ROTATIONAL BLOCH MODES IN THE DECAGONAL PENROSE LATTICE

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Abstract. The paper roughly outlines the progress made in the author's UROP programme as of 27/10/23. An elastic plate is loaded with spring-mass scatterers within a decagon in an aperiodic lattice. Certain frequencies supporting rotationally symmetric modes of displacement are located, which align with the properties of an associated periodic lattice. This provides evidence suggesting the existence of a radial Bloch's Theorem for quasicrystals.

1. Introduction and Setup

1.1. Motivation.

A mathematical language exists for wave propagation in periodic lattices, underpinned by Bloch's Theorem, which dictates that waves in periodically equivalent positions differ only by a phase factor. This results in physical phenomena such as conductance in metals, and the metal-insulator transition.

Current research aims to formulate a similar language for aperiodic structures. The Penrose Tiling is the most famous of such, and is composed of "kite" and "dart" quadrilateral tiles. Most importantly, in certain constructions, the lattice exhibits a five-fold rotational symmetry about a central point. It has been previously established that the interface between an aperiodic tiling and its reflection guides wavefronts along the boundary [1].

It is natural to consider whether axes of rotational symmetry produce similar behaviour - in particular if such a structure will support 'Whispering gallery modes' at certain frequencies, where a wave radially propagates with periodicity in the radial unit cell [2].

1.2. Model.

The author uses a base code written by Richard Wiltshaw[3], that calculates the displacement field of a thin elastic plate governed by the Kirchoff-Love model from a single monopole source at a given position.

In addition, mass-springs with uniform mass and phase are patterned on the plate in specified arrangements as to cause the desired scattering effects.

1.3. Configurations.

The paper focuses on Penrose lattices - while there are various ways of generating such tilings, the method of interest generates a 5-fold rotational symmetry

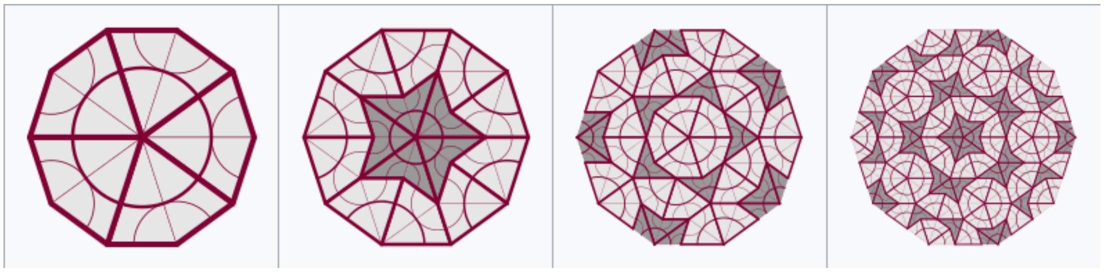

FIGURE 1. Successive deflations of the P2 Sun configuration.

about a single point, namely the P2 Sun initial condition followed by a sequence of Robinson triangle deflations.

Note that there are 10 triangular cells that are alternately reflected in the axes of rotational symmetry. Placing spring-masses at the vertices of the Robinson triangles generates an aperiodic lattice in the form of a decagon with five-fold rotational symmetry.

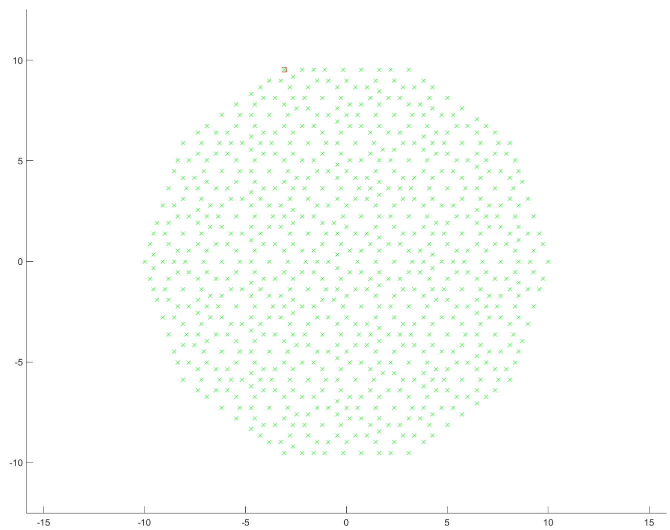

FIGURE 2. P2 Sun lattice after 5 deflations

Tests are also performed on a simple periodic lattice consisting of 11 equally spaced concentric decagons of spring-masses, where the n^{th} decagon has n spring masses on each side.

