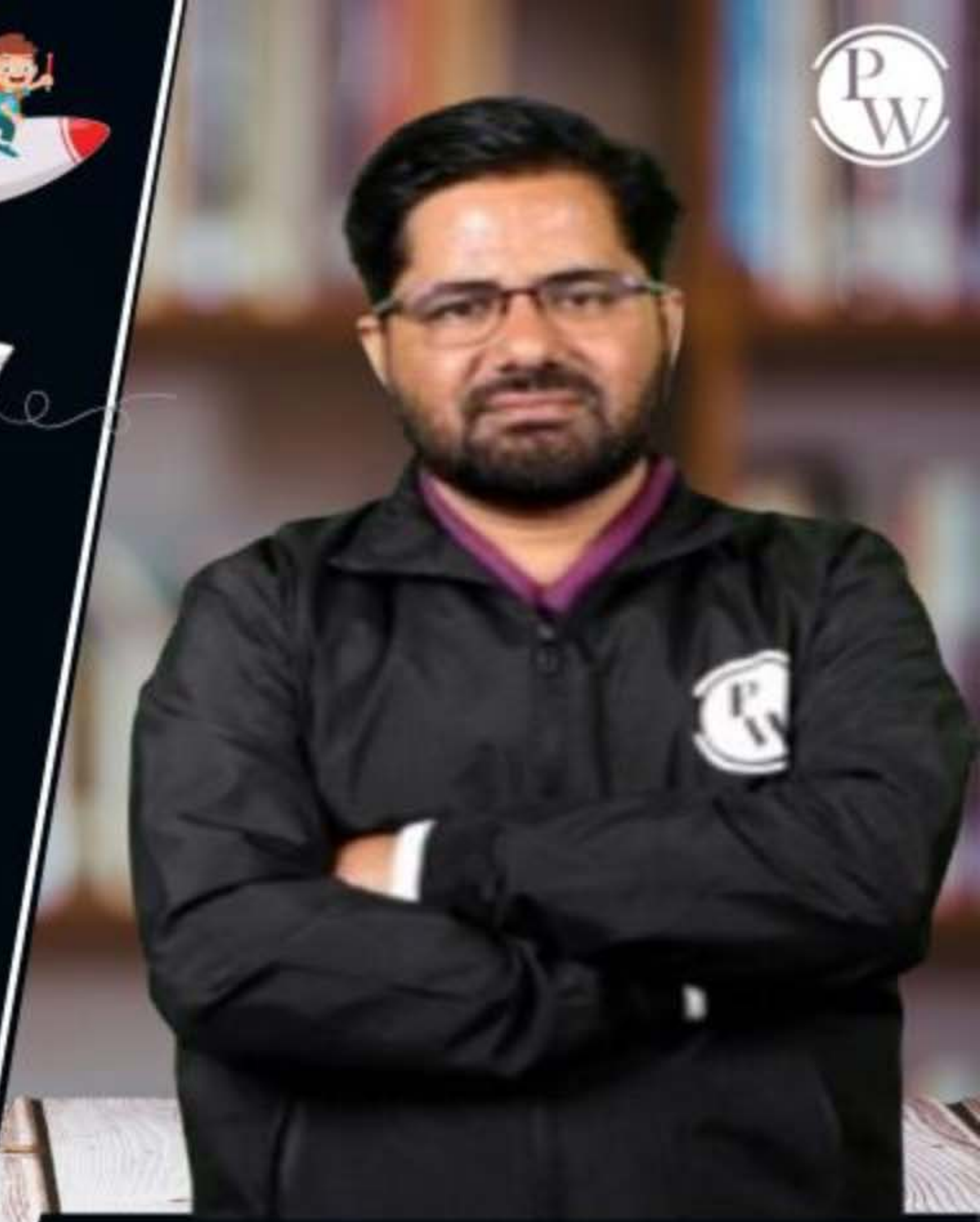
DATA SCIENCE
&
ARTIFICIAL INTELLIGENCE

& CS/IT

Calculus and Optimization

Lecture No. 08



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Recap of previous lecture

Topic

DERIVATIVES & THEIR TYPES

(PART-1)

Topics to be Covered

Topic

DERIVATIVES & their Types

(Part 2)

Types of Questions

1. Based on ordinary Derivative exist in case of curve $y = f(x)$
2. " " Partial Derivative " " " of surface $z = f(x, y)$
3. " " Total Derivative if $z = f(x, y)$, $x = x(t)$, $y = y(t)$
   i.e $z \longrightarrow (x, y) \longrightarrow$ 't' alone
4. " " Chain Rule of Partial Derivatives, if $z = f(x, y)$, $x = x(r, s)$, $y = y(r, s)$
   i.e $z \longrightarrow (x, y) \longrightarrow (r, s)$
5. " " Jacobian if $(u, v) \longrightarrow (x, y)$
6. " " Euler Theorem: if $f(x, y)$ is Homogeneous funcⁿ then we can use E. Th.

