Aayush Bagnet
BCT A Roll no. 004

First order and First Degree Differential Equation

15.4 Homogeneous Differential Equation of First Order

Some questions

(a) $\frac{dy}{dx} = \frac{x+y}{x}$

soln: Here, $\frac{dy}{dx} = \frac{x+y}{x}$

This is homogeneous differential equation so,

let,

$$y = vx$$

Diff wrt x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then,

$$v + x\frac{dv}{dx} = \frac{x+vx}{x}$$

or, $v + x\frac{dv}{dx} = 1+v$

or, $dv = \frac{dx}{x}$

Integrating both sides

$v = \log x + c$

Now $v = \frac{y}{x}$

or, $\frac{y}{x} = \log x + c$

$y = x(\log x + c)$.

(b) $\frac{dy}{dx} = \frac{2x+y}{x}$

soln: Here

Given, $\frac{dy}{dx} = \frac{2x+y}{x}$

This is homogeneous differential eq^n

so put $y = vx$

Differentiating w.r.t. x,

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now,

$v + x\frac{dv}{dx} = \frac{2x + vx}{x}$

or, $v + x\frac{dv}{dx} = (2+v)$

or, $x\frac{dv}{dx} = 2$

or, $dv = 2\frac{dx}{x}$

Integrating both sides,

$\int dv = 2\int \frac{dx}{x}$

or, $v = 2\log(x) + c$

Put $v = y/x$

or, $\frac{y}{x} \; 2\log(x) + c$

$y = x(\log x^2 + c)$,

(c) $\frac{dy}{dx} = \frac{2y-x}{x}$

solu^n Here, $\frac{dy}{dx} = \frac{2y-x}{x}$

This is homogeneous differential equation,

so, put $y = vx$

Diff w.r.t x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then,

$$v + x\frac{dv}{dx} = \frac{2vx - x}{x}$$

or, $$v + x\frac{dv}{dx} = 2v - 1$$

or, $$x\frac{dv}{dx} = v - 1$$

or, $$\frac{dv}{v-1} = \frac{dx}{x}$$

Integrating both sides,

$$\int \frac{dv}{v-1} = \int \frac{dx}{x}$$

or, $$\log(v-1) = \log(x) + c$$

or, put $v = y/x$

$$\log\left(\frac{y}{x} - 1\right) = \log(x) + \log(c)$$

or, $$\log\left(\frac{y-x}{x}\right) = \log(xc)$$

or, $$\frac{y-x}{x} = xc$$ or $$y - x = x^2c$$

or, $$y = x^2c + x$$

$$\therefore y = x(xc+1)$$

(d) $$\frac{dy}{dx} = \frac{xy}{x^2+y^2}$$

soln. Here, $$\frac{dy}{dx} = \frac{xy}{x^2+y^2}$$

This is homogeneous differential equation, so put $y = vx$, then

Diff w.r.t x,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}, \text{ then,}$$

we have,

$$v + x\frac{dv}{dx} = \frac{x \times vx}{x^2 + v^2x^2}$$

or, $$v + x\frac{dv}{dx} = \frac{x^2 v}{x^2(1+v^2)}$$

or, $$v + x\frac{dv}{dx} = \frac{v}{1+v^2}$$

or, $$x\frac{dv}{dx} = \frac{v}{1+v^2} - v$$

or, $$x\frac{dv}{dx} = \frac{v - v(1+v^2)}{1+v^2}$$

or, $$x\frac{dv}{dx} = \frac{v - v - v^3}{1+v^2}$$

or, $$x\frac{dv}{dx} = \frac{-v^3}{1+v^2}$$

or, $$\frac{(1+v^2)dv}{v^3} = -\frac{dx}{x}$$

Integrating both sides

$$\int \frac{1+v^2}{v^3}\,dv = -\int \frac{dx}{x}$$

or, $$\int \frac{1}{v^3}\,dv + \int \frac{1}{v}\,dv = -\log x + c$$

or, $$-\frac{1}{2v^2} + \log v = -\log x + c$$

Put $v = y/x$

$$-\frac{1}{2\frac{y^2}{x^2}} + \log\left(\frac{y}{x}\right) = -\log x + c$$

or, $$\log\left(\frac{y}{x}\right) + \log(x) = \frac{x^2}{2y^2} + c$$

or, $$\log\left(\frac{y}{x} \times x\right) = \frac{x^2}{2y^2} + c$$ $$\log(y) = \frac{x^2}{2y^2} + c$$

(e) $\frac{dy}{dx} = \frac{2x+y}{x}$

solⁿ: Here, $\frac{dy}{dx} = \frac{2x+y}{x}$

The given eqⁿ is homogeneous differential eqⁿ
so, put $y = vx$ then,
Diff. w.r.t. x,
$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then,
$v + x\frac{dv}{dx} = \frac{2x+vx}{x}$

or, $v + x\frac{dv}{dx} = 2+v$

or, $x\,dv = 2dx$

or, $dv = \frac{2dx}{x}$

Integrating both sides
$\int dv = 2\int \frac{dx}{x}$

or, $v = 2\log x + c$

or, $\frac{y}{x} = \log x^2 + c$

$y = x(\log x^2 + c)$

(f) $2xy\frac{dy}{dx} = x^2+y^2$

solⁿ: Here, $2xy\frac{dy}{dx} = x^2+y^2$

or, $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

It is homogeneous differential equation

so, $y = vx$

Differentiating w.r.t. x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then,

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x^2v}$$

or, $$v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

or, $$x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

or, $$x\frac{dv}{dx} = \frac{1+v^2-2v^2}{2v}$$

or, $$\frac{2v\,dv}{1-v^2} = \frac{dx}{x}$$

or, $$-\int \frac{-2v\,dv}{1-v^2} = \int \frac{dx}{x}$$

or, $-\log(1-v^2) = \log(x) + c$

or, $-\log(1-v^2) = \log x + \log c$

or, $\log(1-v^2)^{-1} = \log(xc)$

or, $\cancel{x^2 - x = xc}$ or, $\frac{1}{1-v^2} = xc$

or, $\cancel{\frac{y^2}{x^2} - 1 = xc}$

or, $$\frac{1}{1-\frac{y^2}{x^2}} = xc$$

or, $$\frac{x^2}{x^2-y^2} = xc$$

or, $x = c(x^2-y^2)$ //

(g) $$\frac{dy}{dx} = \frac{x+y}{x-y}$$

soln. Here, $$\frac{dy}{dx} = \frac{x+y}{x-y}$$

It is homogeneous differential equation.

