DATA SCIENCE
&
ARTIFICIAL INTELLIGENCE
& CS/IT

Calculus and Optimization

Lecture No. 09

Recap of previous lecture

Topic

DERIVATIVES & THEIR TYPES

(PART-2)

Topics to be Covered

Topic

MAXIMA & MINIMA

(Part 1)

Types of Questions

RECAP

1. Based on ordinary Derivative exist in case of curve $y = f(x)$
2. " " Partial Derivative " " " of surface $z = f(x, y)$
3. " " Total Derivative if $z = f(x, y)$, $x = x(t)$, $y = y(t)$

   i.e $z \longrightarrow (x, y) \longrightarrow$ 't' alone
4. " " Chain Rule of Partial Derivatives, if $z = f(x, y)$, $x = x(r, s)$, $y = y(r, s)$

   i.e $z \longrightarrow (x, y) \longrightarrow (r, s)$
5. " " Jacobian if $(u, v) \longrightarrow (x, y)$
6. " " Euler Theorem: if $f(x, y)$ is Homogeneous func^n then we can use E. Th.

Q. HWQ. if $V = V(x, y)$ where $x+y = 2e^{\theta}\cos\phi$, $x-y = 2ie^{\theta}\sin\phi$

then evaluate $\frac{\partial V}{\partial \theta}$ and $\frac{\partial V}{\partial \phi} = ?$

Sol:
$$\left.\begin{aligned} x+y &= 2e^{\theta}\cos\phi \\ x-y &= 2ie^{\theta}\sin\phi \end{aligned}\right\} \quad \begin{aligned} 2x &= 2e^{\theta}(\cos\phi + i\sin\phi) \\ 2y &= 2e^{\theta}(\cos\phi - i\sin\phi) \end{aligned} \Rightarrow \begin{aligned} x &= e^{\theta}(e^{i\phi}) = e^{\theta+i\phi} \\ y &= e^{\theta}(e^{-i\phi}) = e^{\theta-i\phi} \end{aligned}$$

$\because V \longrightarrow (x, y) \longrightarrow (\theta, \phi)$ so Quest is Based on Chain Rule.

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\left(\frac{\partial x}{\partial \theta}\right) + \frac{\partial v}{\partial y}\left(\frac{\partial y}{\partial \theta}\right) = V_x\left(e^{\theta+i\phi}\right) + V_y\left(e^{\theta-i\phi}\right) \quad \text{Ans}$$

$$\frac{\partial v}{\partial \phi} = \frac{\partial v}{\partial x}\left(\frac{\partial x}{\partial \phi}\right) + \frac{\partial v}{\partial y}\left(\frac{\partial y}{\partial \phi}\right) = V_x\left(ie^{\theta+i\phi}\right) + V_y\left(-ie^{\theta-i\phi}\right) \quad \text{Ans}$$

JACOBIAN → RECAP

if $u = u(x,y,z)$, $v = v(x,y,z)$, $w = w(x,y,z)$

is $(u,v,w) \longrightarrow (x,y,z)$

then Derivative of (u,v,w) with respect to (x,y,z) is called Jacobian & it is defined as

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Note ① if $(u,v) \longrightarrow (x,y)$ then

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

② if $(x,y) \longrightarrow (u,v)$ then $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = J'$

③ $JJ' = 1 \Rightarrow J' = \frac{1}{J}$ = Reciprocal of J

Application of Jacobian → Let $u = u(x,y)$ & $v = v(x,y)$ are two functions

Functionally Dependent funcⁿ → u & v are functionally dependent if there exist Mathematical Relation b/n them & it's condition is $J = 0$

Functionally Independent funcⁿ → u & v are called functionally independent if there D.N.E any Relation b/n them. & it's Condition is $J \neq 0$

Q ⇒ If $u = \sin^{-1}x + \sin^{-1}y$ & $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ then $\frac{\partial(u,v)}{\partial(x,y)} = ?$

(a) 1

(b) 4

(c) 0 ✓

(d) $\sin u$

(M-I) $\because (u,v) \longrightarrow (x,y)$ so

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ ? & ? \end{vmatrix} = \ldots\ldots = 0$$

(M-II) By observation, we can write,

$u = \sin^{-1}x + \sin^{-1}y$

$u = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

$u = \sin^{-1}(v) \Rightarrow v = \sin u$

i.e. ∃ Relationship b/w u & v

or u & v are Dependent

$\Rightarrow J = 0$

MAXIMA MINIMA

T-1 → Increasing Dec func^n

T-2 → Max-Min of Curve $y = f(x)$

T-3 → Max Min of Surface $z = f(x, y)$

(*) I^{st} Derivation
- → Geometrical Significance → To know the Slope of Tangent
- → Application → To know the Inc or Dec Nature of func^n.

(*) 2^{nd} Derivatives
- → Geometrical Significance → To know Concavity of func^n
- → Application → To know Inflexion Point, Point of Max & Point of minima

INCREASING & DECREASING FUNCTION

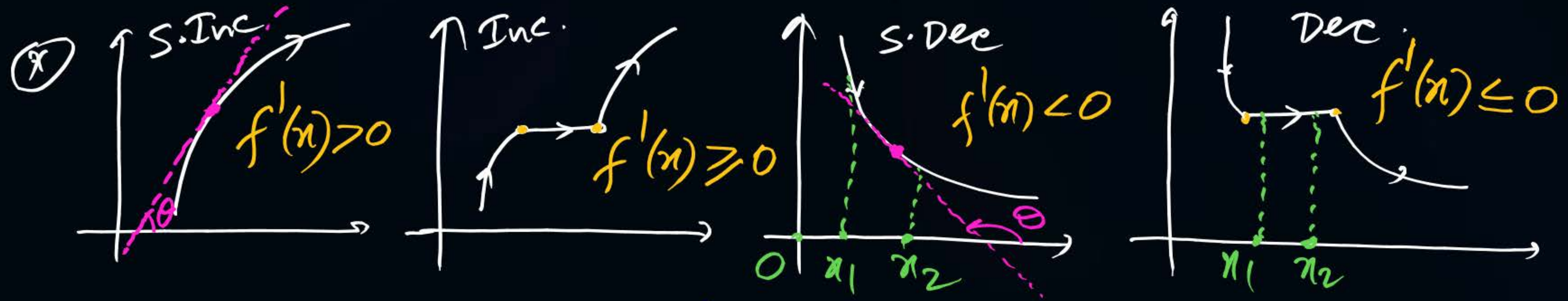
w.k. that for $y = f(x)$, $\frac{dy}{dx} = \boxed{f'(x) = \tan\theta} =$ slope of tangent at any Random point x on $f(x)$

(1) for Acute angle tangent, $f'(x) > 0$

(2) " obtuse " " , $f'(x) < 0$

(3) for Horizontal tangent, $f'(x) = 0$

(4) for Vertical Tangent, $f'(x) =$ D.N.E



eg: if $f(x_1) > f(x_2)\ \forall\ x_1 < x_2$; where $x_1, x_2 \in D_f$ then $f(x)$ is S.Dec.

