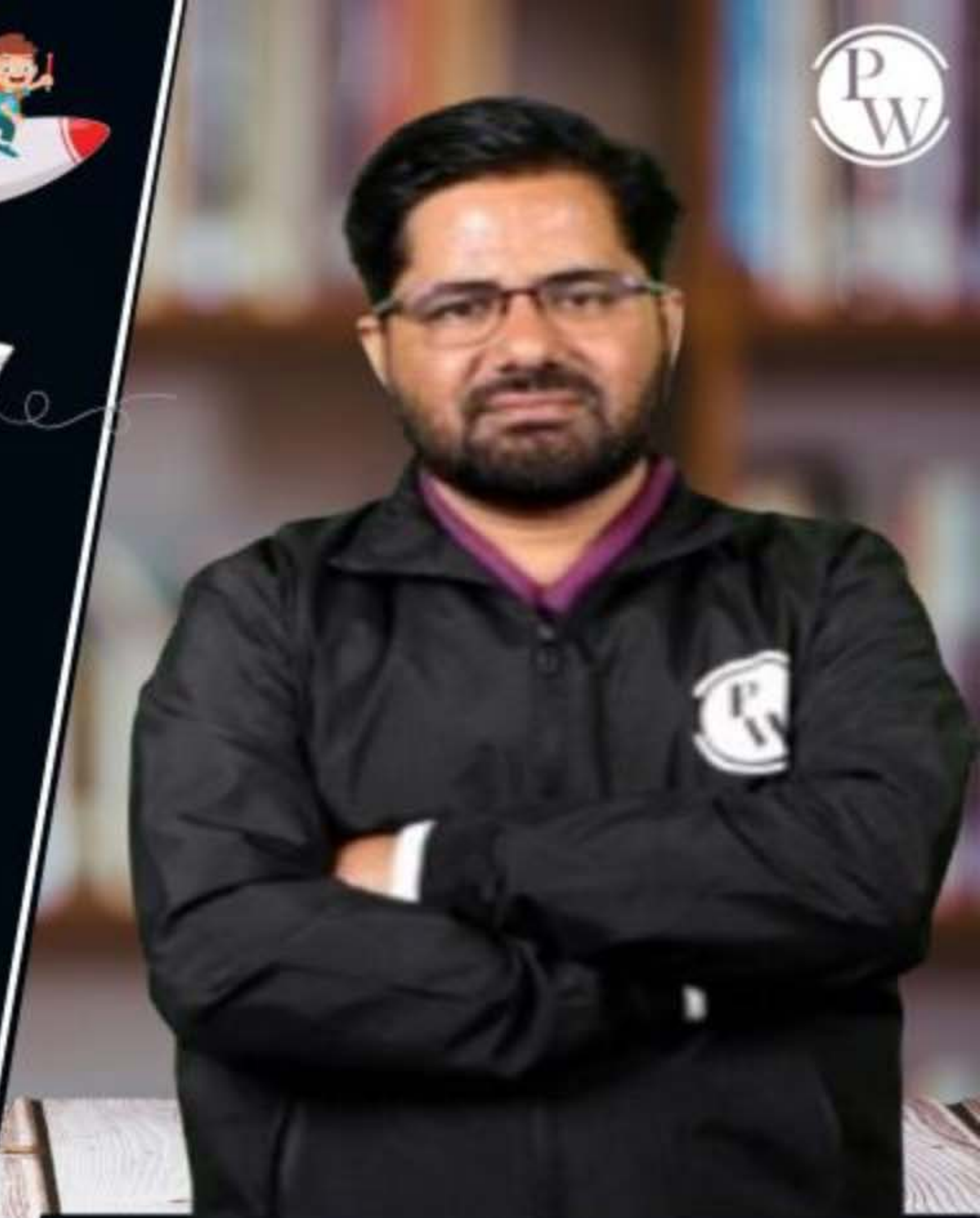
# DATA SCIENCE
# &
# ARTIFICIAL INTELLIGENCE

& CS/IT

## Calculus and Optimization

Lecture No. 06



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# Recap of previous lecture

**Topic**

TAYLOR & MACLAURIN SERIES

# Topics to be Covered

**Topic**

MEAN VALUE THEOREMS

1. Lagrange's M.V.Th. of Differentials
2. Rolle's M.V.Th.
3. Cauchy's M.V.Th.
4. L-M.V.Th for Integrals.

Q. Find points of Discontinuity and Non Diff points of $f(x) = \frac{x - |x-1|}{x}$

HW Q. [Ans: Discont at $x=0$ only & Non Diff at $x = 0$ & $1$]

Sol: $f(x) = \frac{x - |x-1|}{x}$, $D_f = R - \{0\} \Rightarrow$ At $x=0$ $f(x)$ is Discont & Non Diff

Now $f(x) = \begin{cases} \frac{x-(1-x)}{x}, & x<1 \\ 1, & x=1 \\ \frac{x-(x-1)}{x}, & x>1 \end{cases} = \begin{cases} 2-\frac{1}{x}, & x<1 \\ 1, & x=1 \\ 1/x, & x>1 \end{cases} \Rightarrow f'(x) = \begin{cases} 1/x^2, & x<1 \\ -1/x^2, & x>1 \end{cases}$

LHL $=1$, RHL $=1$, $f(1)=1$

Hence Cont at $x=1$

LHD $= 1$, RHD $= -1$

So Not Diff at $x=1$

Q. In the power series Expansion of $f(x)=\frac{x-1}{x+1}$ about $x=1$, 3rd term will be?

(a) $(x-1)^2/2$

(b) $(x-1)^2/4$

(c) $(x-1)^3/8$ ✓

(d) $(x-1)^3/4$

M-I using Conventional Approach of Taylor Series in the Nbd of 1.
But it is not feasible during exam time bcoz calculation of Derivatives at $x=1$ will become tedious

M-II Put $x-1=t$ or $x=t+1$ then

$$f(x)=\frac{x-1}{x+1}=\frac{t}{t+1+1}=\frac{t+2-2}{t+2}=1-\frac{2}{t+2}=1-\frac{1}{\left(\frac{t}{2}+1\right)}$$

$$=1-\left(1+\frac{t}{2}\right)^{-1}=1-\left\{1-\left(\frac{t}{2}\right)+\left(\frac{t}{2}\right)^2-\left(\frac{t}{2}\right)^3+\cdots\right\}$$

$$=\frac{t}{2}-\frac{t^2}{4}+\frac{t^3}{8}-\cdots$$

$$f(x)=\frac{(x-1)}{2}-\frac{(x-1)^2}{4}+\frac{(x-1)^3}{8}-\cdots$$

$\because$

$$(1-x)^{-1}=1+x+x^2+x^3+x^4+\cdots$$

$$(1+x)^{-1}=1-x+x^2-x^3+\cdots$$

Q. Power series Expansion of $f(x) = \frac{\sin x}{x-\pi}$ when $\underbrace{|x-\pi| < \epsilon}$ will be ?

$x \to \pi$ or $x$ lies in the Nbd of $\pi$.