Homogeneous func^n :-

eg: $f(x,y) = x^5 + 2x^3y^2 + 4y^5 + xy^4$

$$f(\lambda x,\lambda y) = (\lambda x)^5 + 2(\lambda x)^3(\lambda y)^2 + 4(\lambda y)^5 + (\lambda x)(\lambda y)^4$$

$$= \lambda^5\left[x^5 + 2x^3y^2 + 4y^5 + xy^4\right]$$

$$= \lambda^5 \cdot f(x,y)$$

So $f(x,y)$ is H. func^n of degree 5.

eg $f(x,y) = \dfrac{x^4+y^4}{\sqrt{x}-\sqrt{y}}$

$$f(\lambda x,\lambda y) = \frac{(\lambda x)^4 + (\lambda y)^4}{\sqrt{\lambda x} - \sqrt{\lambda y}} = \frac{\lambda^4}{\sqrt{\lambda}}\left[\frac{x^4+y^4}{\sqrt{x}-\sqrt{y}}\right]$$

$f(\lambda x,\lambda y) = \lambda^{\frac{7}{2}} \cdot f(x,y)$ is H. func^n of degree 3.5

eg $f(x,y) = x^3 + xy^2 + \sin(x^3)$

$$f(\lambda x,\lambda y) = \lambda^3x^3 + (\lambda x)(\lambda^2y^2) + \sin(\lambda^3x^3)$$

$$= \lambda^3\left[x^3 + xy^2 + \frac{1}{\lambda^3}\sin(\lambda^3x^3)\right]$$

$\neq \lambda^n \cdot f(x,y)$ So Not Homog func^n!

Homogeneous func^n: Consider a func^n $\boxed{U = U(x, y)}$ then this func^n is called Homog. func^n of degree n, if each term is of same degree (n).

Trick: if $\boxed{U(\lambda x, \lambda y) = \lambda^n . U(x, y)}$ then U is called Homog func^n of degree 'n'. where $n \in R$.

eg $U = x^3 + 4xy^2 + 5x^2y$

H. func^n of degree $= 3$

eg $U = \dfrac{x^2 + y^2}{\sqrt{x} + \sqrt{y}}$

Homog func^n of degree $= 1.5$

Euler theorem for Homog. func^n →

if u is Homog. func^n of degree n then $\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n.u}$

Note: ① $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$

② **Sp. Note:** if $\boxed{Z=u+v}$ where u & v are Homog. func^n of degree n_1 & n_2 But Z is Non Homog. func^n then.

(i) $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\boxed{n_1u+n_2v}$

(ii) $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=\boxed{n_1(n_1-1)u+n_2(n_2-1)v}$

Proof ① $\boxed{z = u + v}$ —— ①

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \quad \& \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n_1 u \quad \text{—— ②}$$

$$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \& \quad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n_2 v \quad \text{—— ③}$$

Now, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)$

$$= \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) + \left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$$

$$= \boxed{n_1 u + n_2 v}$$ Hence Proved

Proof ② Similarly we can show that,

$$x^2 z_{xx} + 2xy z_{xy} + y^2 z_{yy} = n_1(n_1 - 1)u + n_2(n_2 - 1)v$$

eg $u = x^5 + 4x^2y^3 + \log(x^5)$.

Non homog. func^n : $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

$$f(\lambda x, \lambda y) = (\lambda x)^5 + 4(\lambda x)^2(\lambda y)^3 + \log(\lambda^5 x^5)$$

$$= \lambda^5 \underbrace{\left[x^5 + 4x^2y^3 + \frac{1}{\lambda^5}\log(\lambda^5 x^5)\right]}$$

$\neq \lambda^n f(x,y)$ so Non Homog.

