DATA SCIENCE
&
ARTIFICIAL INTELLIGENCE
& CS/IT

Calculus and Optimization

Lecture No. 10

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic Maxima - Minima (Part 1)

Topics to be Covered

Topic MAXIMA & MINIMA

(Part 2)

MAXIMA MINIMA

RECAP

T-1 → Increasing Dec. func^n

T-2 → Max-Min of Curve $y = f(x)$

T-3 → Max Min of Surface $z = f(x, y)$

(*) Ist Derivation
- → Geometrical Significance → To know the Slope of Tangent
- → Application → To know the Inc or Dec Nature of func^n.

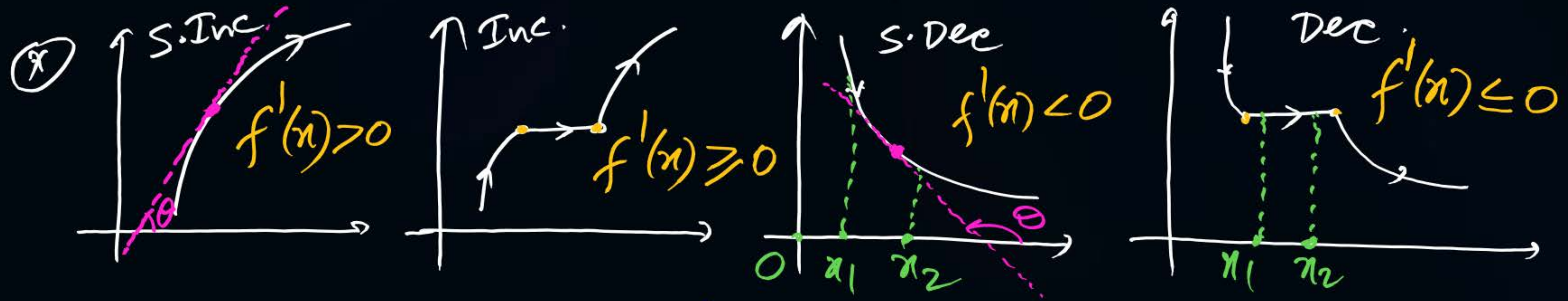
(*) 2nd Derivatives
- → Geometrical Significance → To know Concavity of func^n
- → Application → To know Inflexion Point, Point of Max & Point of minima

RECAP

INCREASING & DECREASING FUNCTION

w.k. that for $y = f(x)$, $\frac{dy}{dx} = \boxed{f'(x) = \tan\theta} =$ slope of tangent at any Random point x on $f(x)$

1. for Acute angle tangent, $f'(x) > 0$
2. " obtuse " " , $f'(x) < 0$
3. for Horizontal tangent, $f'(x) = 0$
4. for Vertical Tangent, $f'(x) =$ D.N.E



eg: if $f(x_1) > f(x_2)\ \forall\ x_1 < x_2$; where $x_1, x_2 \in D_f$ then $f(x)$ is S.Dec.

eg: if $f(x_1) \geq f(x_2)$ " " " " " " " " " Decreasing

(*) Monotonic func's → Those functions which are either S.I or S.D are called Monotonic functions

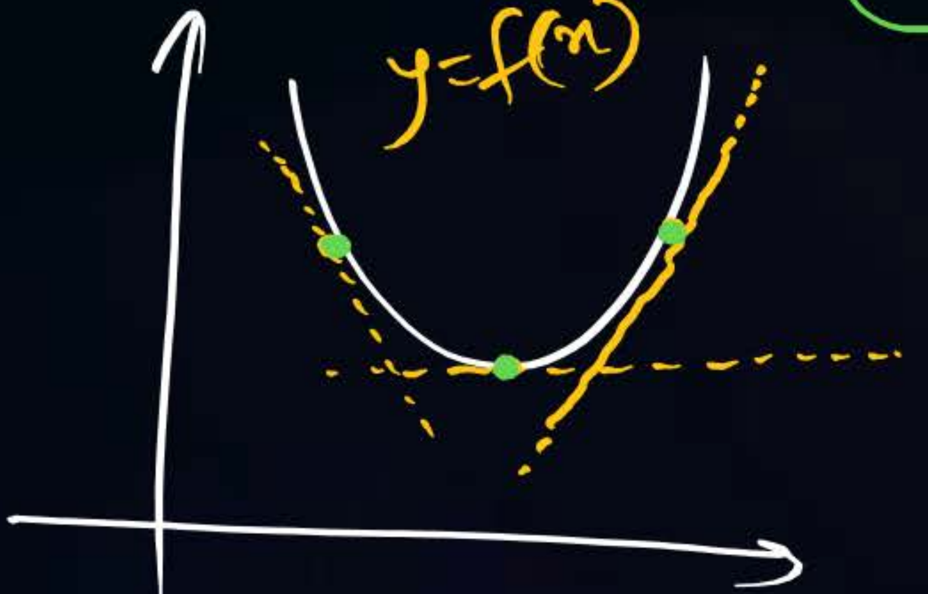
(*) Advice → It is advisable to first write the Domain of function while solving Questions Based on Inc/Dec functions.

RECAP

Information: RECAP

Concave upward Curve:

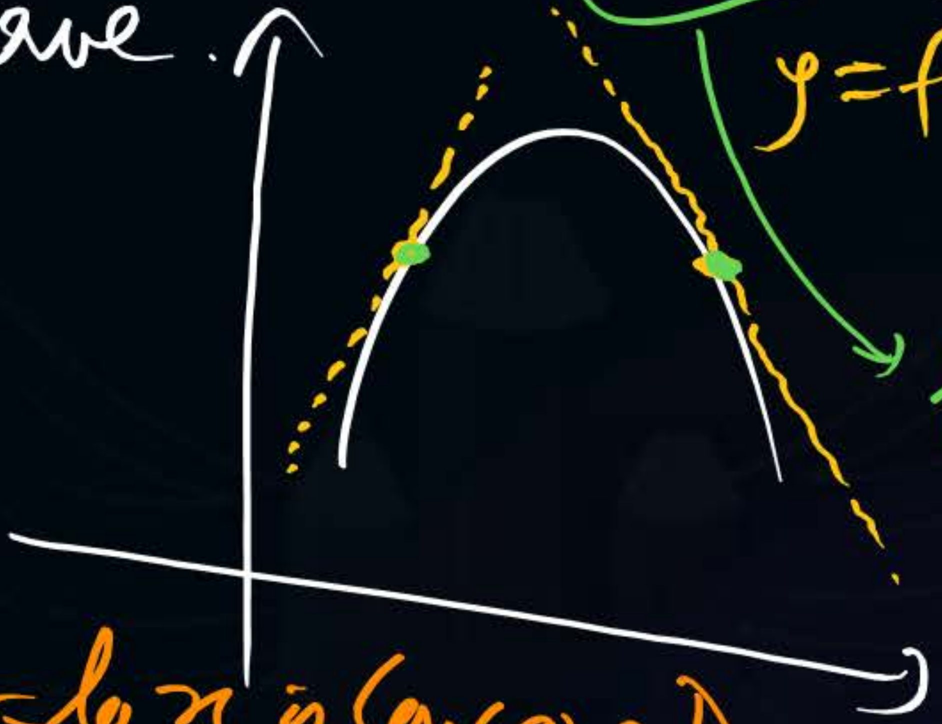
if curve lies above the tangent always then curve is called concave upward curve $\Rightarrow f''(x) > 0$



eg $f(x) = e^x$ is concave upward curve

Concave Downward Curve

if curve lies below the tangent always then curve is called concave Downward curve. $\Rightarrow f''(x) < 0$

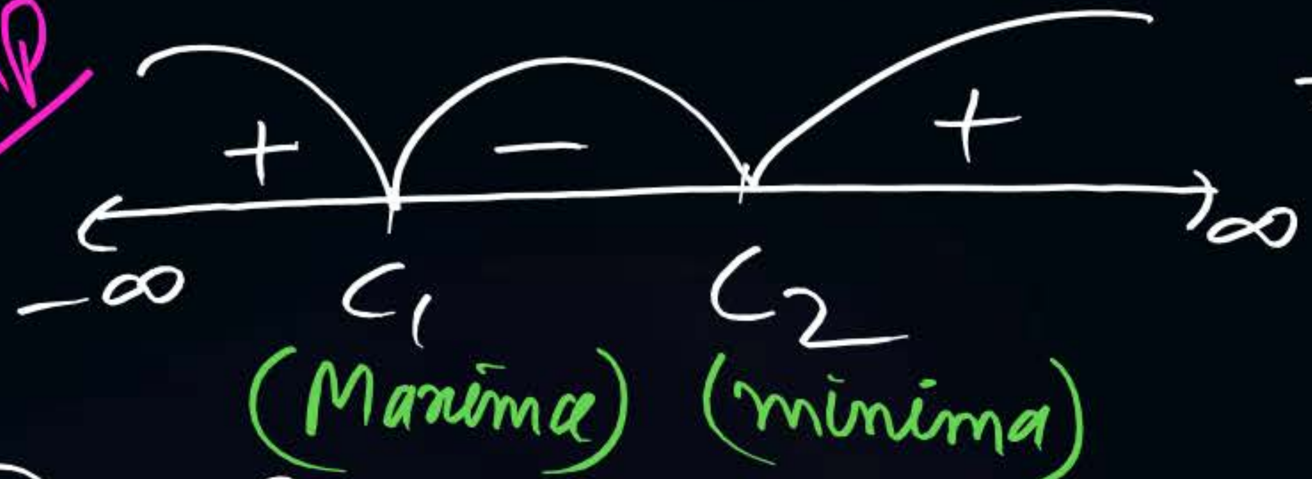


eg $f(x) = \log_e x$ is Concave Downward Curve

Shortcuts: Put $\boxed{f'(x)=0}$ & Try to find T. Points (say $x = c_1, c_2$)

