MS&E 246 Final Report

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Exectutive Summary

In *MS&E 246: Financial Risk Analytics*, our team analyzed a data set of roughly 150,000 loans backed by the US Small Business Administration (SBA) between 1990 and 2014. In doing so, we aimed to implement and test models of the risk and loss of loan default. This report summarizes our findings from exploratory data analysis, details our approaches to modeling loan default probability and loss, and presents our methods of estimating the loss distributions of tranches backed by a portfolio of loans.

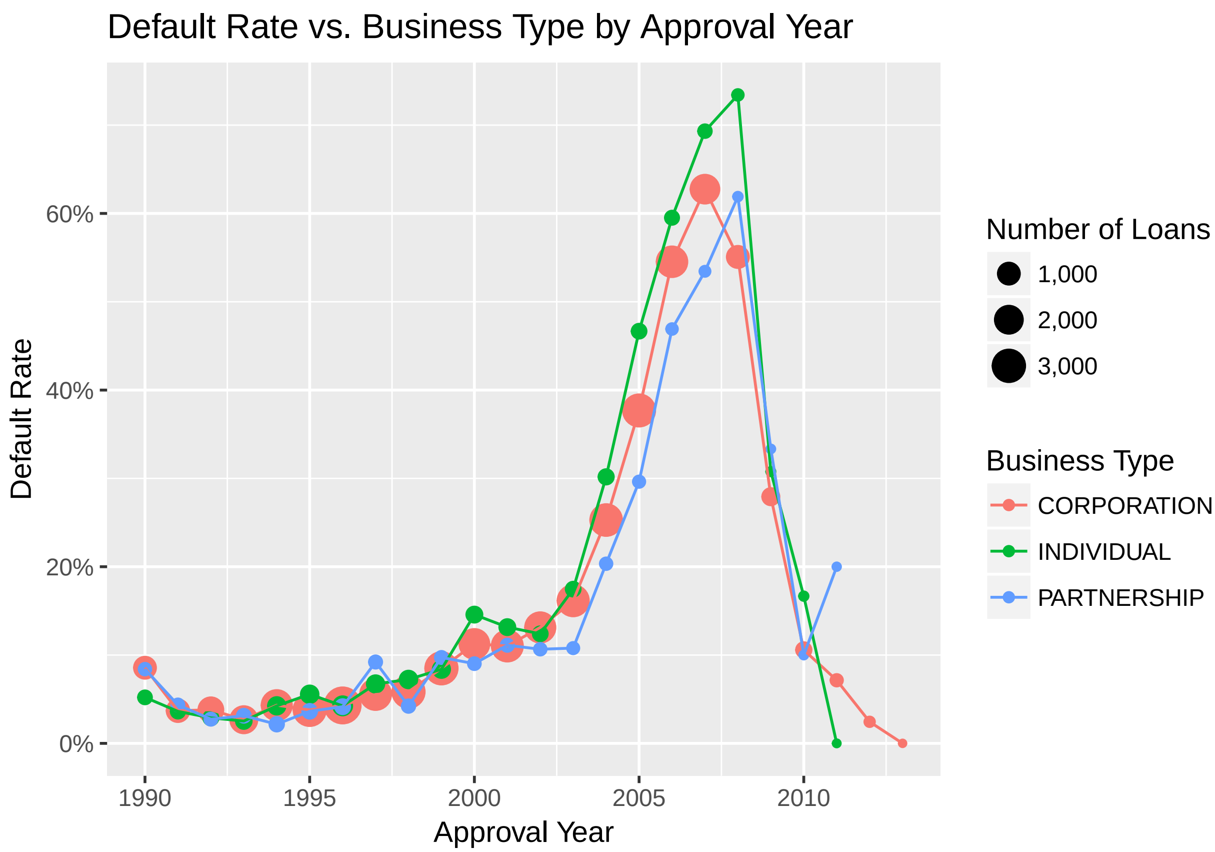
Exploratory Data Analysis

Prior to model building, we explored the data to detect patterns that may provide signal for models of loan default. Because we first aimed to build binary response models of default probability, we excluded "Exempt" loans from our exploratory analysis. Subsequently, we examined the relationship between default rates and the predictor variables, including `Business Type`, `Loan Amount`, `NAICS Code`, and `Loan Term`, among others.

Further, we collected additional predictor variables such as monthly `GDP`, `Crime Rate`, and `Unemployment Rate` by State, as well as macroeconomic predictors such as monthly measures of the `S&P 500`, `Consumer Price Index`, and 14 other volatility market indices (see "Data Cleaning" section for data collection details). We include insights from exploratory analysis of these measures as well.

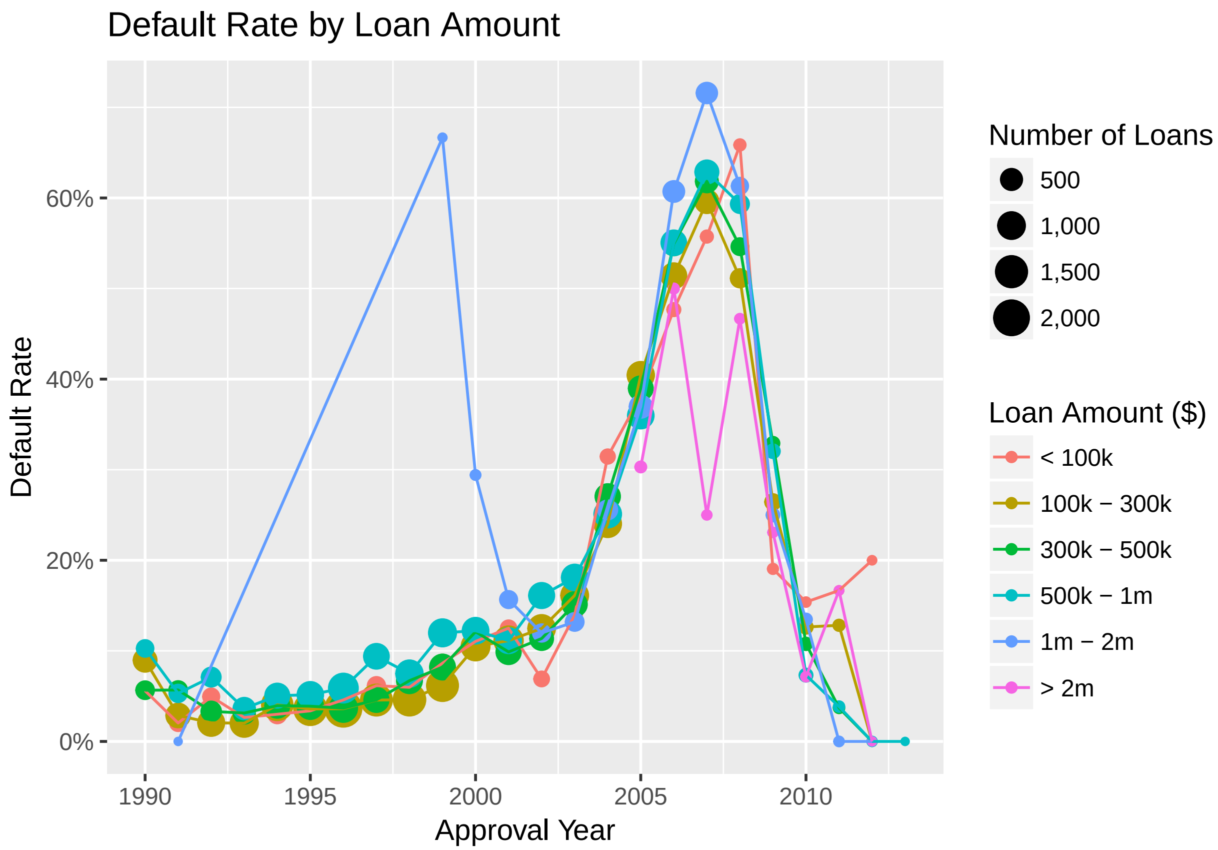
Default Rate vs. Business Type

First, we examined the relationship between default rate and `Business Type` by loan approval year. As shown on the plot below, we observe an interaction effect between these three features, such that default rates spiked for loans that were approved around the Great Recession (approximately 2006- 2009). Further, the different trajectories of the 3 curves implies the "Individual" `Business Type` suffered greater default rates than corporations and partnerships. Although corporations constitute a greater share of the data set, as evidenced by the greater mass in the red circles, they exhibit medium default risk, as compared to the other business types. Taken together, this plot reveals business types were affected differently by the recession, offering useful signal for subsequent modeling.



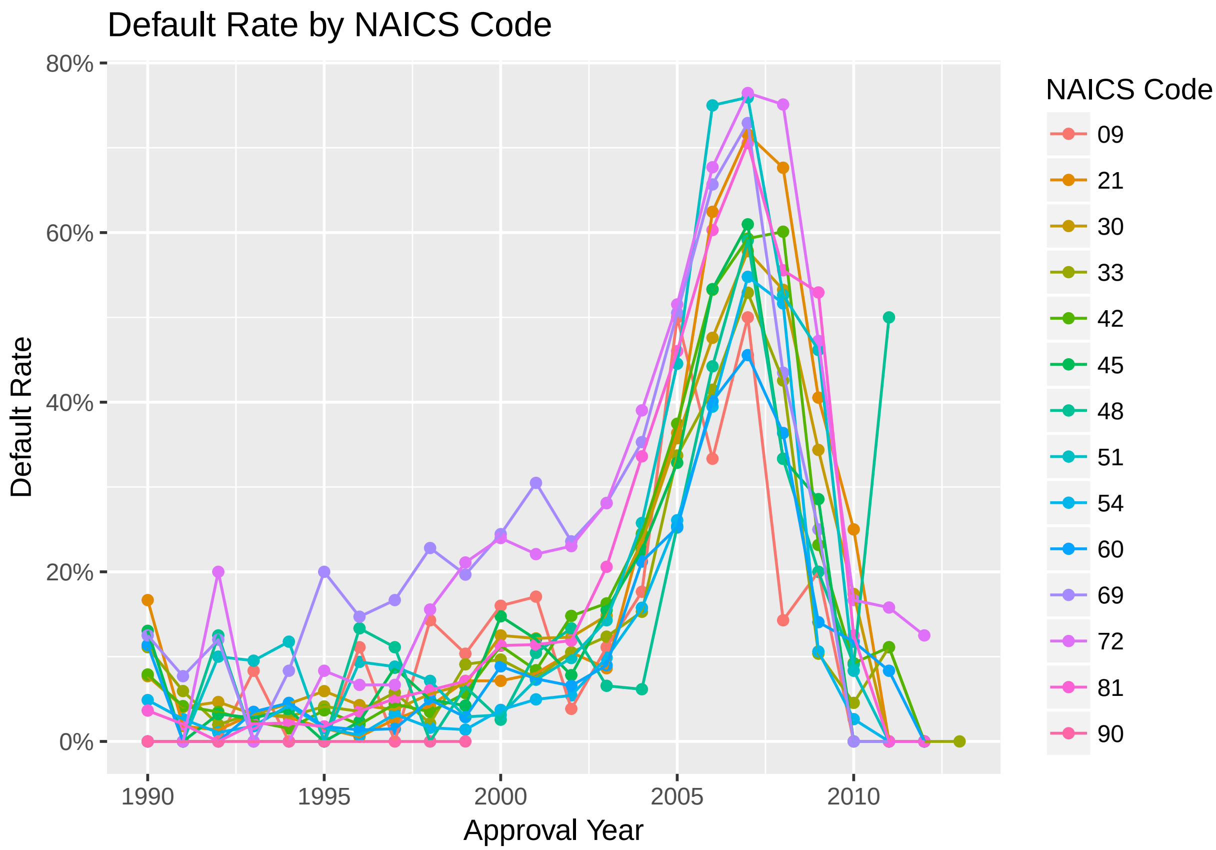
Default Rate by Loan Amount

Second, we examined whether we would observe a similar time-dependent interaction effect between default rate and `Loan Amount`. The plot below reveals that loans of all sizes approved around the Great Recession faced the greatest default rates. However, loans of sizes \$500k-\$1m and \$1m-\$2m appear to have experienced larger default rates over time compared to smaller loans of size \$100k-\$300k and \$300k-\$500k. The spiking behavior of \$1m-\$2m loans in 1999 and of loans greater than \$2m seem to be due to small sample sizes, as depicted by circle diameter. Overall, since loans of different sizes have different default rate patterns over time, we would also expect the `Loan Amount` feature to offer predictive power.