eg: if $f(x_1) \geq f(x_2)$ " " " " " " " " " Decreasing

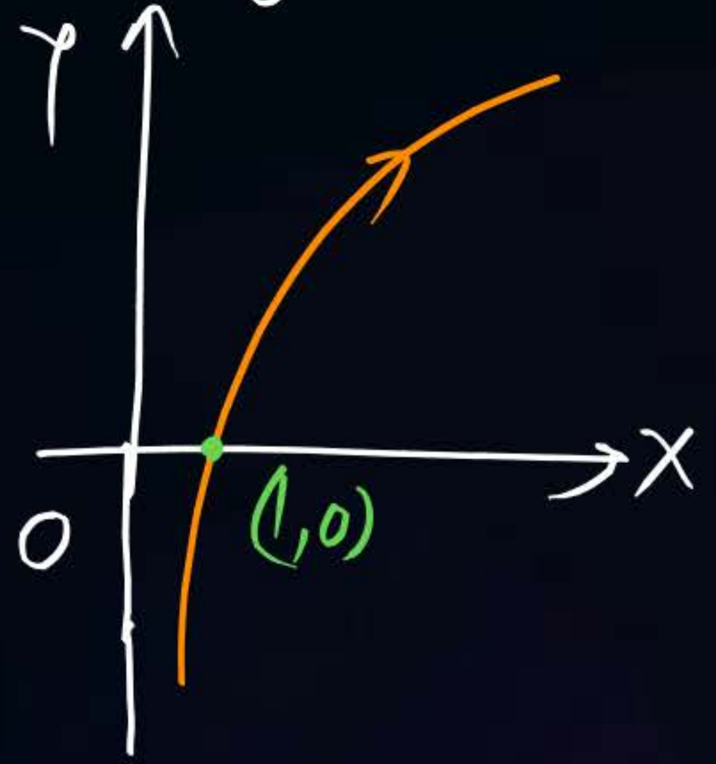
(*) Monotonic func's → Those functions which are either S.I or S.D are called Monotonic functions

(*) Advice → It is advisable to first write the Domain of function while Solving Questions Based on Inc / Dec functions.

Eg: $f(x) = \log_a x$ is S. Inc

where $a > 1$ $\quad D_f = (0, \infty)$

M-I (Using Graph) →



M-II

$f(x) = \log_a x = \dfrac{\log_e x}{\log_e a}$

$\boxed{f'(x) = \dfrac{1}{x \log_e a}}$

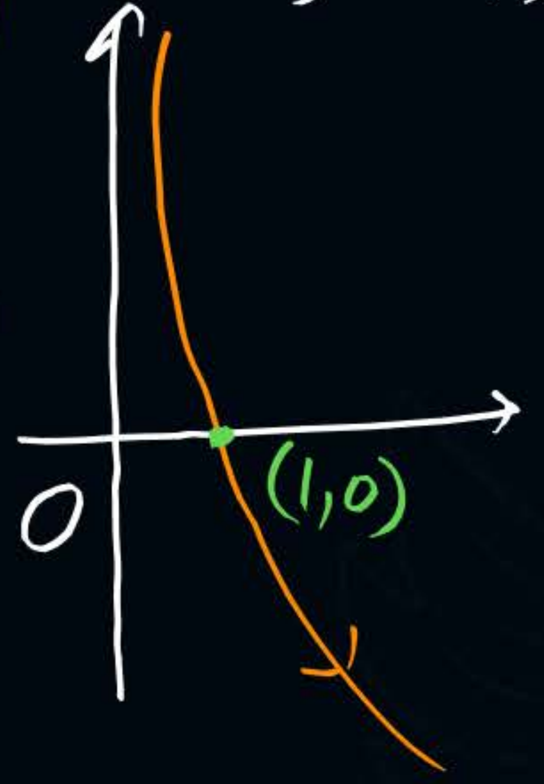
$= \dfrac{1}{(+ve)(+ve)} > 0$

So $f(x)$ is S. Inc.

eg $f(x) = \log_a x$ is S. Dec

where $0 < a < 1$ $\quad D_f = (0, \infty)$

M-I (Using Graph) → M-II $f(x) = \log_a x = \dfrac{\log_e x}{\log_e a}$



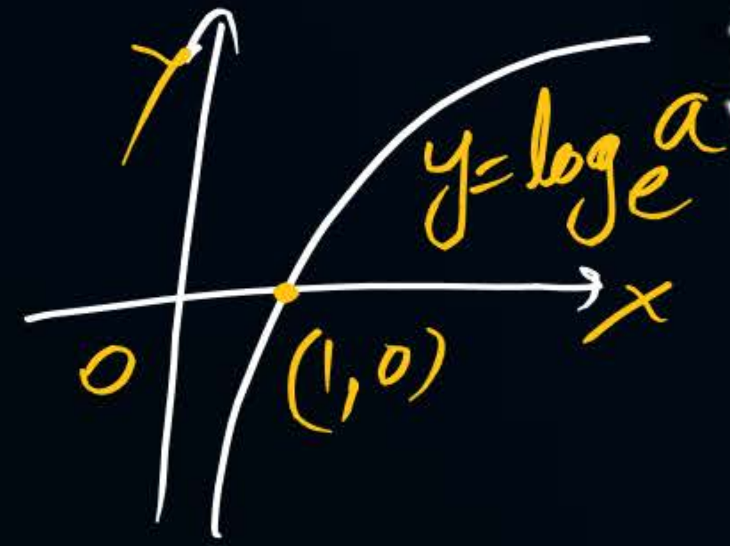
$\boxed{f'(x) = \dfrac{1}{x \log_e a}}$

$= \dfrac{1}{(+ve)(-ve)} < 0$

So $f(x)$ is S. Dec.

Note:

(1) $a > 1$ then $\boxed{\log_e a = ? = +ve}$ Learn



$\because$ let $a = 2$, $\log_e 2 = 0.693 = +ve$.

(2) $0 < a < 1$, $\boxed{\log_e a = ? = -ve}$ Learn.

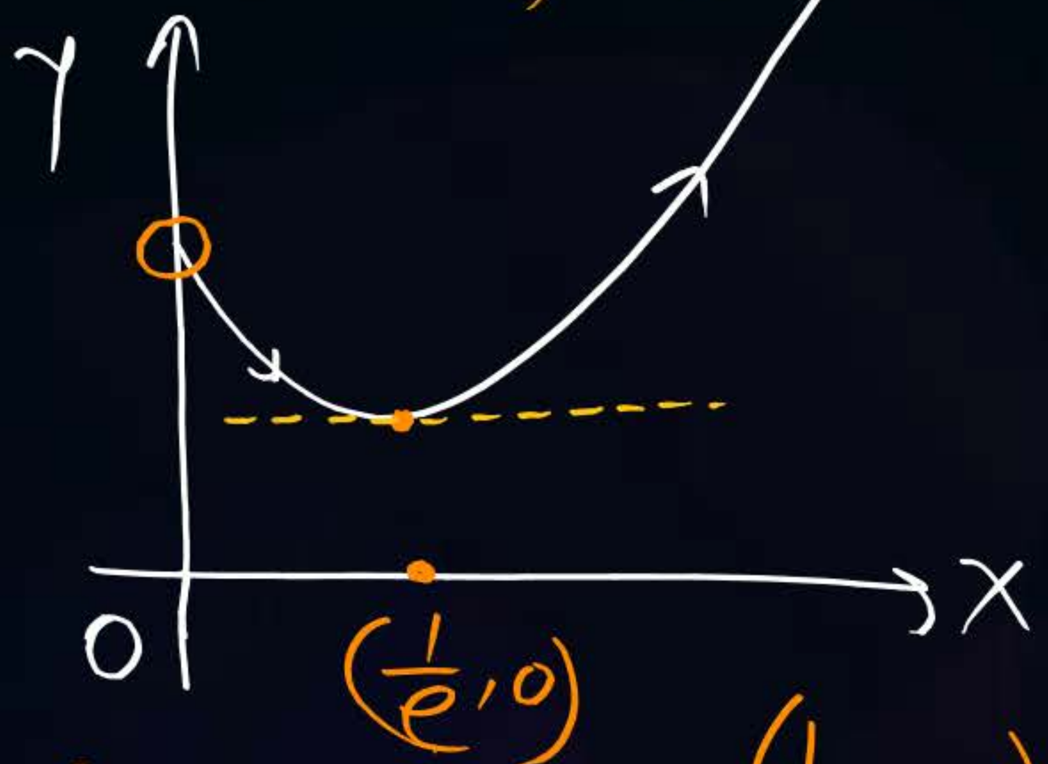
$\because$ let $a = \frac{1}{2}$, $\log_e\left(\frac{1}{2}\right) = \log_e(2)^{-1} = -1 \log_e 2 = -0.6932$

$= -ve$.