Sol (M-I) using fundamental formula of Taylor series $\to$ Not very easy.

(M-II) Let $f(x) = \frac{g(x)}{x-\pi} = \frac{\sin x}{x-\pi}$ where $g(x) = \sin x$

Now will will try to find T.S. Exp of $g(x)$ in the Nbd of $\pi$.

$$g(x) = g(\pi) + (x-\pi)g'(\pi) + \frac{(x-\pi)^2}{2!}g''(\pi) + \frac{(x-\pi)^3}{3!}g'''(\pi) + \cdots$$

$$\sin x = 0 + (x-\pi)(\cos x)_{x=\pi} + \frac{(x-\pi)^2}{2!}(-\sin x)_{x=\pi} + \frac{(x-\pi)^3}{3!}(-\cos x)_{x=\pi} + \cdots$$

$$= 0 + (x-\pi)(-1) + 0 + \frac{(x-\pi)^3}{3!}(+1) + \cdots$$

$$f(x) = \frac{\sin x}{x-\pi} = -1 + \frac{(x-\pi)^2}{3!} + \cdots$$

(III) $f(x) = \frac{\sin x}{x-\pi}$ about $x = \pi$, Put $x - \pi = t$ when $x \to \pi$, $t \to 0$

$$f(x) = \frac{\sin(\pi + t)}{t} = \frac{-\sin t}{t} = \frac{-\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right]}{t}$$

$$f(x) = -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + \frac{t^6}{7!} - \cdots$$

$$\frac{\sin x}{x-\pi} = -1 + \frac{(x-\pi)^2}{3!} - \frac{(x-\pi)^4}{5!} + \frac{(x-\pi)^6}{7!} - \cdots \text{ Ans.}$$

Q: if $\boxed{f(x) = x^3 + 8x^2 + 15x - 24}$ then $f\left(\frac{11}{10}\right) = ?$ using T.S. Exp Method.

(a) 0

(b) 3.5111 ✓

(c) 5.312

(d) 2.179

$\frac{11}{10} = 1 + \frac{1}{10}$ i.e. $\frac{11}{10}$ lies in the Nbd of 1.

$x = a + h$ where $a = 1$ & $h \to 0$

i.e. we will find T.S. Exp of $f(x)$ in the Nbd of $x = 1$

$f(1) = 0$, $f'(x) = 3x^2 + 16x + 15$, $f''(x) = 6x + 16$, $f'''(x) = 6$

$f'(1) = 34$, $f''(1) = 22$, $f'''(1) = 6$

$$f(x) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + 0 + 0 + \dots$$

$$f(x) = 0 + (x-1)(34) + \frac{(x-1)^2}{2!}(22) + \frac{(x-1)^3}{3!}(6) + 0 + 0$$

So $f(1.1) = 0.1 \times 34 + \frac{(0.1)^2}{2}(22) + \frac{(0.1)^3}{6}(6) = 3.511$

# Mean Value Theorems (New chapter.)

**Note:**

(1) Slope of line passing through $(x_1, y_1)$ & $(x_2, y_2)$ is $m = \frac{y_2 - y_1}{x_2 - x_1}$

(2) In general, slope of tangent at any Random point is

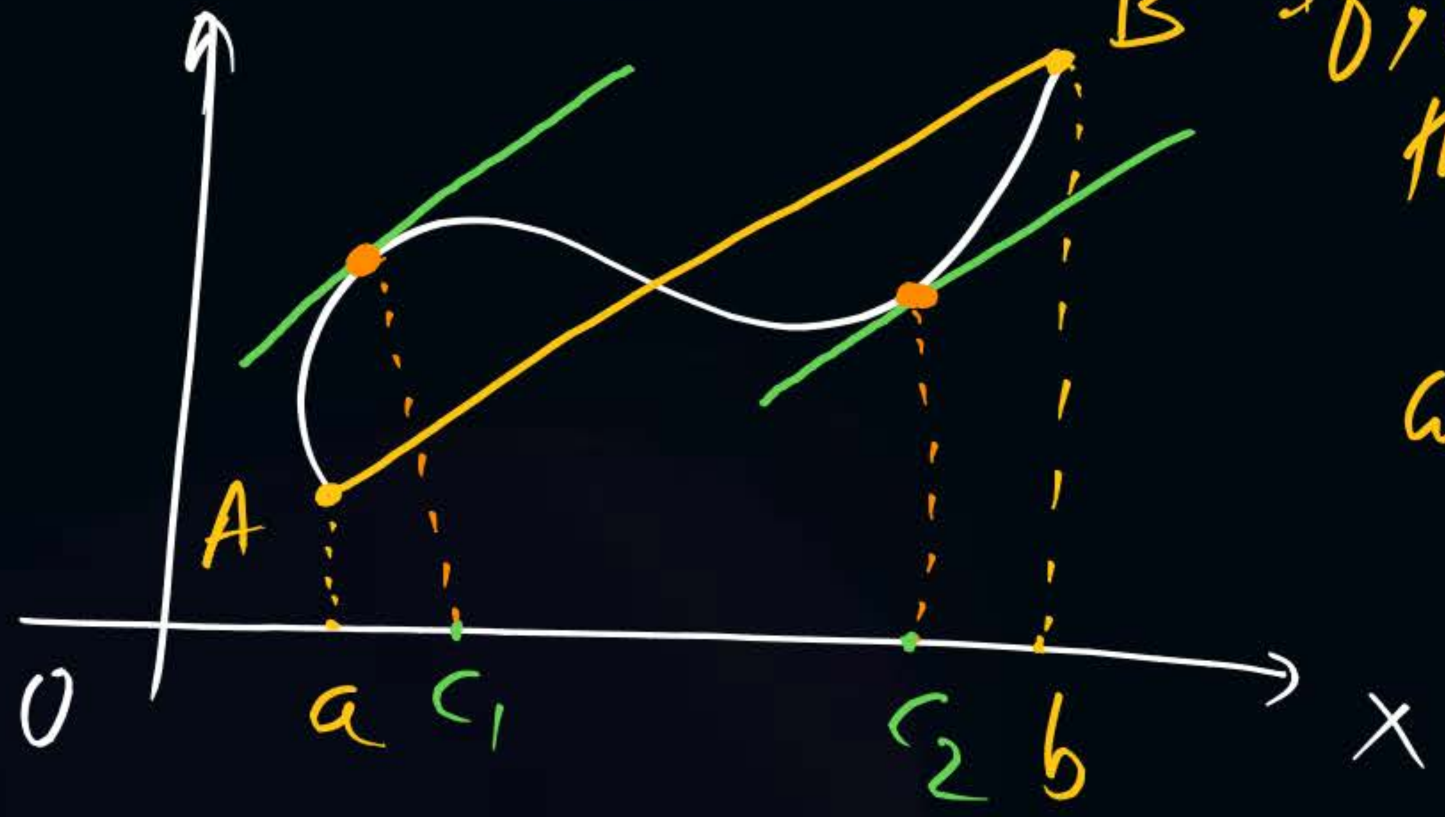
$$m = \tan\theta = f'(x)$$

(3) Slope of Horizontal line is $m = \boxed{f'(x) = \tan 0 = 0}$

(4) Elementary func$^n$ – All polynomial func$^n$, Exp func$^n$, log func$^n$, Trig. Function & Inverse Trig func$^n$ are called E-Functions.

