CS & DA

DPP: 2

CALCULUS AND OPTIMIZATION

Q1 A 2m ladder leans against a vertical wall. If the top of the ladder begins to slide down the wall at the rate 25 cm/sec. then the rate (in cm/sec.) at which the bottom of the ladder slides away from the wall on the horizontal ground when the top of the ladder is 1 m above the ground is:

(A) $25\sqrt{3}$ 　　　　　　(B) 25

(C) $\frac{25}{\sqrt{3}}$ 　　　　　　(D) $\frac{25}{3}$

Q2 The percentage error in calculating the volume of a cubical box if an error of 1% is made in measuring the length of edges of the cube is :

(A) 1% 　　　　　　(B) 2%

(C) 3% 　　　　　　(D) 6%

Q3 Find the tangent line to f (x) = 7x + 4e^x at x = 0.

$y - 11x - 4 = 0$

Q4 Differentiate $f(t) = \frac{1+5t}{\ln(t)}$

Q5 Find $\frac{\partial w}{\partial x}$ for the following function.

$$w = \cos\left(x^2 + 2y\right) - e^{4x - z^4 y} + y^3$$

Q6 Find all the 1st order partial derivatives of the following function.

$$f(x, y, z) = 4x^3y^2 - e^z y^4 + \frac{z^3}{x^2} + 4y - x^{16}$$

Q7 Prove that

f (x, y) = x³ - 2x²y + 3xy² + y³

is homogeneous; what is the degree? Verify Euler's Theorem for f .

Q8 If $v(x, y) = \log\left(\frac{x^2+y^2}{x+y}\right)$, prove that

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1$$

Q9 **Prove that**

g(x, y) = x log (y/x)

is homogenous, what is the degree ? Verify Euler's theorem for g.

Q10 The jacobian of p, q, r, w. r. t x, y, z given = p = x + y + z, q = y + z, r = z is _____.

(A) 0 　　　　　　(B) 1

(C) 2 　　　　　　(D) −1

Q11 Given $u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$ then the value of $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ is _____.

(A) 4 　　　　　　(B) − 4

(C) 0 　　　　　　(D) 1

Q12 The function f(x) = 2log (x−2) − x² + 4x + 1 increase on the interval

(A) (1, 2) 　　　　　　(B) (2,3)

(C) (5/2, 3) 　　　　　　(D) (2, 4)

Q13 The function $f(x) = \frac{x^2-1}{x^2+1}$:

(A) is concave for $x \in \left[\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$

(B) is convex for $x \in \left[\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$

(C) has 2 points of inflexion

(D) is concave for $\left[\frac{1}{\sqrt{3}}, \infty\right)$

Q14 Number of point of inflections of the function:

$$f(x) = x^4 - 18x^2 + 9$$

(A) 1 　　　　　　(B) 2

(C) 3 　　　　　　(D) 4

Q15 The function f(x) = 2x³ − 3x² − 36x + 2 has its maxima at

(A) x = − 2 only

(B) x = 0 only

(C) x = 3 only

(D) both x = −2 and x = 3

Q16 The maximum value of f(x) = x³ −9x² + 24x + 5 in the interval [1, 6] is

Q17 The minimum value of the function $f(x) = x^3 - 3x^2 - 24x + 100$ in the interval

[-3, 3] is

Q18 Suppose that the amount of money in a bank account after t year is

given by $A\left(t\right) = 2000 - 10te^{5-\frac{t^2}{8}}$

The minimum and maximum amount of money in the account during the first 10 year that it is open occur respectively at:

(A) T = 2, t = 0   (B) T = 2, t = 10

(C) T = 0, t = 2   (D) T = 10, t = 0

Q19 The function: f(x) =x² ln (3x) +6 has:

(A) No critical point

(B) Only one critical point

(C) Only one stationary point

(D) Infinitely many critical points

Q20 The function f(x,y)=x²y−3xy+2y+x has

(A) No local extremum

(B) One local minimum but no local maximum

(C) One local maximum but no local minimum

(D) One local minimum and one local maximum

Q21 Find the local minima of the function f(x , y) = 2x² + 2xy + 2y² − 6x

(A) (2,1)   (B) (−2,1)

(C) (2,0)   (D) (2,−1)

Q22 Find the critical points of the function f(x, y) = y² − x² and check for the presence of any saddle point.

Q23 Find the critical points of the function f(x, y) = x²y² − 5x² − 5y² − 8xy and check for the presence of any saddle point.

Q24 Calculate the perpendicular distance between origin and a point P on surface of curve $z^2 = 3xy + 4$

Q25 A jet of an enemy is flying along the curve y =x² + 2. A soldier is placed at the point (3,2). What is the nearest distance between the soldier and the jet?

Q26 A Square piece of tin of side 24 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to from a box. What should be the side of the square to be cut off so that the volume of the box is maximum ? Also find this maximum volume.

Q27 Manufacturer can sell x items at a price of rupees $\left(5 - \frac{x}{100}\right)$ each. The cost price is Rs. $\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to earn maximum profit.