For the control model, the paper considers a lattice consisting of a triangular segment populated with a uniformly random distribution of masses, that is then flipped and rotated in a similar manner to the Penrose model.

A decagon with a uniformly random distribution of masses is also considered to observe the pure effect of the geometrical enclosure.

All lattices share the same rotationally symmetric properties and possess 761, 780 and 760 masses respectively, which are considered sufficiently similar. A unit source is arbitrarily placed in the top left vertex of the decagon; due to rotational symmetry, this is equivalent to sources at alternating vertices, although

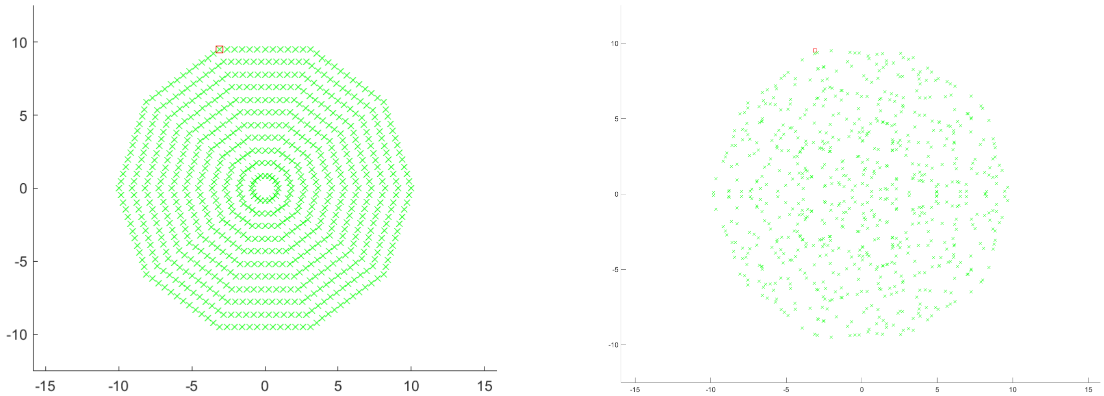

Figure 3. Periodic and random lattices with shared rotational symmetry.

the adjacent vertices may exhibit differing behaviour. The absolute value of the displacement wavefield is plotted for each frequency.

2. Main results

2.1. Observations.

All configurations with rotational symmetry displayed rotationally symmetric modes at certain frequencies, apart from the non-symmetric random configuration. At higher frequencies, $\Omega > 1.5$, the wave visibly propagates through the periodic and Penrose structures, while the wave is quickly dissipated in both of the random structures.

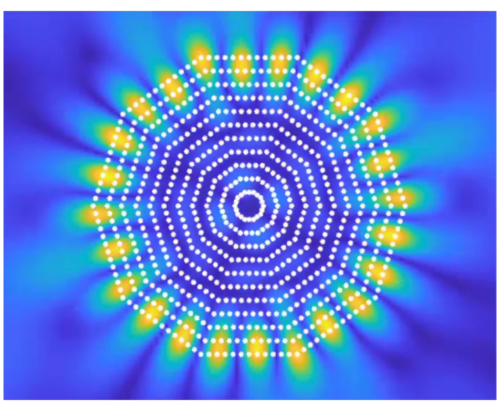

Figure 4. $\Omega = 0.45889$

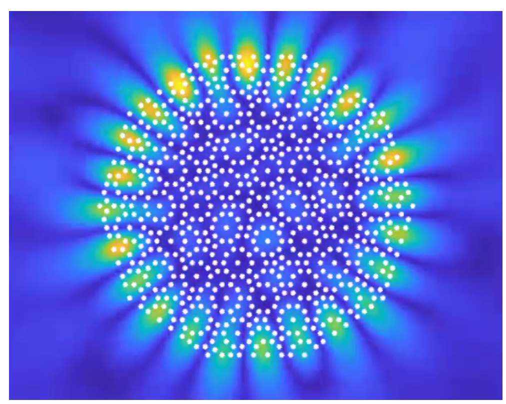

Figure 5. $\Omega = 0.45769$

Figure 6. Fabry-Pérot resonances of the Periodic and Penrose tilings.

