



DS & AI  
CS & IT

Linear Algebra

Lecture No. 08



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# Recap of previous lecture



Topic

- ① Homogeneous system
- ② Basics of Eigen Values

# Topics to be Covered

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Topic

PROPERTIES of EIGEN VALUES

Def<sup>n!</sup> Consider Sq. Mat Anxn, then Non Zero Vector  $x$  is called Eigen vector, corresponding to Eigen value  $\lambda$  (Real / Complex / zero) if we are able to find a relationship of the type,

LHS is the Multi of Two Matrices = RHS is the Scalar Multi in a Mat  
(Tough) (Easy)

④ Here we are considering Homogeneous system as follows

$$AX = \lambda X \Rightarrow AX - \lambda X = 0 \Rightarrow (A - \lambda I)X = 0$$

So it will satisfy all the prop of Homog system.

\* Consider  $AX = \lambda X$

$$(A - \lambda I)X = 0 \quad \text{---} \quad (1)$$

$$MX = 0$$

Non zero Eigen vector

Non zero solution

$\Rightarrow \infty \text{ R.S.}$

$$\text{rank}(M) < n \text{ or } |M| = 0$$

$$\Rightarrow \text{rank}(A - \lambda I) < n \text{ or } |A - \lambda I| = 0$$

So Necessary Condition for the existence  
of Non zero Eigen Vector is

$$\text{rank}(A - \lambda I) < n \text{ or } |A - \lambda I| = 0 \quad \text{---} \quad (1)$$

Characteristic Eqn of A  $\rightarrow$

equation (1) is called C.Eq of A & Roots of

this equn ie values of  $\lambda$  are called

E. Values / E. Roots / Char Values

/ Char Roots / Latent Roots / Sp. Values.

## PROPERTIES of E. Values → Let $A_{m \times n}$ having EigenValues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$

- ① T. Number of E-Values of  $A$  = order of  $A$  (whether diff or Repeated)
- ② Sum of E-Values = Trace( $A$ ) ie  $\lambda_1 + \lambda_2 + \dots + \lambda_n = \text{Tr}(A)$
- ③ Product of E-Values = Det( $A$ ) ie  $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \dots \cdot \lambda_n = |A|$
- ④ (Zero is an E.Value of  $A$ ) iff ( $A$  is singular)
- ⑤ Number of Non zero E.Values of  $A$   $\leq S(A)$  ie we must have at least two eigen values as 0, 0  
eg if  $S(A_{6 \times 6})=4$  then No of Non zero E-Values  $\leq 4$
- ⑥ If sum of all the elements in each Row (or each Column) is unique constant  $K$  then that Constant  $K$  will be one of the EValue of  $A$ .

⑦ Don't use E-Operations in a given Mat while calculating E-Values

But we can apply 3<sup>rd</sup> type of E-operations in it's C-Eqn.

⑧ E-Values of U.T.M, L.T.M, Diag Mat, scalar Mat, Identity Mat are just the diagonal elements.

e.g.  $A = \begin{bmatrix} 2 & 4 & -1 & 3 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ ,  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ ,  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\lambda = 2, 0, -3, -1$        $\lambda = 2, -1, 1$        $\lambda = 2, -3, 4$        $\lambda = 1, 1, 1$

⑨ If  $\lambda$  is an Eigen Value of  $A$  then to find Eigen Value of any algebraic expression formed by  $A$ , we can Replace  $A$  with  $\lambda$  in that expression. i.e  $(A \rightarrow \lambda)$

→ ⑩ Equivalent Matrices have same Rank but may be different E-Values

⑩ Let  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$  are the E values of  $A_{n \times n}$  then

