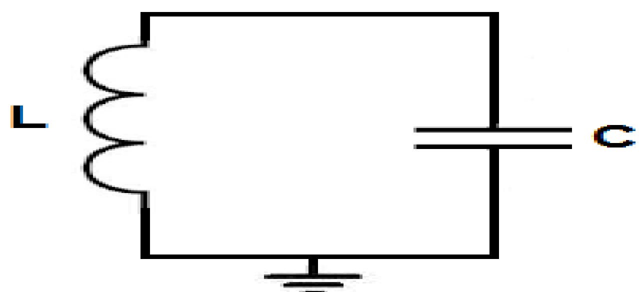
Electromagnetic oscillation

A Coupled Oscillation of the electric field and the magnetic field that constitute a single electromagnetic field is called electromagnetic oscillation. Electromagnetic oscillation propagates as electromagnetic wave consists of oscillating electric and magnetic fields. Electric and magnetic fields oscillate perpendicular to each other and propagate in the direction perpendicular to both.

L-C oscillation (Free Oscillation):-

The electric current and the charge on the capacitor in the circuit undergo electrical LC oscillations when a charged capacitor is connected to an inductor. The electrical energy stored in the capacitor is its initial charge which is maximum.



The inductor contains zero energy. When the switch is turned on, the current in the circuit starts increases and the charge on the capacitor keeps decreasing. The current induced in the circuit produces a magnetic field in the inductor.

The total energy at that instant in an oscillating L-C circuit is the sum of electric and magnetic energy.

$$i.e. U = U_{electric} + U_{magnetic}$$

$$i.e. U = \frac{q^2}{2C} + \frac{LI^2}{2} \dots\dots\dots (1)$$

Since, the circuit has no resistance which indicate, there is no energy loss and the total energy is constant with time.

$$\therefore \frac{dU}{dt} = 0 \dots\dots\dots (2)$$

$$or, \quad \frac{d}{dt}\left(\frac{q^2}{2C} + \frac{LI^2}{2}\right) = 0$$

$$or, \quad \frac{2q}{2C}\frac{dq}{dt} + \frac{2LI}{2}\frac{dI}{dt} = 0$$

$$or, \quad \frac{q}{C}.I + LI\frac{dI}{dt} = 0$$

$$or, \quad \frac{q}{C} + L\frac{dI}{dt} = 0$$

$$or, \quad \frac{q}{C} + L\frac{d}{dt}(\frac{dq}{dt}) = 0$$

$$or, \quad \frac{q}{C} + L\frac{d^2q}{dt^2} = 0$$

$$or, \quad \frac{d^2q}{dt^2} + \frac{q}{LC} = 0 \dots\dots\dots (3)$$

This is the differential equation that, describes the oscillation of resistance less L-C circuit. The solution of this equation is;

$$q = q_0 \sin(wt + \emptyset)$$

Now, comparing this equation with standard differential equation of SHM;

$$i.e. \quad \frac{d^2y}{dt^2} + w^2 y = 0$$

$$\therefore w = \frac{1}{\sqrt{LC}}$$

$$\therefore f = \frac{1}{2\pi\sqrt{LC}} \dots\dots\dots (4)$$

Hence an LC circuits oscillate with constant frequency given by, above equation.

Oscillation of Electric and magnetic energy in an LC circuit:-

The energy stored as the electric field in the capacitor at any time t is;

$$U_E = \frac{q^2}{2C} = \frac{[q_0 \sin(wt + \emptyset)]^2}{2C} = \frac{q_0^2}{2C}\, sin^2(wt + \emptyset)$$

$$(U_E)\, max = \frac{q_0^2}{2C} \dots\dots\dots (5)$$

Similarly, the energy store as magnetic field in the inductor at the same instant is,