(1) then check sign of $f'(x)$ as follows;

RECAP



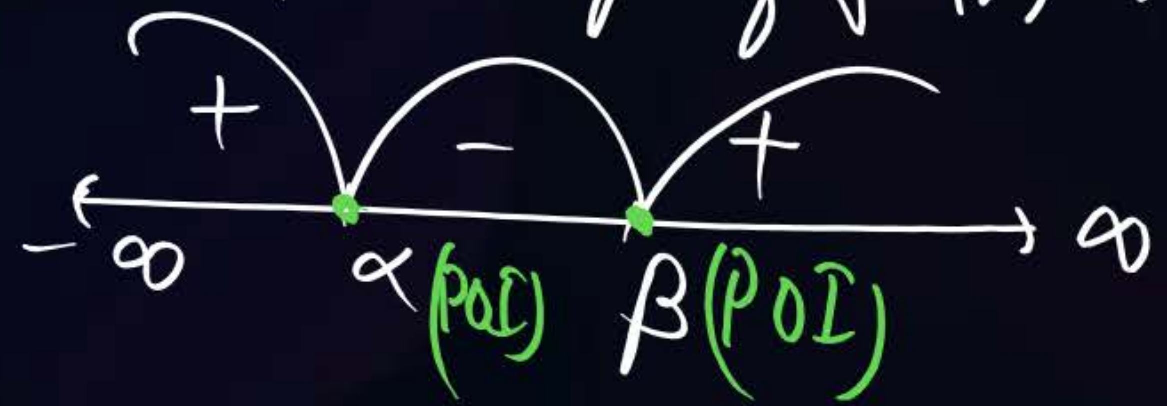
$\Rightarrow f(x)$ Inc in $(-\infty, c_1] \cup [c_2, \infty)$

$f(x)$ Dec in $[c_1, c_2]$

(2) To find the Interval where $f(x)$ is Concave up or Concave Downward →

Put $\boxed{f''(x)=0}$ & Try to find x (say $x = \alpha, \beta$)

then check the sign of $f''(x)$ as follows;



$\Rightarrow f(x)$ is Concave upward in $(-\infty, \alpha] \cup [\beta, \infty)$

$f(x)$ is Concave Downward in $[\alpha, \beta]$

Saddle point / point of Inflexion →

"Those points where curve changes it's concavity are called points of Inflexion"

Note: (1) those points where we are getting Neither Maxima, Nor minima are called Points of Inflexion (F)

RECAP

(2) if ($x = \alpha$ is the point of Inflexion) $\Longrightarrow$ (At $x = \alpha$ we will get N.M.N.M)

$\not\Longleftarrow$

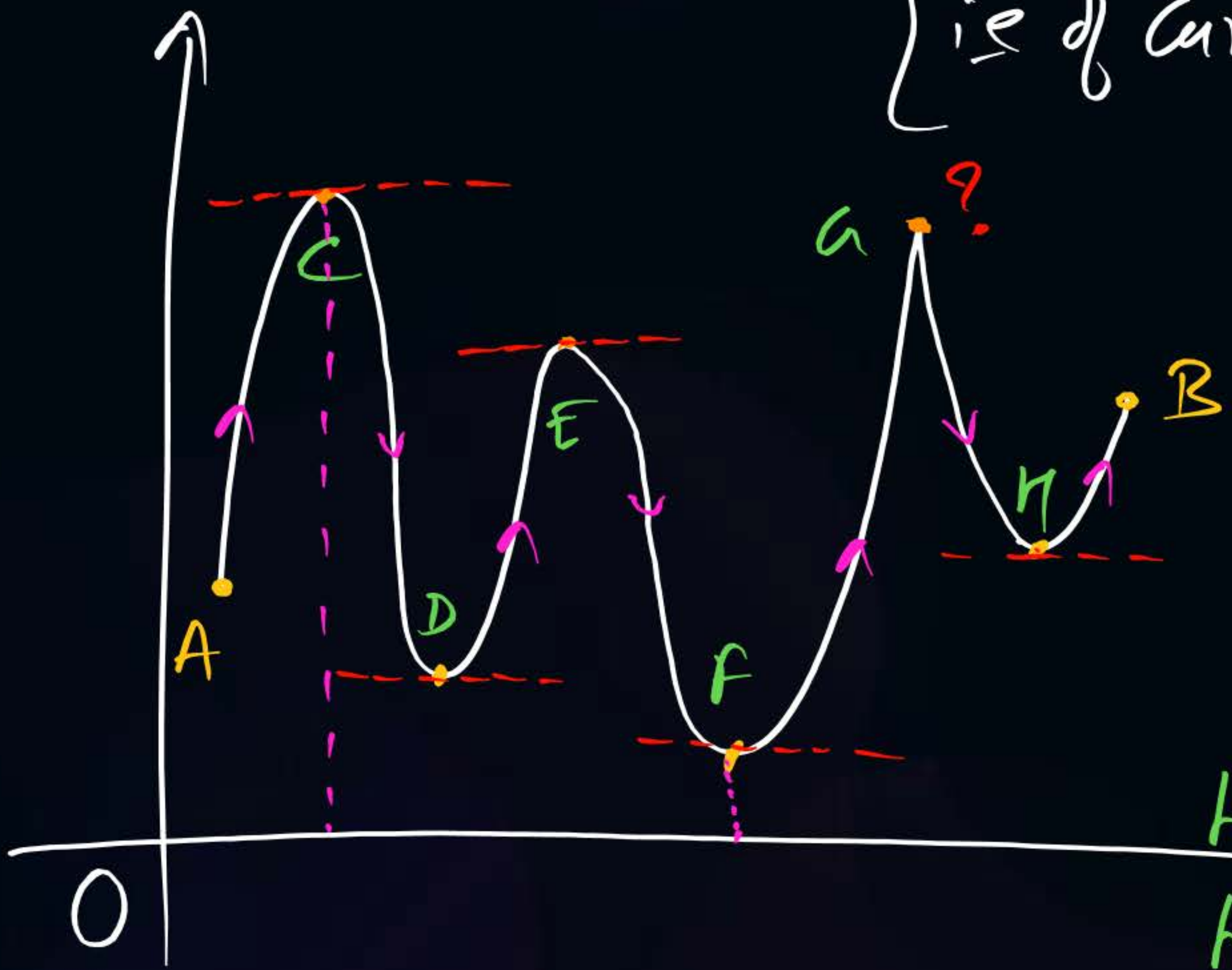
Shortcut: Put $f''(x) = 0$ & Try to get x (Say it is $x = \alpha$)

if $f'''(\alpha) \neq 0 \Rightarrow x = \alpha$ is point of Inflexion

& if $f'''(\alpha) = 0 \Rightarrow$ we can't say anything about α.

MAXIMA-MINIMA of funcⁿ of Single Variable

[i.e of curve $y = f(x)$.] → 12^{th} class.



Point ≃ x

Value ≃ y

Local Max points → C, E, G, B

Local Max Values → $f(C), f(E), f(G), f(B)$

Absolute Maxima/Global → $f(C)$

Local min Points → A, D, F, H

Local Min Values → $f(A), f(D), f(F), f(H)$

Absolute Minima/Global: $f(F)$

(1) Necessary Condition for Maxima-Minima → $\boxed{f'(x) = 0 \text{ or Not Defined}}$ — (1)

(2) Stationary Points: eg C, D, E, F, H

those points where tangent is horizontal are called stationary points.

i.e at stationary points, $f'(x) = 0$

(3) Turning Points/critical Points → eg C, D, E, F, H, G

those points where tangent is either horizontal or Not defined as called T. Points. i.e at T. Points, $f'(x) = 0$ or undefined

→ All the Stationary Points are Turning Points But converse is not Necessarily True (T)

→ By solving eqn (1) we can find Turning Points.

(4) Extreme Points / Optimal Points → eg $A, \underbrace{C, D, E, F, G, H}_{\text{T. Points}}, B$

those points where we get either Maxima or Minima are called E-Points

→ Corner Points are also Extreme points (True)

But take care; Corner points are not critical Points.

(5) Points of Inflexion / saddle points →

those points where curve changes it's concavity are called saddle points

(6) Maxima & Minima occurs alternately (True)

(7) Sufficient Condn for Maxima-Minima → f(x) must satisfy either 1st Derivative test or 2nd Derivative test.

(8) 1^{st} Derivative test :→ Let $x = c$ is the Turning point.

(it is obtained by solving eqn (1)) i.e. $f'(c) = 0$ or undefined.

Now we will check the sign of $f'(c)$ as follows, M. Imp.

