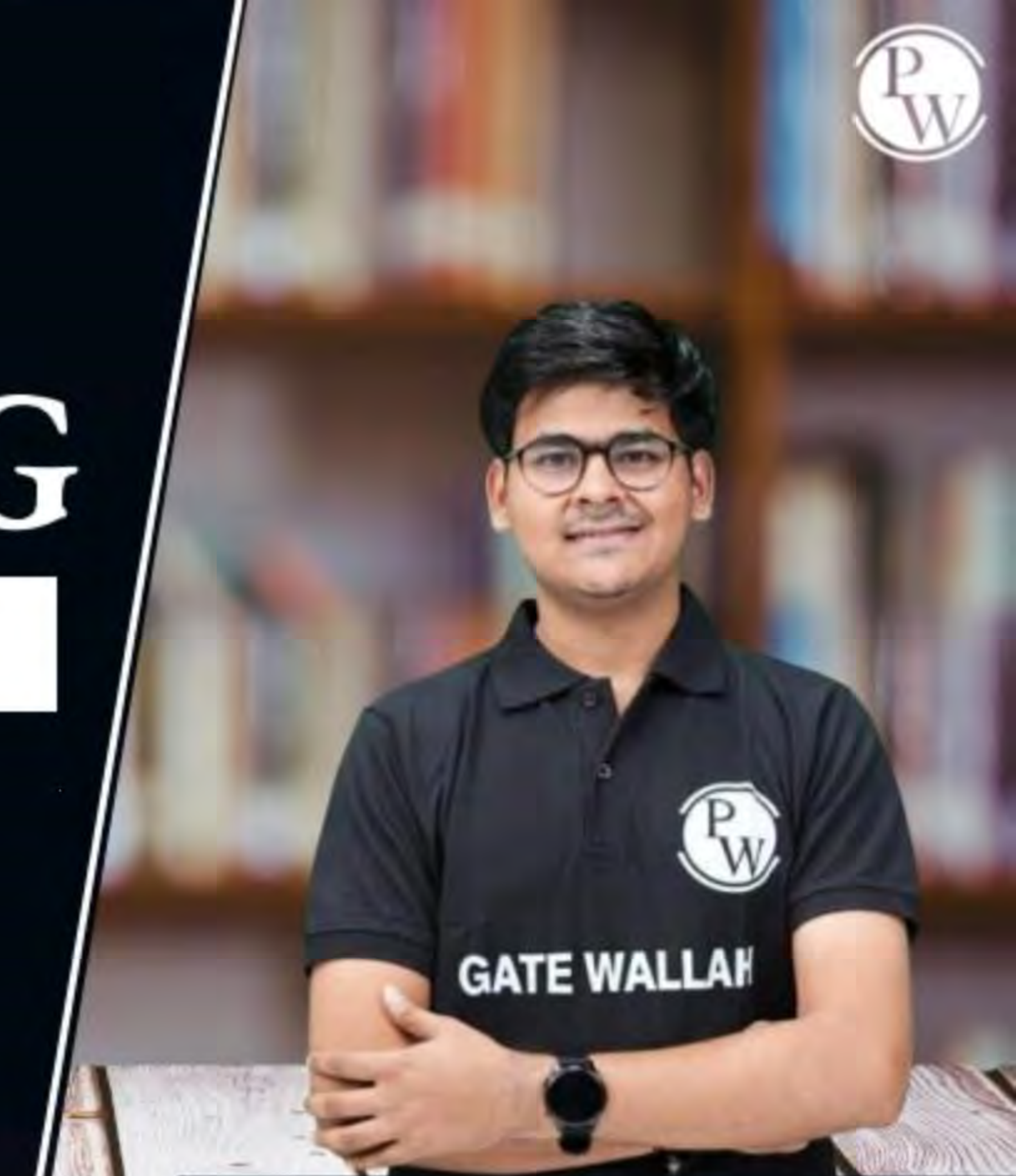
CS & IT ENGINEERING

Algorithms

Analysis of Algorithms

Lecture No.- 04



By- Aditya sir

Recap of Previous Lecture

Topic Big Notations (O, Ω, Θ)

Topic Problem Solving

Topics to be Covered

Topic Small Natation's

Topic Properties of Asymptotic Notations

Topic Problem solving

About Aditya Jain sir

1. Appeared for GATE during BTech and secured AIR 60 in GATE in very first attempt - City topper
2. Represented college as the first Google DSC Ambassador.
3. The only student from the batch to secure an internship at Amazon. (9+ CGPA)
4. Had offer from IIT Bombay and IISc Bangalore to join the Masters program
5. Joined IIT Bombay for my 2 year Masters program, specialization in Data Science
6. Published multiple research papers in well known conferences along with the team
7. Received the prestigious excellence in Research award from IIT Bombay for my Masters thesis
8. Completed my Masters with an overall GPA of 9.36/10
9. Joined Dream11 as a Data Scientist
10. Have mentored 12,000+ students & working professions in field of Data Science and Analytics
11. Have been mentoring & teaching GATE aspirants to secure a great rank in limited time
12. Have got around 27.5K followers on Linkedin where I share my insights and guide students and professionals.