So, put $y = vx$

Differentiating w.r.t. x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then,

$$v + x\frac{dv}{dx} = \frac{x+vx}{x-vx}$$

or, $$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

or, $$x\frac{dv}{dx} = \frac{1+v}{1-v} - v$$

or, $$x\frac{dv}{dx} = \frac{1+v-(1-v)v}{1-v}$$

or, $$x\frac{dv}{dx} = \frac{1+v-v+v^2}{v}$$

or, $$\frac{1-v}{1+v^2}dv = \frac{dx}{x}$$

or, $$\frac{dv}{1+v^2} - \frac{v}{1+v^2}dv = \frac{dx}{x}$$

or, Integrating both sides

$$\int\frac{dv}{1+v^2} - \frac{1}{2}\int\frac{2v}{1+v^2}dv = \int\frac{1}{x}dx$$

or, $$\tan^{-1}v - \frac{1}{2}\log(1+v^2) = \log x + c$$

or, $$\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left(1+\frac{y^2}{x^2}\right) = \log x + c$$

(h) $$\frac{dy}{dx} = \frac{y}{x} - \sin^2\frac{y}{x}$$

Soln: Here $$\frac{dy}{dx} = \frac{y}{x} - \sin^2\frac{y}{x}$$

It is homogeneous differential equation

So, put $y = vx$

Diff w.r.t x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then

$$v + x\frac{dv}{dx} = \frac{vx}{x} - \sin^2\frac{vx}{x}$$

or, $$v + x\frac{dv}{dx} = v - \sin^2 v$$

or, $$x\frac{dv}{dx} = -\sin^2 v$$

or, $$\frac{dv}{-\sin^2 v} = \frac{dx}{x}$$

or, $$-\operatorname{cosec}^2 v \, dv = \frac{dx}{x}$$

Integrating both sides

$$\int -\operatorname{cosec}^2 v \, dv = \int \frac{dx}{x}$$

or, $$\cot v = \log x + c$$

or $$\cot\left(\frac{y}{x}\right) = \log x + c$$

Ex:-21

Solve the following differential equations

1. $x + y\frac{dy}{dx} = 2y$

Solu^n: Here, $x + y\frac{dy}{dx} = 2y$

or, $$y\frac{dy}{dx} = 2y - x$$

or $$\frac{dy}{dx} = \frac{2y - x}{y}$$

It is homogeneous differential equation so,

put $y = vx$, then
Diff w.r.t x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then

$$v + x\frac{dv}{dx} = \frac{2vx - x}{vx}$$

or, $$v + x\frac{dv}{dx} = \frac{(2v-1)x}{vx}$$

or, $$v + x\frac{dv}{dx} = \frac{2v-1}{v}$$

or, $$x\frac{dv}{dx} = \frac{2v-1}{v} - v$$

or, $$x\frac{dv}{dx} = \frac{2v-1-v^2}{v}$$

or, $$\frac{v\,dv}{-(v^2-2v+1)} = \frac{dx}{x}$$

or, $$\frac{v\,dv}{(v^2-2v+1)} = -\frac{1}{x}dx$$

or, Integrating both sides

$$\int \frac{v\,dv}{(v^2-2v+1)} = -\int \frac{1}{x}dx$$

or, $$\int \frac{1}{2}\frac{(2v-2+2)}{(v^2-2v+1)}dv = -\log(x) + c$$

or, $$\frac{1}{2}\left[\int \frac{2v-2}{(v^2-2v+1)}dv + 2\int \frac{1}{(v-1)^2}dv\right] = -\log(x) + c$$

or, $$\frac{1}{2}\left[\log(v^2-2v+1) + 2\int \frac{1}{(v-1)(v-1)}dv\right] = -\log(x) + c$$

or, $$\frac{1}{2}\left[\log(v-1)^2 + 2\int (v-1)^{-2}dv\right] = -\log(x) + c$$

or, $$\frac{1}{2}\left[2\log(v-1) + 2\frac{(v-1)^{-2+1}}{-2+1}\right] = -\log(x) + c$$

or, $$\frac{1}{2}\left[2\log(v-1) - 2(v-1)^{-1}\right] = -\log(x) + c$$

or, $\log(v-1) + \log x = \frac{1}{(v-1)} + c$

Put $v = y/x$

$\log\left(\frac{y}{x} - 1\right) + \log x = \frac{1}{\left(\frac{y}{x} - 1\right)} + c$

or, $\log\left(\frac{y-x}{x} \times x\right) = \frac{x}{y-x} + c$

$\therefore \log(y-x) = \frac{x}{y-x} + c_{//}$

2. $(x^2 - y^2)\,dx + 2xy\,dy = 0$

soln: Here, $(x^2-y^2)\,dx + 2xy\,dy = 0$

or, $2xy\,dy = -(x^2-y^2)\,dx$

or, $\frac{dy}{dx} = \frac{-(x^2-y^2)}{2xy}$

It is homogeneous differential eqn. so,

Put $y = vx$

Diff. w.r.t. x we get

$\frac{dy}{dx} = v + x\frac{dv}{dx}$, then,

we have,

$v + x\frac{dv}{dx} = -\frac{(x^2 - v^2x^2)}{2x \times vx}$

or, $v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x^2 v}$

or, $v + x\frac{dv}{dx} = \frac{x^2(v^2-1)}{2x^2 v}$

or, $x\frac{dv}{dx} = \frac{v^2-1}{2v} - v$

or, $x\frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v}$

or, $\frac{2v\,dv}{-(v^2+1)} = \frac{dx}{x}$

or, $\frac{2v\,dv}{v^2+1} = -\frac{1}{x}dx$

Integrating both sides,

$\int \frac{2v\,dv}{v^2+1} = -\int \frac{1}{x}dx$

or, $\log(v^2+1) = -\log(x) + \log(c)$

or, $\log(v^2+1) = \log\left(\frac{c}{x}\right)$

Put $v = y/x$,

$\log\left(\frac{y^2}{x^2}+1\right) = \log\left(\frac{c}{x}\right)$

or, $\frac{y^2+x^2}{x^2} = \frac{c}{x}$

$\therefore x^2+y^2 = cx$ //

3. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

soln: Here, $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

or, $\frac{dy}{dx} = \frac{y^2}{x^2} - \frac{y}{x}$

or, $\frac{dy}{dx} = \frac{y^2-xy}{x^2}$

It is homogeneous differential equation,

put $y = vx$, then,

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then we have,

$v + x\frac{dv}{dx} = \frac{v^2x^2 - x \times vx}{x^2}$

or, $v + x\frac{dv}{dx} = \frac{x^2(v^2-v)}{x^2}$

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or, $v + x\frac{dv}{dx} = v^2 - v$

or, $x\frac{dv}{dx} = v^2 - 2v$

or, $\frac{dv}{v(v-2)} = \frac{dx}{x}$

Integrating both sides,

$\int \frac{dv}{v(v-2)} = \int \frac{dx}{x}$

using partial fraction on left side,

$\int -\frac{1}{2(v)} + \frac{1}{2(v-2)} \, dv = \log x + c$

or, $-\frac{1}{2}\left[\int \frac{1}{(v-2)} dv - \int \frac{1}{v} dv\right] = \log x + c$

or, $\frac{1}{2}\left[\log(v-2) - \log v\right] = \log x + \log c$

or, $\frac{1}{2}\left[\log\left(\frac{v-2}{v}\right)\right] = \log(xc)$

or, $\cancel{\frac{1}{2}} \log\left(\frac{v-2}{v}\right)^{1/2} = \log(xc)$

or, $\frac{v-2}{v} = x^2 c$ $\quad [c^2 = c \text{ (constant)}]$

Put $v = y/x$

or, $\frac{\frac{y}{x} - 2}{\frac{y}{x}} = x^2 c$

or, $\frac{y - 2x}{y} = x^2 c$

or, $y - 2x = cx^2y$ //

4. $(x^2+y^2)\,dx = (x^2 + xy)\,dy$

soln Here,

$(x^2+y^2)\,dx = (x^2+xy)\,dy$

or, $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{x^2+xy}$

It is homogeneous equation so,

put $y = vx$

or, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now,

$v + x\dfrac{dv}{dx} = \dfrac{x^2+v^2x^2}{x^2+x.vx}$

or, $v + x\dfrac{dv}{dx} = \dfrac{x^2(1+v^2)}{x^2(1+v)}$

or, $x\dfrac{dv}{dx} = \dfrac{1+v^2 - v(1+v^2)}{1+v}$

or, $x\dfrac{dv}{dx} = \dfrac{1+v^2-v-v^2}{1+v}$

or, $\dfrac{(1+v)\,dv}{(1-v)} = \dfrac{dx}{x}$

or $\displaystyle\int \frac{1+v}{1-v}\,dv = \int \frac{dx}{x}$

or, $\cancel{\displaystyle\int \frac{dv}{1-v} + \int \frac{v}{1-v}\,dv = \log(x)+c}$ or, $\displaystyle\int -1 + \frac{2}{1-v}\,dv = \log x + c$

