DS & AI

CS & IT

Linear Algebra

Lecture No. 05

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Recap of previous lecture

Topic

RANK (part 1)

Topics to be Covered

Topic

→ RANK (part 2)

→ LD / LI vectors.

QUICK RECAP :-

1. Singular Mat if $|A| = 0$
2. Non Sing Mat if $|A| \neq 0$
3. Invertible Mat if A^{-1} exist & $A^{-1} = \frac{adj A}{|A|}$
4. Real Mat if $\bar{A} = A$ or $A^{\theta} = A^{T}$
5. Complex Mat if $\bar{A} \neq A$
6. Symm Mat if $A^{T} = A$
7. Skew Symm Mat if $A^{T} = -A$
8. Hermitian Mat if $A^{\theta} = A$
9. Skew Herm Mat if $A^{\theta} = -A$
10. Idempotent if $A^{2} = A$
11. Involutary if $A^{2} = I$
12. Nilpotent if $A^{k} = 0$
13. orthogonal Mat if $AA^{T} = I$ or $A^{-1} = A^{T}$
14. unitary Mat if $AA^{\theta} = I$ or $A^{-1} = A^{\theta}$
15. U.T.M $A = [a_{ij}]_{n \times n}$ where $a_{ij} = 0 \ \forall \ i > j$
16. L.T.M $A = [a_{ij}]_{n \times n}$ where $a_{ij} = 0 \ \forall \ i < j$
17. Diag Mat $A = [a_{ij}]_{n \times n}$ where

$$a_{ij} = \begin{cases} 0 & , i \neq j \\ \text{atleast one element is Non Zero} & , i = j \end{cases}$$

RECAP

RANK

(*) Submatrix $\rightarrow$ By deleting some Rows or Some Columns or Both, the Matrix obtained is called Submatrix.

Defn of Rank: "It is the order of Non singular Submatrix of Highest order that can exist in a given Mat"

Defn In Books:

if $\boxed{\rho(A_{6\times 7}) = 4}$ then

- $\exists$ at least one Non Singular submatrix of order 4×4
- Every Square submatrix of order 5×5 & 6×6 are Singular

Echelon Form RECAP (Triangular Form) →

Any Mat $A_{m \times n}$ is said to be in Echelon Form if

① Number of Zeros before the 1^{st} Non Zero element in a Row should be in an Increasing order in the subsequent Rows.

② Every Zero Row (if exist) should occur at the bottom of a Mat.

Note: ① $\boxed{\rho(\text{Echelon form}) = \text{Number of Non Zero Rows.}}$

② Any Mat can be converted into an E-form by using E-operations.

③ It is advisable to apply only E-Row operations while converting given Mat into an E-Form. (as per our syllabus)

RECAP

Flowchart of converting given Mat into an E-Form →

1. Make a_{11} unity (Not compulsory but advisable)
2. Make all the elements of C_1 (that lies below a_{11}) Zero by using E-Row operation
3. Make a_{22} unity (Not compulsory but advisable)
4. Make all the elements of C_2 (that lies below a_{22}) Zero by " " "
5. Make a_{33} unity & so on - - - - -

Note: Take care, In E-Form, $a_{21} = $ Zero.

Eg: $A = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$ then $\rho(A) = \underline{?}$

$$\xrightarrow[\substack{R_3 \to R_3 - R_2 \\ R_2 \to R_2 - R_1}]{R_4 \to R_4 - R_3} \begin{bmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_2]{R_4 \to R_4 - R_2} \begin{bmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4} = A_1$$

By observation, $\rho(A_1) =$ two

$\because A \sim A_1 \Rightarrow \rho(A) =$ two $\underline{\text{Ans}}$

Sol: $A = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{bmatrix}_{4\times 4} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & -1 & -2 & -3 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \xrightarrow[R_4 - 3R_1]{R_3 - 2R_1} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & +6 \end{bmatrix}$

$\xrightarrow[R_4 + 2R_2]{R_3 + R_2} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ = Echelon form

So $\rho(A) =$ Two.

Qu. (NET) $A = \begin{bmatrix} 2 & & & & & \\ & -1 & & \text{Zero} & & \\ & & 0 & & & \\ & & & 3 & & \\ & \text{Zero} & & & 4 & \\ & & & & & 0 \end{bmatrix}_{6\times 6}$ or $A = \text{diag}(2, -1, 0, 3, 4, 0)$ then $\rho(A) = ?$

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 6} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_5} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 6}$$

So $\rho(A) =$ No. of Non Zero Rows $= 4$.