Q28 Evaluate $\int x^2 \sin x \, dx$

Q29 $\int \frac{dx}{2\sqrt{x}(x+1)}$

Q30 Evaluate: $\int \frac{dx}{(x-1)(x-2)(x-3)}$

Q31 Evaluate: $\int\limits_{2}^{7} \frac{\sqrt{x}dx}{\sqrt{x}+\sqrt{9-x}}$

Q32 Evaluate: $\int\limits_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+e^x)(1+x^2)}$

Q33 Evaluate: $\int\limits_{-a}^{a} \sqrt{\frac{a-x}{a+x}}dx$

Q34 Evaluate: $\int\limits_{0}^{\pi} \frac{xdx}{1+\cos^2 x}$

Q35 Evaluate: $\int\limits_{0}^{\pi} \frac{dx}{1+2\sin^2 x}$

Q36 Let $5f\left(x\right) + 4f\left(\frac{1}{x}\right) = \frac{1}{x} + 3, x > 0$ Then 18 $\int\limits_{1}^{2} f\left(x\right)dx$ is equal to:

(A) 10 log 2 − 6

(B) 10 log 2 + 6

(C) 5 log 2 − 6

(D) 5 log 2 − 3

Q37 $If \, F\left(x\right) = \int\limits_{x}^{x^2} \sqrt{\sin t}dt, \text{ then find } F'\left(x\right)$

Q38

Evaluate: $\displaystyle\lim_{x\to\infty} \frac{\left(\int_0^x e^{t^2}\,dt\right)^2}{\int_0^x e^{2t^2}\,dt}$

Q39 Arc length of the curve $y = x^{3/2}$, z = 0 from (0,0,0) to (4, 8, 0) is-

(A) $\frac{8}{27}\left(10^{3/2} + 1\right)$

(B) $\frac{8}{27}\left(10^{3/2} - 2\right)$

(C) $\frac{8}{27}\left(10^{3/2} - 1\right)$

(D) $\frac{8}{27}\left(10^{3/2} + 2\right)$

Q40 Let f be an increasing, differentiable function. If the curve y =f(x) passes through (1, 1) and has length $L = \int_1^2 \sqrt{\left(1 + \frac{1}{4x^2}\right)}\,dx, 1 \le x \le 2,$ then the curve is-

(A) $y = \ln\left(\sqrt{x}\right) - 1$ (B) $y = 1 - \ln\left(\sqrt{x}\right)$

(C) $y = \ln\left(1 + \sqrt{x}\right)$ (D) $y = 1 + \ln\left(\sqrt{x}\right)$

Q41 The volume of the solid of revolution of $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$ about x-axis between x= 0 and x= b is-

(A) $\frac{\pi a^3}{8}\left(e^{2b/a} - e^{-2b/a}\right) - \frac{\pi a^2 b}{2}$

(B) $-\frac{\pi a^3}{8}\left(e^{2b/a} - e^{-2b/a}\right) + \frac{\pi a^2 b}{2}$

(C) $-\frac{\pi a^3}{8}\left(e^{2b/a} - e^{-2b/a}\right) - \frac{\pi a^2 b}{2}$

(D) $\frac{\pi a^3}{8}\left(e^{2b/a} - e^{-2b/a}\right) + \frac{\pi a^2 b}{2}$

Answer Key

Q1	(C)
Q2	(C)
Q3	5.9459.
Q4	$\dfrac{5\ln(t) - \frac{1}{t} - 5}{[\ln(t)]^2}$
Q5	$-2x\sin\left(x^2 + 2y\right) - 4\,e^{4x - z^4 y}$
Q6	$\dfrac{\partial f}{\partial y} = f_z = -e^z y^4 + \dfrac{3z^2}{x^2}$
Q7	3
Q8	1
Q9	1
Q10	(B)
Q11	(A)
Q12	(B, C)
Q13	(A, C)
Q14	(B)
Q15	(A)
Q16	25
Q17	28
Q18	(A)
Q19	(B, C)
Q20	(A)

|	
Q21	(D)
Q22	(0,0)
Q23	Hence, at (±1, ±1) also we have saddle points.
Q24	2
Q25	$\sqrt{5}$
Q26	1024cm^3
Q27	240
Q28	$-x^2\cos x + 2x\sin x + 2\cos x + c$
Q29	$\tan^{-1}\left(\sqrt{x}\right) + c$
Q30	$\ln\left(\dfrac{\sqrt{x^2 - 4x + 3}}{|x - 2|}\right) + c$
Q31	$I = \dfrac{5}{2}$
Q32	$\dfrac{\pi}{3}$
Q33	$a\left[\dfrac{\pi}{2} - 0\right] = \pi a$
Q34	$\dfrac{\pi^2}{2\sqrt{2}}$
Q35	$2\dfrac{1}{\sqrt{3}}\left[\tan^{-1} t\sqrt{3}\right]_0^\infty = \dfrac{2}{\sqrt{3}} \times \dfrac{\pi}{2} = \dfrac{\pi}{\sqrt{3}}.$
Q36	(A)
Q37	$F'(x) = 2x \cdot \sqrt{\sin x^2} - 1 \cdot \sqrt{\sin x}$
Q38	0
Q39	(C)
Q40	(D)
Q41	(D)
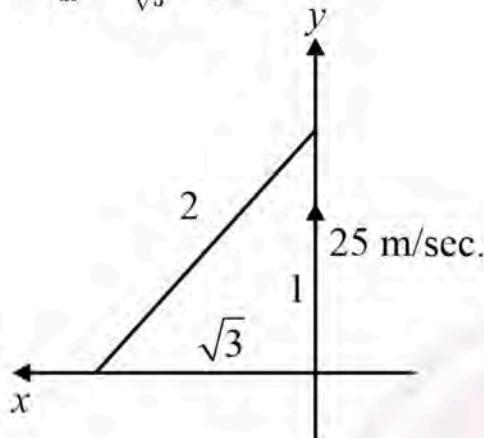
Hints & Solutions

Q1 Text Solution:

$x^2 + y^2 = 4$

$x \times \dfrac{dx}{dt} + y \dfrac{dy}{dt} = 0$

$\Rightarrow \sqrt{3}\dfrac{dx}{dt} - 1\,(25) = 0 \left(\dfrac{dy}{dt} = -25\right)$

$\Rightarrow \dfrac{dx}{dt} = \dfrac{25}{\sqrt{3}}\,cm/\sec.$

Q2 Text Solution:

Let x be the length of an edge of the cube and y be its volume. Then, $y = x^3$. Let Δx be the error in x and y be the corresponding error in Δy.

