DS & AI
CS & IT

Linear Algebra

Lecture No. 10

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Recap of previous lecture

Topic

BASICS of EIGEN VECTORS

Topics to be Covered

Topic

- EIGEN VECTORS (Continued)
- Diagonalisation
- Miscellaneous

EIGEN VECTORS

Def^n: Consider Sq. Mat $A_{n\times n}$ then Non Zero Vector X is called Eigen vector, corresponding to Eigen value λ (Real/Complex/Zero) if we are able to find a relationship of the type,

$$\boxed{AX = \lambda X}$$

λ = Eigen value
X = Eigen vector.

LHS is the Multi of Two Matrices (Tough) = RHS is the Scalar Multi in a Mat (Easy)

(*) Here we are considering Homogeneous system as follows

$$AX = \lambda X \Longrightarrow AX - \lambda X = 0 \Longrightarrow \boxed{(A - \lambda I)X = 0}$$

So it will satisfy all the prop of Homog System.

PROPERTIES of E.VECTORS →

① If X is an E-vector of A then (KX) will also be an E-vector

$$AX = \lambda X \qquad \Longleftrightarrow \qquad A(KX) = \lambda(KX)$$

$$\boxed{(A-\lambda I)X = 0}$$ E.vector

$$K(A-\lambda I)X = K\cdot 0$$

$$\boxed{(A-\lambda I)(KX) = 0}$$ E vector

i.e "we are free to multiply or divide with any constant in case of E vector.

② E-Vectors corresponding to different E Values of Symm Mat are orthogonal

$\Downarrow$

Hence L.I also.

Q if $A_{3\times3}$ s.t $A^T = A$ (= Symm.) and a, b are the E-Values & Corresponding E-vectors are $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ then for $a \neq b$ (different EValues) the value of

(PYQ)

$x_1y_1 + x_2y_2 + x_3y_3 = \underline{?}$

By above property $X \cdot Y = 0 \Rightarrow X^T Y = 0$

$\Rightarrow x_1y_1 + x_2y_2 + x_3y_3 = 0$

③ For Different E-Values, Corresponding E-Vectors are also LI

④ If E-Value Repeats (then Headache will start) then Corresponding E Vectors may be LI or may be LD.

⑤ <u>MODAL MATRIX</u> :→ Matrix formed by E-Vectors is Called Modal Mat

Let $A_{3\times3}$ is the given Mat & X_1, X_2, X_3 are the E-Vectors then

Modal Mat is $P = [X_1 \; X_2 \; X_3]$

⑥ **Algebraic Multiplicity (A.M)** Number of times particular eigen value repeats is called it's A.M

⑦ **Geometric Multiplicity (G.M)** Number of L.I E vectors for particular E value λ is called it's G.M

⑧ **Shortcut of Finding G.M (for λ)** → let λ is the Repeated E value then

$$\underbrace{\text{No. of L.I E. Vectors for } \lambda}_{\text{GM of } \lambda} = \boxed{\text{order} - \rho(A - \lambda I)}$$

(9) Short cut method of finding E. Vectors (if options are given) →

" λ should be unique in all comparisions & there should not be any Invalid Result "

(10) $\boxed{\text{E. Vectors of } A \text{ \& } A^m \text{ would be Same}}$

i.e if (λ, X) is an Eigen pair of A

then (λ^m, X) is an Eigen Pair of A^m.

(11) **Positive Definite Mat** → If All the E Values are +ve then A is Called +ve Definite.

ie $\lambda > 0$

Negative Definite → If all the E Values are −ve then A is Called −ve Definite.

Idefinite Mat → If E Values are +ve as well as −ve then A is Called Indefinite

Semi +ve Definite → If E Values are 0 and +ve then A is semi +ve definite

ie $\lambda \geq 0$

Spectral Radius → Largest Magnitude of Eigen Values is Called Spectral Radius.

Analysis (PODCAST) →.

① $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

$\lambda = 6,\ X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix}$ ---- ∞ Evectors exist.

But all are LD on X_1

$\lambda = 2,\ X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1/5 \\ -1/5 \end{bmatrix}$ --- ∞ Evectors exist.

But all are LD on X_2

② Here X_1 & X_2 are LI (By observation)

Actually we are getting two Ind families of E. vectors. & if we are taking one member from F_1 & one member from F_2 then these are LI

i.e for Mat A, $\exists$ two LI eigen vectors at a time

③ $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, $\begin{cases} \lambda = 6, & X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \lambda = 2, & X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{cases}$

= Symm Mat.