or, $\cancel{-\log(1-v)}$ or, $\displaystyle\int -1\,dv + 2\int \frac{1}{1-v}\,dv = \log x + c$

or, $-v - 2\log(1-v) = \log(xc)$

or, $-v = \log(xc) + \log(1-v)^2$

or, $-v = \log(xc(1-v)^2)$

Restoring x and y,

$e^{-y/x} = xc\left(\dfrac{x-y}{x}\right)^2$

$e^{-y/x} = \dfrac{c(x-y)^2}{x}$

$\therefore (x-y)^2 = cxe^{-y/x}$

5. $(x-y)x\,dy = y(x+y)dx$

Soln: Here, $(x-y)x\,dy = y(x+y)dx$

$$\frac{dy}{dx} = \frac{y(x+y)}{x(x-y)}$$

This is homogeneous equation.

put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now,

$$v + x\frac{dv}{dx} = \frac{vx(x+vx)}{x(x-vx)}$$

or, $$v + x\frac{dv}{dx} = \frac{vx^2(1+v)}{x^2(1-v)}$$

or, $$x\frac{dv}{dx} = \frac{v(1+v) - v(1-v)}{(1-v)}$$

or, $$x\frac{dv}{dx} = \frac{v(1+v-1+v)}{1-v}$$

or, $$x\frac{dv}{dx} = \frac{v \times 2v}{1-v}$$

or, $$\frac{1-v}{2v^2}dv = \frac{dx}{x}$$

or, $$\frac{1}{2}\int \frac{1-v}{v^2}dv = \int \frac{dx}{x}$$

or, $$\frac{1}{2}\left[\int \frac{1}{v^2}dv - \int \frac{1}{v}dv\right] = \int \log(x) + c$$

or, $$\frac{1}{2}\left[-\frac{1}{v} - \log v\right] = \log(x) + c$$

or, $\cancel{-\frac{1}{2v}} - \cancel{\frac{1}{2}\log v} = \cancel{\log(xc)}$ or, $-\frac{1}{v} - \log(v) = 2\log(xc)$

or, $\cancel{-\frac{1}{2v}} \cancel{=} \cancel{\log(xc)} + \cancel{\log v^{1/2}}$ or, $-\frac{1}{v} = (\log x^2 c) + \log v$

or, $-\frac{x}{y} = \log\left(x^2 c \times \frac{y}{x}\right)$

or, $-\frac{x}{y} = \log(xcy)$

$\therefore$ $xcy = e^{-x/y}$

6. $(x^2+y^2)\,dy = xy\,dx$

Soln. Here,

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2}$$

This is homogeneous equation so put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now,

$$v + x\frac{dv}{dx} = \frac{x^2 v}{x^2+x^2v^2}$$

or, $v + x\frac{dv}{dx} = \frac{x^2 v}{x^2(1+v^2)}$

or, $x\frac{dv}{dx} = \frac{v}{1+v^2} - v$

or, $x\frac{dv}{dx} = \frac{v - v - v^3}{1+v^2}$

or, $\frac{1+v^2}{v^3}\,dv = -\frac{dx}{x}$

or, $\frac{1}{v^3}\,dv + \frac{1}{v}\,dv = -\frac{dx}{x}$

Integrating both sides.

$$\int \frac{1}{v^3}\,dv + \int \frac{1}{v}\,dv = -\int \frac{dx}{x}$$

$$-\frac{1}{2v^2} + \log v = -\log x + c$$

or, $$\frac{1}{2v^2} + \log v = -\log\left(\frac{c}{x}\right)$$

or, $$\frac{1}{2v^2} = \log\left(\frac{c}{x}\right) - \log v$$

or, $$\frac{1}{2v^2} = \log\left(\frac{c}{xv}\right)$$

Putting $v = \frac{y}{x}$

or, $$\frac{x^2}{2y^2} = \log\left(\frac{c}{x \times y/x}\right)$$

or, $$e^{x^2/2y^2} = \frac{c}{y}.$$

7. $x^2 dy + y(x+y)\,dx = 0$

Soln Here,

$$x^2 dy + (xy + y^2)dx = 0$$

or, $$x^2 dy = (-xy - y^2)dx$$

or, $$x^2\frac{dy}{dx} = -xy - y^2$$

or, $$\frac{dy}{dx} = \frac{-xy - y^2}{x^2}$$

This is homogeneous equation.

put $y = vx$

differentiating w.r.t. x,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then,

$$v + x\frac{dv}{dx} = \frac{-x(vx) - v^2x^2}{x^2}$$

or, $$v + x\frac{dv}{dx} = \frac{x^2(-v - v^2)}{x^2}$$

or, $v + x\frac{dv}{dx} = -v - v^2$

or, $\frac{dv}{-(v^2+2v)} = \frac{dx}{x}$

or, $\frac{-dv}{v(v+2)} = \frac{dx}{x}$

Integrating both sides

$\int \frac{-dv}{v(v+2)} = \int \frac{dx}{x}$

Using partial fraction in left integral: $\frac{-1}{v(v+2)} = \frac{A}{v} + \frac{B}{v+2}$

$\int -\frac{1}{v2} dv + \int \frac{1}{(v+2)2} dv = \log x + c$

$\frac{-1}{v(v+2)} = \frac{A(v+2) + Bv}{v(v+2)}$

$\therefore A = -\frac{1}{2}, \; B = \frac{1}{2}$

or $\frac{1}{2}(-\log(v) + \log(v+2)) = \log xc$

or $\log\left(\frac{v+2}{v}\right) = \log(x^2c)$

or $1 + \frac{2}{v} = x^2c$

or, $1 + \frac{2}{y/x} = x^2c$

or $1 + \frac{2x}{y} = x^2c$

or, $2x + y = x^2cy$

or, $x^2y = \frac{1}{c}(y + 2x)$

$\therefore x^2y = c(y + 2x)$

8. $x\frac{dy}{dx} = y - \sqrt{x^2+y^2}$

solⁿ: Given,

$x \cdot \frac{dy}{dx} = y - \sqrt{x^2+y^2}$

or, $\frac{dy}{dx} = \frac{y-\sqrt{x^2+y^2}}{x}$

or, This is homogeneous equation

put $y = vx$

Differentiating w.r.t. x

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then,

$v + x\frac{dv}{dx} = \frac{vx - \sqrt{x^2+v^2x^2}}{x}$

or, $v + x\frac{dv}{dx} = \frac{vx - x\sqrt{v^2+1}}{x}$

or, $v + x\frac{dv}{dx} = \frac{x(v-\sqrt{v^2+1})}{x}$

or, $v + x\frac{dv}{dx} = v - \sqrt{v^2+1}$

or, $\frac{-dv}{\sqrt{v^2+1}} = \frac{dx}{x}$

Integrating both sides,

$-\int \frac{dv}{\sqrt{v^2+1}} = \int \frac{dx}{x}$

$-\log(v+\sqrt{v^2+1}) = \log x$

or, $\log\left(\frac{1}{v+\sqrt{v^2+1}}\right) = \log xc$

or, $\frac{1}{v+\sqrt{v^2+1}} = xc$

or, $\frac{1}{\frac{y}{x}+\frac{\sqrt{x^2+y^2}}{x}} = xc$

or, $\frac{x}{y+\sqrt{x^2+y^2}} = xc$

or, $\frac{1}{y+\sqrt{x^2+y^2}} = c$

$\therefore y+\sqrt{x^2+y^2} = c'$

9. $x \sin \frac{y}{x}\, dy = \left(y \sin \frac{y}{x} - x\right) dx$

Sol^n: or, $\frac{dy}{dx} = \frac{\left(y \sin \frac{y}{x} - x\right)}{x \sin y/x}$