Then,

$\dfrac{\Delta x}{x} \times 100 = 1$ (given)

$\Rightarrow \dfrac{dx}{x} \times 100 = 1$ [as dx $\cong \Delta x$] ...(i)

We have to find $\dfrac{\Delta y}{y} \times 100$

Now, $y = x^3 \Rightarrow \dfrac{dy}{dx} = 3x^2$

$\therefore \quad dy = \dfrac{dy}{dx}dx \Rightarrow$ **dy = 3x² . dx**

$\Rightarrow \dfrac{dy}{y} = \dfrac{3x^2}{y}dx$

$\Rightarrow \dfrac{dy}{y} = \dfrac{3x^2}{x^3}dx$ [as y = x³]

$\Rightarrow \dfrac{dy}{y} = 3\dfrac{dx}{x}$

$\Rightarrow \dfrac{dy}{y} \times 100 = 3\left(\dfrac{dx}{x} \times 100\right) = 3$ [using (i)]

$\Rightarrow \dfrac{\Delta y}{y} \times 100 = 3$ [as dy = Δy]

So there is 3% error in calculating the volume of the cube.

Q3 Text Solution:

We know that the derivative of the function will give us the slope of the tangent line so we'll need the derivative of the function.

f' (x) = 7x ln (7) + 4ex

Now all we need to do is evaluate the function and the derivative at the point in question.

f (0) = 5, f' (0) = ln (7) + 4 = 5.9459.

Q4 Text Solution:

$f\left(t\right) = \dfrac{g(t)}{h(t)}$

$differentiating\ by\ u\sin g\ the\ u/v\ formula$ –

$f'\left(t\right) = \dfrac{h(t).g'(t) - g(t).h'(t)}{h^2(t)}$

$f'(t) = \dfrac{5\ln(t) - (1+5t)\left(\frac{1}{t}\right)}{[\ln(t)]^2} = \boxed{\dfrac{5\ln(t) - \frac{1}{t} - 5}{[\ln(t)]^2}}$

Q5 Text Solution:

$\dfrac{\partial w}{\partial x} = w_x = -2x\sin\left(x^2 + 2y\right) - 4$

$e^{4x - z^4 y}$

Q6 Text Solution:

$\dfrac{\partial f}{\partial x} = f_x = 12x^2y^2 - \dfrac{2z^3}{x^3} - 16x^{15}$

$\dfrac{\partial f}{\partial y} = f_y = 8x^3y - 4e^zy^3 + 4$

$\dfrac{\partial f}{\partial y} = f_z = -e^zy^4 + \dfrac{3z^2}{x^2}$

Q7 Text Solution:

f (x, y) = x³ – 2x²y +3xy² + y³

Apply x = λx and y = λy

f (λx, λy) = (λx)³ – 2(λx)²(λy) +3(λx)(λy)² + (λy)³

f (λx, λy) = λ³x³ – 2λ³(x²y) +3λ³xy² + λ³y³

f (λx, λy) = λ³(x³ – 2x²y +3xy² + y³)

So it is a homogeneous function of degree 3.

Given : x³ – 2x²y +3xy² + y³

∂F/∂x = 3x²–2(2x)y+3(1)y²+0

∂F/∂x = 3x²–4xy+3y²

x(∂F/∂x) = x(3x²–4xy+3y²) ---(1)

Given : $x^3 - 2x^2y + 3xy^2 + y^3$

$\partial F/\partial y = 0 - 2x^2(1) + 3x(2y) + 3y^2$

$\partial F/\partial y = -2x^2 + 6xy + 3y^2$

$y(\partial F/\partial y) = y(-2x^2 + 6xy + 3y^2)$ ---(2)

(1) + (2)

$x(\partial F/\partial x) \quad + \quad y(\partial F/\partial y) \quad = \quad x(3x^2 -$ $4xy + 3y^2) + y(-2x^2 + 6xy + 3y^2)$

$= 3x^3 - 4x^2y + 3xy^2 - 2x^2y + 6xy^2 + 3y^3$

$= 3x^3 - 6x^2y + 9xy^2 + 3y^3$

$= 3(x^3 - 2x^2y + 3xy^2 + y^3)$

$= P\,F(x, y)$

Hence it is proved.

Q8 Text Solution:

Given that,

$v(x, y) = \log\left(\dfrac{x^2 + y^2}{x + y}\right)$

Change into an exponential function.

Let $e^v = \dfrac{x^2 + y^2}{x + y} = f(x, y)$

$f(x, y) = \dfrac{\lambda^2 x^2 + \lambda^2 y^2}{\lambda x + \lambda y}$

$\Rightarrow f(x, y) = \dfrac{\lambda^2 (x^2 + y^2)}{\lambda(x + y)} f$ is a homogeneous

fuction of degree 1.