(M-I) $A_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}_{4\times 4} =$ Diag Mat

$=$ Non Singular $\Rightarrow \rho(A) =$ Four.

Q. $A = \begin{bmatrix} 2 & -1 & 4 & 2 \\ 1 & 3 & 0 & -2 \\ -3 & -2 & -4 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix}_{4\times 4} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 & 0 & -2 \\ 2 & -1 & 4 & 2 \\ -3 & -2 & -4 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix} \xrightarrow[R_3+3R_1]{R_2-2R_1} \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & -7 & 4 & 6 \\ 0 & 7 & -4 & -6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$

(a) 1

(b) 2

(c) 3

(d) 4

$\xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & -7 & 4 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix} \xrightarrow{R_2+3R_4} \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & -1 & 7 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix}$

$\xrightarrow{R_4+2R_2} \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & -1 & 7 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 15 & 33 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & -1 & 7 & 15 \\ 0 & 0 & 15 & 33 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ = Echelon Form = B

$\rho(B) = 3 \because A \sim B \Rightarrow \rho(A) = 3$.

The rank of the matrix $\begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 4 & 2 & 3 & 1 \\ 3 & 12 & 24 & 2 \end{bmatrix}$

(a) 3 (b) 1

(c) 2 (d) 4

$$\xrightarrow[R_4 - 3R_1]{R_3 - 4R_1} \begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 0 & -14 & -29 & -27 \\ 0 & 0 & 0 & -19 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & -14 & -29 & -27 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -19 \end{bmatrix} = \text{E-Form.}$$

$\rho(A) = $ Four.

M-II

$$|A| = (-3)\begin{vmatrix} 1 & 4 & 7 \\ 4 & 2 & 1 \\ 3 & 12 & 2 \end{vmatrix} = (-3)\begin{vmatrix} 1 & 4 & 7 \\ 4 & 2 & 1 \\ 0 & 0 & -19 \end{vmatrix}$$

$$= (-3)\left[(-19)(2-16)\right] \neq 0 \text{ i.e. } (A \text{ is Non sing})_{4\times 4} \Rightarrow \rho(A) = 4$$

2015 / 2M

if $A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}_{5\times 5} \xrightarrow[R_4 \leftrightarrow R_5]{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_5 \to R_5 + R_1 + R_2 + R_3 + R_4} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 4$

(a) 5

(b) 4

(c) 3

(d) 2

M-II

$$\xrightarrow{R_4 + R_1} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_4 + R_2} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_4 + R_3} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_5 + R_4} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \text{E-Form}$$

so $\rho(A) = 4$

M III

WRONG APP

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}_{5\times 5} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \rho(A) = 5$$

But $a_{21} \neq 0$??

Applying elementary transform to a matrix its rank ______.

(a) increases
(b) decreases
(c) does not change
(d) None of the above

Properties of Rank :→

1. $\rho(A_{m\times n}) \geq 1$ & $\rho(A_{m\times n}) \leq \min\{m,n\}$ eg $\rho(A_{6\times 4}) \leq 4$

2. $\rho(\text{Null Mat}) = 0$ (defined) ie $\boxed{1 \leq \rho(A_{m\times n}) \leq \min\{m,n\}}$

3. if $A_{n\times n}$ s.t A is Non Singular then $\rho(A) = n$

4. $\rho(A+B) = \rho(A)+\rho(B)$ (F) $\left[\because \rho(A+B) \leq \rho(A)+\rho(B) \text{ (T)}\right]$

5. $\rho(A) = \rho(A^T) = \rho(A^\theta) = \rho(A^{-1}) = \rho(AA^T) = \rho(AA^\theta)$

⇒ ~~$\rho(A)$ & ρ(orthogonal Mat) are equal.~~ ISSEY BADA PAAP NAHI HOO SAKTA.

($\because$ If $AA^T = I$ then A is called O-Mat)

(6) if A & B are two Matrices s.t AB is defined then

$\boxed{\rho(AB) \le \min\{\rho(A), \rho(B)\}}$ eg if $\rho(A_{5\times 6}) = 4$ & $\rho(B_{6\times 7}) = 3$

then $\rho(AB)_{5\times 7} \le \min\{4, 3\} \Rightarrow \rho(AB) \le 3$

ie

RANK of the product can never exceed their individual Ranks.

(7) $\rho(\text{Row Mat}) = \rho(A_{1\times n}) = 1$

$\rho(\text{Column Mat}) = \rho(B_{n\times 1}) = 1$

$\rho(\text{Row} \times \text{Column}) = \rho(AB)_{1\times 1} = 1 \text{ or } 0$

$\rho(\text{Column} \times \text{Row}) = \rho(BA)_{n\times n} = 1 \text{ or } 0$ (using prop. 6).

