Nash Equilibria of the Symmetric Continuous and Discrete Blotto game in the case of 3 Battlefields

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Contents

1 Introduction

Blotto games refer to a class of two-person constant sum games, in which two colonels simultaneously distribute their forces over $K \geq 2$ independent battlefields, and the difference in force over each battlefield is considered. They were first introduced in two forms in 1921 by Borel [Bor53] (which in the same paper introduced the concept of pure and mixed strategies).

In its symmetric continuous form, both players A, B select the respective vectors in the K-dimensional simplex:

$$\mathbf{a}, \mathbf{b} \in \Delta_K, \quad \Delta_n = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1,\ x_i \in [0,1] \right\} \tag{1}$$

In the Plurality game [LP02], each player's payoff is the difference between battlefields won and lost, i.e.:

$$P(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^{K} \operatorname{sgn}(\mathbf{a}_i - \mathbf{b}_i), \quad \operatorname{sgn}(x) = \begin{cases} -1, & x < 1 \\ 0, & x = 0 \\ 1, & x > 1 \end{cases} \tag{2}$$

In the game's discrete form, we keep the same payoff function, but restrict players' choices to a finite set of rational lattice points within the K-dimensional unit simplex, corresponding to an integer partitioning of N into K pieces, $N, K \in \mathbb{N}, N \geq K$:

$$\mathbf{a}, \mathbf{b} \in \Delta_K^N, \quad \Delta_K^N \coloneqq \left\{ \mathbf{x} \in \mathbb{Q}^n : \sum_{i=1}^{K} x_i = 1,\ x_i \in \left\{ 0, \frac{1}{N}, ..., \frac{N-1}{N}, 1 \right\} \right\} \tag{3}$$

For the rest of this paper, we examine the Blotto game with symmetric budgets in both discrete and continuous contexts for $K = 3$.

2 The Continuous Blotto game

2.1 Pure strategies

Define a pure strategy as a point in the K-simplex, formally:

Definition 1. *A pure strategy of player is defined as:*

$$\mathbf{x}_i \in \Delta_K \tag{4}$$

We also define an optimal strategy and Nash Equilibrium in the two-player symmetric zero-sum game:

Definition 2. *G is an optimal strategy iff:*

$$\forall \tilde{\mathbf{x}} \in \Delta_K : P(\tilde{\mathbf{x}}, G) \leq 0 \tag{5}$$

Definition 3. (G_1, G_2) *is Nash equilibrium iff* G_1, G_2 *are optimal strategies.*

We first prove the non-existence of pure strategy Nash equilibria in the above Blotto game[1]:

Theorem 2.1. *The continuous Blotto game, $K \geq 3$, has no pure strategy Nash equilibria.*

Proof. By contradiction, we assume the existence of some optimal pure strategy:

$$\exists \tilde{\mathbf{x}} \in \Delta_K : \forall \mathbf{x} : P(\tilde{\mathbf{x}}, \mathbf{x}) \geq 0 \tag{6}$$

Order the co-ordinates of $\tilde{\mathbf{x}}$ in terms of magnitude, i.e. find some permutation, p, of $\{1, ..., K\}$, such that $\tilde{\mathbf{x}}_{p_1} \leq \tilde{\mathbf{x}}_{p_2} \leq ... \leq \tilde{\mathbf{x}}_{p_k}$.

Construct a new vector $\tilde{\mathbf{y}}$:

$$\tilde{\mathbf{y}}_{\mathbf{p_i}} = \begin{cases} \tilde{\mathbf{x}}_{p_i} + \frac{\tilde{\mathbf{x}}_{p_k}}{2}, & i = 1, 2 \\ 0, & i = k \\ \tilde{\mathbf{x}}_{p_i}, & \text{otherwise} \end{cases} \tag{7}$$

By considering the respective maximising cases $\tilde{\mathbf{x}} = (\frac{1}{K}, ..., \frac{1}{K})$ and $\tilde{\mathbf{x}} = (0, \frac{1}{K-1} ..., \frac{1}{K-1})$, one can show that $\tilde{\mathbf{y}}_{\mathbf{p_1}} \leq \frac{3}{2K} < 1$ and $\tilde{\mathbf{y}}_{\mathbf{p_2}} \leq \frac{3}{2(K-1)} < 1$.
We also have that $\sum_{j=1}^{K} \tilde{\mathbf{y}}_{\mathbf{j}} = \sum_{i=1}^{K-1} \tilde{\mathbf{x}}_{\mathbf{p_i}} + 2\frac{\tilde{\mathbf{x}}_{p_k}}{2} = 1$.
Hence $\tilde{\mathbf{y}} \in \Delta_K$ is also a pure strategy.
We finally obtain the contradiction:

$$\begin{aligned} P(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) &= \sum_{i=1}^{K} \operatorname{sgn}(\tilde{\mathbf{x}}_{p_i} - \tilde{\mathbf{y}}_{p_i}) \\ &= \operatorname{sgn}(-\frac{\tilde{\mathbf{x}}_{p_k}}{2}) + \operatorname{sgn}(-\frac{\tilde{\mathbf{x}}_{p_k}}{2}) + \operatorname{sgn}(0) + ... + \operatorname{sgn}(0) + \operatorname{sgn}(\tilde{\mathbf{x}}_{p_k}) \\ &= -1 \\ &< 0 \end{aligned} \tag{8}$$

□

In fact, there are infinitely many pure strategies, $\mathbf{x}$, that triumph over any given pure strategy, $\mathbf{x}'$. In the case of $K = 3$, by considering the three winning cases, we obtain three triangular regions within Δ_3:

$$\mathbf{x} \in \Delta_K : \begin{array}{rll} (i) & \mathbf{x}_1 > \mathbf{x}'_1, & \mathbf{x}_2 > \mathbf{x}'_2 \\ (ii) & \mathbf{x}_1 > \mathbf{x}'_1, & \mathbf{x}_3 > \mathbf{x}'_3 \\ (iii) & \mathbf{x}_2 > \mathbf{x}'_2, & \mathbf{x}_3 > \mathbf{x}'_3 \end{array} \tag{9}$$

Note that the boundary of these triangular regions contain the drawing pure strategies against $\mathbf{x}$.