- (i) E values of  $A^T$  are  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$  i.e  $A$  &  $A^T$  have Same E values
- (ii) E values of  $A^m$  are  $\lambda_1^m, \lambda_2^m, \lambda_3^m, \dots, \lambda_n^m$ ; ( $m \in \mathbb{N}$ )
- (iii) E values of  $\bar{A}^l$  are  $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \dots, \frac{1}{\lambda_n}$  (Provided  $|A| \neq 0$ )
- (iv) E values of  $(\text{adj } A)$  are  $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}, \dots, \frac{|A|}{\lambda_n}$  (When  $|A| \neq 0$ )

$$\text{Prop 10(i)}: Ax = \lambda x \rightarrow \textcircled{1}$$

$$A(Ax) = A(\lambda x)$$

$$A^2x = \lambda(Ax)$$

$$A^2x = \lambda(\lambda x)$$

$$A^2x = \lambda^2 x$$

ie  $\lambda^2$  is an E value of  $A^2$   
Similarly  $\lambda^3, \dots, \lambda^n$

$$\text{Prop 10(ii)}: Ax = \lambda x \rightarrow \textcircled{1}$$

if  $|A| \neq 0 \Rightarrow A^{-1}$  exist

$$A^{-1}(Ax) = A^{-1}(\lambda x)$$

$$Ix = \lambda(A^{-1}x)$$

$$\lambda(A^{-1}x) = Ix$$

$$A^{-1}x = \left(\frac{1}{\lambda}\right)x$$

$\frac{1}{\lambda}$  is an E value of  $A^{-1}$ .

$$\text{Prop 10(iv)}: Ax = \lambda x \rightarrow \textcircled{1}$$

$$\text{Now } A^{-1}x = \left(\frac{\text{adj } A}{|A|}\right)x$$

$$\left(\frac{1}{\lambda}\right)x = \left(\frac{\text{adj } A}{|A|}\right)x$$

$$\left(\frac{|A|}{\lambda}\right)x = (\text{adj } A)x$$

$$\text{or } (\text{adj } A)x = \left(\frac{|A|}{\lambda}\right)x$$

ie  $\frac{|A|}{\lambda}$  is an E value of  $(\text{adj } A)$

This Prop holds when  $A$  is Non sing.

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$$\text{Q} \quad A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{adj } A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

$\lambda = 2, -3, 0$

$$\lambda = -6, 0, 0.$$

EValue of A are  $2, -3, 0$

" . . . adj A are  $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}$

$$= \frac{0}{2}, \frac{0}{-3}, \frac{0}{0}$$

Not applicable

e.g.  $A = \begin{bmatrix} -2 & -1 & 3 & 4 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

U.T.M.  
 $|A| = -24$

then

- E values of  $A$  are  $\lambda = -2, 1, 4, 3$
- " "  $A^T$  are same
- " "  $A^3$  are  $-8, 1, 64, 27$
- " "  $\bar{A}^1$  are  $\frac{1}{2}, 1, \frac{1}{4}, \frac{1}{3}$
- " "  $\text{adj } A$  are  $\frac{-24}{-2}, \frac{-24}{1}, \frac{-24}{4}, \frac{-24}{3}$

e.g.  $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Diag Mat.

$E$  values of  $A$  are  $1, -2, 0, 4$ .

" of  $A^T$  are  $1, -2, 0, 4$

" of  $A^2$  are  $1, 4, 0, 16$

" of  $\bar{A}^1$  are  $1, -\frac{1}{2} \text{ (ND)}, \frac{1}{4}$  (Blunder)

"  $\bar{A}^1 = \text{DNE}$  so case of  $E$  values will not arise

$$\text{eg } A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}, \quad \lambda = 3, 1, 1$$

$$\text{Tr}(A) = 5$$

$$|A| = 3$$

$$\text{eg } A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix} = (10)(10)(10)(10)$$

is one EV value is  $\lambda = 10$   
(Prop 5)

$$\text{eg } A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}_{2 \times 2}, \quad \lambda = i, -i$$

$$\text{Tr} = 0$$

$$|A| = 1$$

$$\text{eg } A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}_{3 \times 3}, \quad \lambda = 0, 3, 15$$

$$\text{Tr} = 18$$

$$|A| = 0$$

$S(A) = 2 = \text{No. of NonZero EV values}$   
(Prop 5)

