T)** $\because A$ is Hermitian
- (c) If $A^{-1} = A^T$ then $|\lambda_i| = 1 \forall i$ **(T)** $\because A$ is O-Mat
- (d) $\{\lambda_i^m, x_i^m\}$ is an E-pair of A^m ? **(F)** $\because X^m = DNE$
- (e) If $A = A^{-1}$ then the Eigen Value of A is 1 **(T)** $\because A$ is Involutory Mat

$\because X = \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix}_{n \times 1}$ so $X^2 = X$ $X_{m \times 1} X_{n \times 1} = N.D.$

E-Value of A is λ & A^m is λ^m **(T)**
 E-vector of A & A^m are same **(T)**

Explanation: For Involutory Mat A ,

$$A^2 = I$$

$$\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

Note: for Idempotent Mat A ,

$$A^2 = A$$

$$\Rightarrow \lambda^2 = \lambda$$

$$\lambda(\lambda - 1) = 0 \Rightarrow \lambda = 0 \text{ or } 1$$

Note: Real E Values of an
or Orthogonal Mat are 1 or -1 (only)

For any O-Mat A , we have

$$AA^T = I$$

$$(\lambda)(\lambda) = 1$$

$$\lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

$$\therefore |\lambda| = 1 \text{ (justified)}$$

& for complex E values, $|\lambda| = 1$.

The eigen values of $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ are _____

- (a) Purely imaginary (b) Zero
(c) Real (d) None of the above

M-IV

$$|A| = 0 - (i)(-i) = +i^2 = -1$$

$$\Rightarrow \lambda = -1, 1 \text{ (Real)}$$

M-I $\because A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ = Skew Symm.
(But Comp. Skew Symm)
 $\Rightarrow \lambda = \text{Purely Imaginary}$

M-II Now $A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ = Hermitian also.
 $\lambda \in \mathbb{R}$

M-III

C. Equⁿ is $|A - \lambda I| = 0$

$$\begin{vmatrix} 0-\lambda & +i \\ -i & 0-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (i)(-i) = 0$$

$$\lambda^2 + i^2 = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1 \text{ (Real Nos)}$$

Q. A & B are Non Zero sq. Matrices Then $AB = O \Rightarrow$

Famous Doubt

- (a) B is Singular
- (b) A is Singular
- (c) A & B are orthogonal
- (d) A & B are singular

Case I: let A is Non singular then A^{-1} exist

$$AB = O$$

$$A^{-1}(AB) = A^{-1}O$$

$$B = O \text{ which is false ATQ.}$$

So our assumption is wrong i.e. A must be sing

Case II: let B is Non sing then B^{-1} exist.

$$AB = O$$

$$(AB)B^{-1} = O \cdot B^{-1}$$

$$A = O \text{ (false)}$$

So again our assumption is wrong hence B must be sing

THANK - YOU

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