By Euler' Theorem,

$x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 1 \times f$

Putting $f = e^v$

$\Rightarrow x\dfrac{\partial e^v}{\partial x} + y\dfrac{\partial e^v}{\partial y} = e^v$ exists.

Using chain rule–

$\Rightarrow xe^v\dfrac{\partial v}{\partial x} + ye^v\dfrac{\partial v}{\partial y} = e^v$

Hence Proved.

$\Rightarrow x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} = \dfrac{e^v}{e^v} = 1.$

Q9 Text Solution:

$g(x, y) = x \log(y/x)$

Applying $x = \lambda x$ and $y = \lambda y$

$g(\lambda x, \lambda y) = \lambda x \log(\lambda y/\lambda x)$

$g(\lambda x, \lambda y) = \lambda x \log(y/x)$

It is a homogenous function of degree 1.

$\partial F/\partial x = x(x/y)(-yx^{-2}) + \log(y/x)(1)$

$\partial F/\partial x = x(x/y)(-y/x^2) + \log(y/x)$

$\partial F/\partial x = -1 + \log(y/x)$ -----(1)

$\partial F/\partial y = x(x/y)(1/x)$

$\partial F/\partial y = (x/y)$ -----(2)

(1) + (2)

$x(\partial F/\partial x) + y(\partial F/\partial y) = x[-1 + \log(y/x)] + y(x/y)$

$x(\partial F/\partial x) + y(\partial F/\partial y) = -x + x\log(y/x) + x$

$x(\partial F/\partial x) + y(\partial F/\partial y) = x\log(y/x)$

Hence it is proved.

Q10 Text Solution:

We have to find

$J = \dfrac{\partial(p,q,r)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial p}{\partial x} & \dfrac{\partial p}{\partial y} & \dfrac{\partial p}{\partial z} \\ \dfrac{\partial q}{\partial x} & \dfrac{\partial q}{\partial y} & \dfrac{\partial q}{\partial z} \\ \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} & \dfrac{\partial r}{\partial z} \end{vmatrix}$

But $p = x + y + z$, $q = y + z$, $r = z$ (taking partial derivative)

$J = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$

$\left(\dfrac{\partial p}{\partial x} = 1, \dfrac{\partial p}{\partial y} = 1, \dfrac{\partial p}{\partial z} = 1, \dfrac{\partial q}{\partial x} = 0, \dfrac{\partial q}{\partial y} = 1, \dfrac{\partial q}{\partial z}\right.$

$= 1, \dfrac{\partial r}{\partial x} = 0, \dfrac{\partial r}{\partial y} = 0, \dfrac{\partial r}{\partial z} = 1,\Big)$

$\dfrac{\partial r}{\partial y} = 0, \dfrac{\partial r}{\partial z} = 1$

On expanding we get

$J = 1(1 - 0) = 1$

Thus $J = 1$.

Q11 Text Solution:

By Data

$u = \dfrac{yz}{x}, v = \dfrac{zx}{y}, w = \dfrac{xy}{z}$

$J = \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$

$= \begin{vmatrix} \dfrac{-yz}{x^2} & \dfrac{z}{x} & \dfrac{y}{x} \\ \dfrac{z}{y} & \dfrac{-zx}{y^2} & \dfrac{x}{y} \\ \dfrac{y}{z} & \dfrac{x}{z} & \dfrac{-xy}{z^2} \end{vmatrix}$

$= \dfrac{-yz}{x^2}\left\{\left(\dfrac{-zx}{y^2}\right)\left(\dfrac{-xy}{y^2}\right) - \left(\dfrac{x}{z}\right)\left(\dfrac{x}{y}\right)\right\}$

$- \left(\dfrac{z}{x}\right)\left\{\left(\dfrac{z}{y}\right)\left(\dfrac{-xy}{x^2}\right) - \left(\dfrac{y}{z}\right)\left(\dfrac{x}{y}\right)\right\}$

$+ \dfrac{y}{x} - \left\{\left(\dfrac{z}{y}\right)\left(\dfrac{x}{z}\right) - \left(\dfrac{y}{z}\right)\left(\dfrac{-zx}{y^2}\right)\right\}$

$= \dfrac{-yz}{x^2} - \left\{\dfrac{x^2}{yz} - \dfrac{x^2}{yz}\right\} - \dfrac{z}{x}\left\{\dfrac{-x}{z} - \dfrac{x}{y}\right\}$

$+ \dfrac{y}{x}\left\{\dfrac{x}{y} + \dfrac{x}{y}\right\} = 0 + 1 + 1 + 1 = 4$

Therefore $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 4.$

Q12 Text Solution:

f (x) = 2 log (x −2)−x²+4x+1

$\Rightarrow f'(x) = \frac{2}{x-2} - 2x + 4$

$f'(x) = \frac{2}{x-2} - 2x + 4$

$\frac{2-(2x-4)(x-2)}{x-2}$

$\frac{2-(2x^2-4x-4x+8)}{x-2}$

$\frac{-2x^2+8x-6}{x-2}$

$x - 2 = 0, x = 2$

$-2x^2 + 8x - 6 = 0$

$x^2 - 4x + 3 = 0$

$\left(x - 1\right)\left(x - 3\right) = 0$

$x = 1, 3$

Since,

$\therefore f'(x) > 0$

$\Rightarrow -2(x-1)(x-3)(x-2) > 0$

$\Rightarrow (x-1)(x-2)(x-3) < 0$

$\Rightarrow x \in (-\infty, 1) \cup (2, 3).$

$$\begin{array}{ccccc} - & + & - & + & \\ -\infty & 1 & 2 & 3 & \infty \end{array}$$

