

**DS & AI
CS & IT**

Linear Algebra

Lecture No. 02



By- Dr. Puneet Sharma Sir

Recap of previous lecture



Topic

BASICS of Determinants



Topics to be Covered



Topic

BASICS of MATRICES



Remaining Portion of Det -

Analysis: if $A_{n \times n}$ then $\boxed{A^{-1} = \frac{\text{adj } A}{|A|}}$ where $(\text{adj } A)_{n \times n}$ & $|A|$ is any const.
(PODCAST)

Now, $AA^{-1} = \frac{A \text{adj } A}{|A|} \Rightarrow I = \frac{A \text{adj } A}{|A|} \Rightarrow A \text{adj } A = |A| \cdot I_n$

Now $A(\text{adj } A) = |A| \cdot I_n = |A| \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{n \times n} = \begin{bmatrix} |A| & 0 & 0 & \dots & 0 \\ 0 & |A| & 0 & \dots & 0 \\ 0 & 0 & |A| & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & |A| \end{bmatrix}_{n \times n} = \text{Scalar Mat.}$

∴ $|A \text{adj } A| = \underbrace{|A| \cdot |A| \cdot |A| \dots |A|}_{n \text{ times}} = |A|^n$ **learn**

& $\text{Tr}(A \text{adj } A) = |A| + |A| + |A| + \dots + |A| = n|A|$ **learn**

Q. If $A_{n \times n}$ then Evaluate $|adj A|, |cof A|, |\bar{A}'| = ?$

Proof: w.k. that $|A(adj A)| = |A|^n$

$$|A| \cdot |adj A| = |A|^n$$

$$\boxed{|adj A| = |A|^{n-1}}$$

Now again w.k. that, $|B| = |B^T|$

$$\text{So } |cof A| = |(cof A)^T| = |adj A| = |A|^{n-1}$$

$$\text{ie } \boxed{|cof A| = |A|^{n-1}}$$

$$\text{Now, } \bar{A}' = \frac{adj A}{|A|}$$

$$\Rightarrow |\bar{A}'| = \left| \frac{(adj A)}{|A|} \right| = \frac{1}{|A|^n} \cdot |adj A|$$

$$|\bar{A}'| = \frac{|A|^{n-1}}{|A|^n} = \frac{1}{|A|}$$

$$\text{Hence } \det \bar{A}' = \frac{1}{\det A}$$

Imp point: if $A_{n \times n}$ then $\underbrace{|\text{adj adj} \dots \text{adj} A|}_{r \text{ times}} = |A|^{(n-1)^r}$ Learn.

eg if $|A|_{4 \times 4} = 5$ then evaluate $|\text{adj adj adj} A| = ?$

(a) 5^9

$n=4, r=3$

☒ (b) 5^{27}

So $|\text{adj adj adj} A| = |A|^{(n-1)^r}$

(c) 5^4

$= (5)^{(4-1)^3} = (5)^{3^3} = 5^{27}$

(d) 5^3

Note:

$(a^b)^c = a^{bc}$, $(a)^b = (a)^{\underbrace{b \times b \times b \dots \times b}_{c \text{ times}}}$

MATRIX

→ "It is a rectangular arrangement of $m \times n$ numbers"

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

$1 \leq i \leq m$
 $1 \leq j \leq n$

H-lines = Rows
V-lines = Columns.

→ Sq. Mat - Defⁿ (1) No. of Rows = No. of Columns.

Defⁿ (2) If in a Mat, Diag exist, then it Must be Square

Defⁿ (3) if \forall element, \exists corresponding element then Mat is Square Mat.

eg Sq Mat $A = [a_{ij}]_{n \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}_{n \times n}$

$1 \leq i \leq n$
 $1 \leq j \leq n$

- ① for Diagonal elements, $i = j \forall i \neq j$
- ② for upper Diag elements, $i < j \forall i \neq j$
- ③ for Lower " " , $i > j$ " "
- ④ for off Diag elements, $i \neq j$ " "
- ⑤ Corresponding elements are a_{ij} & a_{ji}

$$\text{Trace}(\text{sq Mat}) = \sum a_{ii}$$

ie $\text{Tr}(A) = \text{sum of diag elements}$

eg $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$\text{Tr}(A) = 3$ & $|A| = -16$

Some Special types of sq. Matrices →



eg $\begin{bmatrix} 2 & 4 & -1 & 3 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

U.T.M

Tr = -2, Det = 8

$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 2 & -1 & 3 & -3 \end{bmatrix}$

L.T.M

Tr = 0, Det = 0

$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Diag. Mat.