$\because X_1 \cdot X_2 = (1)(-1) + (1)(1)$
$= 0$

i.e. X_1 & X_2 are orthogonal vectors $\Longrightarrow$ Hence they are LI also.

i.e. for different eigen values of symm Mat, corresponding E. vectors are <u>orthogonal</u> as well as LI

(4) In this Mat, $X_1 \cdot X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-1)(1) + (1)(1) = 0$

is X_1 & X_2 are orthogonal Vectors.

(5) one pair of L.I, E Vectors of $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ will be

(HW)

(a) $\begin{bmatrix} -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \end{bmatrix}$ these are E. Vector But L.D

(b) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ these are E. Vector but L.D

(c) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ these are L.I but 2nd is not an E. Vector.

(d) ✓ $\begin{bmatrix} 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ these are eigen Vector as well as L.I

$\because AX = \lambda X$

$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 14 \\ 14 \end{bmatrix} = \begin{bmatrix} 2\lambda \\ 3\lambda \end{bmatrix}$

$\lambda = 7, \lambda = \frac{14}{3}$ (not unique)

As it is not an E. Vector

eg: For $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(1) $\lambda = 3, 2, 2$

(Already solved).

$\lambda = 3,\ X = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ i \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/3 \end{bmatrix} \ldots\ldots \infty$ E.Vexist

$\lambda = 2,\ X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix} \ldots\ldots \infty$ E.vector exist

(2) X_1 & X_2 are LI (By observation)

(3) AMof '3' = one, GMof '3' = one

AMof '2' = two, GMof '2' = one

(M-II) $GMof(\lambda = 2) = order - \rho(A - \lambda I)$

$= 3 - \rho(A - 2I)$

$GMof(\lambda = 2) = 3 - \rho \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= 3 - 2 =$ one

If $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

① $(\lambda = 3, 1, 1)$

(Already Solved)

$\lambda = 3$, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \end{bmatrix}$ ---- ∞ EV exist.

$\lambda = 1$

$X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ - - - - - - " "

$X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$ - - - - - " "

② Here X_1, X_2, X_3 are LI $\because$ Modal Mat is Non singular where $P = [X_1 X_2 X_3]$

③ AM of '3' = one, GM of '3' = one

AM of '1' = two, GM of '1' = two

M-II $GM\,of\,(\lambda = 1) = order - \rho(A - \lambda I)$

$= 3 - \rho(A - I)$

$GM\,of\,(\lambda = 1) = 3 - \rho \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $(\because |P| \neq 0)$

$= 3 - 1 = 2$ Ans

Q. Find AM & GM of λ for $A = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$, $\alpha \neq 0$.

Sol: $\because$ A is U.T.M so $\lambda = 0, 0, 0$ Now, AM of '0' = three.

$$GM \text{ of } (\lambda = 0) = \boxed{\text{order} - \rho(A - \lambda I)}$$

$$= 3 - \rho(A - 0I) = 3 - \rho(A) = 3 - 1 = 2 \text{ Ans}$$

Verification :- E. Vector for $(\lambda = 0) \rightarrow$

$$AX = \lambda X$$
$$(A - \lambda I)X = 0$$
$$(A - 0I)X = 0$$
$$AX = 0$$

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha x_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\because \alpha \neq 0$, $x_3 = 0$

let $x_1 = k_1$, $x_2 = k_2$ then

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ 0 \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ k_2 \\ 0 \end{bmatrix}$$

$$X = k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

i.e. for $\lambda = 0$, X_1 & X_2 are E. Vector

Note:

① $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has <u>three</u> LI Evectors, <u>one</u> for $\lambda = 3$ & <u>two</u> for $\lambda = 1$

② $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ has <u>two</u> LI Evectors, <u>one</u> for $\lambda = 3$ & <u>one</u> for $\lambda = 2$

③ $A = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has <u>two</u> LI Evectors & <u>Both</u> for $\lambda = 0$

$(\alpha \neq 0)$

$\because$ for $\lambda = 0$

$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} \cdots \infty$ Evectors exist

$X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} \cdots \infty$ Evectors exist.