Q13 Text Solution:
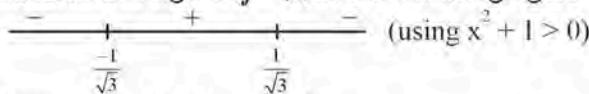
Differentiate w.r.t x to get :

$f'(x) = \frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$

Differentiate again to get :

$f''(x) = 4\frac{(x^2+1)^2 - x(4x)(x^2+1)}{(x^2+1)^4}$

$= 4\frac{(x^2+1)(1-3x^2)}{(x^2+1)^4} = \frac{-4(\sqrt{3x}-1)(\sqrt{3x}+1)}{(x^2+1)^3}$

Observe the sign of f'''(x) in the following figure

$$\begin{array}{ccc} - & + & - \\ \frac{-1}{\sqrt{3}} & & \frac{1}{\sqrt{3}} \end{array} \text{ (using } x^2 + 1 > 0)$$

Convex   Convex   Convex

From figure, points of inflection are x = $\pm\frac{1}{\sqrt{3}}$

Intervals of concavity are x $\in \left[\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$

Intervals of convexity are

$x \in \left(-\infty, \frac{-1}{\sqrt{3}}\right] \cup \left[\frac{1}{\sqrt{3}}, \infty\right).$

Q14 Text Solution:

$f(x) = x^4 - 18x^2 + 9$

$f'(x) = 4x^3 - 36x$

$f''(x) = 12x^2 - 36$

$f'''(x) = 24x$

Points of inflection are −

$f''(x) = 0, x = \sqrt{3}, -\sqrt{3}$

$f'''(\sqrt{3}), f'''(-\sqrt{3})$ *are both not equal to zero*

Thus both are the points of inflection.

Q15 Text Solution:

f(x) = 2x³ − 3x² − 36x + 2

f'(x) = 0

6x² − 6x − 36 = 0

x² − x − 6 = 0

(x − 3) (x + 2) = 0

x = 3 , −2

f'(x) = 12x − 6

at x = −2   f"(x) < 0

at x = 3   f"(x) > 0

Thus maximum at x=−2

Q16 Text Solution:

f(x) = x³ −9x²+ 24x+ 5

$f'(x) = 3x^2 - 18x + 24$

f(x) = 0

$3x^2 - 18x + 24 = 0$

$x^2 - 6x + 8 = 0$

$(x-4)(x-2) = 0$

x = 2 , 4

$f''(x) = 8x - 18$

$f''(x) = $ at $x = 2$

$12 - 18 = 0$

thus maximum

$f''(x)$is $= -$ve at $x = 2$

at x = 6 thus

f(x) = $8 - 9 \times 4 + 24 \times 2 + 5$

$=8 - 36 + 48 + 5$

$= 13 + 48 - 36$

$= 13 + 12 = 25$

Q17 Text Solution:

$f(x) = x^3 - 3x^2 - 24x + 100$

$f'(x) = 3x^2 - 6x - 24$

$f'(x) > 0$

$3x^2 - 6x - 24 = 0$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$
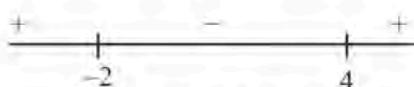
x = 4, − 2

$f''(x) = 6x - 6$

$6 \times 4 - 6 > 24 - 6 < 0 \rightarrow max$

$f''(x) = 6x - 6$

$= 6x - 2 - 6 > -12 - 6 = 16 < 0$

$(-\infty, -2) \rightarrow -fx \text{ is } + ve(\uparrow)$

$(4, \infty) \rightarrow fx \text{ is } + ve(\uparrow)$

(- 2, 1) it is decreasing

Thus b/w (-2 ,4) the fx is decreasing

thus at x = 3 it will be min

$f(x) = 27 - 27 - 24 \times 3 + 100$

$= -72 + 100$

= 28

Q18 Text Solution:

We'll first need the derivative so we can find the critical points.

$A'(t) = -10e^{5-\frac{t^2}{8}} - 10te^{5-\frac{t^2}{8}}\left(-\frac{t}{4}\right)$

$= 10e^{5-\frac{t^2}{8}} - \left(-1 + \frac{t^2}{4}\right)$

Therefore the derivative will only by zero where,

$-1 + \frac{t^2}{4} = 0 \Rightarrow t^2 = 4$

$\Rightarrow t = \pm 2$

However only **t** = 2 is actually in the interval so that is only critical point that we'll use. Now for absolute maxima. We have, Max (**A** (0), **A** (2), **A** (10)} i.e., Max (2000, 199.66, 1999.94} On comparing all these values we found, the maximum amount in the account will be Rs. 2000 which occurs at **t** = 0 and similarly absolute minimum amount in the account will be Rs.199.66 which occurs at the 2 year mark.

Q19 Text Solution:

$f'(x) = 2x \ln(3x) + x^2\left(\frac{3}{3x}\right)$

= 2x ln (3x) + x

= x (2 ln (3x) + 1)

The derivative will only be zero if,

2 ln (3x) +1 = 0

$\ln(3x) = -\frac{1}{2}$

Take exponent on both sides to get :

$e^{\ln}(3x) = e^{-1/2}$

$3x = e^{-1/2}$

$x = \frac{1}{3}e^{-1/2} = \frac{1}{3\sqrt{e}}$

There is only one critical point of **f** (**x**)

i/e. $x = \frac{1}{3\sqrt{e}}$

Which is also a stationary point of **f** (**x**).