Tr = 4, Det = 0

$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Scalar Mat

Tr = 20
Det = 5^4

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Identity Mat

Tr = 4
Det = 1

(*) Diag Mat: $A = [a_{ij}]_{n \times n}$ s.t. $a_{ij} = \begin{cases} \text{at least one diag element should be Non Zero, } i=j \\ 0, & i \neq j \end{cases}$

(*) To Find Determinant of U.T.M, L.T.M, Diag Mat, Scalar Mat, Identity Mat,
Imp just Multiply the diag elements is Det = Product of Diag elements.

Q $|A| = \begin{vmatrix} 0 & 1 & 2 & 0 \\ -1 & 0 & 4 & 0 \\ -2 & -4 & 0 & 6 \\ 0 & 0 & -6 & 0 \end{vmatrix} = ?$

(a) 0

(b) 16

☒ (c) 36

(d) 21

Given Mat is neither U.T.M, Nor L.T.M, not a Triag Mat

So we can not use Shortcut Method. as discussed in Previous slide.

Hence we will follow Conventional approach.

$$|A| = \dots \text{HW} \dots = 36$$

A matrix $A = [a_{ij}]_{n \times n}$ is said to be lower triangular if

- (a) $a_{ij} = 0$ for $i > j$
- ☒ (b) $a_{ij} = 0$ for $i < j$
- (c) $a_{ij} = 0$ for $i \geq j$
- (d) $a_{ij} = 0$ for $i \leq j$

eg $A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3} = \text{L.T.M.}$

$a_{12} = a_{13} = a_{23} = 0$

ie $a_{ij} = 0 \forall i < j$ is L.T.M

Qe If $A = [a_{ij}]_{n \times n}$ s.t $a_{ij} = 0 \forall i > j$ then A is U.T.M

Singular Mat: If $|A| = 0$ then A is called Singular Mat

Non singular Mat: If $|A| \neq 0$ then A is called Non singular Mat

Invertible Mat \rightarrow If A^{-1} exist then A is called Invertible Mat

& it is defined as $A^{-1} = \frac{\text{adj } A}{|A|}$ or $\frac{(\text{Cof } A)^T}{|A|}$

N. Condition for a Matrix to be invertible is $|A| \neq 0$ is A must be Non sing.

Sp. Conclusion: w.k. that, $AA^{-1} = A^{-1}A = I$

so if we have,

$$AB = BA = I$$

\Rightarrow $\begin{cases} \text{Inverse of } A=B \text{ i.e. } A^{-1}=B \\ \text{Inverse of } B=A \text{ i.e. } B^{-1}=A \end{cases}$

i.e. Both are the Inverses of each other.

eg If $A_{n \times n}$ s.t. $A^2 = I$ then $A^{-1} = ?$

(a) I (b) A

(c) O (d) Not possible

Sol: Given, $A \cdot A = I \Rightarrow A^{-1} = A$ is A is self Invertible
on Comparison with,
 $(Mat)(Mat)^{-1} = I$ or A is inverse of itself.

eg: If $M^4 = I$ then Various Conclusions are?

(i) $M \cdot M^3 = I \Rightarrow M^{-1} = M^3$

(ii) $M^2 \cdot M^2 = I \Rightarrow (M^2)^{-1} = M^2$

(iii) $M^3 \cdot M = I \Rightarrow (M^3)^{-1} = M$

$M^{-1} = ?$
 $(M^2)^{-1} = ?$
 $(M^3)^{-1} = ?$

Shortcut Method of finding Inverse of 2×2 Mat \rightarrow



$$\text{if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{eg } A = \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}, \text{ then } A^{-1} = ? = \frac{1}{(14)} \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$$

$$\text{eg } A = \begin{bmatrix} 3+4i & -i \\ i & 3-4i \end{bmatrix} \text{ then } A^{-1} = ? = \frac{1}{24} \begin{bmatrix} 3-4i & +i \\ -i & 3+4i \end{bmatrix}$$