Also evaluate, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ??$

$\because$ u is Non Homog func^n so we can't apply E. Th.
Hence Do yourself by conventional approach.

eg $u = \log x - \log y$

$$u(x,y) = \log\left(\frac{x}{y}\right)$$

$$u(\lambda x, \lambda y) = \log\left(\frac{\lambda x}{\lambda y}\right)$$

$$= \lambda^0 \log\left(\frac{x}{y}\right)$$

is Homog func^n of degree $n=0$

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ? = 0 \times u = 0$

(ii) $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = ? = 0(0-1)u$
$= 0$

Q. if $z = \dfrac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}$ then

① $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = ? = nz = -\dfrac{1}{12}z$

② $x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy} = ?$

$\because z = z(x, y)$ so

$$z(\lambda x, \lambda y) = \frac{\lambda^{\frac{1}{4}}}{\lambda^{\frac{1}{3}}}\left[\frac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}\right]$$

$$= \lambda^{-\frac{1}{12}} \cdot z(x, y)$$

So Homog func^n of degree $n = -\dfrac{1}{12}$

$$= n(n-1)z$$

$$= -\frac{1}{12}\left(-\frac{1}{12} - 1\right)z$$

$$= \frac{13}{144}z$$

Q. if $u = \log_e\left(\frac{x^2+y^2}{x+y}\right)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ?$

(a) 1 (b) e^u

(c) 0 (d) $\log u$

Sol: $\because U(\lambda x, \lambda y) \neq \lambda^n U(x,y)$ ie U is Non Homog func^n & we can't apply E. Theorem.

Let, $\boxed{e^u = v} = \frac{x^2+y^2}{x+y}$ & V is Homog func^n of degree $n=1$

Hence Applying E. Th for v,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n.v$$

$$x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = 1 \cdot e^u$$

$$x(e^u)\frac{\partial u}{\partial x} + y(e^u)\frac{\partial u}{\partial y} = e^u$$

ie $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{e^u}{e^u} = 1$ Ans

Q. if $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ?$

(a) $\sin 2u$

(b) $\tan u$

(c) $\cot u$

(d) $\cos 2u$

$\because u(\lambda x, \lambda y) \neq \lambda^n u(x, y)$, i.e. u is Non Homog (so Can't Apply E. Th

Let $\boxed{\tan u = v} = \frac{x^3+y^3}{x+y}$, Here v is Homog func^n of $n = 2$

So $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n.v$

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2.\tan u$$

$$x(\sec^2 u)\frac{\partial u}{\partial x} + y(\sec^2 u)\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = \text{-----} = \sin 2u$$

Q If $Z = x^2y^4\sin^{-1}\left(\frac{x}{y}\right) + \log x - \log y$ then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = ?$

(a) $6Z$

(b) 0

(c) $6u$

(d) $6(u+v)$

$Z = \underbrace{x^2y^4\sin^{-1}\left(\frac{x}{y}\right)}_{u} + \underbrace{\log\left(\frac{x}{y}\right)}_{v}$, Z is Non Homog func^n but u & v are Homog func^n of degree $n_1 = 6$, $n_2 = 0$ resp.

ie $\boxed{Z = u + v}$ So Applying Standard Result,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n_1 u + n_2 v$$

$$= 6u + 0.v = 6u \quad \text{Ans}$$

(ii) Also evaluate, $x^2 Z_{xx} + 2xy Z_{xy} + y^2 Z_{yy} = ?$

$$= n_1(n_1-1)u + n_2(n_2-1)v$$

$$= 6(6-1)u + 0(0-1)v$$

$$= 30u \quad \text{Ans}$$

Q. If $\boxed{u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)}$ then $x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = \underline{?} = nu = 0 \times u = 0$

Sol: $\because u(\lambda x, \lambda y) = \lambda^{0} \cdot u(x, y)$

So u is homog func^n of $n = 0$

Blunder

(ii) Also evaluate $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = ? = 0(0-1)u = -u$?