Q. Find the Interval in which $\boxed{f(x) = x^x}$ Increases & Decreases.

M-I (Using Graph) →

$y = x^x$; $D_f = (0, \infty)$



$f(x)$ is S.Inc in $\left(\frac{1}{e}, \infty\right)$

$f(x)$ is S.Dec in $\left(0, \frac{1}{e}\right)$

M-II (w/o Graph) →

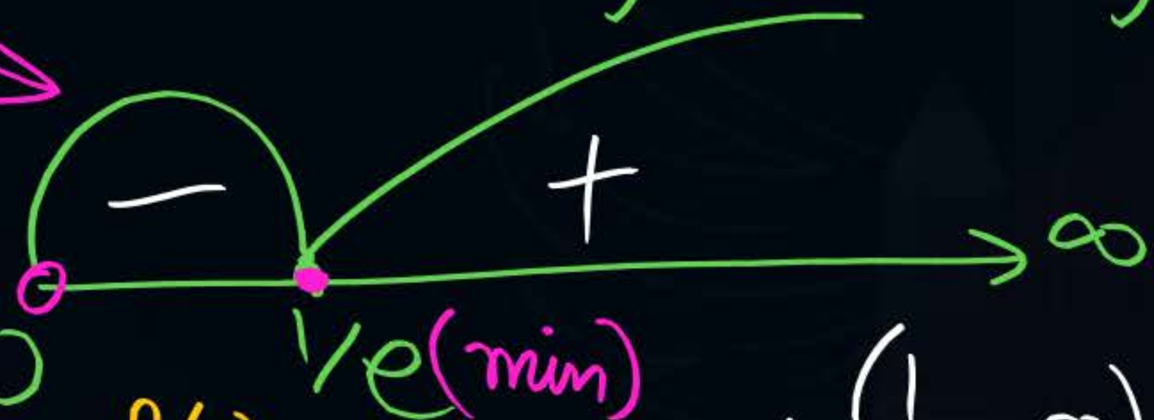
$f'(x) = x^x(1 + \log_e x)$

$f'\left(\frac{1}{e^2}\right) = -ve$

$f'(e) = +ve$

Turning points are; Put $f'(x) = 0$

$x^x(1 + \log x) = 0 \Rightarrow \log_e x = -1 \Rightarrow x = e^{-1}$



So $f(x)$ is S.Inc in $\left(\frac{1}{e}, \infty\right)$

& $f(x)$ is S.Dec in $\left(0, \frac{1}{e}\right)$

MSQ

Q Find the Interval of Increasing & Decreasing for $y = \dfrac{\log_e x}{x}$, $D = (0, \infty)$

(a) Increases in $(0, e]$

(b) Decreases in $(0, e]$

(c) Increases in $[e, \infty)$

(d) Decreases in $[e, \infty)$

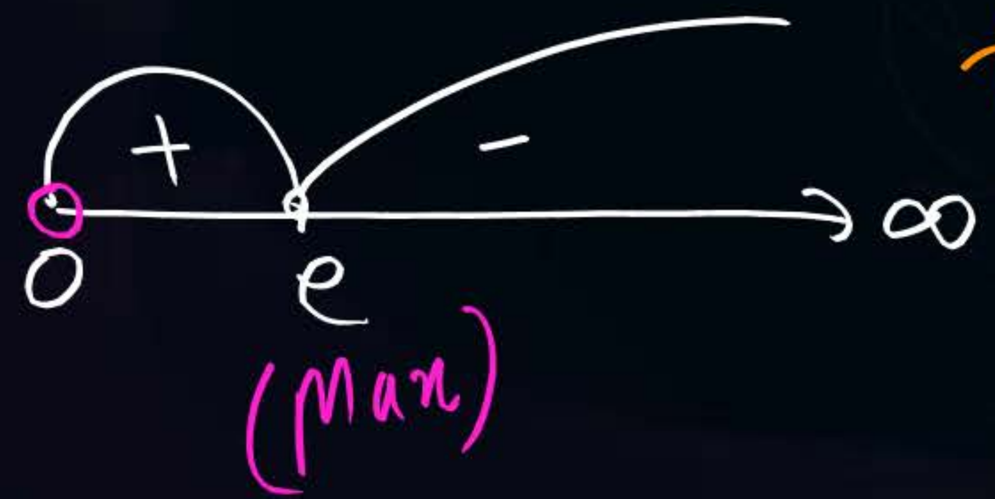
$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \log x\,(1)}{(x)^2} = \frac{1 - \log_e x}{x^2}$$

Turning Point: $f'(x) = 0 \Rightarrow x = e$

$$\boxed{f'(x) = \frac{1 - \log_e x}{x^2}}$$

$f'(1) = +ve$

$f'(e^2) = -ve$



$f(x)$ S.Inc in $(0, e)$

$f(x)$ Inc in $(0, e]$

$f(x)$ S.Dec in (e, ∞)

