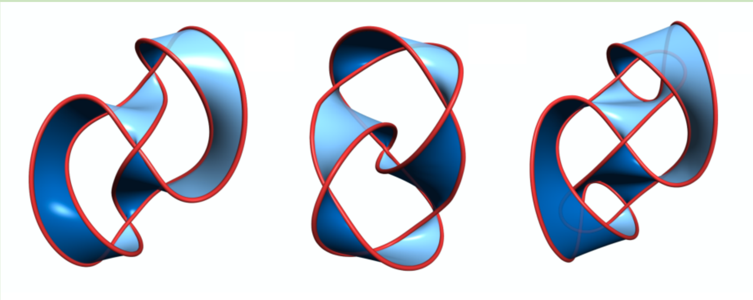

Visualising Knots using Seifert Surfaces

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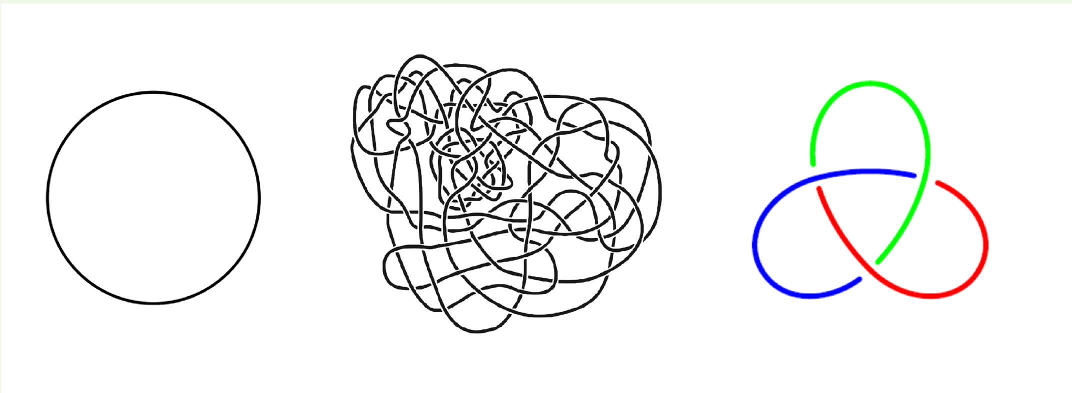
Abstract

Knot theory is a rich subfield of topology that studies the properties of closed curves in 3 dimensions. One major tool used are Seifert surfaces, which allow for the computation of several knot invariants. In this poster we review their properties and their construction using Seifert's Algorithm and Milnor's Fibration theorem.

Background

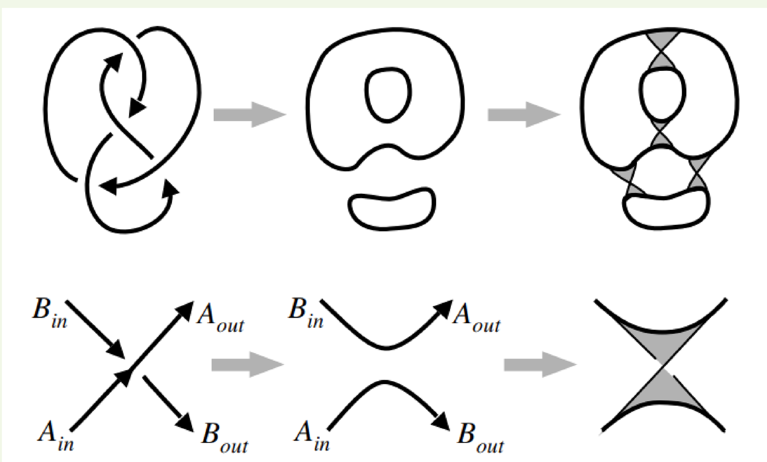
A **Knot** $K \in \mathbb{R}^3$ is a subset of points homeomorphic to a circle. Intuitively, we can imagine a knotted loop of string that can be deformed without tearing. [4]
Two knots are **Equivalent**, $K_1 \sim K_2$, if there exists an orientation-preserving homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K_1) = K_2$. For example, the Unknot is equivalent to Haken's Gordian Knot but not equivalent to the Trefoil knot:



A function from the set of all knots to any set S, $\mathcal{I} : \tilde{K} \to S$, is a **Knot Invariant** iff $K_1 \sim K_2 \Rightarrow \mathcal{I}(K_1) = \mathcal{I}(K_2)$.
An example would be tricolourability - let a **Strand** of a knot be a section from one undercrossing to another. A knot is **Tricolourable** iff every strand can be coloured from one of three colours such that at each crossing all strands are either the same colour or all different. The Trefoil knot is tricolourable while the Unknot is not, as seen above.

Seifert Surfaces

Given a knot K, a **Seifert Surface** is a compact oriented manifold with boundary K. An important result in the field is that every knot has an associated Seifert surface, proven using the Seifert Algorithm:



We can also derive the **Seifert Matrix** of a knot K using its Seifert surface, from which we can calculate several other knot invariants such as the Alexander Polynomial and the Knot Signature. [3]

The Seifert Algorithm: Proof

In 1934, Seifert published an elegant proof of the lemma that every knot has a Seifert surface, detailing a simple algorithm as follows [1]:

1. Assign an orientation to the knot by placing coherent directional arrows on each strand, noting the sign of the crossing - a right handed overlap is positive and vice versa.
2. Splice each crossing: consider the overlapping of two strands A and B - cut each strand and connect the ends of A to the ends of B.
3. We have now eliminated all crossing points and are left with non-intersecting closed curves, offset any nested curves and span each one with a disc.
4. Connect each disc using a right/left-handed twisted band at each crossing point - this is to preserve orientability. This produces a connected Seifert surface.