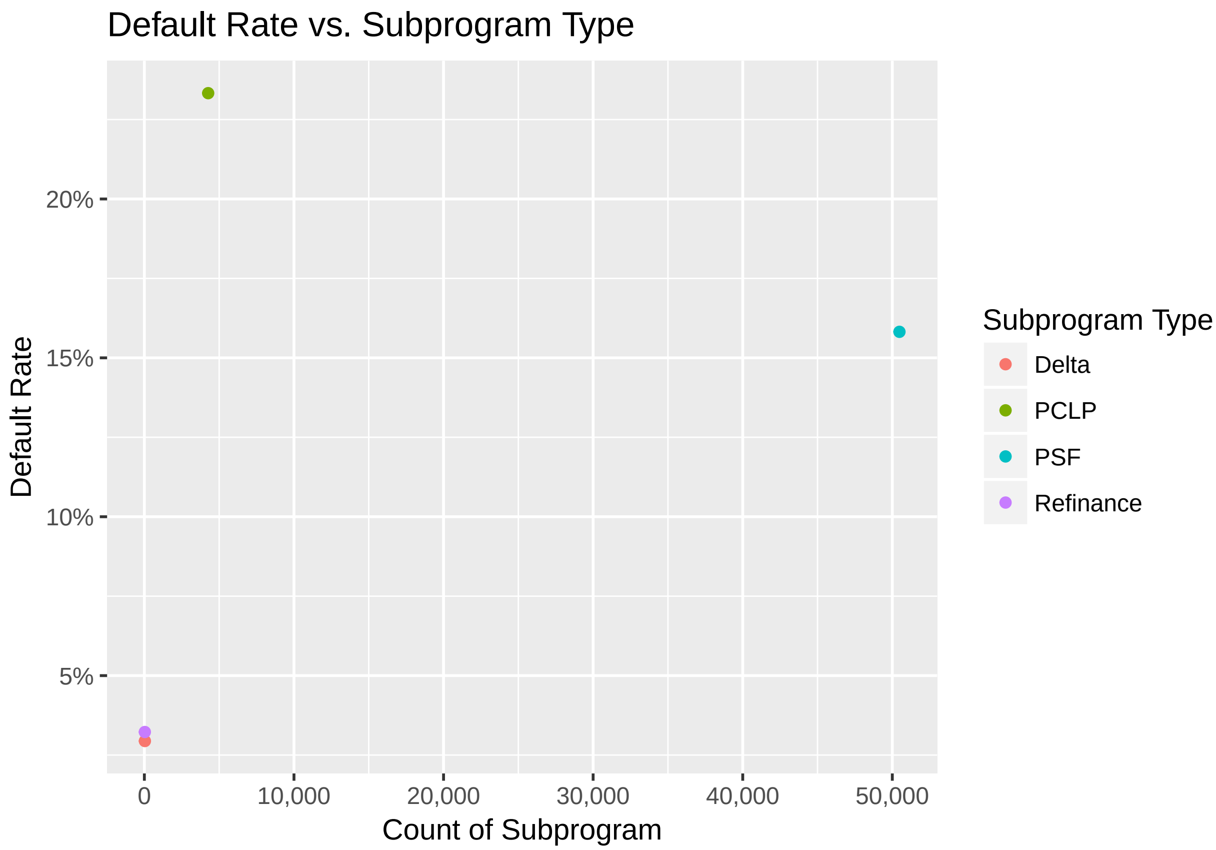
Default Rate by NAICS Code

Third, we hypothesized different economic sectors would exhibit different default rates over time. In turn, we extracted the North American Industry Classification System (NAICS) code for each loan and truncated it to the first two digits, which represents broad industry classes such as "Agriculture" and "Manufacturing." The following plot shows the default rate for loans of each truncated NAICS code approved in each year between 1990-2014. We observe considerable variance in default rates between sectors; for instance, code 72, corresponding to "Accommodation & Food Services", has one of the highest default rates even before the recession. However, code 54, corresponding to "Professional, Scientific, and Technical Services," consistently has one the lowest default rates. These patterns are consistent with intuition, and underscore the value of including the truncated NAICS code as a predictive feature of defaulting.



Default Rate by Subprogram Type

Fourth, we compared the default rates between different loan subprogram types. The plot below shows the default rates of the different loan subprograms versus their respective counts in the data. We observe that the PSF subprogram is the most common and has medium default risk. However, loans in the Premier Certified Lenders Program (PCLP) are less common, but have higher default risk. This suggests `Subprogram Type` offers useful signal for predicting default risk. Lastly, the loans belonging to the Delta and Refinance subprograms are highly uncommon and have low default risk. In order to reduce to the dimensionality of the feature space, we collapsed these two factor levels into "Other."



State GDP vs. Default Rate

- Make plot here

Modeling Default Probability

Building upon our exploratory data analysis, we constructed two types of predictive models of loan default probability: binary response models and the Cox Proportional Hazards model. Here, we present our approach to fitting both model types, including data cleaning, feature engineering, feature selection, hyper-parameter optimization, and evaluation.

Binary Response Models

First, we built binary response models of small-businesses defaulting on loans, which estimate the probability that a given loan *ever* defaults. To do so, we implemented a machine learning pipeline that:

1. Performs feature engineering;
2. Splits the data into train and test sets;
3. Normalizes continuous features;
4. Selects features using recursive feature elimination;
5. Trains binary response predictive models.

Lastly, we evaluated the performance of these models on resampled partitions of the training data, and on a held-out test set in terms of AUC, sensitivity, and calibration.

Feature Engineering

Building on insights derived from exploratory data analysis, we engineered the following features from the raw data:

- `NAICS_code`: truncated to the first two digits of the NAICS code;
- `subprogram`: condensed infrequent factor levels into "other" category;
- `approval_year`: extracted year from loan approval date-time object.
- `SameLendingState`: created flag for whether borrower received loan from in-state;
- `MultiTimeBorrower`: created flag for whether loan recipient is a multi-time borrower;
- `ThirdPartyLender` created flag for whether borrower received third party aide.

In effect, these features represent dimensionality reduction of factors with many levels. For instance, there are 1,239 unique NAICS six-digit NAICS codes in the raw data, yet only 25 unique 2-digit codes. Although we lose fine-grained detail by truncating the NAICS code, we aimed to optimize our models by reducing variance introduced by high dimensionality. After applying such dimension reductions, we eliminated extraneous variables, such as the Borrower's Zip Code and the Project's State, which were used to engineer features.

In addition to constructing features from the raw data, we also incorporated data from external sources, including monthly State-based measures of crime rate, GDP, and unemployment rate. We also joined in time-varying risk factors, including monthly snapshots of the `S&P 500`, `Consumer Price Index`, and 14 other volatility market indices.

- BEN: Fill in where the data came from and any other important info

Data Splitting

We randomly partitioned the data into 70% training and 30% test sets. This approach does not implement a time-based split, but rather a random sampling of observations over the entire 1990-2014 window. We adopted this splitting approach because we were interested in capturing the signal of the Great Recession within our models. Further, we did not create a validation set because we performed feature selection and hyper-parameter optimization using cross-validation on the training set.

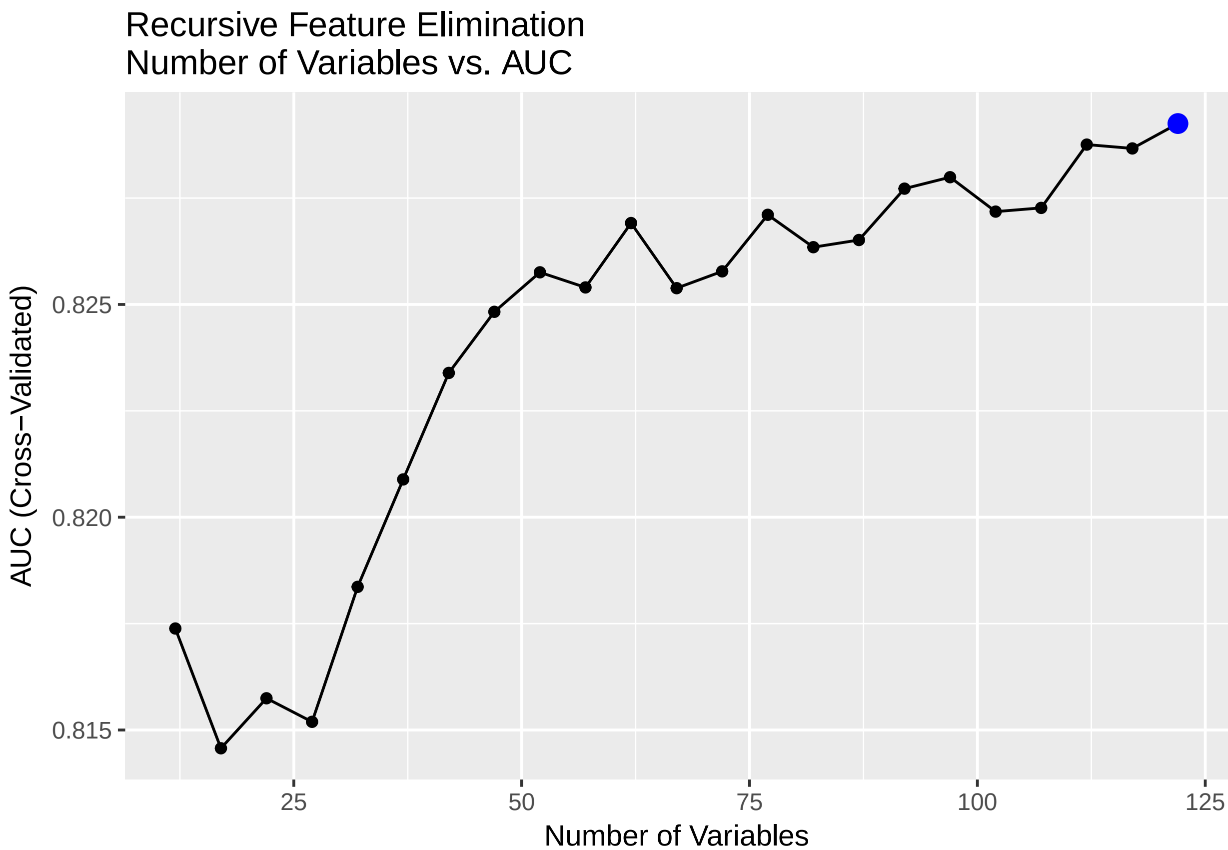
Data Preprocessing

After engineering features and joining in external data sources, we applied several preprocessing steps to our main data frame. First, we centered and scaled the continuous predictors to apply regularization techniques during the modeling phase. Doing so adjusted for variables being on different scales; for example, `Gross Approval` varies in dollar amounts from $30,000 to $4,000,000, whereas `Term in Months` ranges from 1 to 389. Second, we applied a filter to remove features with near zero variance to eliminate predictors that do not offer meaningful signal.

Feature Selection

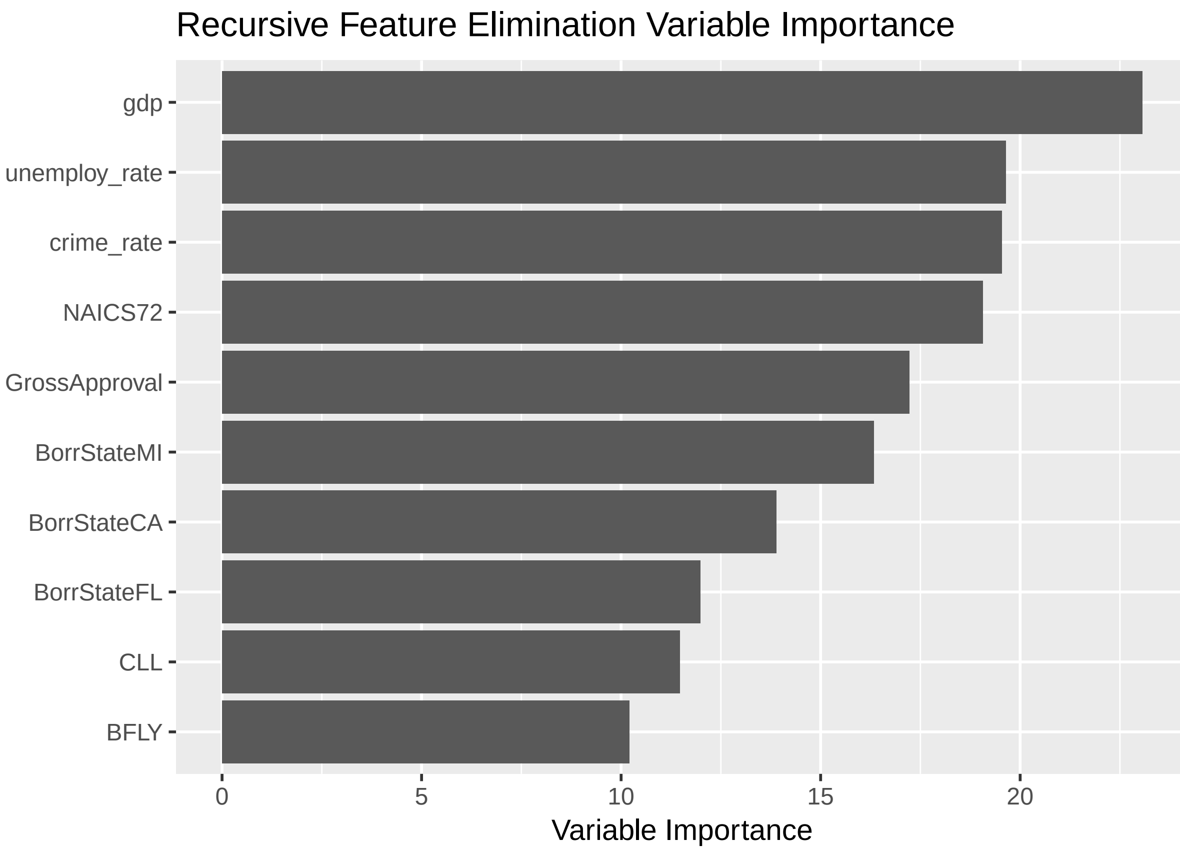
To perform feature selection, we used recursive feature elimination with 10-fold cross-validation. This method uses random forests to iteratively remove variables with low variable importance, as measured by mean increase in out-of-bag area-under-the-curve (AUC). In other words, variables that do not contribute to significant improvements in AUC are eliminated. We performed a grid search over the number of potential features to determine how many features to include. Note that factors were converted to separate dummy variables using a one-hot encoder.

The following plot shows that recursive feature selection chose 122 variables because AUC is maximized (see plot below). In effect, all variables were kept because they offered predictive power regarding loan defaults.



The importances of the top 10 selected features are shown in the plot below. We observe that State GDP, a monthly time-dependent risk factor, is the most important feature, meaning it led to the greatest average increase in AUC across cross-validation iterations. State unemployment rate and crime rate are also highly important, suggesting local time-dependent risk factors are the most predictive of whether a loan defaults.