Q1: if $A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}_{4\times 4}$ then $\rho(A) = ?$

$\because |A| = \ldots\ldots\ldots = 5 \neq 0$.

i.e. A is Non Singular of 4×4

$\Rightarrow \rho(A) = 4$ **Ans** (using Prop 3)

$$|A| = \begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} \xrightarrow{C_1 \to C_1 + (C_2 + C_3 + C_4)} \begin{vmatrix} 5 & 1 & 1 & 1 \\ 5 & 2 & 1 & 1 \\ 5 & 1 & 2 & 1 \\ 5 & 1 & 1 & 2 \end{vmatrix} = 5 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$$

$$\xrightarrow[R_3 - R_1,\ R_4 - R_1]{R_2 - R_1} = 5 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 5\left[(1)\right] = 5$$

Q. if $\rho(A)=3$ where $A=\begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}_{4\times4}$

then No. of different Values of μ will be?

(a) 1 = one

(b) 2 = two

(c) 3 = three

(d) All of above

$\rho(A)=3 \Rightarrow |A|_{4\times4}=0$

$$\begin{vmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{vmatrix}=0$$

$\mu^3-6\mu^2+11\mu-6=0$

$\mu^3-6\mu^2+11\mu-6=0$

$(\mu-1)(\mu-2)(\mu-3)=0$

$\mu=1,2,3$

Q. if $\underbrace{\rho(A_{m\times n})=n}_{n\le m}$, $\underbrace{\rho(B_{n\times p})=p}_{p\le n}$ then $\rho(AB)=?$ (a) m (b) n

(c) p (d) $n+p$

$$\boxed{p\le n\le m} \Rightarrow \rho(AB)_{m\times p}\le p$$

Let $P = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ and $Q = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ be two matrices. Then the rank of P + Q is 2.

$$P+Q = \begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 8 & 8 & 8 \end{bmatrix}_{3\times 3} \Rightarrow \text{Consider } A_1 = \begin{bmatrix} 0 & -1 \\ 8 & 9 \end{bmatrix}_{2\times 2} \Rightarrow A_1 \text{ is non sing.}$$

So $\rho(A) = $ two.

$$|P+Q| = 0-(-1)[64-80]+(-2)[64-72]$$

$$= +(1)[-16] - 2[-8]$$

$$= -16+16 = 0$$

i.e. $(P+Q)$ is singular $\Rightarrow \rho(P+Q) \neq 3$

Qs (2015) If $A=\begin{bmatrix} p & q \\ r & s \end{bmatrix}$, $B=\begin{bmatrix} p^2+q^2 & rp+sq \\ pr+qs & r^2+s^2 \end{bmatrix}$ s.t. $\rho(A)=N$ then $\rho(B)=$?

(a) N

(b) $N-1$

(c) $2N$

(d) N^2

$\because A.A^T=\begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} p & r \\ q & s \end{bmatrix}=\begin{bmatrix} - & - \\ - & - \end{bmatrix}=B$

w.k. that $\rho(A)=N$

$\Rightarrow \rho(AA^T)=N$

or $\rho(B)=N$

M-II let $p=2, q=-3, r=1, s=4$

$A=\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 13 & -10 \\ -10 & 17 \end{bmatrix}$

$|A|\neq 0$, $|B|\neq 0$

$\rho(A)=2=N$ $\rho(B)=2=N$

again, $p=2, q=0, r=4, s=0$.

$A=\begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 4 & 8 \\ 8 & 16 \end{bmatrix}$ $\Rightarrow \rho(A)=1=N$

$\rho(B)=1=N$ Ans

Q: if $A = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}_{1\times3}$, $B = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}_{3\times1}$ then $\rho(AB) = ?$ & $\rho(BA) = ?$

$$AB = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} (8-2+3) \end{bmatrix} = \begin{bmatrix} 9 \end{bmatrix}_{1\times1} \Rightarrow \rho(AB) = \text{one}$$

$$BA = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 8 & -4 & 12 \\ 4 & -2 & 6 \\ 2 & -1 & 3 \end{bmatrix}_{3\times3} \neq O \text{ so } \rho(BA) = \text{one}$$

$$\because BA = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \\ 8 & -4 & 12 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3} \Rightarrow \rho(BA) = \text{one}$$

Qs (CE) if $X=[x_1\ x_2\ x_3 \ldots x_n]'$ is an ordered n-tuple Non Zero vector

s.t $XX'=V$ then $\rho(V)=$?