$$\underline{\text{Sol:}} \quad |A| = (3+4i)(3-4i) - (i)(-i)$$

$$= \dots \dots \dots$$

$$= 24 \because |A| \neq 0 \Rightarrow A^{-1} \text{ exist}$$

Shortcut Method of finding Inverse of 3×3 Mat \rightarrow



eg $A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 0 & 3 \\ -3 & 1 & 4 \end{bmatrix}$ then $A^{-1} = ? = \frac{\text{adj } A}{|A|} = \frac{(\text{Cof } A)^T}{|A|} = \frac{1}{35} \begin{bmatrix} -3 & 10 & -6 \\ -17 & 10 & 1 \\ 2 & 5 & 4 \end{bmatrix}$

Sol: $|A| = \dots = 35$

$\therefore |A| \neq 0 \Rightarrow A^{-1}$ exist.

1	-2	2	1	-2
2	0	3	2	0
-3	1	4	-3	1
1	-2	2	1	-2
2	0	3	2	0

$\text{Cof } A = ? = (\text{adj } A)^T = \begin{bmatrix} -3 & -17 & 2 \\ 10 & 10 & 5 \\ -6 & 1 & 4 \end{bmatrix}$

Top Row of $A^{-1} = ? = \begin{bmatrix} -3 & 10 & -6 \end{bmatrix} \times$
 $= \begin{bmatrix} \frac{-3}{35} & \frac{10}{35} & \frac{-6}{35} \end{bmatrix} \checkmark$

If $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$. Then top row of R^{-1} is

(a) $[5 \ 6 \ 4]$

(b) $[5 \ -3 \ 1]$

(c) $[2 \ 0 \ -1]$

(d) $\begin{bmatrix} 2 & -1 & \frac{1}{2} \end{bmatrix}$

$$|R| = 1[2+3] + (-1)[6-2] = 5-4 = 1$$


$$R^{-1} = \frac{\text{adj } R}{|R|} = \frac{1}{(1)} \begin{bmatrix} 5 & -3 & 1 \\ - & - & - \\ - & - & - \end{bmatrix}$$

$$\begin{array}{c|ccc} 1 & 0 & -1 & 1 & 0 \\ \hline 2 & 1 & -1 & 2 & 1 \\ 2 & 3 & 2 & 2 & 3 \\ 1 & 0 & -1 & 1 & 0 \\ 2 & 1 & -1 & 2 & 1 \end{array}$$

(ii) $\text{Cof of } a_{21} = ? = -3$ Ans

$$\therefore \text{Cof } A = \begin{bmatrix} 5 & - & - \\ -3 & - & - \\ - & - & - \end{bmatrix}$$

P8 $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -6 \\ -10 & -14 & 6 \end{bmatrix}$ then $A^{-1} = ?$, $\text{adj} A = ?$

Sol $\because |A| = \dots = 0$ 
 $\therefore A$ is singular so $A^{-1} = \text{DNE}$

HW

(ii)

	1	2	3	1	2
4	5	-6	4	5	
-10	-14	6	-10	-14	
1	2	3	1	2	
4	5	-6	4	5	

$$\text{adj} A = (C(A))^T = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$$

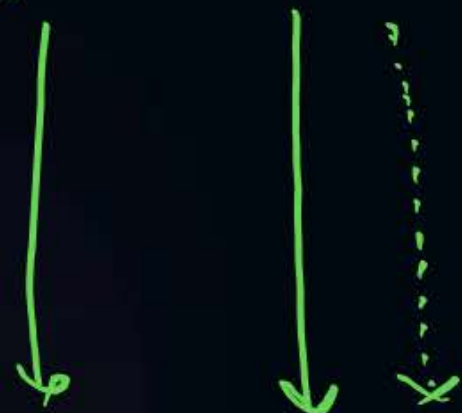
Methods of finding Inverse of $4 \times 4, 5 \times 5, 6 \times 6 \dots$ Matrices



Not in a Syllabus by Conventional Approach.



