If $U = f(x,y,z)$ show that ...

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Q.N2) If $U = \log(x^3 + y^3 + z^3 - 3xyz)$, then show that

(i) $\dfrac{\delta U}{\delta x} + \dfrac{\delta U}{\delta y} + \dfrac{\delta U}{\delta z} = \dfrac{3}{x+y+z}$

(ii) $\left(\dfrac{\delta}{\delta x} + \dfrac{\delta}{\delta y} + \dfrac{\delta}{\delta z}\right)^2 U = \dfrac{-9}{(x+y+z)^2}$

Solution here.

(i)

Given, $U = \log(x^3 + y^3 + z^3 - 3xyz)$ —— (1)

Now, diff. (1) partially w.r.t. x.

$$\frac{\delta U}{\delta x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} \times (3x^2 - 3yz) \text{ —— (2)}$$

Again, diff (1) partially w.r.t. y

$$\frac{\delta U}{\delta y} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} \times (3y^2 - 3xz) \text{ —— (3)}$$

And, diff (1) partially w.r.t. z

$$\frac{\delta U}{\delta z} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} \times (3z^2 - 3xy) \text{ —— (4)}$$

Adding (2), (3), (4), we get.

$$\frac{\delta U}{\delta x} + \frac{\delta U}{\delta y} + \frac{\delta U}{\delta z} = \frac{3x^2 - 3yz + 3y^2 - 3xz + 3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$= 3 \times \frac{\cancel{[x^2 + y^2 + z^2 - xy - yz - zx]}}{(x+y+z)\cancel{[x^2 + y^2 + z^2 - xy - yz - zx]}}$$

$$= \frac{3}{x+y+z} \quad \#$$

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(ii) Given LHS $= \left[\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right]^2 U$

$= \left[\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right]\left[\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right] U$

$= \left[\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right] \times \frac{3}{x+y+z}$

$= \left[3 \frac{\delta}{\delta x} \frac{1}{x+y+z}\right] + \left[3 \frac{\delta}{\delta y} \frac{1}{x+y+z}\right] + \left[3 \frac{\delta}{\delta z} \frac{1}{x+y+z}\right]$

$= \left[3 \frac{\delta}{\delta x} (x+y+z)^{-1}\right] + \left[3 \frac{\delta}{\delta y} (x+y+z)^{-1}\right] + \left[3 \frac{\delta}{\delta z} (x+y+z)^{-1}\right]$

$= \left[3 \times \frac{\delta (x+y+z)^{-1}}{\delta (x+y+z)} \times \frac{\delta (x+y+z)}{\delta x}\right] + \left[3 \times \frac{\delta (x+y+z)^{-1}}{\delta (x+y+z} \times \frac{\delta (x+y+z)}{\delta y}\right] + \left[3 \times \frac{\delta (x+y+z)^{-1}}{\delta (x+y+z)} \times \frac{\delta (x+y+z)}{\delta z}\right]$

$= 3\left[\frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2}\right]$

$= \frac{-9}{(x+y+z)^2}$

$=$ RHS #

Q.N.4) If $x = r\cos\theta$, $y = r\sin\theta$, prove that $\frac{\delta^2 r}{\delta x^2} + \frac{\delta^2 r}{\delta y^2} = \frac{1}{r}\left[\left(\frac{\delta r}{\delta x}\right)^2 + \left(\frac{\delta r}{\delta y}\right)^2\right]$

Solution here

Given : $x = r\cos\theta$ — (1)

$y = r\sin\theta$ — (2)

Squaring and adding we get,

$x^2+y^2 = r^2$ (ie) $r = \sqrt{x^2+y^2}$ —— (3)

Now,

diff (3) partially w.r.t. x

$$\frac{\delta}{\delta x} r^2 = \frac{\delta}{\delta x}(x^2+y^2)$$

or, $$2r\frac{\delta r}{\delta x} = 2x$$

$$\therefore \frac{\delta r}{\delta x} = \frac{x}{r} \quad \text{——— (4)}$$

Again, diff (4) partially w.r.t. x, we get

$$\frac{\delta}{\delta x}\left(\frac{\delta r}{\delta x}\right) = \frac{\delta}{\delta x}\left(\frac{x}{r}\right)$$

or, $$\frac{\delta^2 r}{\delta x} = \frac{r - x\frac{\delta r}{\delta x}}{r^2}$$

or, $$\frac{\delta^2 r}{\delta x} = \frac{r - x \times \frac{x}{r}}{r^2} \quad \left[\text{From (4)}\right]$$

$$\therefore \frac{\delta^2 r}{\delta x} = \frac{r^2 - x^2}{r^3}$$

(ie) $$\frac{\delta^2 r}{\delta x} = \frac{y^3}{r^3} \quad \text{——— (5)}$$

diff (3) partially w.r.t. y. We get

$$\frac{\delta r}{\delta y} = \frac{y}{r} \quad \text{——— (6)}$$

Again, diff (6) w.r.t. y, We get.