$f(x)$ Dec in $[e, \infty)$

Analysis: (1) $f'(x) = \frac{1-\log x}{x^2}$, f

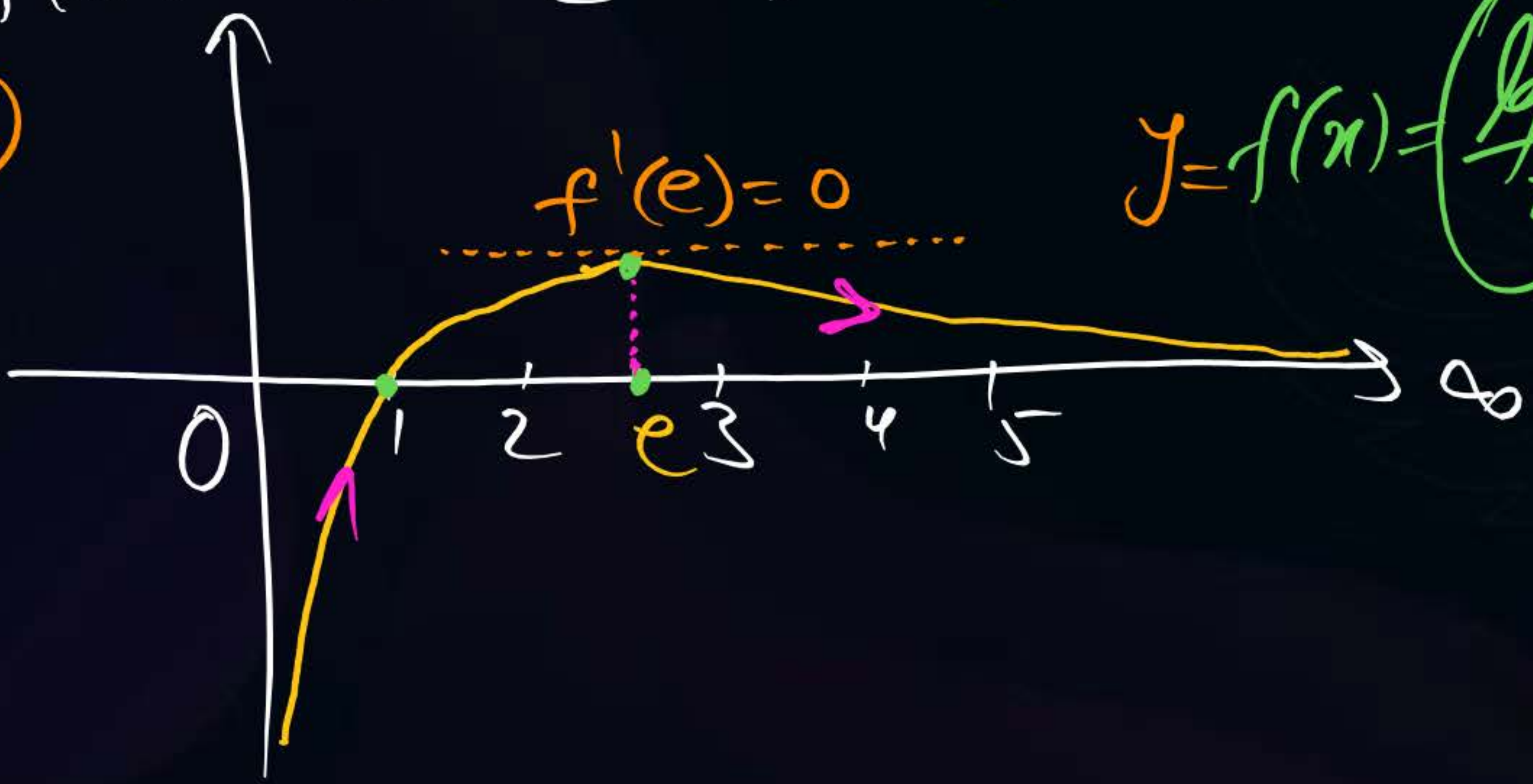
$f(x)$ is S.Inc in $(0, e)$

$f(x)$ is Inc in $(0, e]$ $\because f'(e) = 0$

$f(x)$ is S.Dec in (e, ∞)

$f(x)$ is Dec in $[e, \infty)$ $\because f'(e) = 0$

(2)



$y = f(x) = \frac{\log x}{x}$, $D_f = (0, \infty)$

$f(1) = \frac{0}{1} = 0$

Dom $= (-\infty, \infty)$

Q $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$, then find the interval in which $f(x)$

MSQ Increases and Decreases?

(a) $f(x)$ Inc in $(1,2) \cup (3,\infty)$

(b) $f(x)$ Dec in $(-\infty,1) \cup (2,3)$

(c) $f(x)$ Inc in $x < -1$ or $2 < x < 3$

(d) $f(x)$ Dec in $1 < x < 2 \cup x > 3$

$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

$f'(x) = 4x^3 - 24x^2 + 44x - 24$

$= 4(x^3 - 6x^2 + 11x - 6)$

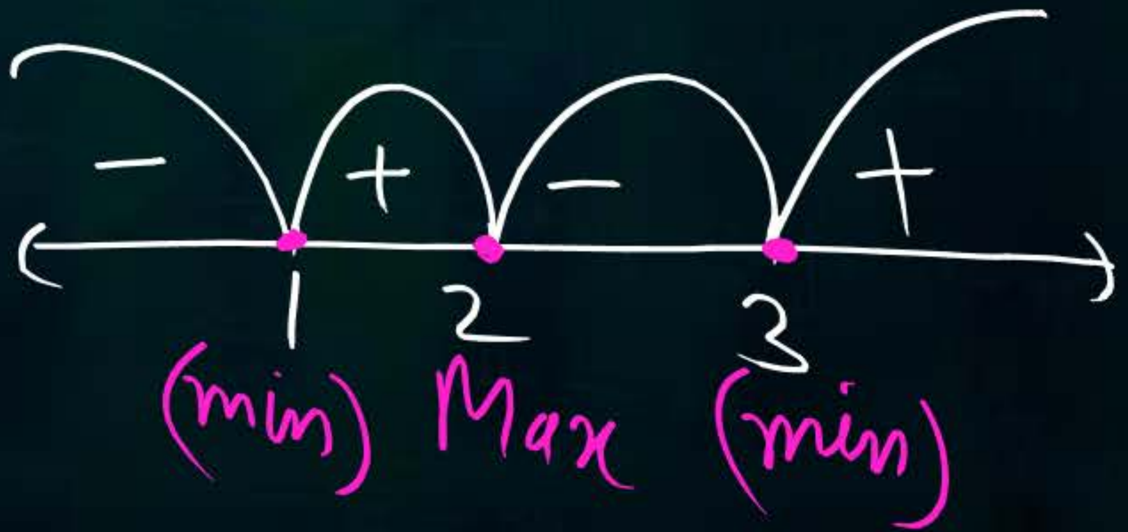
$\boxed{f'(x) = 4(x-1)(x-2)(x-3)}$

$f(0) = -ve$

$f(1.5) = +ve$

$f(2.5) = -ve$

$f(4) = +ve$



$f(x)$ Inc in $[1,2] \cup [3,\infty)$

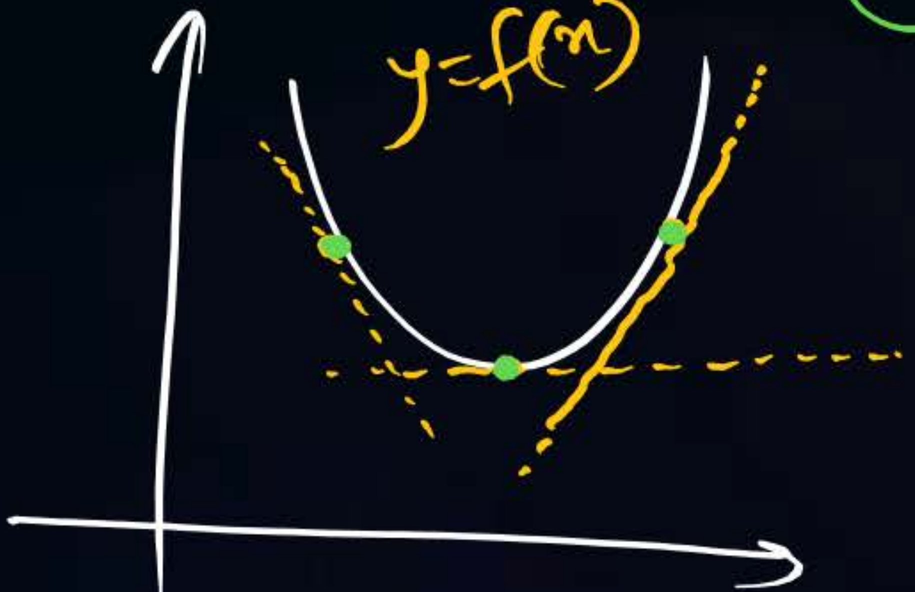
$f(x)$ Dec in $(-\infty,1] \cup [2,3]$

Information :

