CS & IT ENGINEERING

Algorithms

Analysis of Algorithms

Lecture No.- 03

By- Aditya sir

Recap of Previous Lecture

Topic

Topic

Background

Asymptotic Notations

Topics to be Covered

Topic

Topic

Topic

Asymptotic Notations

Practice.

About Aditya Jain sir

1. Appeared for GATE during BTech and secured AIR 60 in GATE in very first attempt - City topper
2. Represented college as the first Google DSC Ambassador.
3. The only student from the batch to secure an internship at Amazon. (9+ CGPA)
4. Had offer from IIT Bombay and IISc Bangalore to join the Masters program
5. Joined IIT Bombay for my 2 year Masters program, specialization in Data Science
6. Published multiple research papers in well known conferences along with the team
7. Received the prestigious excellence in Research award from IIT Bombay for my Masters thesis
8. Completed my Masters with an overall GPA of 9.36/10
9. Joined Dream11 as a Data Scientist
10. Have mentored 12,000+ students & working professions in field of Data Science and Analytics
11. Have been mentoring & teaching GATE aspirants to secure a great rank in limited time
12. Have got around 27.5K followers on Linkedin where I share my insights and guide students and professionals.

Telegram

@ADITYASIR_PW

Telegram Link for Aditya Jain sir: https://t.me/AdityaSir_PW

Topic : Asymptotic Notations

Types of Asymptotic Notations:

Asymptotic Notations

(L+T) (1) Big Notations

(i) UB : Big Oh → O

(ii) LB : Big Omega → Ω

(2) Small Notations (L)

(iii) UB : Small oh → o

(iv) LB : Small omega → ω

(v) Theta : θ → Tight Bound

(T)

$\boxed{O, \Omega, o, \omega, \theta}$

Topic : Asymptotic Notations

Let 'f' and 'g' be functions from the set g integers/real to real number;

1. **Big-Oh(O):** Upper Bound (UB)

- $f(n) = O(g(n))$ if there exists some constant $c > 0$ and $n_0 \geq 0$ such that $f(n) \leq c^*(g(n))$, whenever $n \geq n_0$.

Topic : Asymptotic Notations

Example:

(1) Order of Magnitude

$f(n) = n^2 + n + 1$

$1 + n \leq n^2 + n^2$

$n = 2 \qquad 1 + 2 \leq 2^2 + 2^2$

$n = 3 \qquad 1 + 3 \leq 3^2 + 3^2$

$1 \leq n^2$

$1 + n^2 \leq n^2 + n^2$

$1 + n + n^2 \leq n^2 + n^2 + n^2$

$1 + n + n^2 \leq 3n^2$

$f(n) \leq c * g(n)$

Hence,

$f(n) = O(g(n))$, $c > 3, n \geq n_0, n_0 \geq 1$

$1 + n + n^2 = O(n^2)$

Topic : Asymptotic Notations

- Whenever we determine the upper bound and lower bound, we should find that function 'g' which is closest to the given function.

Example:

$$\text{Nagpur} \xrightarrow[\text{(Non-stop)}]{\text{Flight}} \text{Delhi}$$

$$\text{UB}\begin{cases} \text{Loose Upper bound} \begin{bmatrix} P1 \rightarrow < 1 \text{ year} \\ P2 \rightarrow < 1 \text{ week} \end{bmatrix} \\ \text{Tight UB} \begin{bmatrix} P3 \rightarrow < 5 \text{ hr} \end{bmatrix} \end{cases}$$

Topic : Asymptotic Notations

Example2: Purchasing a Car

F1 : > 50 Rs
F2 : > 5000 Rs → Loose LB

Tight LB ← $\boxed{\text{F3 : > 5L Rs}}$ ✓

Topic : Asymptotic Notations

Shortcut:

Dominating term → Highest term with rate of growth with increasing value of n

Example:

(i) $f(n) = n^2 + n + 1$

$f(n) = O(n^2)$

(ii) $f(n) = 5n^3 + 8n + 7$

$f(n) = O(5n^3) = \underline{O(n^3)}$ ✓

Topic : Asymptotic Notations

(1) Step-count method

$$T(n) = 4n^2 + 8n + 8$$

$$T(n) = O(4n^2)$$

$$T(n) = O(n^2)$$

(2) Order of magnitude

$$T(n) = n^2 + n + 1$$

$$T(n) = O(n^2)$$

Topic : Asymptotic Notations

2. **Big-Omega(Ω):** Lower Bound (LB)

- $f(n) = \Omega(g(n))$ if there exists some constant 'c' and 'n_0' such that $f(n) \geq c^*(g(n))$, whenever $c > 0$, $n \geq n_0$, $n_0 \geq 0$