Telegram

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Topic : Analysis of Algorithms

1.) Big Oh (UB)

$f(n) = O(g(n))$

If $f(n) \le c^*g(n)$, some $c > 0$, $n \ge n_0$, $n_0 \ge 1$

2.) Big Omega (Ω)

$f(n) = \Omega(g(n))$

If $f(n) \ge c^*g(n)$, some $c > 0$, $n \ge n_0$, $n_0 \ge 1$

3.) Theta (θ)

$C1^*(g(n)) \le f(n) \le C2^*(g(n))$

$f(n) = \theta(g(n)) \longrightarrow C1^*(g(n)) \le f(n) \le C2^*(g(n))$

$f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

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$$f(n)=\prod_{i=1}^{n}(1)=\underbrace{1*1*1*1\cdots*1}_{\text{(n times)}}=1 \Rightarrow O(1) \rightarrow \textit{Constant TC}$$

Summation

$$\sum_{i=1}^{n} 1 \Rightarrow \underbrace{1+1+1+1+\cdots+1}_{\text{(n times)}} = n$$

$$\Rightarrow n$$

Σ

Π

Topic : Analysis of Algorithms

#Q. $f(n)=\prod_{i=1}^{n}(i)$

A. f(n)= O(n) ✗

B. f(n) = O(n^2) ✗

C. f(n) = O(n^3) ✗

D. f(n) = O(n^n) ✓

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$$f(n) = \prod_{i=1}^{n} (i) = 1 \times 2 \times 3 \times 4 \ldots\ldots n$$

$$= \boxed{n!}$$

$$f(n) = O(n!)$$

$$= O(n^n)$$

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$n! \text{ vs } n^n$

$n \leq n \quad , n \geq 2$

$n*(n-1) * (n-2) \leq n * n * n$

$n * (n-1) \ldots\ldots 1 \leq n*n*n* \ldots\ldots\ldots\ldots *n$ (n times)

$n! \leq (n^n)$

$\boxed{n! = O(n^n)}$

$n! < n^n$

$n! = O(n^n)$

Is $n! = \Omega(n^n)$?

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#Q. Is $n! = \Omega(n^n)$?

A True

B False

Imp. Note

$n! = O(n^n)$

and $n! \neq \Omega(n^n)$

Hence, $n! \neq \Theta(n^n)$

$n*(n-1) \times (n-2) \ldots\ldots\ldots 1 \geq c * n^n$?

Never

$f(n) \geq c*g(n) \rightarrow \times$

$$\boxed{n! \neq \Omega(n^n)}$$

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#Q. $f(n)=\prod_{i=1}^{n}\left[\log(i)\right]$

$= \log(1) * \log(2) \text{ ....} \log(n)$

$= 0 * \log(2) \text{ ....} \log(n)$

$= 0 \rightarrow \boxed{O(1)}$

A) f(n)= O(logn)

B) f(n) = ~~o(logn)~~ O(1)

C) f(n) = Ω (logn)

D) f(n) = ~~ω(logn)~~ Ω(n)

37.5%

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#Q. $f(n) = \sum_{i=1}^{n} \log(i)$

A $\cancel{f(n) = O(\log n)}$ $f(n) = O(n\log n)$

B $\cancel{f(n) = o(\log n)}$ $f(n) = O(1)$

C $\cancel{f(n) = \Omega(\log n)}$ $f(n) = \Omega(n^2)$

D $\cancel{f(n) = \omega(\log n)}$ $f(n) = \Omega(n^n)$

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Sol. $f(n) = \sum_{i=1}^{n} [\log(i)]$

$= \log(1) + \log(2) + \log(3) \ldots\ldots + \log(n)$

$= \log(1*2*3*\ldots\ldots\ldots*n)$

$f(n) = \log(n!)$.