~~Ques~~ if E-values of  $A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$

are 4 & 8 then  $x+y = ?$

- (a) -4 (b) 6 (c) 10 (d) 14

let  $\lambda_1$  &  $\lambda_2$  are the E-Values of A

then  $\lambda_1 + \lambda_2 = \text{Tr}(A)$  &  $\lambda_1 \cdot \lambda_2 = |A|$

$$4+8=2+y \quad \& \quad (4)(8)=2y-3x$$

$$y=10$$

$$32=2(10)-3x$$

$$3x=-12$$

$$x=-4$$

$$x+y=6 \quad \underline{\text{Ans}}$$

~~Ques~~ if Trace & Det of  $A_{2 \times 2}$  are P W

-2 & -35 resp then  $\lambda_1 + \lambda_2 = ?$

- (a) 12 (b) -12 (c) 2 (d) -2

w.k. that  $\lambda_1 + \lambda_2 = \text{Tr}(A) = -2$

(ii) Also find E values

w.k. that,

$$\lambda_1 + \lambda_2 = \text{Tr}(A) \quad \& \quad \lambda_1 \cdot \lambda_2 = |A|$$

$$\lambda_1 + \lambda_2 = -2 \quad \& \quad \lambda_1 \cdot \lambda_2 = -35$$

$$\lambda_1 = 5, \lambda_2 = -7$$

Consider a  $2 \times 2$  square matrix

$$A = \begin{bmatrix} \sigma & x \\ \omega & \sigma \end{bmatrix}$$

where  $x$  is unknown. If the eigen values of the matrix  $A$  are  $(\sigma + j\omega)$  and  $(\sigma - j\omega)$ , then  $x$  is equal to

- (a)  $+j\omega$
- (b)  $-j\omega$
- (c)  $+\omega$
- (d)  $-\omega$

$$ATB, \quad \lambda_1 = \sigma + j\omega \text{ & } \lambda_2 = \sigma - j\omega$$

$$\lambda_1 + \lambda_2 = \text{Tr}(A)$$

$$2\sigma = 2\sigma \\ (\text{identity})$$

$$\lambda_1 \cdot \lambda_2 = |A|$$

$$\sigma^2 - \omega^2 = \sigma^2 - \omega^2$$

$$\omega^2 = \sigma^2 - \omega^2$$

$$\omega = -\frac{\omega^2}{\omega} = -\omega$$

$$j = \sqrt{-1} = j$$

$$j^2 = -1 = j^2$$

~~Ques~~ If  $A_{3 \times 3}$  &  $|A-I|=0$ ,  $\text{Tr}(A)=13$ ,  $|A|=32$  then  $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = ?$

a) 12

$\because A_{3 \times 3}$  & its E values are  $\lambda_1, \lambda_2, \lambda_3$

b) 13

ATQ,  $|A-I| = 0$

c) 81

d) 80

$|A-\lambda I| = 0$

$\Rightarrow \lambda_1 = 1$

$$\begin{aligned} & \text{ATQ, } |A-I| = 0, \text{ Tr}(A) = 13, |A| = 32 \\ & \lambda_1 + \lambda_2 + \lambda_3 = 13, \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 32 \\ & 1 + \lambda_2 + \lambda_3 = 13, 1(\lambda_2)(\lambda_3) = 32 \\ & \lambda_2 + \lambda_3 = 12, \lambda_2 \lambda_3 = 32 \end{aligned}$$

$$\therefore \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1^2 + 8^2 + 4^2 = 81 \quad \lambda_2 = 8, \lambda_3 = 4$$

If P and Q are two sq. Matrices of same order s.t  $PQ=QP=I$  then  
 0 is an eigen value of ?