(1993) The E. vectors of $A = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\alpha \neq 0$ is/are $(\lambda = 0, 0, 0)$

MS8

(a) $(\alpha\ 0\ 0)'$ (b) $(0\ 0\ 1)'$ (c) $(0\ \alpha\ 0)'$ (d) $(0\ 0\ \alpha)'$

Sol: Taking (a): $AX = \lambda X$

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda\alpha \\ 0 \\ 0 \end{bmatrix}$$

$\lambda\alpha = 0$, $0 = 0$, $0 = 0$

$\lambda = 0$ unique, Valid

(a) ✓

Taking (b), $AX = \lambda X$

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \lambda \end{bmatrix}$$

$\alpha = 0$, $0 = 0$, $\lambda = 0$

Invalid, Valid, unique

So (b) ✗

Taking (c) ✓

Taking (d) ✗

Q. one of the EVector of $A=\begin{bmatrix}5&0&0&0\\0&5&5&0\\0&0&2&1\\0&0&3&1\end{bmatrix}$ will be ?

(a) $\begin{bmatrix}1\\-2\\3\\0\end{bmatrix}$ (b) $\begin{bmatrix}1\\-2\\0\\0\end{bmatrix}$, (c) $\begin{bmatrix}2\\3\\0\\-1\end{bmatrix}$, (d) $\begin{bmatrix}1\\-1\\2\\3\end{bmatrix}$

(ii) Also find Corresponding E. Value.

Taking (a): $AX=\lambda X$

$$\begin{bmatrix}5&0&0&0\\0&5&5&0\\0&0&2&1\\0&0&3&1\end{bmatrix}\begin{bmatrix}1\\-2\\3\\0\end{bmatrix}=\lambda\begin{bmatrix}1\\-2\\3\\0\end{bmatrix}$$

$$\begin{bmatrix}5\\5\\6\\9\end{bmatrix}=\begin{bmatrix}\lambda\\-2\lambda\\3\lambda\\0\end{bmatrix}\Rightarrow \underbrace{\lambda=5,\ \lambda=-\frac{5}{2},\ \lambda=2}_{\text{Not unique}},\ \underbrace{9=0}_{\text{Invalid.}}$$

Taking (b) $AX=\lambda X$

$$\begin{bmatrix}5&0&0&0\\0&5&5&0\\0&0&2&1\\0&0&3&1\end{bmatrix}\begin{bmatrix}1\\-2\\0\\0\end{bmatrix}=\lambda\begin{bmatrix}1\\-2\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}5\\-10\\0\\0\end{bmatrix}=\begin{bmatrix}\lambda\\-2\lambda\\0\\0\end{bmatrix}$$

$$\underbrace{\lambda=5,\ \lambda=5}_{\text{unique}},\ \underbrace{0=0,\ 0=0}_{\text{Valid}}$$

so (b) ✓

(ii) Corresponding E Value is $\lambda=5$

HW8 ① Find the E Values & E Vectors of $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ $\lambda = 1, x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\lambda = 2, x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Gate / ESE

OR

② If one pair of LI E.vectors of $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ a \end{bmatrix}$ & $\begin{bmatrix} 1 \\ b \end{bmatrix}$ then $a+b = ?$

OR

③ one pair of L.I, E vectors of $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$?

ⓐ × $\begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ these are LD

ⓑ × $\begin{bmatrix} -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$ " " LD

ⓒ × $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ these are LI but $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is not an E Vector $\because AX \neq \lambda X$

✓ ⓓ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ " " LI as well as E Vectors

(ii) one pair of L.I. E.Vectors of $A=\begin{bmatrix}1&2\\0&2\end{bmatrix}$ is $\begin{bmatrix}1\\a\end{bmatrix}$ & $\begin{bmatrix}1\\b\end{bmatrix}$ then $a+b=?$

(a) 1

(b) 2

(c) 0.5

(d) 0.6

M-I Separately find E. Vectors using conventional Approach & then Compare.

$A=\begin{bmatrix}1&2\\0&2\end{bmatrix}$

$\lambda=1,\ X=\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\a\end{bmatrix}\Rightarrow a=0$

$\lambda=2,\ X=\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1\\b\end{bmatrix}=\begin{bmatrix}1\\1/2\end{bmatrix}\Rightarrow b=\frac{1}{2}$

ie $a+b=\frac{1}{2}$

M-II

$AX_1=\lambda_1X_1$ & $AX_2=\lambda_2X_2$

$\begin{bmatrix}1&2\\0&2\end{bmatrix}\begin{bmatrix}1\\a\end{bmatrix}=1.\begin{bmatrix}1\\a\end{bmatrix}$ & $\begin{bmatrix}1&2\\0&2\end{bmatrix}\begin{bmatrix}1\\b\end{bmatrix}=2\begin{bmatrix}1\\b\end{bmatrix}$