(5) All E-func$^n$ are Continuous as well as Differentiable in their Domain.

L.M.V.T $A(a, f(a)), B(b, f(b))$




If, $[a, b]$ Cont as well as Diff

then $\exists$ at least one point $c \in (a, b)$

where $\boxed{\text{tangent is } \parallel^r \text{ to chord } AB}$

ie slope of tangent = slope of chord AB

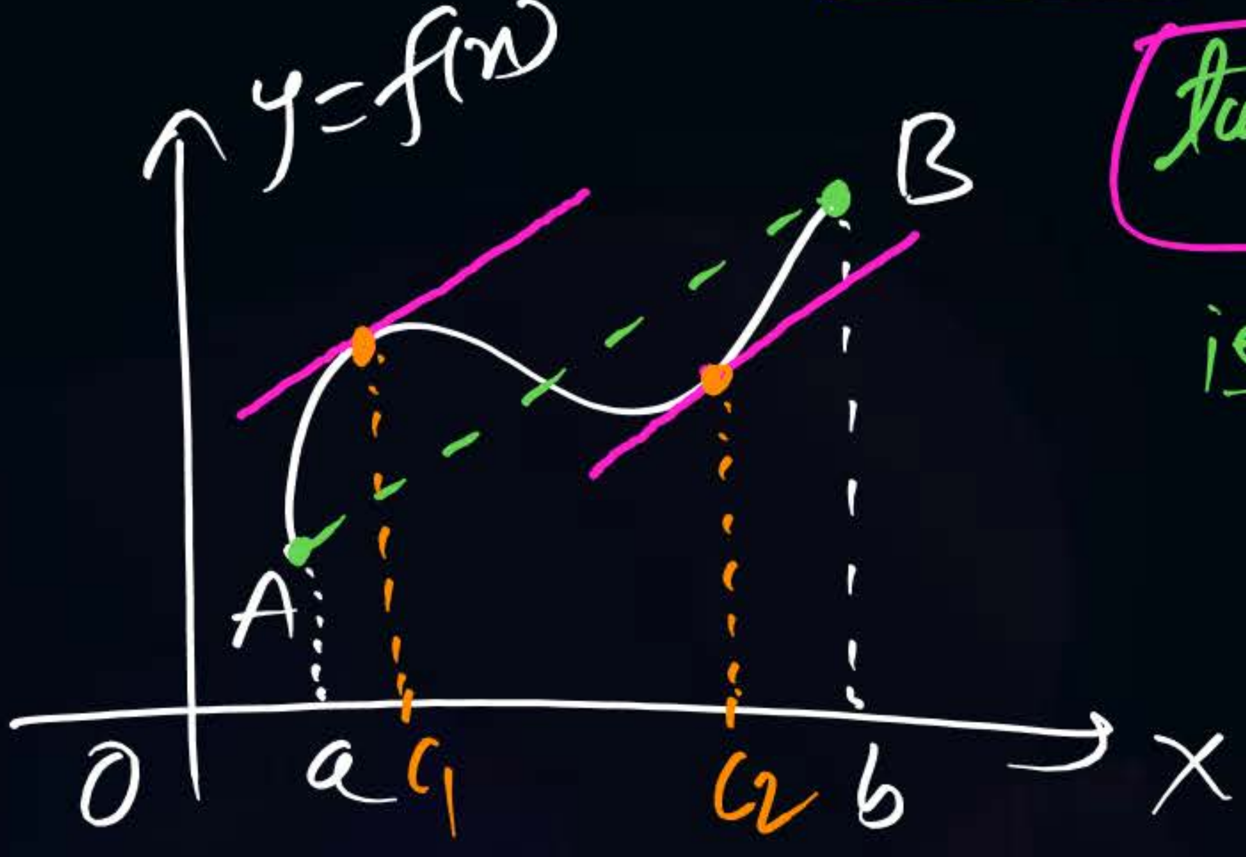
$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

(1) Lagrange's M.V.Th – Let $f(x)$ is defined in $[a, b]$ s.t (i) $f(x)$ is Cont in $[a, b]$, (ii) $f(x)$ is Diff in $(a, b)$ then $\exists$ at least one $c$ in b/w $a$ & $b$ for which

Tangent at $c$ is $\parallel^r$ to chord AB

i.e slope of Tangent = slope of AB

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$




$A(a, f(a))$, $B(b, f(b))$

Note – Converse of LMVTh is not necessarily True.

Q. Verify L.M.V.Th for $f(x) = x^{1/3}$ in $[-1,1]$ & hence evaluate $c = ?$

Sol: $f'(x) = \frac{1}{3} x^{\frac{1}{3}-1} = \frac{1}{3x^{2/3}}$ i.e. At $x=0$, $f'(0) =$ DNE

So we can't say that $f(x)$ is differentiable throughout in $(-1,1)$

i.e. $2^{nd}$ condition of L.M.V.T is not satisfied

Hence L.M.V.T is not applicable Ans.

(ii) Senseless Question

**Analysis:** (PODCAST) → $f(x) = x^{1/3}$, $[-1,1]$

$f(-1) = (-1)^{1/3} = -1$

$f(1) = (1)^{1/3} = 1$

$$f'(x) = \frac{1}{3x^{2/3}}$$

By L.M.V.T, $\dfrac{f(1)-f(-1)}{1-(-1)} = f'(c)$

$$\frac{1-(-1)}{1-(-1)} = \frac{1}{3c^{2/3}}$$

$$1 = \frac{1}{3c^{2/3}}$$

$$c^{2/3} = \frac{1}{3} \Rightarrow c = \left(\frac{1}{3}\right)^{3/2} = \sqrt{\frac{1}{27}} = \frac{\pm 1}{3\sqrt{3}}$$


i.e. Both values of c lies in b/w $-1$ & $1$

Still our func^n is not Diff.

i.e. Converse of L.M.V.T is not Necessarily True.