(a) n

(b) $n-1$

(c) 1

(d) n^2

$X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ = Column Mat, $X'=[x_1\ x_2\ x_3 \ldots x_n]_{1\times n}$ = Row Mat

$\rho(V)=\rho(XX')_{n\times n}=\rho[\text{Column Mat}\times\text{Row Mat}]_{n\times n}=1.$

(using prop 7)

A be 3×3 matrix and Rank of A^3 is 2. Then rank of A^6 will be :

(a) 2 (b) 3

(c) 1 (d) $\cancel{1 \le 2}$, $1 \le \rho(A^6) \le 2$

$\because A_{3\times3} \Rightarrow (A^3)_{3\times3}$ & $(A^6)_{3\times3}$

$\because \rho(A^3) = 2 \Rightarrow |A^3|_{3\times3} = 0 \Rightarrow |A| = 0 \Rightarrow |A|^6 = 0 \Rightarrow |A^6| = 0$

ie A, A^3, A^6 all are singular $\Rightarrow \rho(A^6) \le 2$

Rank of a skew symmetric matrix cannot be

MCQ. VERY GOOD QUESTION.

(a) 1 (b) 2

(c) 4 (d) 0

$A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}_{3\times 3} \Rightarrow \rho(A) \neq 3$ But $\rho(A) = 2$

$A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}_{2\times 2} \Rightarrow \rho(A) = 2$

$A = \begin{bmatrix} 0 \end{bmatrix}_{1\times 1}$ = Null Mat

$\boxed{\text{i.e skew symm Mat of order } 1\times 1 = \text{DNE}}$

eg $A = \begin{bmatrix} 0 & -1 & -2 & 0 \\ 1 & 0 & -4 & 0 \\ 2 & 4 & 0 & 6 \\ 0 & 0 & -6 & 0 \end{bmatrix}_{4\times 4}$

$\because |A| = 36 \Rightarrow \rho(A) = 4$

eg $A = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{bmatrix}_{4\times 4}$

$\because |A| = 0 \Rightarrow \rho(A) = 2$

& $\boxed{\rho(\text{skew symm})_{n\times n} \geq 2}$

Linearly Dependent & Linearly Independent Vectors →

L.D vectors → vectors are called L.D if ∃ Linear Relation b/w them.

L.I Vectors → " " " L.I if there DNE any Linear Relation b/w them.

Linear Combination of vectors → Let $X_1, X_2, X_3, \ldots, X_r$ are the given Vectors
(Linear Relationship) & $k_1, k_2, k_3, \ldots, k_r$ are the scalars (Const)
then the Relationship of the type $\boxed{k_1X_1 + k_2X_2 + k_3X_3 + \ldots + k_rX_r = 0}$ — (1)
is called Linear combination of vectors.

Sp. Note: $X = \begin{bmatrix} \\ \\ \end{bmatrix}_{n\times 1}$ then $X^2 = X.X = \begin{bmatrix} \vdots \end{bmatrix}_{n\times 1}\begin{bmatrix} \vdots \end{bmatrix}_{n\times 1} =$ N.D

i.e if Relationship exist it is always in Linear Form.

Methods of Checking the Nature of Vectors $\rightarrow$

Consider the given vectors are $X_1, X_2, X_3, \ldots\ldots, X_r$

then Construct a Matrix A as follows; $A = [X_1\, X_2\, X_3 \ldots\ldots X_r]$ $\neq$ Row Mat

(M-I) General Method (always applicable) $\rightarrow$

(i) if $\rho(A) =$ No. of vectors $\Rightarrow$ Vectors are LI

(ii) if $\rho(A) <$ " " " $\Rightarrow$ " " LD

(M-II) Tricky Method (applicable only when A is Sq Mat) $\rightarrow$

(i) if $|A| \neq 0 \Rightarrow$ Vectors are LI

(ii) if $|A| = 0 \Rightarrow$ " " LD

Note: If there are two vectors X_1 & X_2 then No Need to use G. Method or T. Method, only use observation method

i.e Consider X_1 & X_2 are given vectors.

if $\boxed{X_1 = KX_2}$ for any K then vectors are LD (for Non Zero K)

& if $X_1 \neq KX_2$ (i.e K = DNE) then " " LI

eg $X_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, X_2 = \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}$, eg $X_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, X_2 = \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix}$

