Chapter 4

Three Dimensional Graphics

Three Dimensional Transformation

- **Translation**

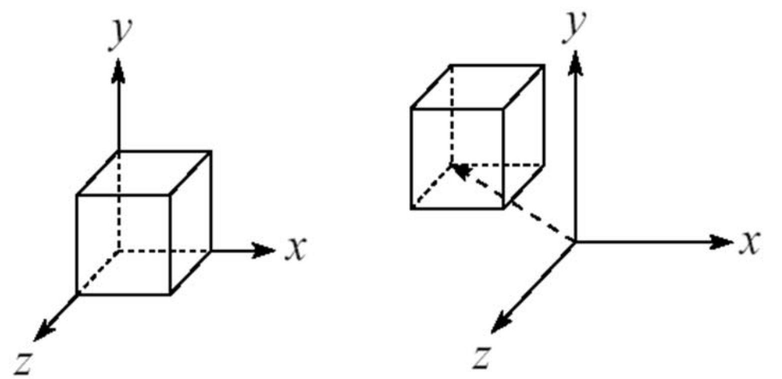
In a three dimensional homogenous coordinate representation, a point is translated from position P = (x,y,z) to position P' = (x',y',z') by following operation

$x' = x + t_x \qquad y' = y + t_y \qquad z' = z + t_z$

where the pair (t_x, t_y, t_z) is called the *translation vector.*

In matrix form,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix}$$



$$P' = P \bullet T$$

Three Dimensional Transformation

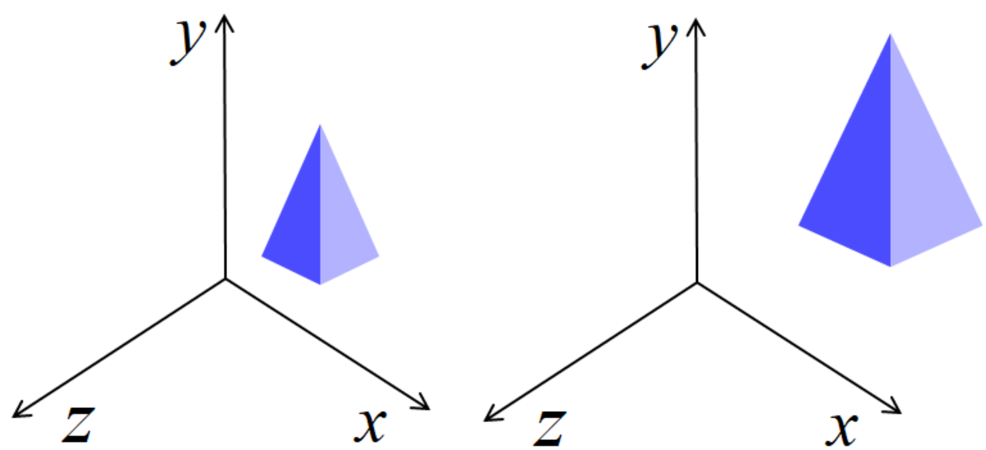
- **Scaling**

The three dimensional homogeneous coordinate representation of scaling about origin is

$x' = x.s_x$ $y' = y.s_y$ $z = z.s_z$

In Matrix form,

$$\mathbf{P}' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{S} \cdot \mathbf{P}$$

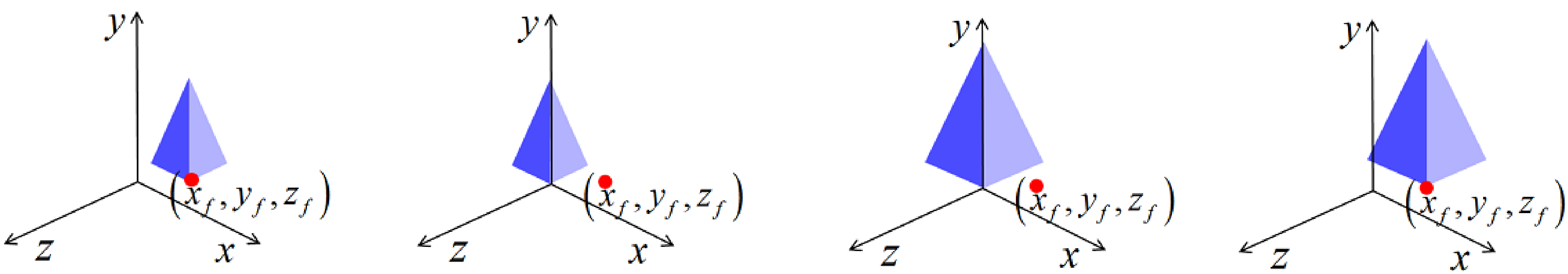


Three Dimensional Transformation

- **Scaling**

Scaling with respect to any fixed position (x_f, y_f, z_f) can be represented with following transformation sequence