S/2 Note:

$$\text{if}\left[f(x)\text{ is Cont \& Diff}\right] \underset{\not\Longleftarrow}{\overset{\text{LMVTh.}}{\Longrightarrow}} \left[\exists\, c \in (a,b) \text{ s.t } \frac{f(b)-f(a)}{b-a} = f'(c)\right]$$

In previous PODCAST, it is possible to find c

But it does not imply that f(x) is cont & differentiable.

Q. Find the point on the curve $y = \sqrt{x-2}$ in $[2,3]$ where tangent is $\parallel^r$ to the chord joining end points of the curve ?

(a) $\left(\frac{10}{4}, \frac{1}{2}\right)$

(b) $\left(\frac{9}{4}, 1\right)$

(c) ✓ $\left(\frac{9}{4}, \frac{1}{2}\right)$

(d) $\left(\frac{5}{2}, \frac{5}{2}\right)$

$f(x)$ is cont in $[2,3]$ $\because$ we have no problem creating point.

$f'(x) = \frac{1}{2\sqrt{x-2}}$ = exist everywhere in $(2,3)$

So $f(x)$ is Diff in $(2,3)$ so we can apply L.M.V.Th.

$a=2$, $f(2)=0 \Rightarrow A(2,0)$ & $B(3,1)$

$b=3$, $f(3)=1$ & slope of $AB = \frac{1-0}{3-2} = 1$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\left(\frac{1}{2\sqrt{x-2}}\right)_{x=c} = \frac{f(3)-f(2)}{3-2}$$

$$\frac{1}{2\sqrt{c-2}} = \frac{1-0}{3-2}$$

$$\sqrt{c-2} = \frac{1}{2}$$

$$c-2 = \frac{1}{4}$$

$$c = \frac{9}{4} = 2.25$$

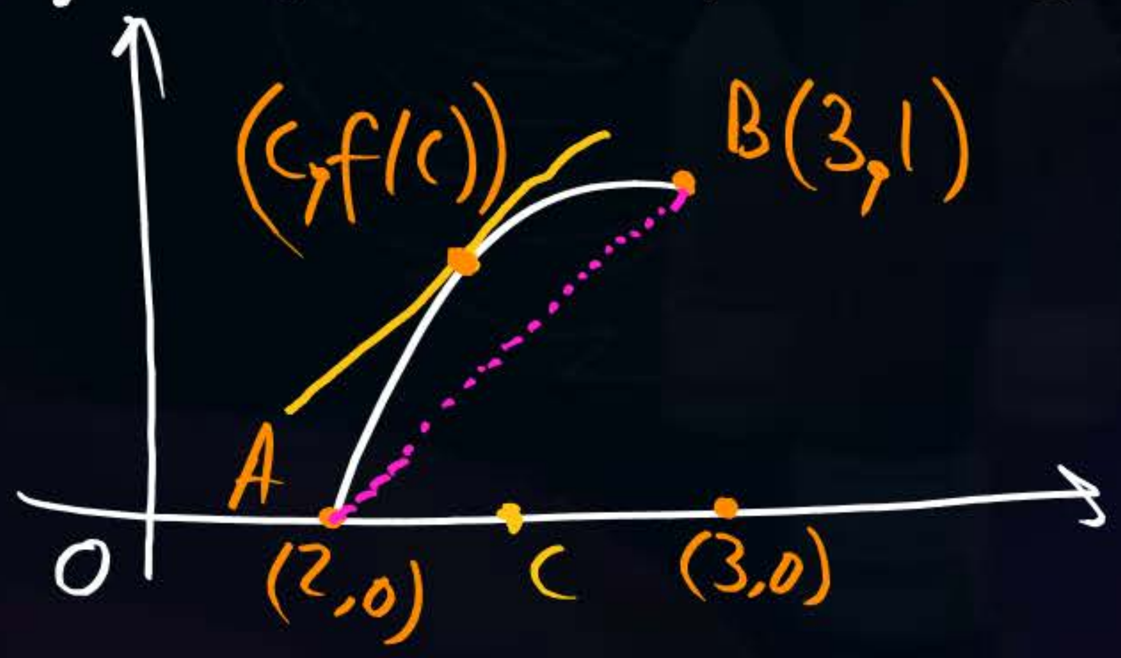
is $c$ lies in b/w 2 & 3 ☺

$$f(c) = \sqrt{c-2}$$

$$f\left(\frac{9}{4}\right) = \sqrt{\frac{9}{4}-2} = \frac{1}{2}$$

So Req Point $= (c, f(c)) = \left(\frac{9}{4}, \frac{1}{2}\right)$

Analysis: $y = \sqrt{x-2}$, $[2,3]$



Q. Consider the function $f(x)=\sqrt{x-2}$ is defined in $(2,3)$ then at least, at one point in this interval $\frac{dy}{dx}$ equal to ___?___ $=1$.