[1]Relaxing to the asymmetric, heterogeneous case, one can find optimal pure strategies [HL17]

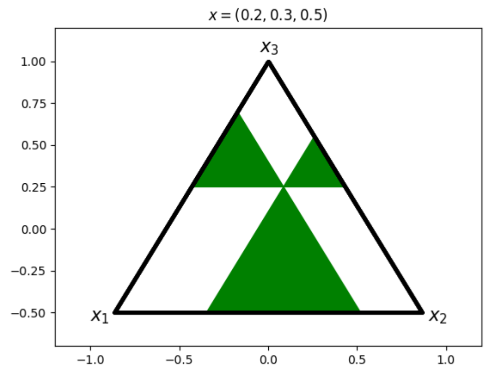


Figure 1: Loci of $\mathbf{x}'$ in Δ_3 w.r.t $\mathbf{x} = (0.2, 0.3, 0.5)$

Via this geometric intuition, we observe that when $\mathbf{x} = \mathbf{e}_i$, all of the interior of the simplex are dominant strategies, while $\mathbf{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ results in 3 smaller congruent triangular regions.

Using simple trigonometry, we can calculate the proportion of the simplex filled by this 'winning region' - this corresponds to the probability that $\mathbf{x}$ loses against a randomly selected pure strategy on a uniform distribution over Δ_3:

$$P_{\text{loss}}(\mathbf{x}) = \frac{\frac{1}{\sqrt{3}} \sum_{i=1}^{3} \mathbf{x}_i^2}{\frac{1}{\sqrt{3}}} = \mathbf{x}_1^2 + \mathbf{x}_2^2 + \mathbf{x}_3^2 \tag{10}$$

Minimising the above w.r.t $g(\mathbf{x}) = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 - 1 = 0$ using Lagrange multipliers:

$$\nabla P_{\text{loss}} = \lambda \nabla g \implies \lambda = 2\mathbf{x}_1 = 2\mathbf{x}_2 = 2\mathbf{x}_3 \implies \mathbf{x} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \tag{11}$$

Although we might expect that an optimal mixed strategy would prefer more "central" pure strategies in the simplex, this is not the case. We will see that variation in location in the simplex, and hence winning regions, is rewarded, even if each strategy in the distribution has a suboptimal P_{loss}.

2.2 Mixed strategies and Characterisation of Equilibria

Following [Kva07], we define a mixed strategy in the continuous case as as a joint distribution function in the K-simplex:

Definition 4. *A mixed strategy of player i is defined as:*
$G_i : \Delta_K \to [0, 1]$
The associated probability density function is:

$g_i : \Delta_K \rightarrow \mathbb{R}, \quad g_i = \frac{\partial^K G}{\partial x_1 ... \partial x_K}$

We also define the univariate marginals of a mixed strategy:

Definition 5. *Define the marginal distribution function of battlefield i w.r.t a mixed strategy, G_i:*
$G_{ij} : [0,1] \rightarrow [0,1], \quad G_{ij}(x) = \int_0^x (\int_0^1 ... \int_0^1 g_i(\mathbf{z}) d\mathbf{z}_{-j}) dz_j$
$(= \int_0^1 ... \int_0^1 g_i(\mathbf{z}) dz_1 ... dz_{j-1} dz_{j+1} ... dz_K)$
The associated marginal density functions are:
$g_{ij} : [0,1] \rightarrow \mathbb{R}, \quad g_{ij} = \frac{dG_{ij}}{dx_j}$

We now prove an important result from [LP02] for $K = 3$ that characterises optimal mixed strategies from their marginals:

Theorem 2.2. *Consider the symmetric continuous Blotto game, $K = 3$. If the marginals of G_i are uniformly distributed on $\left[0, \frac{2}{3}\right]$, then it is an optimal strategy.*

Proof. Consider some G_i, with arbitrary i s.t. $\forall j \in \{1,2,3\} : G_{ij} = U\left(0, \frac{2}{3}\right)$. Fix some pure strategy $\tilde{\mathbf{x}} \in \Delta_K$, and randomly sample a strategy $\mathbf{x}$ from G_i. Consider the expected sign[2] of the difference between the k^{th} co-ordinate of $\tilde{\mathbf{x}}$ and $\mathbf{x}$:

$$P(\tilde{\mathbf{x}}_k > \mathbf{x}_k) = \begin{cases} 1, & \tilde{\mathbf{x}}_k > \frac{2}{3} \\ \frac{3}{2}\tilde{\mathbf{x}}_k, & \tilde{\mathbf{x}}_k \leq \frac{2}{3} \end{cases} \tag{12}$$

$$\begin{aligned} \therefore E[\text{sgn}(\tilde{\mathbf{x}}_k - \mathbf{x}_k)] &= E[\tilde{\mathbf{x}}_k > \mathbf{x}_k] \\ &= \begin{cases} 1 \cdot 1 + 0 \cdot -1 = 1, & \tilde{\mathbf{x}}_k > \frac{2}{3} \\ \frac{3}{2}\tilde{\mathbf{x}}_k \cdot 1 + (\frac{3}{2}\tilde{\mathbf{x}}_k) \cdot -1 = 3\tilde{\mathbf{x}}_k - 1, & \tilde{\mathbf{x}}_k \leq \frac{2}{3} \end{cases} \end{aligned} \tag{13}$$

We now define the subsimplicial set Hex(K):

Definition 6. $Hex(K) = \{\mathbf{x} \in \Delta_K : \mathbf{x} \in [0, \frac{2}{K}]\}$

We now bound the expected payoff of $\tilde{\mathbf{x}}$ against G_i.
Consider the case $\tilde{\mathbf{x}} \notin Hex(3)$; WLOG by the Δ_3 condition:

$$\tilde{\mathbf{x}}_1 > \frac{2}{3}, \quad \tilde{\mathbf{x}}_2 + \tilde{\mathbf{x}}_3 < \frac{1}{3} \implies \tilde{\mathbf{x}}_2 \leq \frac{2}{3}, \tilde{\mathbf{x}}_3 \leq \frac{2}{3} \tag{14}$$

[2] The boundary of Fig. 1, i.e. the set of possible draws, is atomless, hence the probability of a draw is 0. For the rest of the paper, we implement a tie-breaking rule s.t. in the event of a tie, the winner is decided randomly and fairly.

