Computer Science & DA

Calculus and Optimization

DPP 01 Discussion Notes

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[MCQ]

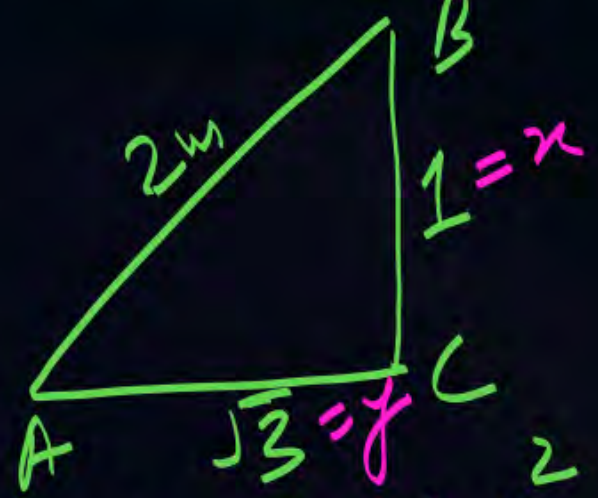
#Q. A 2m ladder leans against a vertical wall. If the top of the ladder begins to slide down the wall at the rate 25 cm/sec. then the rate (in cm/sec.) at which the bottom of the ladder slides away from the wall on the horizontal ground when the top of the ladder is 1 m above the ground is:

A $25\sqrt{3}$

B 25

C $\frac{25}{\sqrt{3}}$

D $\frac{25}{3}$



$4 = 1 + AC^2$

$AC = \sqrt{3}$

$x^2 + y^2 = 4 \longrightarrow (1)$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$(1)(-25) + \sqrt{3}\left(\frac{dy}{dt}\right) = 0$

$\frac{dy}{dt} = \frac{25}{\sqrt{3}}$

[MCQ]

#Q. The percentage error in calculating the volume of a cubical box if an error of 1% is made in measuring the length of edges of the cube is :

A. 1%

B. 2%

C. 3%

D. 6%

Absolute Error $= dx$

Relative " $= \frac{dx}{x}$

% " $= \frac{dx}{x} \times 100$

$V = x^3$ — (1)

$dV = 3x^2\,dx$

$\frac{dV}{V} = \frac{3x^2}{V}\,dx$

$\frac{dV}{V} = \frac{3}{x}\,dx$

$\frac{dV}{V} \times 100 = 3\left(\frac{dx}{x} \times 100\right) = 3 \times 1\%$

$= 3\%$

[NAT]

7^x

#Q. Find the tangent line to $f(x) = 7x + 4e^x$ at $x = 0$.

At $x=0$, $y = f(0) = 7^0 + 4e^0 = 5$

so point of tangent $= (x, y) = (0, 5) = P$

slope of tangent $m = \left(\dfrac{dy}{dx}\right)_P = \left(7^x \cdot \log_e 7 + 4e^x\right)_{P(0,5)} = \log 7 + 4$

eq^n of tangent $y - y_1 = m(x - x_1)$

$y - 5 = (\log 7 + 4)(x - 0)$ Ans

[NAT]

#Q. Differentiate $f(t) = \dfrac{1 + 5t}{ln(t)}$

$$f'(t) = \frac{\log t\,(0+5) - (1+5t)\left(\frac{1}{t}\right)}{(1+\log t)^2} = \frac{5\log t - \frac{1}{t} + 5}{(1+\log t)^2}$$

[NAT]

$\frac{\partial w}{\partial x}$

#Q. Find $\frac{\delta w}{\delta x}$ for the following-

$$w = \cos(x^2 + 2y) - e^{4x - z^4 y} + (y^3)$$

$$\frac{\partial w}{\partial x} = -\sin(x^2+2y)(2x+0) - e^{4x-z^4y}(4-0) + 0$$

$$= -2x\sin(x^2+2y) - 4e^{4x-z^4y}$$

//

[NAT]

#Q. Find all the 1st order partial derivatives of the following function.

$$f(x,y,z) = 4x^3y^2 - e^z y^4 + \frac{z^3}{x^2} + 4y - x^{16}$$

$$\frac{\partial f}{\partial x} = (12x^2)y^2 - 0 + z^3\left(\frac{-2}{x^3}\right) + 0 - 16x^{15}$$

$$\frac{\partial f}{\partial y} = 4x^3(2y) - e^z(4y^3) + 0 + 4 - 0$$

$$\frac{\partial f}{\partial z} = 0 - y^4(e^z) + \frac{3z^2}{x^2} + 0 - 0$$

[NAT]

#Q. Prove that
$f(x, y) = x^3 - 2x^2y + 3xy^2 + y^3$
is homogeneous; what is the degree? Verify Euler's Theorem for f .