(Maxima) (minima) (NNNM) (NMNM)

Explanation :-

DEVELOPING NEURONS →

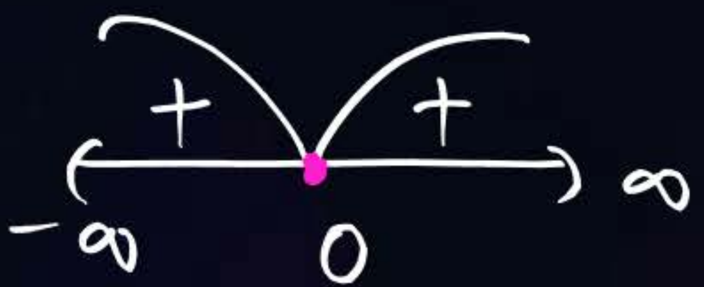
eg $y=f(x)=x^3$, $\boxed{f'(x)=3x^2}$, $f''(x)=6x$, $f'''(x)=6$

Putting $f'(x)=0$

$3x^2=0$

$x=0$

So $\boxed{\text{T. Point is } x=0}$



NMNM.

Putting $f''(x)=0$

$6x=0$

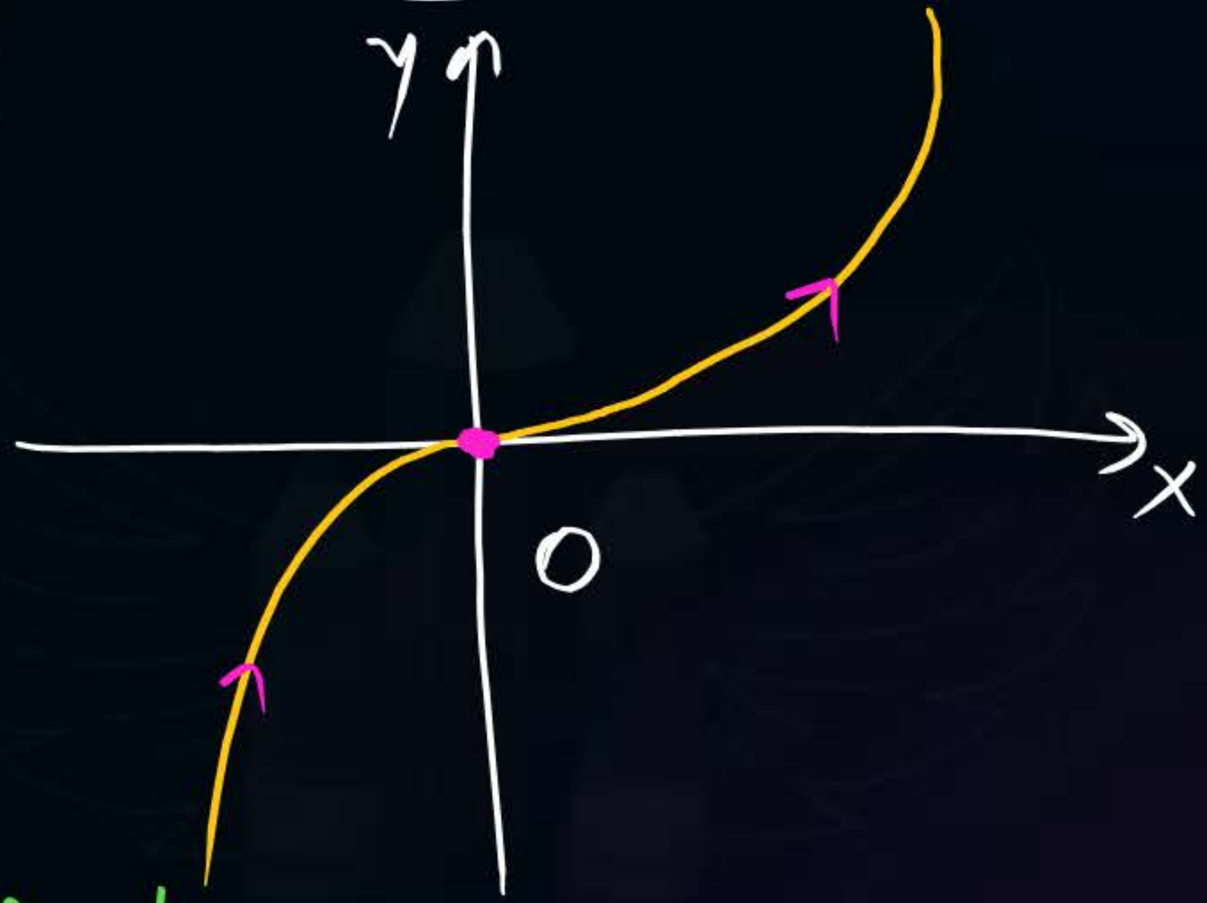
$x=0$

So $f'''(0)=\left[6\right]_{x=0}=6$

i.e. $f'''(0)\neq 0$

$\therefore$ $\boxed{x=0 \text{ is P.O.I}}$

M-II $y=x^3$



(*) Inflexion Points may or may not be a Turning Points.

(9) **2nd Derivative test** → Let $x = c$ is critical point then

$$\boxed{f'(c) = 0}$$

& if $f^{n}(c) = 0$ & $f^{n+1}(c) \neq 0$;
n = even then
$x = c$ is point of Inflexion

$f''(c) > 0$

($x = c$ is Point of Minima)

⇓

$f(x)$ is minimum at $x = c$

$f''(c) = 0$

↓

$f'''(c) \neq 0$

($x = c$ is saddle point)

$f''(c) < 0$

($x = c$ is Point of Maxima)

⇓

$f(x)$ is Maximum at $x = c$

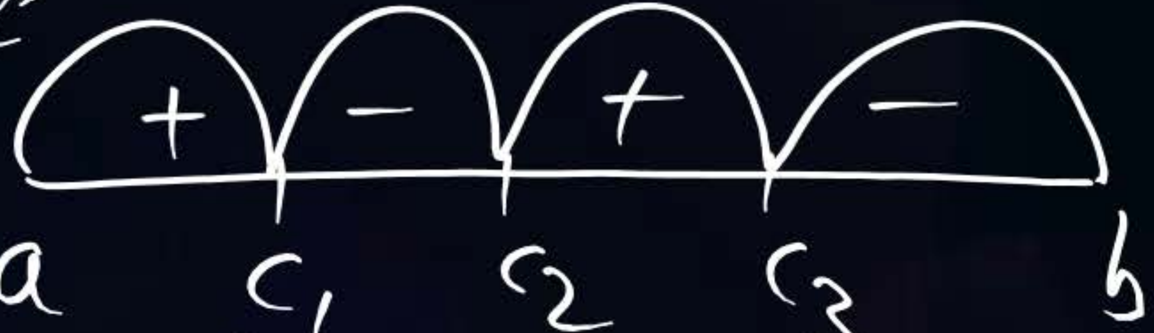
(10) If we want to check the Nature of Corner points, then we have (No) shortcut Method 😦

it's Nature can be calculated by using following theorem only

" $\boxed{\text{Maxima \& Minima occurs Alternately}}$ "

for eg let $f(x)$ is defined in $[a,b]$ & it's T. Points are $x = c_1, c_2, c_3$

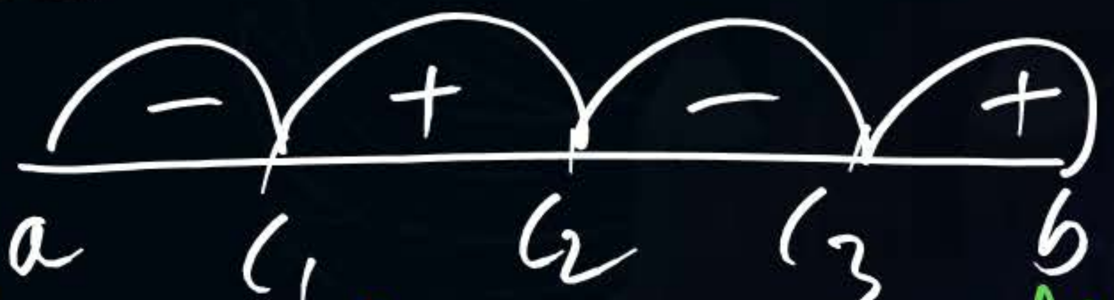
Case I:



min. Max min Max min

Sign of $f'(x)$

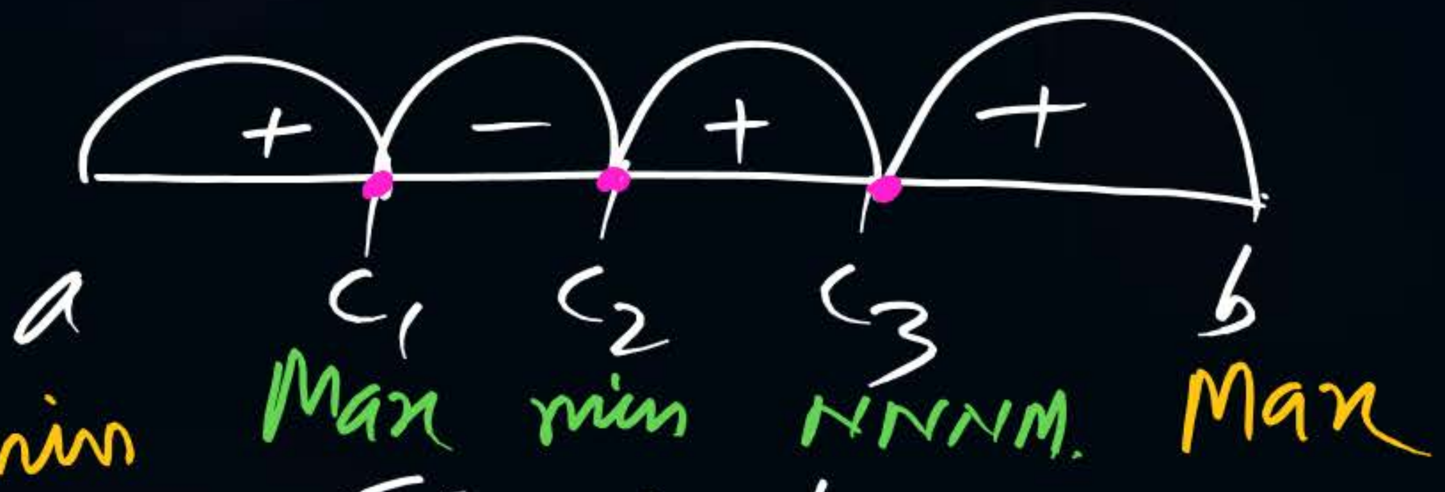
Case II:



Max. (min) Max min Max

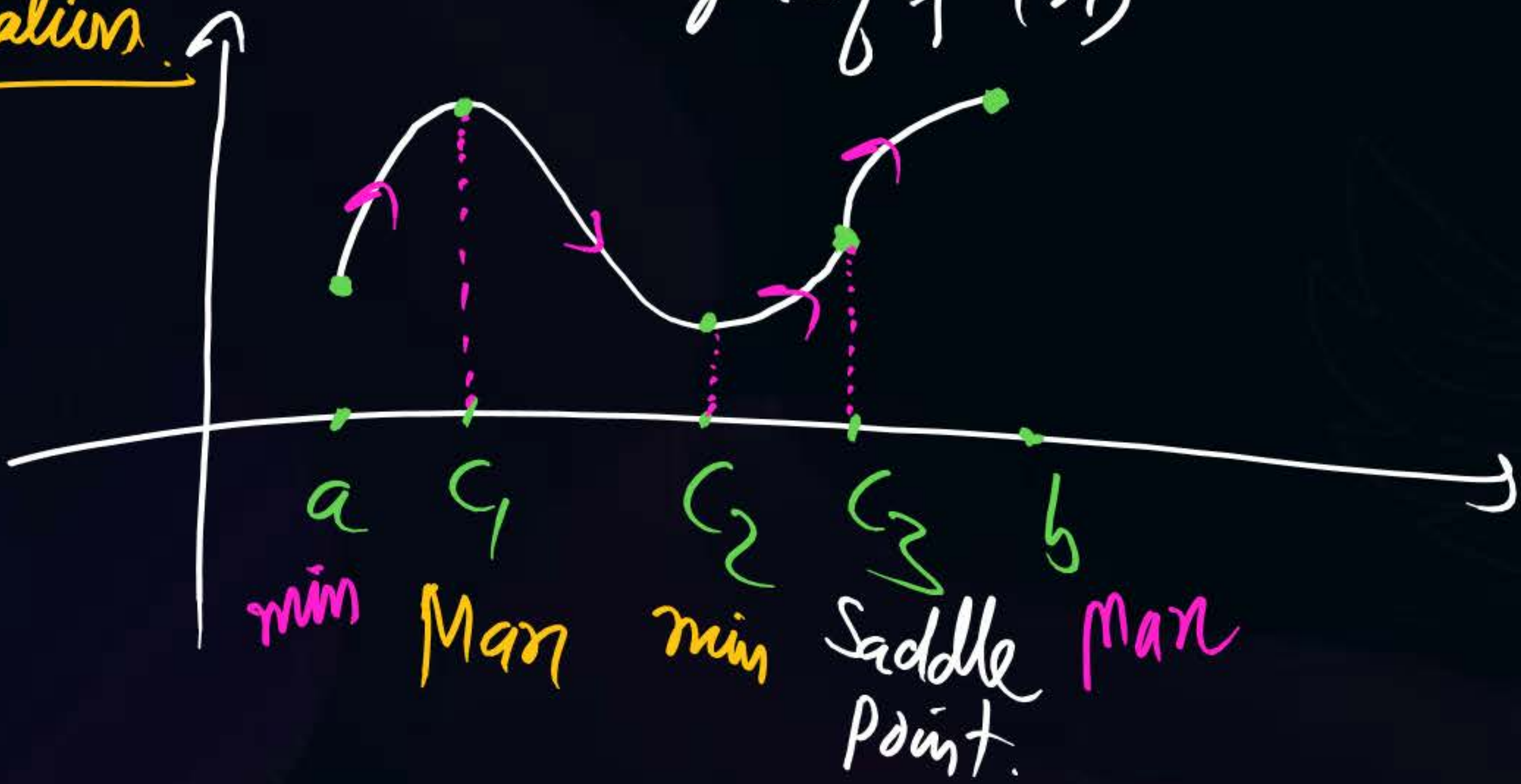
sign of $f'(x)$

Case III :



Sign of $f'(x)$

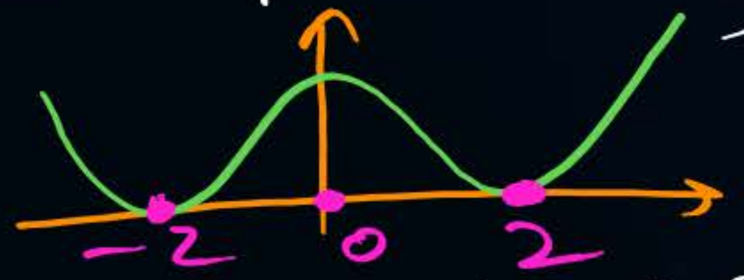
Explanation:



(1) An n^{th} degree polynomial Bends at Most $(n-1)$ times

So it has at Most $(n-1)$ extremas (Maxima & Minima both)

eg $y=(x^2-4)^2$ (it is polynomial of degree 4)



$\Rightarrow$ it has three extremas (one Local Maxima & two L. Minima)

eg: A cubic polynomial with Real Coefficients has

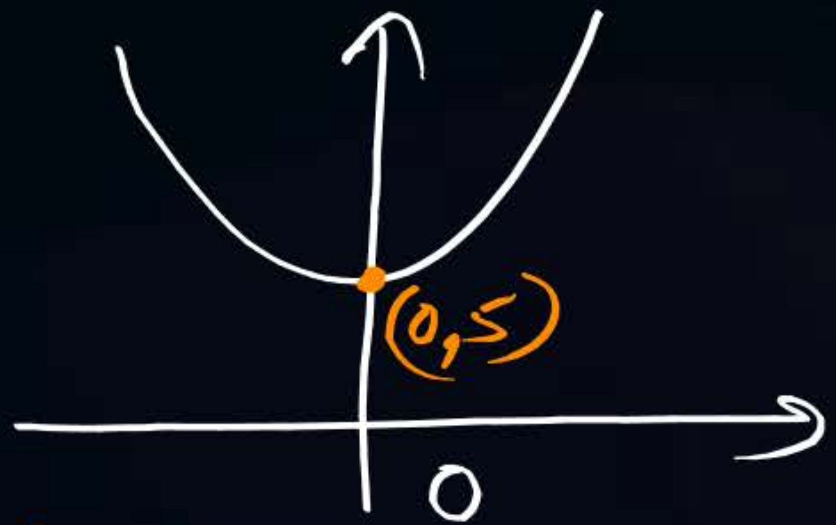
Gate at Most two extremas & three zero crossings.

(Roots)

Questions Based on type I :- (Common sense Based Questions)

Q1:- find Maxima & Minima of (1) $y = x^2 + 5$ (2) $y = \sin 3x + 4$, (3) $y = |x+3|$