- (a) P But not Q
- (b) Q But not P
- (c) Both P & Q
- (d) Neither P nor Q

$$\because PQ=QP=I \Rightarrow \begin{matrix} P^{-1} = Q \\ Q^{-1} = P \end{matrix}$$

i.e Both P & Q are Invertible.

$$\text{or } |P| \neq 0 \text{ & } |Q| \neq 0.$$

i.e Both are Non singular

so Zero can't be eigen value of any one Mat.

Q-8 If one of the EValue of  $A = \begin{vmatrix} 40 & -16 & -24=0 \\ -11 & 30 & -19=0 \\ 26 & 24 & -50=0 \end{vmatrix}$  is 0 then other EVales will be?

(a)  $\lambda - 20$

(b)  $20 - \lambda$

(c)  $20 + \lambda$

(d) 0

so By Prop 6, one EVale is  $\lambda = 0$

Now  $\lambda_1 + \lambda_2 + \lambda_3 = \text{Tr}(A)$

$$0 + \lambda + \lambda_3 = 20$$

$$\lambda_3 = 20 - \lambda$$

(M-II)  $|A| = \begin{vmatrix} 40 & -16 & -24 \\ -11 & 30 & -19 \\ 26 & 24 & -50 \end{vmatrix}$

$$\begin{aligned} C_1 &\rightarrow C_1 + C_2 + C_3 \\ &= \begin{vmatrix} 0 & -16 & -24 \\ 0 & 30 & -19 \\ 0 & 24 & -50 \end{vmatrix} \\ |A| &= 0 \Rightarrow \lambda = 0 \end{aligned}$$

Prop 4

~~Ques~~ one of the E-Value of  $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$  will be? @S ⑥ o  
~~P W~~  
~~✓ 10~~ ④ -1

(M-I) using Prop ⑥, one of the E Value of  $A$  is = 10 An

(M-II) C Equ'n of  $A$ ,  $|A| - \lambda I = 0$

$$\begin{vmatrix} (1-\lambda) & 2 & 3 & 4 \\ 2 & (3-\lambda) & 4 & 1 \\ 3 & 4 & (1-\lambda) & 2 \\ 4 & 1 & 3 & (2-\lambda) \end{vmatrix} = 0$$

$$(10-\lambda) \begin{vmatrix} 2 & 3 & 4 \\ (3-\lambda) & 4 & 1 \\ 4 & (1-\lambda) & 2 \end{vmatrix} = 0$$

$$(10-\lambda) \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \\ 1 & 4 & 1-\lambda & 2 \\ 1 & 2 & 3 & (2-\lambda) \end{vmatrix} = 0 \Rightarrow \lambda = 10$$

$$1 \rightarrow 1 + (2 + 3 + 4)$$

 Q: The E-Values of  $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{4 \times 4}$  are?  $\lambda = 4, ?, ?, ?$

By Prop 6 one E-Value is  $\lambda = 4$ .

$\therefore f(A) = \text{one} \& \text{Number of Non Zero E-Values of } A \leq f(A)$  (Prop 5)

i.e. No of Non Zero E-Values of  $A \leq 1$  ( $\leftarrow$  one)

$A$  has only one Non Zero E-Value which is  $\lambda = 4$

so Remaining E Values are  $\lambda = 0, 0, 0$ .

overall for  $A$ ,  $\lambda = 4, 0, 0, 0$ .

(M-II)

Also find E. Values by Conventional approach ?  $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

C Eqn's is  $|A - \lambda I| = 0$

$$\begin{vmatrix} (1-\lambda) & 1 & 1 \\ 1 & (1-\lambda) & 1 \\ 1 & 1 & (1-\lambda) \end{vmatrix} = 0$$

$$(4-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 1 & (1-\lambda) & 1 \\ 1 & 1 & (1-\lambda) \end{vmatrix} = 0$$

$$R_2 - R_1, R_3 - R_1, R_4 - R_1$$

$$Q \rightarrow Q + (c_2 + c_3 + c_4)$$

$$(4-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(-\lambda^3) = 0$$

$$\begin{vmatrix} (4-\lambda) & 1 & 1 & 1 \\ (4-\lambda) & (1-\lambda) & 1 & 1 \\ (4-\lambda) & 1 & (1-\lambda) & 1 \\ (4-\lambda) & 1 & 1 & (1-\lambda) \end{vmatrix} = 0$$

$$(4-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$\lambda = 4, 0, 0, 0.$$