By observation f is H.f of degree $3 = n$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f} \quad (1)$$

$$LHS = x\left(3x^2 - 4xy + 3y^2\right) + y\left(-2x^2 + 6xy + 3y^2\right)$$

$$= \text{-------------} = 3f = RHS$$

[NAT]

#Q. If $v(x,y) = \log_e\left(\frac{x^2+y^2}{x+y}\right)$, prove that $\boxed{x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1}$

Sol: $\because V(\lambda x, \lambda y) \neq \lambda^n . V(x,y)$ So V is H.Homog

$\log_e a = b$

$\boxed{a = e^b}$

Let $\boxed{e^v = u} = \frac{x^2+y^2}{x+y}$ —— (1)

$\because u(\lambda x, \lambda y) = \lambda^1 u(x,y)$ So u is Homog f^n of $n=1$

So Applying E.T for 'u' i.e $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

$x\frac{\partial}{\partial x}(e^v) + y\frac{\partial}{\partial y}(e^v) = 1 . e^v$

$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1 . \frac{e^v}{e^v} = 1$

[NAT]

#Q. Prove that
g(x, y) = x log (y/x)
is homogenous, what is the degree ? Verify Euler's theorem for g.

$$g(\lambda x, \lambda y) = (\lambda x)\log\left(\frac{\lambda y}{\lambda x}\right) = \lambda\left[x \cdot \log\frac{y}{x}\right] = \lambda^1\left[g(x,y)\right]$$

∴ g is H.f of $n=1$

E.Th is $x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = 1.g$ —— (1)

$$\frac{\partial g}{\partial x} = x\left(\frac{1}{y/x}\cdot\frac{-y}{x^2}\right) + \log\left(\frac{y}{x}\right)[1]$$

$$= -1 + \log\left(\frac{y}{x}\right)$$

$$\frac{\partial g}{\partial y} = x\left[\frac{1}{y/x}\cdot\frac{1}{x}\right] = \frac{x}{y}$$

$$\text{LHS of (1)} = x\left[-1+\log\frac{y}{x}\right] + y\left[\frac{x}{y}\right]$$

$$= x\log\frac{y}{x} = 1.g(x,y)$$

$$= \text{RHS}$$

[NAT]

#Q. The jacobian of p, q, r, w. r. t x, y, z given , p = x + y + z, q = y + z, r = z is ________.

$$J = \frac{\partial(p,q,r)}{\partial(x,y,z)} = \begin{vmatrix} p_x & p_y & p_z \\ q_x & q_y & q_z \\ r_x & r_y & r_z \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \quad \underline{\text{Ans}}$$

[MCQ]

#Q. Given $u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$ then the value of $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ is _____.

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

A) 4

B) −4

C) 0

D) 1

$$J = \frac{1}{x^2y^2z^2} \begin{vmatrix} -yz & zx & xy \\ zy & -zx & xy \\ yz & xz & -xy \end{vmatrix}$$

$R_2 \to R_2 + R_1$ & $R_3 \to R_3 + R_1$

$$= \frac{1}{x^2y^2z^2} \begin{vmatrix} -yz & zx & xy \\ 0 & 0 & 2xy \\ 0 & 2zx & 0 \end{vmatrix}$$

$$= \frac{1}{x^2y^2z^2}(-4x^2yz) = \frac{-4}{yz}$$

[MSQ]

$D_f : x > 2$ or $(2, \infty)$

#Q. The function $f(x) = 2\log(x-2) - x^2 + 4x + 1$ increase on the interval

$$f'(x) = \frac{2}{x-2} - 2x + 4 = \frac{2 - 2(x-2)^2}{(x-2)} = \frac{2 - 2[x^2 + 4 - 4x]}{x-2}$$

(A) (1, 2)

(B) (2, 3)

(C) (5/2, 3)

(D) (2, 4)

$$= \frac{2 - 2x^2 - 8 + 8x}{x-2}$$

$$= -2\left[\frac{x^2 - 4x + 3}{x-2}\right]$$

$$f'(x) = -2\left[\frac{(x-3)(x-1)}{x-2}\right]$$

Sign of $f'(x)$

T. Points are $x = 1, 3, 2$

But Acc to Dom. T.P are $x = 3$.