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$$\frac{\delta}{\delta y}\left(\frac{\delta r}{\delta y}\right) = \frac{r - y \cdot \frac{\delta r}{\delta y}}{r^2}$$

$$\frac{\delta^2 r}{\delta y^2} = \frac{r - y \cdot \frac{y}{r}}{r^2}$$

$$\therefore \frac{\delta^2 r}{\delta y^2} = \frac{r^2 - y^2}{r^3}$$

(ie) $\frac{\delta^2 r}{\delta y^2} = \frac{x^2}{r^3}$ ——— (7)

Now adding (5) and (7) We get

$$\frac{\delta^2 r}{\delta x^2} + \frac{\delta^2 r}{\delta y^2} = \frac{x^2 + y^2}{r^3}$$

$$= \frac{1}{r}\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right]$$

$$= \frac{1}{r}\left[\left(\frac{\delta r}{\delta x}\right)^2 + \left(\frac{\delta r}{\delta y}\right)^2\right]$$

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Q.N. 3) If $U = f(x,y,z)$ show that $x^2 U_{xx} = y^2 U_{yy} = z^2 U_{zz}$.

Solution here:-

Given; $U = f(x,y,z)$ —— (1)

diff (1) partially w.r.t. x.

$$\frac{\delta u}{\delta x} = \frac{\delta}{\delta x} f(x,y,z)$$

or $$\frac{\delta u}{\delta x} = f'(x,y,z) \times yz \text{ —— (2)}$$

Again, diff (2) w.r.t x (partially)

$$\frac{\delta}{\delta x}\left(\frac{\delta u}{\delta x}\right) = \frac{\delta}{\delta x} f'(x,y,z) \cdot yz$$

or $$\frac{\delta^2 u}{\delta x^2} = f''(x,y,z) \times (yz)^2$$

$\therefore$ $U_{xx} = (yz)^2 f''(x,y,z)$

So, $x^2 U_{xx} = (xyz)^2 f''(x,y,z)$ —— (3)

diff (1) partially w.r.t. y.

$$\frac{\delta u}{\delta y} = \frac{\delta}{\delta x} f(x,y,z)$$

or $$\frac{\delta u}{\delta y} = f'(x,y,z) \cdot xz \text{ — (4)}$$

Again, diff (4) w.r.t y (Partially)

$$\frac{\delta}{\delta y}\left(\frac{\delta U}{\delta y}\right) = \frac{\delta}{\delta y} f'(x,y,z) \cdot xz$$

or $$\frac{\delta^2 U}{\delta y^2} = f''(x,y,z) \cdot (xz)^2$$

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$\therefore U_{yy} = f''(x,y,z)\,(xz)^2$

So, $y^2 U_{yy} = (x \cdot y \cdot z)^2 f''(x,y,z)$ — (5)

diff (1) partially w.r.t. z

$$\frac{\delta U}{\delta z} = \frac{\delta}{\delta z} f(x,yz)$$

$$= f'(x,y,z) \times xy \quad \text{— (6)}$$

Again diff (6) w.r.t. z (Partially)

$$\frac{\delta}{\delta z}\left(\frac{\delta U}{\delta z}\right) = \frac{\delta}{\delta z} f'(x,y,z) \times xy$$

$$\text{or } \frac{\delta^2 U}{\delta z^2} = f''(x,y,z) \times (xy)^2$$

$$\therefore U_{zz} = f''(x,y,z)\,(xy)^2$$

$$\text{So, } z^2 U_{zz} = (x \cdot y \cdot z)^2 f''(x,y,z) \quad \text{— (7)}$$

Hence, from (3), (5) and (7)

$$x^2 U_{xx} = y^2 U_{yy} = z^2 U_{zz} = f''(x,y,z) \cdot (xyz)^2$$

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Q.N 11) State and prove Euler's theorem on homogenous function of two independent variables.

Solution here.

Statement : If $U = f(x,y)$ be a homogenous function of degree n in x and y then $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = nV$.

Proof : Given $U = f(x,y)$ be a homogenous function of two independent variables x and y of degree n.

Then by definition,

$$U = x^n \phi\left(\frac{y}{x}\right) \quad \text{———} (1)$$

diff (1) partially w.r.t x.

$$\frac{\delta u}{\delta x} = \frac{\delta}{\delta x}\left[x^n \phi\left(\frac{y}{x}\right)\right]$$

$$\text{or } \frac{\delta u}{\delta x} = n x^{n-1} \phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$\therefore \frac{\delta u}{\delta x} = -x^{n-2} y\, \phi'\left(\frac{y}{x}\right) + n x^{n-1} \phi\left(\frac{y}{x}\right) \quad \text{———} (2)$$

Again, diff (2) partially w.r.t y.

$$\frac{\delta u}{\delta y} = x^n \phi'\left(\frac{y}{x}\right) \times \frac{1}{x}.$$