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(WRONG APP)

$$\begin{array}{l}
 A = \left[ \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right] \xrightarrow{\substack{R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1}} \left[ \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] = A_1 \\
 \lambda = 4, 0, 0, 0
 \end{array}$$

i.e we have justified that equivalent Matrices have different E values

~~Ques~~

If  $A_{2 \times 2}$  s.t  $a_{11} = a_{12} = a_{21} = 1$  &  $a_{22} = -1$  then

E. Values of  $A^{19}$  are ?

(a)  $\pm \sqrt{2}$

(b)  $\pm 2$

(c)  $\pm 1$

(d)  ~~$\pm 512\sqrt{2}$~~

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

C.Eq is  $|A - \lambda I| = 0$

$$\boxed{\lambda^2 - 2 = 0} \Rightarrow \lambda = \pm \sqrt{2}$$

E values of A are  $\sqrt{2}$  &  $-\sqrt{2}$

" "  $A^{19}$  are  $(\sqrt{2})^{19}$  &  $(-\sqrt{2})^{19}$

$$= (\sqrt{2})^{18}(\sqrt{2}) \& (-\sqrt{2})^{18}(-\sqrt{2})$$

$$= 2^9 \sqrt{2} \& 2^9 (-\sqrt{2})$$

$$= 512\sqrt{2} \& -512\sqrt{2}$$

$$\text{Q: } A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & \frac{-1+i\sqrt{3}}{2} & 0 \\ 3 & 4 & \frac{-1-i\sqrt{3}}{2} \end{bmatrix} \text{ Then } \text{Tr}(A^{102}) = ?$$

$$\text{Sol: } A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & \omega & 0 \\ 3 & 4 & \omega^2 \end{bmatrix} = L.T.M$$

$$\omega = \frac{-1+i\sqrt{3}}{2}, \omega^2 = \frac{-1-i\sqrt{3}}{2}$$

$$\omega^3 = 1, 1 + \omega + \omega^2 = 0, \bar{\omega} = \omega^2$$

$$\bar{\omega} = \omega$$

E Values of A are 1,  $\omega$ ,  $\omega^2$

" "  $A^{102}$  are  $1^{102}$ ,  $\omega^{102}$ ,  $(\omega^2)^{102}$

$$= 1, \omega^{102}, \omega^{204}$$

$$= 1, (\omega^3)^{34}, (\omega^3)^{68} = 1, 1, 1^{34}, 1^{68}$$

E Values of  $A^{102}$  are 1, 1, 1

$$\text{Tr}(A^{102}) = 1 + 1 + 1 = 3$$

Let A be a  $3 \times 3$  matrix with Eigen values -1, 1,

0. Then  $\underbrace{|A^{100} + I|}_B$  is \_\_\_\_.

E Values of A are -1, 1, 0

" "  $A^{100}$  are  $(-1)^{100}, (1)^{100}, (0)^{100}$

" "  $A^{100}$  are = 1, 1, 0

Let  $B = \underbrace{A^{100} + I}_{\text{Using Prop 9}}$

$$\rightarrow 1+1=2$$

$$\rightarrow 1+1=2$$

$$\rightarrow 0+1=1$$

E Values of B are 2, 2, 1

$$\therefore |B| = (2)(2)(1) = 4$$

Q If  $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$  &  $B = A^2 - 2A + 3I$  then find the Product of the Eigenvalues of  $B$ ?