[MSQ]

#Q. The function $f(x) = \dfrac{x^2-1}{x^2+1}$: $= f'(x) = \dfrac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2} = \dfrac{2x^3+2x-2x^3+2x}{(\quad)^2}$

$$f'(x) = \frac{4x}{(x^2+1)^2}$$

$$f''(x) = \frac{(x^2+1)^2(4) - 4x[2(x^2+1)(2x)]}{(x^2+1)^4}$$

A. is concave for $x \in \left[\dfrac{-1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right]$

B. is convex for $x \in \left[\dfrac{-1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right]$

C. has 2 points of inflexion

D. is concave for $\left[\dfrac{1}{\sqrt{3}}, \infty\right)$

$$= 4\left[\frac{(x^2+1)\{(x^2+1)-x(4x)\}}{(\quad)^4}\right] = 4\frac{(1-3x^2)}{(x^2+1)^3} = \frac{-12\left(x^2-\frac{1}{3}\right)}{(\quad)^3} = \boxed{\frac{-12\left(x-\frac{1}{\sqrt{3}}\right)\left(x+\frac{1}{\sqrt{3}}\right)}{(x^2+1)^3}}$$

$$\underset{\text{Convex}}{-} \quad -\frac{1}{\sqrt{3}} \quad \underset{\text{Concave}}{+} \quad \frac{1}{\sqrt{3}} \quad \underset{\text{Convex}}{-}$$

Sign of $f''(x)$

[MCQ]

#Q. Number of the points of inflexion of the function:
$f(x) = x^4 - 18x^2 + 9$

$+ \quad - \quad +$

$-\sqrt{3} \quad \sqrt{3}$

$f''(x)$

$f'(x) = 4x^3 - 36x$

$f''(x) = 12x^2 - 36 = 12(x^2 - 3) = \boxed{12(x-\sqrt{3})(x+\sqrt{3})}$

A 1

B 2

C 3

D 4

[MCQ]

#Q. The function $f(x) = 2x^3 - 3x^2 - 36x + 2$ has its maxima at

$$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = \boxed{6[(x-3)(x+2)]}$$

So T. Points are $x = -2, 3$

$+ \quad - \quad +$

-2 (Max) $\quad 3$ (min)

A. $x = -2$ only

B. $x = 0$ only

C. $x = 3$ only

D. both $x = -2$ and $x = 3$

[NAT]

#Q. The maximum value of $f(x) = x^3 - 9x^2 + 24x + 5$ in the interval $[1, 6]$ is

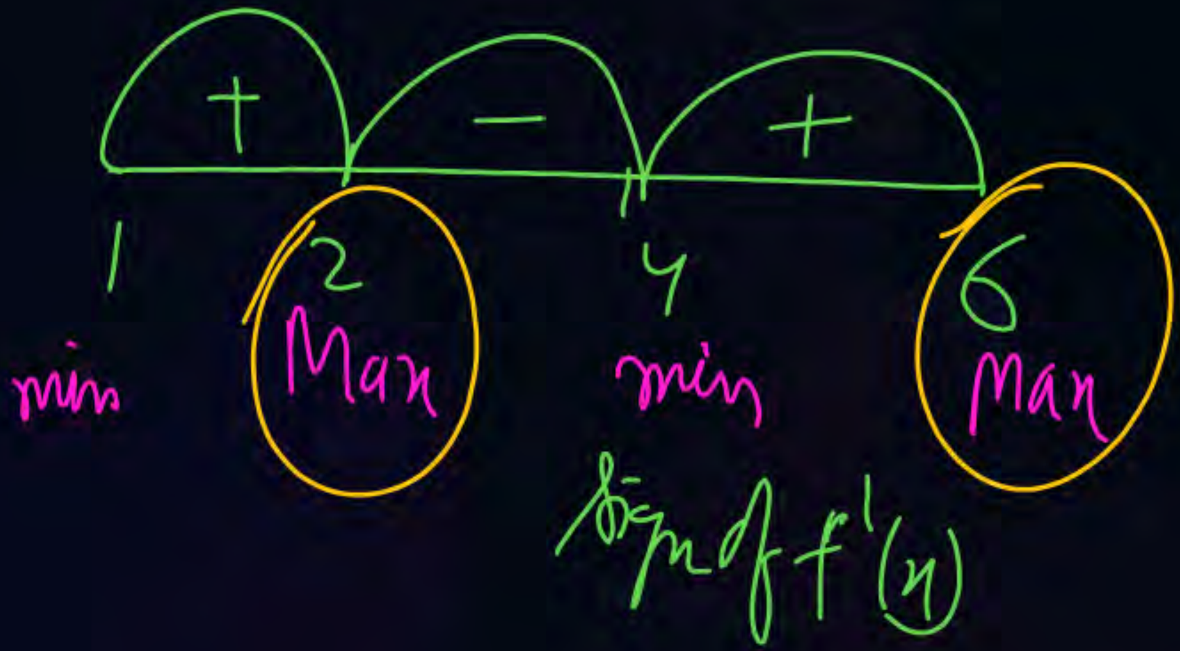
$$f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = \boxed{3(x-4)(x-2)}$$