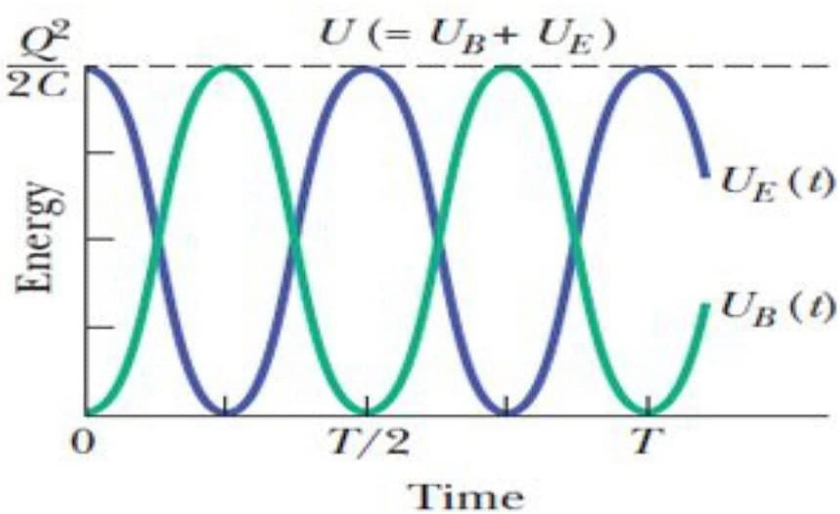
$$U_B = \frac{LI^2}{2} = \frac{1}{2}L(\frac{dq}{dt})^2 = \frac{L}{2}\ [wq_0 \cos(wt + \emptyset)]^2$$

$$U_B = \frac{q_0^2}{2}.L.w^2\ cos^2(wt + \emptyset) = \frac{q_0^2}{2}.L.\frac{1}{LC}\ cos^2(wt + \emptyset)$$

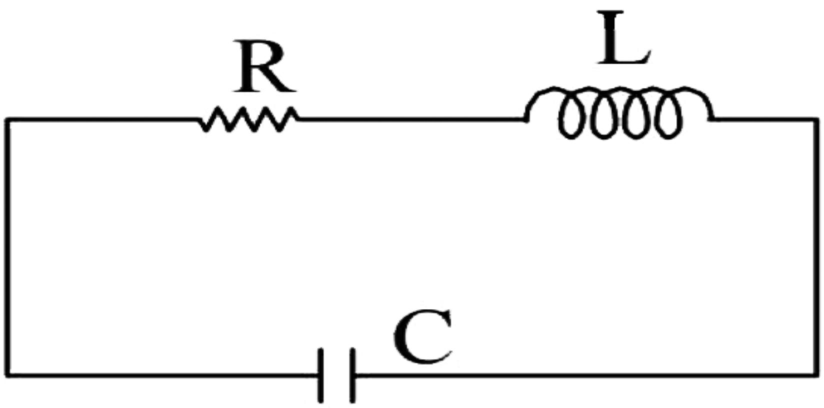
$$U_B = \frac{q_0^2}{2C}\ cos^2(wt + \emptyset)$$

$$(U_B)max = \frac{q_0^2}{2C} \dots\dots\dots.(6)$$

From equation (5) and (6) we can say that maximum value of electric and magnetic energy stored in LC circuit is equal and at any instant the sum of electric and magnetic energy is equal to $\frac{q_0^2}{2C}$ a constant.



L-C oscillation with resistance (Damped oscillation):-



Consider a circuit consists of an inductor with inductance 'L', a capacitance with capacitor 'C' and resistor of resistance 'R' as shown in figure. Consider a capacitor is given a charge Q_o initially, due to the presence of resistance a part of energy is converted in to a thermal energy. Because of this loss of energy the oscillation of charge, current and potential difference continuously decreases in amplitude. Such oscillations are called damped oscillation.

Now, Using Kirchhoff's voltage law;

$$L\frac{dI}{dt} + IR + \frac{Q}{C} = 0$$

$$or, \qquad L\frac{d}{dt}\left(\frac{dQ}{dt}\right) + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

$$or, \qquad L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \ldots\ldots\ldots(1)$$

This is the differential equation for damped LCR oscillation.

Comparing this equation with differential equation for mechanical damped harmonic oscillator equation;

$$or, \qquad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \ldots\ldots\ldots (2)$$

We get;

$$m = L, \qquad b = R, \qquad K = 1/C$$

Therefore, the angular frequency of damped oscillation will be

$$w = \sqrt{\frac{K}{m} - \frac{b^2}{4m^2}} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \ldots\ldots\ldots (3)$$