The importance of NAICS code 72, corresponding to "Accommodation & Food Services", is consistent with our exploratory data analysis finding that the sector is especially risk prone. Borrower States such as Michigan, California, and Florida also offer predictive power regarding defaulting. Lastly, the importances of the Collar Index (CLL) and Iron Butterfly Index (BFLY) imply market volatility measures also improve the discrimination of loan defaults.

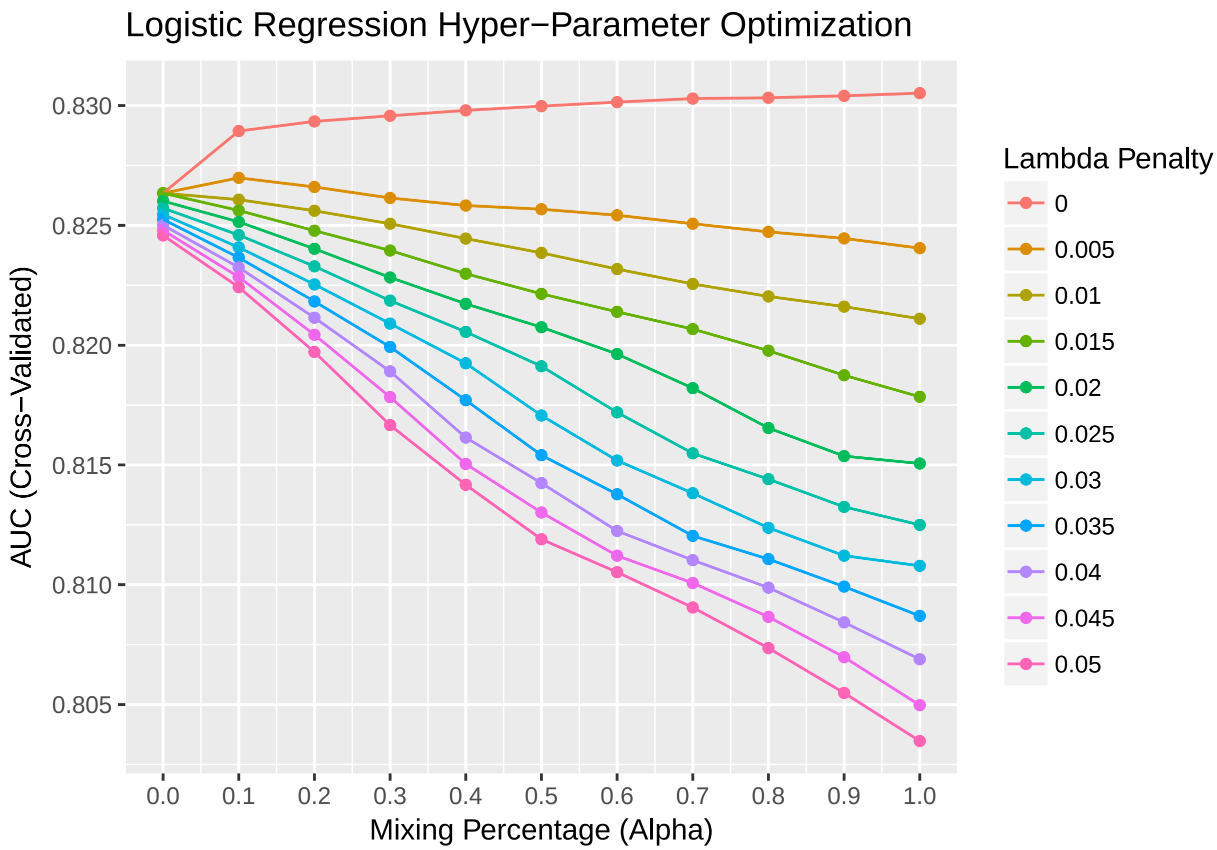


Model Fitting

Using these selected features, we fit models predicting the binary outcome of whether a small business defaults on a loan. We constructed linear and nonlinear models, including a logistic regression model with the elastic net penalty, a random forest classifier, and a gradient boosting machine classifier. To tune hyper-parameters, we used 10-fold cross-validation with the one standard error rule, which selects parameters that obtain the highest cross-validated AUC within one standard error of the maximum. For each model type, we performed a grid search over the hyper-parameters to ensure optimal selection.

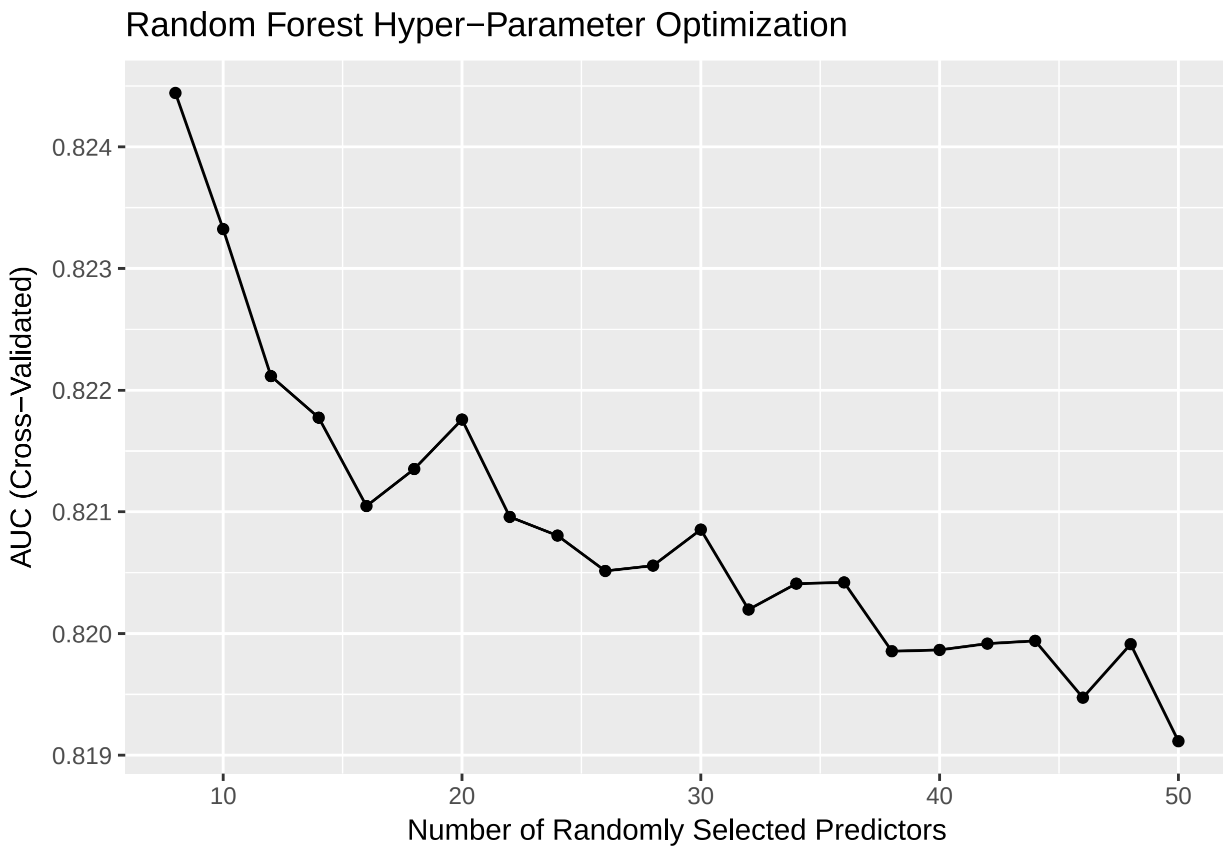
Logistic Regression with Elastic Net

First, AUC was used to select the optimal logistic regression model with an elastic net penalty using the one standard-error rule. As shown in the plot below, the final values used for the model were `alpha` $= 0.1$ and `lambda` $= 0$, indicated by the spike in the red curve at `alpha` $= 0.1$. This implies the optimal model used the ridge penalty more than the LASSO penalty with minimal regularization.



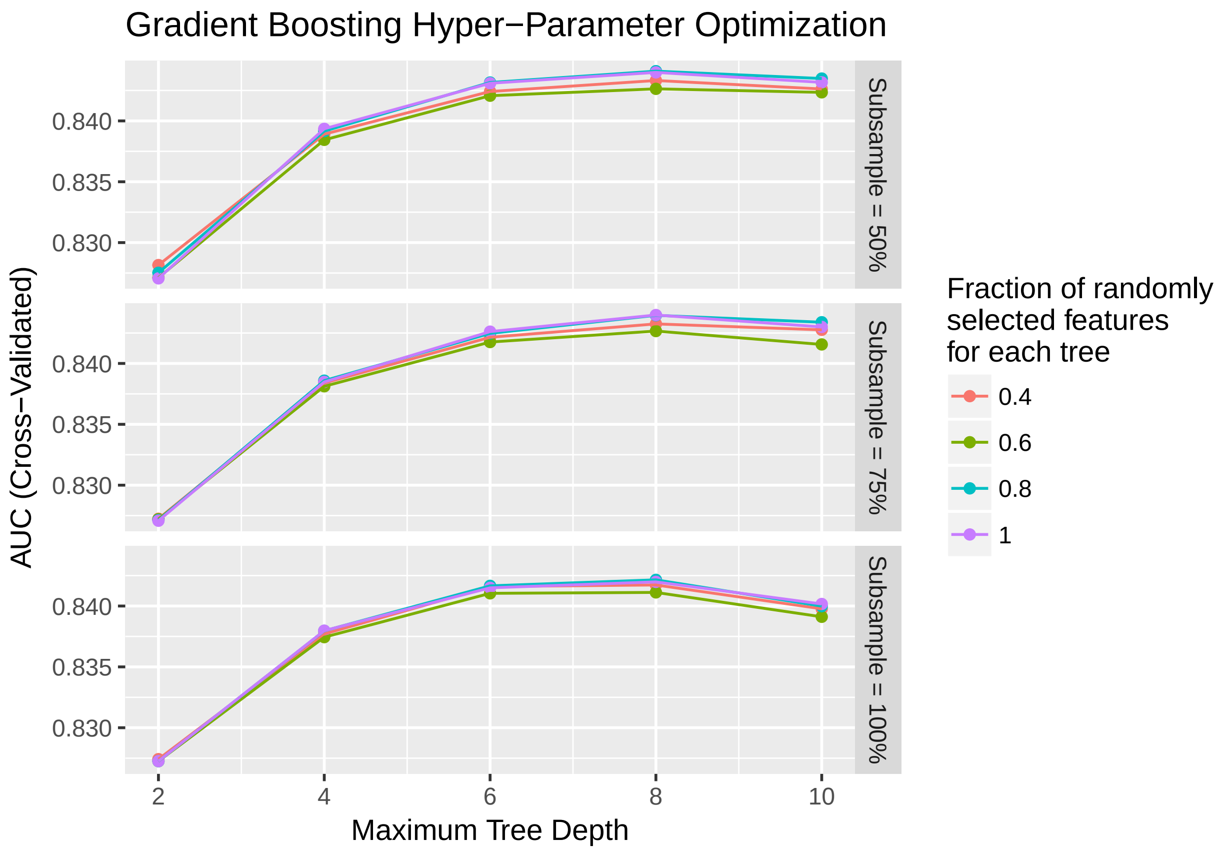
Random Forest Classifier

Second, AUC was used to select the optimal random forest model, which selected `mtry` = 8 as the best parameter. This means 8 random predictors were chosen to build each tree of the random forest. The plot below shows steadily declining AUC as the number of randomly chosen predictors increases, indicating that the optimal model is sparsest.

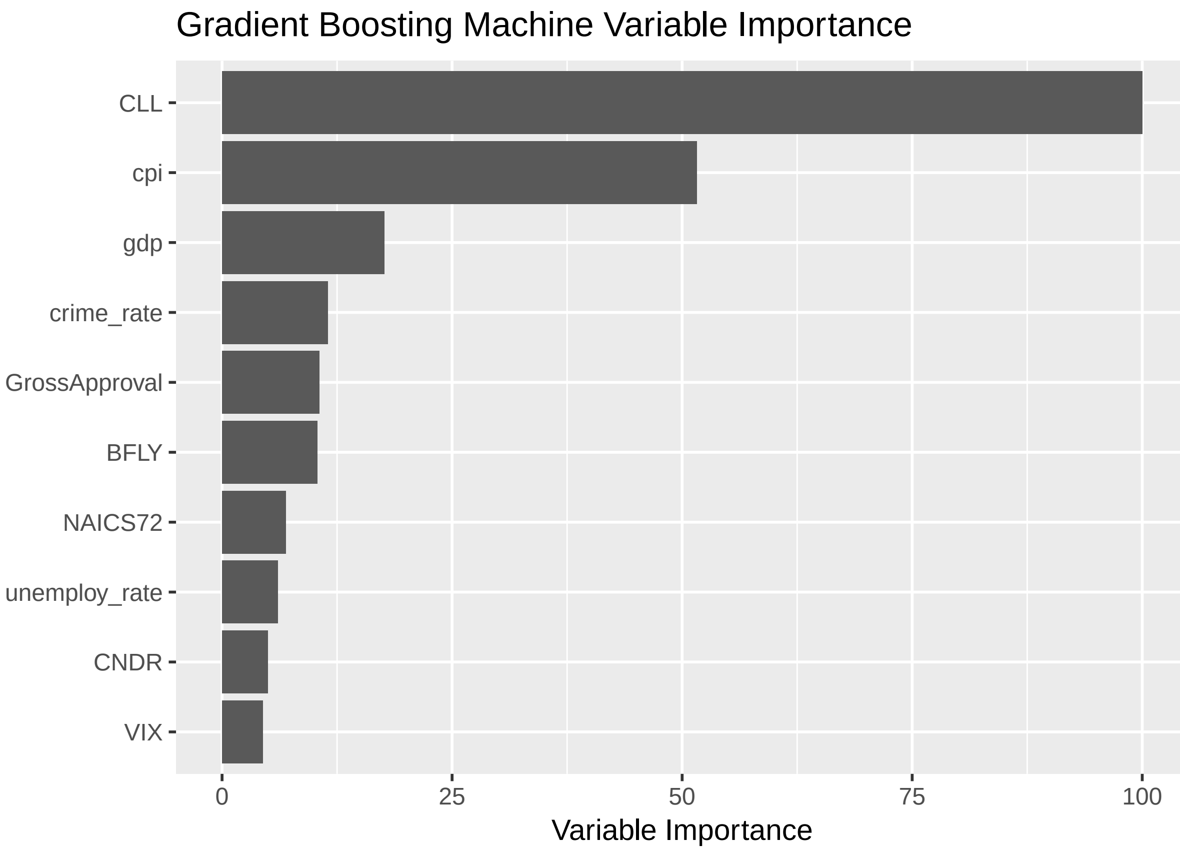


Gradient Boosting Machine Classifier

Third, AUC was similarly used to select the optimal gradient boosting machine (GBM) model. The final values used for the model were `nrounds` = 100, `max_depth` = 6, `eta` = 0.03, `gamma` = 0, `colsample_bytree` = 0.4, `min_child_weight` = 1 and `subsample` = 0.5. This means that the tuning procedure utilized a learning rate of 0.03 and a minimum loss reduction of 0, resulting in the optimal model with 100 trees of maximum depth 6 that subsamples 50% of the observations and 40% of the features for each tree. This combination of optimal hyper-parameters is shown by the spike of the red curve in the first subplot at the maximum tree depth of 6.