$= 0$ Ans

Q. If $\boxed{u = x^3 y^2 \sin^{-1}\left(\frac{y}{x}\right)}$ then Evaluate?

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ?$ (ii) $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = ?$

(a) u

(b) 5

(c) $5u$

(d) $20u$

$\because u(\lambda x, \lambda y) = \lambda^5 u(x,y)$ i.e. u is Homog func^n of $n=5$

(i) $x u_x + y u_y = nu = 5u$

(ii) $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 5(5-1)u = 20u$

Derivative of an Implicit func^n :

Consider $f(x,y) = c$ —(1) then $\boxed{\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}}$

Proof : Diff both sides

$$df = d(c)$$

$$df = 0$$

Now using the S. Result of T.D in LHS

$$\left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy = 0 \Rightarrow \frac{dy}{dx} = -\left(\frac{\partial f/\partial x}{\partial f/\partial y}\right) = -\frac{f_x}{f_y}$$

Q: If $x^3+y^3+3xy=1$ then evaluate $\frac{dy}{dx}=?$

M-I

$$\frac{d}{dx}[x^3]+\frac{d}{dx}(y^3)+3\frac{d}{dx}(xy)=\frac{d}{dx}(1)$$

$$3x^2+\frac{d}{dy}(y^3)\frac{dy}{dx}+3\left[x\frac{dy}{dx}+y(1)\right]=0$$

$$3y^2\frac{dy}{dx}+3x\frac{dy}{dx}=-3x^2-3y$$

$$(y^2+x)\frac{dy}{dx}=-(x^2+y)$$

$$\frac{dy}{dx}=-\left(\frac{x^2+y}{y^2+x}\right)$$

M-II Let $f\equiv x^3+y^3+3xy-1=0$

$$\frac{\partial f}{\partial x}=3x^2+3y \quad \& \quad \frac{\partial f}{\partial y}=3y^2+3x$$

Now, $\boxed{\frac{dy}{dx}=-\left(\frac{\partial f/\partial x}{\partial f/\partial y}\right)}$

MAG

$$=-\left[\frac{3x^2+3y}{3y^2+3x}\right]=-\left(\frac{x^2+y}{y^2+x}\right)$$

Q: If $x^2+xy+y^2=3$ then $\frac{dy}{dx}=$??

(M-I) Let $\boxed{f \equiv x^2+xy+y^2-3}$

$$\frac{\partial f}{\partial x}=2x+y \quad \& \quad \frac{\partial f}{\partial y}=x+2y$$

$$\boxed{\frac{dy}{dx}=-\left(\frac{f_x}{f_y}\right)}=-\left(\frac{2x+y}{2y+x}\right)$$

(M-II) Using Conventional Approach

→ Do yourself.

Total Derivative →

→ if $u = u(x, y, z)$ then $\boxed{du = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z}\right)dz}$

→ if $u = u(x, y, z)$, where $x = x(t)$, $y = y(t)$, $z = z(t)$

i.e. $u \longrightarrow (x, y, z) \longrightarrow t$ alone.

T.D is $du = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z}\right)dz$

$$\frac{du}{dt} = \left(\frac{\partial u}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial u}{\partial y}\right)\frac{dy}{dt} + \left(\frac{\partial u}{\partial z}\right)\frac{dz}{dt}$$

Total Derivative: → if $w = f(x, y, z)$ where $x = x(t),\ y = y(t),\ z = z(t)$

i.e $w \longrightarrow (x, y, z) \longrightarrow$ 't' alone.

Standard Result is: $\boxed{dw = \left(\frac{\partial w}{\partial x}\right) dx + \left(\frac{\partial w}{\partial y}\right) \cdot dy + \left(\frac{\partial w}{\partial z}\right) \cdot dz}$ → Learn by ❤

& Total Derivative of w with resp to t is given as.

$$\frac{dw}{dt} = \left(\frac{\partial w}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial w}{\partial y}\right)\frac{dy}{dt} + \left(\frac{\partial w}{\partial z}\right)\frac{dz}{dt}$$

Note: if $w \longrightarrow (x, y, z) \longrightarrow x$ alone then $\frac{dw}{dx} = \left(\frac{\partial w}{\partial x}\right) + \left(\frac{\partial w}{\partial y}\right) \cdot \frac{dy}{dx} + \left(\frac{\partial w}{\partial z}\right) \cdot \frac{dz}{dx}$ //

Halka Question:

Q: $d(x.y) = ?$

(a) $dx\,dy$

(b) $dx + dy$

(c) $x\,dx + y\,dy$

(d) $x\,dy + y\,dx$

(M-I) using product formula:

$d(xy) = x\,d(y) + y\,d(x)$

$= x\,dy + y\,dx$

(M-II) using T.D concept,

let $u = xy$

$$\boxed{du = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy}$$

$= (y)\,dx + (x)\,dy$

$=$ (d) Ans

Note:

$\frac{d}{dx}(xy) = x\frac{d}{dx}(y) + y\frac{d}{dx}(x)$

$= x\frac{dy}{dx} + y\cdot(1)$

$\frac{d}{dy}(xy) = x\frac{d}{dy}(y) + y\frac{d}{dy}(x)$

$= x(1) + y\frac{dx}{dy}$

$d(xy) = x\,d(y) + y\,d(x)$

$= x\,dy + y\,dx$

Q: if $u = x^2 - y^2 + 2\sin yz$ then

$\frac{\partial u}{\partial x} = ? = 2x - 0 + 0 = 2x$

$\frac{\partial u}{\partial y} = ? = 0 - 2y + 2\cos yz \frac{\partial}{\partial y}(yz)$

$= -2y + 2z\cos(yz)$

$\frac{\partial u}{\partial z} = ? = 0 - 0 + 2\cos(yz)\frac{\partial}{\partial z}(yz)$

$= 2y\cos(yz)$

Q: if $y = e^x$ then

$\frac{dy}{dx} = ? = e^x$

Q: if $z = \ln x$ then

$\frac{dz}{dx} = ? = \frac{1}{x}$

Q. if $u = x^2 - y^2 + z\sin yz$ where $y = e^x$, $z = \ln x$ then $\frac{du}{dx} = ?$

Explicit func^n

Explicit func^n.

Sol: $u \longrightarrow (x, y, z) \longrightarrow x$ alone.

i.e Quest is Based on T.D Concept

So $\boxed{du = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z}\right)dz}$

$$\frac{du}{dx} = \left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial u}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial u}{\partial z}\right)\frac{dz}{dx}$$

$$= (2x) + (-2y + z^2\cos yz)(e^x) + (2y\cos yz)\left(\frac{1}{x}\right)$$

Q. Find the Derivative of x^2y w.r.to x where x & y are connected by the relation $x^2+xy+y^2=1$

Sol: Let $u = x^2y$ —— ① & $\underbrace{x^2+xy+y^2=1}_{f(x,y)=c \text{ Implicit func}^n}$ —— ②

T.D of u is as follows,

$$\boxed{du = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y} = -\left(\frac{2x+y}{x+2y}\right)$$

$$\Rightarrow \frac{du}{dx} = \left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial u}{\partial y}\right)\frac{dy}{dx}$$

$$= (2xy) + (x^2)\left\{-\left(\frac{2x+y}{x+2y}\right)\right\}$$

Analysis:

(1) if $y = f(x)$ then $\frac{dy}{dx} = \boxed{\frac{d}{dx} f(x)} = f'(x)$ = ordinary Derivative

(2) if $z = f(x,y)$ → $\frac{\partial z}{\partial x} = P.D$

→ $\frac{\partial z}{\partial y} = P.D$

→ $\frac{dz}{dx} = \boxed{\frac{d}{dx} f(x,y)} = T.D.$

Q $z = x^2 y \Rightarrow \frac{\partial z}{\partial x} = 2xy,\ \frac{\partial z}{\partial y} = x^2,\ \frac{dz}{dx} = \frac{d(x^2 y)}{dx}$ = use T.D. Concept

& $dz = ? = d(x^2 y) = x^2 dy + y\, d(x^2) = x^2 dy + y(2x\, dx)$

Given, $z(x, y) = e^{x-2y}$, where $x(t) = e^t$ and $y(t) = e^{-t}$. All the variables are real. The total differential $\frac{dz}{dt}$ is ?

(a) $-z(x + 2y)$
(b) $-z(x - 2y)$
(c) $z(x + 2y)$
(d) $z(x - 2y)$

$Z = e^{x-2y}$, $x = e^{t}$, $y = e^{-t}$

$\because Z \longrightarrow (x, y) \longrightarrow t$ alone

is Questn Based on T.D Concept.