~~(a) 81~~ (M-I)  $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow \lambda_A^2 = 6$  (Do yourself/HW)

(b) 30

(c) 12

(d) 144

Using Prop 9:

$$B = (A^2 - 2A + 3I) \Rightarrow \lambda_B = (6)^2 - 2(6) + 3(1) = 27$$

$$\lambda_B = (2)^2 - 2(2) + 3(1) = 3$$

$\therefore$  Req. Product =  $27 \times 3 = 81$

Verification:  $B = A^2 - 2A + 3I = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - 2 \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 12 & 15 \end{bmatrix} = \begin{bmatrix} 27 & 0 \\ 0 & 27 \end{bmatrix}$

$\therefore$  E-Values of  $B$  are 27 & 3 & Product =  $|B| = 81$

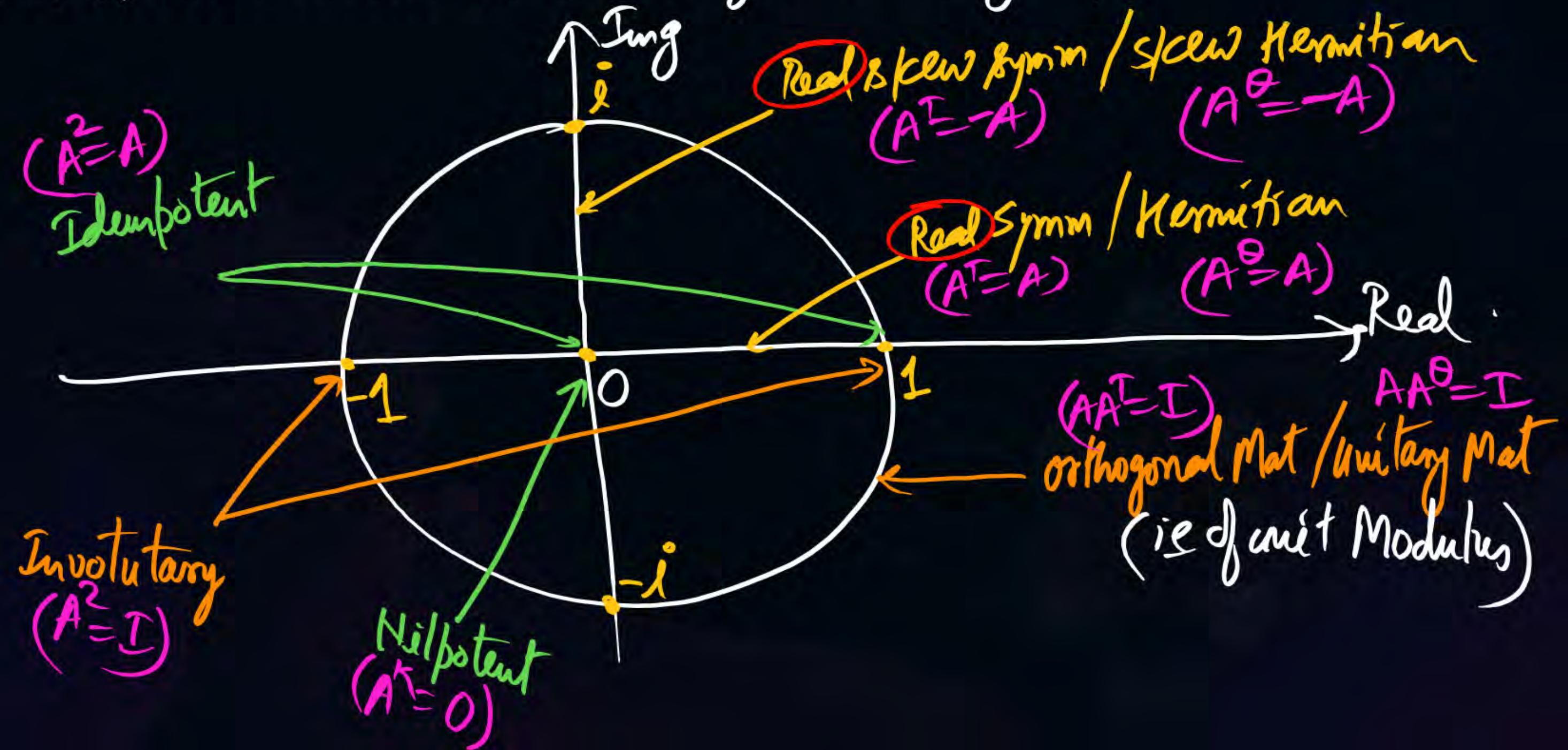
Let A be a  $4 \times 4$  matrix with real entries such that  $-1, 1, 2, -2$  are its Eigen values. If  $B = A^4 - 5A^2 + 5I$  then trace of  $A + B$  is \_\_\_\_\_.