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$$\therefore \frac{\delta v}{\delta y} = x^{n-1}\phi'\left(\frac{y}{x}\right) \quad \text{---} (3)$$

Multiplying eqn (2) by x and (3) by y and adding we get.

$$x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta y} = -x^{n-1}y\phi'\left(\frac{y}{x}\right) + n\, x^{n}\phi\left(\frac{y}{x}\right)$$
$$+ x^{n-1}y\phi'\left(\frac{y}{x}\right)$$

$$= n x^{n}\phi\left(\frac{y}{x}\right)$$

$$\therefore x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta y} = n x^{n}\phi\left(\frac{y}{x}\right)$$

$$(i.e.)\ x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta y} = nV$$

Proved

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Q.N. 12) (iii) $U = \frac{x^2y^2}{x^3+y^3}$

Or, $U = \frac{x^4 \left(\frac{y}{x}\right)^2}{x^3\left[1+(y/x)^3\right]}$

(i.e) $U = x\,\phi(y/x)$ — (1) which, homogenous ~~eq^n~~ function of degree 1.

then by Euler's theorem,

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 1.U.$$

diff (1) partially w.r.t. x and y resp we get.

$$\therefore x\frac{\delta u}{\delta x} = x\,\phi'\left(\frac{y}{x}\right)\left(\frac{y}{-x^2}\right) + 1.\phi\left(\frac{y}{x}\right)$$

$$= \frac{-y}{x}\phi'\left(\frac{y}{x}\right) + \phi\left(\frac{y}{x}\right) \quad —(2)$$

$$y\frac{\delta u}{\delta y} = \cancel{y\,\phi'\left(\frac{y}{x^2}\right)}\, y.x\,\phi'\left(\frac{y}{x}\right)\frac{1}{x} = y\phi'\left(\frac{y}{x}\right) \quad -(3)$$

Adding (2) and 3 we get.

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = x\,\phi\left(\frac{y}{x}\right) = $$

(i.e) $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = U$ #

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$u = e^{x^2+xy+y^2}$

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Q.N. 12) iv) $V = \frac{x^3 y}{x^2+y^2}$

We have, $U = \frac{x^3 y}{x^2+y^2} = \frac{x^4 (y/x)}{x^2[1+(y/x)^2]} = x^2 \phi\left(\frac{y}{x}\right)$ — (1)

Hence, (1) is a homogenous function of degree to 2.

by Eulers theorem

$$x\frac{\delta u}{\delta x} + \frac{y\delta u}{\delta y} = 2V$$

diff (1) partially w.r.t. x and y resp we get

$$\frac{\delta u}{\delta x} = x^2 \phi'\left(\frac{y}{x}\right) \cdot \left(\frac{y}{-x^2}\right) + 2x\phi\left(\frac{y}{x}\right)$$

$$= -y\phi'\left(\frac{y}{x}\right) + 2x\phi\left(\frac{y}{x}\right)$$

$$\therefore x\frac{\delta u}{\delta x} = -xy\,\phi'\left(\frac{y}{x}\right) + 2x^2\phi\left(\frac{y}{x}\right) \quad \text{— (2)}$$

$$\frac{\delta u}{\delta y} = x^2\,\phi'\left(\frac{y}{x}\right) \cdot \frac{1}{x} = x\phi'\left(\frac{y}{x}\right)$$

$$\therefore y\frac{\delta u}{\delta x} = xy\,\phi'\left(\frac{y}{x}\right) \quad \text{— (3)}$$

Adding (2) and (3) we get

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 2x^2\phi\left(\frac{y}{x}\right) = 2U \quad \#$$

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Q.N. 17) $U = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ show that $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = -\frac{1}{2}\cot U$

We have, $U = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$\therefore \cos U = \frac{x+y}{x^{1/2}+y^{1/2}} = \frac{x\left(1+\frac{y}{x}\right)}{x^{1/2}\left[1+\left(\frac{y}{x}\right)^{1/2}\right]}$

$\therefore \cos U = x^{\frac{1}{2}}\phi\left(\frac{y}{x}\right)$ —— (1)

Thus $\cos U$ is homogenous function of degree $\frac{1}{2}$. Then by eulers theorem, $x\frac{\delta \cos U}{\delta x} + y\frac{\delta \cos U}{\delta y} = \frac{1}{2}\cos U$

Diff (1) partially w.r.t x and y resp we get

$\sin U\frac{\delta u}{\delta x} = x^{\frac{1}{2}}\phi'\left(\frac{y}{x}\right)\left(\frac{y}{-x^2}\right) + \frac{1}{2}x^{-\frac{1}{2}}\phi\left(\frac{y}{x}\right)$

$= -x^{-3/2}y\,\phi'\left(\frac{y}{x}\right) + \frac{1}{2}x^{-\frac{1}{2}}\phi\left(\frac{y}{x}\right)$ — (2)

$\sin U\frac{\delta u}{\delta y} = x^{\frac{1}{2}}\phi'\left(\frac{y}{x}\right).\frac{1}{x}$

$= x^{-\frac{1}{2}}\phi'\left(\frac{y}{x}\right)$ —— (3)

Multiplying eqn (2) and (3) by x and y resp, we get.