$\begin{bmatrix}1+2a\\2a\end{bmatrix}=\begin{bmatrix}1\\a\end{bmatrix}$ & $\begin{bmatrix}(1+2b)\\2b\end{bmatrix}=\begin{bmatrix}2\\2b\end{bmatrix}$

$\Rightarrow \left.\begin{matrix}1+2a=1\\2a=a\end{matrix}\right\}\Rightarrow a=0$ & $1+2b=2\Rightarrow b=\frac{1}{2}$

ie $a+b=\frac{1}{2}$

Analysis:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$\lambda = 1,\ X = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \cdots\cdots \infty$

$= \begin{bmatrix} 1 \\ a \end{bmatrix}$

$\lambda = 2,\ X \begin{bmatrix} 2 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdots\cdots \infty$

$= \begin{bmatrix} 1 \\ b \end{bmatrix}$

$\begin{bmatrix} 1 \\ a \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow a = 0$

$\begin{bmatrix} 1 \\ b \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \Rightarrow b = \frac{1}{2}$

For the matrix $A = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ one of the eigen values is equal to -2 which of the following is an eigen vector?

(a) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

Talig (a) $\begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = -2\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} \\ \\ 1 \end{bmatrix} \neq \begin{bmatrix} \\ \\ -2 \end{bmatrix}$ (a) ✗

(b) $\begin{bmatrix} \\ \\ 0 \ 0 \ 1 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} = -2\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} \\ \\ -1 \end{bmatrix} \neq \begin{bmatrix} \\ \\ 2 \end{bmatrix}$ (b) ✗

(c) $\begin{bmatrix} \\ \\ 0 \ 0 \ 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} \\ \\ 3 \end{bmatrix} \neq \begin{bmatrix} \\ \\ -6 \end{bmatrix}$ (c) ✗

The vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is an eigen vector of

$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ then corresponding eigen value of A is

(a) 1 (b) 2

(c) 5 (d) –1

$$AX = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ - \\ - \end{bmatrix} = 5 \begin{bmatrix} 1 \\ - \\ - \end{bmatrix}$$

$$AX = 5X \Rightarrow \lambda = 5$$

Similar Matrices :- Two Matrices A & B are said to similar if

$\exists$ a relationship of the type, $\boxed{P^{-1}AP = B}$

where P is any invertible Matrix

& this Transformation from A to B is called Similarity Transformation

Note ① Similar Matrices have same Trace same Det, same C-Eq.

same E.Values and same Rank

But Equivalent Matrices have only same Rank & other properties may be different

② Similarity Relation b/n Matrices is an Equivalence relation

i.e Reflexive, Symmetric as well as Transitive.

Diagonalisation → Mat A is said to be diagonalisable if $\boxed{\text{it is similar to Diag Mat}}$

& this process of converting A into a Diag Mat is called Diagonalisation.

Consider $A_{3\times 3}$ & let it's E-Values are $\lambda_1, \lambda_2, \lambda_3$

& E-vectors are X_1, X_2, X_3

then Modal Mat is $P = [X_1\ X_2\ X_3]$ & let $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

& let $|P| \neq 0$.

Let us try to Calculate,

$$P^{-1}AP = [P^{-1}][A][P] = \cdots\cdots = D$$

i.e. $\boxed{P^{-1}AP = \text{Diag. Mat.}}$ i.e. A is similar to Diag Mat

this is very special Property of Mat & this property is called Diagonalisation

Application ① with the help of this concept we can find A as follows.

$$\boxed{P^{-1}AP = D} \Rightarrow P(P^{-1}AP)P^{-1} = P.D.P^{-1} \Rightarrow A = P.D.P^{-1} = ?$$

② Necessary Condition for Diagonalisation →

"No. of L.I E.Vectors of A = order of A"

eg Consider $A_{3\times3}$ & let it has 3.E vector X_1, X_2, X_3

& let these are L.D then Modal Mat $P = [X_1 X_2 X_3]$ will

become Singular $(|P| = 0)$ & P^{-1} = DNE

ie whole concept of Diagonalisation will become meaningless.