$$\begin{array}{l} A \xrightarrow{\lambda_A = -1} \\ A \xrightarrow{\lambda_A = 1} \\ A \xrightarrow{\lambda_A = -2} \\ A \xrightarrow{\lambda_A = 2} \end{array}$$
$$B = A^4 - 5A^2 + 5I$$
$$\begin{aligned} (-1)^4 - 5(-1)^2 + 5(1) &= 1 - 5 + 5 = 1 \\ (1)^4 - 5(1)^2 + 5(1) &= 1 - 5 + 5 = 1 \\ (-2)^4 - 5(-2)^2 + 5(1) &= 16 - 20 + 5 = 1 \\ (2)^4 - 5(2)^2 + 5(1) &= 16 - 20 + 5 = 1 \end{aligned}$$

$$\begin{aligned} \text{Tr}(A+B) &= \text{Tr}(A) + \text{Tr}(B) \\ &= (-1+1-2+2) + (1+1+1+1) = 4 \end{aligned}$$

# ⑪ Short cut Method to Learn Various Th based on E Values →

Consider a unit circle centered at origin i.e.  $x^2+y^2=1$



$$\text{if } A = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

= Real Symm.

$\lambda \in \mathbb{R}$

$$\text{, if } A = \begin{bmatrix} 2 & -3 & 4+i \\ -3 & 5 & 1 \\ 4+i & 1 & 0 \end{bmatrix}$$

= Complex Symm.

it is not necessary that  $\lambda$  is Real.

$$\text{if } A = \begin{bmatrix} 0 & -3 & 4 \\ 3 & 0 & 1 \\ -4 & -1 & 0 \end{bmatrix}$$

= Real & skew symm

$\lambda \in$  Purely Imag Number

$$\text{, if } A = \begin{bmatrix} 0 & -3 & 4i \\ 3 & 0 & 2-i \\ -4i & -2+i & 0 \end{bmatrix}$$

= Comp Skew Symm.

it is not necessary that  $\lambda$  is Purely Imag

Q If E-Values of  $A = \begin{bmatrix} 2 & 5+i & -3 \\ x & -1 & y \\ -3 & y & 3 \end{bmatrix}$  are all Real Nos then  $x = ?$

- (a)  $5+i$
- (b)  $5-i$
- (c) 0
- (d) Both a & b

Real E-Values  $\Rightarrow$  Either A is Real Symm  $\times$  ( $\because a_{12} = \text{Complex No}$ )  
 or A is Hermitian ✓

Corresponding elements are Conjugates of each other.

$$y = 5-i$$

Ques If  $A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}_{4 \times 4}$  is an O-Mat then  $(AA^T)^{-1} = ? = (\mathbb{I})^{-1} = \mathbb{I} = I_{4 \times 4}$

(i) which of the following can not be the EValuee of A ?

a)  $-\frac{1}{2} + i\frac{\sqrt{3}}{2} \Rightarrow |\lambda| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1 \quad \checkmark$

b)  $-1 \Rightarrow |\lambda| = 1 \quad \checkmark$

c)  $\frac{\sqrt{3}}{2} + i\frac{1}{2} \Rightarrow |\lambda| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1 \quad \checkmark$

~~d)~~  $\frac{1}{4} + i\frac{\sqrt{3}}{4} \Rightarrow |\lambda| = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2} = \sqrt{\frac{1}{16} + \frac{3}{16}} = \sqrt{\frac{1}{4}} = \frac{1}{2} \quad \text{No}$

e) None

Q) If  $\{\lambda_i, x_i\}$  is an E-pair of  $A_{n \times n}$  then which is false?