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$$-x \sin u \frac{\delta u}{\delta x} - y \sin u \frac{\delta u}{\delta y} = -x^{-\frac{1}{2}} \phi'\left(\frac{y}{x}\right) + \frac{1}{2} \times x^{-\frac{1}{2}} \phi\left(\frac{y}{x}\right) + x^{\frac{1}{2}} \phi'\left(\frac{y}{x}\right)$$

$$\text{or} \quad -\sin u \left[x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y}\right] = \frac{1}{2} x^{\frac{1}{2}} \phi\left(\frac{y}{x}\right)$$

$$\text{or} \quad -\sin u \left[x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y}\right] = \frac{1}{2} \cos u$$

$$\text{or} \quad x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = -\frac{1}{2} \frac{\cos u}{\sin u}$$

$$\text{or} \quad x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = -\frac{1}{2} \cot u \quad \#$$

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Q.N.1) Find the Second order ~~der~~ partial derivative of $V = e^{x^2+xy+y^2}$

Solution here

Given, $U = e^{(x^2+xy+y^2)}$ — (1)

by Euler's

diff (1) partially w.r.t. x.

$$\frac{\delta u}{\delta x} = e^{(x^2+xy+y^2)} \times 2x + \cancel{x}\, y$$

$$\therefore \frac{\delta u}{\delta x} = \cancel{f3x - 3x.} e \quad (2x+y)\, e^{(x^2+xy+y^2)} \quad \text{— (2)}$$

diff (1) partially w.r.t. y

$$\frac{\delta u}{\delta y} = e^{(x^2+xy+y^2)} \times 2y + x$$

$$\therefore \frac{\delta u}{\delta y} = (2y+x)\, e^{(x^2+xy+y^2)} \quad \text{—— (3)}$$

Again.

diff (2) partially w.r.t. x.

we ge

$$\frac{\delta}{\delta x}\left(\frac{\delta u}{\delta x}\right) = \frac{\delta}{\delta x} (2x+y).\, e^{(x^2+xy+y^2)}$$

$$\text{or, } \frac{\delta^2 U}{\delta x^2} = (2x+y) \times \frac{\delta e^{(x^2+xy+y^2)}}{\delta x} + \left[\frac{\delta}{\delta x}(2x+y)\right] \times e^{x^2+xy+y^2}$$

$y^2)$

$$= e^{(x^2+xy+y^2)} \times (2x+y) \times (2x+y) + 2 \cdot e^{x^2+xy+y^2}$$

$$= e^{x^2+xy+y^2}\left[(2x+y)^2 + 2\right]$$

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diff eqⁿ (3) partially w.r.t. x.

$$\frac{\delta^2 u}{\delta x.\delta y} = \frac{\delta\, e^{(x^2+xy+y^2)}}{\delta x} \times (x+2y)$$

$$= \frac{\delta\, e^{x^2+xy+y^2}}{\delta x^2+xy+y^2} \times \frac{\delta\, x^2+xy+y^2}{\delta x} \times (x+2y) + \frac{\delta}{\delta x}(x+2y)$$

$$e^{x^2+xy+y^2}$$

$$= e^{x^2+xy+y^2} \times (2x+y)(x+2y) + e^{x^2+xy+y^2}$$

diff (2) partially w.r.t. y.

$$\frac{\delta u}{\delta y \delta x} = \frac{\delta}{\delta y}\left(e^{x^2+xy+y^2}\right)(2x+y)$$

$$= e^{x^2+y^2+xy} \cdot (x+2y) \times (2x+y) + e^{x^2+xy+y^2}$$

$$= \left(e^{x^2+xy+y^2}\right)(x+2y)(2x+y) + e^{x^2+xy+y^2}$$

Again we have.

diff (3) partially w.r.t. y.

$$\frac{\delta^2 u}{\delta y^2} = \frac{\delta}{\delta y} e^{x^2+xy+y^2}(x+2y)$$

$$\therefore \frac{\delta^2 u}{\delta y^2} = e^{x^2+xy+y^2}\left[(x+2y)^2+2\right]$$

$$\therefore \frac{\delta^2 u}{\delta x^2} = \frac{\delta^2 u}{\delta x^2} \text{ and } \frac{\delta^2 u}{\delta x \delta y} = \frac{\delta^2 u}{\delta y \delta x}.$$