T-Points are $f'(x) = 0 \implies x = 2, 4$

$$f(2) = 8 - 36 + 48 + 5 = 25 \; ??$$

$$f(6) = 216 - 324 + 144 + 5 = 41$$

Ans



[NAT]

#Q. The minimum value of the function $f(x) = x^3 - 3x^2 - 24x + 100$ in the interval [-3, 3] is

$$f'(x) = 3x^2 - 6x - 24$$

$$= 3(x^2 - 2x - 8) = \boxed{3(x-4)(x+2)}$$

172
54

T. Points $\Rightarrow x = -2, 4$

But Acc to given interval, only T.P is $x = -2$

$$f(-3) = -27 - 27 + 72 + 100 = 118$$

$$f(3) = 27 - 27 - 72 + 100 = 28$$ Ans

[MCQ]

#Q. Suppose that the amount of money in a bank account after t year is given by $\boxed{A(t) = 2000 - 10te^{5-\frac{t^2}{8}}}$

$$\Rightarrow \frac{dA}{dt} = 0 - 10\left[e^{5-\frac{t^2}{8}}(1) + t\left\{e^{5-\frac{t^2}{8}}\left(-\frac{2t}{8}\right)\right\}\right]$$

The minimum and maximum amount of money in the account during the first 10 year that it is open occur respectively at:

$$= -10\,e^{5-\frac{t^2}{8}}\left[1 - \frac{t^2}{4}\right]$$

$$= +\frac{10}{4}\,e^{5-\frac{t^2}{8}}\left(t^2 - 4\right)$$

A T = 2, t = 0

B T = 2, t = 10

C T = 0, t = 2

D T = 10, t = 0

$A(0) = 2000$

$A(10) = < 2000$

$-$ $+$

0 Max, 2 min, 10 Max

$$\frac{dA}{dt} = \frac{10}{4}e^{(\ )}(t-2)(t+2)$$

$\Rightarrow t = 2, -2$

[MSQ]

#Q. The function: $f(x) = x^2 \ln(3x) + 6$ has:

$D_f = (0, \infty)$

$f'(x) = 0$

$x^2\left(\frac{1}{3x}\cdot 3\right) + \log(3x)\cdot(2x) = 0$

$x + 2x\log 3x = 0 \Rightarrow$

$x(1 + 2\log 3x) = 0 \Rightarrow$ $x = 0$ ✗ ; $1 + 2\log 3x = 0$

A. No critical point

B. Only one critical point

C. Only one stationary point

D. Infinitely many critical points

$\log_e (3x)^2 = -1$

$3x^2 = e^{-1} \Rightarrow x^2 = \frac{1}{3e}$

$x = \pm\frac{1}{\sqrt{3e}}$

$x = \frac{1}{\sqrt{3e}}$

[MCQ]

#Q. The function $f(x,y) = x^2y - 3xy + 2y + x$ has

A. No local extremum

B. One local minimum but no local maximum

C. One local maximum but no local minimum

D. One local minimum and one local maximum

$f_x = 2xy - 3y + 1$

$f_{xx} = 2y$

$f_y = x^2 - 3x + 2$

$f_{yy} = 0$

$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x^2 - 3x + 2)$

$= 2x - 3$

$rt - s^2 = 0 - (2x-3)^2$

< 0

[MCQ]

#Q. Find the local minima of the function $f(x, y) = 2x^2 + 2xy + 2y^2 - 6x$

$f_x = 4x + 2y - 6$, $f_y = 2x + 4y$, $f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2x + 4y) = 2$

$f_{xx} = 4 = r$, $f_{yy} = 4 = t$ $\qquad s$

A (2, 1)

B (–2, 1)

C (2, 0)

D (2, –1)

$f_x = 0$ & $f_y = 0$

$$\left.\begin{aligned} 4x + 2y &= 6 \\ 2x + 4y &= 0 \end{aligned}\right\} \Rightarrow x = 2,\ y = -1$$

$rt - s^2 = (4)(4) - (2)^2$

> 0

$r = 4 > 0$

[NAT]

#Q. Find the critical points of the function $f(x, y) = y^2 - x^2$ and check for the presence of any saddle point.

$f_x = 0, \quad f_y = 0$

$-2x = 0, \quad 2y = 0$

$x = 0, \quad y = 0$

$P(0,0)$ is th C. Point.

$r = f_{xx} = -2$

$t = f_{yy} = 2$

$s = f_{xy} = 0$

$rt - s^2 = (\ )(\ ) - (\ )^2$

$= (-2)(2) - (0)^2$

< 0

[NAT]

#Q. A jet of an enemy is flying along the curve $y = x^2 + 2$. A soldier is placed at the point (3, 2). What is the nearest distance between the soldier and the jet?