- (a) If Non zero E vector of A if  $\rho(A - \lambda I) < n$  T ( $\because$  it is OR condition)
- (b) If  $A^D = A$  then  $\lambda_i \in \mathbb{R} \forall i (T) \because A$  is Hermitian
- (c) If  $\bar{A}' = \bar{A}^T$  then  $|\lambda_i| = |\lambda_i^*| \forall i (T) \because A$  is O-Mat
- (d)  $\{\lambda_i^m, x_i^m\}$  is an E-Pair of  $A^m$  (F)  $\because X^m = D^{-1}E$
- (e) If  $A = \bar{A}'$  then the Eigen Value of A is  $1 (T) \because A$  is Involuntary Mat

$$\rightarrow \because X = \begin{bmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}_{n \times 1} \text{ so } X^2 = X \quad \begin{matrix} X \\ m \times 1 \end{matrix} = N.D. \quad \left| \begin{array}{l} \text{E-Value of A is } \lambda \text{ & } A^m \text{ is } \lambda^m (T) \\ \text{E-Vectors of A & } A^m \text{ are same (T)} \end{array} \right.$$

Explanation: For Involutory Mat  $A$ ,

$$\begin{aligned} A^2 &= I \\ \Rightarrow \lambda^2 &= 1 \Rightarrow \lambda = \pm 1 \end{aligned}$$

Note: for Idempotent Mat  $A$ ,

$$\begin{aligned} A^2 &= A \\ \Rightarrow \lambda^2 &= \lambda \\ \lambda(\lambda-1) &= 0 \Rightarrow \lambda = 0 \text{ or } \end{aligned}$$

Note: Real EV values of an orthogonal Mat are  $1$  or  $-1$  (only)

for any O-Mat  $A$ , we have

$$AA^T = I$$

$$(\lambda)(\lambda) = 1$$

$$\lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

$$\therefore |\lambda| = 1 \text{ (justified)}$$

{ for Complex EV values,  $|\lambda| = 1$ .

The eigen values of  $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$  are  $|A| = 0 - (\lambda)(-\lambda) = +\lambda^2 = -1 \Rightarrow \lambda = -1, 1$  (Real).

- (a) Purely imaginary
- (b) Zero
- (c) Real
- (d) None of the above

M-I  $\because A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$  is ~~Skew symm.~~  
~~(But Comp. Skew symm)~~  
 $\Rightarrow \lambda = \text{Purely Imag}$

M-II Now  $A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$  is Hermitian also.  
 $\lambda \in \mathbb{R}$ .

M-III C. Equ'n is  $|A - \lambda I| = 0$

$$\begin{vmatrix} 0-\lambda & -i \\ -i & 0-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (\lambda)(-\lambda) = 0$$

$$\lambda^2 + \lambda^2 = 0$$

$$2\lambda^2 = 0$$

$$\lambda^2 = 0$$

$$\lambda = \pm 1 \quad (\text{Real Nos})$$

If A & B are Non zero sq. Matrices then  $AB = O \Rightarrow$

Famous Doubt

(a) B is singular

(b) A is singular

(c) A & B are orthogonal

(d) A & B are singular

sq. Matrices then  $AB = O \Rightarrow$

Case I: let A is Non singular then  $A^{-1}$  exist

$$AB = O$$

$$A^{-1}(AB) = A^{-1}O$$

$$B = O \text{ which is false } A \neq O$$

so our assumption is wrong ie A must be sing

Case II: let B is Non sing then  $B^{-1}$  exist.

$$(AB)B^{-1} = O \cdot B^{-1}$$

$$A = O \text{ (false)}$$

so again our assumption is wrong hence B must be sing



# THANK - YOU

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