VIJAY

DATA SCIENCE & ARTIFICIAL INTELLIGENCE

& CS

Calculus and Optimization

Lecture No. 02

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Recap of previous lecture

Topic

FUNCTIONS & GRAPHS - 1

Topics to be Covered

Topic

FUNCTIONS & GRAPHS - 2

RECAP

Types of functions

→ ALGEBRAIC function
1) Polynomial func^n
2) Rational func^n
3) Irrational func^n
4) Piecewise func^n
- Mod func^n
- Signum func^n
- G.I.F.
- L.I.f
- F.P.f

→ TRANSCEDENTAL function
1) Exponential func^n
2) log function.
3) Trigonometric func^n
4) Inverse Trig. functions.

G.I.f = Greatest Integer func^n (Floor func^n)
LIf = Least Integer func^n (Ceiling func^n)
FPF = Fractional Part func^n.

Polynomial: It's Domain is $(-\infty, \infty)$ & Degree $= 0, 1, 2, 3, 4, 5 \ldots$

& it's Definition is same at all points in the Domain of $y = f(x)$

eg $y = k$ (Constant poly) $\simeq$ degree $= 0$

$y = ax + b$ (Linear poly) $\simeq$ Dg $= 1$

RECAP

$y = ax^2 + bx + c$ (Quad. poly) $\simeq$ Dg $= 2$

$y = ax^3 + bx^2 + cx + d$ (Cubic poly) $\simeq$ Dg $= 3$

$\vdots$

Sp Note - $y = |x| = \begin{cases} -x, & x < 0 \\ +x, & x > 0 \end{cases}$, $D_f = (-\infty, \infty)$. It's not a poly bcoz it's Def^n is not unique at all points in the Domain.

Even func^n: if $f(-x) = f(x) \Rightarrow f(x)$ is called an Even func^n

& it's Graph is symmetrical about Y axis

RECAP

odd func^n :- if $f(-x) = -f(x) \Rightarrow f(x)$ is called an odd func^n.

& it's Graph is symmetrical about origin i.e (I $\Leftrightarrow$ III & II $\Leftrightarrow$ IV)

NENO func^n :- if $\left.\begin{array}{l} f(-x) \neq f(x) \\ \& \ f(-x) \neq -f(x) \end{array}\right\}$ then $f(x)$ is called NENO func^n.

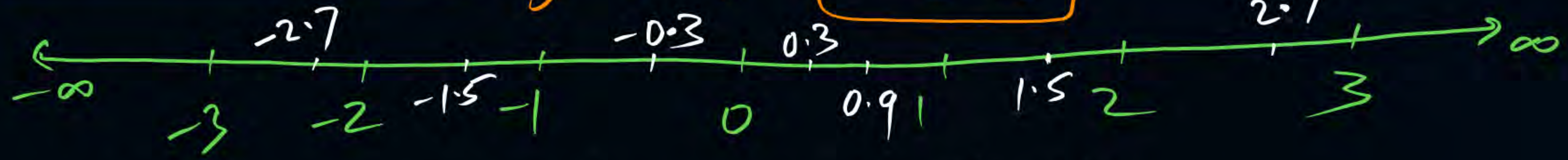
it's Graph is neither symmetrical about Y axis, nor about origin.

PIECEWISE func^n → If func^n is defined by Multiple subfunction s.t, Domain of each subfunction is different then function is called piecewise func^n

for eg, Mod func^n, Signum func^n, G.I.F, L.I.F, Fractional part func^n etc

RECAP

Fractional Part func^n : $y = \{x\} = \boxed{x - \lfloor x \rfloor}$



$\{0.3\} = 0.3$

$\{0.3\} = 0.3 - \lfloor 0.3 \rfloor$
$= 0.3 - 0$
$= 0.3$

$\{-0.3\} = -0.3 - \lfloor -0.3 \rfloor$
$= -0.3 - (-1)$
$= 0.7$

$\{1.5\} = 0.5$

$\{-1.5\} = 0.5$

$\{2\} = 0$

$\{-2\} = 0$

$\{2.7\} = 0.7$

$\{2.7\} = 2.7 - \lfloor 2.7 \rfloor$
$= 2.7 - 2$
$= 0.7$

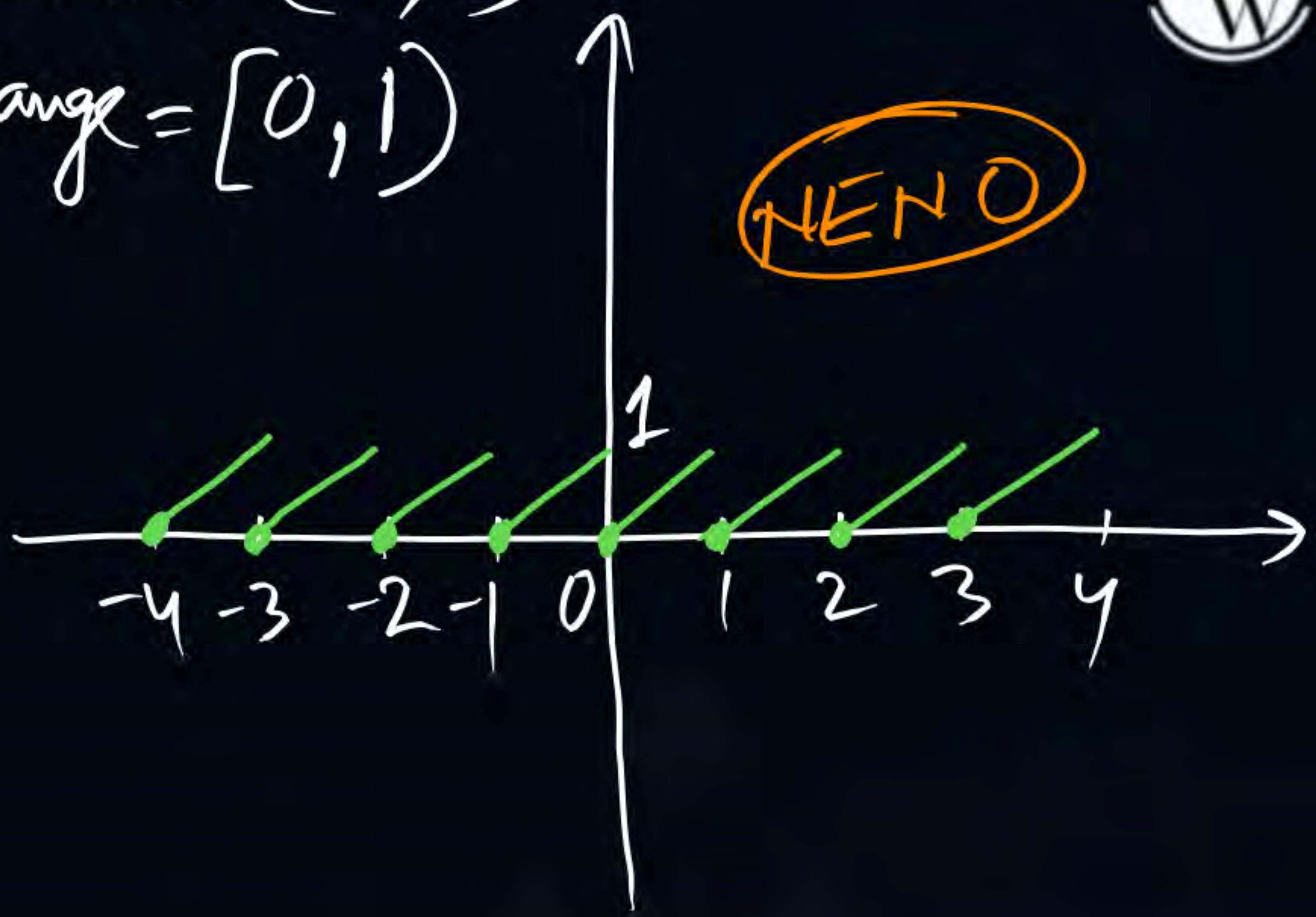
$\{-2.7\} = -2.7 - \lfloor -2.7 \rfloor$
$= -2.7 - (-3)$
$= 0.3$

<u>Def^n:</u>

$$y = \{x\} = \boxed{x - [x]} = \begin{cases} \vdots \\ x+2, & -2 \le x < -1 \\ x+1, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ x-1, & 1 \le x < 2 \\ x-2, & 2 \le x < 3 \\ x-3, & 3 \le x < 4 \\ \vdots \end{cases}$$

Domain $= (-\infty, \infty)$

Range $= [0, 1)$



NENO

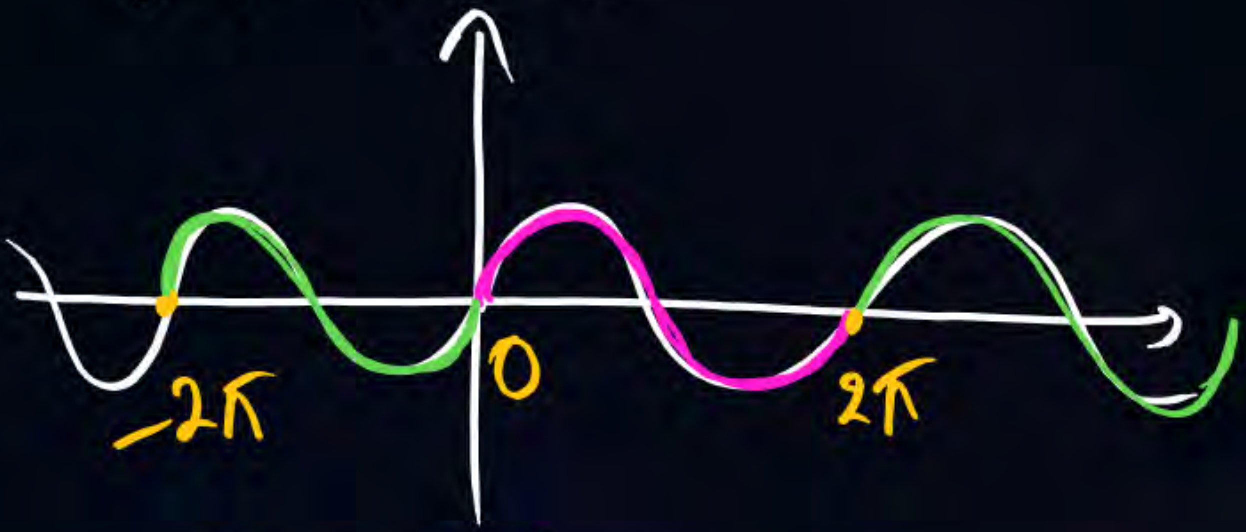
Periodic func^n $\rightarrow$ $f(x)$ is called periodic func^n of period T if

$$\boxed{f(x+T)=f(x)}$$

for eg $\sin(x+2\pi)=\sin x$ so period of $\sin x$ is 2π

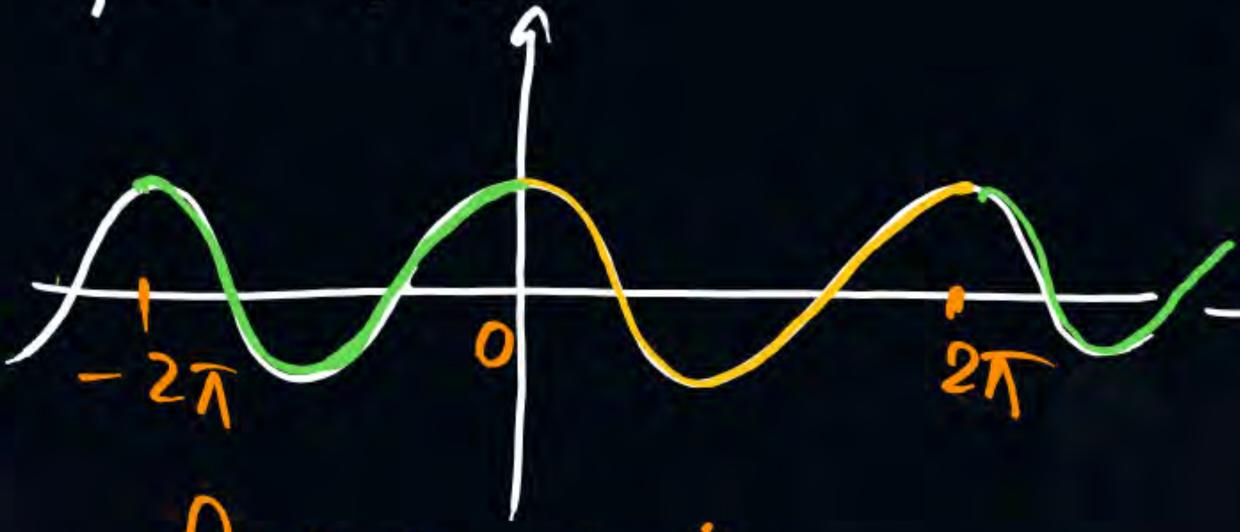
& $\tan(x+\pi)=\tan x$ is period in $\tan x$ is π