The solution of equation (2) is

$$x = Ae^{-bt/2m}\sin(wt + \emptyset)$$

Therefore the solution of equation (1) is given by

$$Q = Q_0 e^{-Rt/2L}\sin(wt + \emptyset) \ldots\ldots\ldots (4)$$

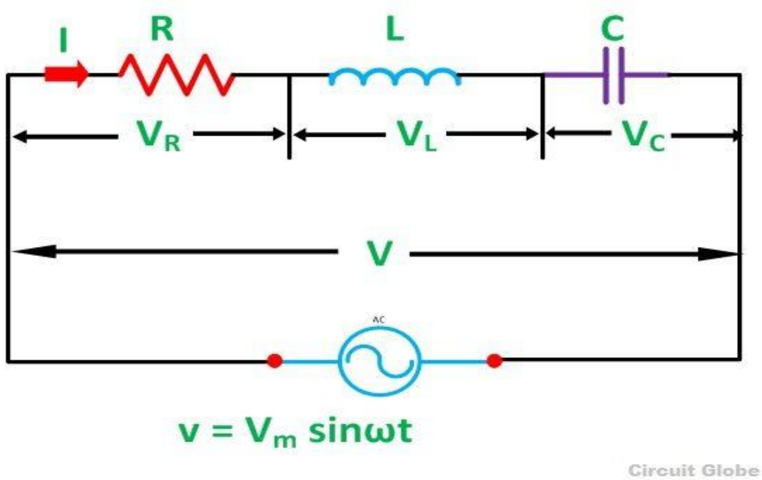
Therefore, oscillation of damped (LCR) oscillation is a sinusoidal oscillation with an exponentially decaying amplitude $Q_0 e^{-Rt/2L}$.

Equation (3) shows that the angular frequency of the damped oscillation is always less than the angular frequency of un-damped oscillation.

From equation (3),

1. When $\frac{1}{LC} > \frac{R^2}{4L^2}$, w is positive, the discharge of capacitor is oscillatory (**under damped**).
2. When $\frac{1}{LC} = \frac{R^2}{4L^2}$, w is zero, the discharge is non-oscillatory (**critically damped**).
3. When $\frac{1}{LC} < \frac{R^2}{4L^2}$, w is negative, the discharge is non-oscillatory (**over damped**).

Forced Electromagnetic oscillation:-



Since, there is always some resistance present in an electric circuit. So some energy is converted in to heat energy causing damped oscillation. Therefore, the periodic input of power is given to bring it to behave as an oscillatory circuit. Such oscillations are called forced electromagnetic oscillation. Figure sows LCR series circuit with an AC frequency generator of emf.

$$V = V_0 sinwt$$

Now, Using KVL;

$$L\frac{dI}{dt} + IR + \frac{Q}{C} = V_0 sinwt$$

$$or, \qquad L\frac{d}{dt}\left(\frac{dQ}{dt}\right) + R\frac{dQ}{dt} + \frac{Q}{C} = V_0 sinwt$$

$$or, \qquad L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_0 sinwt \dots\dots\dots (1)$$

This is the differential equation for forced LCR oscillation.

Comparing this equation with differential equation of force mechanical oscillation;

$$or, \qquad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + Kx = F_0 sinwt \dots\dots\dots (2) we\ get,$$

$$m = L, \qquad b = R, \qquad K = \frac{1}{C}, \quad F_0 = V_0$$

The solution of equation (2) is;

$$x = A\sin(wt + \emptyset)$$

Then, solution of equation (1) becomes;

$$Q = Q_0 \sin(wt + \emptyset) \dots\dots\dots (3)$$

$$\text{Since, } A = \frac{F_0/m}{\sqrt{(w_0{}^2 - w^2)^2 + (\frac{bw}{m})^2}}$$

$$\therefore Q_0 = \frac{V_0/L}{\sqrt{(w_0{}^2 - w^2)^2 + (\frac{Rw}{L})^2}}$$

Differentiating equation (3) w.r.t. time. Current is given by

$$I = \frac{dQ}{dt} = Q_0 w \cos(wt + \emptyset)$$

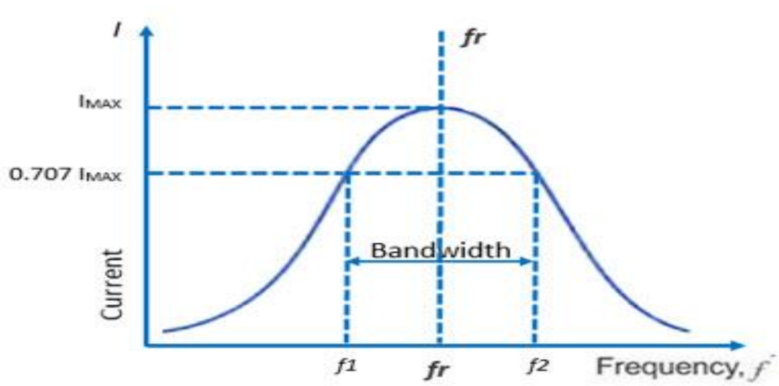
This can be written as, $I = I_0 \cos(wt + \emptyset)$

Where, $I_0 = Q_0 w = \frac{V_0}{Z} \dots \dots \dots (4)$

Here, $Z = \sqrt{R^2 + (X_L - X_c)^2}$ is the impedance of LCR circuit.