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Q.N.-18) $U = \operatorname{Cosec}^{-1}\left(\frac{x^{\frac{1}{x}}+y^{\frac{1}{2}}}{x^{1/3}+y^{1/3}}\right)$

or, $\operatorname{Cosec} U = \frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}$

or, $\operatorname{Cosec} U = \frac{x^{\frac{1}{2}}\left[1+\left(\frac{y}{x}\right)^{\frac{1}{2}}\right]}{x^{\frac{1}{3}}\left[1+\left(\frac{y}{x}\right)^{\frac{1}{3}}\right]}$

or, $\operatorname{Cosec} U = x^{\frac{1}{6}}\phi\left(\frac{y}{x}\right)$

Thus, Cosec U is a homogenous function of degree $\frac{1}{6}$
Then by Euler's

$$x\frac{\delta \operatorname{Cosec} U}{\delta x} + y\frac{\delta \operatorname{Cosec} U}{\delta y} = \frac{1}{6}\operatorname{Cosec} U.$$

diff. partially w.r.t. x and y resp we get.

$$-\operatorname{Cosec} U \operatorname{Cot} U \frac{\delta u}{\delta x} = x^{\frac{1}{6}}\phi'\left(\frac{y}{x}\right)\cdot\left(\frac{y}{-x^2}\right) + \frac{1}{6}x^{-\frac{5}{6}}\phi\left(\frac{y}{5}\right)$$

$$= -x^{-11/6}y\phi'\left(\frac{y}{x}\right) + \frac{1}{6}x^{-5/6}\phi\left(\frac{y}{x}\right) - (2)$$

$$-\operatorname{Cosec} U \operatorname{Cot} U \frac{\delta u}{\delta y} = x^{1/6}\phi\left(\frac{y}{x}\right)\cdot\frac{1}{x}$$

$$= x^{-5/6}\phi\left(\frac{y}{x}\right) \text{ --- } (3)$$

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Multiplying (2) and (3) by x and y resp and adding we get

$$-x^{-5/6}\phi'\left(\frac{y}{x}\right)+\frac{1}{6}x^{\frac{1}{6}}\phi\left(\frac{y}{x}\right)+x^{\frac{5}{6}}y\phi'\left(\frac{y}{x}\right)$$

$$= x\operatorname{Cosec} U \operatorname{Cot} U \frac{\delta U}{\delta x} + y \operatorname{Cosec} U \operatorname{Cot} U \frac{\delta U}{\delta y}$$

or, $$-\operatorname{Cosec} U \operatorname{Cot} U\left(x\frac{\delta U}{\delta x}+y\frac{\delta U}{\delta y}\right)=\frac{1}{6}x^{\frac{1}{6}}\phi\left(\frac{y}{x}\right)$$

$$=\frac{1}{6}\operatorname{Cosec} U$$

$$\therefore x\frac{\delta U}{\delta x}+y\frac{\delta U}{\delta y}=\frac{1}{6}\frac{\operatorname{Cosec} U}{-\operatorname{Cosec} U \operatorname{Cot} U}$$

$$=\frac{1}{-6}\frac{1}{\operatorname{Cot} U}$$

$$=\frac{-1}{6}\operatorname{Tan} U$$

$$\therefore x\frac{\delta U}{\delta x}+y\frac{\delta U}{\delta y}=\frac{-1}{6}\operatorname{Tan} U \ \#$$

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Q.nl. 12 (vii) Q. $U = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$

We have, $U = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$

Since U is a homogenous fraction of degree 0. Then by Euler's theorem

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 0 \quad , \; U = 0$$

diff (1) partially w.r.t x and y resp. We get

$$\frac{\delta u}{\delta x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right)$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2} \quad \text{--- (2)}$$

$$\frac{\delta u}{\delta y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \left(\frac{x}{-y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

$$= -\frac{x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2} \quad \text{--- (3)}$$

Multiplying (2) and (3) by x and y resp and adding we get

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{(x^2+y^2)} - \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{(x^2+y^2)}$$

$$\therefore x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 0 \quad \#$$

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Q.N 12) (v) $U = x^n \tan^{-1}(y/x)$ — (1)

We have $U = x^n \tan^{-1}(y/x)$ which is homogenous function of degree n.

Then by Eulers theorem

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = nU$$

Diff (1) partially w.r.t x and y resp we get,

$$\frac{\delta u}{\delta x} = x^n \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{-y}{x^2}\right) + n x^{n-1} \tan^{-1}\frac{y}{x}$$

$$= \frac{-x^n y}{x^2+y^2} + n x^{n-1} \tan^{-1}\frac{y}{x} \quad \text{—— (2)}$$

$$\frac{\delta u}{\delta y} = x^n \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x^{n+1}}{x^2+y^2} \quad \text{—— (3)}$$

Multiplying (2) and (3) by x and y resp and adding

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \frac{-x^{n+1}y}{x^2+y^2} + n x^n \tan^{-1}\frac{y}{x} + \frac{x^{n+1}y}{x^2+y^2}$$

$$\therefore x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = nU \; \#$$

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Q.N 12) vi) $U = x f(y/x)$

Given $U = x f(y/x)$ — (1) which is homogenous function of degree 1.