Examining the variable importance of the final GBM model, we observe the most important feature for predicting defaults is the Collar Index (CLL), which is "designed to provide investors with insights as to how one might protect an investment in S&P 500 stocks against steep market declines" (CBOE). Other important features include the national consumer price index (CPI), State GDP, crime, and unemployment rates, loan amount, and Chicago Board Options Exchange (CBOE) indices including the Butterfly Index (BFLY), the Iron Condor Index (CNDR), and the Volatility index (VIX). Such variables are "important" because they lead to the greatest improvements to cross-validated AUC across boosting iterations.



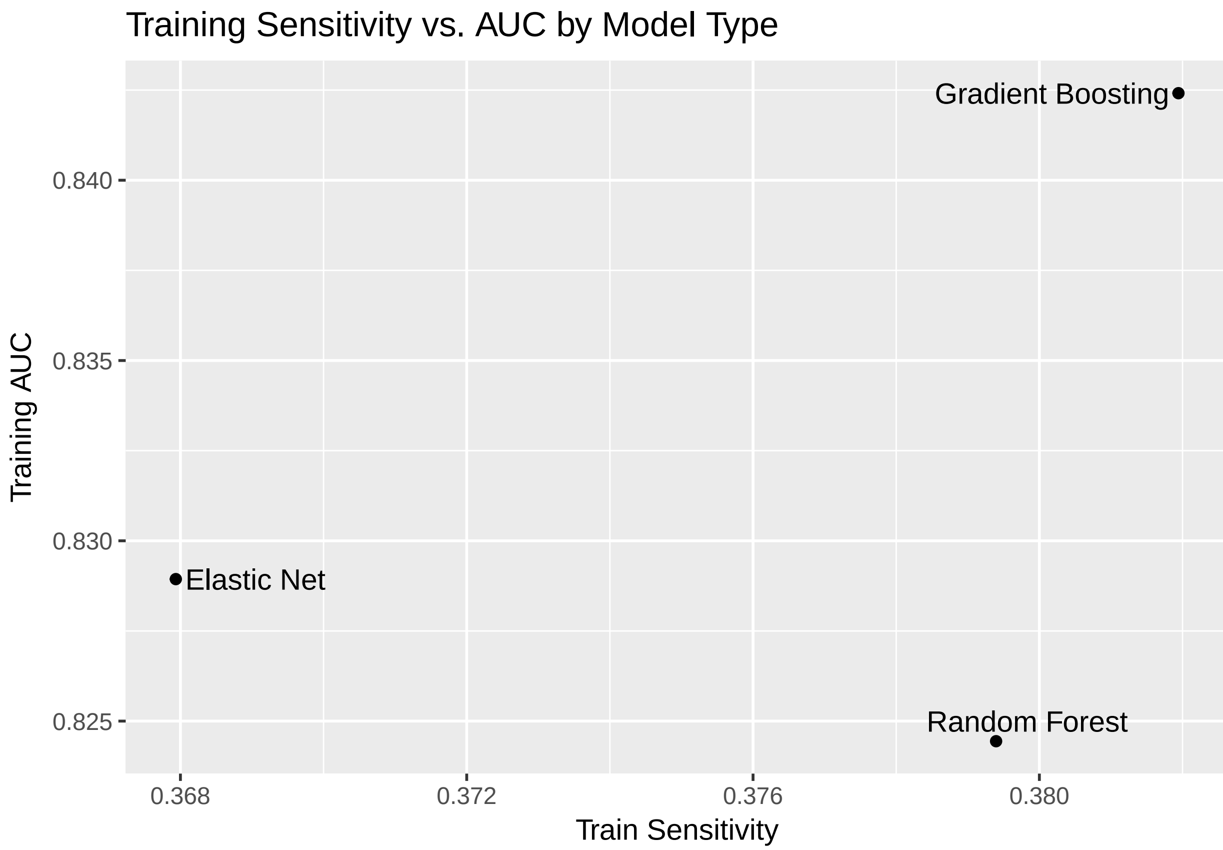
Model Evaluation

After we optimized the hyper-parameters of our models, we evaluated the models using in-sample and out-of-sample metrics, including AUC, sensitivity, ROC curves, and calibration. To do so, we used these "best" models to predict loan defaults in the training and test sets.

In-Sample Evaluation

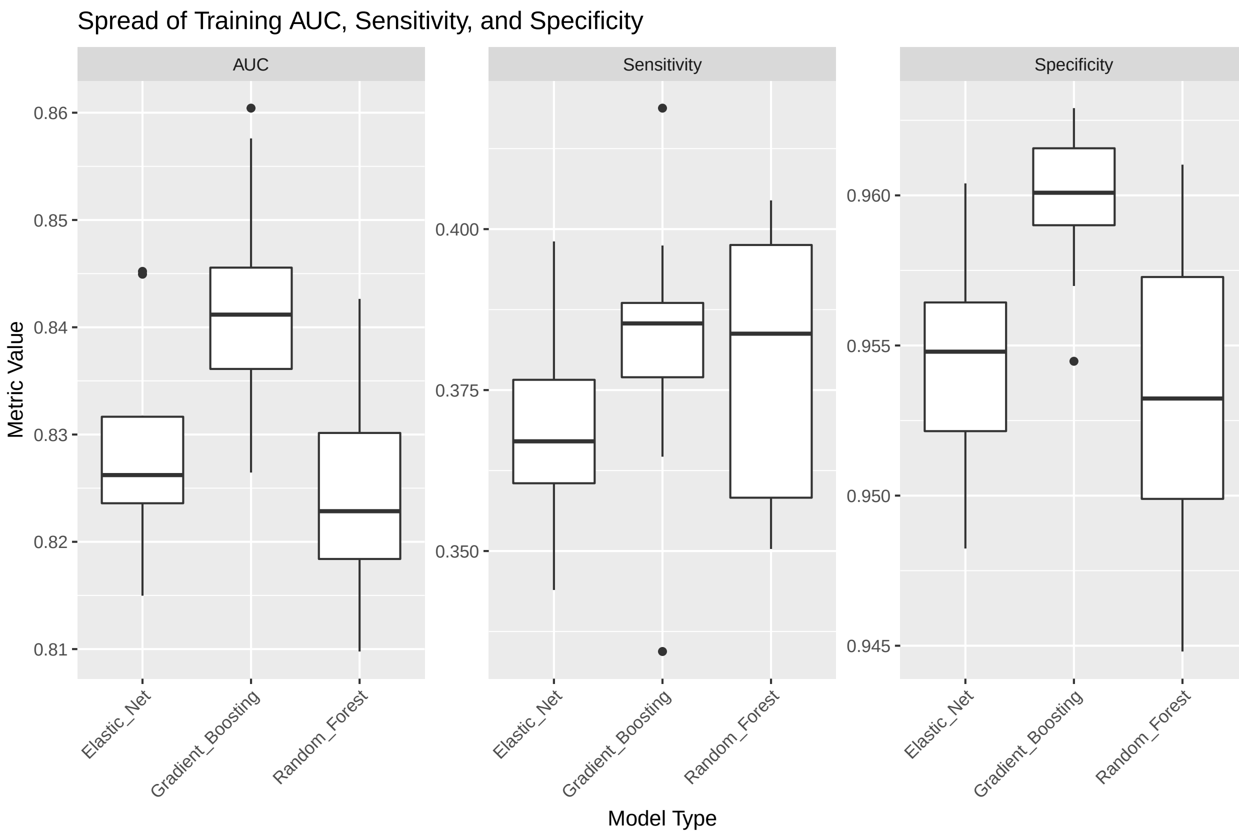
Training AUC and Sensitivity of Best Models

The following plot compares averaged **training** area under the ROC curve and sensitivity across the model types with optimized parameters. We observe that the gradient boosting machine classifier has the highest AUC and sensitivity, whereas the logistic regression model with the elastic net penalty performs the worst.



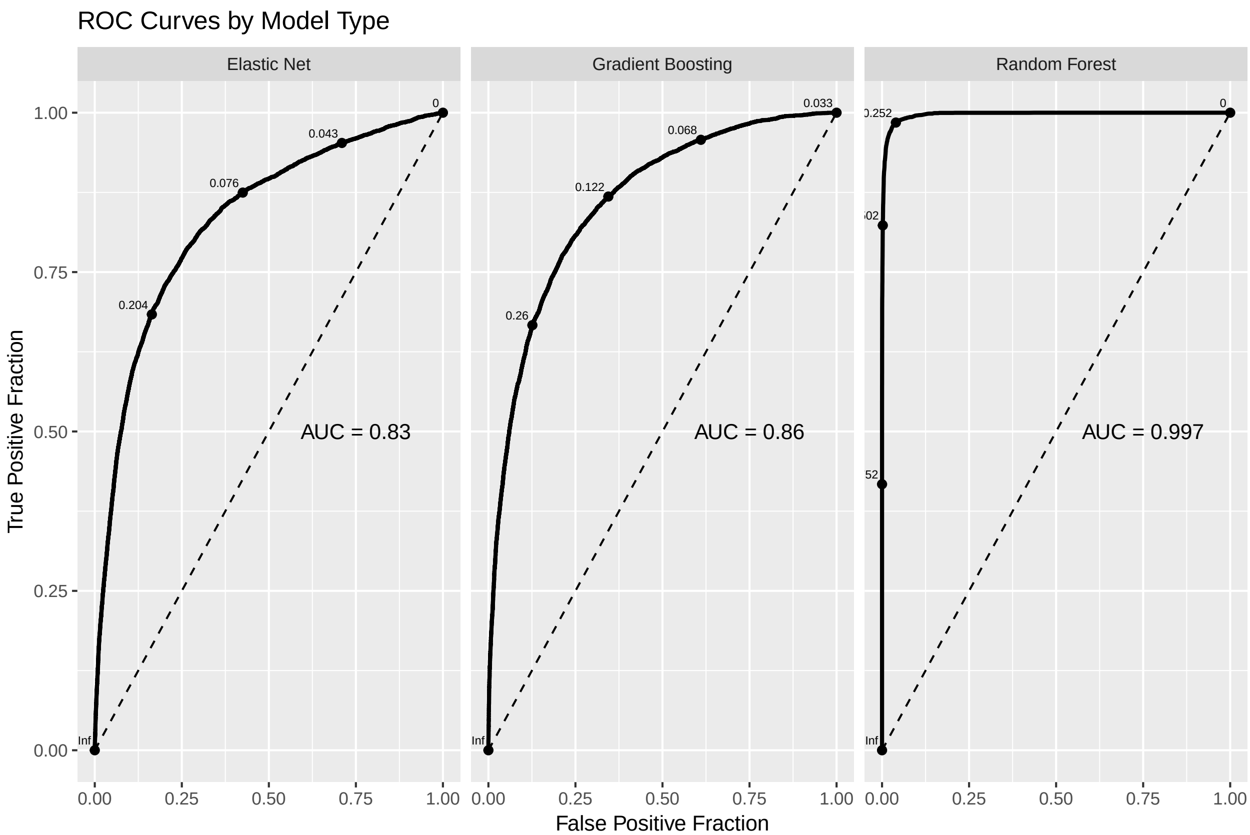
Distribution of Resampled Training AUC, Sensitivity, and Specificity

To examine the spread of **training** area under the ROC curve, sensitivity, and specificity across model types, we leverage the resampled data generated during the cross-validation of model fitting to plot their respective distributions. In the following plot, we observe that the GBM classifier has the highest median AUC, sensitivity, and specificity, as well as the smallest spread. Although the random forest classifier has comparable sensitivity, it exhibits enormous variance compared to the other models, suggesting it is prone to overfitting. For this reason, the logistic regression classifier (a linear model) outperforms the random forest classifier (a non-linear model) in terms of AUC and specificity.



