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Assignment - 3

[Ashwani kr. Chaudhary]
[019 BCT-A]

1a) $\int_0^{\pi/2} \log (a^2 \cos^2\theta + b^2 \sin^2\theta) \, d\theta$

$$I = \int_0^{\pi/2} \log (a^2\cos^2\theta + b^2\sin^2\theta) \, d\theta \quad \text{---(i)}$$

assuming $\alpha, \beta > 0$

diff. both side w. r. t. a we have

$$\frac{dI}{da} = \int_0^{\pi/2} \frac{1}{a^2\cos^2\theta + b^2\sin^2\theta} \, 2a\cos^2\theta \, d\theta$$

$$\frac{dI}{da} = 2a \int_0^{\pi/2} \frac{d\theta}{a^2 + b^2 \tan^2\theta}$$

let $\tan\theta = t$

$\sec^2\theta = dt$

also,

$\theta \rightarrow (0, \pi/2)$

we get $t \rightarrow (0, \infty)$

$$= 2a \int_0^{\infty} \frac{1}{a^2 + b^2t^2} \frac{dt}{1+t^2}$$

$$= \frac{2a}{b^2 - a^2} \int_0^{\infty} \left(\frac{b^2}{a^2+b^2t^2} - \frac{1}{1+t^2} \right) dt$$

$$= \frac{2a}{b^2 - a^2} \int_0^{\infty} \left(\frac{b^2}{a^2+b^2t^2} - \frac{1}{1+t^2} \right) dt$$

$$= \frac{2a}{b^2-a^2}\left[\frac{b^2}{a.b}\tan^{-1}\left(\frac{bt}{a}\right) - \tan^{-1} t\right]_0^{\infty}$$

$$= \frac{\cancel{2a}}{b^2-a^2}\left(\frac{b-1}{\cancel{a}}\right)\frac{\pi}{\cancel{2}}$$

$$= \frac{\pi}{a+b}.$$

again,

$$dI = \frac{\pi}{a+b}\, da$$

Integrating

$$I = \pi \log (a+b) + C \quad \text{---(ii)}$$

putting $a = b$ in eqn (i), we get

$$I = \int_0^{\pi/2} \log a^2 \, d\theta$$

$$= 2 \log a . \frac{\pi}{2} \qquad = \pi \log a$$

$$a > 0$$

Using this eqn (ii) we get

$$\pi \log a = \pi \log 2a + C$$

$$C = -\pi \log 2$$

using (ii) with this value of C

$$I = \pi \log (a+b) - \pi \log 2.$$

$$= \pi \log \frac{(a+b)}{2}$$

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b) $\int_0^\pi \log(1+a\cos x)\dfrac{dx}{\cos x}$

solⁿ let

$$I = \int_0^\pi \frac{\log(1+a\cos x)}{\cos x}\,dx$$

diff. w.r.t. a we have

$$dI = \int_0^\pi \frac{1}{1+a\cos x}\cdot\frac{\cos x\,dx}{\cos x}$$

$$= \int_0^\pi \frac{dx}{1+a\cos x} = \int_0^\pi \frac{dx}{1+a\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)}$$

$$= \int_0^\pi \frac{\sec^2 x/2\,dx}{(1+a)+(1-a)\tan^2 x/2}$$

$$= \frac{1}{1-a}\int_0^\pi \frac{\sec^2 x/2\,dx}{\frac{1+a}{1-a}+\tan^2 x/2}$$

let $\tan\frac{x}{2} = c$

then,
$$\frac{1}{2}\sec^2\frac{x}{2}\,dx = dc$$

also,

$$x \to (0,\pi)$$

we get $(c \to 0,\infty)$

diff.... w.r.t c.

$$\frac{dI}{da} = \frac{2}{1-a}\int_0^\infty \frac{dc}{\frac{1+a}{1-a}+c^2}$$

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$$= \frac{2}{1-a} \cdot \frac{1}{\sqrt{\frac{1+a}{1-a}}} \left[\tan^{-1} \frac{c}{\sqrt{\frac{1+a}{1-a}}} \right]_0^{\infty}$$

$$= \frac{2\cancel{6}}{\sqrt{1-a^2}} \cdot \frac{\pi}{2}$$

$$\frac{dI}{da} = \frac{\pi}{\sqrt{1-a^2}}$$

$$dI = \frac{\pi}{\sqrt{1-a^2}} \, da$$

Integrating

$$I = \pi \sin^{-1} a + c \quad \text{— (i)}$$

When $a = 0$

from eq^n (i), $I = 0$

So, using value in eq^n (ii)

$$I = \pi \sin^{-1} a + c$$

$$= \pi \sin^{-1} a$$ #

2. $$\int_0^{\pi/2} \frac{1}{a^2\sin^2x+b^2\cos^2x}\,dx = \frac{\pi(a^2+b^2)}{4a^3b^3}$$