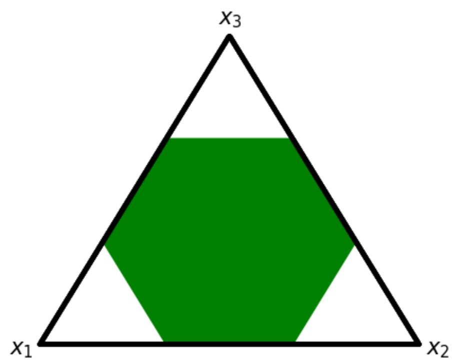


Figure 2: Projection of the regular hexagon, Hex(3), in Δ_3.

$$\begin{aligned}
\therefore P(\tilde{\mathbf{x}}, \mathbf{x}) &= 1 + (3\tilde{\mathbf{x}}_2 - 1) + (3\tilde{\mathbf{x}}_3 - 1) \\
&< 1 + 3 \cdot \frac{1}{3} - 2 \\
&< 0
\end{aligned} \tag{15}$$

Note the above is true if any co-ordinate exceeds $\frac{2}{3}$ by symmetry.

We now consider the complement case $\tilde{\mathbf{x}} \in \mathrm{Hex}(3)$:

$$\tilde{\mathbf{x}}_1 \leq \frac{2}{3}, \tilde{\mathbf{x}}_2 \leq \frac{2}{3}, \tilde{\mathbf{x}}_3 \leq \frac{2}{3} \tag{16}$$

$$\begin{aligned}
\therefore P(\tilde{\mathbf{x}}, \mathbf{x}) &= (3\tilde{\mathbf{x}}_1 - 1) + (3\tilde{\mathbf{x}}_2 - 1) + (3\tilde{\mathbf{x}}_3 - 1) \\
&\leq 3 \cdot \frac{1}{3} - 3 \\
&\leq 0
\end{aligned} \tag{17}$$

Hence the expected payoff of the pure strategy $\tilde{\mathbf{x}}$ against G_i is non-positive. By generality of $\tilde{\mathbf{x}}$, G_i and player i, we conclude the proof.

□

Indeed, the above generalises for $K > 2$ as in [LP02][3]:

Lemma 2.3. *Consider a continuous Blotto game, $K > 2$. If the marginals of G_i are uniformly distributed on $\left[0, \frac{2}{K}\right]$, then it is an optimal strategy.*

The reverse statement is far more complex to prove:

Theorem 2.4. *Consider a continuous Blotto game, $K > 2$. If a mixed strategy is optimal, then the marginals of G_i are uniformly distributed on $\left[0, \frac{2}{K}\right]$.*

[3] For $K = 2$ we have the related result that if $g_i(x) = g_i(1 - x)$, then the mixed strategy is optimal (Kvasov 2006)

It was not until 2006 that Roberson proved this without relying on the connectedness of the support[4] of G_i [Rob06]. We will briefly cover the framework of this proof.

Definition 7. $C : [0,1]^N -> [0,1]$ *is an N-Copula if* C *is a joint distribution function over the* N*-dimensional unit hypercube* $[0,1]^N$ *with uniform marginals.*

Crucially, copulas map between joint distributions and their marginals, and quantify their dependence [Van23]:

Theorem 2.5 (Sklar's Theorem (1983))**.** *Given a* K*-variate joint distribution* G_i *with marginals* G_{ij}*, there exists an N-Copula* C *such that* $\forall \mathbf{x} \in \mathbb{R}$:
$G_i(\mathbf{x}) = C(G_{i1}(x_1), ..., G_{i1}(x_n))$.

In addition, given an N-Copula C *and univariate distributions* G_{ij}*, then* G_i *as defined above has marginals* G_{ij}*.*

Roberson first considers the equilibria of K two-player independent, identical and simultaneous all-pay auctions corresponding to each battlefield, without the constraint of the unit sum of resources. In the symmetric case, the equilibrium distribution of bids are K independent uniform distributions on [0,1] [HR89].

We now impose the constraint that there exists a K-Copula such that the support of G_i, as defined in Sklar's theorem, is contained in Δ_K - Roberson shows the existence of sufficient K-Copula via the existence of a sufficient joint distribution with the equilibrium marginal distributions.

Finally, Roberson creates a bijection and hence equivalence between the set of equilibrium marginal distributions of the K-Blotto game, and a unique set of equilibrium distribution functions of the above all-pay auction game. Note that due to the Copula constraint, the set of joint equilibrium distributions of the Blotto game must be a strict subset of the set of joint distributions that share marginals with the auction game.

In fact, Roberson generalises this result to the asymmetric budget case in which player A and B possess X_A, X_B units of force to be distributed amongst the battlefields - following a similar argument we result in some affine transformation of the uniform marginals in the symmetric case in terms of K, X_A and X_B, depending on the magnitude of the ratio $\frac{X_A}{X_B}$.

2.3 Examples

Although we know that all optimal strategies have uniform marginals on $\left[0, \frac{2}{3}\right]$, there are infinitely many ways in which we can construct such a joint distribution function. We start with a simple observation from the domain of each margin:

[4] I.e. the set $\{\mathbf{x} \in \Delta_K : g_i(\mathbf{x}) > 0\}$

Lemma 2.6. *The support of any optimal strategy, G_i, is a subset of Hex(3).*

We first start with the disc solution proposed by Borel [Bor38] [LP02]:

Lemma 2.7. *Construct the joint distribution G_i^{disc} as follows:*
Inscribe a circle inside Δ_3, such that it lies within $Hex(3)$. Construct a hemisphere above this circular base. Sample a point P from a uniform distribution on the surface of hemisphere, and let P' be its projection down onto Δ_3. Return the vector $H = (h_1, h_2, h_3)$, where h_i is the perpendicular shortest distance from P' to the i^{th} side of Δ_3.
Then G_i^{disc} is an optimal strategy.

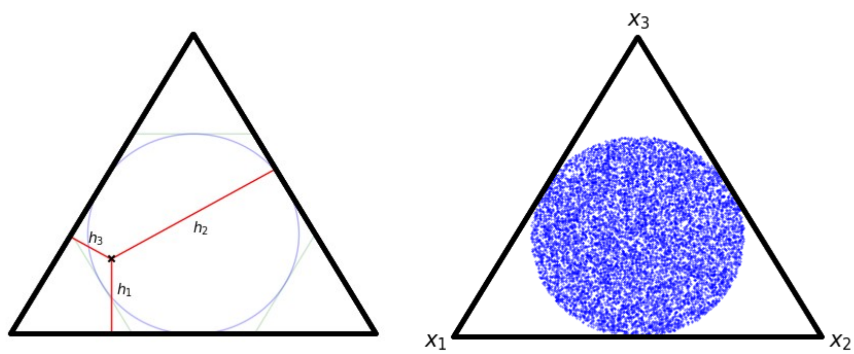


Figure 3: A visual construction of G_i^{disc}, and a sample of 10000 points.