It is homogeneous differential equations

put $y = vx$

Differentiating w.r.t. x

$\frac{dy}{dx} = v + \frac{x\,dv}{dx}$

$\frac{dv}{dx} \times x + v = \frac{vx \sin \frac{vx}{x} - x}{x \sin vx/x}$

or, $\frac{x\,dv}{dx} + v = \frac{x(v \sin v - 1)}{x \sin v}$

or, $x \frac{dv}{dx} + v = \frac{v \sin v - 1}{\sin v}$

or, $x \frac{dv}{dx} = \frac{v \sin v - 1 - v \sin v}{\sin v}$

or, $x \frac{dv}{dx} = \frac{-1}{\sin v}$

or, $-\sin v\, dv = \frac{dx}{x}$

Integrating both sides

or, $\int -\sin v\, dv = \int \frac{dx}{x}$

or, $\cos v + c = \log x$

$\therefore \cos\left(\frac{y}{x}\right) + c = \log x$ //

10 $(1+e^{x/y})dx + e^{x/y}(1-x/y)dy = 0$

soln: or, $(1+e^{x/y})dx = -e^{x/y}(1-x/y)dy$

or, $\frac{dx}{dy} = -\frac{e^{x/y}(1-x/y)}{1+e^{x/y}}$

Which is homogeneous differential equation on x

so, put $x = vy$

Differentiating w.r.t. y

$\frac{dx}{dy} = v + y\frac{dv}{dy}$

Then,

$v + y\frac{dv}{dy} = \frac{-e^{vy/y}(1 - vy/y)}{1+e^{vy/y}}$

or, $v + y\frac{dv}{dy} = \frac{-e^{v}(1-v)}{1+e^{v}}$

or, $y\frac{dv}{dy} = \frac{-e^{v} + ve^{v}}{1+e^{v}} - v$

or, $y\frac{dv}{dy} = \frac{-e^{v} + ve^{v} - v - ve^{v}}{1+e^{v}}$

or, $y\frac{dv}{dy} = \frac{-v - e^{v}}{1+e^{v}}$

or, $\frac{1+e^{v}\,dv}{-(v+e^{v})} = \frac{dy}{y}$

Integrating both sides

$\int\frac{dy}{y} = -\int\frac{1+e^{v}}{v+e^{v}}dv$

or, $\log y = -\log(v+e^{v}) + c$

or, $\log y = \log\left(\frac{c}{v+e^{v}}\right)$

or, $y \times (v+e^{v}) = c$

or, $y \times \left(\frac{x}{y} + e^{x/y}\right) = c$ $\quad\therefore x + ye^{x/y} = c$

11. $\frac{dy}{dx} = \frac{3xy+y^2}{3x^2}$

Soln: It is homogeneous differential equation

put $y = vx$

Differentiating w.r.t x

$\frac{dy}{dx} = v + \frac{dv}{dx} \times x$

Then,

$x\frac{dv}{dx} + v = \frac{3x(vx)+(vx)^2}{3x^2}$

or $x\frac{dv}{dx} + v = \frac{3x^2v+v^2x^2}{3x^2}$

or $x\frac{dv}{dx} + v = \frac{3v+v^2}{3}$

or $x\frac{dv}{dx} = \frac{3v+v^2-3v}{3}$

or, $x\frac{dv}{dx} = \frac{v^2}{3}$

or. $\frac{3\,dv}{v^2} = \frac{dx}{dx}$

Integrating both sides.

$3\int \frac{dv}{v^2} = \int \frac{dx}{x}$

or $3\left(-\frac{1}{v}\right) = \log x + c$

or, $-\frac{3}{v} = \log x + c$

or $\log x + \frac{3}{v} = c$

or, $\log x + \frac{3x}{y} = c$

$\therefore$ $3x + y\log x = cy$

Ex 24

Solve the following differential equations

1. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

soln. Here,

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$$

or, $\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\cdot\frac{1}{x} = \frac{1}{x^2}$

Put $v = \frac{1}{y}$

or, $\frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

or, $\frac{1}{y^2}\frac{dy}{dx} = -\frac{dv}{dx}$

Then,

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x^2}$$

or $\frac{dv}{dx} - \frac{v}{x} = -\frac{1}{x^2}$ .... ①

which is linear in v. now,

$$I.F = e^{\int -1/x\,dx} = e^{-\log(x)} = e^{\log(x^{-1})} = \frac{1}{x}$$

Multiplying I.F on both sides of eqn ①

$$v \times I.F = \int -\frac{1}{x^2} \times \frac{1}{x}\,dx$$

or $v \times \frac{1}{x} = -\int \frac{1}{x^3}\,dx$

or $\frac{v}{x} = -\frac{x^{-3+1}}{-3+1} + c$

or, $\frac{1}{xy} = \frac{1}{2x^2} + c$

or $\frac{1}{xy} = \frac{1+2x^2c}{2x^2}$

or $2x = y + 2x^2cy$

$2x = cx^2y + y$

2. $(1-x^2)\frac{dy}{dx} + xy = xy^2$

soln. Here,

$(1-x^2)\frac{dy}{dx} + xy = xy^2$

or $\frac{1}{y^2}\frac{dy}{dx} + \frac{x}{(1-x^2)} \cdot \frac{1}{y} = \frac{x}{1-x^2}$

Now, put $v = \frac{1}{y}$

$\frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

Then,

$-\frac{dv}{dx} + \frac{x}{1-x^2} \cdot v = \frac{x}{1-x^2}$

or $\frac{dv}{dx} - \frac{x}{1-x^2} \cdot v = -\frac{x}{1-x^2}$ —①

which is linear in v,

So, $P = \frac{-x}{1-x^2}$, $Q = \frac{-x}{1-x^2}$

Now,

$I.F = e^{\int P dx} = e^{\int \frac{-x}{1-x^2} dx} = e^{1/2 \int \frac{-2x}{1-x^2} dx} = e^{1/2 \log(1-x^2)}$

