DATA SCIENCE
&
ARTIFICIAL INTELLIGENCE

& CS / IT

Calculus and Optimization

Lecture No. 07

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Recap of previous lecture

Topic

MEAN VALUE THEOREMS

Topics to be Covered

Topic

DERIVATIVES & their Types

(Part-1)

→ ordinary Derivative
→ Partial Derivative

→ **Curve** & **Surface**

$y = f(x)$ $\quad$ $z = f(x,y)$

eg $x^2+y^2+z^2=9$

→ **Explicit func^n** & **Implicit func^n**

$y = f(x)$ curve $\quad$ ✓ $f(x,y) = c$ curve

$z = f(x,y)$ surface $\quad$ $f(x,y,z) = c$ surface

Explicit func^n: If it is possible to separate Dependent and Independent Variables then func^n is called Explicit func^n for eg. $y = f(x)$ E. Curve
$z = f(x, y)$ E. Surface

eg $x^3 + y^3 + 4xy^3 = 5$

$$(1+4x)y^3 = 5 - x^3 \Rightarrow y = \left(\frac{5 - x^3}{1+4x}\right)^{1/3} \text{ i.e. } y = f(x)$$

Implicit func^n → If it is not possible to separate Dep and Ind Variables then func^n is called Implicit func^n. eg $f(x, y) = c$, I Curve
$f(x, y, z) = c$, I. Surface

$x^3 + y^3 + 3xy = 1$ i.e. $f(x, y) = c$

Types of Questions

(1) Based on ordinary Derivative exist in case of curve $y = f(x)$

(2) " " Partial Derivative " " " of surface $z = f(x, y)$

(3) " " Total Derivative if $z = f(x, y)$, $x = x(t)$, $y = y(t)$

i.e $z \longrightarrow (x, y) \longrightarrow$ 't' alone

(4) " " Chain Rule of Partial Derivatives, if $z = f(x, y)$, $x = x(r, s)$, $y = y(r, s)$

i.e $z \longrightarrow (x, y) \longrightarrow (r, s)$

(5) " " Jacobian if $(u, v) \longrightarrow (x, y)$

(6) " " Euler Theorem : if $f(x, y)$ is Homogeneous func^n then we can use E. Th.

Ordinary Derivatives → exist in case of curve $y=f(x)$ →

Power formula :→ $\boxed{\frac{d}{dx}(x^a) = a x^{a-1}}$

$$\frac{d}{dx}(K) = K\frac{d}{dx}(x^0) = K\{0 \cdot x^{0-1}\} = 0$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(x^1) = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1$$

$$\frac{d}{dx}(x^2) = 2x^{2-1} = 2x$$

$$\frac{d}{dx}(x^3) = 3 \cdot x^{3-1} = 3x^2$$

Similarly $\frac{d}{dx}(x^4) = 4x^3$, $\frac{d}{dx}(x^5) = 5x^4$.

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -1 \cdot x^{-1-1} = \frac{-1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}(x^{-2}) = -2x^{-2-1} = \frac{-2}{x^3}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-3/2}$$

Similarly $\frac{d^n}{dx^n}(x^n) = n!$ i.e. $\boxed{D^n(x^n) = n!}$

$$\frac{d^{n+1}}{dx^{n+1}}(x^n) = 0 \text{ i.e. } \boxed{D^{n+1}(x^n) = 0}$$

$$\frac{d}{dx}(x^3) = 3x^2$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = -\frac{1}{2}x^{-3/2}$$

(2) Exponential formula: $\boxed{\frac{d}{dx} a^x = a^x \log_e a}$ for eg $\frac{d}{dx}(e^x) = e^x$

Q. if $y = x^a + a^x + a^a + x^x$ then $\frac{dy}{dx} = ?$

Sol: $\frac{dy}{dx} = ax^{a-1} + a^x \log_e a + 0 + x^x(1+\log x)$

(3) $\boxed{\frac{d}{dx}(x^x) = x^x(1+\log_e x)}$

(4) $\boxed{\frac{d}{dx}(\log_e x) = \frac{1}{x}}$

eg $\frac{d^2}{dx^2}(\log_e x) = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

Q. $y = \log_e x + \log_a x + \log_x a + \log_x x + \log_a a$ then $\frac{dy}{dx} = ?$

Sol: $y = \log_e x + \frac{\log_e x}{\log_e a} + \frac{\log_e a}{\log_e x} + 1 + 1$