Proof. By Viviani's theorem[5], $H \in \Delta_3$.
We now prove that the marginals of G_i^{disc} are uniform upon $\left[0, \frac{2}{3}\right]$, by considering the probability that h_i is less than some h for some $i \in \{1, 2, 3\}$.
In the simplex, this is the region defined by the circular segment of height h_i, bounded by the chord parallel to side i of the triangle.
We can extend this segment to the corresponding spherical cap above the simplex also of of height h_i. Since the distribution upon the sphere is uniform, the probability of sampling from this region is a ratio of surface area:

$$P(h_i < h) = \frac{A_{\text{cap}}}{A_{\text{hemisphere}}} = \frac{\pi \cdot \frac{1}{3} \cdot h}{2\pi \cdot \frac{1}{3}^2} = \frac{3}{2}h \tag{18}$$

Since the circular base is bounded by $Hex(3)$, we also know that:

$$P(0 \leq h_1 \leq \frac{2}{3}) = 1 \tag{19}$$

[5]A quick proof is as follows: each h_i is equivalent to the altitudes of each green equilateral triangular region in Fig. 1. By sliding the rightmost region to the top of Δ_3 we see that the aligned sum of the altitudes is equal to the unit altitude of Δ_3.

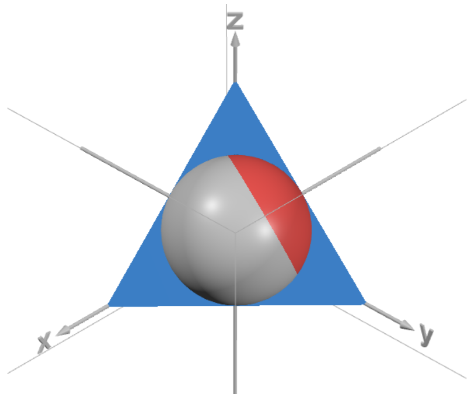


Figure 4: A Desmos render of the spherical cap, $h = 0.4$ and its projection onto Δ_3.

Hence the marginal univariate distribution of h_i is uniform on $\left[0, \frac{2}{3}\right]$. By generality, this is true $\forall i$. Apply Theorem 2.2.

$\square$

By inscribing a circle inside a regular K-gon and measuring the K shortest distances to each side, we can extend the above to $K \geq 3$. In addition, a generalisation to the asymmetric case has been proven using the construction of inscribing a circle inside a tangential irregular K-gon [Tho18].

We also consider the hexagonal construction in Gross and Wagner's paper as follows [GW50]:

Lemma 2.8. *Construct the joint distribution G_i^{Hex} as follows:*
Let $H_{\frac{1}{3}}$ be the set of boundary points of $Hex(3)$.
In general, let H_r be the set of boundary points of the regular hexagon, apothem r, centred at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, oriented such that three of its sides are parallel with that of the boundary of Δ_3.

$$\forall x \in H_r : g_i^{Hex}(x) = \begin{cases} \frac{27\sqrt{3}}{4} r, & r \in [0, \frac{1}{3}] \\ 0, & otherwise \end{cases} \tag{20}$$

When played by both players, G_i^{Hex} is an optimal strategy

Proof. We first verify that G_i^{Hex} is a valid distribution by integrating over its support, Hex(3) - we use the symmetry of each half equilateral triangle cell, within which the density increases linearly:

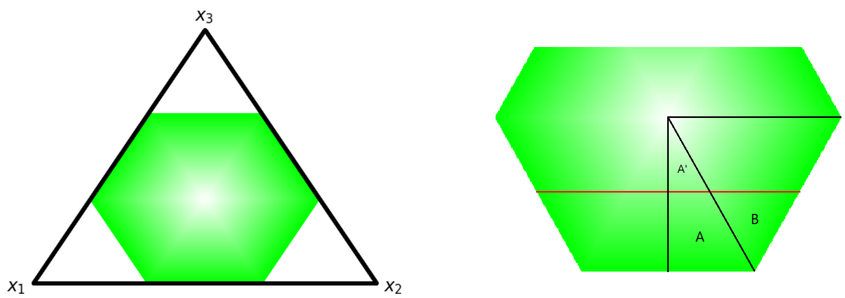


Figure 5: Visualisation of G_i^{Hex} and integration regions A, B in Δ_3

$$\begin{aligned}
\int_{\text{Hex}(3)} G_i^{\text{Hex}} d\sigma &= 12 \int_0^{\frac{1}{3}} \int_0^{\frac{r}{\sqrt{3}}} \frac{27\sqrt{3}}{4} r \, dx \, dr \\
&= 81 \int_0^{\frac{1}{3}} r^2 \, dr \\
&= 1
\end{aligned} \tag{21}$$

We now prove that the marginals of the hexagon are uniform on $\left[0, \frac{2}{3}\right]$. The support of G_i^{Hex} lies within $Hex(3)$ by construction, hence its marginals must lie within $\left[0, \frac{2}{3}\right]$.
WLOG, consider the probability that x_3 is less than some r_0. We derive this by vertically integrating upwards from its base. By symmetry, consider the truncated quadrant above composed of regions A and B respectively.

We note that the above integral can be modified to calculate the integral over the complement of region A:

$$\begin{aligned}
\int_{A'} G_i^{\text{Hex}} d\sigma &= \int_0^{r^0} \int_0^{\frac{r}{\sqrt{3}}} \frac{27\sqrt{3}}{4} r \, dx \, dr \\
&= \frac{27}{4} \int_0^{r_0} r^2 \, dr \\
&= \frac{9}{4} r_0^3
\end{aligned} \tag{22}$$

Hence the integral over region A is:

$$\begin{aligned}
\int_{A} G_i^{\text{Hex}} d\sigma &= \frac{1}{12} - \frac{9}{4} (\frac{1}{3} - r_0)^3 \\
&= \frac{9}{4} r_0^3 - \frac{9}{4} r_0^2 + \frac{3}{4} r_0
\end{aligned} \tag{23}$$

We now consider region B, which corresponds to vertical integration of the equilateral triangle cell from a non-central vertex. We integrate each horizontal slice from $x = \pm\frac{r}{\sqrt{3}}$, through which the density linearly varies from $(1-3r)\frac{9\sqrt{3}}{4}$ to $\frac{9\sqrt{3}}{4}$[6]. Constructing a linear function for each horizontal slice results in the double integral:

$$\begin{aligned}\int_B G_i^{\text{Hex}} d\sigma &= \int_0^{r^0} \int_{-\frac{r}{\sqrt{3}}}^{\frac{r}{\sqrt{3}}} \left(\frac{\frac{r}{\sqrt{3}}+1}{2}\right) \cdot \frac{9\sqrt{3}}{4} \cdot 3r + \frac{9\sqrt{3}}{4}(1-3r)\,dx\,dr \\ &= \int_0^{r_0} \frac{9}{2}r - \frac{27}{4}r^2\,dr \\ &= \frac{9}{4}r_0^2 - \frac{9}{4}r_0^3\end{aligned} \tag{24}$$