$= \sqrt{1-x^2}$

Multiplying by I.F on eqn ① on both sides

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$v \times IF = \int Q \times IF \, dx$

or, $v \times \sqrt{1-x^2}) = \int \frac{-x}{1-x^2} \times \sqrt{1-x^2} \, dx$

or, $\frac{1}{y} \times \sqrt{1-x^2} = \int \frac{-x}{\sqrt{1-x^2}} dx$

or, $\frac{1}{y} \times \sqrt{1-x^2} = \frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}} dx$

or, $\frac{1}{y} \times \sqrt{1-x^2} = \frac{1}{\cancel{2}} \times \cancel{2}\sqrt{1-x^2} + c$

or, $\sqrt{1-x^2} = y\sqrt{1-x^2} + cy$

or, $\sqrt{1-x^2}(1-y) = cy$ //

3. $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$

soln: Here, $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$

or, $\frac{1}{e^y} \frac{dy}{dx} + \frac{1}{e^y} \cdot \frac{1}{x} = \frac{1}{x^2}$

Put $v = \frac{1}{e^y}$

$\frac{dv}{dx} = \frac{de^{-y}}{d-y} \times -\frac{dy}{dx}$

$\frac{dv}{dx} = \frac{1}{e^y} \times -\frac{dy}{dx}$

So,

$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x^2}$

or, $\frac{dv}{dx} - \frac{v}{x} = -\frac{1}{x^2}$ —— ①

which ~~is~~ ~~b~~ linear in v

Now
$P = -\frac{1}{x}$, $Q = -\frac{1}{x^2}$

Now
$I.F = e^{\int P dx} = e^{\int -1/x dx} = e^{-\log x} = \frac{1}{x}$

Multiplying eq^n ① by I.F. in both sides

$v \times I.F = \int Q . IF \, dx$

or, $v \times \frac{1}{x} = \int -\frac{1}{x^2} \times \frac{1}{x} dx$

or $\frac{1}{e^y} \times \frac{1}{x} = -\int \frac{1}{x^3} dx$

or, $\frac{1}{e^y} \times \frac{1}{x} = -\left(\frac{-1}{2x^2}\right) + c$

or $\frac{1}{e^y x} = \frac{1}{2x^2} + c$

or $\frac{1}{x . e^y} = \frac{1 + 2x^2 c}{2x^2}$

or, $2x = e^y (1 + x^2 c)$ //

4 $\frac{dy}{dx} + \frac{y}{x} \log y = \frac{y}{x^2} (\log y)^2$

soln Here, $\frac{dy}{dx} + \frac{y}{x} \log y = \frac{y}{x^2} (\log y)^2$

or, $\frac{1}{y(\log y)^2} . \frac{dy}{dx} + \frac{1}{\log y} . \frac{1}{x} = \frac{1}{x^2}$

Put $v = \frac{1}{\log y}$

$\frac{dv}{dx} = \frac{-1}{(\log y)^2} \times \frac{1}{y} \times \frac{dy}{dx}$

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Now,

$$-\frac{dv}{dx} + v \cdot \frac{1}{x} = \frac{1}{x^2}$$

or, $\frac{dv}{dx} - v \cdot \frac{1}{x} = -\frac{1}{x^2}$ —①

which is linear in v.

So, $P = -\frac{1}{x}$ $Q = -\frac{1}{x^2}$

Now, $IF = e^{\int P dx} = e^{\int -1/x \, dx} = e^{-\log x} = \frac{1}{x}$

Multiplying by I.F. on both sides of eqⁿ ①

$v \times I.F = \int Q \times IF \, dx$

or $v \times \frac{1}{x} = \int -\frac{1}{x^2} \times \frac{1}{x} dx$

or $\frac{1}{\log y} \times \frac{1}{x} = -\int \frac{1}{x^3} dx$

or, $\frac{1}{x \log y} = -\left(\frac{1}{-2x^2}\right) + c$

or $\frac{1}{x \log y} = \frac{1}{2x^2} + c$

or $\frac{1}{x \log y} = \frac{1 + 2x^2 c}{2x^2}$

or, $2x = \log y \,(1 + 2x^2 c)$

$\therefore$ $x = \log y \times \left(\frac{1}{2} + x^2 c\right)$

5. $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

Soln: Here, $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

or $\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{x \cdot 2\sin y\cos y}{\cos^2 y} = x^3$

or $\sec^2 y \frac{dy}{dx} + 2x \cdot \tan y = x^3$

Put $v = \tan y$

$\frac{dv}{dx} = \sec^2 y \frac{dy}{dx}$

Then,

$\frac{dv}{dx} + v \cdot 2x = x^3$ —①

which is linear in v, so,

$P = 2x \qquad Q = x^3$

Now,

$IF = e^{\int Pdx} = e^{\int 2x dx} = e^{2x \cdot x^2/2} = e^{x^2}$

Multiplying eqn ① by IF on both sides

$v \times e^{x^2} = \int x^3 \times e^{x^2} dx$

or, $\cancel{v \times e^{x^2}} \quad x^3 \int e^{x^2}$ put $t = x^2$

$dt = 2x dx$

or $v \times e^{x^2} = \int x \cdot x^2 e^{x^2} dx$

or $v \times e^{x^2} = \frac{1}{2}\int t e^t dt$

or $v \times e^{x^2} = \frac{1}{2}\left(t \int e^t dt - \int\left[\frac{dt}{dt}\int e^t dt\right] dt\right)$

or $v \times e^{x^2} = \frac{1}{2}(e^t \cdot t - e^t) + c$

or, $v \times e^{x^2} = \frac{1}{2}(e^{x^2} \cdot x^2 - e^{x^2} + c)$

or, $e^{x^2}\tan y = \frac{1}{2}e^{x^2}(x^2 - 1) + c$

6. $\cos x \frac{dy}{dx} = y(\sin x - y)$

Sol^n: Here,

$\cos x \frac{dy}{dx} = y \sin x - y^2$

or, $\cos x \frac{dy}{dx} - y \sin x = -y^2$

or, $\frac{dy}{dx} - \frac{y \sin x}{\cos x} = \frac{-y^2}{\cos x}$

or, $\frac{dy}{dx} - y \tan x = -y^2 \sec x$

or, $\frac{-1}{y^2}\frac{dy}{dx} + \frac{\tan x}{y} = \sec x$

put $v = \frac{1}{y}$

$\frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

or, $\frac{dv}{dx} + v \tan x = \sec x$ —(1)

which is linear in v so

$P = \tan x$ $\quad Q = \sec x$

$I.F. = e^{\int P dx} = e^{\int \tan x\, dx} = e^{\log(\sec x)} = \sec x$

Multiplying I.F on both sides of eq^n (1)

$v \times \sec x = \int Q \times \sec x\, dx$

or, $v \times \sec x = \int \sec^2 x\, dx$

or, $\frac{1}{y} \times \sec x = \tan x + c$

$\sec x = y(\tan x + c)$.