Procedure: write $A = I \cdot A$



using E-operation

$$\boxed{I = B \ A} \Rightarrow A^{-1} = B = \underline{A_{adj}}$$

⑧ Addition & Subtraction in Matrix →

this is defined only when Matrices are of Same order

for eg $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & 2 \end{bmatrix}_{2 \times 3}$, $B = \begin{bmatrix} -1 & 3 & 0 \\ 1 & 2 & 5 \end{bmatrix}_{2 \times 3}$, $C = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}_{3 \times 3}$

$$A+B = B+A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 2 & 7 \end{bmatrix}_{2 \times 3}$$

$$A-B = \begin{bmatrix} 3 & -4 & 3 \\ 3 & -2 & -3 \end{bmatrix} \text{ i.e. } A-B \neq B-A$$

$$B-A = \begin{bmatrix} -3 & 4 & -3 \\ -3 & 2 & 3 \end{bmatrix}$$

$$A+C = \text{N.D.} \text{ \& } B+C = \text{N.D.}$$

“Matrix Addition is commutative
But Mat subtraction is
not commutative”

⊗ Matrix Multiplication → is defined when

“Number of Columns in 1st Mat = Number of Rows in 2nd Matrix”

for eg, consider $A_{m \times n}$ & $B_{n \times p}$ then $AB = []_{m \times p}$.

$BA = B_{n \times p} A_{m \times n} = \text{N.D.}$

eg: $A = [1 \ 2 \ -4]_{1 \times 3}$ & $B = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}_{3 \times 1}$ then $AB = ?$ & $BA = ?$

Sol: $AB = [1 \ 2 \ -4]_{1 \times 3} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}_{3 \times 1} = [(2 + 2 + (-12))] = [-8]_{1 \times 1} = -8$ Multi = 3 times
Add = 2 times

$BA = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}_{3 \times 1} [1 \ 2 \ -4]_{1 \times 3} = \begin{bmatrix} 2 & 4 & -8 \\ 1 & 2 & -4 \\ 3 & 6 & -12 \end{bmatrix}_{3 \times 3}$

No. of Multi used = 9 times
No. of Addition used = 0 times

The value of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 3 & 2 & 4 \end{bmatrix} \times \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}$ equals

PYQ

(a) $\begin{bmatrix} 52 \\ -104 \\ 156 \end{bmatrix}$ (b) $[52 \quad -104 \quad 15]$

(c) $\begin{bmatrix} 52 \\ 104 \\ 156 \end{bmatrix}$ (d) None of these

$$AB = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3 \times 1} \begin{bmatrix} 3 & 2 & 4 \end{bmatrix}_{1 \times 3} = \begin{bmatrix} 3 & 2 & 4 \\ 6 & 4 & 8 \\ 9 & 6 & 12 \end{bmatrix}_{3 \times 3}$$

$$(AB)C = \begin{bmatrix} 3 & 2 & 4 \\ 6 & 4 & 8 \\ 9 & 6 & 12 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}_{3 \times 1} \rightarrow \downarrow$$

$$= \begin{bmatrix} (12+12+28) \\ (24+24+56) \\ (36+36+84) \end{bmatrix}_{3 \times 1} = \begin{bmatrix} 52 \\ 104 \\ 156 \end{bmatrix}_{3 \times 1}$$

M-II $BC = []_{1 \times 1}$

$$A(BC) = \begin{bmatrix} - \\ - \\ - \end{bmatrix}_{3 \times 1} \begin{bmatrix} - \end{bmatrix}_{1 \times 1} = \begin{bmatrix} 52 \\ 104 \\ 156 \end{bmatrix}_{3 \times 1}$$

Ex: $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 1 & 2 \end{bmatrix}_{2 \times 3}$ & $B = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 0 & 4 \\ 1 & 2 & -2 \end{bmatrix}_{3 \times 3}$ Then $\begin{cases} AB = ? \\ BA = ? = \text{N.D.} \end{cases}$