$f(x)=\sin x$



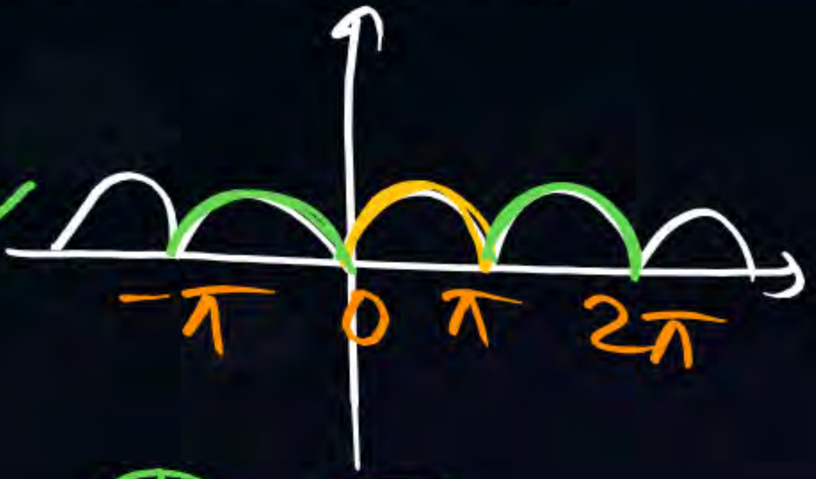
Period $(T)=2\pi$

$f(x)=\cos x$



Periodic having $T=2\pi$

$f(x)=|\sin x|$



$T=\pi$

Periodic.

* $y = \sin x$ = odd func^n

Domain $= (-\infty, \infty)$, Range $= [-1, 1]$, Period $= 2\pi$

* $y = |\sin x|$ = Even func^n

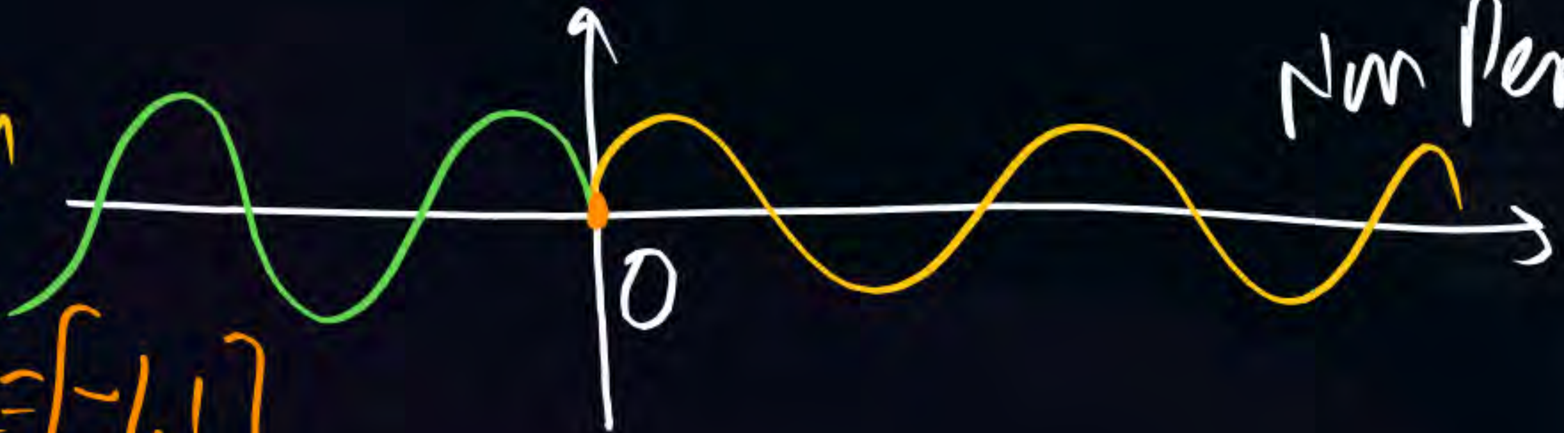
Period $= \pi$

Dom $= (-\infty, \infty)$

Range $= [0, 1]$

* $y = \sin|x|$ = Even func^n

Non Periodic



Domain $= (-\infty, \infty)$, Range $= [-1, 1]$

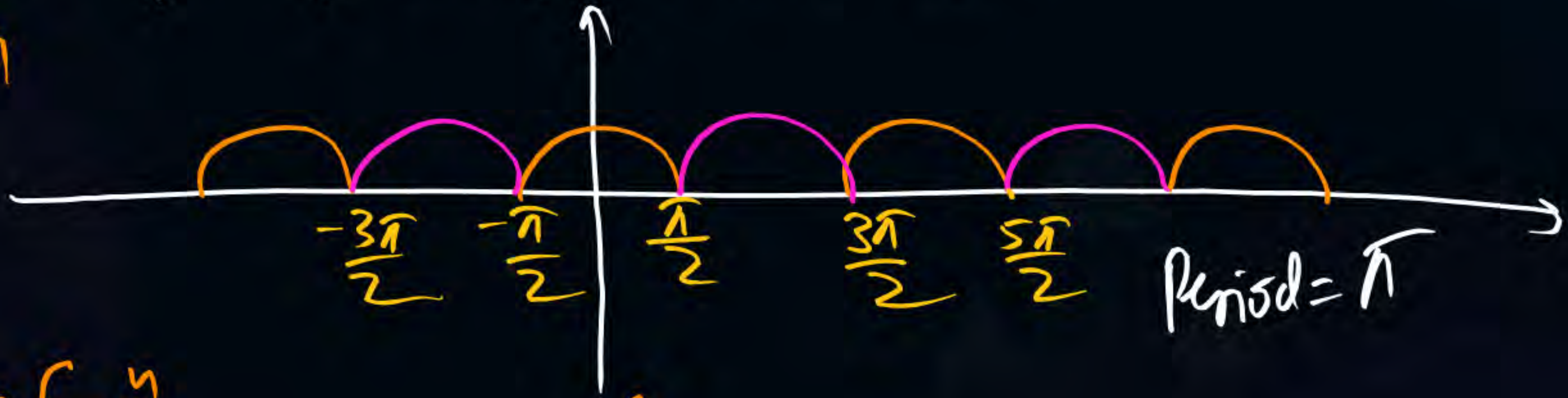
(*) $y = \cos x$ = Even func^n.

Domain = $(-\infty, \infty)$, Range = $[-1, 1]$, Period = 2π

(*) $y = |\cos x|$ = Even func^n

Dom = $(-\infty, \infty)$

Range = $[0, 1]$

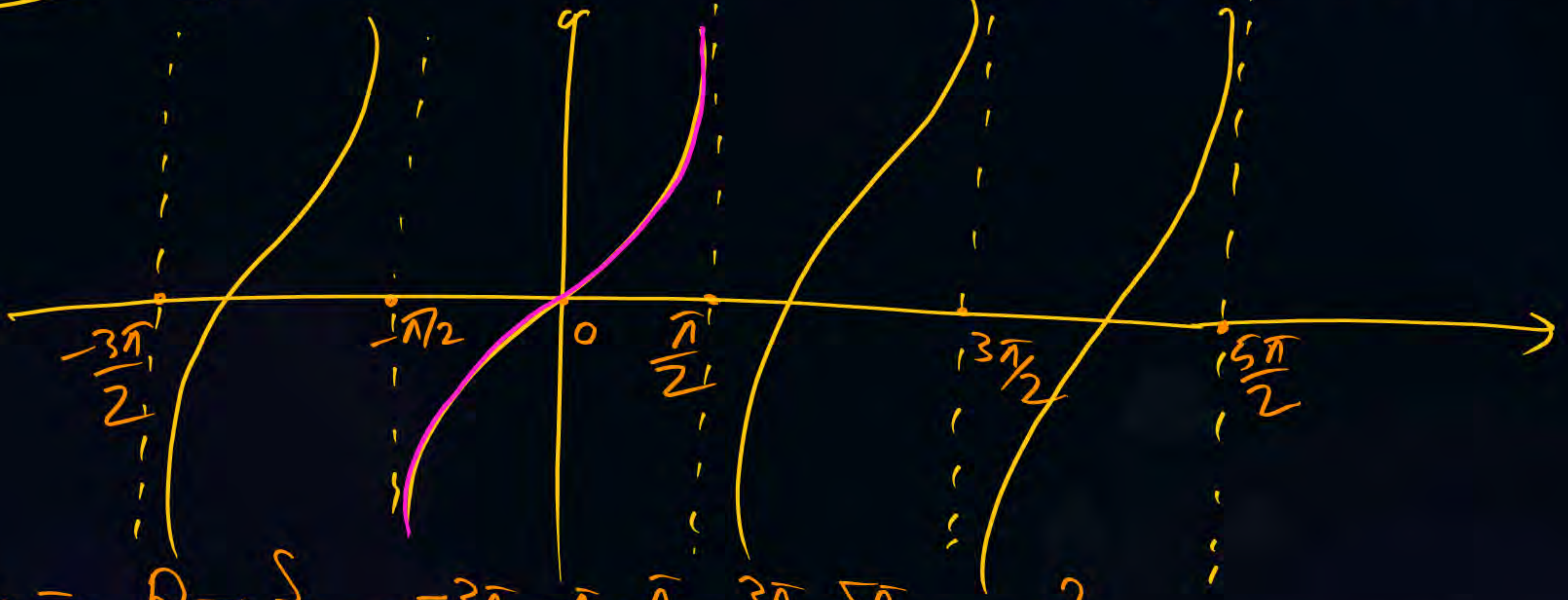


(*) $y = \cos|x|$ = Even func^n

Dom = $(-\infty, \infty)$

Range = $[-1, 1]$

(x) $y = \tan x = \dfrac{\sin x}{\cos x} = \dfrac{\text{odd}}{\text{Even}} = $ odd funcⁿ, Period $= \pi$, Range $= (-\infty, \infty)$



Domain $= R - \left\{ \ldots\ldots -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\ldots \right\}$

