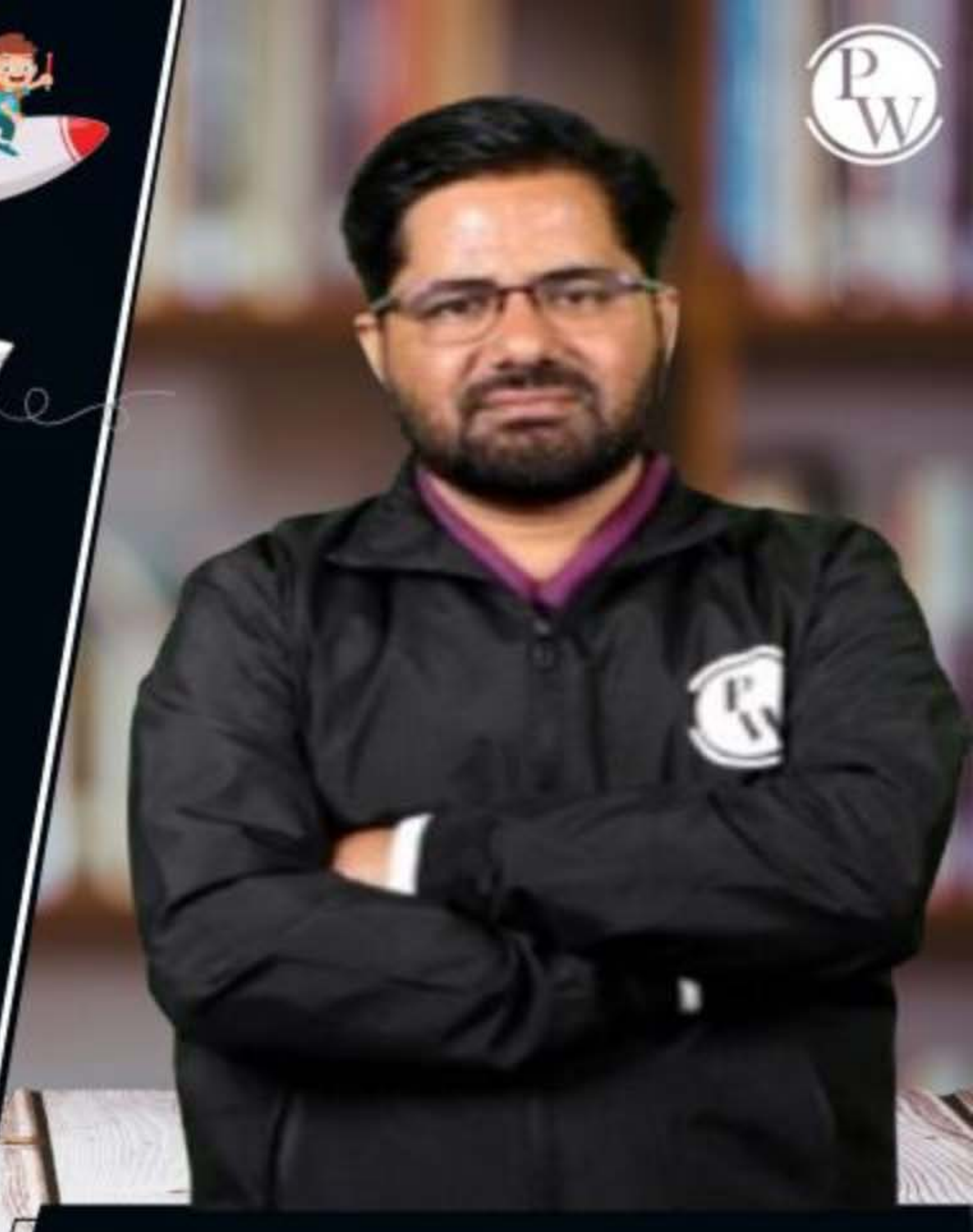
DS & AI
CS & IT

Probability & Statistics

Lecture No. 06



By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

PROBABILITY (Part-1)

(BASIC CONCEPTS)

Topics to be Covered

Topic

PROBABILITY (Part-2)

(PRACTICE QUESTION)

- → General Questions.
- → M.E. Events
- → IND EVENTS

Thumb Rule of this Chapter → Try to avoid making Question by using following words;

"If, what if, AGAR, YADI, TOH, ..."

OR

Don't Try to develop Question (by your little mind until you have a complete understanding of the chapter) & try to solve the Quest.