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 1 & 2 \end{bmatrix}_{2 \times 3} \begin{bmatrix} 2 & -1 & 3 \\ -1 & 0 & 4 \\ 1 & 2 & -2 \end{bmatrix}_{3 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2 \times 3} = \begin{bmatrix} 7 & 5 & -11 \\ 9 & 0 & 12 \end{bmatrix}$$

$$a_{11} = (1) \cdot (2) + (-2) \cdot (-1) + (3) \cdot (1) = 7$$

$$a_{12} = (1) \cdot (-1) + (-2) \cdot (0) + (3) \cdot (2) = 5$$

$$a_{13} = (1) \cdot (3) + (-2) \cdot (4) + (3) \cdot (-2) = -11$$

$$a_{21} = (4) \cdot (2) + (1) \cdot (-1) + (2) \cdot (1) = 9$$

$$a_{22} = (4) \cdot (-1) + (1) \cdot (0) + (2) \cdot (2) = 0$$

$$a_{23} = (4) \cdot (3) + (1) \cdot (4) + (2) \cdot (-2) = 12$$

Number of times, symbol of Multiplication is used to find $AB = 18$ times

Number of times, symbol of addition is used to find $AB = 12$ times

Shortcut: Consider $A_{m \times n}$ & $B_{n \times p}$ then to find $(AB)_{m \times p}$

Number of Multiplications Required = $m \cdot n \cdot p$ & Number of Additions Req = $m(n-1) \cdot p$

eg if $A_{2 \times 3}$ & $B_{3 \times 3}$ then to find AB
 $m=2, n=3, p=3$
 No. of Multi Req = $2 \times 3 \times 3 = 18$ times
 No. of Additions Req = $2(3-1)3 = 12$ times

eg if $A_{3 \times 4}$ & $B_{4 \times 3}$ then to find (AB)
 $m=3, n=4, p=3$
 No. of Multi Req = $3 \times 4 \times 3 = 36$ Times
 No. of Additions Req = $3(4-1)3 = 27$ Times

& to find (BA)
 $B_{4 \times 3} A_{3 \times 4}$ = Defined. $m=4, n=3, p=4$
 No. of Multi Req = $4 \times 3 \times 4 = 48$
 No. of Additions Req = $4(3-1)4 = 32$

MAJEDAR QUESTION → Consider $A_{2 \times 3}$, $B_{3 \times 4}$ & $C_{4 \times 2}$ then find the minimum number of Multiplications & Additions that will be Required to find Matrix product (ABC) ?

Sol: we can find (ABC) either by using

- Case I: $(AB)C$
 - Multi = 40
 - Additions = 28
- Case II: $A(BC)$
 - Multi = 36
 - Addition = 26

ie Min. Multi Required = 36
 Min Addition Required = 26 (obtained by Case II)

Commutative Law

(*)

$$A + B = B + A$$

But $AB \neq BA$

Associative Law

$$A + (B + C) = (A + B) + C$$

& $A(BC) = (AB)C$

Matrix Addition is Commutative as well as Associative.

While Matrix Multiplication is associative but not commutative in general.

Note Although $AB \neq BA$ in general But $\boxed{\text{Tr}(AB) = \text{Tr}(BA)}$ always.

e.g. $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 4 & 0 \end{bmatrix}$

& $B = \begin{bmatrix} 2 & 1 \\ 0 & -2 \\ 1 & 4 \end{bmatrix}$

then

$$AB = \begin{bmatrix} 5 & 17 \\ 2 & -7 \end{bmatrix}$$

$$\text{Tr}(AB) = -2$$

$$BA = \begin{bmatrix} 3 & 0 & 6 \\ -2 & -8 & 0 \\ 5 & 14 & 3 \end{bmatrix}$$

$$\text{Tr}(BA) = -2$$

hence verified

If A & B are two matrices of same order then which of the following is true.

\Downarrow
AB & BA both defined

(a) \times $(A + B)^2 = A^2 + 2AB + B^2$

(b) \times $(A - B)^2 = A^2 - 2AB + B^2$

(c) \checkmark $(A + B)^2 + (A - B)^2 = 2A^2 + 2B^2$

(d) \times $(A + B)(A - B) = A^2 - B^2$

(d) $(A+B)(A-B) = A^2 - AB + BA + B^2$

(a) $(A+B)^2 = (A+B)(A+B)$
 $= A^2 + AB + BA + B^2$

$(AB \neq BA)$

(b) $(A-B)^2 = A^2 - AB - BA + B^2$
 $\quad \quad \quad ?$

(c) $(A+B)^2 + (A-B)^2$

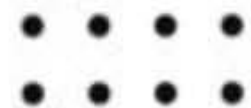
$= (A^2 + B^2 + AB + BA)$

$+ (A^2 + B^2 - AB - BA)$

$= 2A^2 + 2B^2$



Thank
you



Keep Hustling!