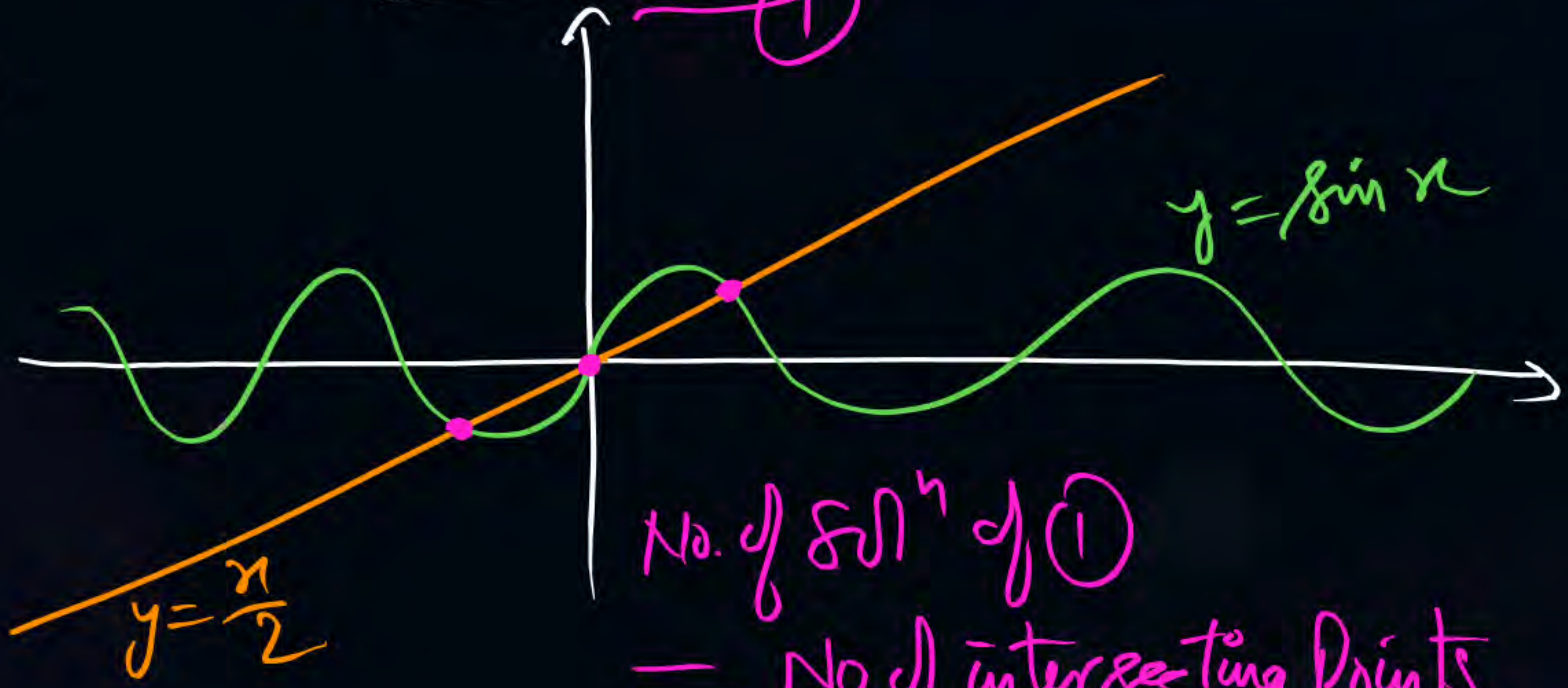
$= R - (2n+1)\frac{\pi}{2}$, $n \in I$

i.e. tan x is not defined at ODD multiples of $\frac{\pi}{2}$

Q. The number of solutions of $\boxed{\sin x = \frac{x}{2}}$ is/are ? = three —— ①

Consider $y = \sin x$

$y = \frac{x}{2}$



No. of soln of ①

= No. of intersecting Points

= three

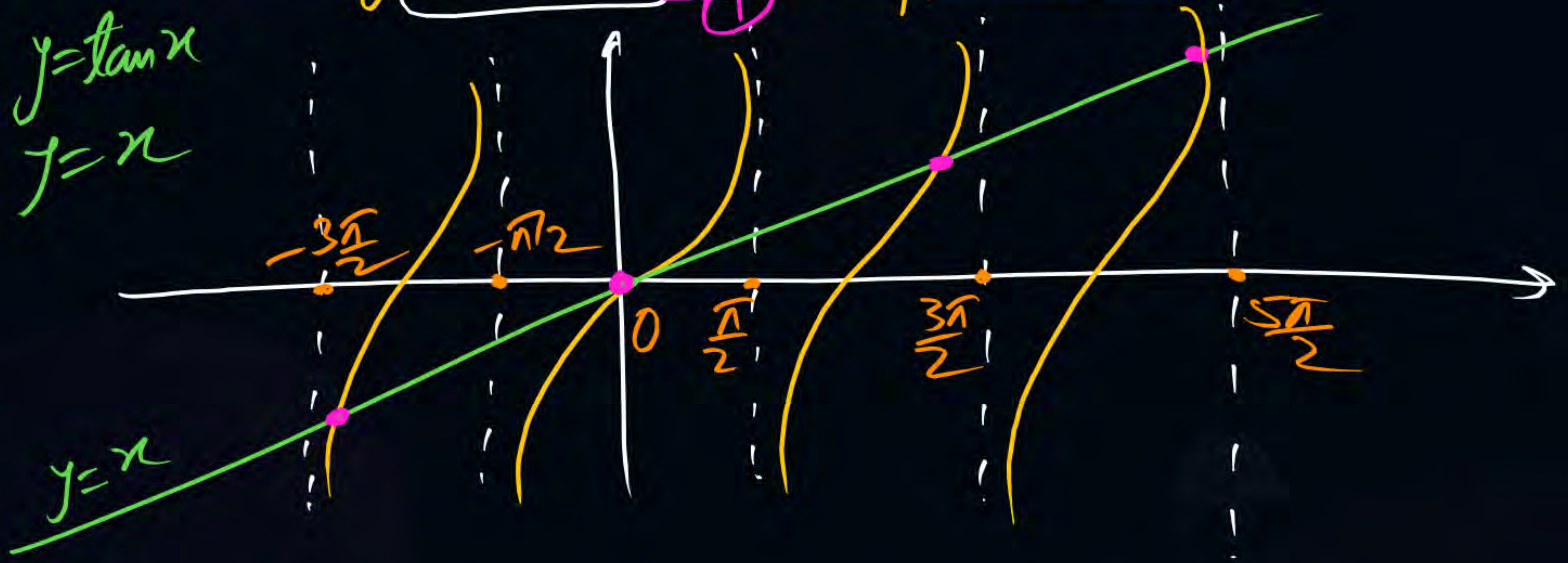
Q: One of the solution of $\boxed{\tan x = x}$ — (1) can be approximated as

(a) 1.57

(b) 3.14

(c) 4.50 ✓

(d) None



(ii) No. of sol^{n}s of (1) = ? = ∞

& sol^{n}s are $x \simeq \ldots\ldots -\frac{3\pi}{2}, 0, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\ldots$

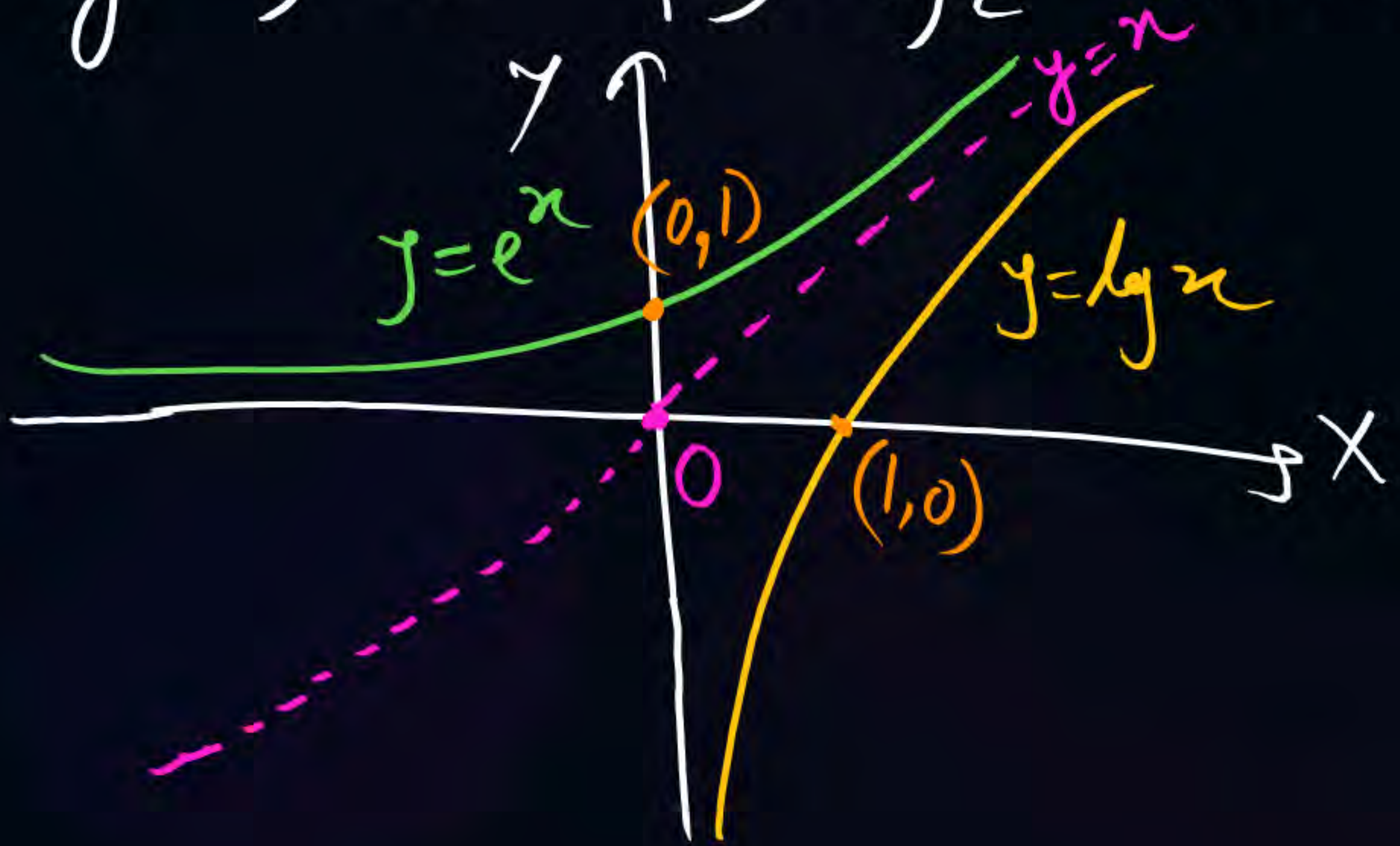
$x \approx \ldots\ldots -4.5, 0, 4.5, 7.5 \ldots\ldots$

INVERSE func^n → If $y = f(x)$ & $y = g(x)$ are Inverse func^n of each other

Then they are symmetrical about the line $y = x$

i.e. $y = x$ will behave like a mirror for $f(x)$ & $g(x)$.