Topic : Asymptotic Notations

Example: $8n^2 + 3n + 5$

$f(n) = 1 + n + n^2 \geq 1 * n^2 \rightarrow \Omega(n^2)$

$f(n) = 1 + n + n^2 \geq 1 * n \rightarrow \Omega(n)$

$f(n) = n + n + n^2 \geq 1 * \sqrt{n} \rightarrow \Omega(\sqrt{n})$

$f(n) = n + n + n^2 \geq 1 * 1 \rightarrow \Omega(1)$

$\Omega(n^2)$

$\boxed{1 + n + n^2 \geq 1 * n^2}$

$f(n) \geq c^*(g(n))$

$f(n) = \Omega(g(n)) = \Omega(n^2)$

Hence,

$1 + n + n^2 = \Omega(n^2)$

Topic : Asymptotic Notations

3. Theta(θ): Tight Bound

- $f(n) = \theta(g(n))$ if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$

  $(c_1 . g(n)) \leq f(n) \leq (c_2 . g(n))$

If $f(n) = O(g(n))$

and

$f(n) = \Omega(g(n))$

$\Rightarrow f(n) = \theta(g(n))$

Topic : Asymptotic Notations

Example: $n^2 + n + 1$

$$1 * n^2 \leq \underline{1 + n + n^2} \leq 3 * n^2$$

$$(c_1 * g(n)) \leq f(n) \leq (c_2 * g(n))$$

$f(n) = O(n^2)$

and

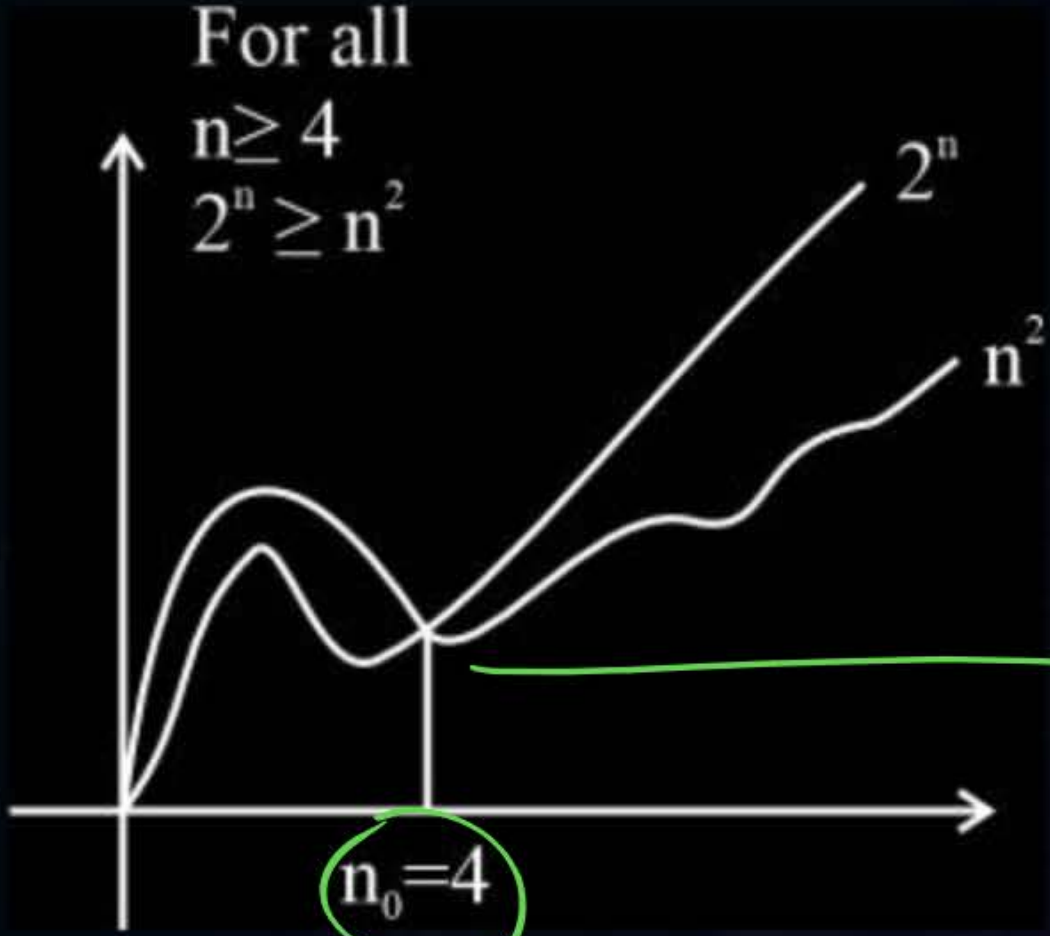
$f(n) = \Omega(n^2)$

$\Rightarrow$ $f(n) = \theta(n^2) \rightarrow$ Tight bound

Topic : Asymptotic Notations

Example: n^2 vs 2^n

n	n^2	2^n
1	1	2
2	4	4
3	9	8
4	16	16
5	25	32
6	36	64
7	49	128

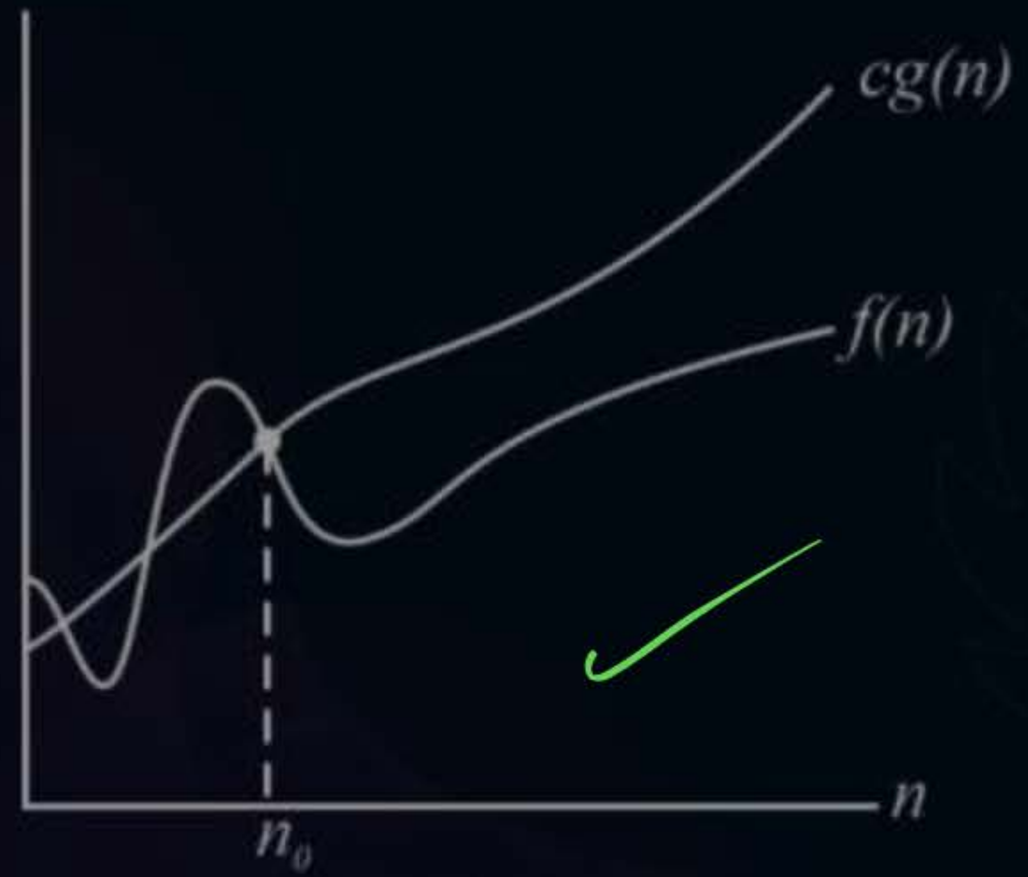


Topic : Asymptotic Notations

Big-Oh Notation:

$$f(n) \leq c * g(n);\ c > 0,\ n \geq n_0,\ n_0 \geq 0$$

$$f(n) = O(g(n))$$



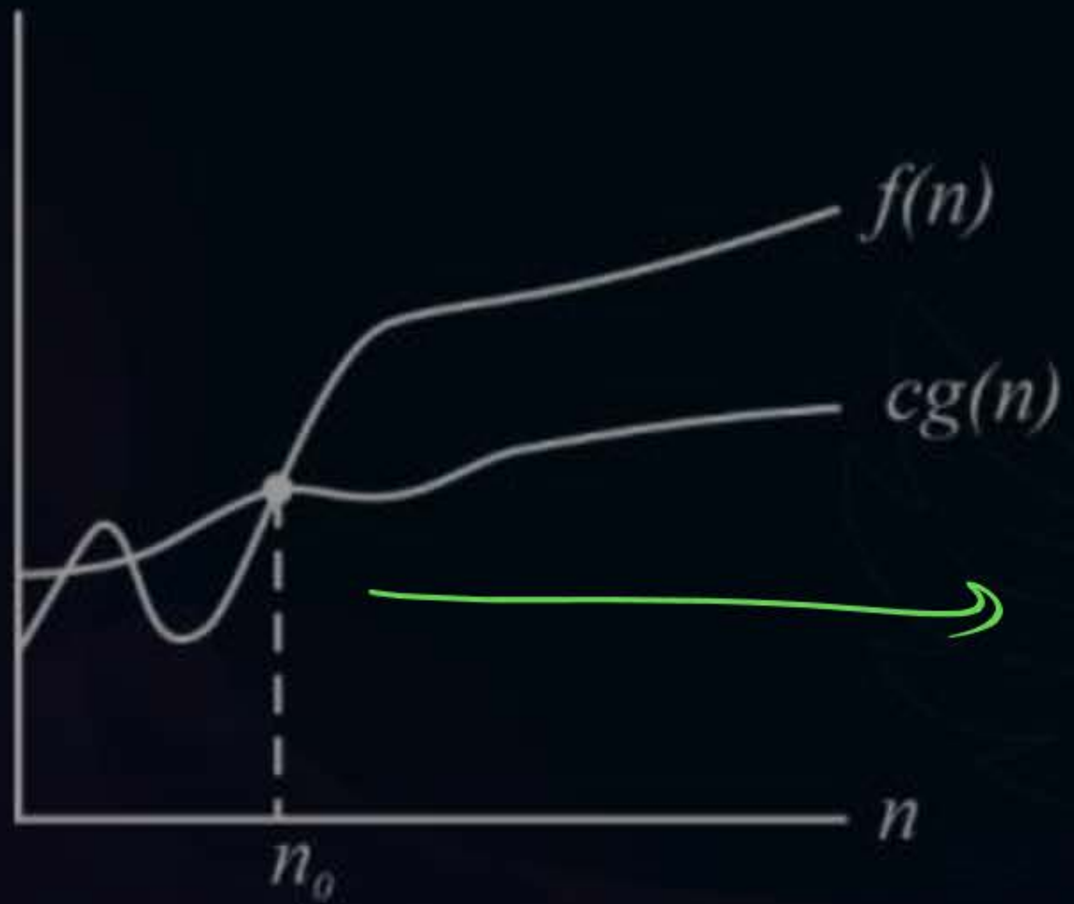
$f(n) = O(g(n))$

Topic : Asymptotic Notations

Big-Omega Notation:

$f(n) \geq c * g(n); c > 0, n \geq n_0, n_0 \geq 0$

$f(n) = \Omega(g(n))$



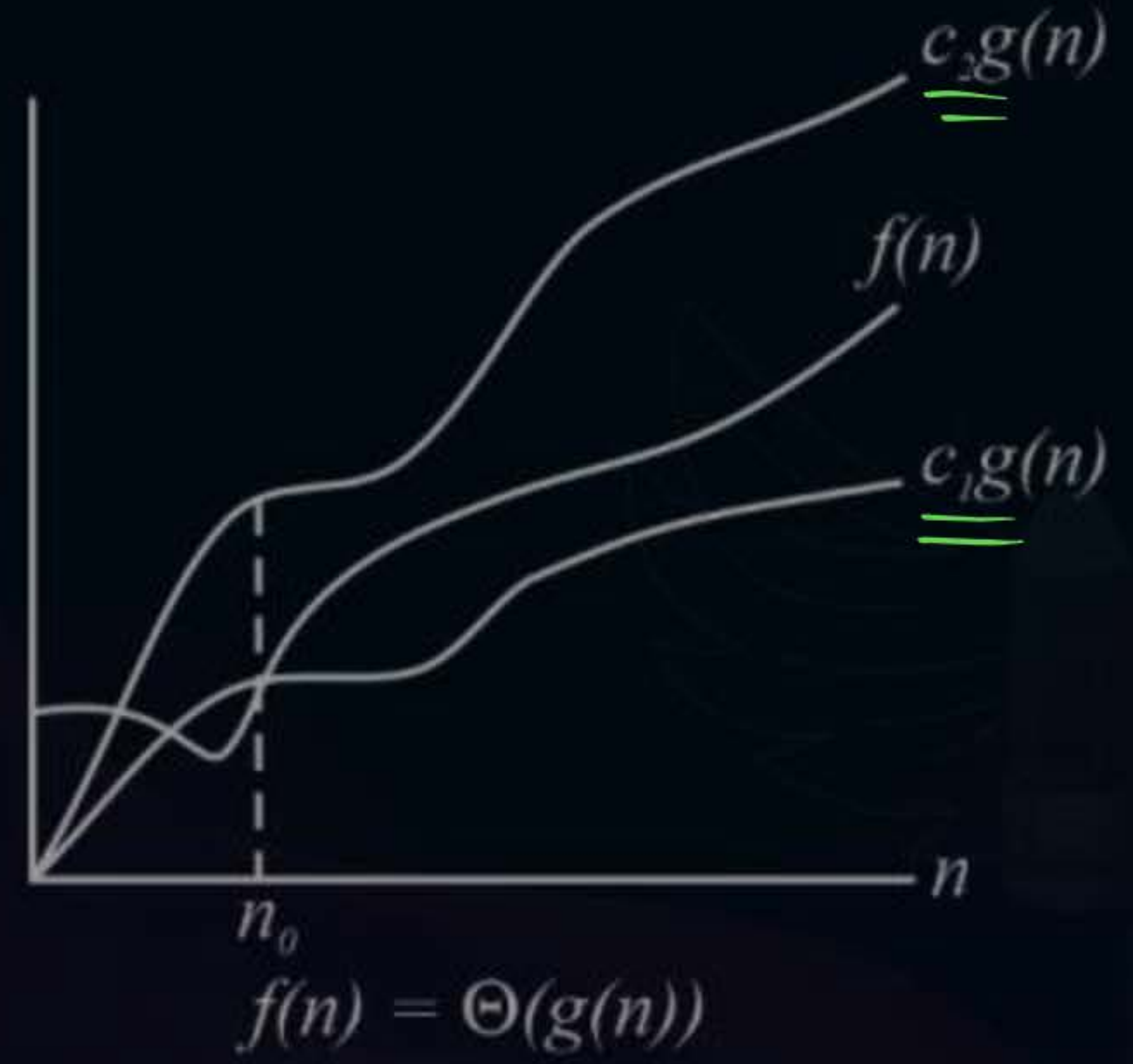
$f(n) = \Omega(g(n))$

Topic : Asymptotic Notations

Theta Notation:

$$(c_1 * g(n)) \le f(n) \le (c_2 * g(n));\ c_1 > 0,\ c_2 > 0,\ n \ge n_0,\ n_0 \ge 0$$

$$f(n) = \theta(g(n))$$



Topic : Asymptotic Notations

Summary:

Asymptotic Notations

Big | Small

O, Ω | θ | o, ω

Topic : Asymptotic Notations

Important observations:

(1) ~~Longer~~ Larger functions are always omega of the smaller functions.

$2^n = \Omega(n^2)$

$2^n = \Omega(n^3)$

e.g.: $n^3 \geq c * n^2$

$n^3 = \Omega(n^2)$

Topic : Asymptotic Notations

Important observations:

(2) Smaller functions are always order (Big Oh) of the larger functions.

$n^2 = O(2^n)$

$n^3 = O(2^n)$

e.g.: $n^2 \leq c * n^3$

$n^2 = O(n^3)$

Topic : Asymptotic Notations

Important observations:

(3) If two functions have equal rate of growth then they are theta of each other.

$f(n) = 8n^2$

$g(n) = 5n^2$

e.g.: $f(n) = \theta(g(n))$

or

$g(n) = \theta(f(n))$

Topic : Asymptotic Notations

Practice Questions:

1) $f(n) = 8n$

O

Ω

Θ

$8n \leq 10n \rightarrow O(n)$

$8n \geq 2n \rightarrow \Omega(n)$

$\Theta(n)$

Topic : Asymptotic Notations

Practice Questions:

(2) $f(n) = 9^{200}$

$9^{200} = \underline{\text{const}}$

$f(n) = O(1)$

$= \Omega(1) = \underline{\Theta(1)}$

Topic : Asymptotic Notations

Practice Questions:

(3) $f(n) = 100n^2 + 500\,n$

$n^2 > n$

$O(n^2)$

$\Omega(n^2)$

$> \Theta(n^2)$

Topic : Asymptotic Notations

Practice Questions:

(4) $f(n) = 100(\log n) + 50 * \sqrt{n}$

$O(\sqrt{n}), \Omega(\sqrt{n})$

$\Theta(\sqrt{n})$

$\log_2 n < \sqrt{n}$

$n = 64$

$\log_2 64 \quad \sqrt{64}$

$6 < 8$

Practice Questions:

(5) $f(n) = 500 * \sqrt{n} + 2n + 100$

$O(n)$

$\Omega(n)$

$\Theta(n)$

Topic : Exponentials

Important Properties:-

For all real a> 0, m, n

$a^0 = 1$

$a^1 = a$

$a^{-1} = 1/a$

$(a^m)^n = a^{mn}$

$(a^m)^n = (a^n)^m$

$a^m * a^n = a^{m+n}$

$$a^{-2} = \frac{1}{a^2}$$

Topic : Exponentials

$(2^m)^n = a^{m*n}$

e.g.: $(2^3)^2 \longrightarrow (2^3)^2 = (2^2)^3$

$(2^3)^2 \downarrow (8)^2 \downarrow 64$

$(2^3)^2 \downarrow 2^{3 \times 2} \downarrow (2^6) \downarrow 64$

$(2^2)^3 \downarrow 4^3 \downarrow 64$

Topic : Exponentials

$a^m \times a^n = a^{(m+n)} \neq a^{(m \times n)}$

e.g.: $2^3 \times 2^2 = 2^{(3+2)}$

$8 \times 4 = 2^5$

$32 = 32$

Topic : Analysis of Algorithms

Logarithmic Properties:

$\log x^y = y \log x$

$\log xy = \log x + \log y$

$\log \log n = \log (\log n)$

$a^{\log_b^x} = x^{\log_b^a}$

$a = b^{\log_b a}$

$\log_b a^n = n.\log_b a$

$\log_b a = \frac{1}{\log_a b}$

$b^{\log_b^a}$

$\log n = \log_{10}^n$

$\log^k n = (\log n)^K$

$\log \frac{x}{y} = \log x - \log y$

$\boxed{\log_b^x = \frac{\log_a^x}{\log_a^b}}$

$\log_c(ab) = \log_c a + \log_c b$

$\log_b^{(1/a)} = -\log_b a$

$a^{\log_b c} = c^{\log_b a}$

Topic : Asymptotic Notations

Example: $\log_2 (16)$

$\Rightarrow \quad \log_2(4 \times 4) = \log_2 4 + \log_2 4 = 2 + 2 = 4$

$\Rightarrow \quad \log_2(2^4) = 4 \log_2 2 = 4$

Topic : Asymptotic Notations

Logarithmic Properties:

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$$

$$\log(a * b) = \log(a) + \log(b)$$

$$\Rightarrow \log_b(a) = \frac{\log_c(a)}{\log_c(b)} = \frac{1}{\frac{\log_c(b)}{\log_c(a)}} = \frac{1}{\log_a(b)}$$

Topic : Asymptotic Notations

Logarithmic Properties:

$$\log^2(n) = [\log(n)]^2 = (\log n) * (\log n)$$

$$\log_b\left(\frac{1}{a}\right) = \log_b(1) - \log_b(a)$$

$$= 0 - \log_b(a)$$

$$= -\log_b a$$

$$\log(x/y) = \log x - \log y$$

AJ Sir GATE $\Longrightarrow$ $77.33/100 \longrightarrow$ AIR 60

-0.66

Topic : Geometric Sum Formula

1) The geometric sum formula for finite terms is given as:

- If $r = 1$: $S_n = n \cdot a$
- If $|r| < 1$: $S_n = \frac{a(1-r^n)}{1-r}$
- If $|r| > 1$: $S_n = \frac{a(r^n-1)}{r-1}$

Where:

- a is the first term
- r is the common ratio
- n is the number of terms

Topic : Asymptotic Notations

Geometric Progression:

$$S = 2^1 + 2^2 + 2^3 + 2^4 = 2 + 4 + 8 + 16 = 10 + 20 = 30$$

Or, $r = 2$ and $n = 4$, $a = 2$

$$\text{Sum} = \frac{a(r^n - 1)}{(r - 1)} = \frac{2(2^4 - 1)}{(2 - 1)} = \frac{2(16 - 1)}{1} = 30$$

Topic : Asymptotic Notations

Example:

$$\sum_{i=1}^{n} 2^i = 2^1 + 2^2 + \cdots + 2^n$$

n terms, $a = 2^1 = 2$, $r = 2$

$$S_n = \frac{a(r^n - 1)}{(r - 1)} = \frac{2(2^n - 1)}{2 - 1} = 2(2^n - 1)$$

Topic : Asymptotic Notations

Example:

$$\sum_{i=1}^{n} \frac{1}{3^i} = \frac{1}{3^1} + \frac{1}{3^2} + \cdots + \frac{1}{3^n}$$

n terms, $a = 1/3^1 = 1/3$, $r = 1/3$

$$S_n = \frac{a(1-r^n)}{(1-r)} = \frac{\frac{1}{3}\left(1 - \frac{1}{3}^n\right)}{1 - \frac{1}{3}} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right) \checkmark$$

Topic : Geometric Sum Formula

2. The geometric sum formula of infinite terms is given as:

if $|r| < 1$ $\qquad S_\infty = \frac{a}{1-r}$

if $|r| > 1$, the series does not converge and it has no sum.