Concave upward Curve :

if curve lies above the tangent always then curve is called concave upward curve

$$\underbrace{\text{concave upward curve}}_{\Downarrow}$$

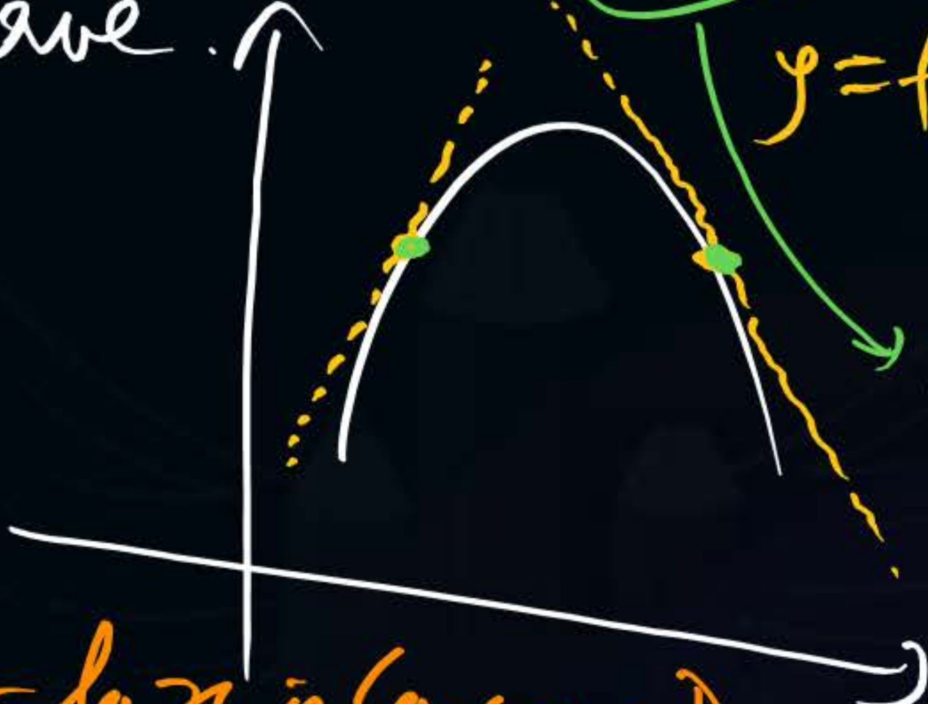


$$f''(x) > 0$$

eg $f(x)=e^x$ is concave upward curve

Concave Downward Curve

if curve lies below the tangent always then curve is called concave downward curve.



$$f''(x) < 0$$

eg $f(x)=\log_e x$ is concave downward curve eg $f(x) = \boxed{y = e^x}, (-\infty, \infty)$

$f''(x) > 0$

Concave up. Curve

M-II

$f'(x) = e^x$

$f''(x) = e^x$

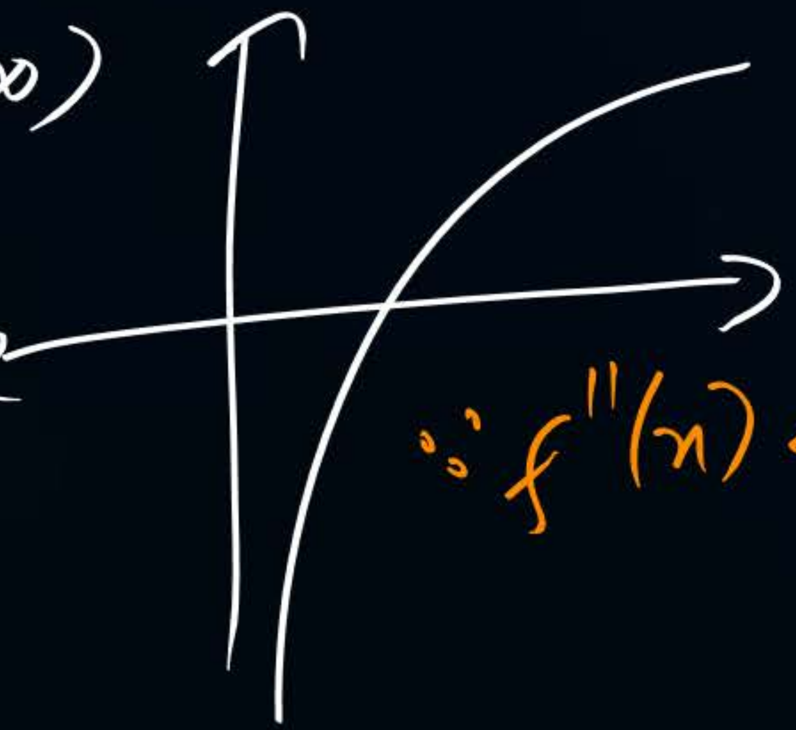
i.e. $f''(x) > 0$ always in Domain

$\Longrightarrow$ $\underline{f(x)}$ is Concave upward Curve

eg $f(x) = e^{-x}$ is also Concave upward

eg $f(x) = \boxed{y = \log_e x}, (0, \infty)$

Concave Down Curve



$\because f''(x) < 0$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

i.e. $f''(x) < 0$ always

$\Rightarrow$ $\underline{f(x)}$ is Concave Downward Curve

eg $f(x) = \log_a x$, $0 < a < 1$ is also Concave up

Q. The function $f(x) = x^4$ is ?

2014 (MSQ)

(a) strictly Increases

(b) strictly Decreases

(c) Concave Downward

(d) Concave upward

$\boxed{f(x) = x^4}$, $(-\infty, \infty)$

$f'(x) = 4x^3 \Longrightarrow$

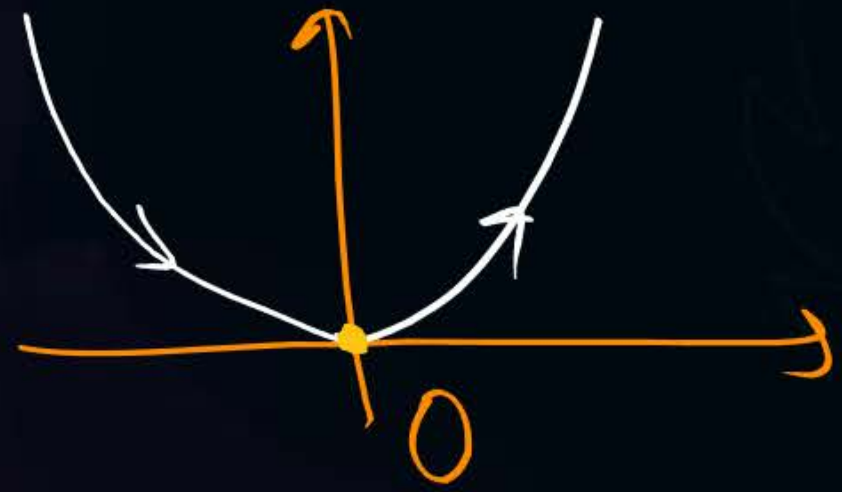
$f(x)$ is S.Inc $(0, \infty)$

$f(x)$ is S.Dec in $(-\infty, 0)$

$f''(x) = 12x^2 > 0$ always $\because$

$\Rightarrow f(x)$ is concave upward in $(-\infty, \infty)$

(M-II) $y = x^4$



Shortcuts: Put $\boxed{f'(x) = 0}$ & Try to find T. Points (Say $x = c_1, c_2$)

(1) then check sign of $f'(x)$ as follows;

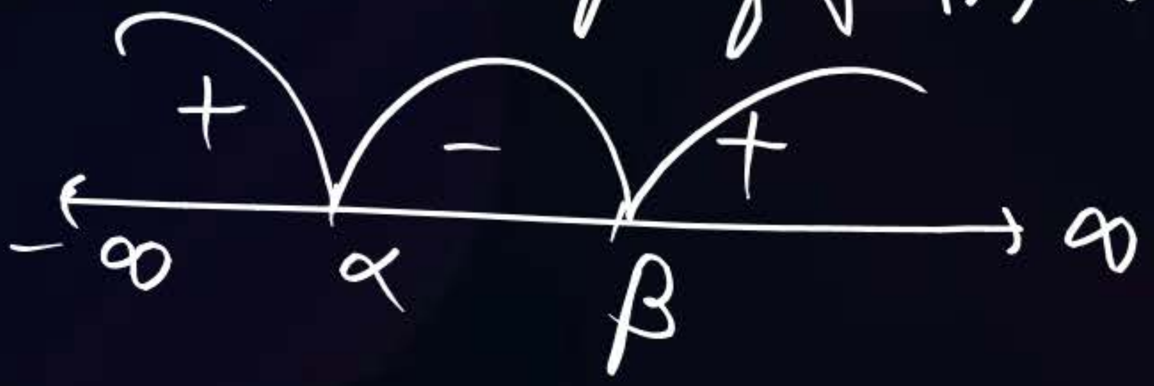
$\Rightarrow f(x)$ Inc in $(-\infty, c_1] \cup [c_2, \infty)$