Sol (1) $y = x^2 + 5$, $(-\infty, \infty)$



Point of Minima is $x = 0$

Minimum Value is $y = 5$

Max. Value = DNE

(2) $y = \sin 3x + 4$

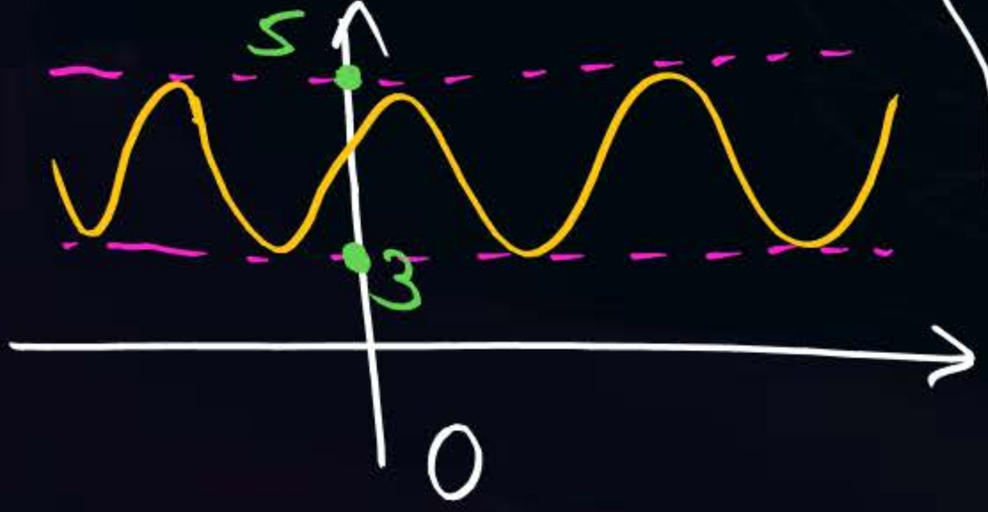
$-1 \le \sin 3x \le 1$

$3 \le (\sin 3x + 4) \le 5$

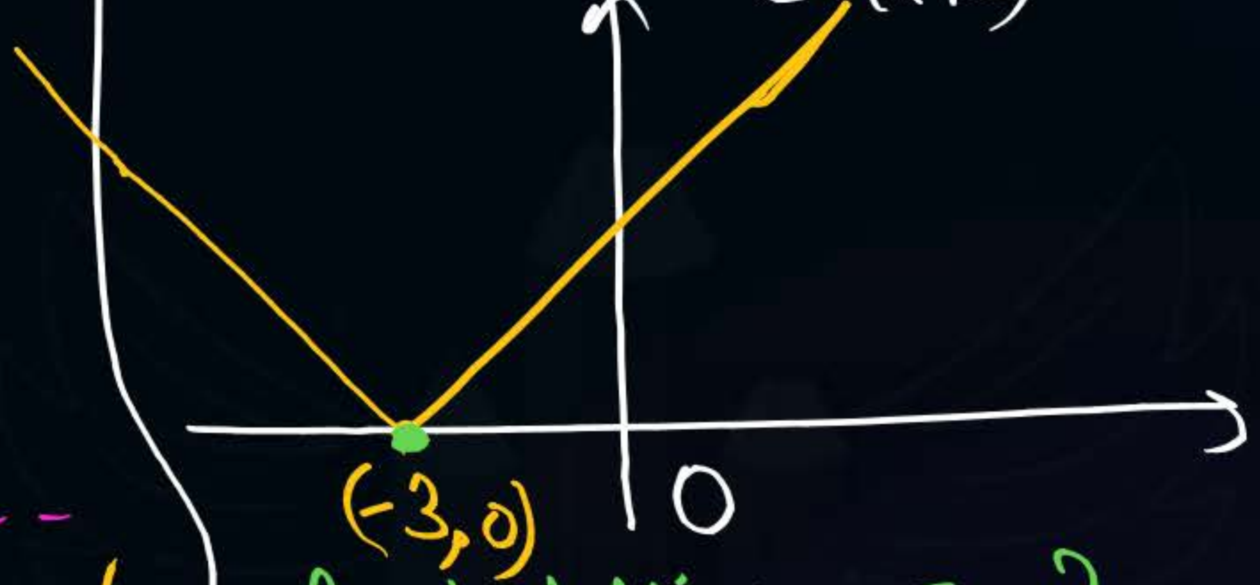
$3 \le y \le 5$

i.e. Min. Value = 3

Max Value = 5



(3) $y = |x+3| = \begin{cases} -(x+3), & x < -3 \\ +(x+3), & x > -3 \end{cases}$



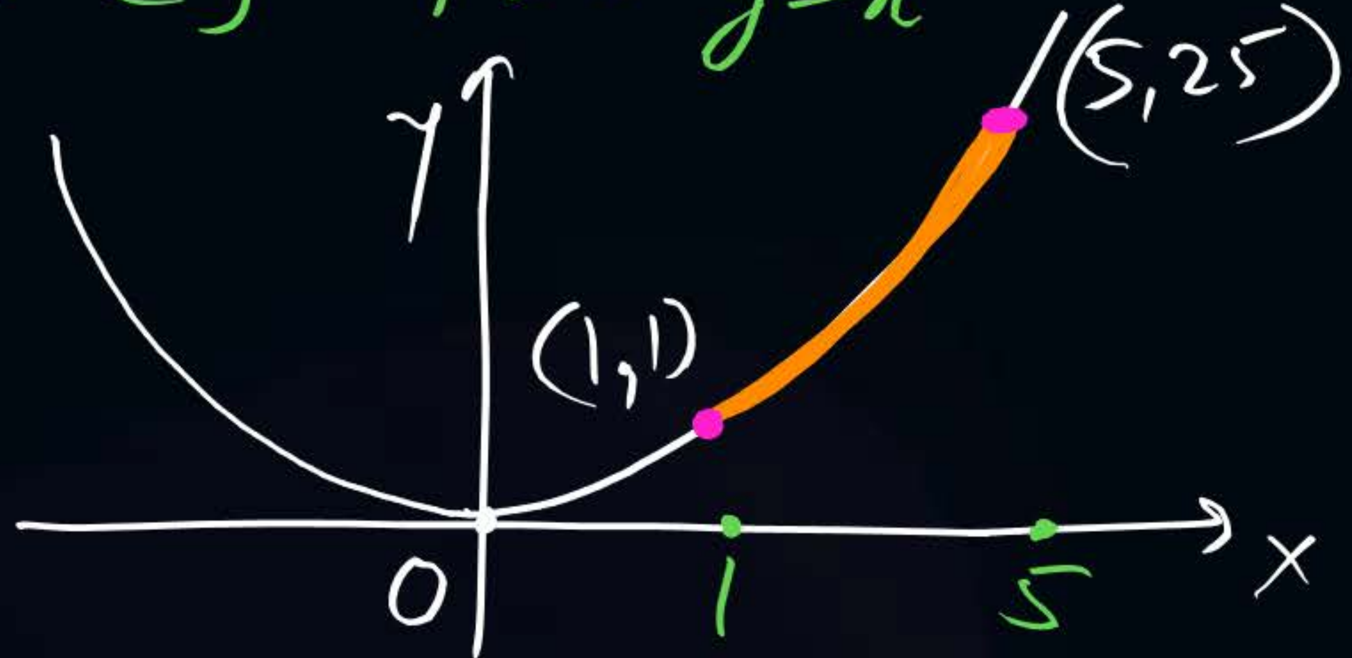
Point of Minima = -3

Min. Value = 0

Max Value = DNE

GATE: Max Value of $f(x)=x^2$ in $[1,5]$ will be ? $=25$

M-I (Using Graph) → $y=x^2$



Min Value $= f(1) = 1$ & it occurs at $x=1$

Max Value $= f(5) = 25$ & " " at $x=5$

M-II (W/O Graph) →

$f(x)=x^2$, $[1,5]$

$\boxed{f'(x)=2x}$

$f'(0^-) = -ve$

$f(0^+) = +ve$

T-Point is $f'(x)=0$

$2x=0$

$x=0$

But $x=0$ is not in given Domain So it is Total WASTE OF TIME

Now, $f(x) = x^2$, $[1,5]$

$f'(x) = 2x$

$\because f'(x) > 0 \ \forall x \in [1,5]$

So $f(x)$ is S-Inc in $[1,5]$

Hence Min & Max Value will occur at $x=1$ & $x=5$ resp.

Min Value $= f(1) = 1$

Max Value $= f(5) = 25$

Q. Find Min & Max Values of $f(x) = x^2$ in $(1,5)$

(a) 1, 5 resp (b) 1, 25 resp

(c) 0, 25 resp (d) DNE, DNE resp.

$\because$ Corner points are not there in the given Domain.

So Min Value occurs in the Right Nbd of 1 which is not defined.

& Max Value will occur in the left Nbd of 5 which is also not defined.

For real values of x, the minimum value of th

function f(x) = exp(x) + exp(−x) is

(a) 2 (b) 1

(c) 0.5 (d) 0

we know that if $x \in R^+$ then

$\boxed{x + \frac{1}{x} \geq 2}$ learn.

eg $1 + \frac{1}{1} = 2$

eg $2 + \frac{1}{2} > 2$

eg $3 + \frac{1}{3} > 2$

eg $0.2 + \frac{1}{0.2} > 2$

eg $0.5 + \frac{1}{0.5} > 2$

eg $1.5 + \frac{1}{1.5} > 2$

....