⇒ Soln,

$$I = \int_0^{\pi/2} \frac{dx}{a^2\sin^2x+b^2\cos^2x}$$

$$= \int_0^{\pi/2} \frac{\sec^2x\,dx}{a^2\tan^2x+b^2}$$

$$= \frac{1}{b}\cdot\frac{b}{a}\left[\tan^{-1}\frac{a\tan x}{b}\right]_0^{\pi/2}$$

$$= \frac{1}{a.b}\left(\frac{\pi}{2}-0\right)$$

$$I = \frac{\pi}{2ab}$$

$$\int_0^{\pi/2} \frac{dx}{a^2\sin^2x+b^2\cos^2x} = \frac{\pi}{2ab} \quad \text{--- (1)}$$

Diff. both side w.r.t. 'a' we get

$$\int_0^{\pi/2} \frac{-2a\sin^2x\,dx}{(a^2\sin^2x+b^2\cos^2x)^2} = \frac{-\pi}{2a^2b}$$

$$\int_0^{\pi/2} \frac{\sin^2x\,dx}{(a^2\sin^2x+b^2\cos^2x)^2} = \frac{\pi}{4a^2b} \quad \text{--- (2)}$$

Again diff. both side (1) w.r.t. (b)

$$\int_0^{\pi/2} \frac{\cos^2x\,dx}{(a^2\sin^2x+b^2\cos^2x)^2} = \frac{\pi}{4ab^2} \quad \text{--- (3)}$$

Adding (2) & (3)

$$\int_0^{\pi/2} \frac{dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{\pi}{4a^2b} + \frac{\pi}{4ab^2}$$

$$= \frac{\pi(a^2+b^2)}{4a^3b^3}$$ #

(3) Evaluate $\int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx$

:- Sol^n,

let $I = \int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)}$ —(1)

diff. w.r.t. sig^n of integration, we get

$$\frac{dI}{da} = \int_0^{\infty} \frac{1}{x(1+x^2)} \cdot \frac{x}{(1+a^2x^2)}\,dx$$

$$= \int_0^{\infty} \frac{1}{(1+x^2)(1+a^2x^2)}\,dx$$

$$= \frac{1}{1-a^2} \int_0^{\infty} \left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right] dx$$

$$= \frac{1}{1-a^2} \left[\tan^{-1}x - a\tan^{-1}(ax)\right]_0^{\infty}$$

$$= \frac{1}{1-a^2} \left[\frac{\pi}{2} - \frac{a\pi}{2}\right]$$

$$= \frac{\pi}{2} \frac{1-a}{1-a^2}$$

$$= \frac{\pi}{2} \frac{1}{1+a}$$

$$\frac{dI}{da} = \frac{\pi}{2} \cdot \frac{1}{1+a}$$

integrating both side w.r.t. (a) we get

$$I = \frac{\pi}{2} \log (a+1) + C \quad -(2)$$

from (1) when $a = 0$

$I = 0$

using $I = 0$, when $a = 0$ in (2)

$$0 = \frac{\pi}{2} \log (1+0) + C$$

$C = 0$

Hence, eqⁿ(2) gives

$$I = \frac{\pi}{2} \log (a+1).$$

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4)(a) $\cancel{E(v)}\int_1^{\infty} \frac{x\,dx}{(1+x^2)^2}$

$= \int_1^{\infty} \frac{x\,dx}{(1+x^2)^2}$

$= \lim_{b\to\infty} \int_1^{b} \frac{x\,dx}{(1+x^2)^2}$

$= \lim_{b\to\infty} \frac{1}{2}\left[\frac{1}{1+x^2}\right]_1^b$

$= \frac{-1}{2} \lim_{b\to\infty} \frac{1}{1+b^2} + \frac{1}{4}$

$= -0 + \frac{1}{4}$

$= \frac{1}{4}$ #

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b) $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx$

Her,

$$= \int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx$$

Since 0 is an interior point of $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx$$

$$= \int_{\infty}^{0} \frac{e^x dx}{1+e^{2x}} + \int_{0}^{\infty} \frac{e^x dx}{1+e^{2x}}$$

$$= \lim_{a\to\infty} \int_{-a}^{0} \frac{e^x}{1+(e^{2x})^2} dx + \lim \int_{0}^{b} \frac{e^x dx}{1+(e^{2x})^2}$$

but $e^x = t$

$e^x dx = dt$

so, $\int \frac{dt}{1+t} = \tan^{-1} t$

$= \tan^{-1}(e^x)$

$$= \lim_{a\to\infty} [\tan^{-1}(e^x)]_{-a}^{0} + \lim [\tan^{-1}(e^x)]_{0}^{b}$$

$$= \lim_{a\to\infty} \left[\frac{\pi}{4} - \tan^{-1}(e^{-a})\right] + \lim_{b\to\infty} \left[\tan^{-1} e^b - \frac{\pi}{4}\right]$$

$$= \cancel{\frac{\pi}{4}} - 0 + \frac{\pi}{2} - \cancel{\frac{\pi}{4}}$$

$$= \frac{\pi}{2} \#$$