$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{\log_e a}\left(\frac{1}{x}\right) + \log_e a\left[\frac{-1}{x(\log x)^2}\right] + 0 + 0$

eg $\frac{d}{dx}\left(\frac{1}{\log_e x}\right) = -\frac{1}{(\log x)^2}\left(\frac{1}{x}\right)$

Q. $y = \log_{10} 10 + \log_x x + \log_{10} x + \log_x 10$ then $\frac{dy}{dx} = ?$

$$y = 1 + 1 + \frac{\log_e x}{\log_e 10} + \frac{\log_e 10}{\log_e x}$$

$$\frac{dy}{dx} = 0 + 0 + \frac{1}{\log_e 10}\left[\frac{1}{x}\right] + \log_e 10\left[\frac{-1}{(\log x)^2} \cdot \frac{1}{x}\right]$$

2 Let $f(x) = e^{-|x|}$, where x is real. The value of $\frac{df}{dx} =$ at $x = -1$ is PYQ

(a) $-e$

(b) e

(c) $\frac{1}{e}$

(d) $-\frac{1}{e}$

$$f(x) = e^{-|x|} = \begin{cases} e^{x}, & x<0 \\ e^{-x}, & x>0 \end{cases}$$

$$f'(x) = \begin{cases} e^{x}, & x<0 \\ -e^{-x}, & x>0 \end{cases}$$

$$f'(-1) = e^{-1} = \frac{1}{e}$$

(5) Chain Rule : $\boxed{\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)}$

eg $\frac{d}{dx}\left(\sin\sqrt{\tan x^3}\right) = ?$

$$= \cos\left(\sqrt{\tan x^3}\right) \cdot \frac{d}{dx}\left(\sqrt{\tan x^3}\right)$$

$$= \cos\left(\sqrt{\tan x^3}\right) \cdot \frac{1}{2\sqrt{\tan x^3}} \cdot \frac{d}{dx}\left(\tan x^3\right)$$

$$= \cos\left(\sqrt{\tan x^3}\right) \cdot \frac{1}{2\sqrt{\tan x^3}} \cdot \sec^2(x^3) \cdot \frac{d}{dx}(x^3)$$

$$= \cos\left(\sqrt{\tan x^3}\right) \cdot \frac{1}{2\sqrt{\tan x^3}} \cdot \sec^2(x^3) \cdot (3x^2)$$

Note :

$$\frac{d}{dx}(xy) = x\frac{d}{dx}(y) + y\frac{d}{dx}(x)$$

$$= x\frac{dy}{dx} + y(1)$$

$$\frac{d}{dy}(xy) = x\frac{d}{dy}(y) + y\frac{d}{dy}(x)$$

$$= x(1) + y\frac{dx}{dy}$$

$$d(xy) = x\,d(y) + y\,d(x)$$

$$= x\,dy + y\,dx$$

eg $\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3) \cdot \frac{dy}{dx}$

$$= (3y^2)\, dy/dx$$

Some More Standard Results :-

(6) $\frac{d}{dx}(\sin x) = \cos x$

(7) $\frac{d}{dx}(\cos x) = -\sin x$

(8) $\frac{d}{dx}(\tan x) = \sec^2 x$

(9) $\frac{d}{dx}(\cot x) = -\csc^2 x$

(10) $\frac{d}{dx}(\csc x) = -\csc x \cot x$

(11) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(12) $\frac{d}{dx}(\sinh x) = \cosh x$ (13) $\frac{d}{dx}(\cosh x) = \sinh x$

(14) Product formula :-

$$\frac{d}{dx}(fg) = fg' + gf'$$

$$\frac{d}{dx}(fgh) = f'gh + fg'h + fgh'$$

(15) Quotient formula :-

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf' - fg'}{(g)^2}$$

or $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

(16) log Diff :→

Q: if $y = x^x$ then $\frac{dy}{dx} = ?$ = ... $= x^x(1+\log x)$

H.W

Q: if $x^y = e^{x-y}$ then $\frac{dy}{dx} = ?$

Sol:
$$\log(x^y) = \log e^{x-y}$$
$$y\log x = (x-y)\log_e e$$
$$y\log x = x-y$$
$$y(1+\log x) = x$$
$$y = \frac{x}{1+\log x}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x}{1+\log x}\right]$$
$$= \frac{(1+\log x)(1) - x\left(\frac{1}{x}\right)}{(1+\log x)^2}$$
$$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