In particular, at certain frequencies one can observe Fabry-Pérot resonances - alternating maxima and minima in a circular geometry around the boundary of the decagon. The higher the frequency, the more maxima and minima are observed[1].

[1] This is expected due to the inverse proportionality between wavelength and frequency.

2.2. **Rotational Analysis.** In order to locate rotational Bloch modes, the displacement field at each frequency is converted into Polar coordinates, with the pole at the centre of the decagon. If the field exhibits some form of rotational symmetry, then the rows of the transformed matrix will repeat with a period corresponding to the "angular wavenumber" of the rotational mode.

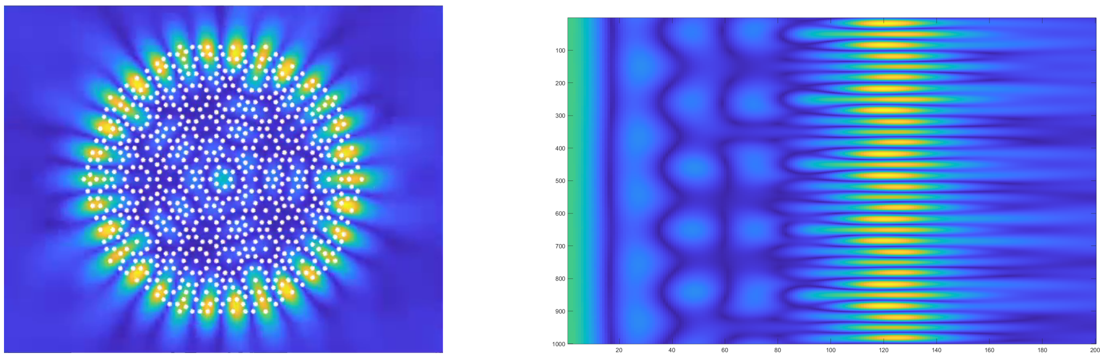

FIGURE 7. Polar 'unwrapping' of Fabry-Pérot resonance at $\Omega = 0.68454$ with five-fold rotational symmetry.

A 2-dimensional Fourier Transform is thus applied to the matrix to locate any radial periodic behaviour.

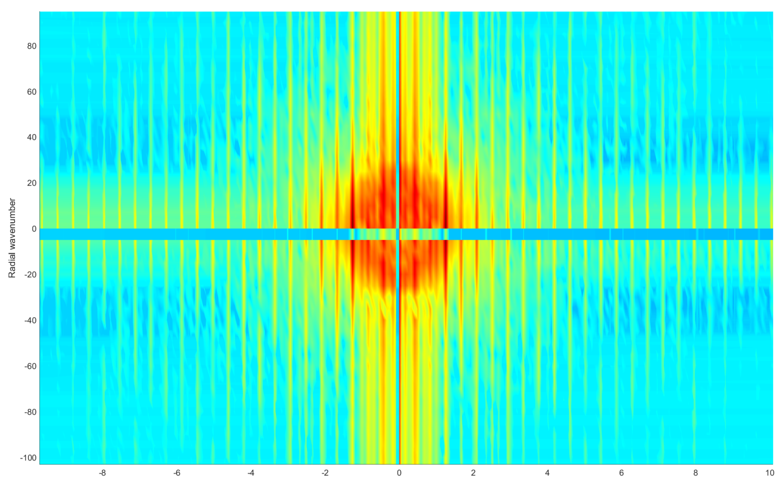

FIGURE 8. Log-absolute value plot of Frequency domain of Figure 7.[2]

In this case, two strong maxima are located at $\frac{2\pi}{5}$ as expected, as well as smaller maxima every $\frac{2\pi}{15}$[3]. As a heuristic for locating frequencies with such periodic behaviour, the maximal absolute value on the x axis in the frequency domain is interpreted as a 'magnitude' of the rotational symmetry of the displacement field.

[2]The horizontal and vertical bands at the origin are numerical artefacts not to be considered.

[3]This may indicate the F-P resonance is composed of eigenfunction states at these angular increments.