Finally, we use a symmetry argument to construct the marginal distribution:

$$2(\int_A G_i^{\text{Hex}} d\sigma + \int_A G_i^{\text{Hex}}) = \frac{3}{4}r_0 \tag{25}$$

$$\begin{aligned}\therefore P(x_3 < r_0) &= \begin{cases} 2\cdot\frac{3}{4}r_0, & 0 \le r_0 < \frac{1}{3} \\ \frac{1}{2} + 2\cdot\frac{3}{4}(r_0 - \frac{1}{3}), & \frac{1}{3} \le r_0 \le \frac{2}{3} \\ 0, & \text{otherwise}\end{cases} \\ &= \begin{cases} \frac{3}{2}r_0, & 0 \le r_0 \le \frac{2}{3} \\ 0, & \text{otherwise}\end{cases}\end{aligned} \tag{26}$$

Hence the marginal univariate distribution of x_3 is uniform on $\left[0, \frac{2}{3}\right]$. By generality, this is true $\forall i \in \{1,2,3\}$. Apply Theorem 2.2. $\square$

We also include an example of a one-dimensional distribution with disconnected support, and its rotationally symmetric average as seen in Weinstein's paper [Wei12]:

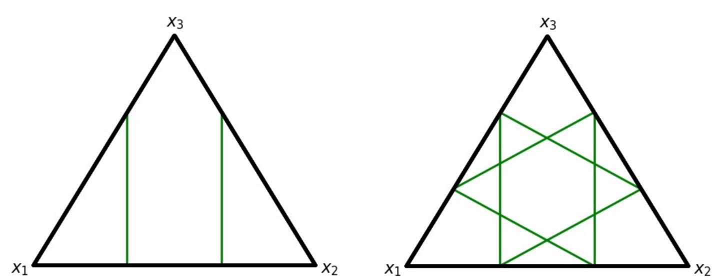


Figure 6: Weinstein's optimal strategies, uniformly distributed on each line segment.

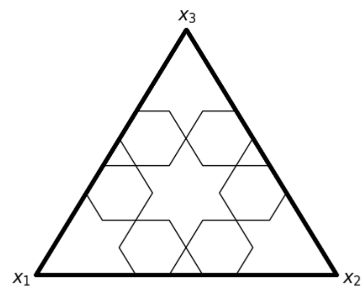
[6]On the left bound, the density linearly varies in $[0, \frac{9\sqrt{3}}{4}]$ at the same rate that r ascends vertically by construction.

One can prove that the marginals of both distributions are uniform using a similar strategy as above.

We now follow the final result in [GW50]:

Lemma 2.9. *There exists countably infinitely many optimal strategies of the continuous symmetric* 3*-Blotto game.*

Proof. Begin with the set $Hex(3) \in \Delta_3$. At each of the 6 vertices, place a smaller hexagon of side-length $\frac{1}{3}^{th}$ of the side-length of $Hex(3)$ as follows:



Consider a set E of 6 optimal strategies; scale each distribution E_i within $Hex(3)$ length-wise by $\frac{1}{3}$, and its density such that $P(x \in E_i) = \frac{1}{6}$.

Place each such that its support lies within one of the hexagons. Call this resultant distribution G_E.

Consider the marginals of G_E - since its support lies within $Hex(3)$, its marginals are bounded by $\left[0, \frac{2}{3}\right]$. By Theorem 2.4, the marginals of each distribution within E are uniform within the hexagon.

Considering the marginal distribution of x_3, we see that any horizontal infinitesimal slice of $Hex(3)$ contains two infinitesimal slices corresponding to two uniform marginal distributions, and zero elsewhere. Integrating upwards yields in a uniform distribution on $\left[0, \frac{2}{3}\right]$. By 3-fold rotational symmetry, we conclude that this is true for all marginals. Applying Theorem 2.2, G_E is optimal.

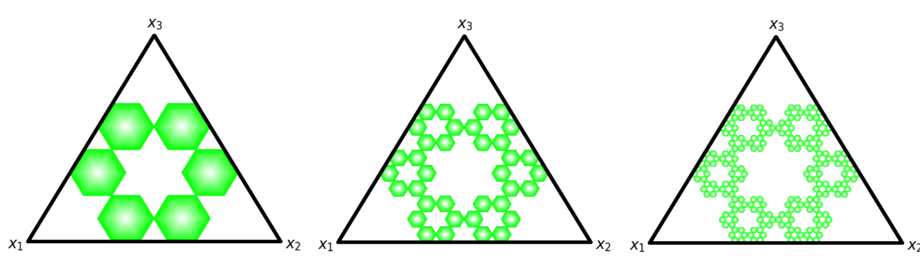


Figure 7: $G^n, n = 1, 2, 3$ for $E = \{G^{\text{Hex}}, ..., G^{\text{Hex}}\}$

Starting with $G^1 \coloneqq G_E$, $E = \{G^{\text{disc}}, ..., G^{\text{disc}}\}$ one may recursively define:

$$\forall n \in \mathbb{N} : G^n = G_{\{G^{n-1}, ..., G^{n-1}\}} \tag{27}$$

Inductively from above, G^n are optimal.

$\square$

Following Kvasov's Decomposition lemma [Kva07], the Cartesian product of two optimal strategies of the K, L-Blotto games is itself an optimal strategy of the $(K + L)$-Blotto game; intuitively the resulting strategy is composed of two uncorrelated groups of battlefields of size K, L respectively.

It thus follows that we can create a family of optimal strategies for each $K > 3$ composed of the optimal strategies within the 2 and 3 Blotto games - hence there exist countably infinite Nash equilibria for $K > 3$.

3 The Discrete Blotto Game

3.1 Initialisation

A significant amount of existing literature studies the continuous Blotto game, in which the constraint of a finite amount of strategies is relaxed. The discrete case has proven to resist such techniques used in the continuous case, and as of this paper, a complete characterisation of Nash equilibria does not exist.