The inductive reactance, $X_L = Lw = 2\pi f L$ and the capacitive reactance, $X_C = \frac{1}{Cw} = \frac{1}{2\pi f C}$.

Resonance:-



Bandwidth of a Series Resonant Circuit

When the applied frequency is equal to natural frequency the amplitude of current in the circuit becomes maximum. Such condition is called resonance. The frequency at which amplitude of current becomes maximum is called resonance frequency.

From equation (4) it is seen that, the current in the circuit will be maximum when Z is minimum, and Z is minimum when $X_L = X_C$.

$$\Rightarrow wL = \frac{1}{wC}$$

$$\Rightarrow w = \frac{1}{\sqrt{LC}} \quad \therefore 2\pi f = \frac{1}{\sqrt{LC}}$$

Hence, Resonance frequency, $f = \frac{1}{2\pi\sqrt{LC}}$

$$\text{At resonance}, w = w_0 = \frac{1}{\sqrt{LC}}$$

$$\therefore \ f_r = \frac{1}{2\pi\sqrt{LC}}$$

Quality Factor:-

It is defined as the ratio of the voltage drop across inductor (L) or across capacitor (C) to the voltage drop across resistor (R) at resonance.

$$i.e. Q = \frac{V_L or\ V_c}{V_R}$$

It is also defined as the ratio of the energy in the inductor or capacitor to the energy dissipated across the resistance.

$$i.e. Q = \frac{Energy\ stored}{Energy\ dissipated}$$

The quality factor is also defined in terms of lower and upper half part of frequencies f_1 and f_2 these frequencies lying below and above the resonance frequency. Where, the power dissipation is half of that at resonance frequency. At these frequencies current in the circuit is $\frac{I_r}{\sqrt{2}}$ where I_r is current at resonance frequency.

$$i.e. Q = \frac{2\pi f_r}{f_2 - f_1}$$

Numerical Examples:-

1. **A radio tuner has a frequency range from 500 KHz to 5 MHz. If its LC circuit has an effective inductance of 400 μH. What is the range of its variable capacitor?**

Solution:-

$$F_1 = 500 \text{ KHz} = 500 \times 10^3\ Hz$$

$$F_2 = 5 \text{ MHz} = 5 \times 10^6\ Hz$$

$$L = 400\ \mu H = 400 \times 10^{-6}\ H$$

$C = ?$

We have; $F_1 = \dfrac{1}{2\pi\sqrt{LC}}$

$$or, C = \frac{1}{4\pi L F_1^2} = \frac{1}{4\pi \times 400 \times 10^{-6} \times (500 \times 10^3)^2}$$

$\therefore C = 2.535 \times 10^{-10}\ F$

Also; $F_2 = \dfrac{1}{2\pi\sqrt{LC}}$

$$or, C = \frac{1}{4\pi^2 L F_2^2} = \frac{1}{4\pi^2 \times 400 \times 10^{-6} \times (5 \times 10^6)^2}$$

$\therefore C = 2.535 \times 10^{-12}\ F$

$\therefore$ Range of variable capacitor is $2.535 \times 10^{-12}\ F\ to\ 2.535 \times 10^{-10}\ F$.

2. A circuit has $L = 10\ mH$ and Capacitance $C = 10\ \mu F$. How much resistance should be added to circuit so that frequency of oscillation will be 1 % less than that of LC oscillation?