Q20 Text Solution:

r=∂²f/∂x²=2y

s=∂²f/∂x∂y=2x−3

t=∂²f/∂y2=0

Since, rt−s²≤0, (if rt−s²< 0 then we have no maxima or minima, if = 0 then we can't say anything).

Maxima will exist when rt−s²>0 and r<0.

Minima will exist when rt−s²>0 and r>0.

As rt−s² is never greater than 0 so we have no local extremum.

Q21 Text Solution:

$f_x(x,y)$ = 4x + 2y - 6=0   (1)

$f_y(x,y)$ = 2x + 4y=0   (2)

On solving (1) and (2) we get,

x=2,y=-1

r=∂²f/∂x²=4

s=∂²f/∂x∂y=2

t=∂²f/∂y²=4

rt−s²=12

As rt−s²>0 and r>0. Thus, (2,-1) is the point of local minima.

Q22 Text Solution:

Let us find the first-order partial derivative with respect to x and y to determine the critical points.

$f_x(x, y) = -2x$

and $f_y(x, y) = 2y$ which exists everywhere.

Clearly, f has a critical point at (0, 0). Let us apply the second-order partial derivative test,

$f_{xx}(x, y) = -2$

$f_{yy}(x, y) = 2$

$f_{xy}(x, y) = 0$

Therefore, the discriminant $D = f_{xx}(0, 0) f_{yy}(0, 0) - [f_{xy}(0, 0)]^2$

$\Rightarrow D = -4 - 0^2 = -4 < 0$

Hence, at (0, 0) f has a saddle point.

Q23 Text Solution:

Let us find the first-order partial derivative with respect to x and y to determine the critical points.

$f_x(x, y) = 2xy^2 - 10x - 8y$

and $f_y(x, y) = 2x^2y - 10y - 8x$ which exists everywhere.

Now, the second equation from first and equating to zero, we get

$2xy (y - x) + 2(y - x) = 0$

$\Rightarrow (y - x)(xy + 1) = 0$

$\Rightarrow$ either x = y or y = −1/x

If x = y, substituting this in any of the partial derivatives, we get

$2x^3 - 18x = 0$

$\Rightarrow x = 0, 3, -3$

And if y = −1/x, substituting this in any of the partial derivatives, we get

$2/x - 10x + 8/x = 0 \Rightarrow 1 - x^2 = 0$

$\Rightarrow x = 1, -1$

Therefore, we get the critical points (0, 0), (3, 3), (−3, −3), (1, −1) and (−1, 1).

Now, $f_{xx}(x, y) = 2y^2 - 10$

$f_{yy}(x, y) = 2x^2 - 10$

$f_{xy}(x, y) = 4xy - 8$

At (0, 0), $D = f_{xx}(0, 0) f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 100 - 64 = 36 > 0$ and $f_{xx}(0, 0) = -10 < 0$

Hence, at (0, 0) we have a local maximum

At (±3, ±3), $D = f_{xx}(\pm3, \pm3) f_{yy}(\pm3, \pm3) - [f_{xy}(\pm3, \pm3)]^2 = 64 - 784 = -720 < 0$

Hence, at (±3, ±3) we have saddle points.

At (±1, ±1), $D = f_{xx}(\pm1, \pm1) f_{yy}(\pm1, \pm1) - [f_{xy}(\pm1, \pm1)]^2 = 64 - 144 =$ a negative quantity

Hence, at (±1, ±1) also we have saddle points.

Q24 Text Solution:

Since perpendicular distance of the point from curve is the minimum distance so maxima-minima concept can be applied here such that distance d is -

$d = \sqrt{x^2 + y^2 + z^2}$

$z^2 = 3xy + 4$

$d = \sqrt{x^2 + y^2 + 3xy + 4}$

$f(x, y) = d^2 = x^2 + y^2 + 3xy + 4$

$\frac{\partial f}{\partial x} = 2x + 3y, r = \frac{\partial^2 f}{\partial x^2} = 2 \; and \; S = \frac{\partial^2 f}{\partial x \partial y}$

$= 3$

$\frac{\partial f}{\partial y} = 2y + 3x$

$\frac{\partial^2 f}{\partial y^2} = 2$

$rt - s^2 = 4 - 3 > 0$

$r = 2 > 0$

$point \; of \; relative \; minima$

$f(x, y) = f(0, 0) = 4$

$d^2 = 4$

$d = 2$

Q25 Text Solution:

Let P(x,y) be the position of jet and the soldier is placed at A(3,2) . Then, the distance between the soldier and jet is given by.

$AP = \sqrt{(x - 3)^2 + (y - 2)^2}$

$= \sqrt{(x - 3)^2 + x^4}$

$[\because \; y = x^2 + 2]$ .... (i)

Let Z = AP^2. Then AP = $\sqrt{Z}$ so (AP)$_{min}$ = $\sqrt{Z_{min}}$ .... (2)

Z = (x − 3)^2 + x^4

$\frac{dz}{dx} = 0 \Rightarrow x = 1$

$\left(\frac{d^2z}{dx^2}\right)_{x=1} = -ve$

so, Z is min at x = 1 & Z_{min} = 5

Hence, AP$_{min}$ = $\sqrt{5}$

Q26 Text Solution:

Let x cum be the length of a side of the square which is cut-off from each corner of the plate.

Then, sides of the box as shown in figure are 24 −2x, 24 −2x and x.