$\Rightarrow P(x, y)$

$A =$

$$AP = \sqrt{(x-3)^2 + (y-2)^2}$$

$$AP = \sqrt{(x-3)^2 + (x^2)^2}$$

let $AP^2 = \boxed{U = (x-3)^2 + x^4}$ —— (1)

where $AP_{min} = \sqrt{U_{min}} = ?$

$$AP_{min} = \sqrt{5} \quad \underline{Ans}$$

$$\frac{du}{dx} = 2(x-3) + 4x^3$$

$$= 4x^3 + 2x - 6 \Rightarrow \frac{du}{dx} = 0 \text{ will be when } x = 1$$

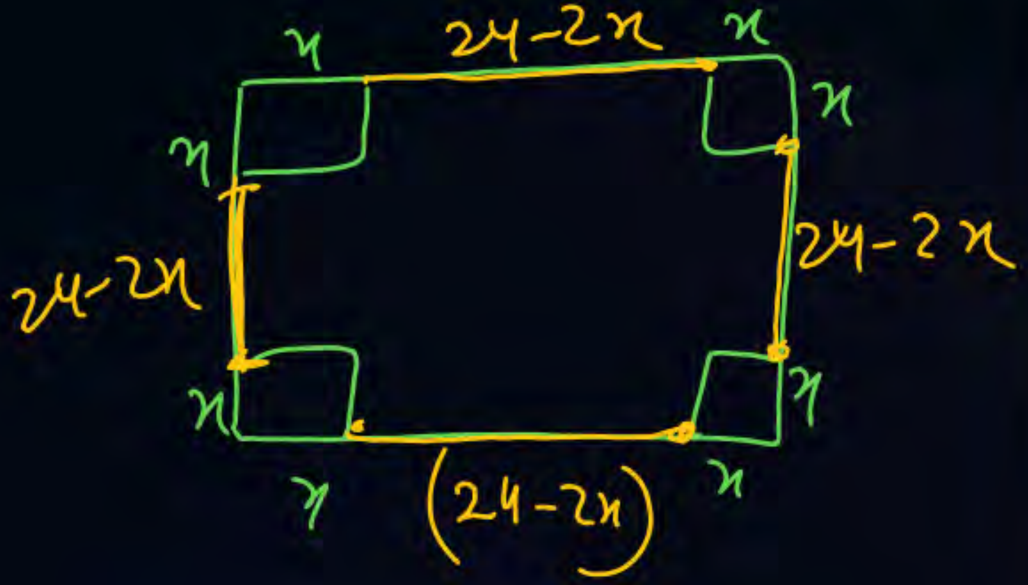
$$\left(\frac{d^2u}{dx^2}\right) = (12x^2 + 2) = +ve$$

$x = 1$

ie U will be min at $x = 1$

$$U_{min} = (1-3)^2 + 1^4 = 5$$

#Q. A Square piece of tin of side 24 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. What should be the side of the square to be cut off so that the volume of the box is maximum ? Also find this maximum volume.



Height = x

$$V = l \times b \times h = (24-2x)(24-2x)x = 16 \times 16 \times 4$$

$$V = (576 + 4x^2 - 96x)x = \boxed{4x^3 - 96x^2 + 576x}$$

for Max. Volume $\frac{dV}{dx} = 0$

$$12x^2 - 192x + 576 = 0$$

$$12(x^2 - 16x + 48) = 0$$

$$12(x-12)(x-4) = 0$$

$x = 4$ & $\cancel{12}$

$$V_{max} = 4 \times 64 - 96 \times 16 + 576 \times 4$$

$$= 1024$$

[NAT]

#Q. Manufacturer can sell x items at a price of rupees $\left(5-\frac{x}{100}\right)$ each.

The cost price is Rs. $\left(\frac{x}{5}+500\right)$. = Total C.P

Find the number of items he should sell to earn maximum profit.

$x = ?$

$P = \text{Total SP} - \text{Total CP}$

$= \left(5-\frac{x}{100}\right).x - \left(\frac{x}{5}+500\right)$

$P = -\frac{x^2}{100} + 5x - \frac{x}{5} - 500$ —— (1)

$\frac{dP}{dx} = \frac{-x}{50} + 5 - \frac{1}{5} = \frac{-x}{50} + \frac{24}{5} = \frac{-x+240}{50}$

$\frac{d^2P}{dx^2} = -\frac{1}{50}$ (<ve) $\Rightarrow$ Maxima

$\frac{dP}{dx} = 0 \Rightarrow x = 240$ Ans

[NAT]