7. $x\frac{dy}{dx} + y = y^2 \log x$

Sol^n Here, $x\frac{dy}{dx} + y = y^2 \log x$

or $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2 \log x}{x}$

or $\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\cdot\frac{1}{x} = \frac{\log x}{x}$

Put $v = \frac{1}{y}$

or $\frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

Now,

$-\frac{dv}{dx} + \frac{v}{x} = \frac{\log x}{x}$

or, $\frac{dv}{dx} - \frac{v}{x} = -\frac{\log x}{x}$ — ①

which is linear in v. So, $P = -\frac{1}{x}$, $Q = -\frac{\log x}{x}$

Now,

I.F. $= e^{\int P dx} = e^{\int -1/x\, dx} = e^{-\log(x)} = \frac{1}{x}$

Multiplying both sides of eq^n ① by I.F. we get,

$v \times I.F = \int Q \times I.F\, dx$

or, $v \times \frac{1}{x} = \int -\frac{\log x}{x} \times \frac{1}{x}\, dx$

or, $v \times \frac{1}{x} = -\int \frac{1}{x^2} \log x\, dx$

Put $t = \log x$ in integral of R.H.S

$dt = \frac{1}{x} dx$

or $v \times \frac{1}{x} = -\int \frac{1}{x} \cdot \frac{1}{x} \log x\, dx$

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or, $v \times \frac{1}{x} = -\int t \cdot dt \times \frac{1}{e^t}$

or, $v \times \frac{1}{x} = -\int e^{-t} \cdot t\,dt$

or, $v \times \frac{1}{x} = -\left[t \int e^{-t} dt - \int \frac{dt}{dt} \int e^{-t} dt \; dt\right]$

or, $v \times \frac{1}{x} = -\left[t \times \frac{e^{-t}}{-1} + \int e^{-t} dt\right]$

or, $v \times \frac{1}{x} = -\left[-e^{-t} \cdot t - e^{-t}\right] + c$

or, $\frac{1}{xy} = e^{-t} + e^{-t} \cdot t + c$

or, $\frac{1}{xy} = e^{-\log x} + e^{-\log x} \cdot \log x + c$

or, $\frac{1}{yx} = \frac{1}{x} + \frac{1}{x} \log x + c$

or, $\frac{1}{xy} = \frac{1 + \log x + cx}{x}$

or, $y(1 + \log x + cx) = 1$

or, $y(\log x + 1) + cxy = 1$

8. $\frac{dy}{dx} = y \tan x - y^2 \sec x$

soln: Here, $\frac{dy}{dx} = y \tan x - y^2 \sec x$

or, $\frac{dy}{dx} - y \tan x = -y^2 \sec x$

or, $-\frac{1}{y^2} \frac{dy}{dx} + \frac{\tan x}{y} = \sec x$

Put $v = 1/y$

$\frac{dv}{dx} = -\frac{1}{y^2} \frac{dy}{dx}$

Then,

$$\frac{dv}{dx} + v\tan x = \sec x \quad —①$$

which is linear in v. So, $P = \tan x$, $Q = \sec x$

Now,

$I.F = e^{\int P dx} = e^{\int \tan x\, dx} = e^{\log(\sec x)} = \sec x$

Multiplying by $I.F. = \sec x$ in eq^n ① on both sides

$I.F \times v = \int Q \times I.F\, dx$

or $\sec x \times v = \int \sec x \times \sec x\, dx$

or $\sec x \times \frac{1}{y} = \int \sec^2 x\, dx$

or $\sec x \times \frac{1}{y} = \tan x + c$

$\therefore$ $\sec x = y\tan x + cy$

9. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

Solu^n: Here, $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

or, $\frac{1}{\sec y}\frac{dy}{dx} - \frac{\tan y}{\sec y} \times \frac{1}{(1+x)} = (1+x)e^x$

or $\cos y \frac{dy}{dx} - \sin y \frac{1}{(1+x)} = (1+x)e^x$

put $v = \sin y$

$\frac{dv}{dx} = \cos y \times \frac{dy}{dx}$

Then,

$$\frac{dv}{dx} - \frac{v}{1+x} = (1+x)e^x \quad —①$$

which is linear in v. So, $P = -\frac{1}{1+x}$ $Q = (1+x)e^x$

Now.

$$I.F = e^{\int P dx} = e^{\int \frac{1}{1+x} dx} = e^{-\log(1+x) dx} = \frac{1}{(1+x)}$$

Multiplying both sides of eqⁿ ① by I.F $\frac{1}{1+x}$ we get,

$$v \times I.F = \int Q \times I.F\, dx$$

or, $\sin y \times \frac{1}{(1+x)} = \int (x\ 1+x) e^x \times \frac{1}{(1+x)} dx$

or, $\frac{\sin y}{(1+x)} = \int e^x dx$

or $\frac{\sin y}{(1+x)} = e^x + c$

$\therefore$ $\sin y = (1+x)(e^x + c)$,,

10. $\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3\cos^2 y$

Solⁿ: Here, $\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3\cos^2 y$

or $\sec^2 y \frac{dy}{dx} + \frac{1}{x}\frac{2\sin y \cos y}{\cos^2 y} = x^3$

or $\sec^2 y \frac{dy}{dx} + \frac{2}{x}\tan y = x^3$

Put $v = \tan y$

$\frac{dv}{dx} = \sec^2 y \frac{dy}{dx}$

Then,

$\frac{dv}{dx} + \frac{2}{x} v = x^3$ —① which is linear in v,

So $P = \frac{2}{x}$, $Q = x^3$

Now,

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{2}{x} dx} = e^{2\log(x)} = x^2$$

Multiplying I.F. $= x^2$ on both sides of eq^n ①

$$\text{I.F.} \times v = \int Q \times \text{I.F} \, dx$$

or, $x^2 \times v = \int x^3 \times x^2 dx$

or, $x^2 \times \tan y = \int x^5 dx$

or, $x^2 \tan y = \frac{x^6}{6} + c$

or, $6x^2 \tan y = x^6 + c$ //

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Ex-23

1. $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2}$

soln Here, the given differential eqn is linear in y, so;

$P = \frac{2x}{1+x^2}$, $Q = \frac{1}{(1+x^2)^2}$

Now,

Integrating factor (IF) $= e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1+x^2)} = 1+x^2$

Multiplying by $I.F = 1+x^2$ on both sides of given equation,

$y \times (1+x^2) = \int Q \times (1+x^2)\, dx$

or, $y(1+x^2) = \int \frac{1}{(1+x^2)^2} \times (1+x^2)\, dx$

or, $y(1+x^2) = \int \frac{1}{1+x^2} dx$

$\therefore y(1+x^2) = \tan^{-1}x + c$

2. $\frac{dy}{dx} + \frac{y}{x^2} = \frac{1}{x^2}$

soln: Given differential eqn is linear in y so,

$P = \frac{1}{x^2}$, $Q = \frac{1}{x^2}$

Now

$I.F = e^{\int P dx} = e^{\int 1/x^2 dx} = e^{\frac{x^{-2+1}}{-2+1}} = e^{-1/x}$

Multiplying both sides of given differential eqn by $IF = e^{-1/x}$

$y \times e^{-1/x} = \int Q \times 1 e^{-1/x} dx$

or, $y \times e^{-1/x} = \int e^{-1/x} \frac{1}{x^2} dx$

Put $t = \frac{1}{x}$ in ~~left~~ right side,

$$dt = \frac{-1}{x^2}\, dx$$

$$y \times e^{-1/x} = \int e^{t} \cdot dt$$

or, $y \times e^{-1/x} = e^{t} + c$

or, $y = e^{1/x}(e^{-1/x} + c)$

or, $y = e^{1/x - 1/x} + e^{1/x} \cdot c$

$\therefore$ $y = 1 + ce^{1/x}$

3. ~~$\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$~~ 3. $(1-x^2)\frac{dy}{dx} - xy = 1$

soln: soln: Here, given differential eq^n is

$$(1-x^2)\frac{dy}{dx} - xy = 1$$

or, $\frac{dy}{dx} - \left(\frac{x}{1-x^2}\right)y = \frac{1}{1-x^2}$ —①

Which is linear in y, so,

$P = \frac{-x}{1-x^2}$, $Q = \frac{1}{1-x^2}$

Now,

$$I.F = e^{\int P dx} = e^{\int \frac{-x}{1-x^2} dx} = e^{\frac{1}{2}\int \frac{-2x}{1-x^2} dx} = e^{\frac{1}{2}\log(1-x^2)} = \sqrt{1-x^2}$$