1. Translate the fixed point to the origin
2. Scale the object relative to the coordinate origin
3. Translate the fixed point back to its original position



Three Dimensional Transformation

- **Scaling**

$$T(x_f, y_f, z_f) \cdot S(s_x, s_y, s_z) \cdot T(-x_f, -y_f, -z_f) = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x_f \\ 0 & 1 & 0 & y_f \\ 0 & 0 & 1 & z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_f \\ 0 & 1 & 0 & -y_f \\ 0 & 0 & 1 & -z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\mathbf{T} \cdot \mathbf{S} \cdot \mathbf{T}^{-1} = \begin{bmatrix} S_x & 0 & 0 & (1-S_x)x_f \\ 0 & S_y & 0 & (1-S_y)y_f \\ 0 & 0 & S_z & (1-S_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Three Dimensional Transformation

- **Rotation**

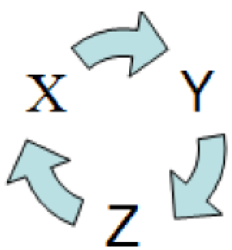
➢ To generate a rotation transformation for an object in 3D space, we require the following:

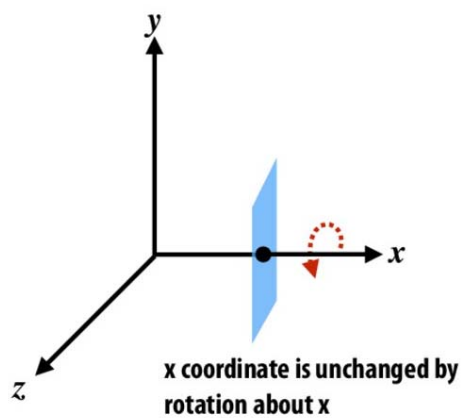
  - Angle of rotation.
  - Pivot point
  - Direction of rotation
  - Axis of rotation

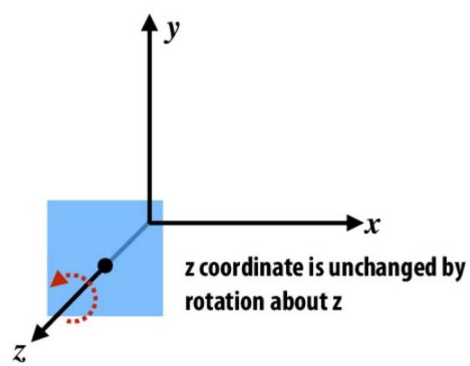
➢ Axes that are parallel to the coordinate axes are easy to handle.

➢ Cyclic permutation of the coordinate parameters x, y and z are used to get transformation equations for rotations about the coordinates

$x \rightarrow y \rightarrow z$







Three Dimensional Transformation

- **Rotation**

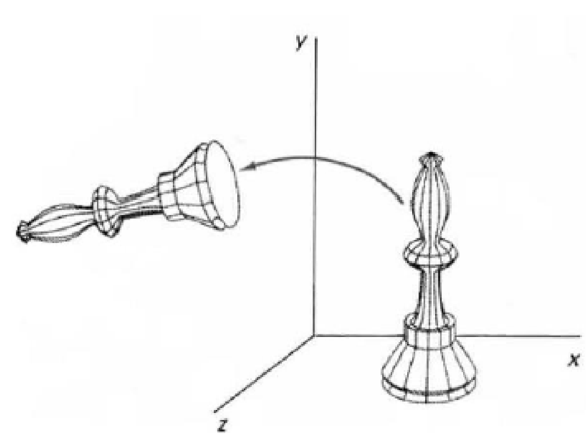
➢ Taking origin as the centre of rotation, when a point P(x, y, z) is rotated through an angle θ about any one of the axes to get the transformed point P'(x', y', z'), we have the following equation for each.