Then by Eulers theorem

$$x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = U$$

diff (1) partially w.r.t. x and y resp.

$$\frac{\delta u}{\delta x} = x f'\left(\frac{y}{x}\right) \cdot \left(\frac{-y}{x^2}\right) + f\left(\frac{y}{x}\right) \times 1$$

$$= -\frac{y}{x} f'\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right) \quad \text{— (2)}$$

$$\frac{\delta u}{\delta y} = x f'\left(\frac{y}{x}\right) \cdot \frac{1}{x} = f'\left(\frac{y}{x}\right) \quad \text{— (3)}$$

Multiplying (2) and (3) by x and y resp and adding

$$x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = -y f'\left(\frac{y}{x}\right) + x f\left(\frac{y}{x}\right) + y f'\left(\frac{y}{x}\right)$$

$$\therefore x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = x f\left(\frac{y}{x}\right) = U \quad \#$$

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Q.N 12) ii) $U = x^n \sin(y/x)$

Given, $U = x^n \sin(y/x)$ — (1) is homogenous function of degree n.

By Euler's theorem

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = nV = n x^n \sin(y/x)$$

diff (1) partially w.r.t. x and y resp., we get.

$$\frac{\delta u}{\delta x} = x^n \cos\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + n x^{n-1} \sin\left(\frac{y}{x}\right)$$

$$= -x^{n-2} y \cos(y/x) + n x^{n-1} \sin(y/x)$$

$$\therefore x\frac{\delta u}{\delta x} = -x^{n-1} y \cos(y/x) + n x^n \sin(y/x) \quad \text{—— (2)}$$

$$\frac{\delta u}{\delta y} = x^n \cos\left(\frac{y}{x}\right) \cdot \frac{1}{x} = x^{n-1} \cos\left(\frac{y}{x}\right)$$

$$\therefore y\frac{\delta u}{\delta y} = x^{n-1} y \cos(y/x) \quad \text{—— (3)}$$

Adding (2) and (3) we get.

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = n x^n \sin\left(\frac{y}{x}\right) = nV$$

#

Q.N. 13) If $U = \sin^{-1}\left(\frac{x^2 y^2}{x+y}\right)$, show that $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 3\tan U$.

Solution here.

Given, $U = \sin^{-1}\left(\frac{x^2 y^2}{x+y}\right)$

or $\sin U = \frac{x^4 \left(\frac{y}{x}\right)^4}{x\ (1+y/x)}$ ______ (1)

$\sin U = x^3 \phi\left(\frac{y}{x}\right)$ which is homogenous function of degree $n = 3$

Then, by Euler's theorem

~~x δ Tan~~ $x\frac{\delta \sin U}{\delta x} + y\frac{\delta \sin U}{\delta y} = 3\sin U$

or $x\frac{\delta \sin U}{\delta U} \times \frac{\delta U}{\delta x} + y\frac{\delta \sin U}{\delta U} \times \frac{\delta U}{\delta y} = 3\sin U$

or $x\cos U\frac{\delta U}{\delta x} + y\cos U\frac{\delta U}{\delta y} = 3\sin U$

or $\cos U\left[x\frac{\delta U}{\delta x} + y\frac{\delta U}{\delta y}\right] = 3\sin U$

or $x\frac{\delta U}{\delta x} + y\frac{\delta U}{\delta y} = 3\frac{\sin U}{\cos U}$

or $x\frac{\delta U}{\delta x} + y\frac{\delta U}{\delta y} = 3\tan U$ #

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Q.N. 14 If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, show that $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \sin 2u$

Solution here.

Given: $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

or, $\tan u = \frac{x^3}{x}\left[\frac{1+(\frac{y}{x})^3}{1-(y/x)}\right]$

or $\tan u = x^2\,\phi(y/x)$ which is a homogenous function of degree 2.

Then by Euler's theorem.

$$x\frac{\delta \tan u}{\delta x} + y\frac{\delta \tan u}{\delta y} = 2\tan u$$

or, $$x\frac{\delta \tan u}{\delta u} \times \frac{\delta u}{\delta x} + y\frac{\delta \tan u}{\delta u} \times \frac{\delta u}{\delta y} = 2\tan u$$

or, $$x\sec^2 u\frac{\delta u}{\delta x} + y\sec^2 u\frac{\delta u}{\delta y} = 2\tan u$$

or, $$\sec^2 u\left[x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}\right] = 2\tan u$$

or, $$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 2 \times \frac{\sin u}{\cos u} \times \frac{1}{\sec^2 u}$$

or, $$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 2\frac{\sin u}{\cancel{\cos u}} \times \cos^{\cancel{2}} u$$

or, $$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \sin 2u$$ #

Q.N.16) If $u = \log x$, show that $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 1$

Solution here

Given: $u = \log x$

Taking log (anti) on both side.

$$e^u = x$$

(i.e) $$e^u = x \cdot \left(\frac{y}{x}\right)^0$$

So, $e^u = x^1 \phi(y/x)$ which is homogenous function of degree 1.