Training ROC Curves

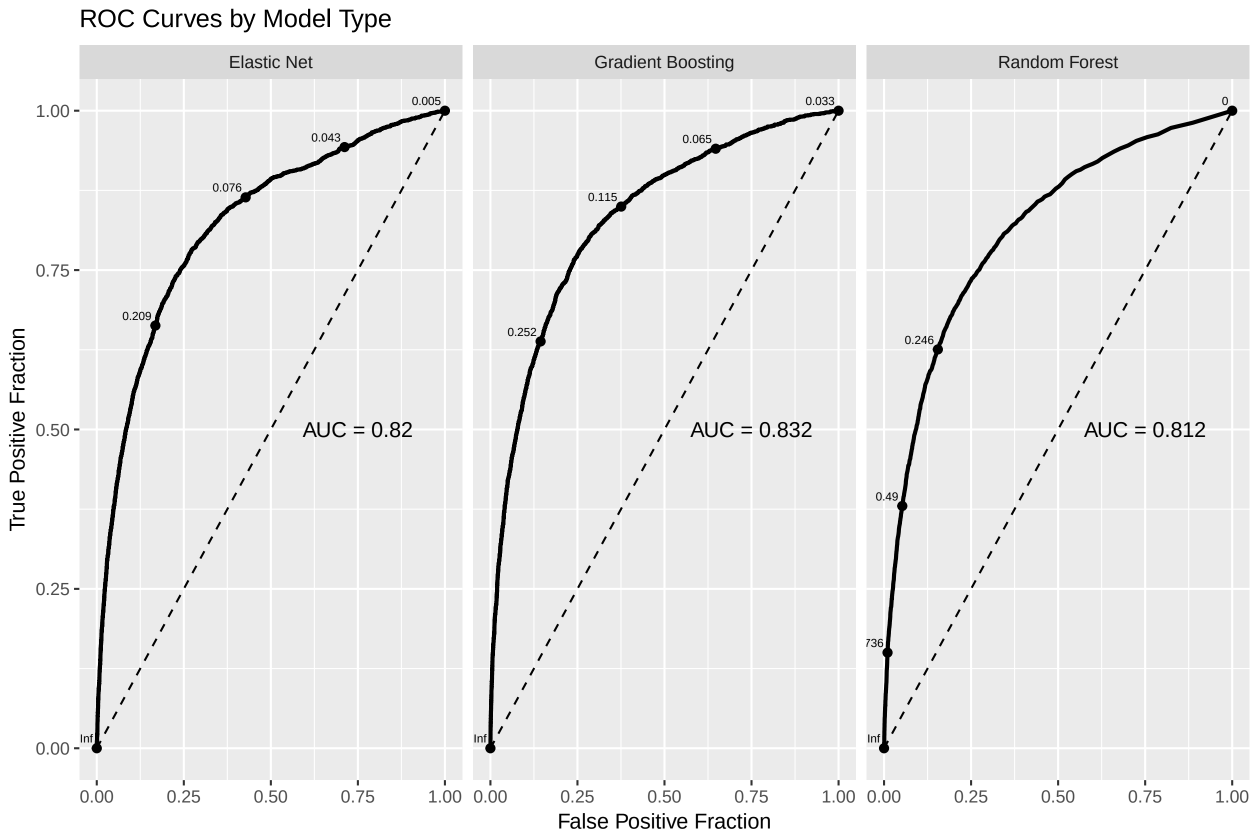
Lastly, we can examine the **training** ROC curves by model type. We observe that the random forest model has a near-perfect ROC curve, which also implies it is overfitting to the training data. The GBM model again performs worse than the random forest model on the training data, but likely because it is avoiding overfitting. The logistic regression model with the elastic net penalty performs the worst.



Out-of-Sample Evaluation

Test ROC Curves

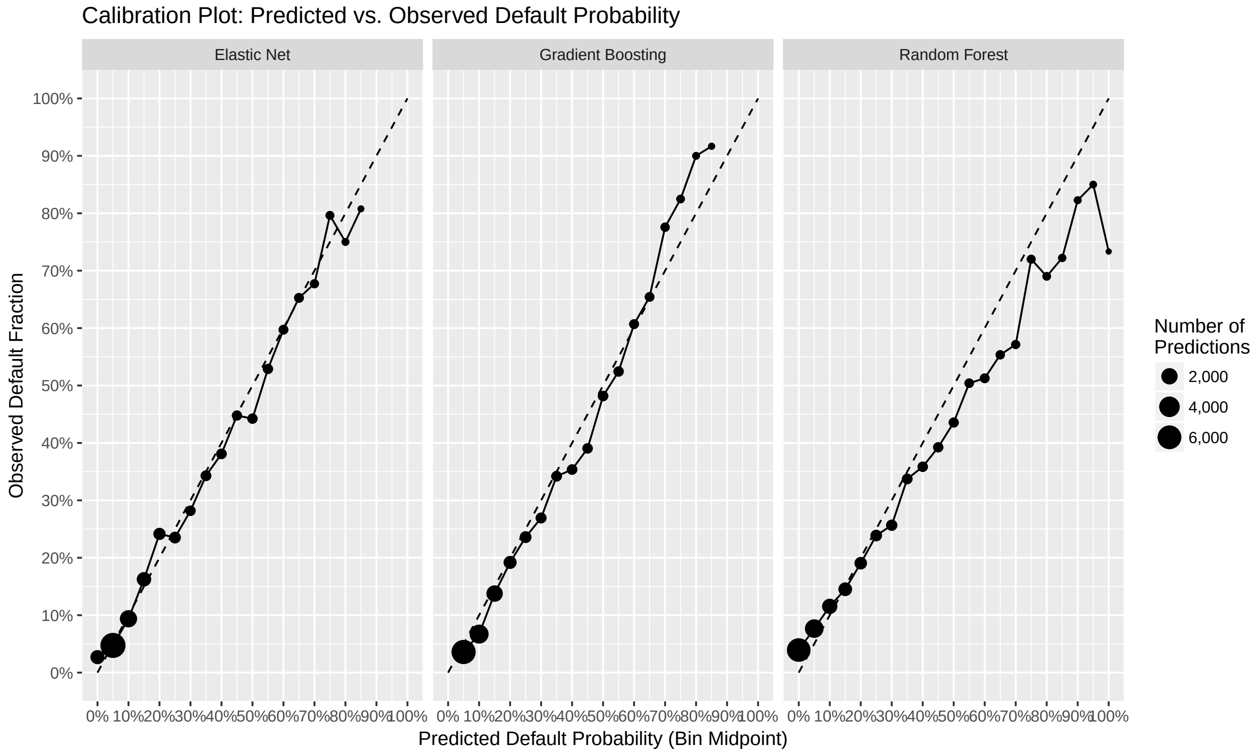
We evaluated our best models on a held-out test set representing 30% of the original data. The ROC curves below reveal that the GBM model performed the best on the test set, followed by the logistic regression model, and finally, the random forest classifier. The weak performance of the random forest classifier is likely due to overfitting on the training set. Nevertheless, all models achieve good performance over "random guessing" baselines.



Test Calibration Plots

Lastly, we evaluated the calibration of the our models' predicted probabilities of loan default against the observed fraction of defaults in the data. A point on the dashed line means that the model's predicted probability of default matched the empirical default rate. Points to the right of the line mean the model overestimated the default probability, whereas points to the left mean the model underestimated the default probability.

The GBM model achieves the best calibration because its points follow the dashed line most closely. The logistic regression model with the elastic net penalty achieves comparable performance; however, the random forest classifier tends to overestimate default probabilities. Again, this weaker performance is likely due to overfitting.



The overfitting of the random forest classifier may be due to the fact that too many features were randomly selected to build trees at each iteration. Our hyper-parameter optimization approach performed a grid search over possible values of `mtry`, representing the number of features randomly chosen to build each tree in the forest. However, our grid may have not been large enough, since the minimum value of `mtry` was chosen. However, computational resources limited our ability to refit models over a larger search space.

Moreover, the gradient boosting machine classifier demonstrated the best performance on the test set in terms of AUC and calibration.

Cox Proportional Hazards Model

Survival analysis gives more detailed information about how the default risk of a loan varies over time. With binary classification, we estimated the probability that a given loan *ever* defaults. With a hazard model, we are able to estimate the probability that a loan defaults between any two points of time in its life.

Model Choice

There exist many specialized Cox models that assume a particular form of the baseline hazard function. The Cox Proportional Hazards Model does not have this requirement. We can see this in the following description of the partial maximum likelihood procedure used to estimate the parameters of the Cox PH model:

The form of the cox model is:

$$h(t) = h_0(t)exp(\beta^T X)$$

Suppose there are r observed death times in the data (all distinct), and that t_j is a death time in the set of possible death times: $R = \{t_1, t_2, ..., t_r\}$.

Then the conditional probability that an individual dies at time t_j given t_j is a time of death in the set R:

$$\frac{P(\text{individual with feature vector } X^{(j)} \text{ dies at } t_j)}{P(\text{one death at } t_j)}$$

$$= \frac{P(T = t_j|X^{(j)}, T \geq t_j)}{P(T = t_j|X^{(k_0)}, T \geq t_j) \cup P(T = t_j|X^{(k_1)}, T \geq t_j) \cup ...P(T = t_j|X^{(k_q)}, T \geq t_j)}$$

Where $k_0, ..., k_q$ correspond to the indices of observations with event times greater than or equal to t_j. Since the probabilities in the denominator are *assumed to be conditionally independent*, the denominator can be expressed as a sum of probabilities. Converting the above to continuous time, we get:

$$= \frac{\lim_{\delta \to 0} \frac{P(T < t_j + \delta|X^{(j)}, T \geq t_j)}{\delta}}{\sum_{i=k_0}^{k_q} \lim_{\delta \to 0} \frac{P(T < t_j + \delta|X^{(i)}, T \geq t_j)}{\delta}}$$

$$= \frac{h_j(t_j)}{\sum_{i=k_0}^{k_q} h_i(t_j)} = \frac{h_0(t_j) exp(\beta^T X^{(j)})}{\sum_{i=k_0}^{k_q} h_0(t_j) exp(\beta^T X^{(i)})} = \frac{exp(\beta^T X^{(j)})}{\sum_{i=k_0}^{k_q} exp(\beta^T X^{(i)})}$$

And we can see that the contribution of any observation to the likelihood function will not be dependent on $h_0(t)$. $\square$

Modifications to the Data

Roughly 95% of loans in the training data set had a term of 20 years. We decided that considering loans with the same term was more appropriate for this analysis (84,949 loans). Within the training data, about 86% of loans were right censored (term did not expire in window, and did not default), about 7% of loans were paid off (term expired in window), and about 7% of loans defaulted within the window (figure 1).

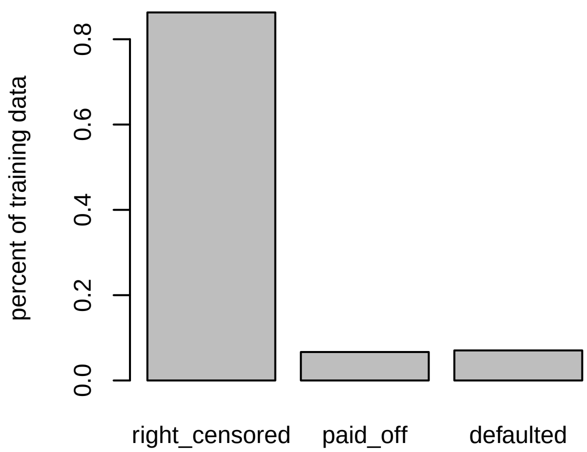


Figure 1: Loans in training data by status

Polynomial terms up to *degree five* were added for all numeric variables. Our intention was to include these features to capture non-linearities in these variables, and conduct feature selection during model fitting (through regularization).

Further, all numeric variables were centered to 0, and scaled by standard deviation.

Missing values were set to 0 and an missing value indicator feature was added for each original variable.

Including expanded categorical variables, polynomials, and missing value dummies, the data had 201 features.

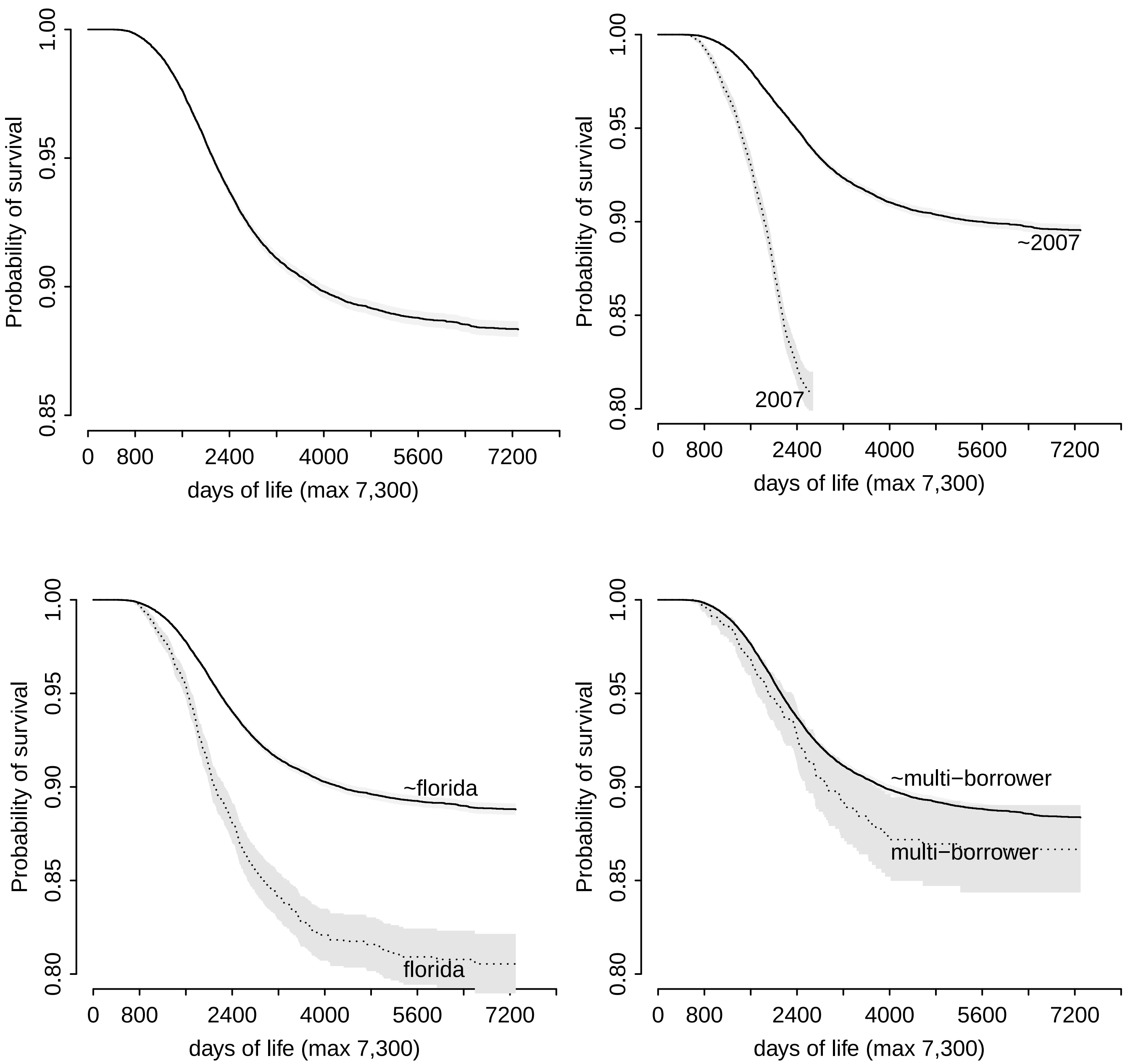
Kaplan-Meier Survival Curves

A Kaplan-Meier curve is a non-parametric estimate of the survival function, $S(t) = P(T > t)$, defined as:

$$\hat{S(t)} = \prod_{t_i \leq t} [1 - \frac{d_i}{n_i}]$$

Where $\{t_1, ..., t_r\}$ are the death times of observations in the data, $\{d_1, ..., d_r\}$ are the number of deaths that occur at those times, and $\{n_1, ..., n_r\}$ are the number of observations remaining in the at-risk population just before those times.