#Q. Evaluate $\int x^2 \sin x\, dx$

$$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - \cdots\cdots$$

$$\int \underset{u}{\underset{\parallel}{x^2}} \cdot \underset{v}{\underset{\parallel}{\sin x}}\, dx = x^2(-\cos x) - 2x(-\sin x) + 2(\cos x) - 0(?) + 0 \cdots$$

[NAT]

#Q. $\int \frac{dx}{2\sqrt{x}(x+1)}$. $= \frac{1}{2}\int \frac{2t\,dt}{t\cdot(t^2+1)} = \int \frac{dt}{t^2+1} = \tan^{-1}(t) + C$

Put $x = t^2$

$dx = 2t\,dt$

$= \tan^{-1}(\sqrt{x}) + C$

[NAT]

#Q. Evaluate $\int \frac{1 \cdot dx}{(x-1)(x-2)(x-3)}$

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Put $x=1$, $1 = A(-1)(-2) \Longrightarrow A = 1/2$

Put $x=2$, $1 = B(1)(-1) \Longrightarrow B = -1$

Put $x=3$, $1 = C(2)(1) \Longrightarrow C = \frac{1}{2}$

$$= \int \left(\frac{1/2}{x-1} + \frac{-1}{x-2} + \frac{1/2}{x-3} \right) dx$$

$$= \frac{1}{2} \log(x-1) - 1 \log(x-2) + \frac{1}{2} \log(x-3)$$

$$= \log(x-1)^{1/2} + \log(x-3)^{1/2} - \log(x-2)$$

$$= \log \left[\frac{\sqrt{(x-1)(x-3)}}{x-2} \right]$$

[NAT]

#Q. Evaluate $\displaystyle I = \int_2^7 \frac{\sqrt{x}\,dx}{\sqrt{x}+\sqrt{9-x}}$ —— (1)

$$= \int_2^7 \frac{\sqrt{9-x}}{\sqrt{9-x}+\sqrt{9-(9-x)}}\,dx = \int_2^7 \frac{\sqrt{9-x}}{\sqrt{9-x}+\sqrt{x}}\,dx \quad \text{—— (2)}$$

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

(1) + (2) $\Longrightarrow$

$$2I = \int_2^7 \left(\frac{\sqrt{x}+\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}}\right)dx = \int_2^7 (1)\,dx$$

$$2I = 5$$

$$I = \frac{5}{2}$$

[NAT]

#Q. Evaluate $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+e^x)(1+x^2)}$

$$I = \int_{-a}^{a} f(x)\,dx = \int_0^a [f(x)+f(-x)]\,dx$$

$$= \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x) = f(x) \\ 0, & \text{if } f(-x) = -f(x) \end{cases}$$

$$f(x) = \frac{1}{(1+e^x)(1+x^2)} \quad \text{--- (1)}$$

$$f(-x) = \frac{1}{(1+e^{-x})(1+x^2)}$$

$$= \frac{e^x}{(1+e^x)(1+x^2)} \quad \text{--- (2)}$$

$$f(x)+f(-x) = \frac{(1+e^x)}{(1+e^x)(1+x^2)} = \frac{1}{1+x^2}$$

$$I = \int_0^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \left[\tan^{-1}(x)\right]_0^{\sqrt{3}}$$

$$= \tan^{-1}(\sqrt{3}) - \tan^{-1}(0)$$

$$= \frac{\pi}{3} \quad //$$

[NAT]

#Q. Evaluate $\int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}dx$ $=\int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}\times\sqrt{\frac{a-x}{a-x}}\,dx=\int_{-a}^{a}\frac{a-x}{\sqrt{a^2-x^2}}\,dx$

$$I=\int_{-a}^{a}\frac{a}{\sqrt{a^2-x^2}}dx-\int_{-a}^{a}\left(\frac{x}{\sqrt{a^2-x^2}}\right)dx$$

$$=a.2\int_{0}^{a}\frac{1}{\sqrt{a^2-x^2}}dx-0$$

$$I=2a\int_{0}^{a}\frac{1}{\sqrt{a^2-x^2}}dx=2a\left(\sin^{-1}\left(\frac{x}{a}\right)\right)_0^a=2a\left[\sin^{-1}(1)-\sin^{-1}0\right]=2a\left(\frac{\pi}{2}\right)$$

$$=\pi a$$

[NAT]

#Q. Evaluate $I = \int_0^{\pi} \frac{x\,dx}{1+\cos^2 x} = \int_0^{\pi} \frac{x\,dx}{1+\cos^2 x}$ —— (1)

$\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$I = \int_0^{\pi} \frac{\pi - x}{1+\cos^2 x}dx = \int_0^{\pi} \frac{\pi}{1+\cos^2 x}dx - I$

$2I = \pi \int_0^{\pi} \frac{dx}{1+\cos^2 x} = 2\pi \int_0^{\pi/2} \frac{dx}{1+\cos^2 x}$