Topic : Analysis of Algorithms

Arithmetic series

$$\sum_{k=14}^{n} k = 1 + 2 + \ldots + + n = \frac{n(n+1)}{2}$$

Geometric series

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 \ldots + x^n = \frac{x^{n+1} - 1}{x - 1} (x \neq 1)$$

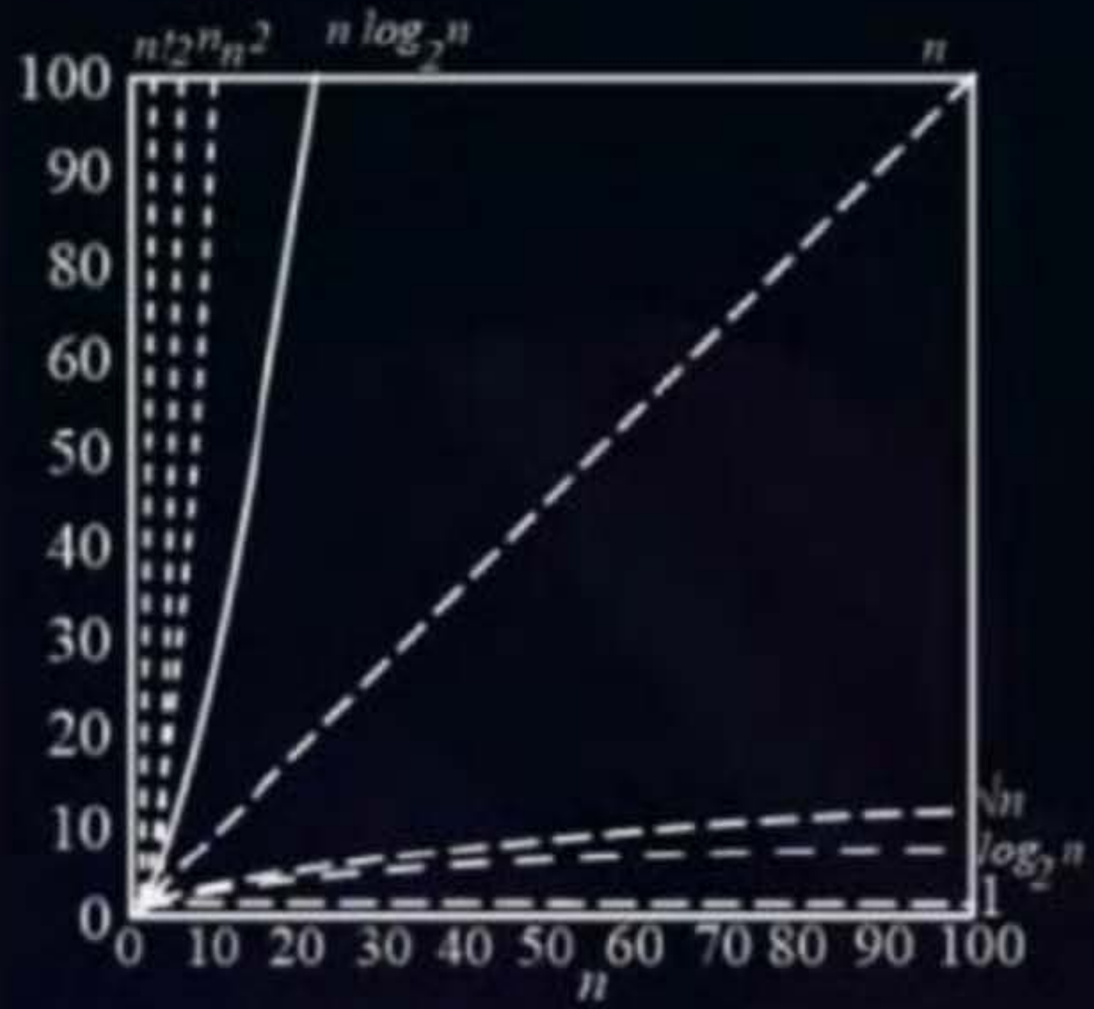
Harmonic series

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \approx \log n$$

Topic : Analysis of Algorithms

Dominance Functions Relation:-

Decreasing < Constant < logarithmic < polynomial < exponential



Topic : Analysis of Algorithms

Practice Questions:

(1) $f(n) = \sum_{a=1}^{n} a$

$1+2+3 \cdots n$

$= \frac{n(n+1)}{2} = \frac{n^2+n}{2} = \underline{\underline{O(n^2)}}$

Practice Questions:

(2) $f(n) = \sum_{a=1}^{n} a^2$

$= 1^2 + 2^2 \cdots\cdots n^2$

$= \left[\dfrac{n(n+1)(2n+1)}{6} \right] = \Theta(n^3)$

Practice Questions:

(3) $f(n) = \sum_{a=1}^{n} a^3$

$= 1^3 + 2^3 + 3^3 \cdots\cdots n^3$

$= \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4} = \frac{n^2(n^2+2n+1)}{4} = \Theta(n^4)$

Topic : Analysis of Algorithms

Practice Questions:

(4) $f(n) = \sum_{a=1}^{n} 1$

$= \underbrace{1+1+1 \cdots +1}_{n \text{ times}} = n$

$= \underline{\Theta(n)}$

Topic : Analysis of Algorithms

Practice Questions:

(5) $f(n) = \sum_{a=1}^{n} 3^a$

$= 3^1 + 3^2 \cdots 3^n$

$a = 3$

$r = 3$

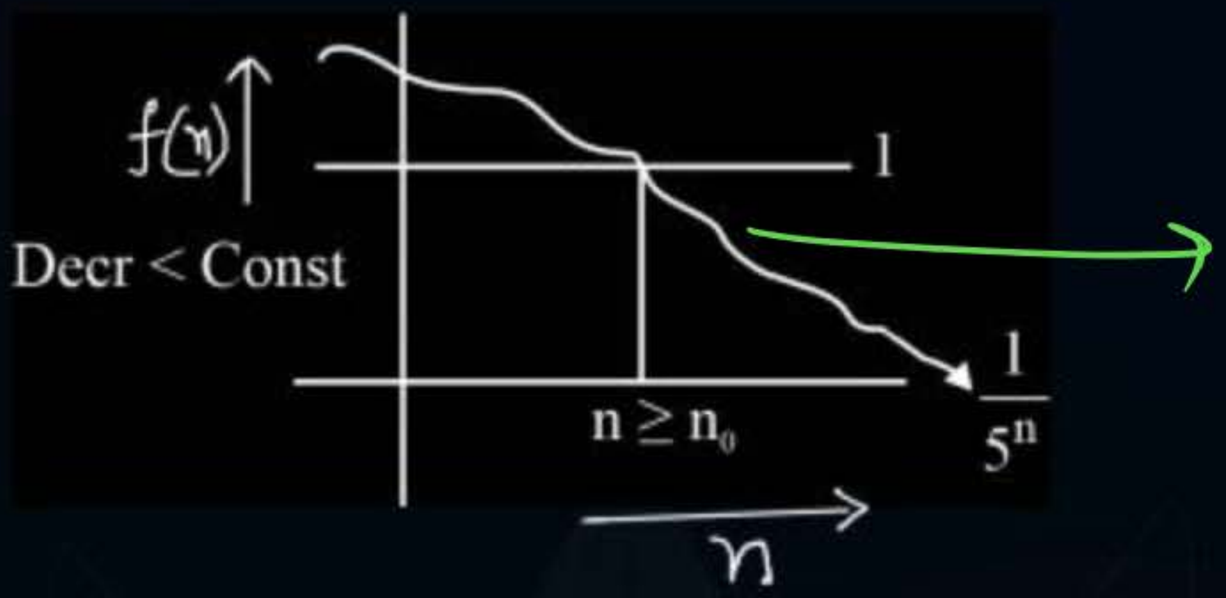
$n = n$

$= \frac{a(r^n - 1)}{r - 1} = \frac{3(3^n - 1)}{(3-1)} = \frac{3}{2}(3^n - 1) = \Theta(3^n)$

Practice Questions: V. V. imp

(6) $f(n) = \sum_{a=1}^{n} \left(\frac{1}{5}\right)^a$

$= \Theta(1)$



$= \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} \cdots \frac{1}{5^n}$

$a = 1/5$
$r = 1/5$
$n = n$

$\left(\frac{a(1-r^n)}{1-r}\right) = \frac{\frac{1}{5}\left(1 - \frac{1}{5^n}\right)}{1 - 1/5} = \frac{1}{4}\left(1 - \frac{1}{5^n}\right)$

Practice Questions:

(7) $f(n) = \sum_{a=1}^{n} n$

$$= \underbrace{n + n + n \cdots + n}_{n \text{ times}}$$

$$= n * n = n^2 \quad = \Theta(n^2)$$

THANK - YOU