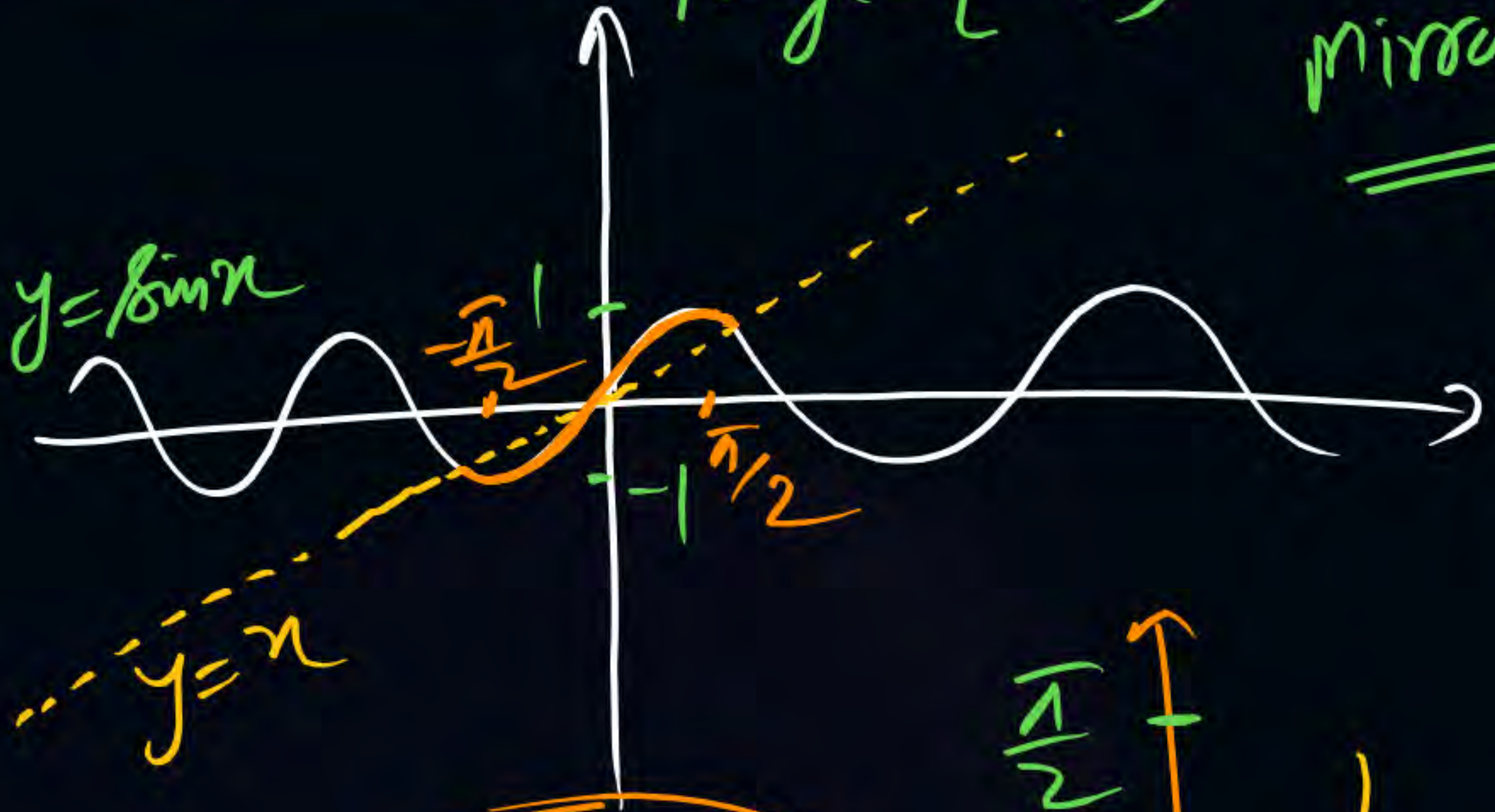
eg $y = e^x$ & $y = \log_e x$ are Inverse func^n of each other.