By L.M.V.Th, we have proved that,

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$f'(c)=\frac{1-0}{3-2}$$

$$\left(\frac{dy}{dx}\right)_{x=c}=1$$

& c lies in the interval $(2,3)$

So our answer is 1

Q. for the function $f(x)=|x|$, Lagrange's Mean Value is not applicable in ?

(a) $1 \leq x \leq 3$

(b) $x<-1$ or $x>1$

(c) $0<x<1$

(d) $-2<x<2$ ✓

w.k. that at $x=0$, $f(x)=|x|$ is not Diff

i.e. 2nd cond^n of L.M.V.Th is not satisfied at $x=0$

Q. for the function $f(x) = \sin\left(\frac{1}{x}\right)$ Lagrange's Mean Value is applicable in ?

(a) $[-3, 3]$ ×

(b) $[-2, 5]$ ×

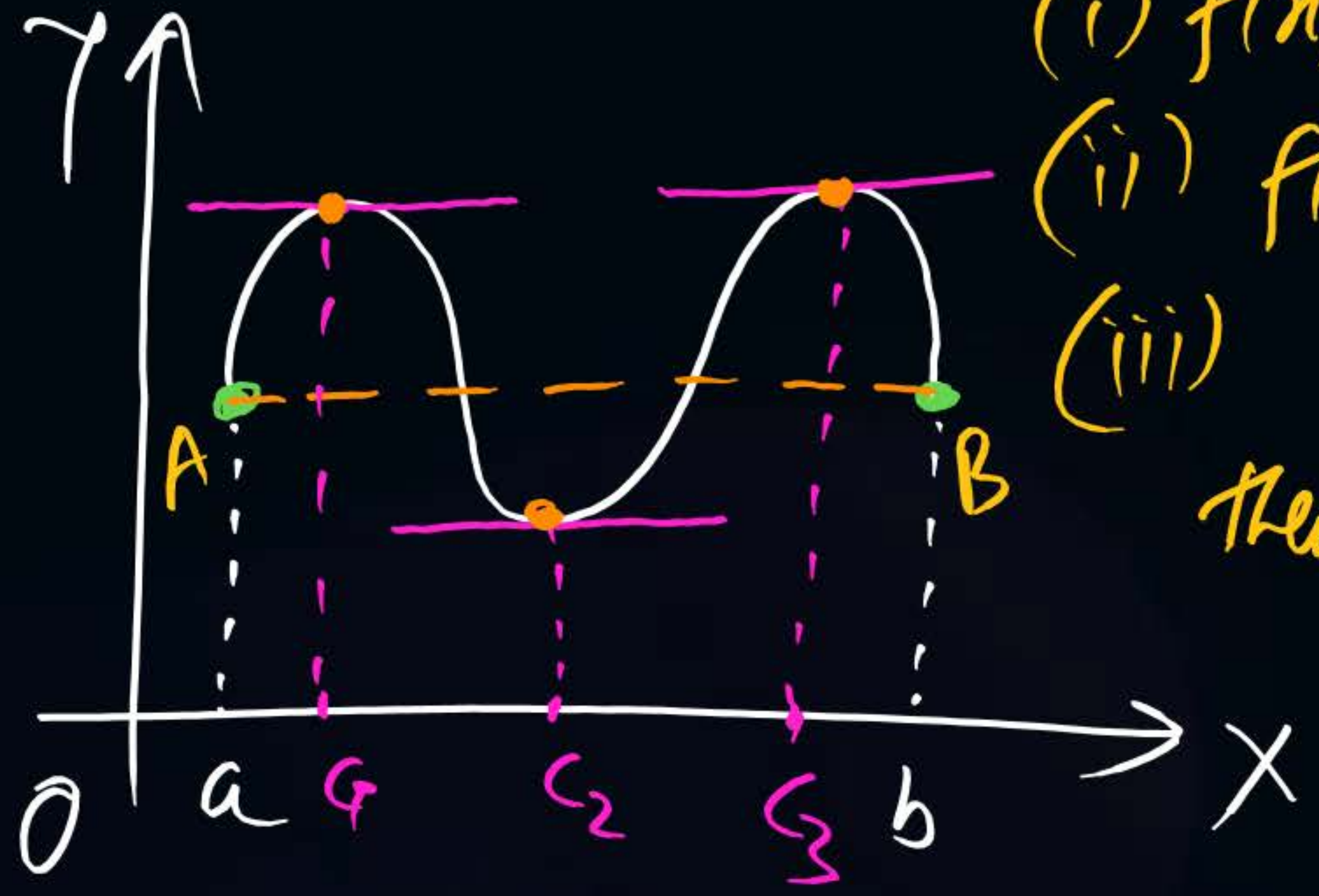
(c) $[2, 3]$ ✓

(d) $[-1, 4]$ ×

Dom $= R - \{0\}$

i.e At $x = 0$ $f(x)$ is Not Continuous.

② Rolle's M.V. Th → Let $f(x)$ is defined in $[a, b]$ s.t




(i) $f(x)$ is Continuous in $[a, b]$

(ii) $f(x)$ is Diff in $(a, b)$

(iii) $f(a) = f(b)$

then $\exists$ at least one point $c$ in b/w $a$ & $b$ for which

$\boxed{\text{tangent is Horizontal}}$

or tangent is $\parallel^r$ to X axis

or $\boxed{f'(c) = 0}$

Q. The ordinate of point on the curve $\boxed{f(x) = \cos x - 1}$ ; $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ where tangent is $\parallel$ to X axis ?

(a) $\pi$

(b) $-2$

(c) $2\pi$

(d) $0$

KHELA HOO GAYA

$\because f(x)$ is an Elementary func$^n$ so it is cont as well as Diff

Now $f\left(\frac{\pi}{2}\right) = -1 = f\left(\frac{3\pi}{2}\right)$ i.e. 3rd condition also satisfied.

So we can use Rolle's Th,

$$f'(c) = 0$$

$$(-\sin x)_{x=c} = 0$$

$$\sin c = 0$$

$$\sin c = \sin n\pi$$

$c = n\pi,\ n \in I$

$c = \ldots\ldots -2\pi, -\pi, 0, \pi, 2\pi, 3\pi \ldots\ldots$

$\because$ only $c = \pi$ lies $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

So $f(c) = \cos c - 1 = \cos \pi - 1 = -1 - 1 = -2$

Q. If for the function $\boxed{f(x) = x^3 + bx^2 + ax}$, where $x \in [1,3]$, Rolle's Theorem holds with $c = 2 + \frac{1}{\sqrt{3}}$ then $a =$ ______ & $b =$ ______ or $a + b = ?$

Sol: $\because$ f(x) is polynomial func$^n$ (i.e an Elementary function) so Cont and Diff both.

& 3$^{rd}$ Cond$^n$ of R.Th is

$$f(a) = f(b)$$

$$f(1) = f(3)$$

$$1^3 + b(1)^2 + a(1) = (3)^3 + b(3)^2 + a(3)$$

$$1 + a + b = 27 + 9b + 3a$$

$$-8b - 2a = 26 \Rightarrow \boxed{a + 4b = -13}$$

By R.Th, $f'(c) = 0$

$$(3x^2 + 2bx + a)_{x=c} = 0$$

$$3c^2 + 2bc + a = 0$$

$$3\left(2 + \frac{1}{\sqrt{3}}\right)^2 + 2b\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$$

$$3\left(2+\frac{1}{\sqrt{3}}\right)^2+2b\left(2+\frac{1}{\sqrt{3}}\right)+a=0$$

$$3\left[4+\frac{1}{3}+\frac{4}{\sqrt{3}}\right]+4b+\frac{2b}{\sqrt{3}}+a=0$$

$$a+\left(\frac{4\sqrt{3}+2}{\sqrt{3}}\right)b=-\left(\frac{13\sqrt{3}+12}{\sqrt{3}}\right) \quad (2)$$

$$a+4b=-13 \quad (1)$$

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$a=?\ ,\ b=?$

(3) Cauchy's M.V.Th → Let $f(x)$ & $g(x)$ are two func$^n$ defined in $[a,b]$ s.t

(i) Both $f(x)$ & $g(x)$ are continuous in $[a,b]$

(ii) " " " " differentiable in $(a,b)$

(iii) $g'(x) \neq 0 \ \forall x \in (a,b)$

Then $\exists$ at least one $c$ in b/w $a$ & $b$ for which

$$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}}$$

(Ratio of Difference of $f(x)$ & $g(x)$ at $a$ & $b$) = (Ratio of slope of tangent of $f(x)$ & $g(x)$ at $c$)

Q. If $f(x) = \log_e x$ & $g(x) = \log_e\left(\frac{1}{x}\right)$ defined in $[1,2]$ then By C.M.V.Th, $c = ?$

(a) 1.0

(b) 1.25

(c) 1.5

(d) Any value b/n 1 & 2

$f(x)$ & $g(x)$ are cont as well as Diff in $(1,2)$

$\because$ these are Elementary func$^n$.