We first construct the payoff matrix, $A \in S^{(n,n)}$, $S = \{-1, 0, 1\}$, for both players. Via a "stars and bars" combinatoric argument[7], one can prove:

$$n = \binom{N + K - 1}{K - 1} = \frac{(N + K - 1)!}{(K - 1)!N!} \tag{28}$$

One can algorithmically generate all n pure strategies by first visiting each ordered K-partition of N in Colex order [Knu11], i.e. starting with $N0...0$: given a partition, we find the next partition by finding the leftmost segment that can be increased without changing successive segments.
An example for $N = 6, K = 3$ is as follows:

$$600, 510, 420, 330, 411, 321, 222 \tag{29}$$

The set of permutations of each of the above partitions yields the set of all n pure strategies[8], scaling by $\frac{1}{N}$ gives $\tilde{\Delta}_K^N$.

[7]Consider N stars and $K - 1$ bars in a row. There are $N + K - 1$ objects, of which we choose the placement of $K - 1$ bars. This corresponds to a partition of the N stars into K sets.

[8]This can proven by showing that Colex order produces partitions in lexicographic order of their conjugate partition, i.e. the reflection of its Ferrers diagram in $y = -x$.

Indexing each pure strategy $\mathbf{x}_i, i \in \{1, 2, ..., n\}$ as above we define each element of A:

$$A_{ij} \coloneqq P(\mathbf{x}_i, \mathbf{x}_j) \tag{30}$$

By Theorem 1.4 in the Course Notes [Str23], we conclude that every discrete N, K-Blotto game must have at least one Nash Equilibrium.

3.2 Optimal Strategies

In a similar vein to before, we prove there does not exist any optimal strategies in the discrete case for $K = 3$:

Theorem 3.1. *The discrete N, K Blotto game, $N > K$, has no pure strategy Nash equilibria.*

Proof. By contradiction, we assume the existence of some optimal pure strategy:

$$\exists \tilde{\mathbf{x}} \in \Delta_K^N : \forall \mathbf{x} : P(\tilde{\mathbf{x}}, \mathbf{x}) \geq 0 \tag{31}$$

Again, we order the co-ordinates of $\tilde{\mathbf{x}}$ in terms of magnitude, i.e. find some permutation, p, of $\{1, ..., K\}$, such that $\tilde{\mathbf{x}}_{p_1} \leq \tilde{\mathbf{x}}_{p_2} \leq ... \leq \tilde{\mathbf{x}}_{p_k}$.

Construct a new vector $\tilde{\mathbf{y}}$:

$$\tilde{\mathbf{y}}_{\mathbf{p_i}} = \begin{cases} \tilde{\mathbf{x}}_{p_i} + 1, & i = 1, 2 \\ \tilde{\mathbf{x}}_{p_k} - 2, & i = k \\ \tilde{\mathbf{x}}_{p_i}, & \text{otherwise} \end{cases} \tag{32}$$

Since $N > K$, by the Pigeonhole principle $\tilde{\mathbf{x}}_{p_k} \geq 2$, so $\tilde{\mathbf{y}} \in \Delta_K^N$.
We finally obtain the contradiction:

$$\begin{aligned} P(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) &= \sum_{i=1}^{K} \operatorname{sgn}(\tilde{\mathbf{x}}_{p_i} - \tilde{\mathbf{y}}_{p_i}) \\ &= \operatorname{sgn}(-1) + \operatorname{sgn}(-1) + \operatorname{sgn}(0) + ... + \operatorname{sgn}(0) + \operatorname{sgn}(2) \\ &< 0 \end{aligned} \tag{33}$$

□

Note that in the case where $N = K$, the strategy of allocating 1 unit to each battlefield is an optimal pure strategy.

By existence of Nash equilibria, it must be the case that each discrete N, K-Blotto game has an optimal mixed strategy, i.e. some distribution defined on Δ_K^N

Hart identified a family of optimal strategies in the N, K discrete Blotto game by first identifying the optimal strategies of the General Lotto game [Har06]:

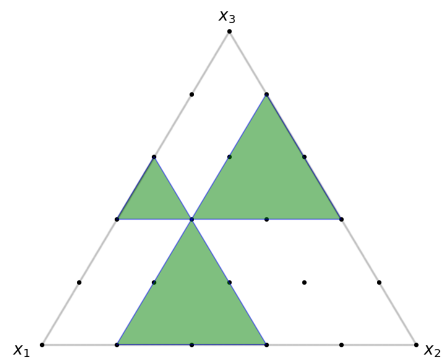


Figure 8: Winning and drawing regions of (2,1,2) in the $(5, 3)$-Blotto game.

Definition 8. *Define the General Lotto game, $\Gamma(a, b)$, as follows: Player A and B choose independent non-negative integer-valued distributions X, Y such that $E[X] = a, E[x] = b$ respectively.*
$P_{lotto}(X, Y) \coloneqq P(X > Y) - P(X < Y)$

Define the uniform distributions on the odd and even numbers up to $2m-1$ and $2m$ respectively:

$$
\begin{aligned}
U_O^m &\coloneqq U(\{1, 3, 5, ..., 2m-1\}) \\
U_E^m &\coloneqq U(\{0, 2, 4, ..., 2m\})
\end{aligned}
\tag{34}
$$

Note that both distributions have expectation m. Crucially, in the General Lotto game where $a = b$, any strategy that is a convex combination of U_O^m, U_E^m is optimal.

We now interpret U_O^m, U_E^m as probability distributions on integer partitions: for some pure strategy $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_K)$ we consider the analogous distribution $X = U(\{\mathbf{x}_1, ..., \mathbf{x}_K\})$. Following [Har06]:

Theorem 3.2. *Consider a symmetric (N, K)-Blotto game, $K > 1$:*

1. *If $N = mK$, $m \in \mathbb{N}$:*
   - *(a) U_O^m is an optimal strategy iff N and K have the same parity.*
   - *(b) U_E^m is an optimal strategy iff N is even.*
2. *If $N = mK + r$, $m, r \in \mathbb{N}$, $1 \le r \le K-1$: then:*

$$\left(1 - \frac{r}{K}\right) U_E^m + \left(\frac{r}{K}\right) U_O^{m+1} \tag{35}$$

is an optimal strategy.

Note that the above cases are due to restrictions on valid integer partitions - in the case that N is divisible by K, then if K and hence N are even, partitions can either be all odd or all even-sized, whereas if K is odd, then the partitions are all odd-sized for odd N, and all even-sized for even N.