Short RECAP

Operation	P&C	Prob	Formula	ME	Ind.
Either or	Add	union	Addition Th	$P(A\cup B) = P(A)+P(B)$	—
AND	Multiply	Intersection	Multi Th	$P(A\cap B) = 0$	$P(A\cap B) = P(A)\cdot P(B)$

Addition Th: $\boxed{P(A\cup B) = P(A)+P(B)-P(A\cap B)}$

for Independency: $P(A\cup B) = P(A)+P(B)-P(A)\cdot P(B)$

for ME : $P(A\cup B) = P(A)+P(B)-0$

Q. (3) Dice are thrown simultaneously then find the prob that at least one die show digit '4' ?

Sol: App II

$S = \{(111), (112), (113), \ldots (116), (211), (212) \ldots (216), (311), \ldots, (666)\}$

$n(S) = \underset{D_1}{6} \times \underset{D_2}{6} \times \underset{D_3}{6} = 6^3 = 216$ Triplets

Unfav Cases $= \{$No Die show digit '4'$\} = \underset{D_1}{5\text{ way}} \times \underset{D_2}{5\text{ way}} \times \underset{D_3}{5\text{ way}} = 125$ Triplets

fav Case = Total − No die show digit 4 $= 216 - 125 = 91$ Triplets

P(at least one die show digit 4) $= \dfrac{\text{fav}}{\text{Total}} = \dfrac{91}{216}$

F.Q 2:- 6 six coins are tossed simultaneously then write it's s. space?

$S = \{(HHHHHH), (HHHHHT), (HHHHTT), (HHHTTT), (HHTTTT), (HTTTTT), (TTTTTT)\}$

${}^6C_0 = 1$, ${}^6C_1 = 6$, ${}^6C_2 = 15$, ${}^6C_3 = 20$, ${}^6C_4 = 15$, ${}^6C_5 = 6$, ${}^6C_6 = 1$

Here we have 7 Combinations & 64 Permutations

$$n(S) = \frac{2}{C_1} \times \frac{2}{C_2} \times \frac{2}{C_3} \times \frac{2}{C_4} \times \frac{2}{C_5} \times \frac{2}{C_6} = 2^6 = 64 \text{ Tuples.}$$

$P(H) = \frac{1}{2} = P(T)$, All coins are Independents $\Rightarrow$ All tosses are Ind.

$S = \{E_1, (E_2 - E_7), (E_8 - E_{22}), (E_{23} - E_{42}), (E_{43} - E_{57}), (E_{58} - E_{63}), E_{64}\}$

these 64 Events (when individually) Taken are M.E

while for coin 1 to coin 6, S.Sp $= \{H, T\}$ i.e $C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = \{H, T\}$

(1) Find the prob that all the outcomes are identical?

Sol: fav Event $(E) = \{(HHHHHH), (TTTTTT)\} \Rightarrow n(E) = 2$

App I

So Req Prob $= P(E) = \frac{n(E)}{n(S)} = \frac{2}{64} = \frac{1}{32}$

App III

Req Prob $= P[(HHHHHH) \cup (TTTTTT)]$

$= P(E_1 \cup E_{64})$

$= P(E_1) + P(E_{64})$ $(\because E_1 \,\&\, E_{64}$ are ME$)$

$= P(HHHHHH) + P(TTTTTT)$

$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$ $(\because$ All tosses are Ind.$)$

$= \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^6 = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

② Find the prob that H & T appears alternately?

App I fav Cases $(E) = \{(HTHTHT), (THTHTH)\} \Rightarrow n(E) = 2$

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{64} = \frac{1}{32}$$

ME

App III Req Prob $= P[(HTHTHT) \text{ or } (THTHTH)]$

$= P(HTHTHT) + P(THTHTH)$ { ∵ these are the two individual outcomes of S.Sp & hence ME }

$= \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^6 = \left(\frac{1}{2}\right)^5$

(∵ H & T are Ind).

Note Find the prob that exactly 3 H occurs?

App II fav Cases $= \{\text{eg } (HHHTTT) \ldots\ldots\} = \frac{6!}{3!\,3!} = {}^6C_3 = 20$ so $P = \frac{20}{64}$

(3) Find the prob that Both H & T appears at least once?

App I unfav outcomes = {(HHHHHH), (TTTTTT)} = 2 so fav outcomes = 64 − 2 = 62

Hence Req Prob = $\frac{\text{fav}}{\text{Total}} = \frac{62}{64}$

(4) Find the prob that H appears at least once?