$x^{\sin y} = y^{\sin x}$, then $\dfrac{dy}{dx}$ is equal to

(a) $\dfrac{x^2 \cos x \log x - y \sin y}{x^2 \cos x \log x - x \sin x}$

(b) $\dfrac{y^2 \cos y \log y - x \sin x}{y^2 \cos y \log y - y \sin y}$

(c) $\dfrac{xy \cos x \cos y - y \sin y}{xy \cos x \cos y - x \sin x}$

(d) $\dfrac{xy \cos x \log y - y \sin y}{xy \log x \cos y - x \sin x}$

$$x^{\sin y} = y^{\sin x}$$

$$\sin y (\log x) = \sin x (\log y)$$

$$\frac{d}{dx}[\sin y (\log x)] = \frac{d}{dx}[\sin x (\log y)]$$

$$\sin y \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin y) = \sin x \frac{d}{dx}(\log y) + \log y \frac{d}{dx}(\sin x)$$

$$\sin y \left(\frac{1}{x}\right) + \log x \left[\cos y \frac{dy}{dx}\right] = \sin x \left(\frac{1}{y}\frac{dy}{dx}\right) + \log y (\cos x)$$

$$\left[\cos y \log x - \frac{\sin x}{y}\right]\frac{dy}{dx} = \cos x \log y - \frac{\sin y}{x}$$

$$\frac{dy}{dx} = \left[\frac{x \cos x \log y - \sin y}{y \cos y \log x - \sin x}\right] \cdot \frac{y}{x}$$

(d)

(17) Differentiation of Infinite Series →

Q: $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots \infty}}}$

then $\frac{dy}{dx} = ?$ Ans: $\frac{\cos x}{2y-1}$

Sol: $y = \sqrt{\sin x + y}$

$$y^2 = \sin x + y$$

$$y^2 - y = \sin x$$

$$\frac{d}{dx}(y^2) - \frac{d}{dx}(y) = \frac{d}{dx}(\sin x)$$

$$2y\frac{dy}{dx} - \frac{dy}{dx} = \cos x$$

$$(2y-1)\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{2y-1}$$ Ans

Q. $y = x^{x^{x^{\cdots \infty}}}$ then $\frac{dy}{dx} = ?$ $\left(= \frac{y^2}{x(1 - y\log x)}\right)$

Sol: $y = x^y$

$\log y = \log(x^y)$

$\log y = y \log x$

$\frac{d}{dx}(\log y) = \frac{d}{dx}[y \cdot \log x]$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = y\left(\frac{1}{x}\right) + \log x\left(\frac{dy}{dx}\right)$

$\left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$

$\frac{(1 - y\log x)}{y}\frac{dy}{dx} = \frac{y}{x}$

$\frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)}$ //

(18) Differentiation of parametric func^n →

Curve	Parametric Coordinates	Equation
(1) Circle	$(a\cos\theta, a\sin\theta)$	$x^2+y^2=a^2$
(2) Parabola	$(at^2, 2at)$	$y^2=4ax$
(3) Ellipse	$(a\cos\theta, b\sin\theta)$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
(4) Hyperbola	$(a\sec\theta, b\tan\theta)$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

(*) $\cos 2\theta = \begin{cases} \cos^2\theta-\sin^2\theta \\ 2\cos^2\theta-1 \\ 1-2\sin^2\theta \end{cases}$

$2\cos^2\theta-1 \longrightarrow \frac{1+\cos 2\theta}{2}=\cos^2\theta$

$1-2\sin^2\theta \longrightarrow \frac{1-\cos 2\theta}{2}=\sin^2\theta$

(*) if $x = f(t)$, $y = g(t)$ then $\boxed{\frac{dy}{dx} = ? = \frac{dy}{dt} \cdot \frac{dt}{dx}} = g'(t)\left(\frac{1}{f'(t)}\right)$

$\frac{dx}{dt} = f'(t)$, $\frac{dy}{dt} = g'(t)$

Q. if $x = at^2$, $y = 2at$ then $\frac{dy}{dx} = ?$

(a) t

(b) $\frac{1}{t}$ ✓

(c) t^2

(d) $-t$

$\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$= (2a)\left(\frac{1}{2at}\right)$

$= \frac{1}{t}$

Q. if $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $\frac{dy}{dx} = ?$

(a) $\cot\frac{\theta}{2}$ ✓ (b) $\tan\frac{\theta}{2}$ (c) $\frac{\theta}{2}$ (d) $\cot\theta$