For expository purposes the following plots show the estimated survival function conditioned on select categorical variables such as a particular year, state, or status, as well as the general survival curve for our loan population. Note that the survival curve was significantly steeper for loans conditioned on these variables (a higher probability of default at all times).



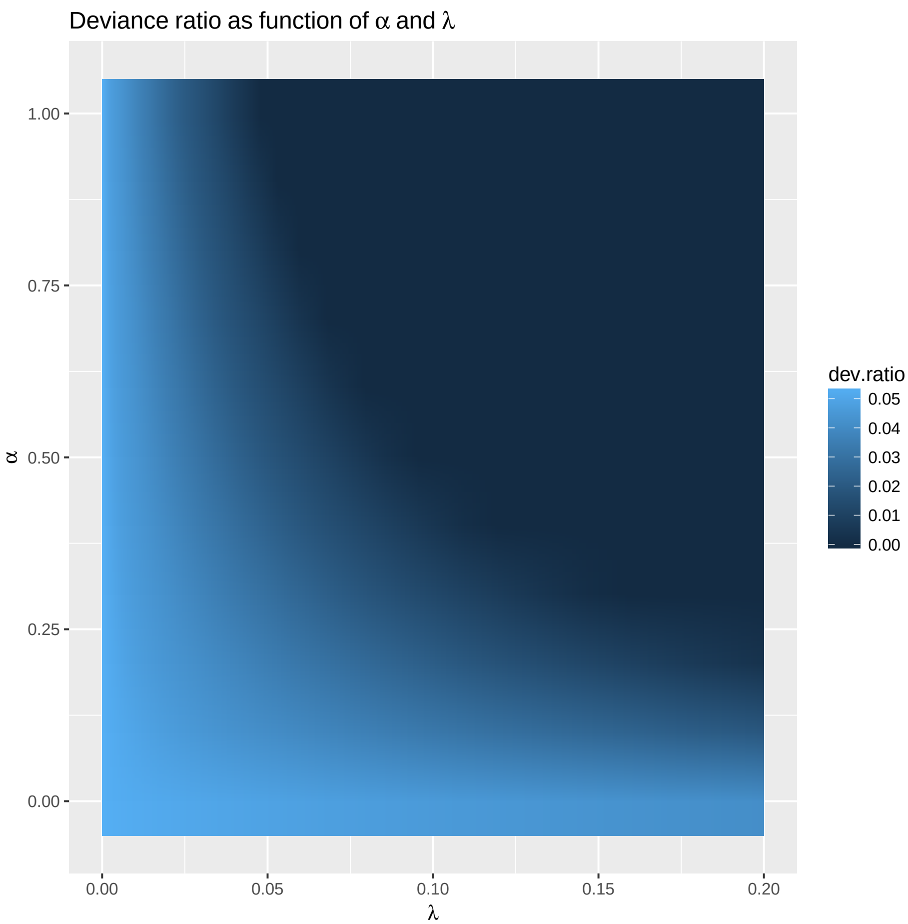
Penalized Cox Proportional Hazards Model

For the purpose of feature selection, we fit a series of penalized Cox models to the training data.

We used an elastic net penalty– a penalty term that is a linear combination of the l_1 and l_2 penalties.

$$\lambda[(1-\alpha)||\beta||_2 + \alpha||\beta||_1]$$

We fit models varying α and λ in the penalty– we selected the model with the largest evaluated value of the likelihood function.

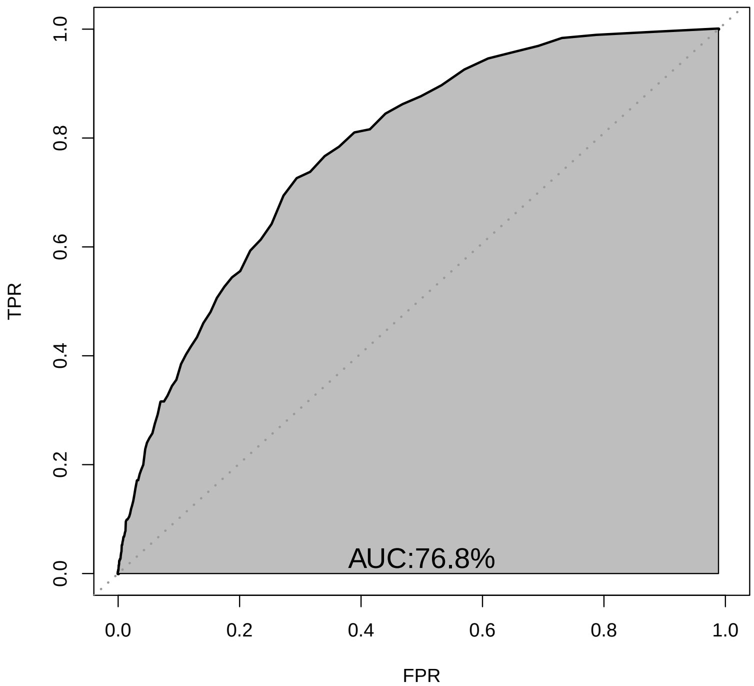


The best model, in terms had a value of λ very close to 0, and α very close to 0 (the ridge penalty). Ninety-seven variables of the original 201 had non-zero coefficients.

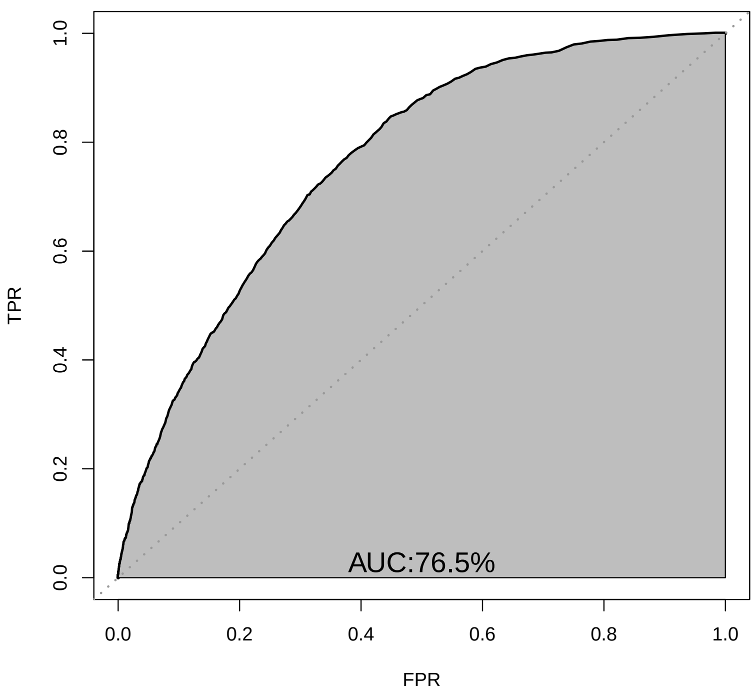
One Year and Five Year Predictions of Default (out of sample)

The below figures show the out of sample performance of the one and five year probabilities estimated by the Cox model:

ROC curve for 1 year ahead default predictions



ROC curve for 5 year ahead default predictions



Modeling Loss at Default

Using our optimal Cox proportional hazards model, we computed the value-at-risk (VaR) and average value-at-risk (AVaR) for a portfolio of 500 randomly chosen loans. Here, we detail our procedure for selecting a loan portfolio, constructing a model for loss at default, simulating the total loss of the portfolio, and computing VaR and AVaR.

Portfolio Selection

To build a model for loss at default, we considered a portfolio of 500 loans selected from the withheld test data set. These loans met the following criteria:

1. Loans that had not defaulted as of 02-01-2010.
2. Loans that were approved before 02-01-2010.
3. Loans less than 15 years old.

These conditions were to ensure that the 500 loans in question were active as of the portfolio construction date, which we determined to be 02-01-2010. The 15 year age limit was so that estimation of 5 year ahead default probabilities would be valid.

Data Cleaning

Before fitting the loss at default model, we cleaned the training set by filtering it to only include defaulted loans and by removing unnecessary features such as `LoanStatus`.

Feature Selection

To select the features used in the loss at default model, we performed recursive feature elimination. We used 5-fold cross validation and the "one standard error rule" to choose the number of features that minimized mean squared error within one standard error of the minimum.

Model Fitting

Using the features selected by recursive feature elimination, we built a random forest model of loss at default. We used 5-fold cross-validation and the one standard error rule to find the optimal number of features to be considered for splitting during construction of each tree.

Model Evaluation

Test Set Prediction

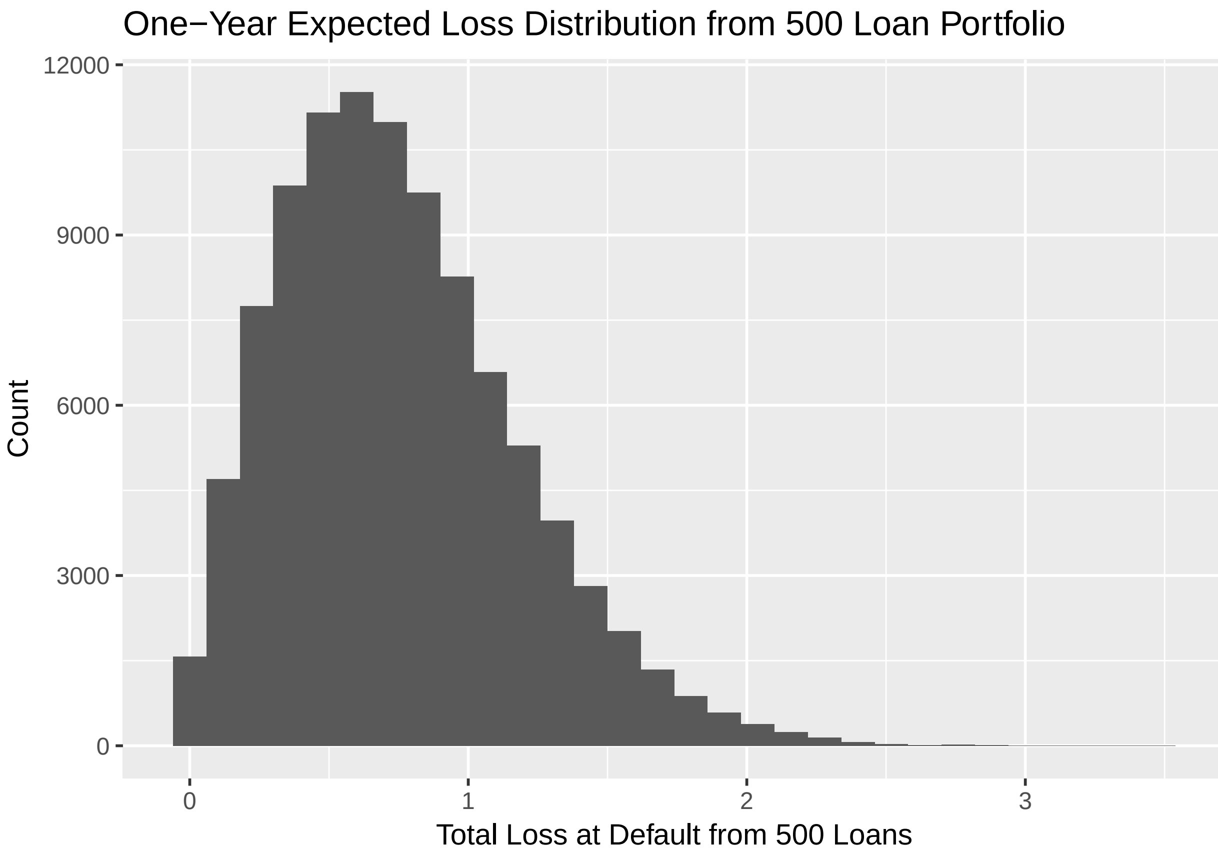
We calculate the expected loss and default probability of each loan in the portfolio of 500 loans by using the model of expected loss and best model of default probability.