App I unfav outcome = {(TTTTTT)} = 1 so fav outcomes = 64 − 1 = 63

Req Prob = $\frac{\text{fav}}{\text{Total}} = \frac{63}{64}$

App III

$$P(\text{at least one } H) = 1 - P(\text{No Head})$$
$$= 1 - P(\text{All Tail}) = 1 - P(\text{all Tail})$$
$$= 1 - P(TTTTTT) = 1 - \left(\frac{1}{2}\right)^6$$

(5) If 1st three outcomes are H, H, H then find the prob of occurring Tail, when coin is tossed again? given statement

App III (M-I) (Books) $\text{Req Prob} = P(T \text{ in } 4^{th} \text{ toss}) = \frac{1}{2}$

App III (M-II) $\text{Req Prob} = P\left[\underset{\text{fix}}{HHH}\ \underset{4^{th}}{T}\ \underset{5^{th}}{\text{something}}\ \underset{6^{th}}{\text{occurs}}\right] = 1^3 \times \frac{1}{2} \times 1^2 = \frac{1}{2}$

App I $\because$ this is a Question Based on Conditional Prob so will be discussed Later

(6) If 1st three outcomes are H, H, H, then find the prob of occurring Tail in Remaining tosses? given statement

App III (M-I) Req Prob $= P\left(\underset{4^{th}}{T}\ \underset{5^{th}}{T}\ \underset{6^{th}}{T}\right) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8}$

App III (M-II) Req Prob $= P\left[\underset{1\ 2\ 3}{H\ H\ H}\ \underset{4^{th}}{T}\ \underset{5^{th}}{T}\ \underset{6^{th}}{T}\right] = 1^3\times\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

App I: will be discussed Later

(7) Find the prob that only 1^{st} two tosses produces Head ?

App I fav Cases $= \{$ only 1^{st} two tosses are Heads $\} = \{ (H\,H\,T\,T\,T\,T) \} = 1$.

$$\text{Req Prob} = \frac{\text{fav}}{\text{Total}} = \frac{1}{64}$$

App III : Req Prob $= P($only 1^{st} two tosses are $H) = P[HHTTTT]$

$$= \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^6$$

(8) Find the prob that 1^{st} two tosses produces head ?

App III Req Prob $= P[\underline{H}\,\underline{H}\ \underline{\text{something}}\ \underline{\text{occurs}}] = \frac{1}{2} \times \frac{1}{2} \times 1^4 = \frac{1}{4}$

App I fav Cases $= \{(\underline{H}\,\underline{H}\,\_\,\_\,\_\,\_) \ldots\ldots\} =$ Not easy to count

App II fav Cases $= \{(\underset{1}{\underline{H}}\ \underset{2}{\underline{H}}\ \underset{3}{\underline{H/T}}\ \underset{4}{\underline{H/T}}\ \underset{5}{\underline{H/T}}\ \underset{6}{\underline{H/T}}) \ldots\ldots\}$

$= 1\text{ way} \times 1\text{ way} \times 2\text{ ways} \times 2\text{ ways} \times 2\text{ ways} \times 2\text{ ways} = 16\text{ ways}.$

So Req Prob $= \frac{\text{fav}}{\text{Total}} = \frac{16}{64} = \frac{1}{4} = \frac{1 \times 1 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2 \times 2} = \frac{1}{2} \times \frac{1}{2} \times 1^4$

Analysis of Part (8) →

$$\text{Req Prob.} = \frac{\text{fav}}{\text{Total}} = \frac{16}{64} = \frac{(1\times1\times2\times2\times2\times2)}{2^6}$$

App II

$$= \frac{1}{2}\times\frac{1}{2}\times\frac{2}{2}\times\frac{2}{2}\times\frac{2}{2}\times\frac{2}{2}$$

$$= \frac{1}{2}\times\frac{1}{2}\times \underbrace{1\times1\times1\times1}$$

App III

$$= P(\underline{H}\ \underline{H}\ \underline{SO}\ \underline{SO}\ \underline{SO}\ \underline{SO})$$

$$= \frac{1}{4}$$

(9) Find the prob that exactly 2 H will occur ?

App I: fav cases = { eg (HTTTHT) ------ } = Not easy to count.

App II: fav cases = { eg (HTTTHT) ------ } $\frac{6!}{2!\,4!} = {}^6C_2 = 15$ cases.

So Req Prob $= \frac{\text{fav}}{\text{Total}} = \frac{15}{64} = \frac{{}^6C_2}{2^6}$

App III: this Q. can be solved by using formula of Binomial Distribution (will be discussed Later).

(10) Find the prob that exactly 3 H will occur ?