Sol: $\frac{dx}{d\theta} = a[1 - \cos\theta]$, $\frac{dy}{d\theta} = a[0 + \sin\theta]$

$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = (a\sin\theta)\left[\frac{1}{a(1-\cos\theta)}\right]$

$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$

(19) Differentiation of func^n w.r.to another func^n →

if $u = f(x)$, $v = g(x)$ then $\frac{du}{dv} = ? = \frac{du/dx}{dv/dx} = \frac{f'(x)}{g'(x)}$

Q. Differentiate x^x w.r.to $x \log x$?

(a) x^x

(b) $x^x(1+\log x)$

(c) $1+\log x$

(d) $x \log x$

Let $u = x^x \Rightarrow \frac{du}{dx} = f'(x) = x^x(1+\log x)$

& $v = x \log x \Rightarrow \frac{dv}{dx} = g'(x) = x\left(\frac{1}{x}\right) + \log x(1) = 1 + \log x$

$$\frac{du}{dv} = \frac{du}{dx} \cdot \frac{dx}{dv} = x^x(1+\log x)\left[\frac{1}{1+\log x}\right] = x^x$$

M-II $\frac{du}{dv} = \frac{f'(x)}{g'(x)} = \frac{x^x(1+\log x)}{(1+\log x)} = x^x$

Q. Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w.r.to $\tan^{-1}x$?, $-1<x<1$

(a) $\tan\theta$ (b) θ (c) 2 (d) 2θ where $\theta=f(x)$

Note: $\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\cos^{-1}x=\frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$

$\frac{d}{dx}(\cot^{-1}x)=\frac{-1}{1+x^2}$

$\sin^{-1}x+\cos^{-1}x=\pi/2$

$\tan^{-1}x+\cot^{-1}x=\pi/2$

Sol: let $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$

let $u=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\sin^{-1}(\sin 2\theta)=2\theta$

& let $v=\tan^{-1}x=\theta$

$$\frac{du}{dv}=\frac{du/d\theta}{dv/d\theta}=\frac{2}{1}=2$$

PARTIAL DIFF. [for $z = f(x,y)$.]

$$\frac{\partial z}{\partial x} = \lim_{h\to 0}\left[\frac{f(x+h,y)-f(x,y)}{h}\right]_{y=\text{Const.}} , \quad \frac{\partial z}{\partial y} = \lim_{K\to 0}\left[\frac{f(x,y+K)-f(x,y)}{K}\right]_{x=\text{Const.}}$$

Note:-

In 2-D.

Equ^n of X axis : $y = 0$

Equ^n of line $\parallel^r$ to X axis : $y = K$

Equ^n of Y axis : $x = 0$

Equ^n of line $\parallel^r$ to Y axis ; $x = h$

In 3D.