Verification: for $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ $\begin{cases} \lambda = 6, & X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \lambda = 2, & X = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{cases}$

let $P = [X_1 \; X_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $P^{-1} = \dfrac{\text{adj } A}{|A|} = \dfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Now $P^{-1}AP = \dfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \ldots\ldots = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} = D$

ie we have verified that $P^{-1}AP = D$ ie A is similar to Diag. Mat.

Use: $\because P^{-1}AP = D$

$P(P^{-1}AP)P^{-1} = PDP^{-1} \Longrightarrow A = P.D.P^{-1} = ?$

Q: if $A_{2\times2}$ s.t $A\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}-2\\2\end{bmatrix}$ & $A\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}6\\6\end{bmatrix}$ then $A=?$

(a) $\begin{bmatrix}-1&1\\1&1\end{bmatrix}$ $\because Tr\neq 8$ ✗

(b) $\begin{bmatrix}4&2\\2&4\end{bmatrix}$ ✓

(c) $\begin{bmatrix}-2&6\\2&6\end{bmatrix}$ $\because Tr\neq 8$ ✗

(d) $\begin{bmatrix}2&0\\0&6\end{bmatrix}$

$A\begin{bmatrix}-1\\1\end{bmatrix}=2\begin{bmatrix}-1\\1\end{bmatrix}$ & $A\begin{bmatrix}1\\1\end{bmatrix}=6\begin{bmatrix}1\\1\end{bmatrix}$

ie $\lambda=2$, $X_1=\begin{bmatrix}-1\\1\end{bmatrix}$ ie $\lambda=6$, $X_2=\begin{bmatrix}1\\1\end{bmatrix}$

$Tr(A)=2+6=8$

$|A|=(2)(6)=12$

$D=\begin{bmatrix}2&0\\0&6\end{bmatrix}$, $P=[X_1,X_2]=\begin{bmatrix}-1&1\\1&1\end{bmatrix}$, $P^{-1}=\frac{-1}{2}\begin{bmatrix}1&-1\\-1&-1\end{bmatrix}$

Now using the concept of Diagonalisation,

$P^{-1}AP=D \Rightarrow A=P.DP^{-1}=\begin{bmatrix}-1&1\\1&1\end{bmatrix}\begin{bmatrix}2&0\\0&6\end{bmatrix}\frac{-1}{2}\begin{bmatrix}1&-1\\-1&-1\end{bmatrix}=\begin{bmatrix}4&2\\2&4\end{bmatrix}$

Q: If $A_{2\times 2}$ s.t. $A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$ & $A\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$ then find $A=$?

(a) $\begin{bmatrix}1&2\\0&1\end{bmatrix}$ $\because Tr\neq 3$

(b) $\begin{bmatrix}1&0\\0&2\end{bmatrix}$

(c) $\begin{bmatrix}1&4\\0&2\end{bmatrix}$

(d) $\begin{bmatrix}1&2\\0&2\end{bmatrix}$

$A\begin{bmatrix}1\\0\end{bmatrix}=1\cdot\begin{bmatrix}1\\0\end{bmatrix}$ & $A\begin{bmatrix}2\\1\end{bmatrix}=2\begin{bmatrix}2\\1\end{bmatrix}$

$\lambda=1,\ X_1=\begin{bmatrix}1\\0\end{bmatrix}$, $\lambda=2,\ X_2=\begin{bmatrix}2\\1\end{bmatrix}$

$Tr(A)=1+2=3$

$|A|=1\times 2=2$

$D=\begin{bmatrix}1&0\\0&2\end{bmatrix}$, $P=[X_1, X_2]=\begin{bmatrix}1&2\\0&1\end{bmatrix}$, $P^{-1}=\frac{1}{(1)}\begin{bmatrix}1&-2\\0&1\end{bmatrix}$

Now, $P^{-1}AP=D \Rightarrow A=P\cdot DP^{-1}=\begin{bmatrix}1&2\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}1&-2\\0&1\end{bmatrix}$

$=\ldots\ldots=\begin{bmatrix}1&2\\0&2\end{bmatrix}$ //

For a given 2 × 2 matrix A, it is observed that

$A\begin{bmatrix}1\\-1\end{bmatrix} = -1\begin{bmatrix}1\\-1\end{bmatrix}$ and $A\begin{bmatrix}1\\-2\end{bmatrix} = -2\begin{bmatrix}1\\-2\end{bmatrix}$ then the matrix A is

$\lambda = -1$ $\lambda = -2$

(a) $A = \begin{bmatrix}2 & 1\\-1 & -1\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}$