App II Req Prob $= \dfrac{\text{fav}}{\text{Total}} = \dfrac{{}^6C_3}{2^6} = \dfrac{\frac{6!}{3!\,3!}}{64} = \dfrac{20}{64}$

App III (using Binomial Distribution) $\longrightarrow$

HAVE PATIENCE

(11) Find the prob that H & T appears equal number of times?

P(getting Head & Tail equal No. of times)

$= P(\text{getting exactly } 3H) = \dfrac{^6C_3}{2^6} = \dfrac{20}{64}$ (same as part 10)

Note: If a coin is tossed 7 times then find the prob that H & T appears equal no of times? (senseless Quest)

$= P(\text{Impossible Event}) = 0$

Q8 A coin is tossed 10 times then find the prob that

(1) exactly 3 H will occur ?

(2) 4^{th} Head will occur in 9^{th} toss ?

$$S = \{(HHHHHHHHHH), (HHHHHHHHHT), \ldots, (TTTTTTTTTT)\}$$

$$n(S) = 2^{10} = 1024 \text{ Tuples}$$

(1) $\text{Fav. Tuples} = {}^{10}C_3 = \dfrac{10!}{3!\,7!} = 120 \text{ Tuples}$

$$\text{Hence Req Prob} = \frac{F}{T} = \frac{{}^{10}C_3}{2^{10}} = \frac{120}{1024}$$

② App II : Req Prob = P(getting 4^{th} Head in 9^{th} toss)

$= P$[getting exactly 3 H Heads in 1^{st} 8 throws] $\times P$(H in 9^{th} toss) $\times P$(S.O. in 10^{th} toss)

(Not sure about the location of H)

$$= \left(\frac{{}^8C_3}{2^8}\right) \times \frac{1}{2} \times 1 = \frac{7}{64}$$

App II Fav Cases = { eg ((exactly 3 H) _ _ _ _ _ _ _ _ H H/T) ..... } $= {}^8C_3 \times 1 \times 2$

1 2 3 4 5 6 7 8, 9 10

So Req Prob $= \frac{f}{T} = \frac{{}^8C_3 \times 1 \times 2}{2^{10}} = \frac{{}^8C_3 \times 1 \times 2}{2^8 \times 2 \times 2} = \frac{{}^8C_3}{2^8} \times \frac{1}{2} \times \frac{2}{2} = \frac{7}{64}$

ANALYSIS

Req Prob = $P(4^{th}$ Head in 9^{th} toss$)$

$= P(\text{exactly } 3H \text{ in } 1^{st}\ 8 \text{ tosses}) \times P(H \text{ in } 9^{th} \text{ toss}) \times P(\text{Something occurs})$

$= \left(\frac{^8C_3}{2^8}\right) \times \left(\frac{1}{2}\right) \times (1) = \frac{7}{64}$ Ans

Link b/w App II & App III :-

$$\text{Req Prob} = \frac{\text{fav}}{\text{Total}} = \underbrace{\frac{^8C_3 \times 1 \times 2}{2^{10}}}_{\text{App II}} = \left(\frac{^8C_3}{2^8}\right) \times \left(\frac{1}{2}\right) \times \left(\frac{2}{2}\right) = \underbrace{\frac{^8C_3}{2^8} \times \frac{1}{2} \times 1}_{\text{App III}} = \frac{7}{64}$$

Four fair coins are tossed simultaneously. The probability that at least one heads and at least one tails turn up is

(a) $\frac{1}{16}$ (b) $\frac{1}{8}$

(c) $\frac{7}{8}$ (d) $\frac{15}{16}$

Total Cases $= \underset{C_1}{2} \times \underset{C_2}{2} \times \underset{C_3}{2} \times \underset{C_4}{2} = 2^4 = 16$

$= 16$ Quadruples.

unfav Quadruples $= \{(HHHH), (TTTT)\}$

$= 2$

fav $= 16 - 2 = 14$

Prob $= \frac{\text{fav}}{\text{Total}} = \frac{14}{16} = \frac{7}{8}$

Three fair cubical dice are thrown simultaneously. The probability that all three dice have the same number on the faces showing up is (up to third decimal place) ______.

(a) $\frac{1}{216}$ (b) $\frac{1}{36}$

(c) $\frac{1}{6}$ (d) $\frac{1}{2}$

Req Prob = P (all the outcomes are identical)

$$= \frac{f}{T} = \frac{6}{6^3} = \frac{1}{6^2} = \frac{1}{36}$$

Two dice each numbered from 1 to 6 are thrown together. Let A and B be two events given by

A : Even number on the first dice

B : Number on the second dice is greater than 4

(i) What is the value of $P(A \cap B)$ and $P(A \cup B)$ respectively?