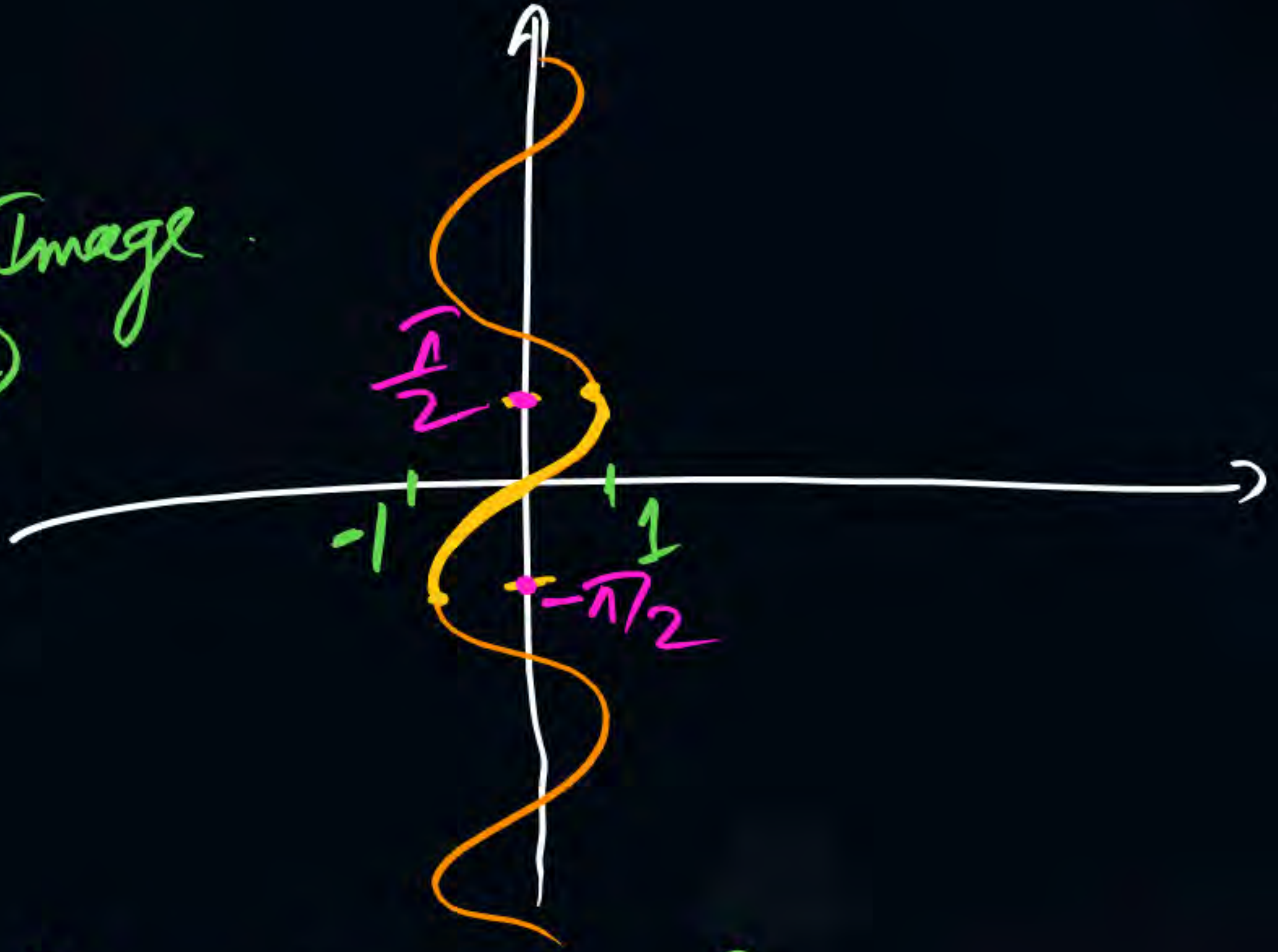
(*) $y = \sin^{-1}x$

Dom $= (-\infty, \infty)$

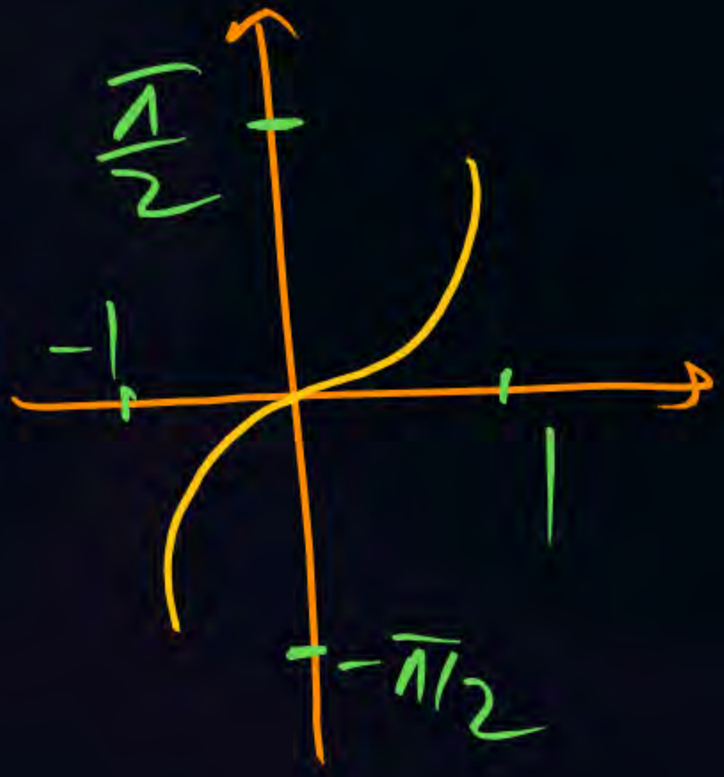
Range $= [-1, 1]$



Mirror Image $\Longrightarrow$



$y = \sin^{-1}x$

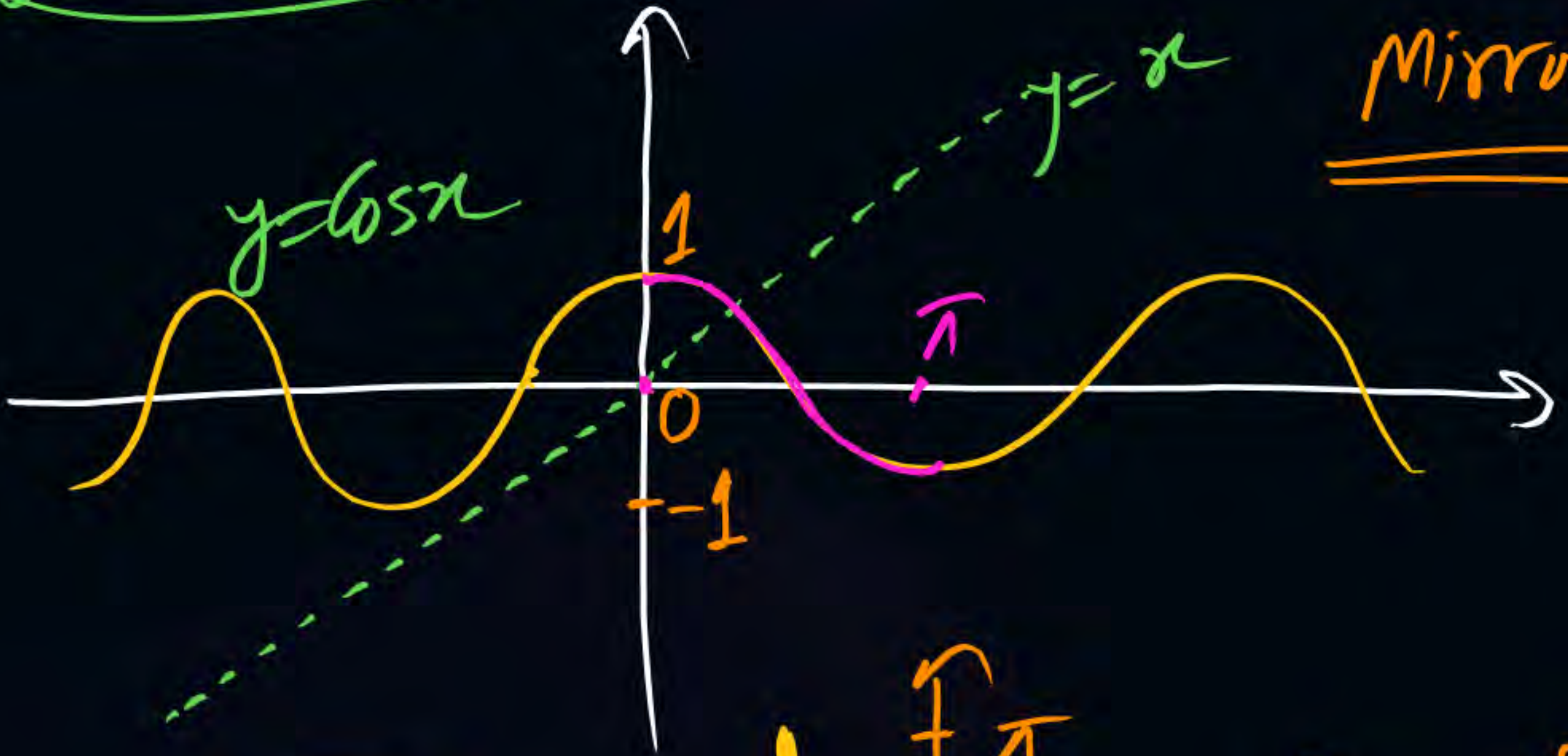


Domain $= [-1, 1]$

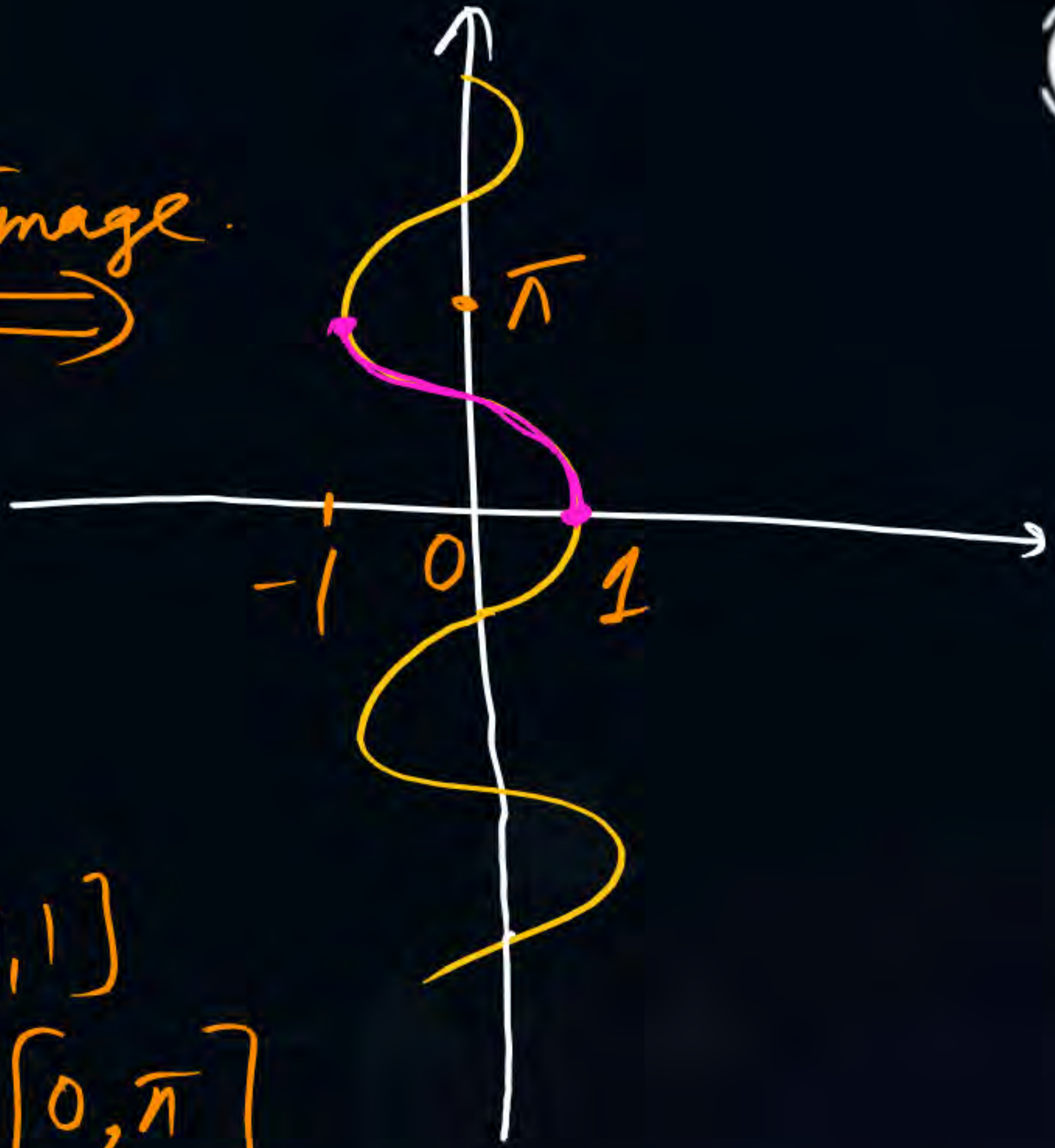
Range $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Odd func^n

$y = \cos^{-1} x$



Mirror Image



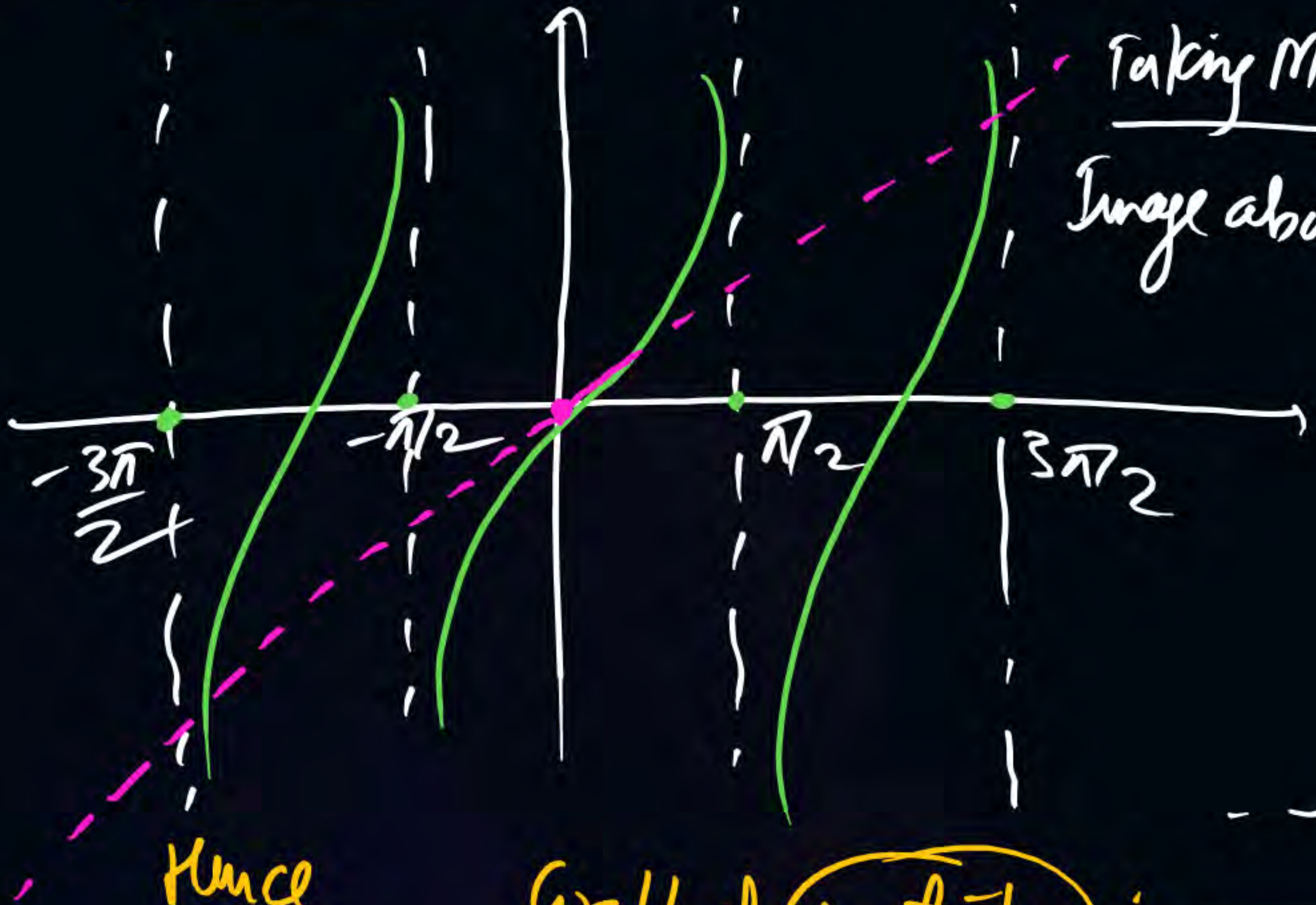
$y = \cos^{-1} x$

Dom $= [-1, 1]$

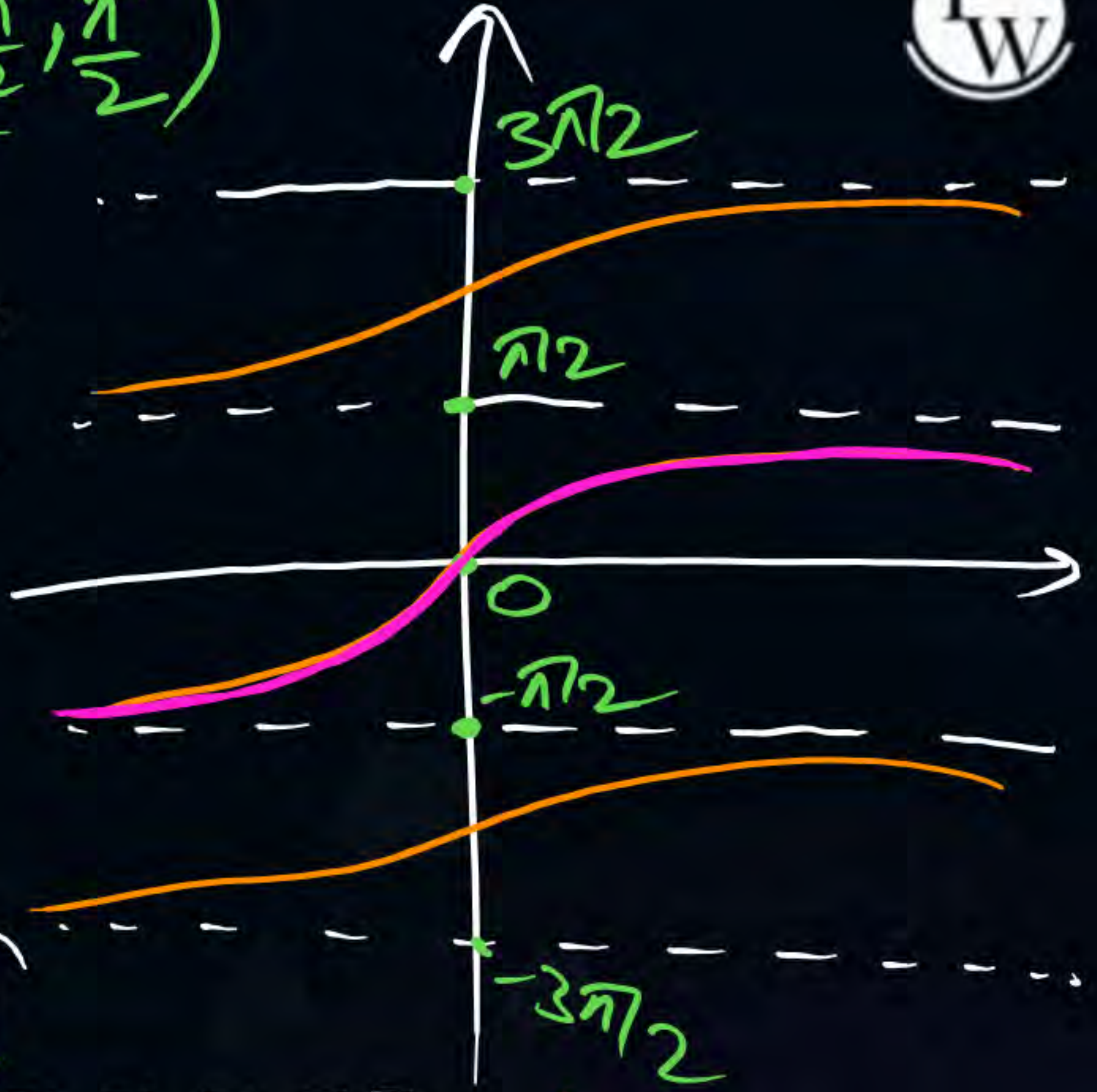
Range $= [0, \pi]$

NENO.