(b) $A = \begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\-1 & -1\end{bmatrix}$

(c) $A = \begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}2 & 1\\-1 & -1\end{bmatrix}$

(d) $A = \begin{bmatrix}1 & -2\\1 & -3\end{bmatrix}$ $Tr \neq -3$

$Tr(A) = (-1) + (-2) = -3$

$|A| = (-1)(-2) = 2$

$\because P^{-1}AP = D \Rightarrow A = P \cdot D \cdot P^{-1}$

$A = \begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix}\begin{bmatrix} ? \end{bmatrix}$

(c)

Some Confusions →

① No. of L.I Row vectors in $A = \rho(A)$

② No. of L.I Column vectors in $A = \rho(A)$

③ No. of $\underbrace{\text{L.I } \boxed{\text{Solution}}}_{\text{Nullity}}$ in Homog. System $(AX=0) = \boxed{\text{No. of Column} - \rho(A)}$

④ No. of $\underbrace{\text{LI } \boxed{\text{Eigen Vector}}}_{\text{GM of } \lambda}$ for $\lambda = \boxed{\text{order} - \rho(A-\lambda I)}$

∵ Here we have Homog system of the type $(A-\lambda I)X=0$

L-U Decomposition

If we want to factorize any square mat into the product of L.T.M & U.T.M then we have following Methods;

① Dolittle Method

write $A = (\text{unit LTM})(\text{U.TM})$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}$$

② CROUT Method

$A = (\text{L.T.M})(\text{unit U.T.M})$

$$A = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}$$

③ Cholesky Method

$$A = LL^{\theta}$$

$$A = L(\overline{L^T})$$

Application

By using L-U Decomposition Concept we can solve S.O.Eq.

Consider $\boxed{AX=B}$ is to be solved i.e our aim is to find $X=\begin{bmatrix} \\ \\ \\ \vdots \\ \end{bmatrix}=?$

write $A=LU$ — (1) → D. Method, C. Method.

$$(LU)X=B$$

$$L(UX)=B$$

$$LY=B \quad \text{where } UX=Y \quad \text{— (3)}$$

— (2)

$$\begin{bmatrix} L.T.M \end{bmatrix}_{3\times3}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Longrightarrow Y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=$$ Possible to solve easily using Forward substitution.

Now By (3),

$$UX=Y$$

$$\begin{bmatrix} U.T.M \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Possible to solve x_3, x_2, x_1 easily using Backward substitution.

Q. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$ is to be decomposed into the product of L.T.M & U.T.M then properly decomposed L & U Matrices are ?

MSQ

(a) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 2 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$

(d) None

(M-I) Using Dollitle Method

$A = (\text{Unit LTM})(\text{U.T.M})$

$A = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} b & c \\ 0 & d \end{bmatrix}$ — (1)

$\begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix} = \begin{bmatrix} b & c \\ ab & ac+d \end{bmatrix}$

$b = 1$, $c = 2$, $ab = 2 \Rightarrow a = 2$

$ac + d = 13$

$(2)(2) + d = 13 \Rightarrow d = 9$

So by (1)

$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 9 \end{bmatrix}$

(a) ✓

M-II Using Crout M :-

$$A = (L.T.M)(\text{unit U.T.M})$$

$$= \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \quad \text{--- (1)}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix} = \begin{bmatrix} a & ad \\ b & bd+c \end{bmatrix}$$

$$\Rightarrow a=1,\ b=2,\ ad=2 \Rightarrow d=2$$

$$bd+c=13 \Rightarrow (2)(2)+c=13 \Rightarrow c=9$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

(b) ✓

M-III Using Cholesky Method :-

write $A = LL^{\theta}$ or $L(\bar{L})^T$

$$A = LL^T \quad (\because A \text{ is Real Mat so will } L \text{ also i.e. } L^{\theta} = L^T)$$

$$A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad \text{--- (1)}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix} = \begin{bmatrix} a^2 & ab \\ ab & b^2+c^2 \end{bmatrix}$$

$$a^2=1,\ ab=2,\ b^2+c^2=13$$

$$a=1 \qquad b=2 \qquad 4+c^2=13$$

$$c^2=9 \Rightarrow c=3$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \text{ so (c)}$$ ✓

In the LU decomposition of the matrix $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$, if the diagonal elements of U are both 1, then the lower diagonal entry l_{22} of L is _____.