$\because -2X_1 = X_2$ $\quad$ $\because X_1 \neq KX_2$ for any K i.e K = DNE

i.e L.D $\quad$ Hence LI

Q Check the Nature of vectors, $\begin{bmatrix}2\\-1\\0\\3\end{bmatrix}, \begin{bmatrix}-1\\2\\3\\1\end{bmatrix}, \begin{bmatrix}-6\\6\\6\\-4\end{bmatrix}$

Sol: M-I

$X_1 = \begin{bmatrix}2\\-1\\0\\3\end{bmatrix}, X_2 = \begin{bmatrix}-1\\2\\3\\1\end{bmatrix}, X_3 = \begin{bmatrix}-6\\6\\6\\-4\end{bmatrix}$

By observation, $X_1 - X_2 = \begin{bmatrix}3\\-3\\-3\\2\end{bmatrix} = -\frac{1}{2}X_3$

i.e. $\boxed{2X_1 - 2X_2 + X_3 = 0}$

Hence X_1, X_2, X_3 are LD

M-II $A = [X_1\ X_2\ X_3]$

$$A = \begin{bmatrix}2 & -1 & -6\\ -1 & 2 & 6\\ 0 & 3 & 6\\ 3 & 1 & -4\end{bmatrix}_{4\times 3}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix}-1 & 2 & 6\\ 2 & -1 & -6\\ 0 & 3 & 6\\ 3 & 1 & -4\end{bmatrix}$$

$$\xrightarrow{\text{H.W.}} \begin{bmatrix}-1 & 2 & 6\\ 0 & a_{21} & a_{22}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$\rho(A) = \text{two} < \text{No. of vectors } (3)$

So vectors are LD

(PYQ) the vectors $(1\ 2\ -1)'$, $(2\ 3\ 4)'$, $(0\ 1\ 2)'$, $(4\ -3\ 2)'$ are ? = LD.

Sol: $x_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, x_3 = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$

Construct $A = [x_1\ x_2\ x_3\ x_4]$

$$= \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 3 & 1 & -3 \\ -1 & 4 & 2 & 2 \end{bmatrix}_{3\times 4}$$

$\because \rho(A) \leq 3$ (By Property of Rank)

ie $\rho(A) < 4$ definitely (By Common Sense)

or $\rho(A) <$ No. of vectors $\Rightarrow$ LD

Q: Find λ for which $\exists$ a linear combination b/w the vectors; $\Rightarrow$ L.D vectors.

$\hat{i}+2\hat{j}+3\hat{k}$, $4\hat{i}+5\hat{j}+6\hat{k}$, $7\hat{i}+\lambda\hat{j}+9\hat{k}$ then $\lambda =$?

Sol: $X_1=\begin{bmatrix}1\\2\\3\end{bmatrix}, X_2=\begin{bmatrix}4\\5\\6\end{bmatrix}, X_3=\begin{bmatrix}7\\\lambda\\9\end{bmatrix}$

$A=[X_1\,X_2\,X_3]=\begin{bmatrix}1&4&7\\2&5&\lambda\\3&6&9\end{bmatrix}_{3\times3}$ = Sq. Mat.

For LD vectors, $|A|=0 \Rightarrow \begin{vmatrix}1&4&7\\2&5&\lambda\\3&6&9\end{vmatrix}=0$ .... $\Rightarrow \lambda=8$

Q: No. of different values of λ for which above vectors are LI will be ? $=\infty$

for $\lambda\neq 8 \Rightarrow$ vectors are LI

i.e. $\lambda\in R-\{8\}$.

Q. The vectors $(1\ 2\ 1)'$, $(2\ 1\ -4)'$, $(3\ -2\ 1)'$ are orthogonal & L.I.

M-I $X_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}$, $X_2 = \begin{bmatrix}2\\1\\-4\end{bmatrix}$, $X_3 = \begin{bmatrix}3\\-2\\1\end{bmatrix}$

$$X_1 \cdot X_2 = X_2 \cdot X_3 = X_3 \cdot X_1 = 0$$

Hence orthogonal.

w.k. that $\boxed{\text{orthogonal vectors} \Longrightarrow \text{LI also}}$ ($\not\Longleftarrow$)

So these vectors are LI also.

VARIOUS Def^n of RANK :-

Def^n $\rho(A_{6\times 7}) = 4$

→ Mat A will have at most 4 LI Row vectors.

→ Mat A " " at most 4 LI Column vectors.

Def^n of Rank : "It is the order of Non singular submatrix of Highest order that can exist in a given Mat"

Def^n In Books :

if $\boxed{\rho(A_{6\times 7}) = 4}$ then

→ $\exists$ at least one Non singular submatrix of order 4×4

→ Every square submatrix of order 5×5 & 6×6 are singular

THANK - YOU

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