(x) $y = \tan^{-1}x$ Domain: $(-\infty, \infty)$, Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



Taking Mirror Image about $y = x$



Hence Graph of $y = \tan^{-1}x$ is

& it is an ODD fnc^n.

Function → If $\forall x \in A$ $\exists$ unique $y \in B$ s.t $f(x) = y$ then f is called func^n from A to B & it is denoted as $f : A \longrightarrow B$

A ↓ Domain

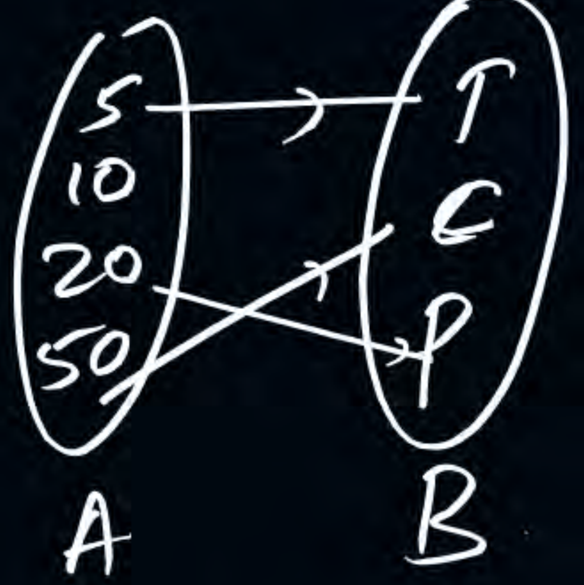
B ↓ Codomain

(*) In $y = f(x)$

y ↓ Dependent Variable (OUTPUT)

x ↓ Independent Variable (INPUT)

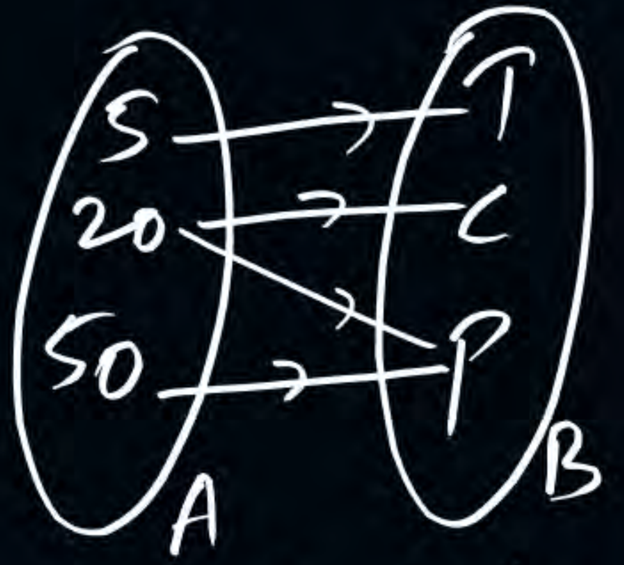
eg:



it is not a func^n

$\because \forall$ is not satisfied

eg



it is not a func^n

$\because$ uniqueness is not satisfied

eg

it is func^n, Dom = {5, 20, 50}

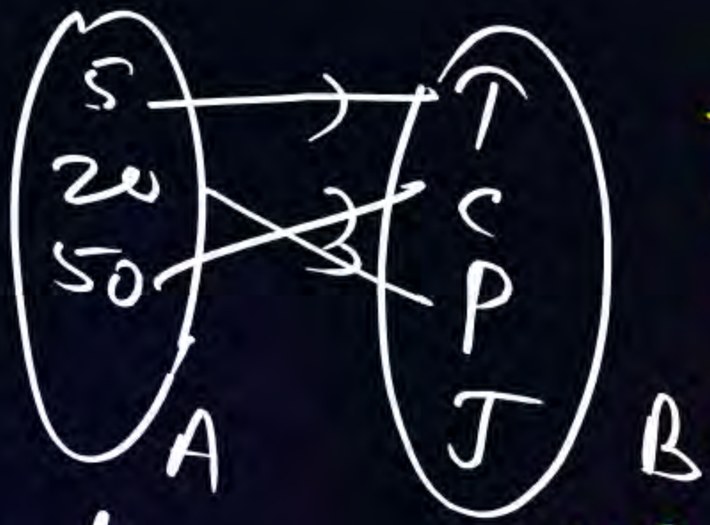
Range = {T, C, P}

Codomain = {T, C, P}

$\because$ Range = Codomain

So f is ONTO

eg



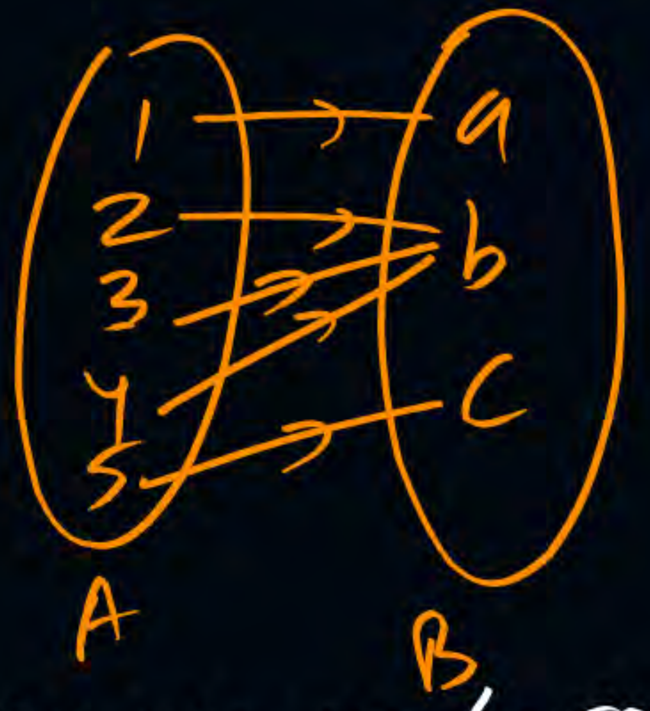
it is also func^n

Dom = {5, 20, 50}

Range = {T, C, P}

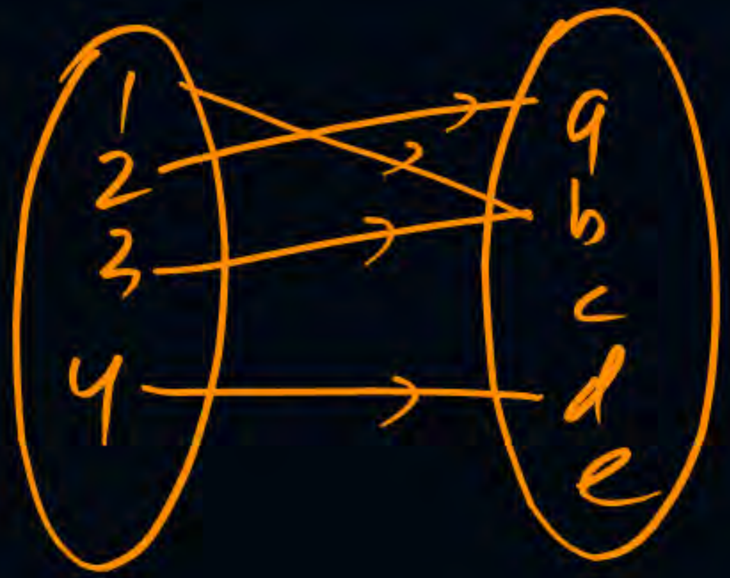
Codomain = {T, C, P, J}

$\because$ Range $\subseteq$ Codomain So f is INTO g

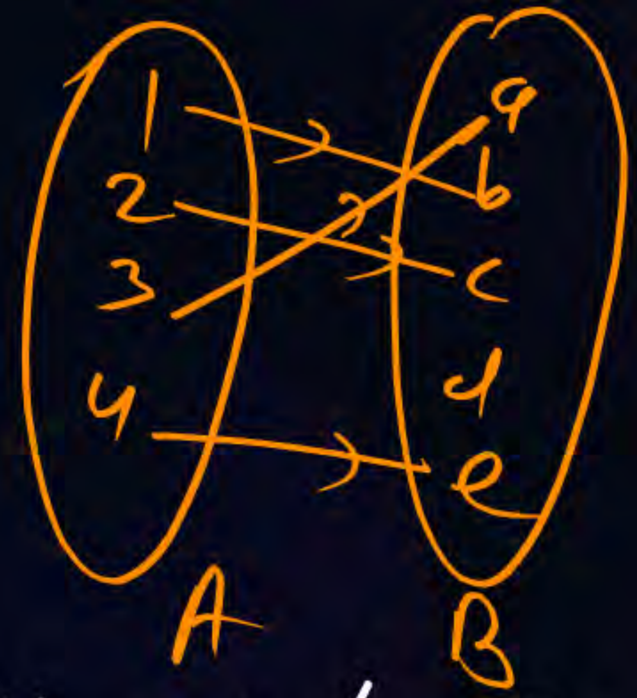


MANYONE/ONTO

ONE-ONE/ONTO

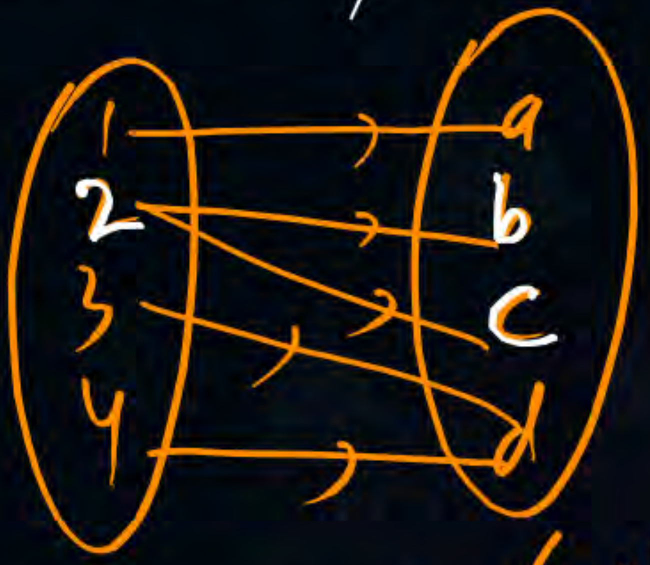


MANYONE/INTO



ONE-ONE/INTO

MANY-ONE/ONTO



ONE-MANY (not a func^n)

X

Domain of $y=f(x) \Rightarrow$ Set of permissible values of x is called Domain

Range of $y=f(x) \Rightarrow$ Set of permissible values of y is called Range

ie Restrictions imposed on Inputs (x) is called Domain

& " " " outputs (y) " " Range.