Solution:-

$L = 10\ mH = 10 \times 10^{-3}\ H$

$C = 10\ \mu F = 10 \times 10^{-6}\ F$

We have; $F_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{10 \times 10^{-3} \times 10 \times 10^{-6}}}$

$= 503.29\ Hz$

From question; $F = F_0 - 1\%$ of F_0

$= 503.29 - \frac{503.29}{100} = 498.51\ Hz$

Then, $F = F_0 - 1\%$ of F_0

$$or, \quad \frac{1}{2\pi}\sqrt{\frac{1}{LC} - (\frac{R}{2L})^2} = 498.51$$

$$or, \quad \sqrt{\frac{1}{LC} - (\frac{R}{2L})^2} = 3130.64$$

$$or, \quad \frac{1}{LC} - (\frac{R}{2L})^2 = 9800924.341$$

$$or, \quad \frac{1}{10 \times 10^{-3} \times 10 \times 10^{-6}} - 9800924.341 = \frac{R^2}{4 \times (10 \times 10^{-3})^2}$$

$$or, \quad R^2 = 79.63$$

$\therefore\ R = 8.92\ \Omega$

3. If 10 mH inductor and two capacitor of 5 μF and 2 μF are given. Find the two resonance frequencies that can be obtained by connecting these elements in different ways.

Solution:-

$$\mathrm{L} = 10\ \mathrm{mH} = 10\ \times 10^{-3}\ \mathrm{H}$$

$$\mathrm{C_1} = 5\ \mu\mathrm{F} = 5 \times 10^{-6}\ \mathrm{F}$$

$$\mathrm{C_2} = 2\ \mu\mathrm{F} = 2 \times 10^{-6}\ \mathrm{F}$$

For series combination;

$$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$= \frac{1}{5 \times 10^{-6}} + \frac{1}{2 \times 10^{-6}} = 700000$$

$$\therefore C_s = 1.42 \times 10^{-6}\ F$$

$$Then;\ \mathrm{F_r} = \frac{1}{2\pi\sqrt{\mathrm{LC}}} = \frac{1}{2\pi\sqrt{10\ \times 10^{-3} \times 1.42 \times 10^{-6}}}$$

$$\therefore\ \mathrm{F_r} = 1335.599\ \mathrm{Hz}$$

For parallel combination;

$$C_p = C_1 + C_2 = 5 \times 10^{-6} + 2 \times 10^{-6} = 7 \times 10^{-6}\ F$$

$$\text{Then;}\ \mathrm{F_r} = \frac{1}{2\pi\sqrt{\mathrm{LC}}} = \frac{1}{2\pi\sqrt{10\ \times 10^{-3} \times 7 \times 10^{-6}}}$$

$\therefore\ \mathrm{F_r} = 601.55\ \mathrm{Hz}$

4. In an oscillating LC circuit, what value of charge expressed in terms of maximum charge is present on the capacitor when the energy is shared equally between the electric and magnetic fields? At what time will this condition occur, assuming the capacitor is fully charged initially? Assume that $\boldsymbol{L = 10\ mH}$ and $\boldsymbol{C = 1\ \mu F}$.

Solution:-

$Here, L = 10\ mH = 1 \times 10^{-3} H, C = 1\mu F = 1 \times 10^{-6} F$

According to question, $U_E = U_B$

Since, $U_E + U_B = U_{max}$

$$2U_E = U_{max}$$

$$U_E = \frac{1}{2} U_{max}$$

$$\frac{q^2}{2C} = \frac{1}{2}\frac{{q_0}^2}{2C}$$

$$\therefore q = \frac{q_0}{\sqrt{2}}$$

Also from question, at $t = 0, q = q_0 \Rightarrow \emptyset = 0$

$$q = q_0\, sin\, wt$$

$$\frac{q_0}{\sqrt{2}} = q_0\, sin\, wt \Rightarrow \sin wt = \frac{1}{\sqrt{2}} = sin\frac{\pi}{4}$$

$$wt = \frac{\pi}{4} \Rightarrow t = \frac{\pi}{4w} = \frac{\pi\sqrt{LC}}{4}$$

$$t = \frac{3.14 \times \sqrt{1 \times 10^{-3} \times 1 \times 10^{-6}}}{4}$$

$$\therefore t = 2.5 \times 10^{-4} sec$$

5. **What resistance 'R' should be connected in series with an inductance 220 mH and capacitance 12 μF for the maximum charge on the capacitor to decay to 99% of its initial value in 50 Cycles.**