4 Regret Minimisation

4.1 Correlated Equilibria

We first define a Correlated ϵ-equilibrium of the discrete Blotto game following [Str23]:

Definition 9. *A joint distribution p_{ij} on Δ_K^N is a Correlated ϵ-equilibrium of the discrete (N,K)-Blotto game with payoff matrix A, $\epsilon > 0$ if:*

$$\forall i, i', j, j' : \sum_k A_{i'k} \leq \sum_k A_{ik}, \quad \sum_k A^T_{i'k} \leq \sum_k A^T_{ik}, \tag{36}$$

The joint distribution is a Correlated equilibrium for $\epsilon = 0$.

Intuitively, both players would not have been better off from switching actions from i to i', or j to j' - neither player has an incentive to deviate from choosing a joint action from p_{ij}.

Note that Nash equilibria correspond to the special case in which p_{ij} is a product measure of independent optimal strategies. However, the set of Correlated equilibria also includes plays which are dependent on each other, such as the Pareto optimum.

Lemma 4.1. *The set of Correlated equilibria is nonempty, closed and convex.*

By existence of Nash equilibria, the CE set must be nonempty. The following can be seen by interpreting the definition of a Correlated equilibria as a set of linear inequalities governing a convex polytope within the n-simplex.

4.2 Regret Matching

We now introduce Hart and Mas-Colell's Regret-Matching algorithm [HM00]:

Definition 10. *Suppose the discrete (N,K)-Blotto game is iteratively played in time, $t = 1, 2, ...,$ where players A and B play actions x_t, y_t drawn from their mixed strategies p^i_t.*
Define variants (a) and (b) of the Regret-Matching algorithm for player A following the repeated procedure:

1. *Let e_j, $j \in \{1, ..., n\}$ be the j^{th} unit vector.*
   Define the SWAP matrix of player i at time t:

$$[W^i_t]_{j,k} = \begin{cases} P(e_k, y_t), & x_t = e_j \\ P(x_t, y_t), & \text{otherwise} \end{cases} \tag{37}$$

2. *Let $1_{(n,n)}$ be the constant $n \times n$ matrix of ones.*
   Define the DIFF matrix of player i at time t:

$$D^i_t = \frac{1}{t} \sum_{\tau=1}^{t} [W^i_\tau - P(x_\tau, y_\tau) \cdot \mathbf{1_{(n,n)}}] \tag{38}$$

3. *Define the REGRET matrix of player i at time t:*

$$[R_t^i]_{j,k} = max([D_t^i]_{j,k}, 0) \tag{39}$$

4. *Calculate p_{t+1}^i:*

   (a) *Let j^* be such that $x_t = e_{j^*}$:*

$$[p_{t+1}^i]_j = \begin{cases} \frac{1}{\mu}[R_t^i]_{j^*,j}, & j \neq j^* \\ 1 - \sum_{j \neq j^*} [p_{t+1}^i]_j, & j = j^* \end{cases} \tag{40}$$

   where μ is set sufficiently large such that $\forall t : p_{t+1}^i \in \Delta_n$.

   (b) *Set p_{t+1}^i to the left eigenvector of R_t^i with eigenvalue 1, corresponding to the invariant probability vector satisfying:*

$$\forall j \in \{1, ..., n\} : \sum_{k=1}^{n} [R_t^i]_{k,j} [p_{t+1}^i]_k = [p_{t+1}^i]_j \sum_{k=1}^{n} [R_t^i]_{j,k} \tag{41}$$

Intuitively, "regret" for past actions is captured by the average payoff that player A would receive, if in the past it had chosen action k whenever they in fact played j.
We then utilise positive regrets to inform our strategy for the next iteration. Treating R_t^i as a stochastic transition matrix between pure strategies, variation (b) utilises its invariant distribution as the next mixed strategy; a stable state that the Markov chain would approach over time if R_t^i did not change. However, it requires an eigenvalue calculation at each iteration.

Variation (a), however exhibits "inertia" - the current strategy is favoured for the next iteration unless there is another strategy that appears better. In addition it exhibits "friction" - there is a non-zero probability in each iteration that the current strategy does not change; this helps to "break out" of local oscillations. Both characteristics are controlled via μ, which also controls the speed of convergence. In addition, the algorithm does not require any computationally expensive operations.

We now state important results from [HM00] connecting Regret-matching and Correlated equilibria:

Lemma 4.2. *If player A follows either variant of the above algorithm, then almost surely:*
$\forall j, k \in \{1, ..., n\},\ j \neq k : \quad [R_t^i]_{j,k} \to \mathbf{0}$ *as* $t \to \infty$

Lemma 4.3. *Define the Empirical distribution, $z_t : \Delta_K^N \to [0, 1]$:*

$$z_t(s) = \frac{1}{t} |\{\tau \leq t : x_t = s\}| \tag{42}$$

Consider a sequence of actions s_t, and $\epsilon \geq 0$.
$\forall j, k \in \{1, ..., n\},\ j \neq k : \lim_{t \to \infty} \sup [R_t^i]_{j,k} \iff$ *the sequence $\{z_t\}$ converges to the set of Correlated ϵ-equilibria.*

Theorem 4.4. *If both players both follow variant (a) or variant (b) of Theorem 10, then z_t converges almost surely to the set of Correlated equilibria as $t \to \infty$.*

The proofs of the above hinge upon Blackwell's Approachability Theorem - this also yields a bound on the speed of convergence: $E([R^i_t]_{j,k}) \approx O\left(\frac{1}{\sqrt{t}}\right)$.

4.3 Implementation

We first generate all K-partitions of N using Codex order and permutations as previously discussed:

```
def partition(n,m):

    a = [0] * (m+1)
    a[0] = n
    a[m] = -1
    actions = []

    while True:
        while True:
            actions.append(a[:-1])
            if a[1] >= a[0] - 1:
                break
            a[0] -= 1
            a[1] += 1

        j = 3
        s = a[0] + a[1] - 1
        while a[j-1] >= a[0] - 1:
            s += a[j-1]
            j += 1

        if j > m:
            return actions
        x = a[j-1] + 1
        a[j-1] = x
        j -= 1

        while j > 1:
            a[j-1] = x
            s -= x
            j -= 1
        a[0] = s

def getActions(S, N):
    p = partition(S,N)
    actions = []
    for i in p:
        for j in set(permutations(i)):
            actions.append(list(j))
    return np.array(actions)

N = len(actions)
actionDict = {".".join([str(j) for j in act]): i for i, act in
    enumerate(actions.tolist())}
```

We now construct the payoff function for one battlefields, and the corresponding payoff matrix:

```
def getUtility(A,B):
    return np.sum(A > B) - np.sum(A < B)

def payoffmatrix():
    global N, actions

    A = np.tile(actions, (N, 1)).reshape((N,N,3))
    B = A.transpose((1,0,2))
    P = np.sum(B > A, axis = 2) - np.sum(B < A, axis = 2)
    return P
```