$f(x)$ Dec in $[c_1, c_2]$

(2) To find the Interval where $f(x)$ is concave up or concave Downward $\longrightarrow$

Put $\boxed{f''(x) = 0}$ & Try to find x (Say $x = \alpha, \beta$)

then check the sign of $f''(x)$ as follows;



$\Rightarrow f(x)$ is concave upward in $(-\infty, \alpha] \cup [\beta, \infty)$

$f(x)$ is concave Downward in $[\alpha, \beta]$

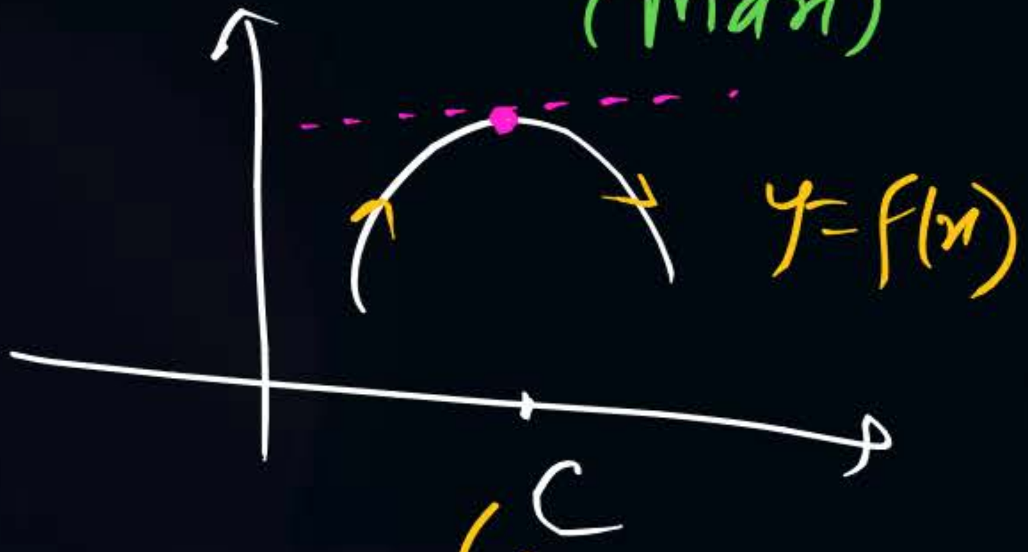
(*) Consider $y = f(x)$ is given func^n & it's T. points is $x = c$

Then sign of $f'(x) \Longrightarrow$

$+ \quad c \quad -$ (Max)

$- \quad c \quad +$ (min)

eg



(Max)

(min)

$+ \quad c \quad +$ (NMNM)

$- \quad c \quad -$ (NMNM)

Saddle point / point of Inflexion →

"Those points where curve changes it's concavity are called points of Inflexion"

Note: (1) Those points where we are getting Neither Maxima, Nor minima are called Points of Inflexion (F)

(2) if ($x = \alpha$ is the point of Inflexion) $\Longrightarrow$ (At $x = \alpha$ we will get N.M. N.M.)

$\not\Longleftarrow$

Shortcut: Put $f''(x) = 0$ & Try to get x (Say it is $x = \alpha$)

if $f'''(\alpha) \neq 0 \Rightarrow x = \alpha$ is point of Inflexion

& if $f'''(\alpha) = 0 \Rightarrow$ we can't say anything about α.