$g(x) = \log\frac{1}{x} = -\log x \Rightarrow g'(x) = -\frac{1}{x}$

$\because g'(x) \neq 0 \; \forall x \in (1,2)$

So all the conditions of C.M.V.Th are satisfied.

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{\ln 2 - \ln 1}{-\ln 2 + \ln 1} = \frac{1/c}{-1/c}$$

$\Rightarrow -1 = -1$ i.e identity.

Q. If $f(x)=\frac{1}{x}$ & $g(x)=\frac{1}{x^2}$ are defined in $[4,6]$ then value of $c$ using Cauchy's M.V.Th ?

(a) 4.5
(b) 5.2
(c) 4.8
(d) 5

At $x=0$, $f(x)$ & $g(x)$ are not cont and not Diff.
But $x=0$ is not in the given domain $\because 0 \notin [4,6]$
i.e. Both $f(x)$ and $g(x)$ are cont as well as Diff in $(4,6)$

Now $g'(x)=\frac{-2}{x^3}$, i.e. $g'(x)\neq 0 \;\forall x\in(4,6)$

So all the conditions of C.M.V.Th are satisfied.

$a=4$, $f(x)=\frac{1}{x}$, $f(4)=1/4$, $f(6)=1/6$, $f'(c)=-1/c^2$

$b=6$, $g(x)=\frac{1}{x^2}$, $g(4)=\frac{1}{16}$, $g(6)=\frac{1}{36}$, $g'(c)=-2/c^3$

80 By C.M.V.Th :

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(6)-f(4)}{g(6)-g(4)} = \frac{-1/c^2}{-2/c^3}$$

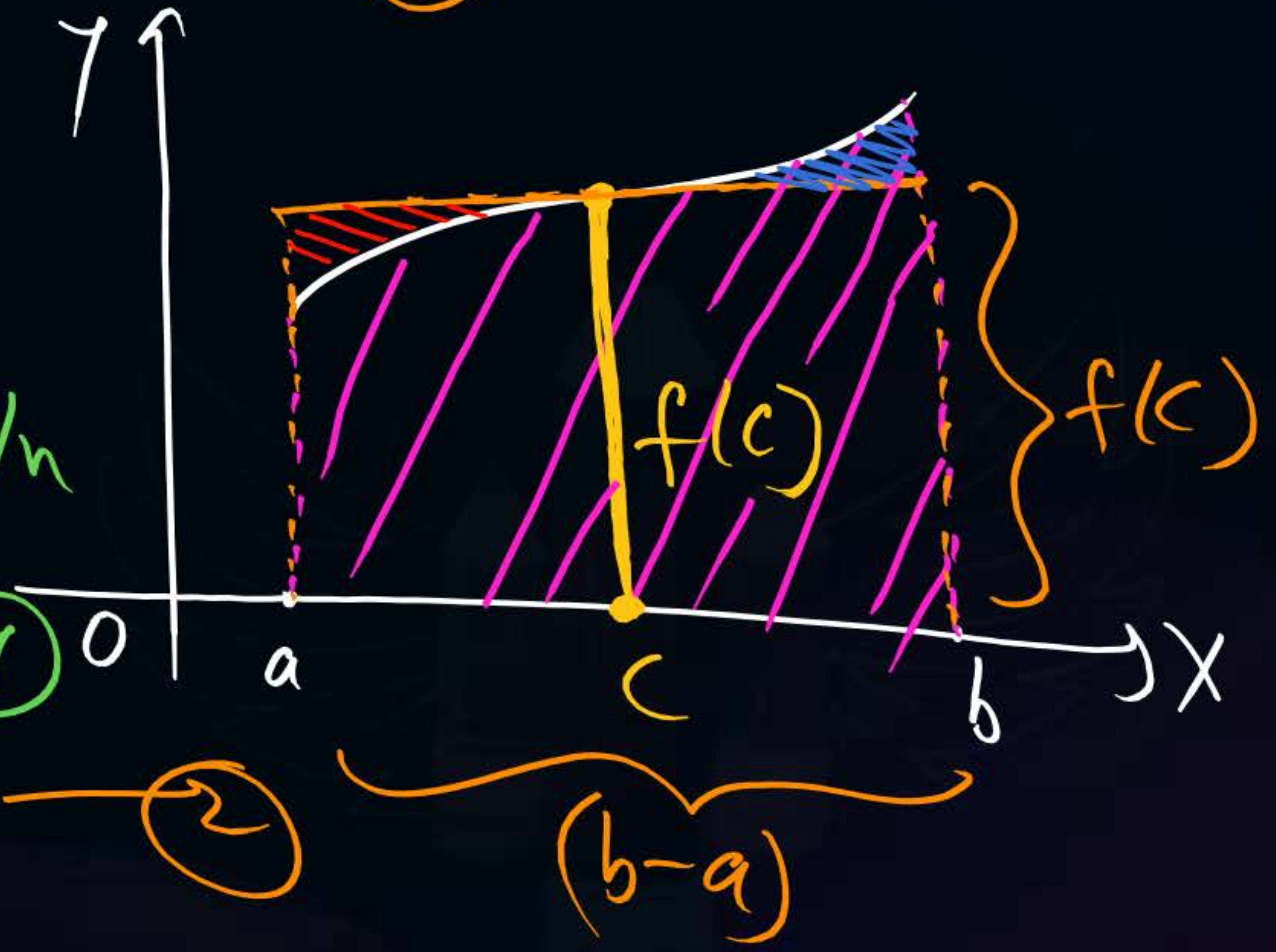
$$\frac{\frac{1}{6}-\frac{1}{4}}{\frac{1}{36}-\frac{1}{16}} = \frac{c}{2} \Rightarrow \frac{1}{\frac{1}{6}+\frac{1}{4}} = \frac{c}{2} \Rightarrow c = 4.8$$