Equ^n of XY plane ; $z = 0$

" " YZ plane ; $x = 0$

" ZX plane ; $y = 0$

Equ^n of plane $\parallel^r$ to XZ plane $\boxed{y = K}$

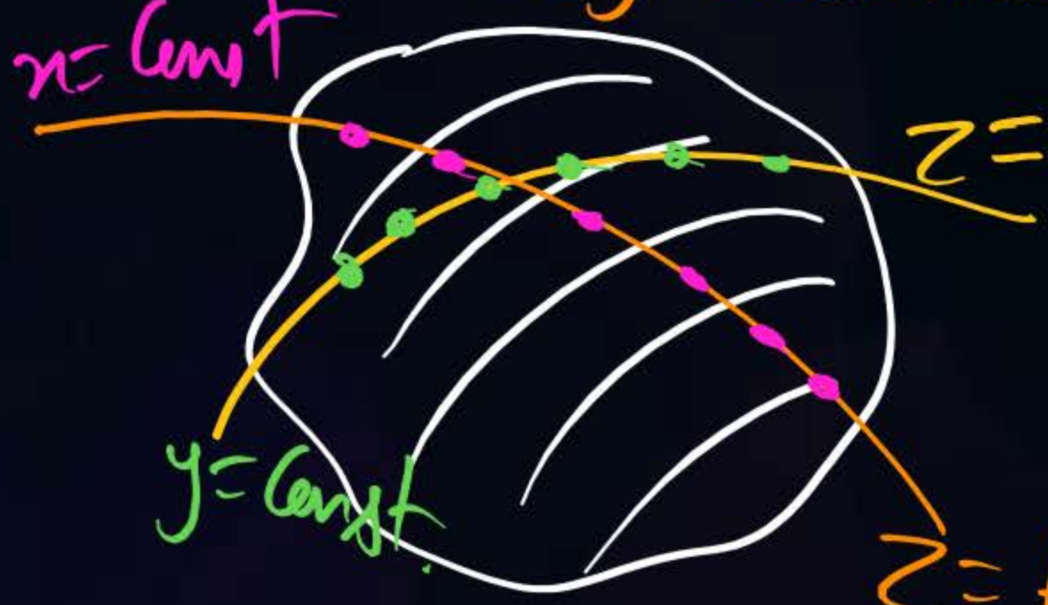
" $\parallel^r$ to YZ plane ; $x = h$

Significance of $\frac{\partial z}{\partial x}$ $\rightarrow$ $z=f(x,y)$

If we cut our surface by the plane $\parallel^r$ to xz plane (i.e. form $y=$ const) then we will get a curve of the type $z=f(x)$ & now

$\frac{\partial z}{\partial x}=$ slope of tangent at any random point on this curve.

similarly we can define $\frac{\partial z}{\partial y}=$?



$z=f(x)$ (curve) $\Rightarrow$ slope of tangent on this curve.

$z=f(y)$ (curve) $\Rightarrow$ slope of tangent on this curve.

Note: $z = f(x,y)$ then $z_x = \frac{\partial z}{\partial x} = f_x = \frac{\partial f}{\partial x}$ all are same

(1) & $z_{xx} = f_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}$

& $z_{xy} = f_{xy} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = ?$

(2) $$\boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}}$$

(3) All the Results that are applicable in case of ordinary derivatives are also valid in case of partial derivatives keeping other variable constant

(4) Don't assume Dependent Variable as Constant, if we are solving Questions Based on Partial Derivatives.

Q. if $r^2 = x^2+y^2+z^2$ then evaluate $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}$

$r = f(\underbrace{x, y, z}_{\text{Ind. Variables}})$

r → Dep. Variable

$$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$\frac{\partial}{\partial r}(r^2) \cdot \frac{\partial r}{\partial x} = 2x + 0 + 0$$

$$(2r)\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ & $\frac{\partial r}{\partial z} = \frac{z}{r}$

Q. PYQ

if $\boxed{r = x^2 + y - z}$ & $\boxed{y^3 + z^3 + yz - xy = 1}$

where x & y are Independent Variables then at $(2,-1,1)$

evaluate $\frac{\partial r}{\partial x} = ?$

(a) 4.0

(b) 2.0

(c) 5.5

(d) 4.5

$r = f(x, y, z)$ & $f(x,y,z) = c$ ie $z = f(x,y)$

or we can say that z is also Dependent Variable

ie $r = x^2 + y - z$

$$\frac{\partial r}{\partial x} = 2x + 0 - \frac{\partial z}{\partial x}$$

$$\frac{\partial}{\partial x}(y^3) + \frac{\partial}{\partial x}(z^3) + \frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}(xy) = \frac{\partial}{\partial x}(1)$$

$$0 + 3z^2 \cdot \frac{\partial z}{\partial x} + y\left(\frac{\partial z}{\partial x}\right) - y(1) = 0$$

$$\frac{\partial z}{\partial x} = \frac{y}{3z^2 + y}$$

$$\frac{\partial r}{\partial x} = 2x - \left[\frac{y}{3z^2 + y}\right]$$

$$\left(\frac{\partial r}{\partial x}\right)_{(2,-1,1)} = 2(2) - \left[\frac{-1}{3(1)^2 - 1}\right] = 4.5$$

Q. if $u = \log_e(x^2+y^2)$ then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = ?$

(a) 2

(b) 0

(c) -1

(d) $\frac{2(x+y)}{(x^2+y^2)}$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\log(x^2+y^2)\right] = \frac{1}{x^2+y^2}\frac{\partial}{\partial x}(x^2+y^2) = \frac{1}{x^2+y^2}(2x+0) = \frac{2x}{x^2+y^2}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right)$$

$$= \left[\frac{(x^2+y^2).(2) - 2x(2x+0)}{(x^2+y^2)^2}\right] = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

Similarly $u_{yy} = \frac{\partial^2 u}{\partial y^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$

So $u_{xx} + u_{yy} = \frac{2(y^2-x^2)}{(x^2+y^2)^2} + \left[\frac{-2(y^2-x^2)}{(x^2+y^2)^2}\right] = 0$