Q. Find the interval in which $f(x) = x^4 - 24x^2 + 11$ is S.Inc, S.Dec, Concave upward & Concave Downward?

MSQ

(a) S.Dec in $(-\infty, -2\sqrt{3}) \cup (0, 2\sqrt{3})$ ✓

(b) S.Inc in $(-2\sqrt{3}, 0) \cup (2\sqrt{3}, \infty)$ ✓

(c) Concave upward in $(-\infty, -2) \cup (2, \infty)$ ✓

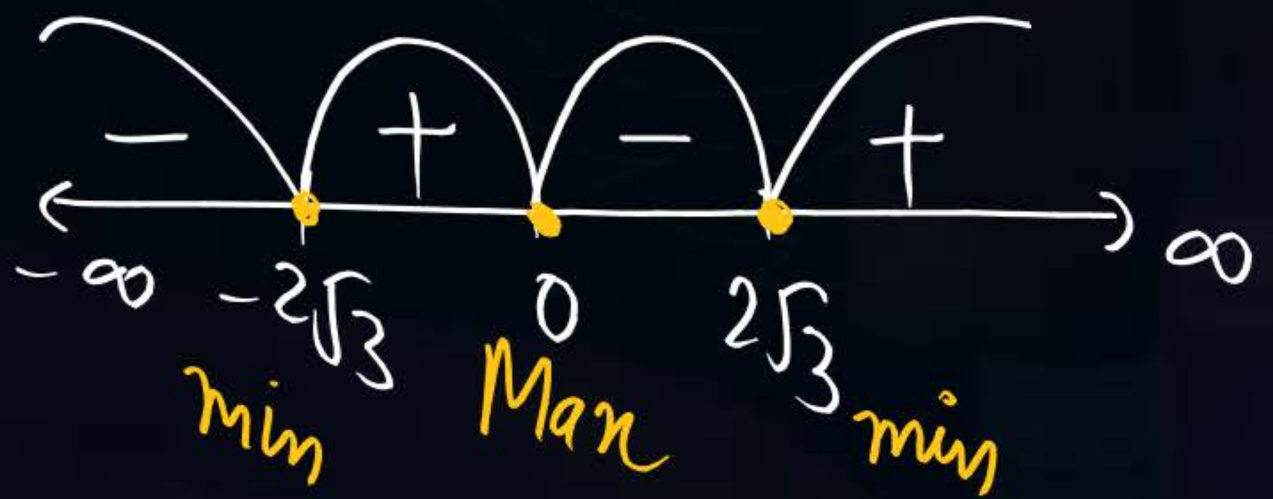
(d) Concave Downward in $(-2, 2)$ ✓

$f(x) = x^4 - 24x^2 + 11$, $(-\infty, \infty)$

$f'(x) = 4x^3 - 48x = 4x(x^2 - 12)$

$$\boxed{f'(x) = 4x(x - 2\sqrt{3})(x + 2\sqrt{3})}$$

T. Points are $x = -2\sqrt{3}, 0, 2\sqrt{3}$



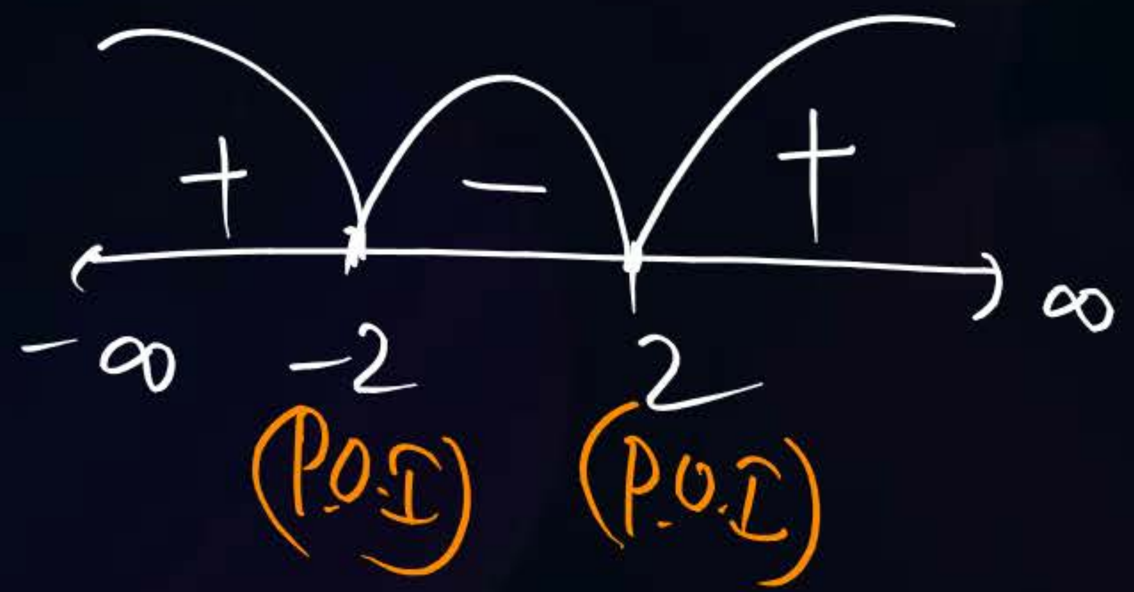
Again, $f(x) = x^4 - 24x^2 + 11$

$$f'(x) = 4x^3 - 48x$$

$$f''(x) = 12x^2 - 48$$

$$= 12(x^2 - 4)$$

$$\boxed{f''(x) = 12(x-2)(x+2)}$$



$f(x)$ is concave up in $(-\infty, -2) \cup (2, \infty)$

" " Concave Down in $(-2, 2)$

Note: $\because$ Concavity changes at $x = -2$ & 2

So these two points are saddle points.

or Points of Inflexion

& At $x = \pm 2$ we will get NMNM

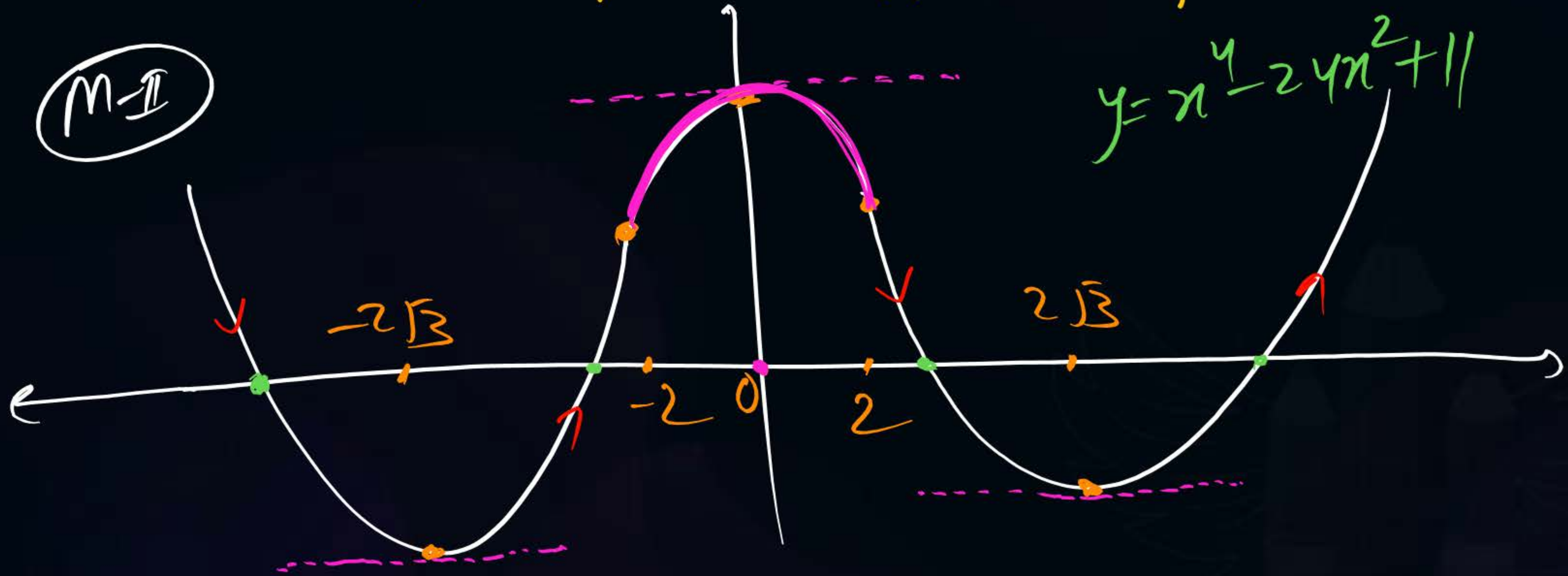
(*) if ($x = \alpha$ is P.O.I) $\Longrightarrow$ (we will get NMNM at α)

$\not\Longleftarrow$

In previous Ex, T-points are $x = -2\sqrt{3}, 0, 2\sqrt{3}$

& Saddle points are $x = -2$ & 2

M-II



Q. The number of Inflexion points of $f(x) = x + 12x^4$ is/are?

(a) 0
(b) 1
(c) 2
(d) 3

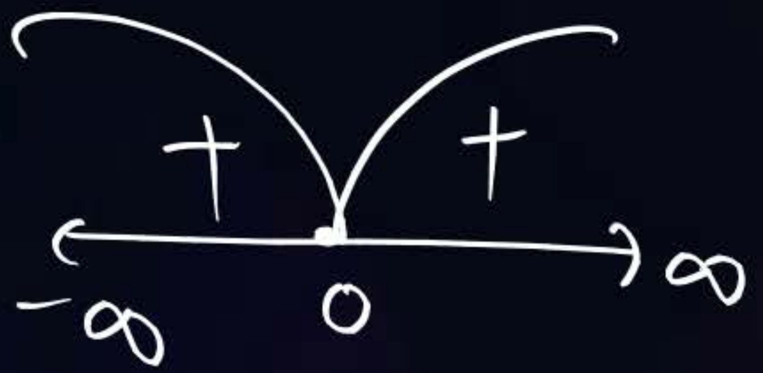
ZERO

M-I $f(x) = x + 12x^4$

$f'(x) = 1 + 48x^3$

$$\boxed{f''(x) = 144x^2}$$

Put $f''(x) = 0 \Rightarrow x = 0$



$f(x)$ is concave upward always.

M-II $f'(x) = 1 + 48x^3$

$f''(x) = 144x^2$, $f'''(x) = 288x$

Putting $f''(x) = 0 \Longrightarrow x = 0$

Now $f'''(0) = (288x)_{x=0} = 0$

i.e. $x = 0$ is not a Point of Inflexion.