We now utilise the framework of the provided Neller and Lanctot's Regret Matching algorithm, which provides a fast and crude approximation of variant (b) [NL13].
In each iteration, we increment a running list regretSum with the difference between the payoffs of each pure strategy against the opponent's last action, and the actual payoff received. The next probability vector is obtained by disregarding negative regrets, and normalising over its positive values.
We improve the authors' implementation by using vectorisation and hash tables to reduce operations performed.

```
def getAction(p):
    global N, actions
    ind = np.random.choice(np.arange(N), p=p)
    return actions[ind,:]

def getStrategy(regretSum, strategySum):
    global N

    strategy = np.copy(regretSum)
    strategy[strategy<0] = 0
    normalisingSum = np.sum(strategy)

    if normalisingSum > 0:
        strategy /= normalisingSum
    else:
        strategy = 1/N * np.zeros(N)
    strategySum += strategy

    return strategy, strategySum

def train(iterations, oppStrategy):
    global N, actions

    strategySum = np.zeros(N)
    regretSum = 1/N * np.ones(N)

    P = payoffmatrix()

    for i in range(iterations):
        strategy, strategySum = getStrategy(regretSum,strategySum)
        myAction = getAction(strategy)
        oppAction = getAction(oppStrategy())
```

```python

        oppInd = actionDict[".".join([str(j) for j in oppAction])]
        actionUtility = P[:,oppInd]
        regretSum += actionUtility - getUtility(myAction,oppAction)

    return strategySum
```

Finally, we return the Empirical distribution z_t:

```python
def getAverageStrategy(strategySum):
    global N

    avgStrategy = np.zeros(N)

    normalisingSum = np.sum(strategySum)
    if normalisingSum > 0:
        avgStrategy = strategySum / normalisingSum
    else:
        avgStrategy = 1/N * np.ones(N)

    return avgStrategy
```

4.4 Results

We now follow Theorem 4.4 and execute the above algorithm on two players over 200000 iterations. We plot the resulting z_t within the simplex with a linear colourmap from green to red:

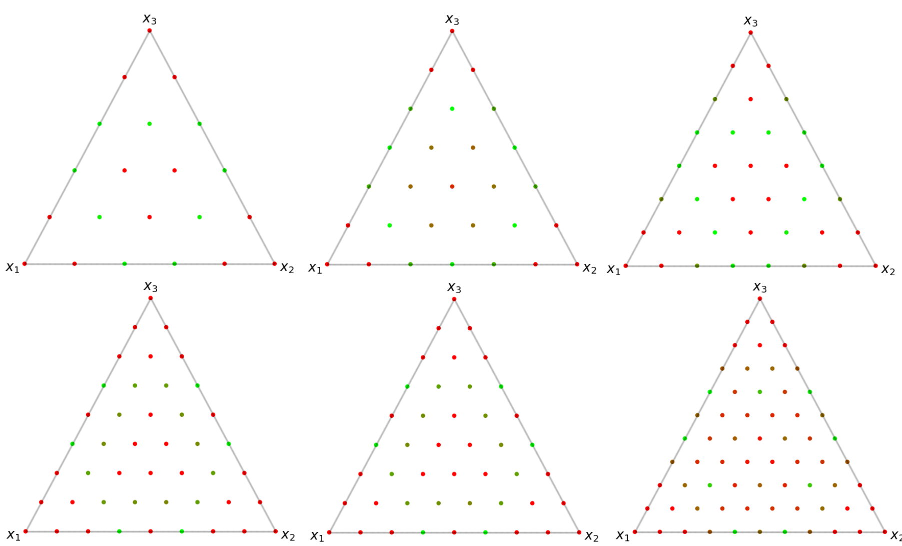


Figure 9: Correlated Equilibrium set in $N, 3$-Blotto game for $N \in \{5, ..., 10\}$

Note that z_t for each player converges to the same CE set, so it suffices to show that of player A. We may immediately draw parallels with the continuous case by observing that the supports of all strategies within the CE set lie within Hex(3).

Upon closer inspection, we observe that the optimal strategies characterised by Theorem 3.2 are contained within each CE set. Consider $N = 6$: applying 1b yields a uniform distribution on lattice points with even-sized partitions, corresponding to the vertices of Hex(3) - however we miss alternate points on Hex(3) corresponding to permutations of $(0, 3, 3), (1, 1, 4)$. However, applying 1a to $N = 9$ produces the entire CE set.

Interpreting the above as a mixed strategy, one can verify whether it is optimal by calculating the right product with the payoff matrix - if it is optimal then every pure strategy should have negative payoff against it, i.e. all entries of the right product are negative.
Interestingly, whilst $N = 5, 6, 7$ all yielded negative vectors, for larger N a small number of entries remained positive - since a larger N results in a slower convergence rate, we conclude the the last three solutions may not have fully converged within the set number of iterations.

We finally consider whether the limiting behaviour of the discrete CE set tends towards the continuous case. Running at $N = 100$ for sufficient iterations reveals a distribution reminiscent of G^{Hex} - however we can observe alternating artefacts within Hex(3) as a result of the discrete nature of U_O^m, U_E^m, but one can hypothesise that as $N \to \infty$, we converge to G^{Hex}.

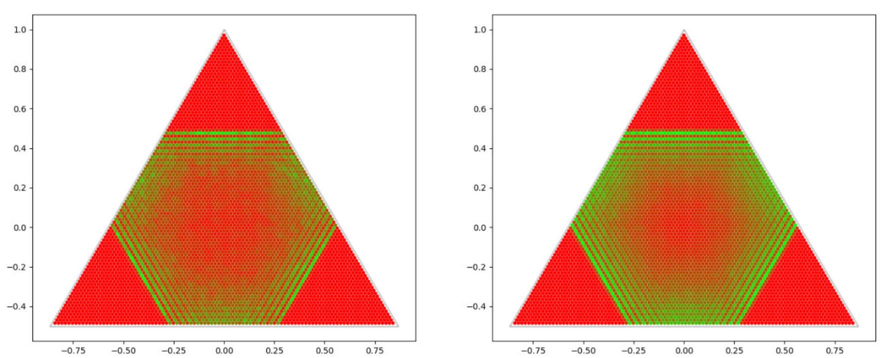


Figure 10: Correlated Equilibrium set in $N, 3$-Blotto game for $N \in \{5, ..., 10\}$

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