Solution:-

$$Here, L = 220 \times 10^{-3} H, \qquad C = 12 \times 10^{-6} F$$

$$t = 50T = 50\left(\frac{2\pi}{w_0}\right) = 100\pi\sqrt{LC} = 0.5104\ sec$$

We have, $Q = Q_0 e^{-Rt/2L} \sin(wt + \emptyset)$

$$Q_{max} = Q_0 e^{-Rt/2L}, where\ Q_0 is\ charge\ at\ t = 0sec$$

$$e^{Rt/2L} = \frac{Q_0}{Q_{max}}$$

$$R = \frac{2L}{t}\ \ln(\frac{Q_0}{Q_{max}})$$

From question, $Q_{max} = 99\%\ of\ Q_0 = 0.99Q_0$

$$\therefore R = \frac{2 \times 220 \times 10^{-3}}{0.5104} \ln(\frac{Q_0}{0.99Q_0})$$

$$= 8.66 \times 10^{-3}\Omega$$

6. A series LCR circuit has inductance 12 mH, Capacitance 1.6 μF and resistance 1.5 Ω i) At what time the amplitude of charge oscillation will be 50 % of its initial value ii) To how many periods of oscillation does this happen.

Solution:-

$$Here, L = 12 \times 10^{-3}H, C = 1.6 \times 10^{-6}F, R = 1.5\Omega$$

i) We have for damped LCR circuit,

$$Q = Q_0 e^{-Rt/2L} \sin(wt + \emptyset)$$

$\therefore$ Amplitude of charge osillation, $Q_m = Q_0 e^{-Rt/2L}$

From question, $Q_m = 50\% of\, Q_0 = \dfrac{Q_0}{2}$

$$\therefore \frac{Q_0}{2} = Q_0 e^{-Rt/2L} \Rightarrow e^{Rt/2L} = 2$$

$$t = \frac{2L}{R}\ln(2) = \frac{2 \times 12 \times 10^{-3} \times \ln(2)}{1.5}$$

$$\therefore t = 0.011\, sec$$

ii) Since, $T = \dfrac{2\pi}{w} = 2\pi\sqrt{LC}$

$$= 2 \times 3.14\sqrt{12 \times 10^{-3} \times 1.6 \times 10^{-6}}$$

$$= 8.7 \times 10^{-4}\, sec$$

$$\therefore \text{Number of periods} = \frac{0.011}{8.7 \times 10^{-4}} = 13$$

Exercise:-

1. Define LC oscillation qualitatively by using necessary circuit and graph.
2. What is LC oscillation? Derive the differential equation of free oscillation and compare its solution with mass spring system.
3. Prove that LC circuit is an analogy of spring mass system. Hence prove that maximum energy stored in the capacitor is equal to maximum energy stored in inductor.
4. Define sharpness of resonance. Derive the relation for current amplitude of forced e-m oscillation.
5. Compare the damped and forced LCR oscillation. Derive the differential equation of forced e-m oscillation and compare it with driven mechanical oscillation.
6. What is resonance? Formulate the differential equation of forced e-m oscillation. Then determine the expression for resonant frequency.
7. Derive the differential equation of the forced oscillation of LCR circuit with an AC source and find the expression for the current amplitude. Hence explain the condition of current resonance in such circuit.
8. LC oscillations are called e-m oscillations, why? Derive the differential equation for damped electromagnetic oscillations and find the amplitude and frequency of that oscillation.

9. Discuss about the damped electromagnetic oscillation. Find the expression for damped frequency. Also discuss about over damping, critical damping and under damping condition.
10. Derive a relation for current flopping in the circuit containing a resistor, an inductor and a capacitor in series with a sinusoidal varying emf. Find the condition for current response.
11. Obtain an expression for current in a driven LCR circuit and discuss how the current leads or lags the applied voltage in phase; (a) when the net reactance in circuit is inductive and (b) when the reactance in circuit is equal to resistance. Illustrate it with the help of a figure.
12. A 2 μF capacitor is charged up to 50 V. The battery is disconnected and 50 mH coil is connected across the capacitor so that LC oscillation to occur. Calculate the maximum value of current in the circuit.
13. A circuit has L = 1.2 mH, C = 1.6 μF and R = 1.5 Ω. (a) After what time t will the amplitude of the charge oscillation drop to one half of its initial value. (b) To how many periods does this correspond?
14. What should be the capacitance of a capacitor in a tuned circuit of frequency 10 MHz having an inductance of 0.01 mH? The resistance of the circuit is negligible?
15. A 20 mH inductor and 600 μF capacitor form an oscillating circuit. What is the peak value of current if the initial charge is 60 μC?

16. A circuit has $L = 5$ mH and $C = 2$ μF. How much resistance must be inserted in the circuit to reduce the resonance frequency by 5 %.

17. Calculate the resonating frequency and quality factor of a circuit having 0.02 μF capacitance, 8 mH inductance and 0.25 Ω resistance.