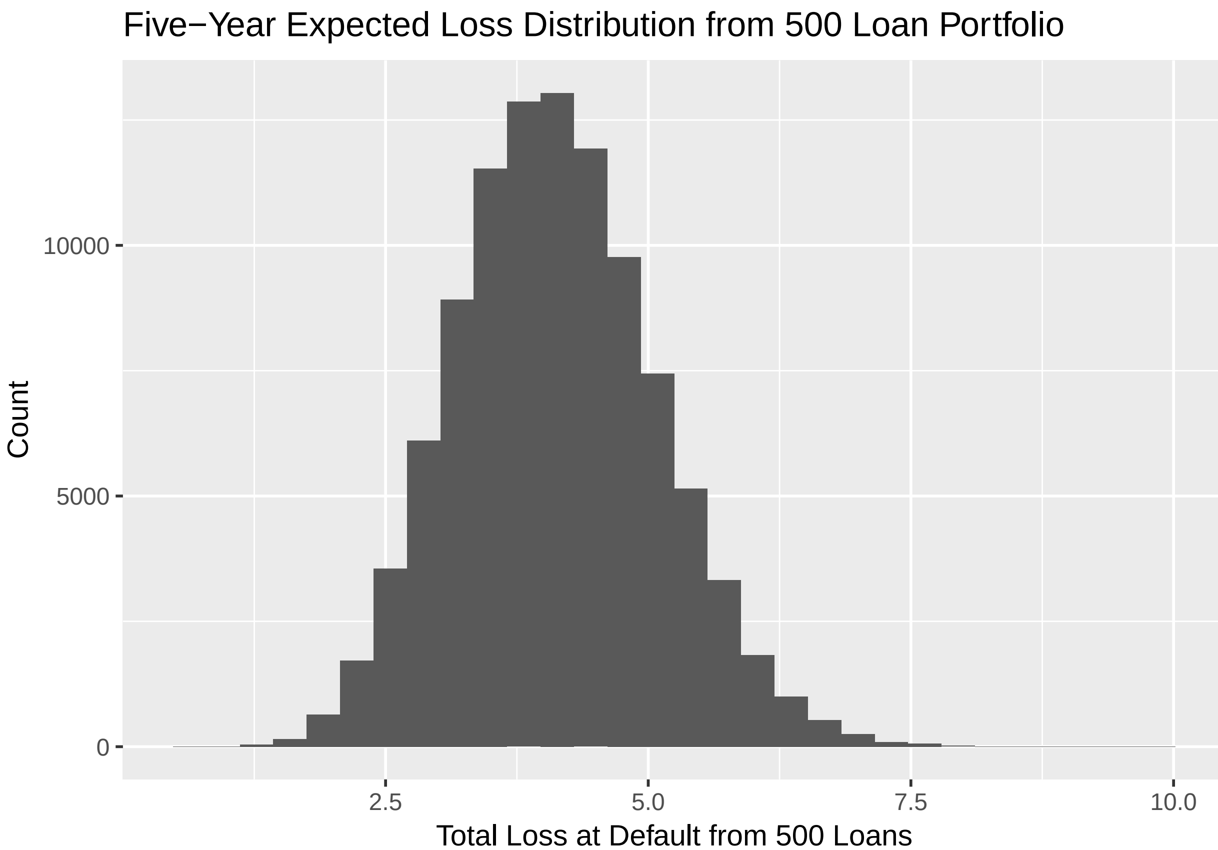
Simulate Distribution of Total Loss

To estimate the value at risk, we generate simulations of the loan losses for the portfolio in batches. For each batch of 10'000 portfolio simulations, we compute the value at risk and expected shortfall and store them. We then take the average value at risk and calculate confidence interval for both metrics.

Plot Expected Loss Distributions

The following plots show the Total loss distribution in percentage of the total porftolio nominal for 100'000 portfolio simulations. Further, we get an average loss of 0.7519071% for the one year ahead period and 4.1222375% for five years.





Compute Value-at-Risk

Following the simulations, the table below shows the VaR results and the 95 and 99% level with a 95% confidence interval for one and five years respectively.

		Confidence Interval	
	Mean	Lower	Upper
1Y VaR 95%	1.5327%	1.5325%	1.5329%
1Y VaR 99%	1.9461%	1.9026%	1.9896%
5Y VaR 95%	5.7504%	5.7501%	5.7508%
5Y VaR 99%	6.4918%	6.4912%	6.4925%

Figure 2: Value at Risk results

Compute Average Value-at-Risk

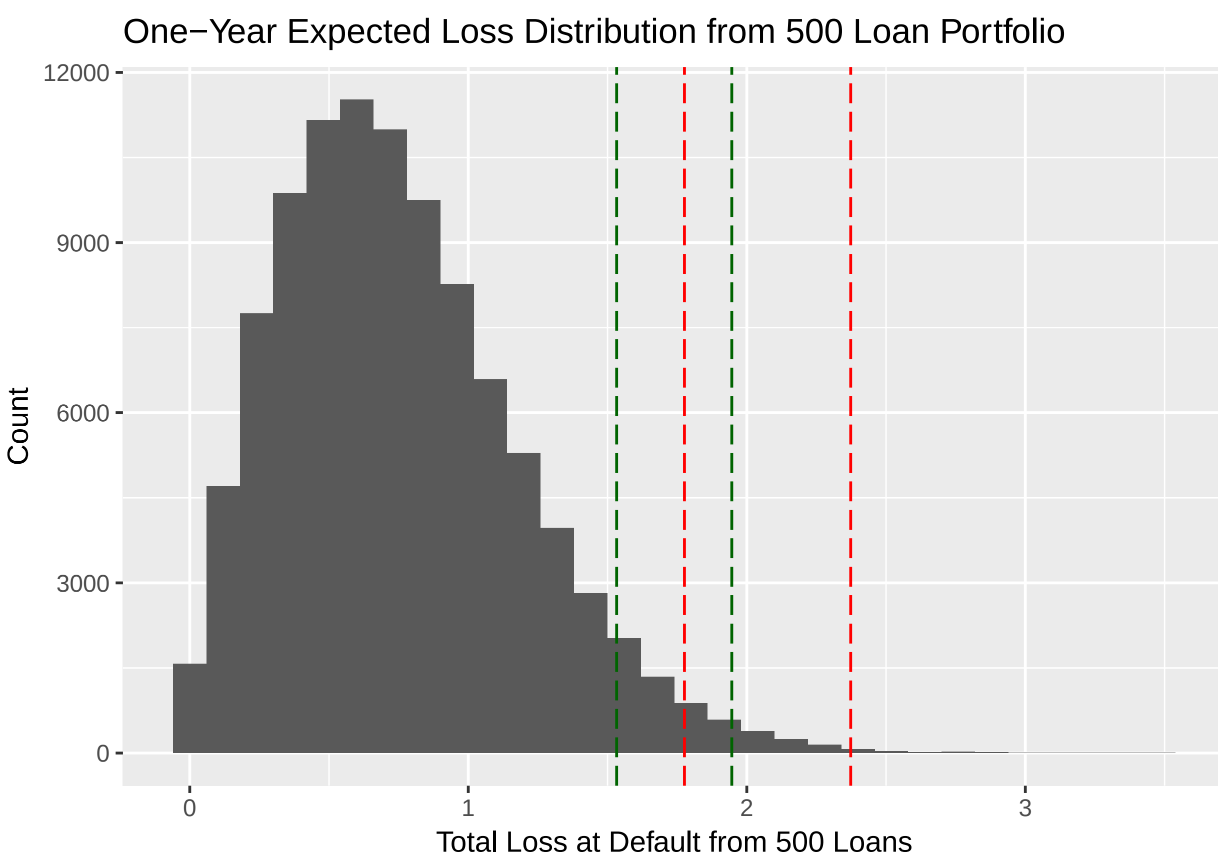
Similarly to the Value at Risk we compute the same metrics for the Average Value at Risk, also named Expected Tail loss. This metric represents the expected loss on the portfolio in the worst 1% and 5% of scenarios respectively. Again, we do this for 1 year and 5 year simulations.

		Confidence Interval	
	Mean	Lower	Upper
1Y ETL 5%	1.7761%	1.7757%	1.7765%
1Y ETL 1%	2.3730%	2.2774%	2.4686%
5Y ETL 5%	6.2023%	6.2018%	6.2029%
5Y ETL 1%	6.8905%	6.8894%	6.8915%

Figure 3: Expected Tail loss results

Interpretation and Risk Analysis

- To be discussed at Thursday meeting.
- I think we should run this pipeline for portfolios from different time periods



Loss Distributions by Tranche

In this section, we will estimate the distribution for the one and five year losses of an investor who has purchased a [5%, 15%] tranche backed by the 500 loan portfolio. In addition, we will investigate the loss distribution of the [15%, 100%] senior tranche.

Portfolio and assumptions

We assume that all active loans whose term length does not expire within the 1- and/or 5-year window are eligible for the tranche. We select from the dataframe of total loans, a subset of active loans that meet this requirement.

Selection of loans for portfolio

Once the dataframe of eligible loans for the portfolio has been created, we select 500 loans uniformly random from the list. We store the 500 loans in a matrix in R.

Determine value of the portfolio of loans

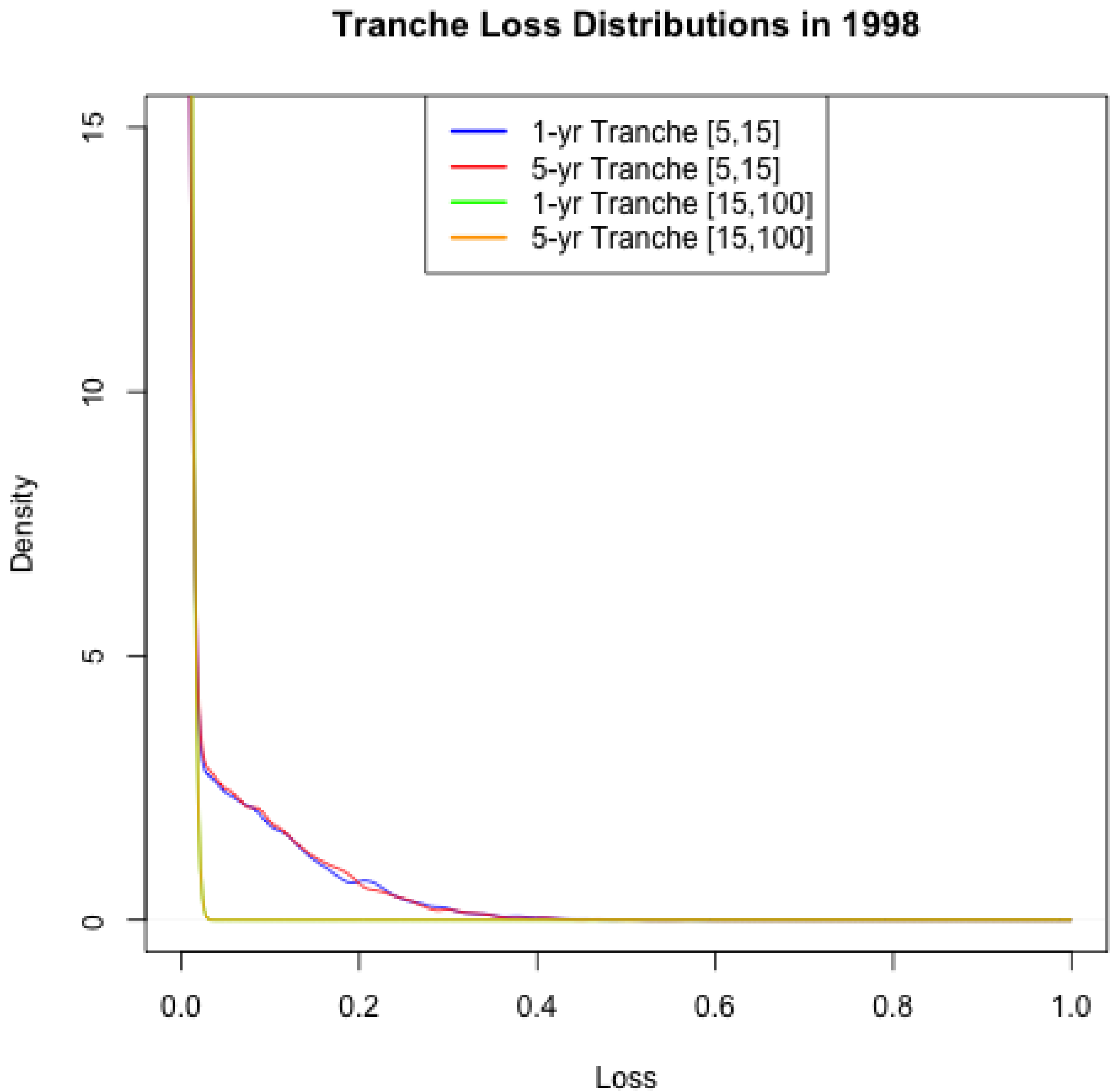
Once we have our loans selected for our portfolio, we determine the value of the portfolio. It may seem intuitive to simply add the value of each loan for the portfolio to determine the value of the tranche. However, this method would not account for different term lengths. For example, a loan for $100,000 over 1-year would be more be more valuable in the 1-year tranche than a 5-year loan for $200,000. We account for this problem by normalizing the value of each loan by the term length. Note, this will ignore minor discrepancies between accrued interest. In addition, we assume that loans either default or are paid in full at the loan termination date. Note, this assumption ignores the possibility of a borrower paying the loan off before the loan due date.