(a) 1/2, 1/6 (b) 1/4, 2/3 (c) 2/3, 1/6 (d) 1/6, 2/3

Total Cases $= \underset{D_1}{6} \times \underset{D_2}{6} = 6^2 = 36$ pair & $\underbrace{\text{Both Dices are Independent.}}$

So A & B are also Ind.

$P(A) = P(\text{Even No of } 1^{st} \text{ Die}) = \frac{3}{6}$

$P(B) = P(\text{No} > 4) = \frac{2}{6}$

$P(A \cap B) = P(A) \cdot P(B) = \frac{3}{6} \times \frac{2}{6} = \frac{1}{6}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

M-II $S = \{(11), (12), (13) \ldots (66)\} \Rightarrow n(S) = 36$ pair

$A = \{(\underline{\text{Even No}}, \underline{\text{S.O}}) \ldots\} = 3\text{ way} \times 6\text{ way} = 18$ pair

So $P(A) = \frac{18}{36} = \frac{1}{2}$

$B = \{(\underline{\text{S.O}}, \underline{\text{No} \geq 5})\} = 6\text{ way} \times 2\text{ way} = 12$ pair So $P(B) = \frac{12}{36} = \frac{1}{3}$

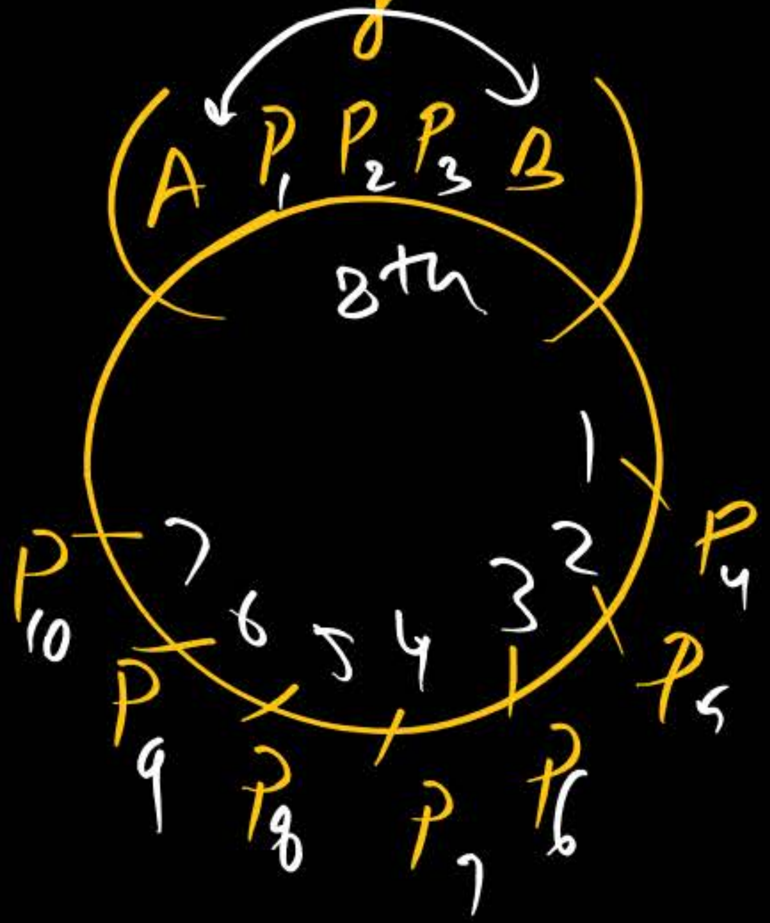
$A \cap B = \{(\underline{\text{Even No}}, \underline{\text{No} \geq 5})\} = 3\text{ way} \times 2\text{ way} = 6$ pair. $P(A \cap B) = \frac{6}{36} = \frac{1}{6}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \ldots = \frac{2}{3}$

A and B stand in a ring with 10 other persons. If the arrangement of the 12 person is at random, find the chance that there are exactly 3 persons between A and B.

Total persons = 12, Total circular arrangements = $(12-1)! = 11!$

fav arrangements = $\left({}^{10}C_3 \times 3!\right) \times 2! \times (8-1)!$



$$\text{Req Prob} = \frac{\text{fav}}{\text{Total}} = \frac{{}^{10}C_3 \times 3! \times 2! \times 7!}{11!}$$

$$= 0.18$$

PYQ

What is the probability that a divisor of 10^{99} is a multiple of 10^{96}?