$f(x) = e^x + e^{-x}$

$f(x) = e^x + \frac{1}{e^x}$ $(\because e^x \in R^+)$

= No + it's Reciprocal

$f(x) \geq 2$

so Min $f(x) = 2$

& Max $f(x)$ = DNE

& Point of Minima is $x = 0$

Doubt: Also find Point of Minima? for $f(x) = e^x + e^{-x}$

Sol: $f(x) = e^x + e^{-x}$

$f'(x) = e^x - e^{-x}$

Put $f'(x) = 0$

$e^x - e^{-x} = 0$

$e^x = e^{-x}$

$e^x = \frac{1}{e^x}$

$e^{2x} = 1$

$\log_e(e^{2x}) = \log_e 1$

$2x(1) = 0$

$x = 0$

Q. TYPE II $f(x) = x^3 - 9x^2 + 24x + 5$, defined in $[1, 6]$ then minimum and maximum values are respectively

(a) 21, 41 ✓

(b) 4, 6 ✗

(c) 21, 25

(d) DNE, DNE ✗

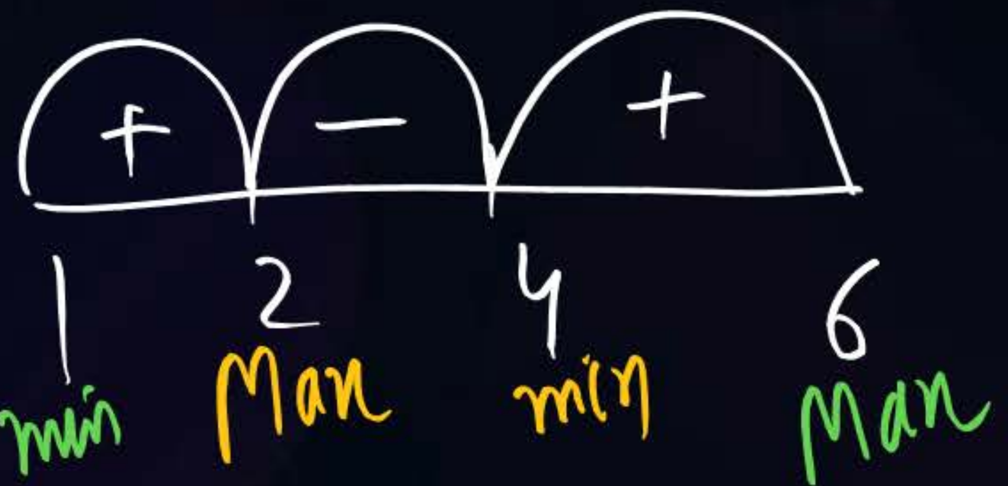
$$\boxed{f(x) = x^3 - 9x^2 + 24x + 5}$$

$$f'(x) = 3x^2 - 18x + 24$$

$$= 3(x^2 - 6x + 8)$$

$$\boxed{f'(x) = 3(x-4)(x-2)}$$

So T. Points are $x = 2$ & 4



L. Min Points are $x = 1$ & 4

L. Min Values $\begin{cases} f(1) = 21 \\ f(4) = 21 \end{cases}$

So Absolute Min. Value $= 21$

L. Max Points are $x = 2$ & 6

L. Max Values $\begin{cases} f(2) = 25 \\ f(6) = 41 \end{cases}$

So Absolute Max Value $= 41$

& it occurs at $x = 6$

Q. $f(x)=x^3-9x^2+24x+5$ then min & max values are resp.

(a) 21, 41

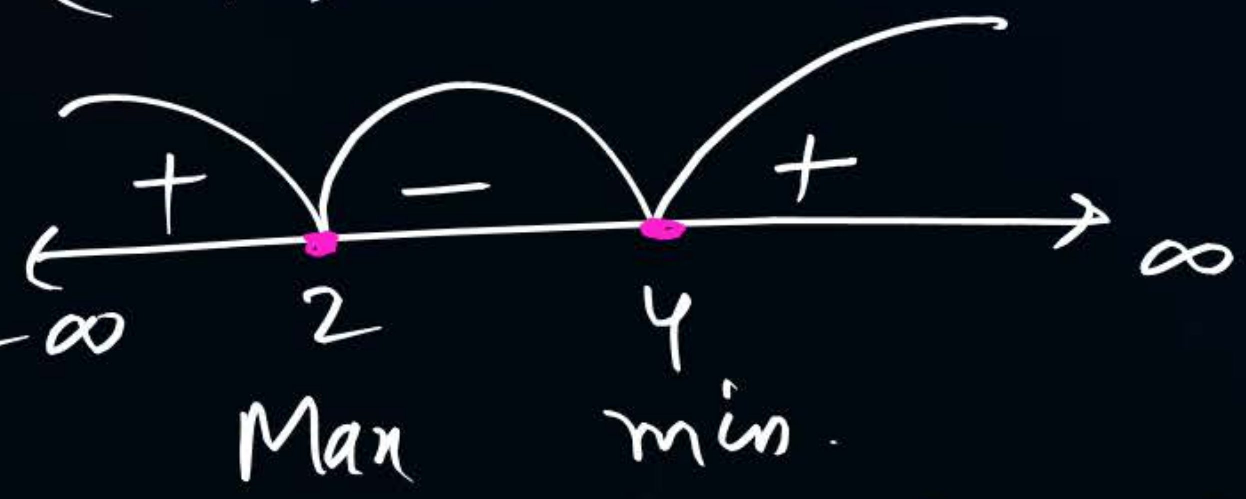
(b) 4, 6

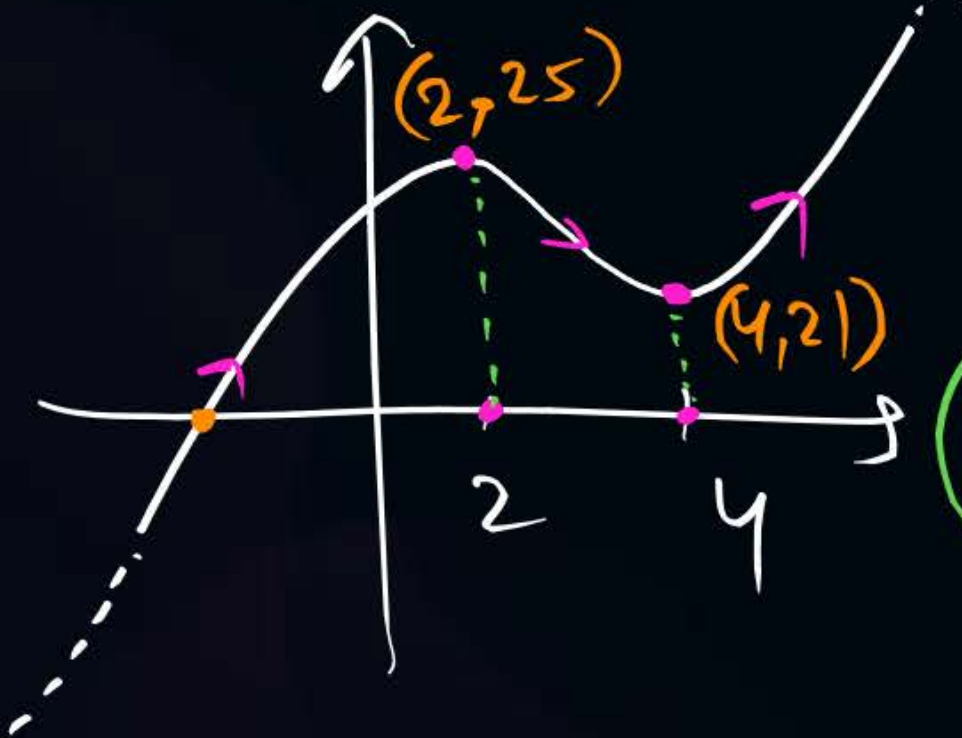
(c) 21, 25

(d) ✓ DNE, DNE

$f(x)=x^3-9x^2+24x+5$; $(-\infty,\infty)$

$$\boxed{f'(x)=3(x-2)(x-4)}$$





Local Max Value $= f(2) = 25$

Local Min Value $= f(4) = 21$

Absolute Min Value = DNE

Absolute Max Value = DNE

Q. The Maximum slope of $y = x^3 - 9x^2 + 24x + 5$ will be? ______

HW

(a) 2

(b) 3

(c) 4

(d) DNE

Q. $f(x) = \frac{4x^2+1}{x}$ then find the Interval in which $f(x)$ Inc & Dec. Also find minimum & Max Value of $f(x)$ if exist?

Sol: $f(x) = \frac{4x^2+1}{x}$, $D_f = R - \{0\}$

$$f(x) = 4x + \frac{1}{x}$$

$$f'(x) = 4 - \frac{1}{x^2} = \frac{4x^2-1}{x^2}$$

$$f'(x) = \frac{(2x-1)(2x+1)}{x^2}$$

$f'(x) = 0$ or undefined

$\Rightarrow x = \pm\frac{1}{2}$ or $x = 0$

But $x = 0 \notin D_f$ so T. Points are $x = \pm\frac{1}{2}$

$f(x)$ Inc in $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$

$f(x)$ Dec in $\left[-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right]$

Local Max Point is $x = -\frac{1}{2}$

" " Value $= f\left(-\frac{1}{2}\right) = \dfrac{4\left(-\frac{1}{2}\right)^2+1}{-1/2} = \dfrac{1+1}{-1/2} = -4$