Note: Vertical line Test → If any Random line $\parallel^r$ to y axis, cuts the graph only at one point, then it is a func^n.

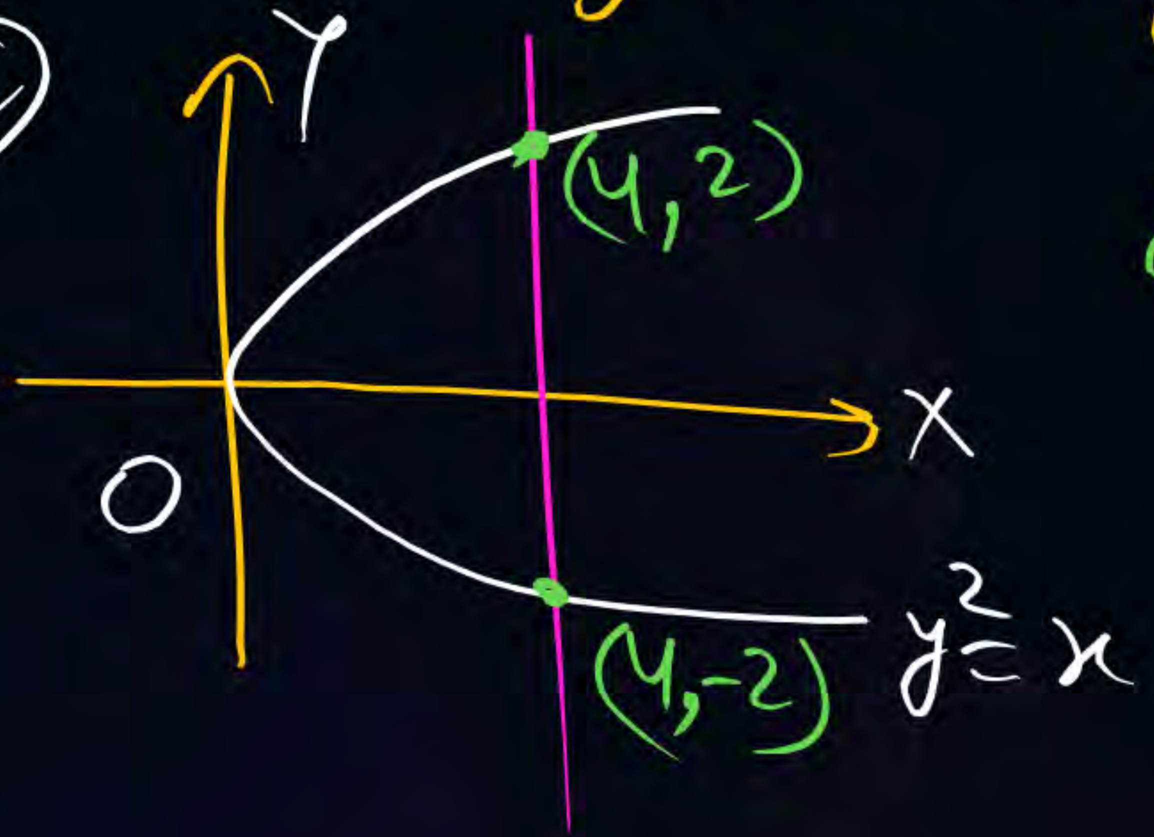
& if this line cuts the graph at more than one point, then it is not a func^n.

eg Take $y = x^2$ then At $x = 4$, $y = 16$, unique so it is func^n.

Take $y^2 = x$, then $y = \pm\sqrt{x}$ so at $x = 4$, $y = \pm\sqrt{4} = \pm 2$

i.e y is not unique for $x = 4$ so it is not a func^n

M-II



for $y = f(x) \Rightarrow f(4) \begin{cases} +2 \\ -2 \end{cases}$??

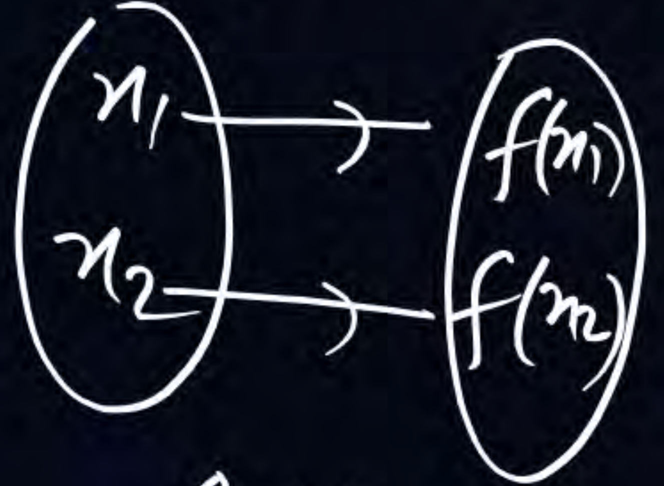
OR we can say that it is one to Many so Not a func^n.

one-one func^n: if $\boxed{x_1 \neq x_2 \iff f(x_1) \neq f(x_2)}$ where $x_1, x_2 \in D_f$

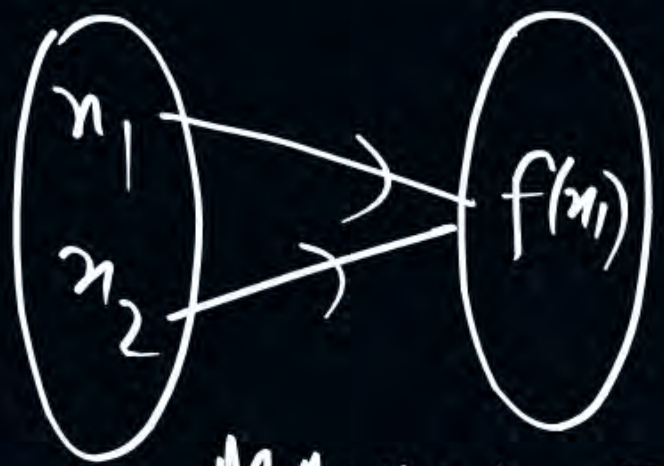
then f is called ONE-ONE i.e. Diff elements have Diff Images.

if $\boxed{x_1 \neq x_2 \iff f(x_1) = f(x_2)}$ then it is MANY ONE

i.e. for MANY ONE func^n, Different elements Same Images.



ON-ONE



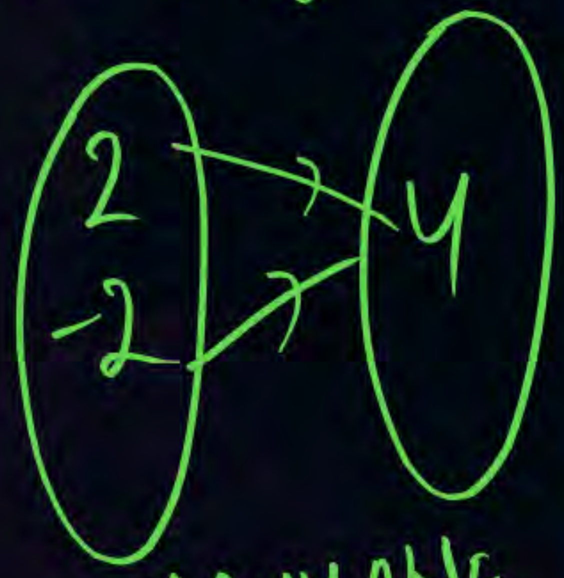
MANY ONE

Horizontal Line Test (Shortcut of checking ONE-ONE OR MANY ONE)

If any Random line $\parallel^r$ to x axis cuts the graph only at one point then it is one-one otherwise MANY-ONE

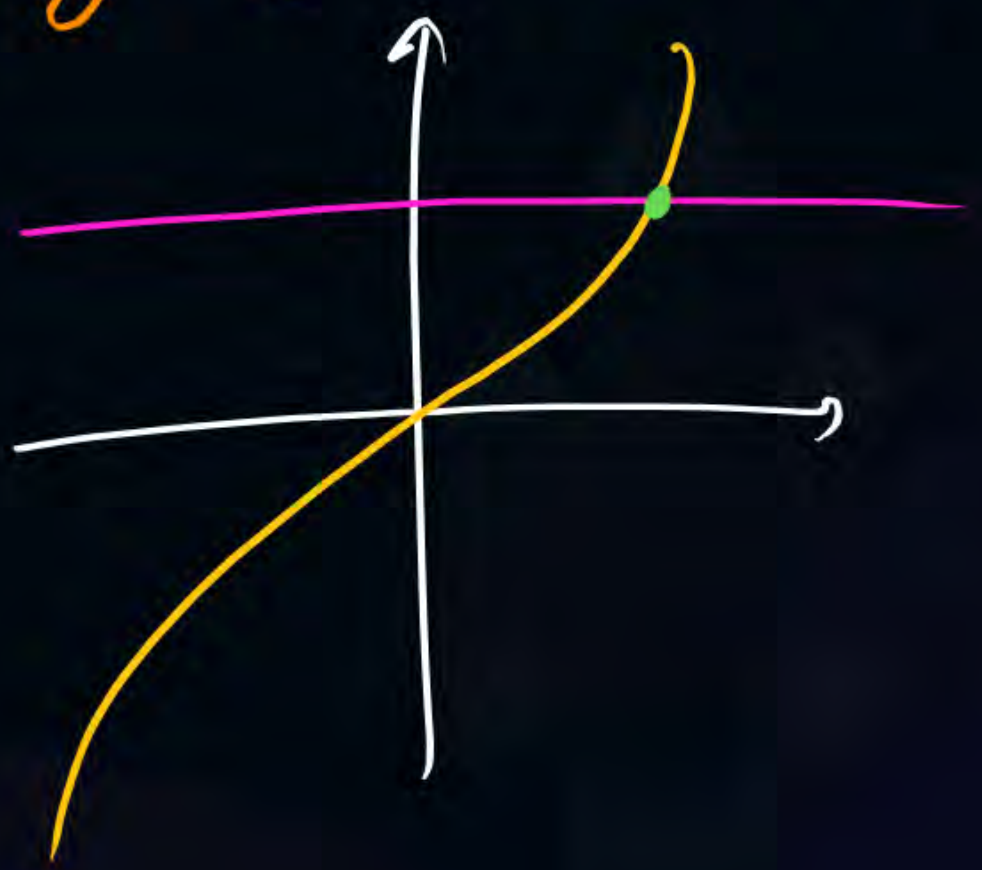
eg $y = x^2$ (M-I) $\because$ At $x = 2,\ y = 4$ & at $x = -2,\ y = 4$

$\therefore$ MANYONE.



MANYONE

eg $y = x^3$ = ONE-ONE



Conclusions: (1) Vertical line test → To check validity of func^n.

(2) Horizontal line test → To check validity of one-one

(3) If $\boxed{\text{Range} = \text{Codomain}}$ → Then func^n is ONTO.

(4) ONE-ONE func^n $\Longrightarrow$ INJECTIVE MAPPING

(5) ONE-ONE/ONTO $\Longrightarrow$ BIJECTIVE MAPPING / one-one correspondence.