$I = \pi \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + 1}dx$

$\int_0^{\alpha} f(x)dx = \begin{cases} 2\int_0^{\alpha/2} f(x)dx, & f(\alpha - x) = f(x) \\ 0, & f(\alpha - x) = -f(x) \end{cases}$

$I = \pi \int_0^{\pi/2} \frac{\sec^2 x}{2+\tan^2 x}dx$

Put $\tan x = t$

$\sec^2 x\,dx = dt$

$= \pi \int_0^{\infty} \frac{dt}{2+t^2}$

$= \pi \left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_0^{\infty} = \frac{\pi}{\sqrt{2}}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{2\sqrt{2}}$

[NAT]

#Q. Evaluate $\int_0^{\pi} \frac{dx}{1+2\sin^2 x}$ $= \int_0^{\pi} \frac{dx}{1+2\sin^2 x} = 2\int_0^{\pi/2} \frac{dx}{1+2\sin^2 x} = 2\int_0^{\pi/2} \frac{\sec^2 x\, dx}{\sec^2 x + 2\tan^2 x}$

$$\int_0^{a} f(x)dx = \begin{cases} 2\int_0^{a/2} f(x)dx, & \text{if } f(a-x) = f(x) \\ 0, & \text{if } f(a-x) = -f(x) \end{cases}$$

$$= 2\int_0^{\pi/2} \frac{\sec^2 x\, dx}{1+3\tan^2 x} = \frac{2}{3}\int_0^{\pi/2} \frac{\sec^2 x\, dx}{\left(\frac{1}{\sqrt{3}}\right)^2 + \tan^2 x}$$

Put $\tan x = t$ $\begin{cases} t = 0 \\ t = \infty \end{cases}$

$\sec^2 x\, dx = dt$

$$= \frac{2}{3}\int_0^{\infty} \frac{dt}{\left(\frac{1}{\sqrt{3}}\right)^2 + t^2} = \frac{2}{3} \frac{1}{(1/\sqrt{3})}\left[\tan^{-1}\left(\frac{t}{1/\sqrt{3}}\right)\right]_0^{\infty}$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{\sqrt{3}}$$

[MCQ]

#Q. Let $5f(x)+4f\left(\frac{1}{x}\right)=\frac{1}{x}+3, x>0$ Then $18\int_1^2 f(x)\,dx$ is equal to:

A. 10 log 2 – 6

B. 10 log 2 + 6

C. 5 log 2 – 6

D. 5 log 2 – 3

$5f(x)+4f\left(\frac{1}{x}\right)=\frac{1}{x}+3 \longrightarrow (1) \Rightarrow 25f(x)+\cancel{20f\left(\frac{1}{x}\right)}=\frac{5}{x}+15$

$5f\left(\frac{1}{x}\right)+4f(x)=x+3$

i.e. $4f(x)+5f\left(\frac{1}{x}\right)=x+3 \longrightarrow (2) \Rightarrow \underline{16}f(x)\underline{+}\cancel{20f\left(\frac{1}{x}\right)}=\underline{4x}+\underline{12}$

$(1)\times 5-(2)\times 4 \Rightarrow$

$9f(x)=\frac{5}{x}-4x+3$

$f(x)=\frac{5}{9x}-\frac{4x}{9}+\frac{3}{9}$

$$I = 18\int_1^2 f(x)\,dx = 18\int_1^2 \frac{1}{9}\left(\frac{5}{x} - 4x + 3\right)dx = 2\int_1^2 \left(\frac{5}{x} - 4x + 3\right)dx$$

$$= 2\Big[5\log x - 4x^2 + 3x\Big]_1^2$$

$$= 2\Big[(5\log 2 - 16 + 6) - (0 - 4 + 3)\Big] = 10\log 2 - 18 \quad \text{Ans}$$

[NAT]

#Q. *If* $F(x)=\int_{x}^{x^2}\sqrt{\sin t}\,dt$, then find $F'(x)$

$$F'(x)=\frac{d}{dx}f(x)=\frac{d}{dx}\int_{x}^{x^2}\sqrt{\sin t}\,.\,dt$$

$$=\frac{d}{dx}(x^2)\sqrt{\sin(x^2)}-\frac{d}{dx}(x)\sqrt{\sin(x)}$$

$$=2x\sqrt{\sin x^2}-\sqrt{\sin x}$$

[NAT]

#Q. Evaluate $\lim_{x\to\infty}\dfrac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt}$ $= \lim_{x\to\infty}\dfrac{\left[\int_0^x e^{t^2}dt\right]^2}{\int_0^x e^{2t^2}dt} = ? = \dfrac{\infty}{\infty}$ form.