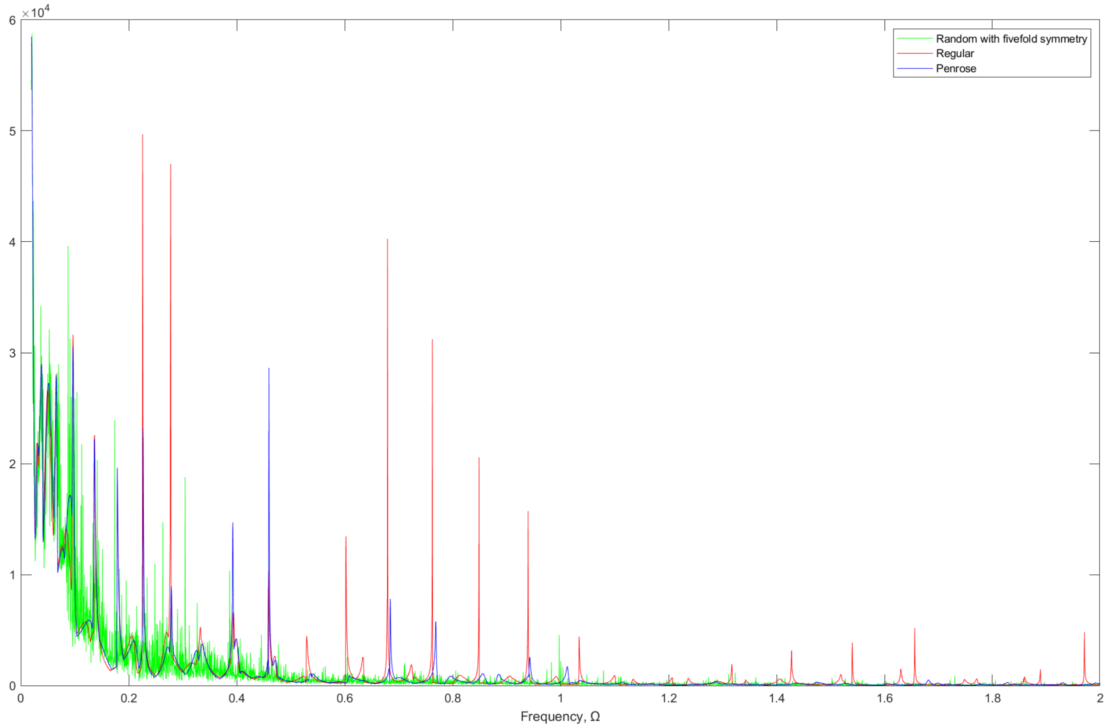

FIGURE 9. Plot of the above 'magnitude' against frequency. Peaks align with rotational modes in the displacement field.

It is clear that the rotational modes of the periodic and Penrose tiling closely align before $\Omega = 1$[4]. Importantly, the random lattice does not exhibit similar behaviour, implicating some shared Bloch-wave characteristics of the periodic and aperiodic case.

Following some of the main Fabry-Pérot resonance peaks we also identify localised modes[4] in the periodic and Penrose structures, represented by a smaller peak in the plot.

FIGURE 10. Localised modes at $\Omega = 0.473295, 0.47569$ in the periodic and Penrose lattices respectively.

[4]Simulations were run on a linear spacing on frequency value, so the proximity of a grid point to the actual resonance frequency may vary - hence the magnitude of the peaks should not be considered.

3. Additional Results

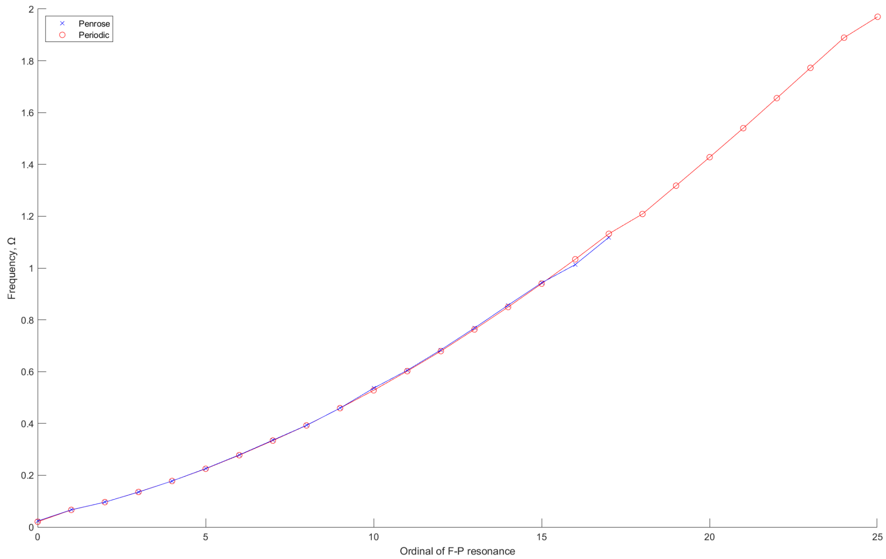

Figure 11. Plot of ordinal of F-P resonance to corresponding frequency - roughly linear as expected.