Let V be the volume of the box.
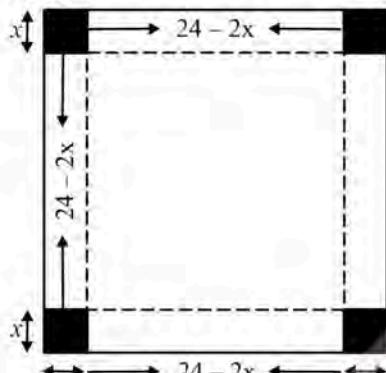
Then,

$$V = (24 - 2x)^2 x = 4x^3 - 96x^2 + 576x$$

$$\Rightarrow \frac{dV}{dx} = 12x^2 - 192x + 576 \text{ and }$$

$$\frac{d^2V}{dx^2} = 24x - 192$$

For maximum or minimum values of V, must have

$$\frac{dV}{dx} = 0$$

$$\Rightarrow 12x^2 - 192x + 576 = 0$$

$$\Rightarrow x^2 - 16x + 48 = 0$$

$$\Rightarrow (x - 12)(x - 4) = 0 \Rightarrow x = 12, 4$$

But, x = 12 is not possible.

Therefore, x = 4

Now, $\left(\frac{d^2V}{dx^2}\right) = 24 \times 4 - 192 < 0$.

Thus, V is maximum when x = 4.

Hence, the volume of the box is maximum when the side of the square is 4 cm.

The maximum volume is , $V = (24 - 8)^2 \times 4 = 1024$ cm^3

Q27 Text Solution:

Suppose x items are sold to maximize the profit P. Then, P=Revenue-Cost

$$\Rightarrow P = x\left(5 - \frac{x}{100}\right) - \left(\frac{x}{5} + 500\right)$$

$$P = \frac{24}{5}x - \frac{x^2}{100} - 500$$

$$\frac{dP}{dx} = \frac{24}{5} - \frac{2x}{100}$$

$$\frac{d^2P}{dx^2} = \frac{-1}{50}$$

The critical numbers of P are given by $\frac{dP}{dx} = 0$

$$\frac{dP}{dx} = \frac{24}{5} - \frac{2x}{100} = 0$$

$$x = 240$$

$$\frac{d^2P}{dx^2} = \frac{-1}{50} < 0$$

Hence, profit P is maximum when 240 items are sold.

Q28 Text Solution:

$$\int x^2 \sin x \, dx$$

$$= x^2 \int \sin x \, dx - \int \left(2x \int \sin x \, dx\right) dx$$

$$= -x^2 \cos + 2\left[x \int \cos x \, dx - \int \left(1 \int \cos x \, dx\right) dx\right]$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$$

Q29 Text Solution:

$$x = t^2 \Rightarrow dx = 2t \, dt$$

$$I = \int \frac{dx}{2\sqrt{x}(x+1)} = \int \frac{2t \, dt}{2t(t^2+1)}$$

$$\int \frac{dt}{1+t^2} = \tan^{-1} t + c$$

$$= \tan^{-1}\left(\sqrt{x}\right) + c$$

Q30 Text Solution:

Put

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$\Rightarrow I = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Put $x = 1$, we get, $A = \frac{1}{2}$

$x = 2$, we get, $B = -1$

$x = 3$, we get, $C = \frac{1}{2}$

So integral, $= \frac{1}{2}\int \frac{dx}{x-1} - \int \frac{dx}{x-2} + \frac{1}{2}\int \frac{dx}{x-3}$

$$= \ln\left(\frac{\sqrt{x^2-4x+3}}{|x-2|}\right) + c$$

Q31 Text Solution:

$$\int_2^7 \frac{\sqrt{x} \, dx}{\sqrt{x} + \sqrt{9-x}}$$

$$I = \int_2^7 \frac{\sqrt{9-x}}{\sqrt{9-x} + \sqrt{9-(9-x)}} dx$$

$$I = \int_2^7 \frac{\sqrt{9-x}}{\sqrt{9-x} + \sqrt{x}} dx$$

Adding (i) and (ii), we get

$$2I = \int_2^7 \left(\frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}} + \frac{\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}}\right) dx$$

$$= \int_2^7 dx = x\Big|_2^7 = 5$$
$$I = \frac{5}{2}$$

Q32   Text Solution:

$$I = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+e^x)(1+x^2)}$$

Here, $f(x) = \frac{1}{(1+e^x)(1+x^2)}$

$$\Rightarrow f(-x) = \frac{1}{(1+e^{-x})\left(1+(-x)^2\right)} = \frac{e^x}{(1+e^x)(1+x^2)}$$

So, $I = \int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^{\sqrt{3}} = \frac{\pi}{3}$

Q33   Text Solution:

$$I = \int_{-a}^a \sqrt{\frac{a-x}{a+x}}dx = \int_{-a}^a \frac{a-x}{\sqrt{a^2-x^2}}dx$$

$$= a\int_{-a}^a \frac{dx}{\sqrt{a^2-x^2}} - \int_{-a}^a \frac{x\,dx}{\sqrt{a^2-x^2}}$$

$$= a\cdot 2\int_0^a \frac{dx}{\sqrt{a^2-x^2}}$$

$$- 0\left(\because \frac{x}{\sqrt{a^2-x^2}}\text{is an odd function}\right)$$

$$= 2a\left[\sin^{-1}\frac{x}{a}\right]_0^a$$

$$\Rightarrow 2a\left[\sin^{-1}(1) - \sin^{-1}(0) = a\right]$$

$$= a\left[\frac{\pi}{2} - 0\right] = \pi a$$

Q34   Text Solution:

$$I = \int_0^\pi \frac{x\,dx}{1+\cos^2 x}$$

$$I = \int_0^\pi \frac{(\pi-x)dx}{1+\cos^2(\pi-x)} = \int_0^\pi \frac{(\pi-x)dx}{1+\cos^2 x}$$