$= \lim_{x\to\infty}\dfrac{\frac{d}{dx}\left[\int_0^x e^{t^2}dt\right]^2}{\frac{d}{dx}\int_0^x e^{2t^2}dt} = \lim_{x\to\infty}\dfrac{2\left[\int_0^x e^{t^2}dt\right]^{2-1}\cdot\frac{d}{dx}\int_0^x e^{t^2}dt}{\frac{d}{dx}\int_0^x e^{2t^2}dt}$

$\lim_{x\to\infty}\dfrac{2\int_0^x e^{t^2}dt\left\{\frac{d}{dx}(x)\, e^{x^2}-0\right\}}{\left\{\frac{d}{dx}(x)\, e^{2x^2}-0\right\}} = \lim_{x\to\infty}\left[\dfrac{2\int_0^x e^{t^2}dt}{e^{x^2}}\right] = \dfrac{\infty}{\infty} = \dfrac{2\left[(1)\cdot e^{x^2}-0\right]}{e^{x^2}\cdot(2x)}$

$\lim_{x\to\infty}\left(\dfrac{1}{x}\right)$

$= 0$ Ans

[MCQ]

#Q. Arc length of the curve $y = x^{3/2}$, $z = 0$ from $(0, 0, 0)$ to $(4, 8, 0)$ is

$4^{\frac{3}{2}} = (64)^{\frac{1}{2}} = 8$

A. $\frac{8}{27}(10^{3/2}+1)$

B. $\frac{8}{27}(10^{3/2}-2)$

C. $\frac{8}{27}(10^{3/2}-1)$ ✓

D. $\frac{8}{27}(10^{3/2}+2)$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}\sqrt{x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{9}{4}x$$

$$\text{length} = \int_{x=a}^{b}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{x=0}^{4}\sqrt{1+\frac{9}{4}x}\,dx$$

$$= \frac{1}{2}\int_0^4 (4+9x)^{\frac{1}{2}}\,dx = \frac{1}{2}\left(\frac{(4+9x)^{3/2}}{\frac{3}{2}(9)}\right)_0^4 = \frac{1}{27}\left[(40)^{3/2} - 4^{3/2}\right]$$

$$= \frac{8}{27}\left[(10)^{3/2} - 1\right]$$

[MCQ]

#Q. Let f be an increasing, differentiable function. If the curve y = f(x) passes through (1, 1) and has length $L = \int_1^2 \sqrt{\left(1+\frac{1}{4x^2}\right)}dx, 1 \le x \le 2,$ then the curve is-

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \Rightarrow \frac{dy}{dx} = \frac{1}{2x}$$

$$\int dy = \int \frac{1}{2x}\,dx + c$$

$$y = \frac{1}{2}\log x + c \Longrightarrow y = \log(\sqrt{x}) + 1$$

At $(1,1) \Rightarrow 1 = 0 + c \Rightarrow c = 1$

A $y = \ln(\sqrt{x}) - 1$

B $y = 1 - \ln(\sqrt{x})$

C $y = \ln(1 + \sqrt{x})$

D $y = 1 + \ln(\sqrt{x})$

[MCQ]

#Q. The volume of the solid of revolution of $y=\frac{a}{2}\left(e^{x/a}+e^{-x/a}\right)$ about x-axis between x= 0 and x= b is-

A $\frac{\pi a^3}{8}\left(e^{2b/a}-e^{-2b/a}\right)-\frac{\pi a^2 b}{2}$

B $-\frac{\pi a^3}{8}\left(e^{2b/a}-e^{-2b/a}\right)+\frac{\pi a^2 b}{2}$

C $-\frac{\pi a^3}{8}\left(e^{2b/a}-e^{-2b/a}\right)-\frac{\pi a^2 b}{2}$

D $\frac{\pi a^3}{8}\left(e^{2b/a}-e^{-2b/a}\right)+\frac{\pi a^2 b}{2}$ ✓

$$V=\int_a^b \pi y^2\,dx=\int_0^b \pi\left[\frac{a}{2}\left(e^{x/a}+e^{-x/a}\right)\right]^2 dx$$

$$=\frac{\pi a^2}{4}\int_0^b\left(e^{\frac{2x}{a}}+e^{-\frac{2x}{a}}+2\right)dx$$

$$=\frac{\pi a^2}{4}\left[\frac{e^{\frac{2x}{a}}}{\left(\frac{2}{a}\right)}+\frac{e^{-\frac{2x}{a}}}{\left(-\frac{2}{a}\right)}+2x\right]_0^b$$

$$=\frac{\pi a^2}{4}\left[\frac{a}{2}\cdot e^{\frac{2b}{a}}-\frac{a}{2}e^{-\frac{2b}{a}}+2b\right]$$

THANK - YOU