Then by Euler's theorem;

$$x\frac{\delta e^u}{\delta x} + y\frac{\delta e^u}{\delta y} = 1\, e^u$$

or, $$x\frac{\delta e^u}{\delta u} \times \frac{\delta u}{\delta x} + y\frac{\delta e^u}{\delta u} \times \frac{\delta u}{\delta y} = e^u$$

or, $$x \cdot e^u \frac{\delta u}{\delta x} + y\, e^u \cdot \frac{\delta u}{\delta y} = e^u$$

or, $$e^u\left[x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}\right] = e^u$$

or, $$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 1$$ #

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Q.N. 19) $\sin U = \dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

We have, $\sin U = \dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \dfrac{x^{\frac{1}{2}} - y^{\frac{1}{2}}}{x^{1/2}+y^{1/2}} = \dfrac{x^{\frac{1}{2}}}{x^{1/2}} \times \dfrac{\left[1 - \left(\frac{y}{x}\right)^{1/2}\right]}{\left[1+\left(\frac{y}{x}\right)^{1/2}\right]}$

or $\sin U = \phi(y/x)$ — (1), which is homogenous function of degree 0

Then by euler's theorem

~~$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}$~~ . $x\dfrac{\delta \sin U}{\delta x} + y\dfrac{\delta \sin U}{\delta y} = 0 . \sin U . = 0$

Diff (1) partially w.r.t. x and y resp. we get

$\cos U \dfrac{\delta u}{\delta x} = \phi'\left(\dfrac{y}{x}\right)\left(\dfrac{y}{-x^2}\right) = -\dfrac{y}{x^2}\phi'\left(\dfrac{y}{x}\right)$ — (2)

$\cos U \dfrac{\delta u}{\delta y} = \phi'\left(\dfrac{y}{x}\right) \cdot \dfrac{1}{x} = \dfrac{1}{x}\phi'\left(\dfrac{y}{x}\right)$ — (3)

Multiplying eqn (2) and (3) by x and y resp, and adding

$x\cos U\dfrac{\delta u}{\delta x} + y\cos U\dfrac{\delta u}{\delta y} = -\dfrac{y}{x}\phi'\left(\dfrac{y}{x}\right) + \dfrac{y}{x}\phi'\left(\dfrac{y}{x}\right)$

or, $\cos U\left[x\dfrac{\delta u}{\delta x} + y\dfrac{\delta u}{\delta y}\right) = 0$

$\therefore x\dfrac{\delta u}{\delta x} + y\dfrac{du}{\delta y} = 0$ #

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Q.N.10) Solution here.

$$Z = \phi(x+ay) + \psi(x-ay) \quad —(1)$$

diff (1) partially w.r.t. x and y resp

$$\frac{\delta z}{\delta x} = \frac{\delta \phi(x+ay)}{\delta (x+ay)} \times \frac{\delta (x+ay)}{\delta x} + \frac{\delta \psi(x-ay)}{\delta (x-ay)} \times \frac{\delta (x-ay)}{\delta x}$$

$$= \phi'(x+ay) + \psi'(x-ay$$

Again,

$$\frac{\delta^2 z}{\delta x^2} = \frac{\delta \phi'(x+ay)}{\delta (x+ay)} \times \frac{\delta (x+ay)}{\delta x} +$$

$$\frac{\delta \psi'(x-ay)}{\delta (x-ay)} \times \frac{\delta (x-ay)}{\delta x}$$

$$= \phi''(x+ay) + \psi''(x-ay) \quad —(2)$$

$$\therefore a^2 \frac{\delta^2 z}{\delta x^2} = a^2 [\phi''(x+ay) + \psi''(x-ay)]$$

$$\therefore \frac{\delta z}{\delta y} = \frac{\delta \phi(x+ay)}{\delta (x+ay)} \times \frac{\delta (x+ay)}{\delta y} + \frac{\delta \psi(x-ay)}{\delta (x-ay)} \times \frac{\delta (x-ay)}{\delta y}$$

$$= \phi'(x+ay) \cdot a + \psi'(x-ay) \cdot (-a)$$

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Again

$$\frac{\delta^2 z}{\delta y^2} = a\left[\frac{\delta \phi'(x+ay)}{\delta(x+ay)} \times \frac{\delta(x+ay)}{\delta y} - \frac{\delta \psi'(x-ay)}{\delta(x-ay)} \times \frac{\delta(x-ay)}{\delta y}\right]$$

$$= a\left[\phi''(x+ay)\, a + \psi''(x-ay) \times a\right]$$

$$= a^2\left[\phi''(x+ay) + \psi''(x-ay)\right]$$

$$= a^2 \frac{\delta^2 z}{\delta x^2} \quad \#$$

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QN.9) Solution here:-

$$U = e^{xyz} \quad -(1)$$

diff (1) partially wrt. z

$$\frac{\delta u}{\delta z} = e^{xyz} \times xy \quad -(2)$$

diff (2) wrt. y

$$\frac{\delta^2 u}{\delta y \cdot \delta z} = e^{xyz} \times xz \cdot xy + x \cdot e^{xyz}$$

$$= e^{xyz} \cdot x^2 zy + x \cdot e^{xyz} \quad -(3)$$

diff (3) wrt x partially.