(a) 1/625 (b) 4/625

(c) 12/625 (d) 16/625

$10^{99} = 2^{99} \times 5^{99}$

So Total factors $= (99+1)\times(99+1)$

$= 100\times 100$

W.K. that $\boxed{10^{99} = 10^{96}\times 10^{3}}$

App II

$\approx 10^{96} \times 2^{3} \times 5^{3}$

$= 10^{96} \times 2^{0,1,2,3} \times 5^{0,1,2,3}$

fav factors $= 1 \times 4$ ways $\times 4$ ways $= 16$ factors.

So Req Prob $= \frac{f}{T} = \frac{16}{100\times 100} = \frac{1}{25\times 25}$

$= \frac{1}{625}$

Independent Events → (Questions Based on Ind Events)

if $\boxed{A \,\&\, B \text{ are Ind}}$ then $\Longleftrightarrow$ $\boxed{P(A \cap B) = P(A) \cdot P(B)}$

A bag contains 5 red and 7 black balls and a second contains 4 blue and 3 green balls. A ball is taken out from each bag. Find the probability that

(i) one ball is red and other blue (a) $\frac{5}{21}$ (b) $\frac{10}{21}$ (c) $\frac{1}{84}$ (d) $\frac{1}{20}$

(ii) one ball is black and other green (a) $\frac{1}{2}$ (b) $\frac{1}{5}$ (c) $\frac{1}{21}$ (d) $\frac{1}{4}$

5R
7B
B_1

4 Bl
3 G
B_2

Both Bags are Ind

(1) $P(R \cap Bl) = \frac{5}{12} \times \frac{4}{7} = \frac{5}{21}$

(2) $P(B \cap G) = \frac{7}{12} \times \frac{3}{7} = \frac{1}{4}$

A fair dice is rolled twice. The probability that an odd number will follow an even number is

(PYQ)

(a) $\frac{1}{2}$

(b) $\frac{1}{6}$

(c) $\frac{1}{3}$

(d) $\frac{1}{4}$

$P(\text{Even} \cap \text{odd}) = \frac{3}{6} \times \frac{3}{6} = \frac{1}{4}$

Both the Throws are Independent

X and Y are two random independent events. It is known that $P(X) = 0.40$ and $P(X \cup Y^c) = 0.7$. Which one of the following is the value of $P(X \cup Y)$? $= P(X) + P(Y) - P(X \cap Y) =$

(a) 0.7 ✓

(b) 0.5

(c) 0.4

(d) 0.3

Do Yourself

Mutually Exclusive Events →

if A & B are ME then $A \cap B = \phi$

$P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B)$

Assertion (A) : The probability of drawing either an ace or a king from a deck of card in a single draw is $\frac{2}{13}$.

$P(A \cup K) = \frac{2}{13}$ (given)

Reason (R) : For two events E_1 and E_2 which are not mutually exclusive, the probability is given by

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2).$$

(a) A and R are true, R is the correct explanation of A

(b) A and R are true but R is not the correct explanation of A

(c) A is true but R is false

(d) A is false but R is true

$$P(A \cup K) = P(A) + P(K) - P(A \cap K)$$

$$= \frac{4}{52} + \frac{4}{52} - 0$$

$\because A \cap K = \phi$

$\Rightarrow A$ & K are ME

$$= \frac{2}{13}$$ i.e. Assertion is also True.

Two dice are tossed. One dice is regular and the other is biased with probabilities P(1) = P(6) = 1/6, P(2) = P(4) = 0 and P(3) = P(5) = 1/3. The probability of obtaining a sum of 4 is

(a) 1/9 (b) 1/12 (c) 1/18 (d) 1/24

Both Die are Ind.