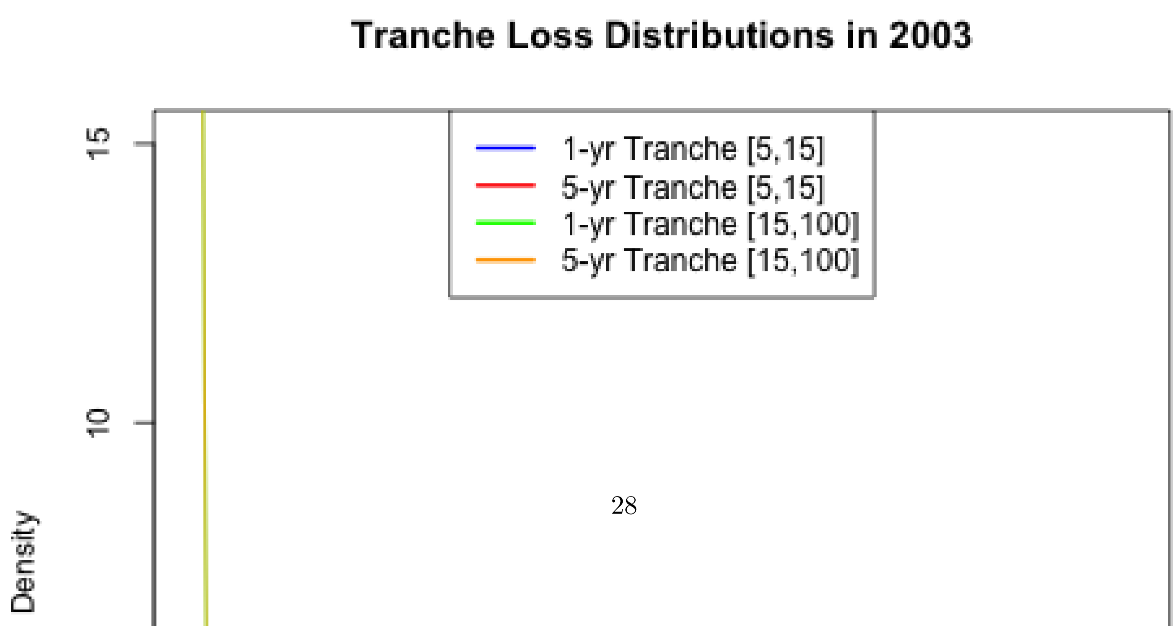
Determine the loss from the portfolio of loans

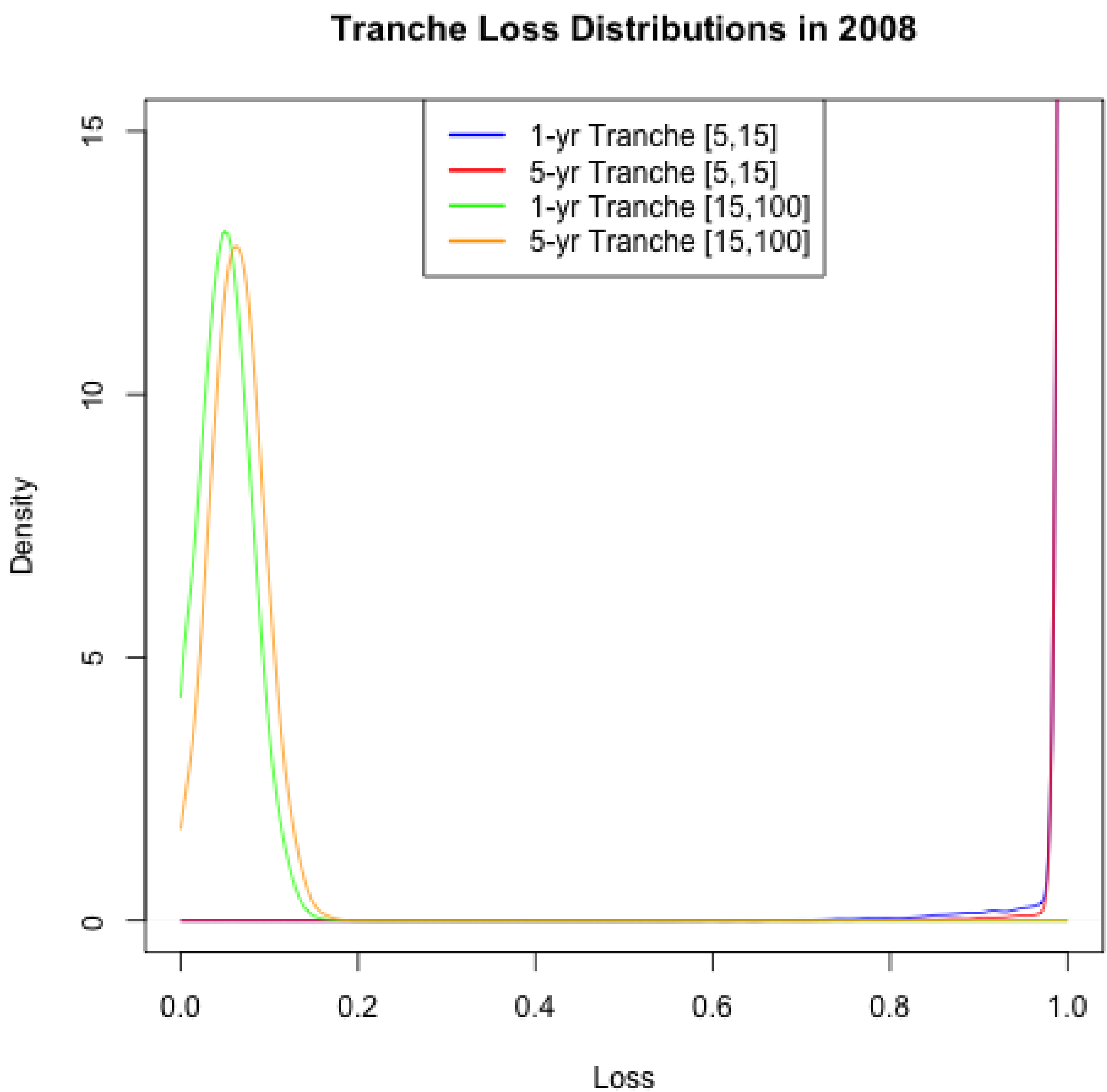
In an identical manner to determining the value of the portfolio, we will determine the loss observed by the portfolio.

Generate loss distribution and plotting

We run this simulation of selected 500 loans uniformly random from the list of active loans 1000 times, and compute the appropriate losses for each tranche. We then plot the approximated distribution using the Kernel Density Estimator (KDE) with bounded [0,1] support.


Tranche Loss Distributions in 1998
1-yr Tranche [5,15]
5-yr Tranche [5,15]
1-yr Tranche [15,100]
5-yr Tranche [15,100]
Density
0
5
10
15
0.0
0.2
0.4
0.6
0.8
1.0
Loss


Tranche Loss Distributions in 2003
1-yr Tranche [5,15]
5-yr Tranche [5,15]
1-yr Tranche [15,100]
5-yr Tranche [15,100]
Density
10
15



Tranche Loss Distributions in 2008
1-yr Tranche [5,15]
5-yr Tranche [5,15]
1-yr Tranche [15,100]
5-yr Tranche [15,100]
Density
Loss
0
5
10
15
0.0
0.2
0.4
0.6
0.8
1.0

Interpretations and Comparison of Distributions

Year	Mean				Min				Max			
1990	49.77%	50.35%	0.00%	0.00%	0.00%	1.36%	0.00%	0.00%	100.00%	100.00%	0.31%	0.20%
1991	21.95%	21.55%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	79.63%	81.34%	0.00%	0.00%
1992	13.01%	13.72%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	75.19%	76.16%	0.00%	0.00%
1993	5.04%	5.35%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	62.51%	54.38%	0.00%	0.00%
1994	3.52%	3.57%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	51.91%	52.93%	0.00%	0.00%
1995	3.77%	3.84%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	53.32%	67.91%	0.00%	0.00%
1996	5.25%	2.76%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	100.00%	52.36%	2.29%	0.00%
1997	3.85%	3.75%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	67.99%	59.71%	0.00%	0.00%
1998	4.25%	4.24%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	56.74%	58.25%	0.00%	0.00%
1999	7.96%	6.83%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	100.00%	62.20%	4.88%	0.00%
2000	9.66%	9.84%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	77.82%	71.38%	0.00%	0.00%
2001	15.81%	16.09%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	100.00%	79.83%	0.63%	0.00%
2002	19.12%	20.07%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	100.00%	85.36%	0.26%	0.00%
2003	24.13%	25.02%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	85.82%	100.00%	0.00%	0.17%
2004	36.42%	37.79%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	100.00%	100.00%	1.14%	1.34%
2005	59.27%	60.94%	0.02%	0.03%	0.00%	0.00%	0.00%	0.00%	100.00%	100.00%	5.53%	5.63%
2006	87.88%	89.64%	0.72%	0.87%	14.98%	19.24%	0.00%	0.00%	100.00%	100.00%	10.07%	10.26%
2007	98.75%	99.32%	3.53%	4.30%	30.11%	49.85%	0.00%	0.00%	100.00%	100.00%	15.58%	17.88%
2008	99.66%	99.90%	5.17%	6.45%	57.01%	58.92%	0.00%	0.00%	100.00%	100.00%	18.60%	20.43%
2009	99.20%	99.86%	4.28%	6.24%	40.07%	61.65%	0.00%	0.00%	100.00%	100.00%	17.38%	18.46%
2010	97.69%	N/A	2.92%	N/A	27.42%	N/A	0.00%	N/A	100.00%	N/A	16.55%	N/A
2011	89.64%	N/A	1.03%	N/A	17.27%	N/A	0.00%	N/A	100.00%	N/A	11.09%	N/A
2012	60.71%	N/A	0.05%	N/A	0.00%	N/A	0.00%	N/A	100.00%	N/A	8.36%	N/A
2013	7.02%	N/A	0.00%	N/A	0.00%	N/A	0.00%	N/A	69.05%	N/A	0.00%	N/A

I-year Tranche [5%, 15%]
I-year Tranche [15%, 100%]
5-year Tranche [5%, 15%]
5-year Tranche [15%, 100%]

Year	Median				Q1				Q3			
1990	49.80%	50.40%	0.00%	0.00%	40.20%	41.60%	0.00%	0.00%	59.20%	59.30%	0.00%	0.00%
1991	21.20%	20.90%	0.00%	0.00%	11.30%	11.90%	0.00%	0.00%	31.10%	30.20%	0.00%	0.00%
1992	10.90%	11.90%	0.00%	0.00%	1.90%	2.70%	0.00%	0.00%	20.50%	21.50%	0.00%	0.00%
1993	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	8.20%	8.70%	0.00%	0.00%
1994	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	4.60%	4.80%	0.00%	0.00%
1995	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	5.30%	5.40%	0.00%	0.00%
1996	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	4.70%	2.50%	0.00%	0.00%
1997	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	5.30%	5.20%	0.00%	0.00%
1998	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	6.30%	6.40%	0.00%	0.00%
1999	2.20%	2.20%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	12.00%	11.60%	0.00%	0.00%
2000	6.20%	6.90%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	15.60%	16.20%	0.00%	0.00%
2001	13.60%	14.50%	0.00%	0.00%	3.90%	4.90%	0.00%	0.00%	24.20%	24.60%	0.00%	0.00%
2002	17.40%	18.80%	0.00%	0.00%	7.30%	8.80%	0.00%	0.00%	27.90%	29.50%	0.00%	0.00%
2003	23.20%	24.10%	0.00%	0.00%	13.00%	13.90%	0.00%	0.00%	33.90%	35.10%	0.00%	0.00%
2004	35.90%	37.20%	0.00%	0.00%	24.70%	26.20%	0.00%	0.00%	47.30%	48.70%	0.00%	0.00%
2005	58.80%	60.50%	0.00%	0.00%	46.10%	47.70%	0.00%	0.00%	72.10%	74.00%	0.00%	0.00%
2006	93.30%	96.50%	0.00%	0.00%	79.00%	82.00%	0.00%	0.00%	100.00%	100.00%	1.00%	1.40%
2007	100.00%	100.00%	3.30%	4.10%	100.00%	100.00%	1.40%	2.20%	100.00%	100.00%	5.30%	6.20%
2008	100.00%	100.00%	5.10%	6.40%	100.00%	100.00%	3.10%	4.40%	100.00%	100.00%	7.10%	8.40%
2009	100.00%	100.00%	4.10%	6.20%	100.00%	100.00%	2.10%	4.10%	100.00%	100.00%	6.20%	8.20%
2010	100.00%	N/A	2.60%	N/A	100.00%	N/A	0.60%	N/A	100.00%	N/A	4.60%	N/A
2011	97.80%	N/A	0.00%	N/A	81.80%	N/A	0.00%	N/A	100.00%	N/A	1.70%	N/A
2012	60.20%	N/A	0.00%	N/A	46.10%	N/A	0.00%	N/A	75.00%	N/A	0.00%	N/A
2013	1.50%	N/A	0.00%	N/A	0.00%	N/A	0.00%	N/A	11.70%	N/A	0.00%	N/A

I-year Tranche [5%, 15%]
I-year Tranche [15%, 100%]
5-year Tranche [5%, 15%]
5-year Tranche [15%, 100%]

See below table

We can see from the approximated density plots and the above table that in the early- to mid-90's, the [5%, 15%] tranche was only slightly more risky than the [15%, 100%] tranche. Almost half the randomly generated portfolios generated a 0% loss in the [5%, 15%] tranche in 1997. The senior tranche was loss-less with 75% certainty the through 2005. However, by 2007, the [5%, 15%] tranche receives almost 100% loss, and the senior tranche received an average of 3-5% loss. From a risk management point of view, an individual who is not entirely risk-averse would likely invest in the [5%, 15%] tranche prior to year 2000. A completely risk-averse individual could still invest in the senior tranche with almost certainly 0% loss. However, after the financial crisis, the [5%, 15%] tranche received 100% loss with 75% certainty, and the senior tranche received an average of 3-5% loss.