Q. If $u = e^{xyz}$ then evaluate $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = ?$ At $(2,-1,0)$

Sol:

$$\frac{\partial u}{\partial z} = e^{xyz} \cdot \frac{\partial}{\partial z}(xyz) = xy\, e^{xyz}(1)$$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial z}\right] = \frac{\partial}{\partial y}\left[\underset{\mathrm{I}}{y}\, \underset{\mathrm{II}}{e^{xyz}}\right]$$

$$= x\left[y \frac{\partial}{\partial y}\left(e^{xyz}\right) + e^{xyz}\frac{\partial}{\partial y}(y)\right]$$

$$= x\left[y\, e^{xyz} \cdot \frac{\partial}{\partial y}(xyz) + e^{xyz}(1)\right]$$

$$= x\left[y \cdot e^{xyz} \cdot (xz) + e^{xyz}\right]$$

$$\frac{\partial^2 u}{\partial y \partial z} = e^{xyz}\left[x^2yz + x\right]$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial^2 u}{\partial y \partial z}\right]$$

$$= \frac{\partial}{\partial x}\left[\underset{\mathrm{I}}{e^{xyz}}\underset{\mathrm{II}}{\left(x^2yz + x\right)}\right]$$

$$= e^{xyz}\left[2xyz + 1\right] + \left(x^2yz + x\right)\left[e^{xyz} \cdot yz\right]$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}\left[2xyz + 1 + x^2y^2z^2 + xyz\right]$$

At $(2,-1,0) \Rightarrow$ Ans $= 1\left[0 + 1 + 0 + 0\right] = 1$

Q. If $z = f(x-by) + \phi(x+by)$ then Evaluate $b^2\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \underline{?= 0}$

(a) 0

(b) 1

(c) -1

(d) $(a^2-b^2)(f''+\phi'')$

$z_x = \frac{\partial z}{\partial x} = f'(x-by)(1-0) + \phi'(x+by)(1+0)$

$= f'(x-by) + \phi'(x+by)$

$z_{xx} = \frac{\partial^2 z}{\partial x^2} = f''(x-by)(1-0) + \phi''(x+by)(1+0)$

$= \boxed{f''(x-by) + \phi''(x+by)}$ —— (1)

$z_y = \frac{\partial z}{\partial y} = f'(x-by)(0-b) + \phi'(x+by)(0+b)$

$= -b f'(x-by) + b\phi'(x+by)$

$z_{yy} = \frac{\partial^2 z}{\partial y^2} = -b f''(x-by)(-b) + b\phi''(x+by)(b)$

$= \boxed{b^2[f''(x-by) + \phi''(x+by)]}$ —— (2)

Analysis $z = f(x-by)$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x-by) = \frac{\partial}{\partial f} f(x-by) \cdot \frac{\partial f}{\partial x} = f' \cdot (1-0)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x-by) = \frac{\partial}{\partial f} f(x-by) \frac{\partial f}{\partial y} = f' (-b)$$

Let $f(x, y) = \dfrac{ax^2 + by^2}{xy}$, where a and b are constants. If $\boxed{\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y}}$ at $x = 1$ and $y = 2$, then the relation between a and b is

(a) $a = \dfrac{b}{4}$ (b) $a = \dfrac{b}{2}$

(c) $a = 2b$ (d) $a = 4b$

$$f(x,y) = \frac{ax^2+by^2}{xy} = a\left(\frac{x}{y}\right) + b\left(\frac{y}{x}\right) \quad (1)$$

$$\frac{\partial f}{\partial x} = a\left[\frac{1}{y}\right] + b\left[\frac{-y}{x^2}\right]$$

$$\left(\frac{\partial f}{\partial x}\right)_{\substack{x=1\\y=2}} = \frac{a}{2} - 2b = \frac{a-4b}{2}$$

$$\frac{\partial f}{\partial y} = a\left(\frac{-x}{y^2}\right) + b\left(\frac{1}{x}\right)$$

$$\left(\frac{\partial f}{\partial y}\right)_{\substack{x=1\\y=2}} = -\frac{a}{4} + b = \frac{-a+4b}{4}$$

ATQ, $f_x = f_y$

$$\frac{a-4b}{2} = \frac{-a+4b}{4}$$

$$2a - 8b = -a + 4b$$

$$3a = 12b$$

$$a = 4b$$

Test Syllabus : till lec 6