$$\frac{\delta^3 u}{\delta x \delta y \delta z} = x^2 zy \cdot \frac{\delta e^{xyz}}{\delta xyz} \times \frac{\delta xyz}{\delta x} + e^{xyz} \times \frac{\delta x^2 yz}{\delta x} + \frac{\delta x}{\delta x} e^{xyz} + x \cdot \frac{\delta e^{xyz}}{\delta xyz} \cdot \frac{\delta xyz}{\delta x}$$

$$= x^2 zy \cdot e^{xyz} \cdot yz + e^{xyz}\, 2yxz + e^{xyz} + x e^{xyz} \cdot yz$$

$$\therefore \frac{\delta^3 u}{\delta x \delta y \delta z} = e^{xyz}\left[1 + 3xyz + x^2 y^2 z^2\right] \#$$

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Q.N. 8) Solution here

$$U = \log\sqrt{x^2+y^2+z^2} \quad -(1)$$

diff (1) partially w.r.t. x

$$\frac{\delta u}{\delta x} = \frac{\delta \log\sqrt{x^2+y^2+z^2}}{\delta\sqrt{x^2+y^2+z^2}} \times \frac{\delta}{\delta x}(x^2+y^2+z^2)^{1/2}$$

$$= \frac{1}{x^2+y^2+z^2} \times \frac{1}{2\sqrt{x^2+y^2+z^2}} \times 2x$$

$$= \frac{x}{\cancel{\sqrt{}}\, x^2+y^2+z^2}$$

$$\frac{\delta^2 U}{\delta x^2} = \frac{\delta}{\delta x}\left(\frac{x}{x^2+y^2+z^2}\right) = \frac{(x^2+y^2+z^2)\frac{\delta x}{\delta x} - x\frac{\delta}{\delta x}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}$$

$$= \frac{x^2+y^2+z^2-2x^2}{(x^2+y^2+z^2)^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$$

Similarly, $\frac{\delta^2 U}{\delta y^2} = \frac{x^2+z^2-y^2}{(x^2+y^2+z^2)^2}$, $\frac{\delta^2 U}{\delta z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$

$$\text{Now, RHS} = (x^2+y^2+z^2)\left[\frac{\delta^2 U}{\delta x^2} + \frac{\delta^2 U}{\delta y^2} + \frac{\delta^2 U}{\delta z^2}\right]$$

$$= \frac{x^2+y^2+z^2}{x^2+y^2+z^2}$$

$$= 1 \; \#$$

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Q.N.7) Solution.

$U = x^2 + y^2 + z^2$

diff w.r.t x (Partial)

$\frac{\delta U}{\delta x} = 2x.$

Similarly

$\frac{\delta U}{\delta y} = 2y$

$\frac{\delta U}{\delta z} = 2z$

Now

$LHS = x U_x + y U_y + z U_z$

$= x \times 2x + y \times 2y + z \times 2z$

$= 2 [x^2 + y^2 + z^2]$

$= 2U$ #

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Q.N.6) Solution here:-

$U = f(r)$ —(1)

diff partially w.r.t x.

$$\frac{\delta u}{\delta x} = f'(r)\frac{\delta r}{\delta x}. \quad —(2)$$

$$\frac{\delta}{\delta x}\left(\frac{\delta u}{\delta x}\right) = f'(r)\frac{\delta r}{\delta x^2} + f''(r)\left(\frac{\delta r}{\delta x}\right)^2 \quad —(3)$$

Similarly.

$$\frac{\delta^2 U}{\delta x^2} = f'(x)\frac{\delta^2 r}{\delta y^2} + f''(r)\left(\frac{\delta r}{\delta y}\right)^2 \quad —(4)$$

Adding (3) and (4)

$$\frac{\delta^2 U}{\delta x^2} + \frac{\delta^2 U}{\delta y^2} = f'(r)\left[\frac{\delta^2 r}{\delta x^2} + \frac{\delta^2 r}{\delta y^2}\right] + f''(r)\left[\left(\frac{\delta r}{\delta x}\right)^2 + \left(\frac{\delta r}{\delta y}\right)^2\right] \quad —(5)$$

$$= f'(r)\left[\frac{y^3}{r^3} + \frac{x^2}{r^3}\right] + f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right]$$

$$= \frac{1}{r}f'(r) + f''(r) \quad \#$$

$$\left[\begin{array}{c} x = r\cos\theta,\ y = r\sin\theta \\ r^2 = x^2 + y^2 \\ \frac{\delta r}{\delta x} = \frac{x}{r},\ \frac{\delta r}{\delta y} = \frac{y}{r} \\ \frac{\delta^2 r}{\delta x^2} = \frac{y^2}{r^3},\ \frac{\delta^2 r}{\delta y^2} = \frac{x^2}{r^3} \end{array}\right]$$