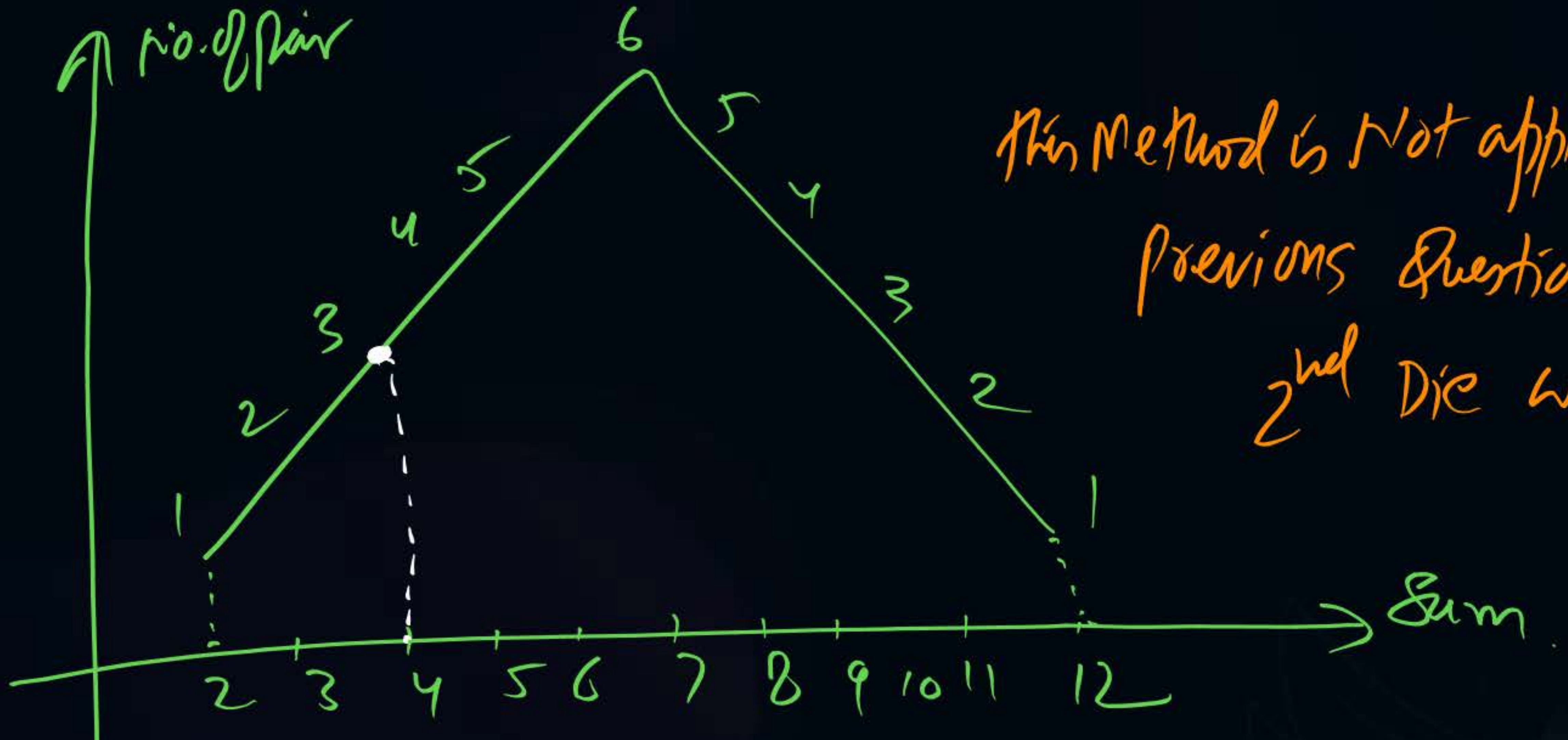
$D_R : P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$

$D_{Biased} : P(1) = P(6) = \frac{1}{6}, P(2) = P(4) = 0, P(3) = P(5) = \frac{1}{3}$

fav Cases = {Sum is 4} = {(1 3) or (2 2) or (3 1) or $\cancel{(4\ 0)}$ or $\cancel{(0\ 4)}$}

$$\text{Req Prob} = P[(1 \cap 3) \cup (2 \cap 2) \cup (3 \cap 1)]$$

$$= \left(\frac{1}{6} \times \frac{1}{3}\right) + \left(\frac{1}{6} \times 0\right) + \left(\frac{1}{6} \times \frac{1}{6}\right) = \frac{1}{18} + \frac{1}{36} = \frac{2+1}{36} = \frac{1}{12}$$



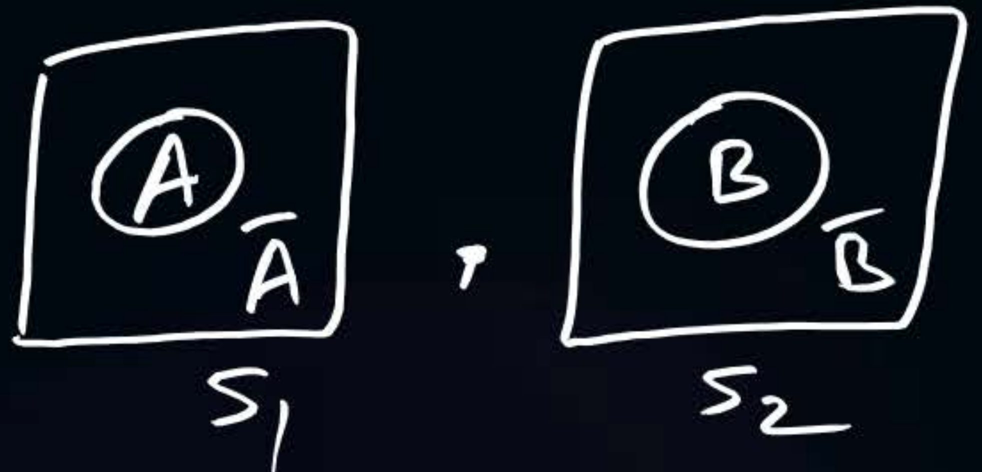
this method is Not applicable in previous Question ∵ 2nd Die was $\boxed{\text{BIASED.}}$