Addition both, we get

$$2I = \int_0^\pi \frac{\pi\,dx}{1+\cos^2 x} \Rightarrow I = \frac{\pi}{2}\int_0^\pi \frac{dx}{1+\cos^2 x}$$

$$= \frac{\pi}{2}\left[\int_0^\pi \frac{dx}{1+\cos^2 x} + \int_0^{\pi/2} \frac{dx}{1+\cos^2(\pi-x)}\right]$$

$$= \pi\int_0^{\pi/2} \frac{\sec^2 x\,dx}{2+\tan^2 x} \quad \text{Put } \tan x = t$$

$$I = \pi\int_0^\infty \frac{dt}{t^2+2}$$

$$= \frac{\pi}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\Big|_0^\infty = \frac{\pi^2}{2\sqrt{2}}$$

Q35   Text Solution:

$$\int_0^\pi \frac{dx}{1+2\sin^2 x} = 2\int_0^{\pi/2} \frac{dx}{1+2\sin^2 x}$$

$$\Bigg($$

$$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \; if \; f(2a-x)$$

$$= f(x)\Bigg)$$

$$= 2\int_0^{\pi/2} \frac{\sec^2 x\,dx}{\sec^2 x+2\tan^2 x} = 2\int_0^{\pi/2} \frac{\sec^2 x\,dx}{1+3\tan^2 x}$$

(Not that in the beginning can not divided numerator and denominator by $\cos^2 x$, as $\cos x = 0$ at $x = \pi/2$)

$$= 2\int_0^\infty \frac{dt}{1+3t^2}, \qquad (\tan x = t)$$

$$= 2\frac{1}{\sqrt{3}}\left[\tan^{-1} t\sqrt{3}\right]_0^\infty = \frac{2}{\sqrt{3}}\times\frac{\pi}{2} = \frac{\pi}{\sqrt{3}}$$

Q36   Text Solution:

$$5f(x) + 4f\left(\frac{1}{x}\right) = \frac{1}{x} + 3 \text{------------------} (1)$$

Replace $x \to \frac{1}{x}$

$$5f\left(\frac{1}{x}\right) + 4f(x) = x + 3 \text{------------------} (2)$$

Multiplyb equation 1 with 5 and multiply equation 2 with 4 and then subtract bith the equation we get-

$$f(x) = \frac{1}{9}\left(\frac{5}{x} - 4x + 3\right)$$

$$\therefore I = 18\int_1^2 \frac{1}{9}\left(\frac{5}{x} - 4x + 3\right)dx$$

$$I' = 18\int \frac{1}{9}\left(\frac{5}{x} - 4x + 3\right)dx$$

$$2\left(5\log x - 2x^2 + 3x\right)$$

$$10\log x - 4x^2 + 6x$$

Putting the limits

$$10\log 2 - 10\log 1 - 4\left(4 - 1\right) + 6\left(2 - 1\right)$$

$$10\log 2 - 12 + 6$$

$$10\log_e 2 - 6$$

Q37   Text Solution:

$$F'(x) = 2x\cdot\sqrt{\sin x^2} - 1\cdot\sqrt{\sin x}$$

Q38   Text Solution:

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$$\lim_{x \to \infty} \frac{\left(\int_0^x e^{t^2} dt\right)^2}{\int_0^x e^{2t^2} dt} \because \left(\frac{\infty}{\infty} form\right)$$

Applying L's Hospital rule:

$$\lim_{x \to \infty} \frac{2 \cdot \int_0^x e^{t^2} dt \cdot e^{x^2}}{1 \cdot e^{2x^2}}$$

$$\lim_{x \to \infty} \frac{2 \cdot \int_0^x e^{t^2} dt}{e^{x^2}} = \lim_{x \to \infty} \frac{2 \cdot e^{x^2}}{2x \cdot e^{x^2}}$$

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Q39 Text Solution:

Arc length $= \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

$\qquad = \int_0^4 \sqrt{1 + \frac{9x}{4}} dx$

$= \frac{8}{27} \left(1 + \frac{9}{4}x\right)^{3/2} \Big|_0^4 = \frac{8}{27} \left(10^{3/2} - 1\right)$

Q40 Text Solution:

$L = \int_1^2 \sqrt{1 + \frac{1}{4x^2}} dx$

$\int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_1^2 \sqrt{1 + \frac{1}{4x^2}} dx$

So, $\frac{dy}{dx} = \frac{1}{2x}$

$\int dy = \frac{1}{2} \int \frac{dx}{x} + c$

$y = \ln \sqrt{x} + c$

y = f(x) passes through (1, 1)

$y = \ln \sqrt{x} + 1 \Rightarrow c = 1$

Q41 Text Solution:

Volume of solid of revolution $V = \int_0^b \pi y^2 dx$

$= \frac{\pi a^2}{4} \int_0^b \left(e^{2x/a} + e^{-2x/a} + 2\right) dx$

$= \frac{\pi a^2}{4} \left[\frac{a}{2} e^{2x/a} - \frac{a}{2} e^{-2x/a} + 2x\right]_0^b$

$= \frac{\pi a^3}{8} \left(e^{2b/a} - e^{-2b/a}\right) + \frac{\pi a^2 b}{2}$

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