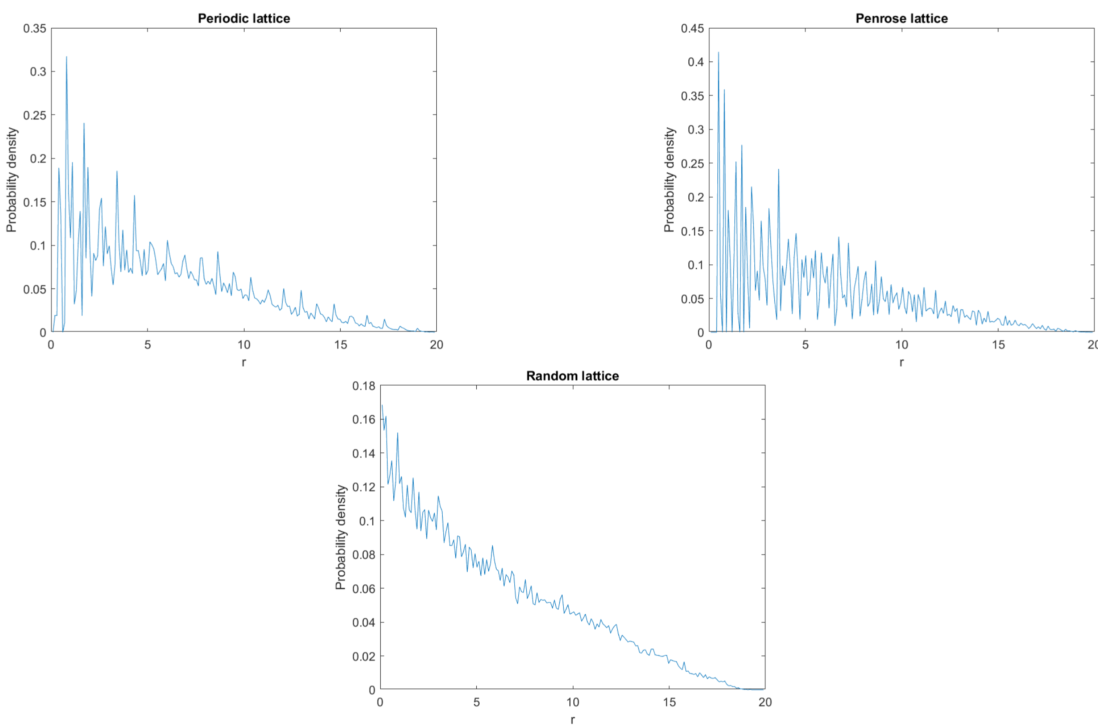

Figure 12. Modified Pair Correlation function of periodic, Penrose and random structures - the 'ruggedness' suggests periodicity.

[5]The full simulation, among others, can be found in the Result folder in the Git repository.