$$P(\text{Sum} = 4) = \frac{3}{36} = \frac{1}{12}$$

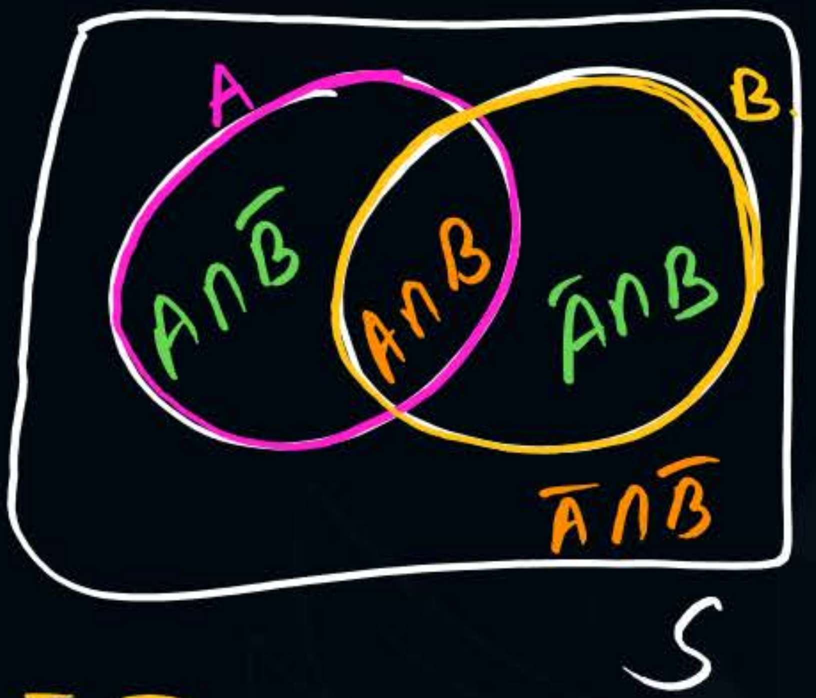
Concept of ME and Independency in a Single Question →

eg Two Persons A & B fire at the target once then write it's s.space ?

Sol:



$\Rightarrow$



A & B are Independent (T)

A & $\bar{B}$ " " (T)

$\bar{A}$ & B " " (T)

$\bar{A}$ & $\bar{B}$ " " (T)

$S = \{\bar{A}\cap\bar{B}, A\cap\bar{B}, \bar{A}\cap B, A\cap B\}$

$= \{E_1, E_2, E_3, E_4\}$

So E_1, E_2, E_3, E_4 are ME Events.

Analysis: Various possibilities are ; $S = \{\underbrace{\bar{A}\cap\bar{B}}_{=E_1}, \underbrace{A\cap\bar{B}}_{=E_2}, \underbrace{\bar{A}\cap B}_{=E_3}, \underbrace{A\cap B}_{=E_4}\}$

(None will hit) or (A hit & B missed) or (A missed & B hit) or (Both hit) = Total possibilities

$$(\bar{A}\cap\bar{B}) \cup \underbrace{(A\cap\bar{B}) \cup (\bar{A}\cap B)}_{\text{only one will hit / Exactly one will hit.}} \cup (A\cap B) = S$$

$$\underbrace{(A\cap\bar{B}) \cup (\bar{A}\cap B) \cup (A\cap B)}_{\text{At least one will hit.}}$$

$$E_1 \cup E_2 \cup E_3 \cup E_4 = S \Rightarrow P(E_1 \cup E_2 \cup E_3 \cup E_4) = P(S)$$

$\because E_1, E_2, E_3, E_4$ are ME so $\underbrace{P(E_1)+P(E_2)+P(E_3)+P(E_4)}_{\text{i.e } P(\text{something occurs}) = 1} = 1$

Thank

you

Keep Hustling!