## Lagrange's Mean Value Theorem for Integrals →

Average height

if $f(x)$ is continuous func$^n$ in $[a,b]$ then $\exists$ c in b/w a & b s.t

$$\boxed{f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx}$$



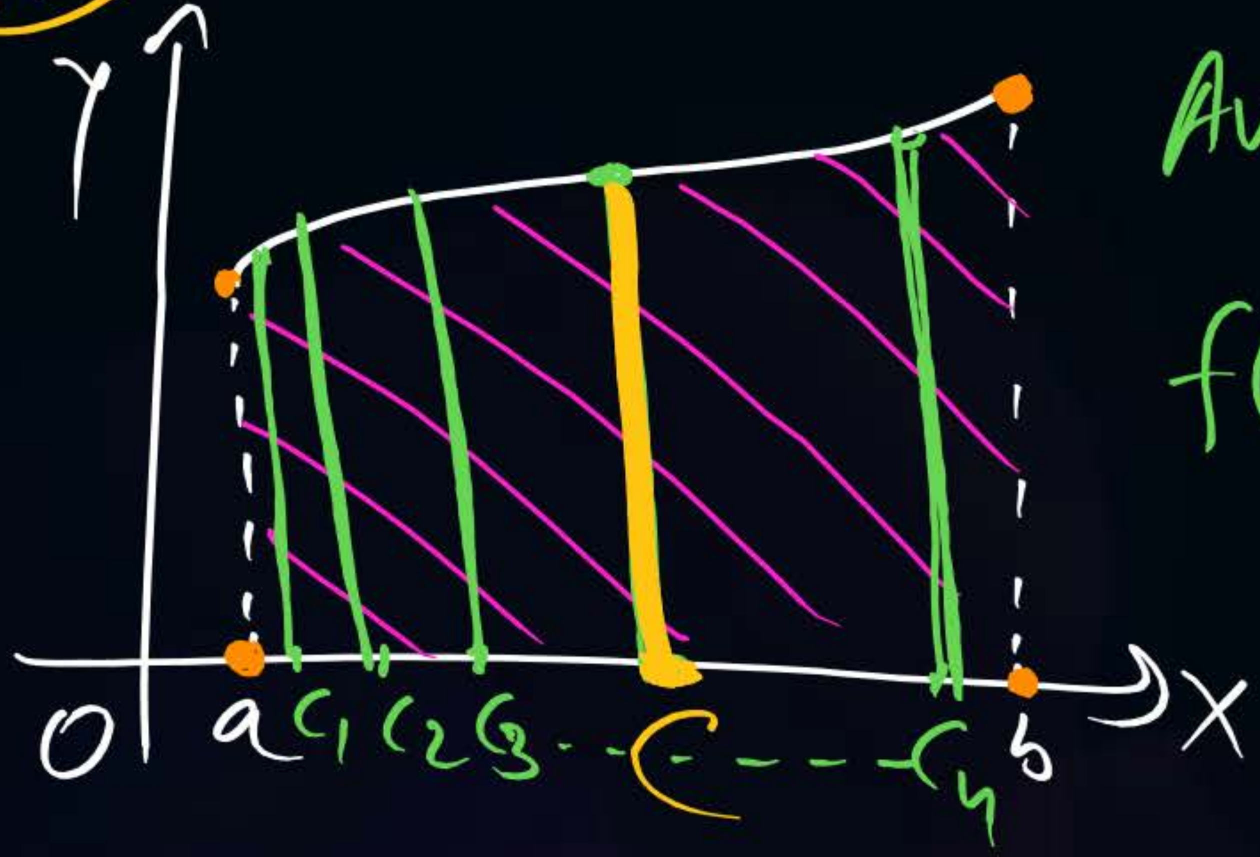

Proof: w.k.that $\int_a^b f(x)\,dx$ = Area under $f(x)$ b/w a & b and X axis — (1)

Area of this Rectangle = length × height

$= (b-a) \times f(c)$ — (2)

By (1) & (2), $(b-a)\times f(c)=\int_a^b f(x)\,dx \Rightarrow \boxed{f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx}$

M-II :→




Av. Height of $f(x)$ is given as,

$$f(c)=\frac{f(c_1)+f(c_2)+f(c_3)+\dots+f(c_n)}{n}$$

$$f(c)=\frac{\int_a^b f(x)\,dx}{\text{length of interval}}=\frac{\int_a^b f(x)\,dx}{b-a}$$

i.e. Av. Height of Curve $=\frac{1}{b-a}\int_a^b f(x)\,dx$

**Note:** ie Average Height of Curve b/n $a$ & $b$ is $= f(c)$

& it occurs at $x = c$ where $c \in (a, b)$

**Verification:** find Av. Height of $y = 2x$ b/n 1 & 6

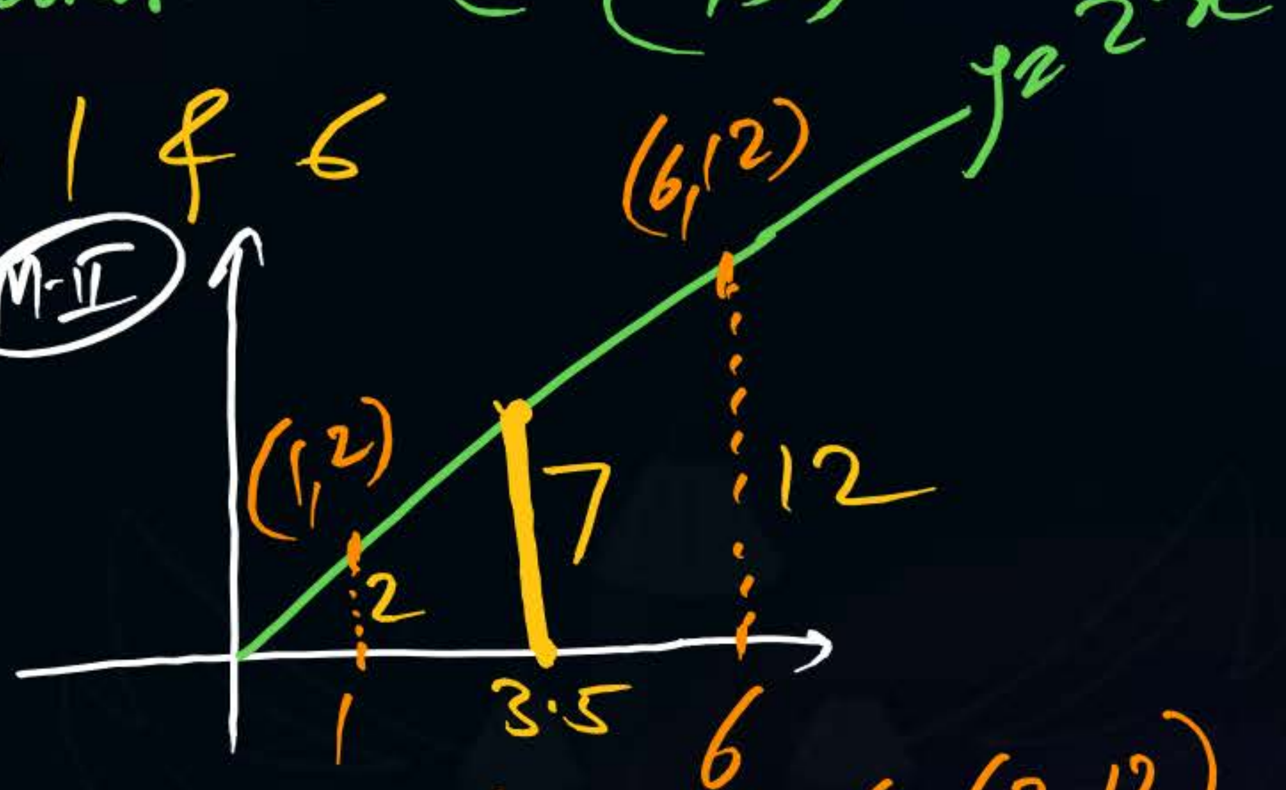
(M-I) Av Height $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