Torus Knots

A **(p,q) Torus knot** is a knot that winds around the surface of a torus in $\mathbb{R}^3$ p times with q revolutions, where p and q are coprime. Let $f : \mathbb{C}^2 \to \mathbb{C}$, $f(z, w) = z^p - w^q$. Let V be the complex hypersurface corresponding to the set of z, w such that $f(z, w) = 0$.
Consider the intersection of V with a 3-sphere of small radius ϵ [2]. We have the following equations:

$$z^p = w^q, \quad |z|^2 + |w|^2 = \epsilon^2$$

Setting $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$ gives:

$$z = ae^{qi\theta}, \quad w = be^{pi\theta + \frac{\pi i\theta}{q}}, \quad a, b \in \mathbb{N}$$

which is in fact the parameterisation of a (p,q) torus knot in S^3 [7].

Milnor Fibrations

A knot K is **Fibered** iff there exists a family of Seifert Surfaces or **Fibers** F_θ, $\theta \in S_1$ that intersect only in the boundary K. [5] Intuitively, we can think of constructing infinitely many shells of surfaces filling up the plane that are all anchored to K. Note that all torus knots are fibered.
Milnor's Fibration Theorem:
Let f be a polynomial of n+1 complex variables, $f(z_1, z_2, ...z_{n+1})$ and $V = f^{-1}(0)$. Let z_0 be a point on V, S_ϵ be a small sphere centered at z_0, and $K = S_\epsilon \cap V$, then the **Milnor Map**:

$$M : S_\epsilon \backslash K \to S^1, \quad M = \frac{f(z_0)}{|f(z_0)|}$$

is a fibration [8].

Printing of a Trefoil Seifert Surface

Since the fiber of a (2,3) torus knot corresponding to $M^{-1}(0)$ exists in 4 dimensions, I used the **Inverse Stereographic Projection** from the pole $(\sqrt{2}, 0, 0, 0)$ [5]:

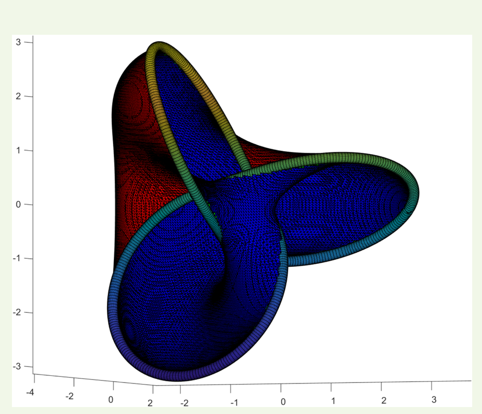
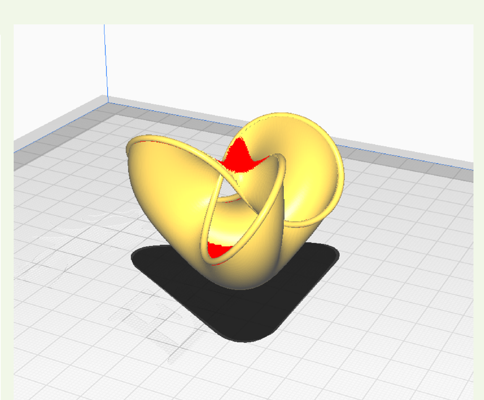
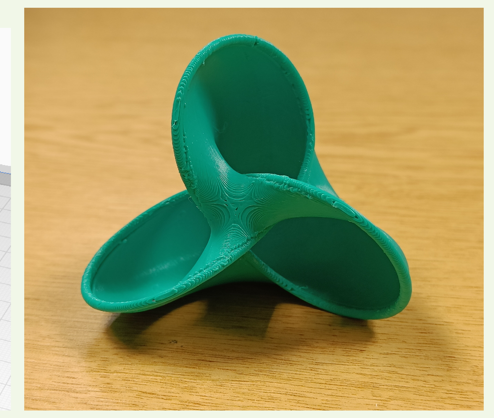
$$x \to \frac{\sqrt{2}(-2 + x^2 + y^2 + z^2)}{2 + x^2 + y^2 + z^2}, \quad y \to \frac{4x}{2 + x^2 + y^2 + z^2}$$

$$z \to \frac{4y}{2 + x^2 + y^2 + z^2}, \quad w \to \frac{4z}{2 + x^2 + y^2 + z^2}$$

to map the intersection of $\Re(z^2 - w^3)$ and a 3-sphere radius $\sqrt{2}$ where $\Im(z^2 - w^3) > 0$ to Cartesian coordinates in $\mathbb{R}^3$. To plot the trefoil boundary, I took real and imaginary parts of K: $(x, y, z, w) = (\cos 2\theta, \sin 2\theta, \cos 3\theta, \sin 3\theta)$, and used the Stereographic Projection:

$$x \to \frac{\sqrt{2}y}{\sqrt{2} - x}, \quad y \to \frac{\sqrt{2}z}{\sqrt{2} - x}, \quad z \to \frac{\sqrt{2}w}{\sqrt{2} - x}$$

to map the knot into $\mathbb{R}^3$. The model was then exported into Ultimaker Cura and printed using an Ender 3 Pro.

Conclusion

Seifert Surfaces are effective tools in knot theory that finds applications in determining numerous knot invariants and are used in studying 3-manifolds. More generally, Knot theory is a rapidly developing field that is used in analysing DNA entanglement, determining the chirality of molecules and quantum computing [2]. In this poster I touched on a few principles of a massive branch of mathematics, and I invite the reader to explore more of the subject.

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