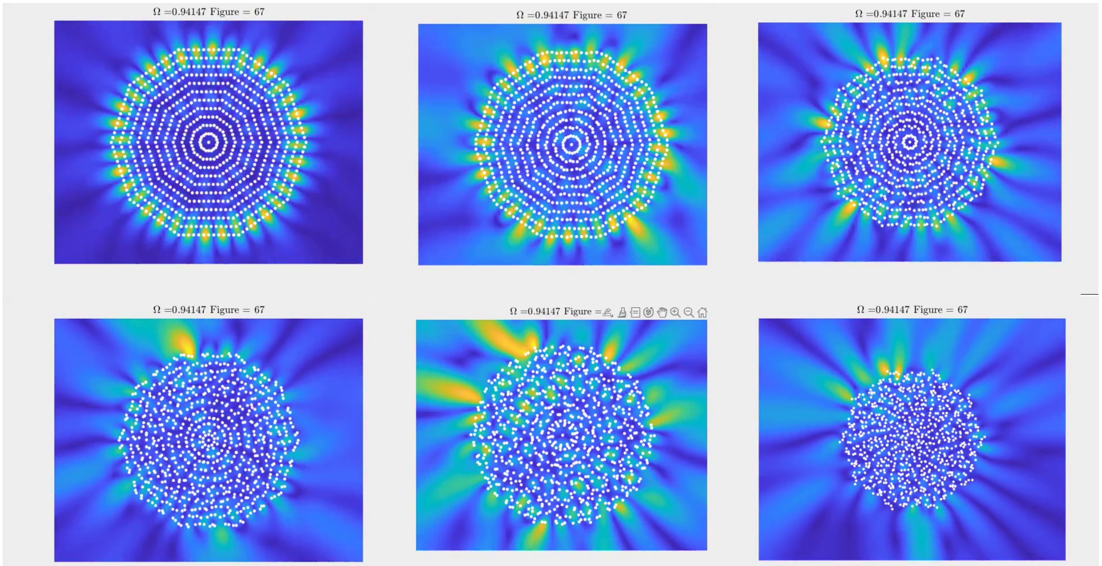

FIGURE 13. Evolution of a F-P resonance in a periodic lattice given a normal perturbation with increments in standard deviation of 0.05 (top left to bottom right) - this demonstrates the metal-insulator transition.[5]

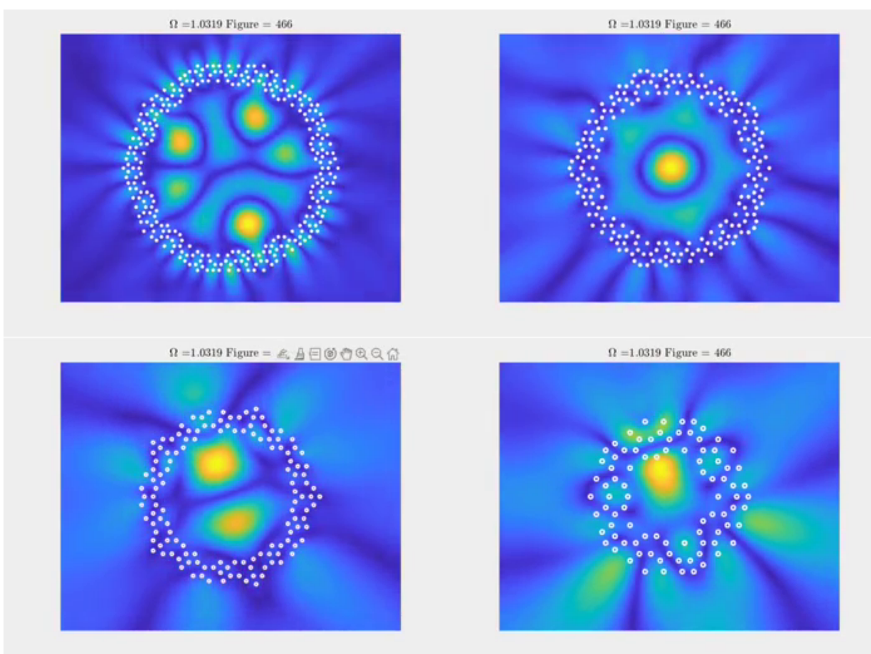

FIGURE 14. Energy trapping states in decagonal annulus layers of the Penrose tiling at a common frequency.

4. Conclusion

In summary, the paper finds evidence that within a decagonal structure, both the periodic and aperiodic lattices exhibit rotational Bloch modes at the same frequencies, while random structures do not.

The author hypothesises that the axes of rotational symmetry and reflection in the aperiodic lattice support radial unit cells that generate periodic wave behaviour. It is possible that resonances occur due the different 'levels' of radii within the periodic and aperiodic structures. It however may be possible that the regular spacing of both the periodic and aperiodic models promote wave propagation, and that 'clusters' of mass-springs in the random case disperse the wave.

Further investigation is required into other geometries, and the qualitative properties of a given lattice that determine its support of such rotational modes.

Acknowledgements

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Github Repository

All code used in the project can be found in the following repository - please contact about any enquiries.

https://github.com/raahweng/penrosequasicrystal

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