$\frac{\partial z}{\partial x} = e^{x} \cdot e^{-2y}$ & $\frac{\partial z}{\partial y} = e^{x} \cdot e^{-2y}(-2)$

$$\boxed{dz = \left(\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial z}{\partial y}\right) dy}$$

$$\frac{dz}{dt} = \left(\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial z}{\partial y}\right)\frac{dy}{dt}$$

$$= \left(e^{x-2y}\right)e^{t} + \left(-2e^{x-2y}\right)\left(-e^{-t}\right)$$

$$= x \cdot Z + 2yZ$$

$$= Z(x + 2y)$$

Type IV Chain Rule of Partial Derivative (change of Variable Concept) $\longrightarrow$

if $w = f(x, y, z)$ where $x = x(r, s, t)$, $y = y(r, s, t)$, $z = z(r, s, t)$

i.e. $w \longrightarrow (x, y, z) \longrightarrow (r, s, t)$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial r}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial r}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial r}\right)$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial s}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial s}\right)$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial t}\right)$$

Short Cut:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\quad}\frac{\quad}{\partial r} + \frac{\partial w}{\quad}\frac{\quad}{\partial r} + \frac{\partial w}{\quad}\frac{\quad}{\partial r}$$

$$= \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial r}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial r}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial r}\right)$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\quad}\frac{\quad}{\partial s} + \frac{\partial w}{\quad}\frac{\quad}{\partial s} + \frac{\partial w}{\quad}\frac{\quad}{\partial s}$$

$$= \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial s}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial s}\right)$$

Similarly, $\frac{\partial w}{\partial t} = ?$

Shortcut to learn above Result: →

$$u \longrightarrow (r, s, t) \longrightarrow (x, y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\;}\frac{\;}{\partial x} + \frac{\partial u}{\;}\frac{\;}{\partial x} + \frac{\partial u}{\;}\frac{\;}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial u}{\partial s}\left(\frac{\partial s}{\partial x}\right) + \frac{\partial u}{\partial t}\left(\frac{\partial t}{\partial x}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\;}\frac{\;}{\partial y} + \frac{\partial u}{\;}\frac{\;}{\partial y} + \frac{\partial u}{\;}\frac{\;}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial y}\right) + \frac{\partial u}{\partial s}\left(\frac{\partial s}{\partial y}\right) + \frac{\partial u}{\partial t}\left(\frac{\partial t}{\partial y}\right)$$

If $u = f(r, s)$ where $r = x + y$, $s = x - y$ then then

$u_x + u_y =$

(a) $2u_r$ (b) $2u_s$

(c) $-2u_r$ (d) $-2u_s$

$u \longrightarrow (r, s) \longrightarrow (x, y)$

is Quest is Based on Chain Rule Concept.

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial u}{\partial s}\left(\frac{\partial s}{\partial x}\right) = u_r(1) + u_s(1)$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial y}\right) + \frac{\partial u}{\partial s}\left(\frac{\partial s}{\partial y}\right) = u_r(1) + u_s(-1)$$

$$u_x + u_y = 2u_r \quad (a)$$

Q. If $v = f(2x-3y, 3y-4z, 4z-2x)$ then $\boxed{6V_x + 4V_y + 3V_z = ?}$

(a) 1
(b) −2
(c) 13
(d) 0

Let $r = 2x-3y$, $s = 3y-4z$, $t = 4z-2x$

$\rightarrow V = f(r, s, t)$ where $(r, s, t) = g(x, y, z)$

i.e. $V \longrightarrow (r, s, t) \longrightarrow (x, y, z)$

$$V_x = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial v}{\partial s}\left(\frac{\partial s}{\partial x}\right) + \frac{\partial v}{\partial t}\left(\frac{\partial t}{\partial x}\right) = V_r(2) + V_s(0) + V_t(-2)$$

$$V_y = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\left(\frac{\partial r}{\partial y}\right) + \frac{\partial v}{\partial s}\left(\frac{\partial s}{\partial y}\right) + \frac{\partial v}{\partial t}\left(\frac{\partial t}{\partial y}\right) = V_r(-3) + V_s(3) + V_t(0)$$

$$V_z = \frac{\partial v}{\partial z} = \frac{\partial v}{\partial r}\left(\frac{\partial r}{\partial z}\right) + \frac{\partial v}{\partial s}\left(\frac{\partial s}{\partial z}\right) + \frac{\partial v}{\partial t}\left(\frac{\partial t}{\partial z}\right) = V_r(0) + V_s(-4) + V_t(4)$$

$$V_x = 2V_r - 2V_t$$

$$V_y = -3V_r + 3V_s$$

$$V_z = -4V_s + 4V_t$$

$$6V_x + 4V_y + 3V_z = (12V_r - 12V_t) + (-12V_r + 12V_s) + (-12V_s + 12V_t)$$

$$= 0$$

Ans

Q. if $V = V(x,y)$ where $x+y = 2e^{\theta}\cos\phi$, $x-y = 2ie^{\theta}\sin\phi$

then evaluate $\frac{\partial V}{\partial \theta}$ and $\frac{\partial V}{\partial \phi} = ?$