$$f(c) = \frac{1}{6-1}\int_1^6 (2x)\,dx = \frac{2}{5}\left(\frac{x^2}{2}\right)_1^6$$

$$f(c) = \frac{36-1}{5} = 7 \text{ Ans}$$

$$\Rightarrow 2c = 7 \Rightarrow c = 3.5 \text{ Ans}$$

(M-II)




ie when $x \in (1,6)$, $y \in (2,12)$

Hence Average $y = \dfrac{2+12}{2} = 7 = f(c)$

& it occurs at $c = 3.5$

eg $X = 3, 4, 5, 6, 7, 8, 9 \Rightarrow \bar{x} = ?$

(M-I) $\bar{x} = \frac{\Sigma x}{N} = \frac{3+4+5+6+7+8+9}{7} = \frac{42}{7} = 6$

(M-II) $\bar{X} = \frac{a+b}{2} = \frac{1^{st}\text{ Point} + \text{Last Point}}{2}$

$= \frac{3+9}{2} = 6$

**Q:** Find the Average Value of $f(x) = x^2$ b/w 1 & 4 ?

**Sol:** $f(c) = \frac{1}{4-1}\int_1^4 (x^2)\,dx = \frac{1}{3}\left(\frac{x^3}{3}\right)_1^4 = \frac{64-1}{9} = 7$

(ii) Also find the coordinates of that point ?

(a) $\pm\sqrt{7}$

(b) $(\sqrt{7}, 7)$ ✓

(c) $(-\sqrt{7}, 7)$

(d) $(2.5, 7)$

**Sol:** $\because f(c) = 7$

$c^2 = 7$

$c = \pm\sqrt{7}$

$\because c = -\sqrt{7} \notin (1, 4)$

So possible value of $c = \sqrt{7}$ ✓

So Req Point $= (c, f(c))$

$= (\sqrt{7}, 7)$

Q. The Mean Value of the function $f(x) = 5x^4 + 2$ b/n $-1$ & $2$ is ?

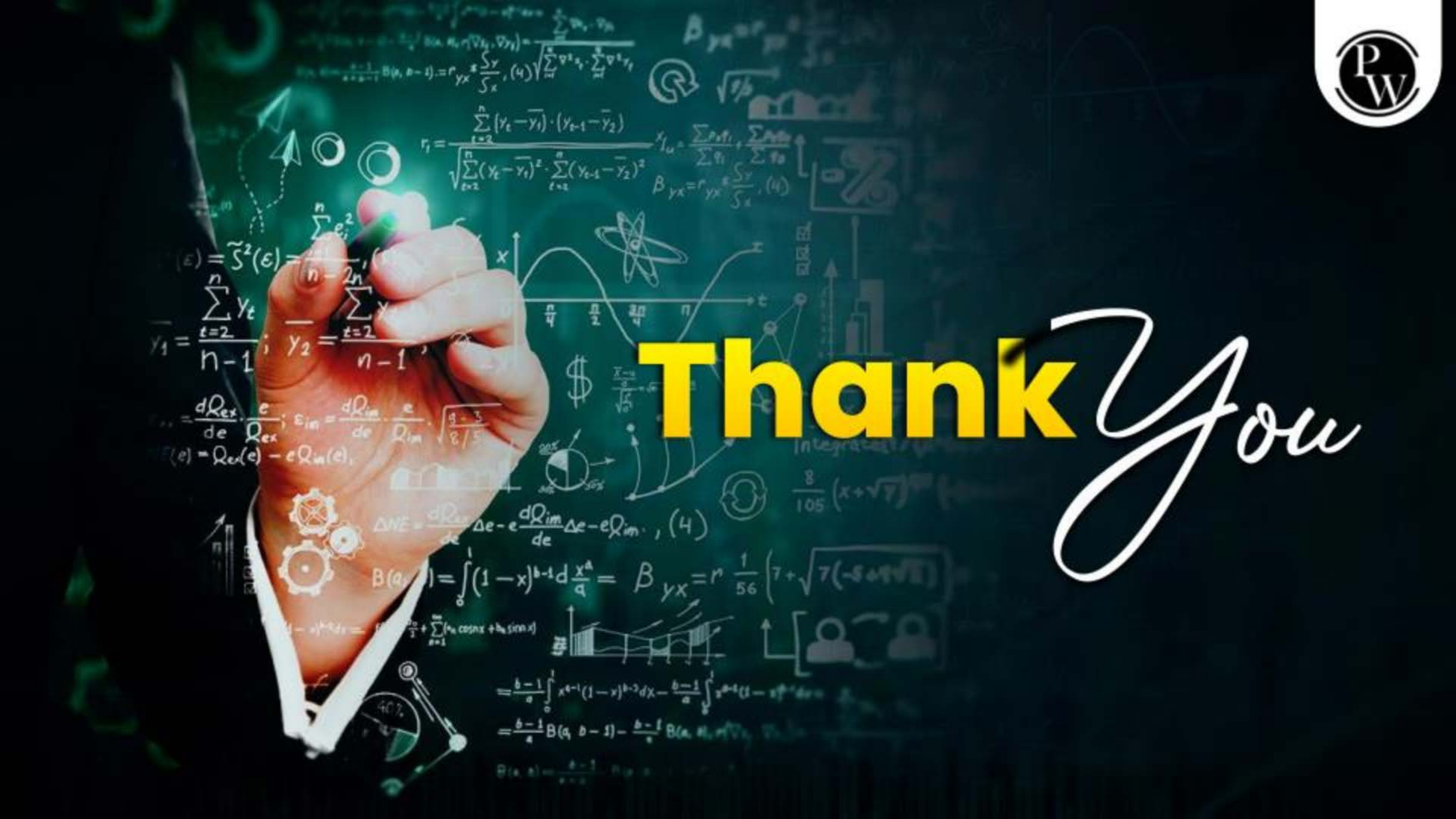
Sol: Av Height of func$^n$ $= \frac{1}{b-a}\int_a^b f(x)\,dx$

$$f(c) = \frac{1}{2-(-1)}\int_{-1}^{2}(5x^4+2)\,dx$$

$$= \frac{1}{3}\left[x^5 + 2x\right]_{-1}^{2}$$

$$= \frac{1}{3}\left[(32+4) - (-1-2)\right]$$

$f(c) = 13$ Ans


Thank You