$= \log(n^n)$

$n! = O(n^n)$

$= n \log n$

$\log a + \log b = \log(ab)$

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Method:- **1.**

Logic

$\text{Log}(n) \leq \log(n)$

$\log(n) + \log(n-1) \leq \log(n) + \log(n)$

$\log(n) + \log(n-1) \ldots\ldots\log(1) \leq \log(n) + \log(n)\ldots\ldots+\log(n) \rightarrow$ n times

$\log(n) + \ldots\log(1) \leq n^* \log(n)$

$\boxed{\log(n!) \leq n^*\log(n)}$ $\Rightarrow \log(n!) = O(n \log n)$

$\log(n!) < n\log n$

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Method:- **2.**

Using Stirling's Approximation

$\Downarrow$

$$n! \approx \sqrt{2\pi n} * \left(\frac{n}{e}\right)^n$$

$$f(n)=\sum_{i=1}^{n}\log(i)=\cancel{\log(n)}\ \log(n!)$$

$$=\log\left(\sqrt{2\pi n}\,(n/e)^{n}\right)\rightarrow \text{using Stirling's Approx.}$$

$$=\log\left(\sqrt{2\pi}\right)+\frac{1}{2}\log(n)+n\left[\log(n)-\log(e)\right]$$

$$=\left[\log\left(\sqrt{2\pi}\right)+\frac{1}{2}\log(n)+n*\log(n)-n*\log(e)\right]$$

$$\boxed{\begin{array}{l} f(n)=O(n\log n)\\ f(n)=\Omega(n\log n)\end{array}} \longrightarrow \boxed{f(n)=\theta(n\log n)}$$

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PYQ

75.44

#Q. $f(n) = \sum_{i=1}^{n} (i^3) = x$, Choice for x is

I. $\theta(n^4)$ ✓ II. $\theta(n^5)$ ✗ III. $O(n^5)$ ✓ IV. $\Omega(n^3)$ ✓

A I, II, II

B II, III, IV

C I, II, III, IV

D I, III, IV ✓

D

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n^2+2n+1)}{4} = \left(\frac{n^4+2n^3+n^2}{4}\right) = x$$

$$O(n^4)$$

$$\Omega(n^4)$$

$$\Omega(n^3)$$

$$\Theta(n^4)$$

Topic : Analysis of Algorithms

The big natation (O, Ω) provides the upper bounded & Lower Bound that may or may not be light.

Tight LB:

(Can be as well as loose bounded)

E.g.1.

Tight LB $\leftarrow \Omega(n) \leftarrow n+10 \rightarrow O(n) \rightarrow$ Tight UB

Loose LB $\left\{ \begin{array}{l} \Omega(\sqrt{n}) \\ \Omega(1) \end{array} \right.$ $\qquad \left. \begin{array}{r} O(n^2) \\ O(n^3) \end{array} \right\} \rightarrow$ Loose UB

2. Small notations always provide bounds that are loose bounds.

↓ (o, ω)

(never tight bound)

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1. Small - Oh Notation → Loose upper Bound

$\Rightarrow f(n)$ is $o(g(n))$ if,

$\Rightarrow f(n) < c^*(g(n))$, for all $c > 0$

When ever $n \geq n_0$

for some $n_0 \geq 0$

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E.g.2. Diff. between o and O

$n < c^*n$

if $c = 1, n < 1 \, ^*n$ ✗

Hence, $n \neq o(n)$

1. $f(n) = n^2 + n + 5$

   $O(n^2) \rightarrow$ Tight Bound

   $o(n^2)$? ✗

$n \leq 3 \; ^* n = O(n)$

$n = O(n^2)$

$n = O(n^3)$

$f(n) = O(n^3)$ ✓

$= O(n^2)$ ✓

$= o(n^2)$ ✗

$= o(n^3)$ ✓

$= o(n\sqrt{n})$ ✗

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$n\sqrt{n}$	n^2
$\cancel{n} * \sqrt{n}$	$\cancel{n} * n$
$\sqrt{n}$ <	n

$n\log n$	n^2
$\cancel{n}\log n$	$\cancel{n} * n$
$\log n$ <	n

2. Small Omega (ω) Notation → Loose lower Bound

⇒ $f(n)$ is $\omega(g(n))$ if,

⇒ $f(n) > c^* g(n)$, For all $c > 0$, whenever $n \geq n_0$, $n_0 >= 0$

Eg.3. Big Omega(Ω) vs small Omega (ω)