Global Maxima = DNE

Local Min Point is $x = \frac{1}{2}$

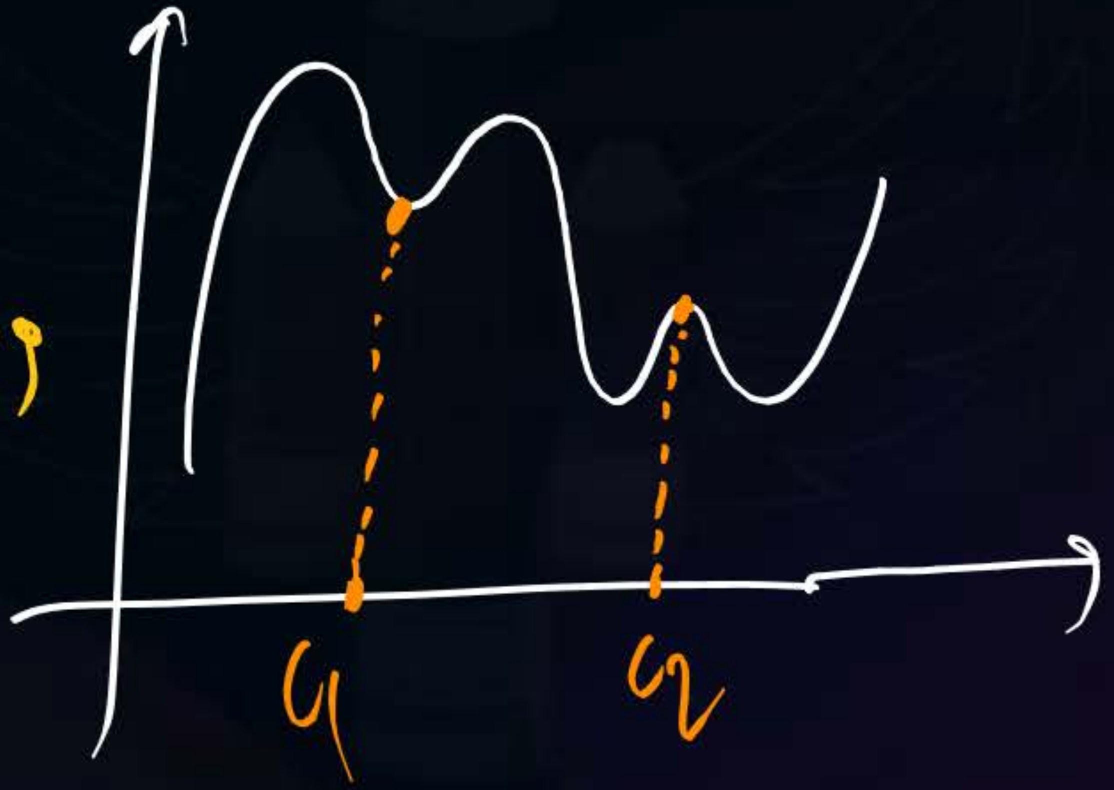
" " Value $= f\left(\frac{1}{2}\right) = \dfrac{4\left(\frac{1}{2}\right)^2+1}{1/2} = \dfrac{1+1}{1/2} = 4$

Global Minima is = DNE

GAZAB KA Conclusion →

"Local Minimum Value may be gretar that Local Max Value"

Here $f(c_1) > f(c_2)$ Hence Verified



Note: $(1)^{1/3} = 1, \omega, \omega^2$ & $(-1)^{1/3} = -1, -\omega, -\omega^2$

Q. The Minimum Value of $y = (x-1)^{2/3}$ occurs at $\underline{x=1}$ & Min Value is $\underline{f(1)=0}$

Sol: $\boxed{f(x) = (x-1)^{2/3}}$, $D_f = (-\infty, \infty)$, $f(0) = (0-1)^{2/3} = [(-1)^2]^{\frac{1}{3}} = (1)^{1/3} = 1$ exist

$$f'(x) = \frac{2}{3}(x-1)^{\frac{2}{3}-1} = \boxed{\frac{2}{3(x-1)^{1/3}}}$$

$$f'(0) = \frac{2}{3(-1)^{1/3}} = \frac{2}{3(-1)} = -ve$$

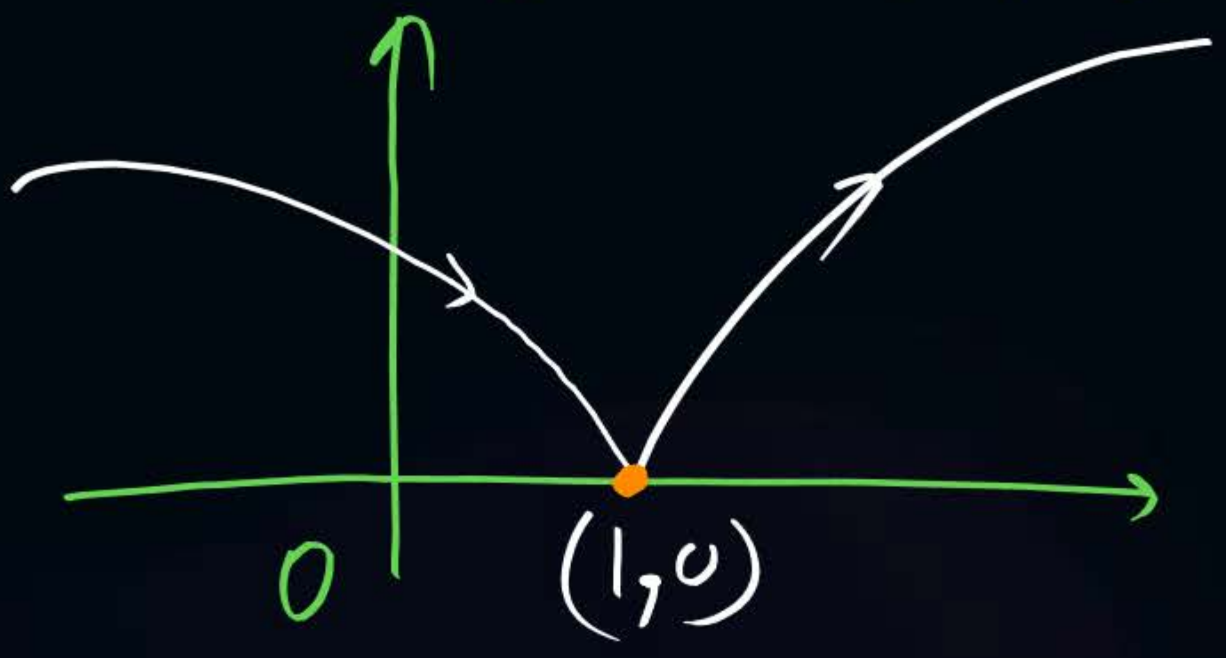
$$f'(2) = \frac{2}{3(1)^{1/3}} = \frac{2}{3(1)} = +ve$$

T-Point: $f'(x) = 0$ or $f'(x) = N.D$

$\Downarrow$ No Value of x exist

$\Downarrow$ $x = 1$ $\because f'(1) = DNE$

Graph: $y=(x-1)^{2/3}, (-\infty,\infty)$



PODCAST: $f(x)=(x-1)^{2/3}$

(1) $f'(x)=\dfrac{2}{3(x-1)^{1/3}}$, $f''(x)=\dfrac{-2}{9(x-1)^{4/3}}$

$\because f''(x)<0$ always

$\Rightarrow f(x)$ is Concave Downward Curve at every point in the Domain of $f(x)$

(2) Solve above Question by using 2nd Derivative test.

Not possible to solve because $f''(1)=$ DNE.

Q. $f(x) = \dfrac{e^{\sin x}}{e^{\cos x}}$, $x \in R$ then Max Value of $f(x)$ is ______

HW

(a) $\dfrac{3\pi}{4}$

(b) $e^{-\sqrt{2}}$

(c) $e^{\sqrt{2}}$

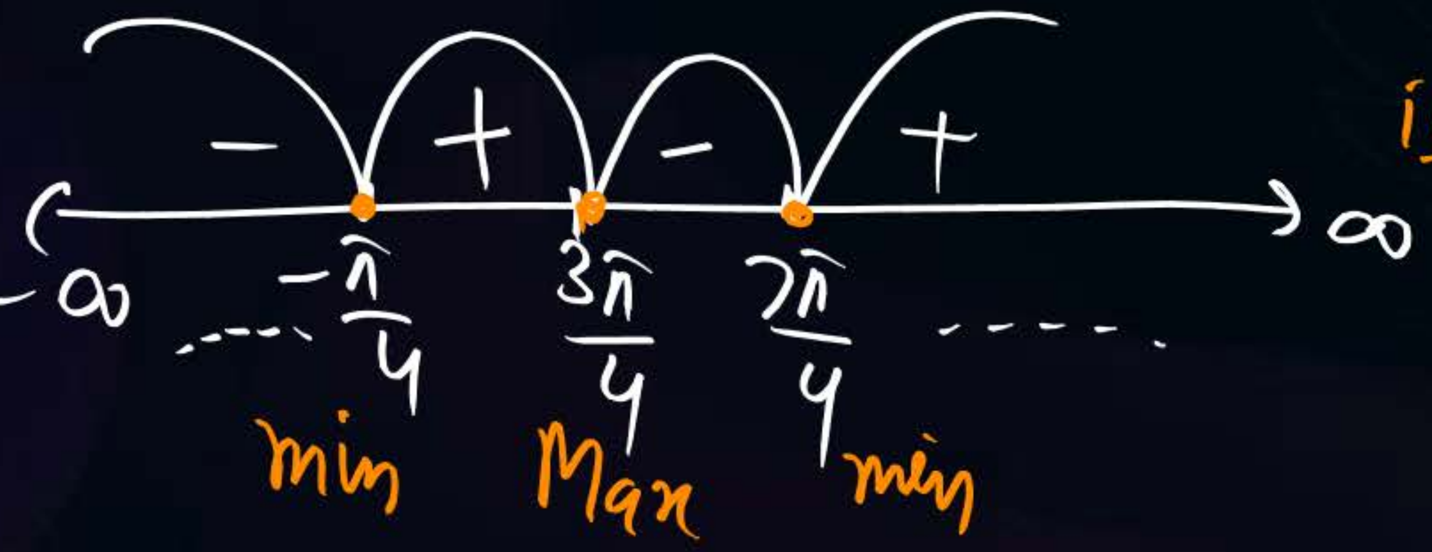
(d) -1

$$\boxed{f(x) = e^{\sin x - \cos x}} \Rightarrow \boxed{f'(x) = e^{\sin x - \cos x}(\cos x + \sin x)}$$