Multiplying both sides of eq^n ① by I.F we get,

$$I.F \times y = \int Q \times I.F \, dx$$

$$y \times \sqrt{1-x^2} = \int \frac{1}{1-x^2} \times \sqrt{1-x^2}\, dx$$

or, $y \times \sqrt{1-x^2} = \int \frac{1}{\sqrt{1-x^2}}\, dx$

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or, $y \times \sqrt{1-x^2} = \sin^{-1}x + c$

4. $\frac{dy}{dx} + 2 \cdot y = 4x$

solⁿ: Given, eqⁿ is linear in y, so,

$P = 2 \qquad Q = 4x$

Now,

$I.F. = e^{\int P dx} = e^{\int 2 dx} = e^{2x}$

Multiplying both sides of given eqⁿ by $I.F = e^{2x}$ we get,

$y \times I.F = \int Q \times I.F dx$

or, $y \times e^{2x} = \int 4x \cdot e^{2x} dx$

or $y \times e^{2x} = 4x \cdot \int e^{2x} dx - \int \left[\frac{d4x}{dx} \left(\int e^{2x} dx \right) \right] dx$

or, $y \times e^{2x} = 4x \cdot \frac{e^{2x}}{2} - \int 4 \frac{e^{2x}}{2} dx$

or, $y \times e^{2x} = 2x \cdot e^{2x} - 2 \times \frac{e^{2x}}{2} + c$

or $y = e^{-2x}(2x \cdot e^{2x} - e^{2x} + c)$

$\therefore y = 2x - 1 + ce^{-2x}$

5. $(1+x)\frac{dy}{dx} - xy = 1-x$

solⁿ: Here, $(1+x)\frac{dy}{dx} - xy = 1-x$

or, $\frac{dy}{dx} - \frac{x}{1+x} y = \frac{1-x}{1+x}$ —①

which is linear in y. so

$P = \frac{-x}{1+x}, \quad Q = \frac{1-x}{1+x}$

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$$I.F = e^{\int P dx} = e^{\int \frac{-x}{1+x} dx} = e^{-\int \frac{1+x}{1+x} - \frac{1}{1+x} dx} = e^{-\left[\int dx - \int \frac{1}{1+x} dx\right]}$$

$$= e^{-[x - \log(1+x)]}$$

$$= e^{\log(1+x) - x}$$

$$= e^{\log(1+x)} \cdot e^{-x}$$

$$= (1+x) \cdot e^{-x}$$

Multiplying by $I.F = (1+x) \cdot e^{-x}$ on both sides of eq^n ①

$$y \times I.F = \int Q \times I.F \, dx$$

or, $y \times (1+x) e^{-x} = \int \frac{1-x}{1+x} \times (1+x) e^{-x} dx$

or, $y \times (1+x) e^{-x} = \int (1-x) e^{-x} dx$

or, $y \times (1+x) e^{-x} = (1-x) \int e^{-x} dx - \int \left[\frac{d(1-x)}{dx} \int e^{-x} dx \right] dx$

or, $y(1+x) e^{-x} = (1-x) \frac{e^{-x}}{-1} - \int -1 \cdot \frac{e^{-x}}{-1} dx$

or, $y(1+x) e^{-x} = (x-1) e^{-x} - \int e^{-x} dx$

or, $y(1+x) e^{-x} = (x-1) e^{-x} + e^{-x} + c$

or, $y(1+x) = e^{x}([x-1] e^{-x} + e^{-x} + c)$

or, $y(1+x) = x - 1 + 1 + c e^{x}$

$\therefore y(1+x) = x + c e^{x}$

6. $(1+x^2) \frac{dy}{dx} + y = e^{\tan^{-1} x}$

Sol^n: Here, $(1+x^2) \frac{dy}{dx} + y = e^{\tan^{-1} x}$

$$\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e^{\tan^{-1} x}}{1+x^2} \quad —①$$

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which is linear in y. so,

$P = \frac{1}{1+x^2}$, $Q = \frac{e^{\tan^{-1}x}}{1+x^2}$,

$I.F = e^{\int P dx} = e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1}x}$

Multiplying eqⁿ ① by $I.F = e^{\tan^{-1}x}$ on both sides,

$I.F \times y = \int Q \times I.F\, dx$

or $y \times e^{\tan^{-1}x} = \int \frac{e^{\tan^{-1}x}}{1+x^2} \times e^{\tan^{-1}x} dx$

Put $a = \tan^{-1}x$

$da = \frac{1}{1+x^2} dx$

or $y \times e^{\tan^{-1}x} = \int e^a \times e^a da$

or $y \times e^{\tan^{-1}x} = \int e^{2a} da$

or $y \times e^{\tan^{-1}x} = \frac{e^{2a}}{2} + c$

or $2y e^{\tan^{-1}x} = e^{2\tan^{-1}x} + c_4$

7. $\frac{dy}{dx} x\cos x + y(x\sin x + \cos x) = 1$

soln. Here, $\frac{dy}{dx} + \frac{y(x\sin x + \cos x)}{x\cos x} = \frac{1}{x\cos x}$ —①

which is linear in y, so,

$P = \frac{x\sin x + \cos x}{x\cos x}$, $Q = \frac{1}{x\cos x}$

$I.F = e^{\int P dx} = e^{\int \frac{x\sin x + \cos x}{x\cos x} dx}$
$= e^{\int \frac{x\sin x}{x\cos x} dx + \int \frac{\cos x}{x\cos x} dx}$
$= e^{\int \tan x\, dx + \int \frac{1}{x} dx}$
$= e^{\log(\sec x) + \log(x)}$
$= e^{\log(x\sec x)}$
$= x\sec x$

Multiplying both sides of eq^n ① by I.F we get

$y \times I.F = \int Q \times IF\, dx$

or $y \times x\sec x = \int \frac{1}{\cancel{x}\cos x} \times \cancel{x}\sec x\, dx$

or, $y \times x\sec x = \int \sec^2 x\, dx$

or $y \times x\sec x = \tan x + c$

$\therefore$ $yx\sec x = \tan x + c$ //

8. $\cos^2 x \frac{dy}{dx} + y = \tan x$

Solu^n: Here, $\cos^2 x \frac{dy}{dx} + y = \tan x$

or, $\frac{dy}{dx} + y\sec^2 x = \frac{\sin x}{\cos^3 x}$ —①

which is linear in y, so,

$P = \sec^2 x$, $Q = \tan x \cdot \sec^2 x$

$I.F = e^{\int P dx} = e^{\int \sec^2 x\, dx} = e^{\tan x}$

Multiplying both sides of eq^n ① by I.F $e^{\tan x}$ we get

$y \times I.F = \int Q \times I.F\, dx$

or, $y \times e^{\tan x} = \int \tan x \cdot \sec^2 x \cdot e^{\tan x} dx$

put,

$a = \tan x$

$da = \sec^2 x\, dx$

so, $y \times e^{\tan x} = \int a \cdot e^a\, da$

or, $y \times e^{\tan x} = a\int e^a\, da - \int\left(\frac{da}{da} \times \int e^a\, da\right) da$