HW

$$V \longrightarrow (x,y) \longrightarrow (\theta,\phi)$$

is Question is Based on chain Rule of Partial Derivative.

$$\frac{\partial V}{\partial \theta} = ?$$

$$\frac{\partial V}{\partial \phi} = ?$$

JACOBIAN: if $u \to (x,y)$ & $v \to v(x,y)$

i.e. $(u,v) \to (x,y)$ then

Derivative of (u,v) w.r.to (x,y) is called Jacobian

& it is defined as $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

(*) if $(u,v,w) \to (x,y,z)$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

(*) if $(u,v) \to (x,y)$ then $\dfrac{\partial(x,y)}{\partial(u,v)} = ? = J^{1} = \dfrac{1}{J}$

w.k. that $\boxed{JJ^{1} = 1}$

i.e J^{1} = Reciprocal of J

JACOBIAN $\rightarrow$ if $u = u(x,y,z)$, $v = v(x,y,z)$, $w = w(x,y,z)$

is $(u,v,w) \rightarrow (x,y,z)$

then Derivative of (u,v,w) with respect to (x,y,z) is called Jacobian & it is defined as

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Note ① if $(u,v) \rightarrow (x,y)$ then

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

② if $(x,y) \rightarrow (u,v)$ then $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = J'$

③ $JJ' = 1$

Q. If $u = 3x+5y$ & $v = 4x-3y$ then $\frac{\partial(u,v)}{\partial(x,y)} = ?$

(a) 29

(b) -29 ✓

(c) 11

(d) -11

$\because (u,v) \longrightarrow (x,y)$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 4 & -3 \end{vmatrix}$$

$$= (3)(-3) - (4)(5)$$

$$= -9 - 20 = -29$$

Q: if $x = u$, $y = u\tan v$, $z = w$ then Derivative of (u,v,w) w.r.to (x,y,z) will be ?

(a) $\dfrac{\cos^2 v}{u}$

(b) $u\cos^2 v$

(c) $u\sec^2 v$

(d) 0

Here $(x,y,z) \longrightarrow (u,v,w)$

then $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$, $J' = \dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ & $JJ' = 1$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \tan u & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u\sec^2 v$$

So $J' = \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{1}{J} = \dfrac{1}{u\sec^2 v} = \dfrac{\cos^2 v}{u}$

eg $\boxed{f(x) = x\sqrt{x}} = x^{3/2}$

$$f'(x) = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}\sqrt{x}$$

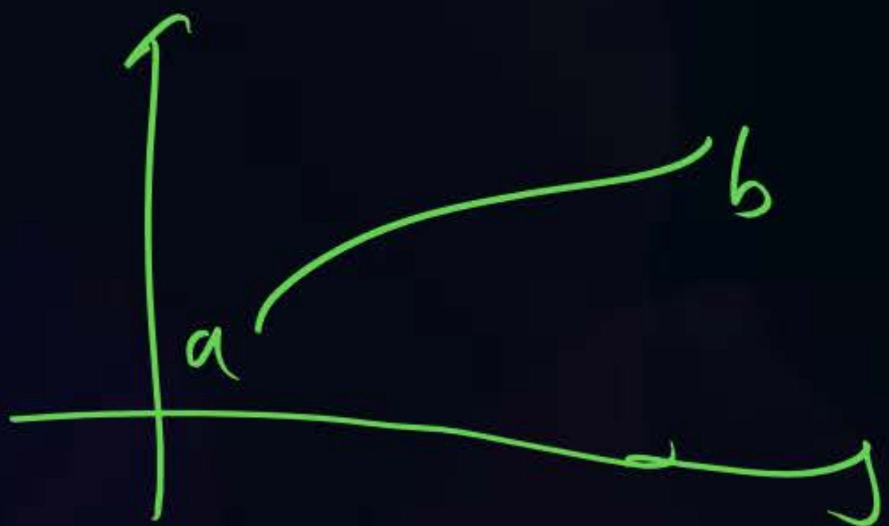
$f'(0) =$ exist.

eg $f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$f'(0) =$ DNE

eg:



$f'(a^+) =$ Check

$f'(b^-) =$ "

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