Hence given $f(x)$ has NO POI

Q. Number of Inflexion points of $\boxed{f(x) = x^4 - 18x^2 + 9}$ is/are ? Two.

(a) 0

(b) 1

(c) 2 ✓

(d) 3

(M-I) $f'(x) = 4x^3 - 36x$

$f''(x) = 12x^2 - 36$

$= 12(x^2 - 3)$

$$\boxed{f''(x) = 12(x - \sqrt{3})(x + \sqrt{3})}$$

(POI) (POI)

$f(x)$ is concave upward in $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

" " " Down in $(-\sqrt{3}, \sqrt{3})$

(M-II) $f'(x) = 4x^3 - 36x$

$f''(x) = 12x^2 - 36$, $f'''(x) = 24x$

Putting $f''(x) = 0 \Rightarrow x = \pm\sqrt{3}$

Now $f'''(\sqrt{3}) = (24x)_{x=\sqrt{3}} \neq 0$

& $f'''(-\sqrt{3}) = (24x)_{x=-\sqrt{3}} \neq 0$

ie Both the points are points of Inflexion

(ii) Also find Max / min of $f(x) = x^4 - 18x^2 + 9$

$$f'(x) = 4x^3 - 36x = 4x(x^2 - 9)$$

$$\boxed{f'(x) = 4x(x+3)(x-3)}$$

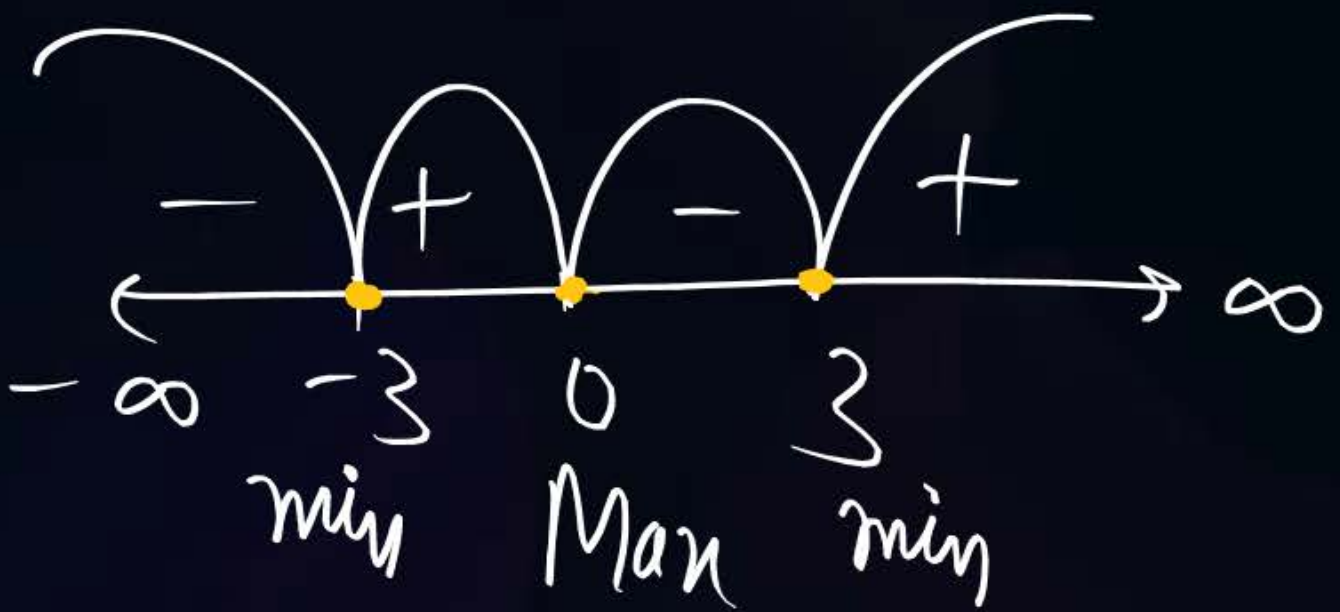
T-Points are $x = -3, 0, 3$

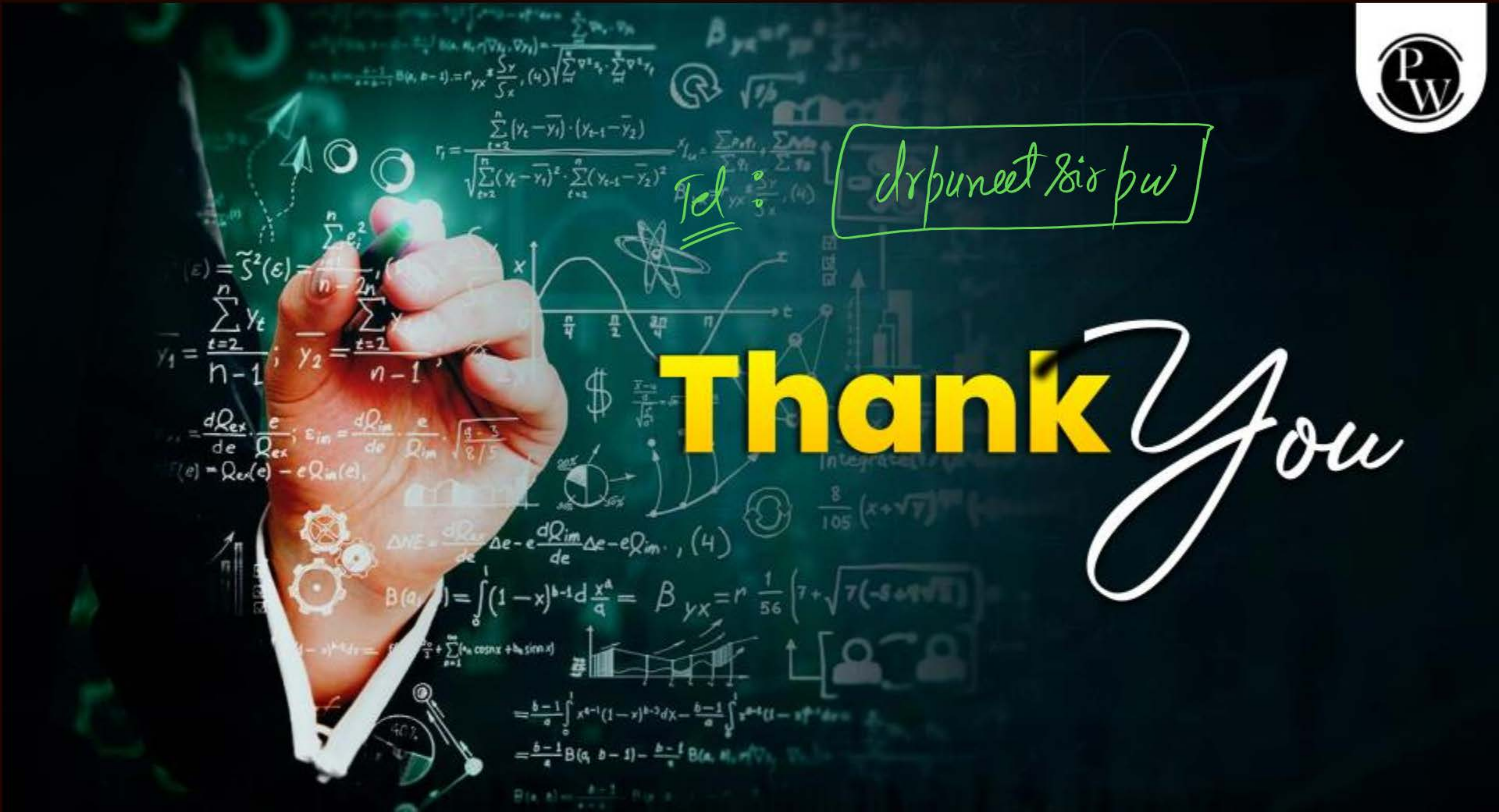
$f'(-5) = -ve$

$f'(-1) = +ve$

$f'(1) = -ve$

$f'(4) = +ve$

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Thank You