$$f(n) = n \Rightarrow f(n) = \Omega(n) \checkmark$$

$$= \Omega(\sqrt{n}) \checkmark$$

$$= \Omega(n^2) \times$$

$$= \omega(n) \times$$

$$= \omega(\sqrt{n}) \checkmark$$

$$= \omega(1) \checkmark$$

Topic : Analysis of Algorithms

Imp. Practice Questions:-

#Q. $2(2^n) = O(2^n)$

$2^{n^2} = O(2^n)$? ✗

$n^2 > n$

$2^{n^2} > 2^n$

$2 \times 2^n = O(2^n)$

True

Imp. Practice Questions:-

#Q. ~~$2(2^n) = O(2^n)$~~

$$2^{(2n)} = O(2^n) \; ?$$

False

$$2^{2n} = 2^{(n+n)}$$

$$= 2^n \times 2^n > 2^n$$

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Imp. Practice Questions

#Q. $2^{n+1} = O(2^n)$

$2^{n+1} = 2^n \times 2^1 = \underline{2 \times 2^n}$ $\underline{\text{True}}$

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Imp. Practice Questions

#Q. If $0 < a < b$ then $n^a = O(n^b)$

True

$n^2 = O(n^3)$

#Q. $2^{\left(n^2\right)} = O(n!)$

$2^{(n^2)} = O(n!) \rightarrow$ False

$2^{n^2} > n^n$

$$2^{n^2} > n^n$$

Take log both sides

$$\log(2^{n^2}) \qquad \log(n^n)$$

$$n^2 \log_2 2 \qquad n * \log_2 n$$

$$n^2 \qquad n \log n$$

$$\cancel{n} * n \qquad \cancel{n} \log n$$

$$n > \log n$$

[MCQ]

Q#. Consider the following functions from positive integer to real numbers

$10, \sqrt{n}, n, \log_2 n, \frac{100}{n}$

The correct arrangement of the above functions in increasing order of asymptotic complexity ?

A $\log_2 n, \frac{100}{n}, 10, \sqrt{n}, n$

B $10, \frac{100}{n}, \sqrt{n}, \log_2 n, n$

C $\frac{100}{n}, 10, \log_2 n, \sqrt{n}, n$

D $10, \frac{100}{n}, \sqrt{n}, n, \log_2 n$

Topic : General Properties of Big Oh Notation

Let $d(n)$, $e(n)$, $f(n)$, and $g(n)$ be functions mapping nonnegative integers to nonnegative reals. Then

1. If $d(n)$ is $O(f(n))$, then $\cancel{ad(n)}$ $a*d(n)$ is $O(f(n))$, for any constant $a > 0$
2. If $d(n)$ is $O(f(n))$ and $e(n)$ is $O(g(n))$, then $d(n) + e(n)$ is $O(f(n) + g(n))$.
3. If $d(n)$ is $O(f(n))$ and $e(n)$ is $O(g(n))$, then $d(n)*e(n)$ is of $O(f(n)*g(n))$
4. If $d(n)$ is $O(f(n))$ and $f(n)$ is $O(g(n))$, then $d(n)$ is $O(g(n))$.
5. If $f(n)$ is a polynomial of degree d (that is, $f(n) = (a_0 + a_1 n + ... + a_d n^d)$ then $f(n)$ is $O(n^d)$, $n > 1$.
6. n^x is $O(a^n)$ for any fixed $x > 0$ and $a > 1$
7. $\log(n^x)$ is $O(\log n)$ for any fixed $x > 0$
8. $\log^x n$ is $O(n^y)$ for any fixed constants $x > 0$ and $y > 0$

Topic : General Properties of Big Oh Notation

1. If $d(n) = O(f(n))$,

   then $a * d(n) = O(f(n))$, $a > 0$

e.g.: $d(n) = n^2 \quad \rightarrow \quad O(n^2)$

$a = 10$

$a * d(n) = 10 * n^2 \quad \rightarrow \quad O(n^2)$

Topic : General Properties of Big Oh Notation

2. $d(n) = 5n + 2 \rightarrow \quad O(n)$

$e(n) = 10n^2 + 7n + 9 \rightarrow \quad O(n^2)$

$\underline{d(n) + e(n)} = (10n^2 + 7n + 9) + (5n + 2) \quad \Rightarrow O(f(n) + g(n))$

$= (10n^2 + 12n + 11) \quad \Rightarrow O(n + n^2)$

$= \underline{\underline{O(n^2)}} \quad \Rightarrow \underline{\underline{O(n^2)}}$

Shortcut: $O(f(n) + g(n)) = \underline{\underline{O(\max(f(n), g(n)))}}$

Topic : General Properties of Big Oh Notation

3. $d(n) = 5n \rightarrow O(n)$

$e(n) = 10n^2 + 2 \rightarrow O(n^2)$

$d(n) * e(n) = (10n^2 + 2) * (5n)$

$= (50n^3 + 10n)$

$= O(n^3)$

$\Rightarrow O(f(n) * g(n))$

$\Rightarrow O(n * n^2)$

$\Rightarrow O(n^3)$

Topic : General Properties of Big Oh Notation

4. $d(n) = O(f(n))$ and $f(n) = O(g(n)) \Rightarrow d(n) = O(g(n))$

e.g.:

$d(n) = O(n^2)$

and $\overset{n^2}{\cancel{f(n)}} = O(n^4)$

then $\underline{d(n) = O(n^4)}$

$f \leq g \leq h \Rightarrow f \leq h$

Topic : General Properties of Big Oh Notation

5. $f(n) = 5 + 10n + 15n^3 + 7n^4$

$f(n) = O(n^4)$

Topic : General Properties of Big Oh Notation

6. $n^x = O(a^n)$, for any $x > 0$ & $a > 1$

$n^x, x > 0 \rightarrow$ Polynomial

$a^n, a > 1 \rightarrow$ Exponential

Poly < Expo

Poly = O(Expo)

Topic : General Properties of Big Oh Notation

7. $f(n) = \log(n^x) = x * \log(n) = O(\log n)$

$f(n) = \log(n^3) = 3 * \log(n)$

$f(n) = \log(n^6) = 6 * \log(n)$

$f(n) = \log(n^{10}) = 10 * \log(n)$

$f(n) = \log(n^{100}) = 100 * \log(n)$

$= O(\log n)$

Topic : General Properties of Big Oh Notation

8. $(\log n)^x$ or $\log^x(n) = O(n^y), x > 0, y > 0$

Logarithmic < Polynomial

$\log = O(\text{poly})$

Topic : General Properties of Big Oh Notation

Practice Question (T/F):

(1) If $0 < x < y$, then $n^x = O(n^y)$

True

$0 < 2 < 3$

$n^2 = O(n^3)$ ✓

Topic : General Properties of Big Oh Notation

Practice Question (T/F):

(2) log (n) is Ω(1/n)

True

$$\left(\frac{1}{2} > \frac{1}{3} > \frac{1}{4}\right)$$

$$\log n > \frac{1}{n}$$

Topic : General Properties of Big Oh Notation

Practice Question (T/F):

(3) 2^{n^2} is $O(n!)$

False

$2^{n^2} > n^n$

$\log()$

$n^2 \quad n\log n$

$n > \log n$

Topic : General Properties of Big Oh Notation

Practice Question (T/F):

(4) $20*n*\log n = O(n \log n)$

True

Topic : General Properties of Big Oh Notation

Practice Question (T/F):

(5) $(n+c)^k \neq O(n^k)$ for some $k > 0$, $c > 0$

False

$(n+c)^k \longrightarrow \Theta(n^k)$

$(n+2)^2 = n^2 + 4n + 4$

$= O(n^2)$

Topic : General Properties of Big Oh Notation

Practice Question (T/F):

(6) n^2 is $O(2^{(\log n)})$

False

$n^2 > 2^{\log n}$

$\log(\,)$

maths

$\boxed{2\log n > \log n}$

Topic : Adding Functions

The sum of two functions is governed by the dominant one, namely:

$O(f(n)) + O(g(n)) \rightarrow O\ (\max(f(n), g(n)))$

$\Omega(f(n)) + \Omega(g(n)) \rightarrow \Omega(\max(f(n), g(n)))$

$\theta(f(n)) + \theta(g(n)) \rightarrow \theta(\max(f(n), g(n)))$

Topic : Adding Functions

Example:

$$f(n) = 5n^2 + 2 \quad \rightarrow \quad O(n^2)$$

$$g(n) = 10n^3 \quad \rightarrow \quad O(n^3)$$

$$O(f(n)) = n^2 \text{ and } O(g(n)) = n^3$$

$$\begin{aligned} n^2 + n^3 &= O(\max(5n^2 + 2, 10n^3)) \\ &= O(10n^3) \\ &= O(n^3) \end{aligned}$$

Topic : Multiplying Functions

$O(f(n)) * O(g(n)) \rightarrow O(f(n) * g(n))$

$\Omega(f(n)) * \Omega(g(n)) \rightarrow \Omega(f(n) * g(n))$

$\theta(f(n)) * \theta(g(n)) \rightarrow \theta(f(n) * g(n))$

Topic : General Properties of Big Oh Notation

Imp. Practice Question (T/F):

(1) $n^2 = O(2^{(2 \log n)})$

Topic : General Properties of Big Oh Notation

Imp. Practice Question (T/F):

(2) $(\log n)^{1/2} = O(\log(\log n))$

Topic : General Properties of Big Oh Notation

Imp. Practice Question (T/F):

(3) $a^n \neq O(n^x)$, for $a > 1, x > 0$

THANK - YOU