T-Points; $f'(x) = 0 \Rightarrow e^{\sin x - \cos x}(\cos x + \sin x) = 0$

$\because e^{\sin x - \cos x} \neq 0$ so $\cos x + \sin x = 0 \Rightarrow \tan x = -1$

$x = \ldots\ldots -\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{7\pi}{4}, \ldots\ldots$



$\therefore x = \dfrac{3\pi}{4}$ is Point of Maxima

& Max Value $= f\left(\dfrac{3\pi}{4}\right) = \ldots\ldots = e^{\sqrt{2}}$

M-II T. Points are $x = \ldots\ldots -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots\ldots$

$f''\left(-\frac{\pi}{4}\right) = \ldots\ldots = +ve$ & $f''\left(\frac{3\pi}{4}\right) = \ldots\ldots = -ve$

$x = -\frac{\pi}{4}$ is point of minima & $x = \frac{3\pi}{4}$ is point of Maxima

Min Value $= f\left(-\frac{\pi}{4}\right) = \left(e^{\sin x - \cos x}\right)_{x=-\frac{\pi}{4}}$

$= e^{-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}}$

$= e^{-\frac{2}{\sqrt{2}}} = e^{-\sqrt{2}}$

Max Value $= f\left(\frac{3\pi}{4}\right) = \left(e^{\sin x - \cos x}\right)_{x=\frac{3\pi}{4}}$

$= e^{\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)}$

$= e^{\frac{2}{\sqrt{2}}} = e^{\sqrt{2}}$

Q. $f(x)=(x-1)^3(x+1)^2$ then Number of Extremas will be ?

(a) 1

(b) 2

(c) 3

(d) 5

(e) 4

$$f'(x)=(x-1)^3\{2(x+1)\}+(x+1)^2\{3(x-1)^2\}$$

$$=(x+1)(x-1)^2\left[2(x-1)+3(x+1)\right]$$

$$\boxed{f'(x)=(x+1)(x-1)^2\left[5x+1\right]}$$

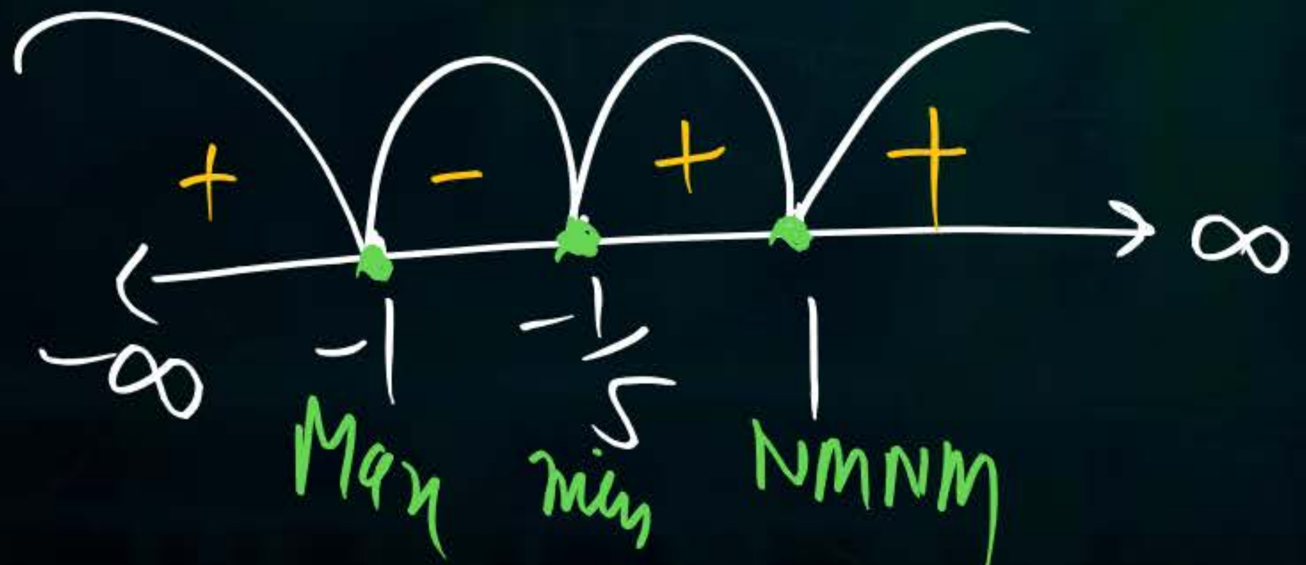
T-Points are $x=-1, -\frac{1}{5}, 1$

$f(-2)=+ve$

$f(-2/5)=-ve$

$f(0)=+ve$

$f(2)=+ve$

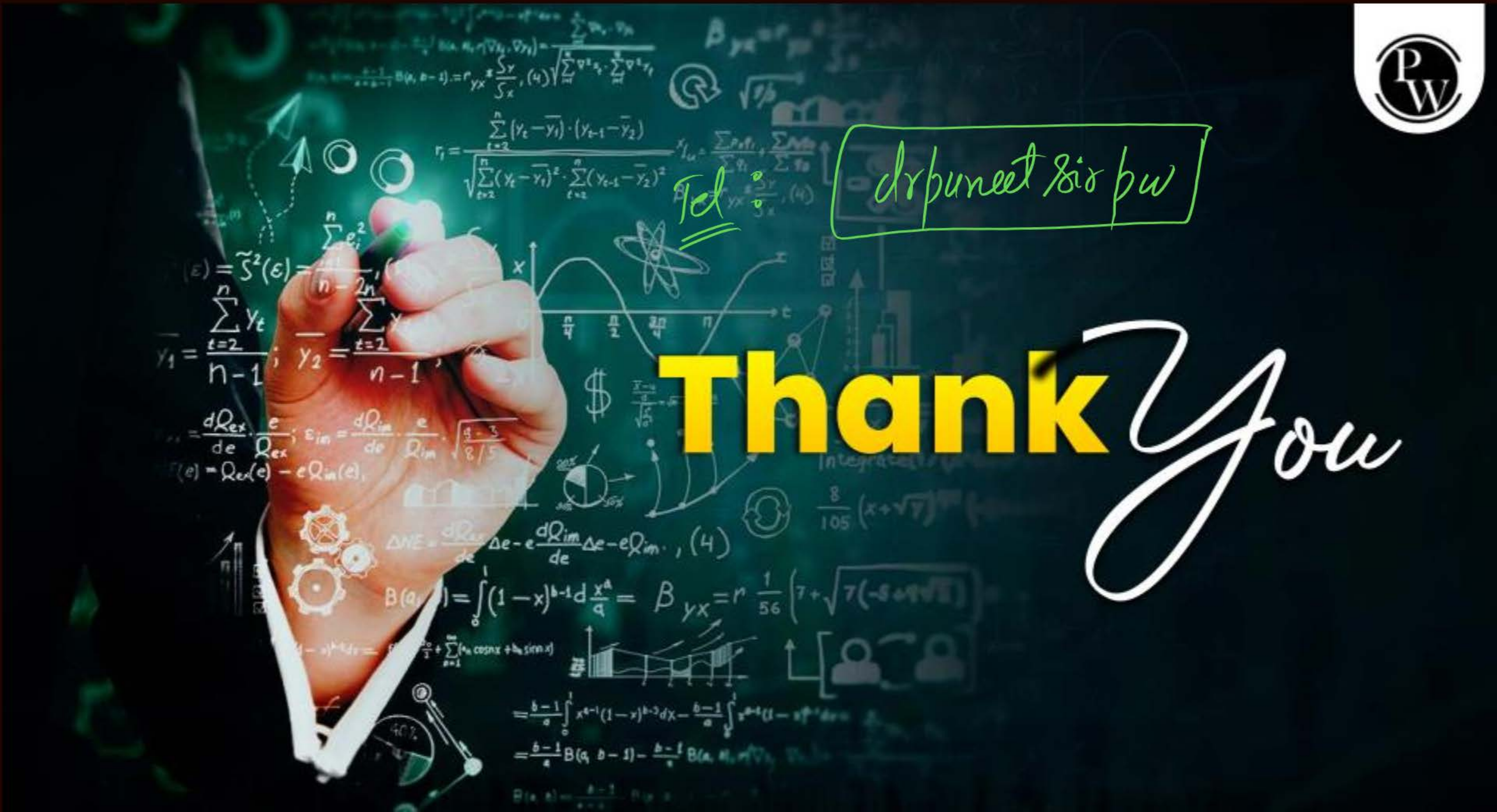


i.e. Number of Extremas = two

Qs Type II A Right Circular Cone of Maximum Volume is to be inscribed in a sphere of Radius 1 mtr then find the height of such cone.

(Application Based Questions)

Ans $h = \frac{4}{3}$



Tel : drpuneet sir pw