Crout Method.

$$A = (L.T.M)(\text{unit } U.T.M)$$

$$\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$$

$$\longrightarrow l_{22} = 5$$

Reflection Matrix → with angle $\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ Learn.

(i) About X axis $(\theta = 0)$

$RM = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(ii) About Y axis $(\theta = \pi/2)$

$RM = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(iii) About $y = x$ $\left(\theta = \frac{\pi}{4}\right)$

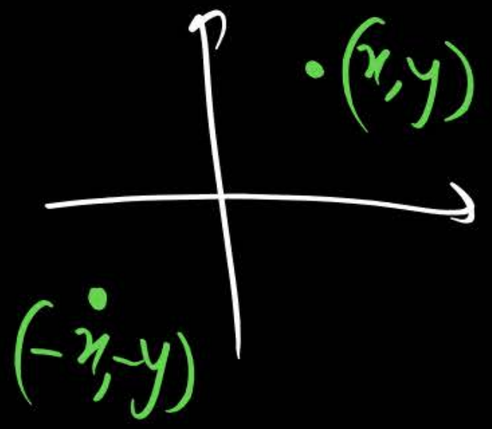
$RF = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(iv) About $y = -x$ $\left(\theta = \frac{3\pi}{4}\right)$

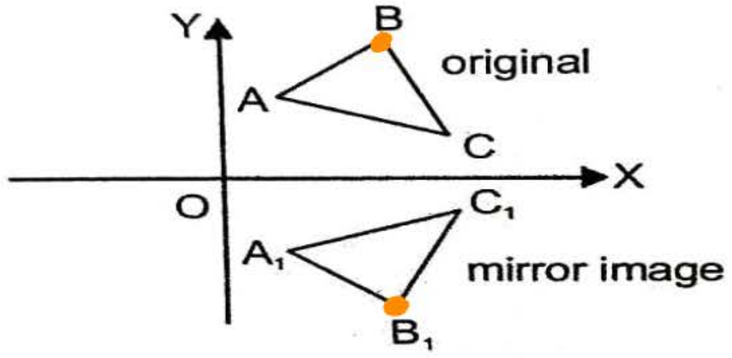
$RF = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(v) Special Case: (About origin) → $R.M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(Above shortcut is not applicable)



The figure shows a shape ABC and its mirror image $A_1B_1C_1$ across the horizontal axis (X –axis). The coordinate transformation matrix that maps ABC to $A_1B_1C_1$ is



(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Let $A(x, y)$

$\Downarrow$

$A_1(x, -y)$

Taking (a)

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \neq \begin{bmatrix} x \\ -y \end{bmatrix}$$

Taking (b)

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} \neq \begin{bmatrix} x \\ -y \end{bmatrix}$$

Taking (d)

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

Taking (c)

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \neq \begin{bmatrix} x \\ -y \end{bmatrix}$$

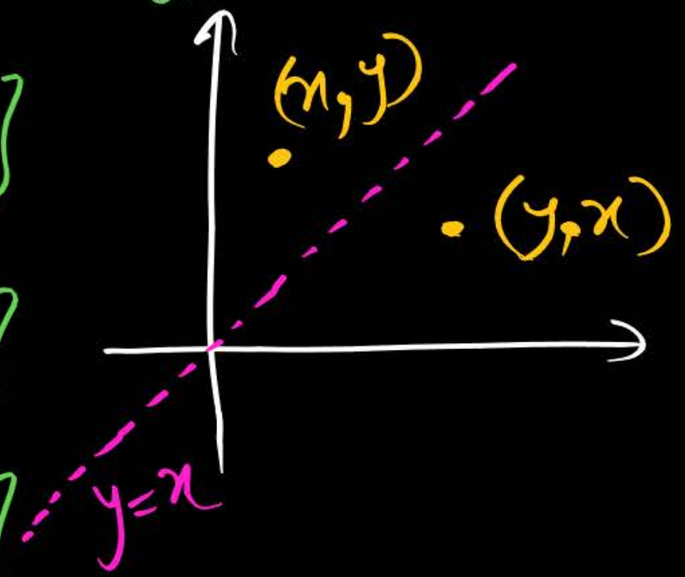
Q. The Transformation Matrix to find mirror image of any point $P\begin{bmatrix} x \\ y \end{bmatrix}$ in x-y plane about the line $y=x$ is ?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$



(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} y \\ x \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

THANK - YOU

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