(6) one one-one/ONTO func^n have Inverse.

i.e if $f(x)$ is one one/ONTO then only $f^{-1}(x)$ exist

$\Longleftrightarrow$

Domain: Set of permissible values of x is called Domain of $y=f(x)$ & there is No shortcut Method to find Domain of given func^n; it can be calculated only by using common sense or by previous knowledge.

eg: find the Domain of following func^n;

① $y=f(x)=\dfrac{1}{x^2-5x+6}$

$$y=\frac{1}{(x-2)(x-3)}$$

At $x=2$ & $x=3$, $y=$ DNE

Dom $= R-\{2,3\}$

② $y=\sqrt{-x^2+5x-6}$

w.k.that, $-x^2+5x-6\geq 0$

$$x^2-5x+6\leq 0$$

$$(x-2)(x-3)\leq 0$$

$$2\leq x\leq 3$$

So Dom $=[2,3]$

CROSS Check

eg $y = \sqrt{-x^2+5x-6}$, Dom = [2, 3]

let us take $x = 1$ then

$y = \sqrt{-1^2+5(1)-6} = \sqrt{-2}$ = Not Real

let us take $x = 5$ then

$y = \sqrt{-5^2+5(5)-6} = \sqrt{-6}$ = Not Real.

ie Permissible values of x lies only in b/w 2 & 3

eg let $x = 2.5$

$y = \sqrt{-(2.5)^2+5(2.5)-6}$

$= \sqrt{-6.25+12.5-6}$

$= \sqrt{-12.25+12.50}$

$= \sqrt{0.25} = 0.5$

ie y is also Real.

ie Valid

Common Sense:

① $\dfrac{1}{f(x)}$; $f(x) \neq 0$

② $\sqrt{f(x)}$; $f(x) \geq 0$

③ $\dfrac{1}{\sqrt{f(x)}}$; $f(x) > 0$

④ $\log_e(f(x))$; $f(x) > 0$

⑤ $\sin(f(x))$; $-\infty < f(x) < \infty$

Qs $y = f(x) = \log(x-3)$,

w.k that, $(x-3) > 0 \Rightarrow x > 3$

So Dom $= (3, \infty)$

Qs $y = f(x) = \sin(x-3)$

w.k. that, $-\infty < (x-3) < \infty$

$-\infty < x < \infty$

i.e Dom $= (-\infty, \infty)$

Eg: $\boxed{y=\sin^{-1}(x-3)}$

or $\sin y = x-3$

W.K. that, $-1 \le \sin y \le 1$

$$-1 \le x-3 \le 1$$

$$-1+3 \le x \le 1+3$$

$$2 \le x \le 4$$

$$\text{Dom} = [2,4]$$

Q: $y = x^x$

$$y = e^{\log_e x^x} = e^{(x \log_e x)}$$

So Domain is $(0,\infty)$ or $x>0$

$y = x \ne e^{\log_e x}$, Dom $= (-\infty, \infty)$

Doubts: ① Evaluate $(-2)^{-2} = ? = \frac{1}{(-2)^2} = \frac{1}{4}$ ✓

② if $y = x^x$ then evaluate $y(-2) = ?$

$\because$ Domain $= (0, \infty)$ so $y(-2) =$ DNE

Def^n of func^n in My Language → "it is special type of Relationship b/w two Variables x & y under certain Restrictions"

& these Restrictions on x are called Domain of func^n.

(*) one-one Mapping $\simeq$ INJECTIVE Mapping

(*) one-one / ONTO Mapping $\simeq$ $\underbrace{\text{BIJECTIVE Mapping}}_{\text{one-one correspondence}}$

(*) ONTO MAPPING $\rightleftharpoons$ SURJECTIVE

eg if, $y = f(x) = x^2$; $\underset{\text{Domain}}{R} \to \underset{\text{Codomain}}{R}$ then find Range = ? $= [0, \infty)$

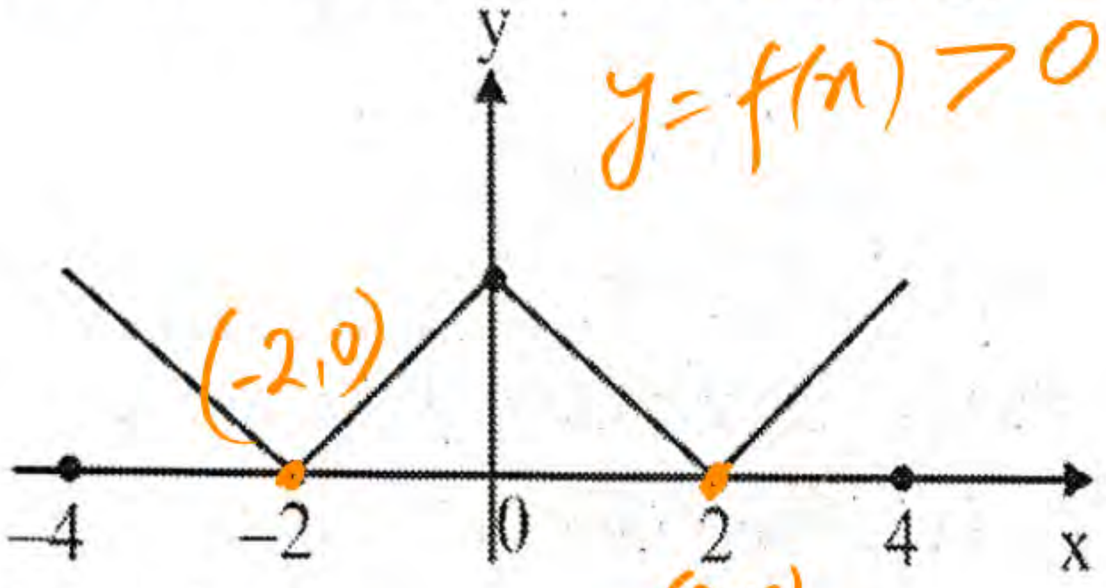
$\because$ Range $\subset$ Codomain $\Rightarrow$ INTO

eg if, $y = f(x) = x^2$; $\underset{\text{Dom}}{R} \to \underset{\text{Codom.}}{R^+}$ then find it's Nature?

& Range $= R^+$ Hence ONTO

[MCQ] [GATE-ME-2023: 1M]

The figure shows the plot of a function over the interval [–4, 4], which one of the options given CORRECTLY identifies the function?



(a) $|2 - x|$

(b) $|2 - |x||$

(c) $|2 + |x||$

(d) $2 - |x|$

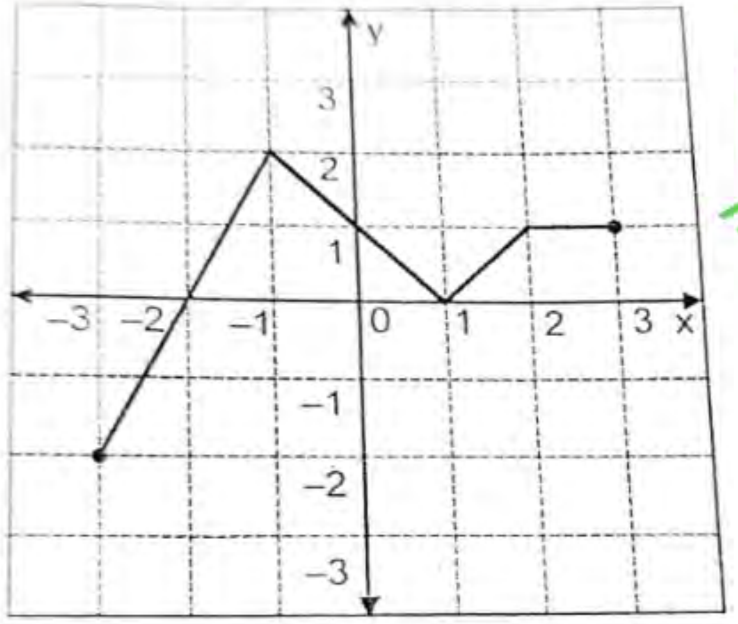
At $x = -2, y = 0$

(a) $y = |2 - x| = |2 - (-2)| = |4| = 4$

(b) $y = |2 - |x|| = |2 - |-2||$
$= |2 - 2| = 0$

So (b)

Which of the following function(s) is an accurate description of the graph for the range(s) indicated?



HWQ

(i) $y = 2x + 4$ for $-3 \leq x \leq -1$

(ii) $y = |x - 1|$ for $-1 \leq x \leq 2$

(iii) $y = ||x| - 1|$ for $-1 \leq x \leq 2$

(iv) $y = 1$ for $2 \leq x \leq 3$

(a) (i), (ii) and (iii) only

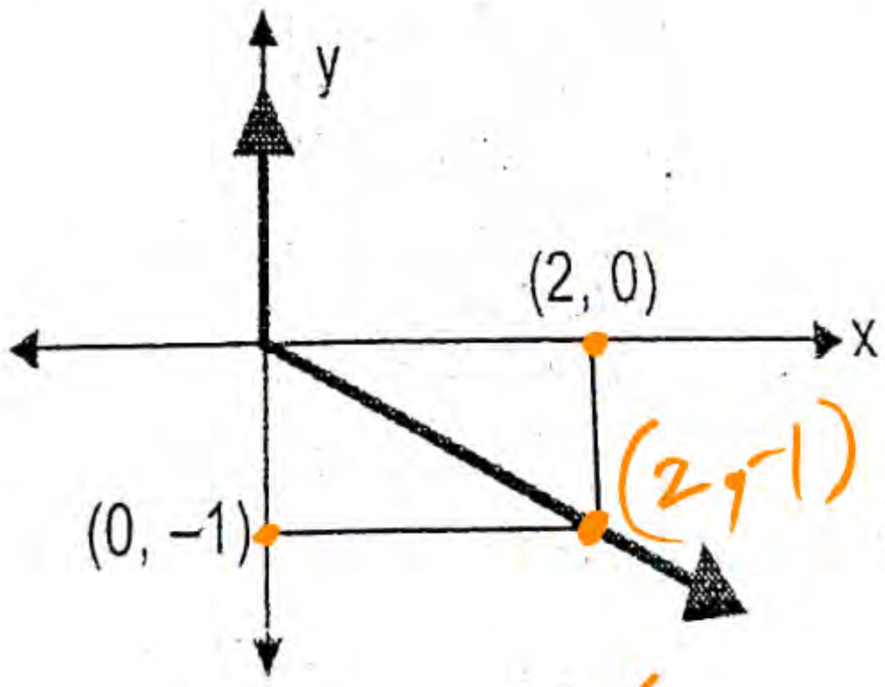
(b) (i), (ii) and (iv) only

(c) (i) and (iv) only

(d) (ii) and (iv) only

[GATE 2018 (CE-MorningSession

Choose the most appropriate equation for the function drawn as a thick line, in the plot below



(a) $x = y - |y|$
(b) $x = -(y - |y|)$
(c) $x = y + |y|$
(d) $x = -(y + |y|)$

[GATE-2015-CS-SET-3; 2 Marks]

$x = 2, y = -1$

Taking (a): $x = y - |y|$

$2 = -1 - |-1|$

$= -1 - (+1)$

$2 = -2$ Not Valid for (a) ✗

Taking (b) $x = -(y - |y|)$

$2 = -(-2)$

$2 = 2$ i.e. Valid for (b) ✓

$y > 0$, $x = -(y - |y|) = -(y - y) = 0$ i.e. y axis

$y < 0$, $x = -(y - |y|) = -(y - (-y)) = -2y$

or $y = -\frac{1}{2}x$

THANK you