or, $y \times e^{\tan x} = a \cdot e^a - \int e^a da$

or $y \times e^{\tan x} = a \cdot e^{a} - e^{a} + c$

or $y \times e^{\tan x} = e^{\tan x}(\tan x - 1) + c$

or $(y - \tan x + 1)e^{\tan x} = c$ //

9. $\sin x \frac{dy}{dx} + y\cos x = x\sin x$

Solu^n:- Here, $\sin x \frac{dy}{dx} + y\cos x = x\sin x$

or, $\frac{dy}{dx} + y \cdot \cot x = x$ —①

which is linear in y, so,

$P = \cot x$, $Q = x$, so,

$I.F = e^{\int P dx} = e^{\int \cot x dx} = e^{\log(\sin x)} = \sin x$

Multiplying both sides of eq^n ① by $\sin x$ we get,

$y \times \sin x = \int Q \times \sin x \, dx$

or $y \sin x = \int x \cdot \sin x \, dx$

or $y \sin x = x\int \sin x \, dx - \int\left(\frac{dx}{dx}\int \sin x \, dx\right)dx$

or, $y \sin x = x(-\cos x) - \int -\cos x \, dx$

or $y \sin x = -x\cos x + \sin x + c$

$\therefore$ $y\sin x + x\cos x - \sin x = c$ //

10. $\frac{dy}{dx} + \cot x \, y = 2\cos x$

sol: which is linear in y, so

$P = \cot x$, $Q = 2\cos x$

Now,

$I.F = e^{\int P dx} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$

Multiplying both sides of given eq^n by I.F we get,

$y \times I.F = \int Q \times I.F \, dx$

$y \times \sin x = \int 2\cos x \times \sin x \, dx$

$y \times \sin x = \int \sin 2x \, dx$

$y \times \sin x = \frac{-\cos 2x}{2} + c$

$\therefore 2y\sin x + \cos 2x = c_1$

11. $\frac{dy}{dx} + y\tan x = \sec x$

sol^n: The given eq^n is linear in y. so

$P = \tan x$, $Q = \sec x$

Now,

$I.F = e^{\int P dx} = e^{\int \tan x dx} = e^{\log(\sec x)} = \sec x$

Multiplying both sides by I.F. on eq^n given,

$y \times I.F = \int Q \times I.F \, dx$

or $y \times \sec x = \int \sec x \times \sec x \, dx$

or $y \sec x = \int \sec^2 x \, dx$

or $y \sec x = \tan x + c$

or $y = \tan x \times \cos x + c\cos x$

$y = \sin x + c \cos x$

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12. $x\log x \frac{dy}{dx} + y = 2\log x$

soln: Here, given, $x\log x \frac{dy}{dx} + y = 2\log x$

or $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x}$ —①

Here eqn ① is linear in y so,

$P = \frac{1}{x\log x}$, $Q = \frac{2}{x}$,

Now

$I.F = e^{\int P dx} = e^{\int \frac{1}{x\log x} dx}$, put $a = \log x$

$da = \frac{1}{x} dx$

$= e^{\int \frac{1}{a} da}$

$= e^{\log(a)}$

$= a$

$= \log x$

Multiplying eqn ① by I.F on both sides we get,

$y \times I.F = \int Q \times IF\, dx$

or, $y \times \log x = \int \frac{2}{x} \log x\, dx$

or $y \times \log x = 2\int \log x \cdot \frac{1}{x} dx$

[put $b = \log x$
$db = \frac{1}{x} dx$,]

or $y \times \log x = 2\int b.db$

or $y \times \log x = 2 \times \frac{b^2}{2} + c$

or $y \times \log x = (\log x)^2 + c$,

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13. $\frac{dy}{dx} + y = \cos x$

Solu: Here the eqn is linear in y, so,

$P = 1, \quad Q = \cos x$

Now,

$I.F = e^{\int P dx} = e^{\int 1 dx} = e^{\int dx} = e^{x}$

Multiplying given eqn by e^x on both sides we get,

$y \times I.F = \int Q \times I.F \, dx$

or $y \times e^{x} = \int \cos x \times e^{x} dx$

or $y \times e^{x} = \cos x \int e^{x} dx - \int \left[\frac{d \cos x}{dx} \int e^{x} dx \right] dx$

or $y \times e^{x} = \cos x \int e^{x} dx + \int \sin x \int e^{x} dx$

or $y \times e^{x} = \cos x \int e^{x} dx + \sin x \int e^{x} dx - \int \left(\frac{d \sin x}{dx} \int e^{x} dx \right) dx$

or, $y \times e^{x} = \cos x . e^{x} + \sin x . e^{x} - \int \cos x . e^{x} dx + c$

or $y \times e^{x} = \cos x . e^{x} + \sin x . e^{x} - y \times e^{x} + c$

or $2ye^{x} = \cos x . e^{x} + \sin x . e^{x} + c$

$\therefore 2y = \cos x + \sin x + ce^{-x}$

14. $(1+y^2) dx = (\tan^{-1} y - x) dy$

Solu: Here, $(1+y^2) dx = (\tan^{-1} y - x) dy$

$\frac{dx}{dy} = \frac{\tan^{-1} y}{1+y^2} - \frac{x}{1+y^2}$

or $\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$ —(1)

which is linear in ~~y~~ x.

$P = \frac{1}{1+y^2}$, $Q = \frac{\tan^{-1}y}{1+y^2}$

Now

$I.F. = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1}y}$

Multiplying both sides by I.F. on eq^n ①

$x \times I.F = \int Q \times I.F \, dy$

or, $x \times e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} \times e^{\tan^{-1}y} dy$

Put $\tan^{-1}y = f$

$\frac{1}{1+y^2} dy = df$

or, $x \times e^{\tan^{-1}y} = \int f.e^f df$

or, $x \times e^{\tan^{-1}y} = f \int e^f df - \int \left(\frac{df}{df} \int e^f df\right) df$

or, $x \times e^{\tan^{-1}y} = f.e^f - \int e^f df$

or, $x \times e^{\tan^{-1}y} = \tan^{-1}y \, . e^{\tan^{-1}y} - e^{\tan^{-1}y} + c$

or, $x.e^{\tan^{-1}y} = (\tan^{-1}y - 1) e^{\tan^{-1}y} + c$

or, $x = (\tan^{-1}y - 1) + c e^{-\tan^{-1}y}$

15. $\frac{dy}{dx} + \frac{y}{x} = x^2$ if $y = 1$ when $x = 1$

Sol^n: Here, $\frac{dy}{dx} + \frac{y}{x} = x^2$ — ①

which is linear in y.

So, $P = \frac{1}{x}$, $Q = x^2$,

$I.F = e^{\int 1/x \, dx} = e^{\log(x)} = x$

Multiplying both sides of eq^n ① by $I.F. = x$,

$I.F \times y = \int Q \times I.F \, dx$

or $y \times x = \int x^2 \times x \, dx$

or $y \times x = \int x^3 dx$

or $yx = \frac{x^4}{4} + c$

when, $x = 1$, $y = 1$, so

$1.1 = \frac{1^4}{4} + c$

or $1 = \frac{1 + 4c}{4}$

or $4 = 1 + 4c$

or $3 = 4c$

$\therefore c = \frac{3}{4}$

so

$yx = \frac{x^4}{4} + \frac{3}{4}$

or $4yx = x^4 + 3$