➢ 3D z-axis rotation equations are expressed in homogenous coordinate form as

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

➢ In Matrix form

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



P' = Rz (θ) . P

Three Dimensional Transformation

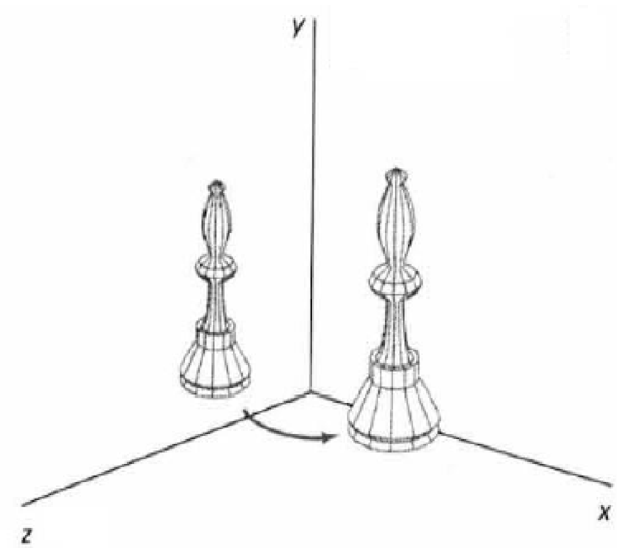
- **Rotation**

➢ 3D y-axis rotation equations are expressed in homogenous coordinate form as

$$x' = z\sin\theta + x\cos\theta$$
$$y' = y$$
$$z' = z\cos\theta - x\sin\theta$$

➢ In Matrix form

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



P' = Ry (θ) . P

Three Dimensional Transformation

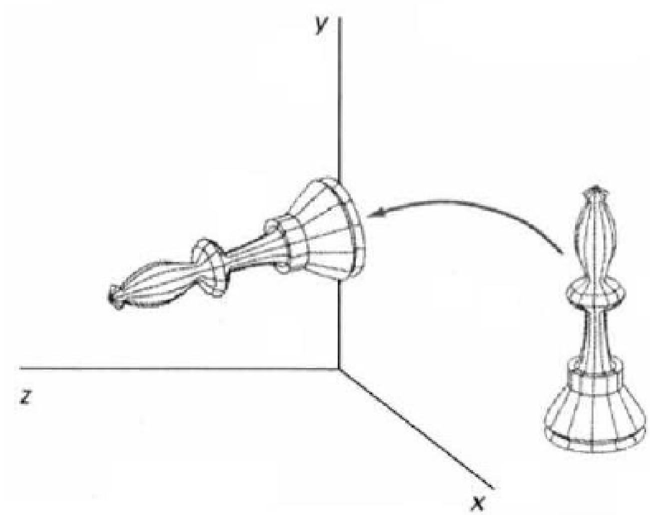
• Rotation

➤ 3D x-axis rotation equations are expressed in homogenous coordinate form as

$x' = x$
$y' = y\cos\theta - z\sin\theta$
$z' = y\sin\theta + z\cos\theta$

➤ In Matrix form

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



P' = Rx (θ) . P

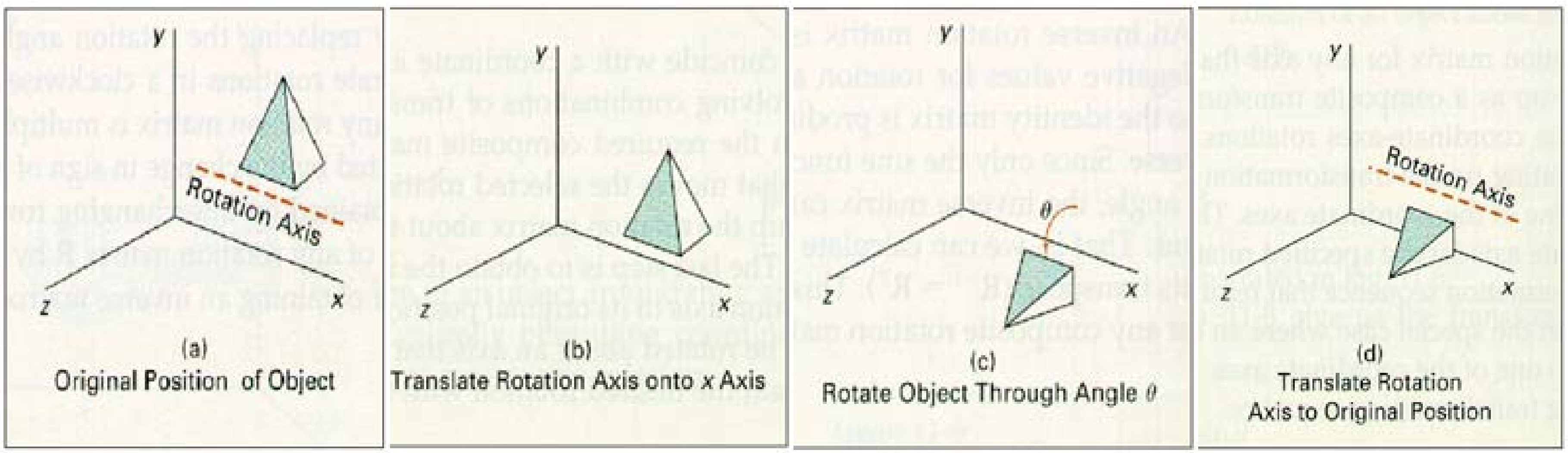
Three Dimensional Transformation

- **Rotation about an axis parallel to one of the coordinate axes:**

➢ Steps:

- Translate object so that rotation axis coincides with the parallel coordinate axis.
- Perform specified rotation about that axis
- Translate object back to its original position.

i.e. $\mathbf{P' = [\, T^{-1} . R_x(\theta) . T \,] . P}$



(a) Original Position of Object

(b) Translate Rotation Axis onto *x* Axis

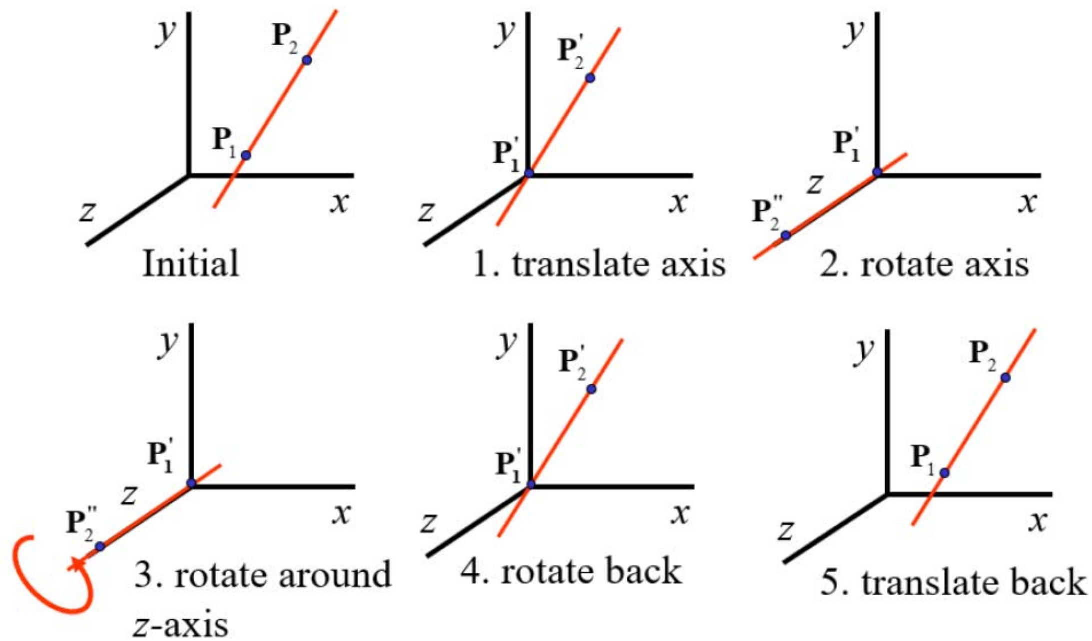
(c) Rotate Object Through Angle *θ*

(d) Translate Rotation Axis to Original Position

Three Dimensional Transformation

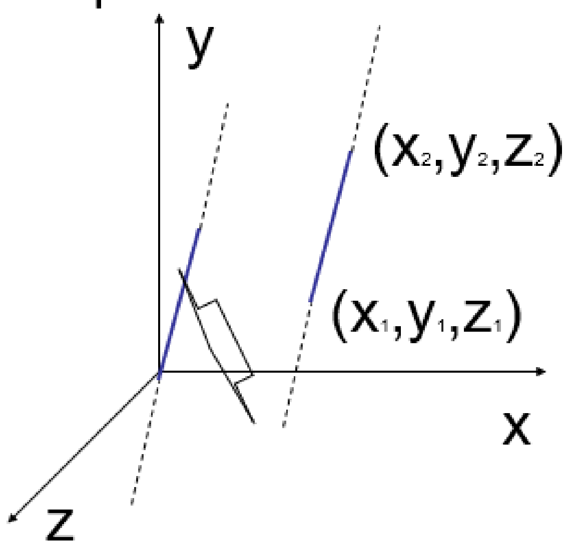
- **Rotation about any arbitrary axis in 3D Space**

- Steps:
  - ➢ Translate the object such that rotation axis passes through the origin.
  - ➢ Rotate the object such that rotation axis coincides with one of Cartesian axes.
  - ➢ Perform specified rotation about the Cartesian axis.
  - ➢ Apply inverse rotation to return rotation axis to original direction.
  - ➢ Apply inverse translation to return rotation axis to original position.



Three Dimensional Transformation

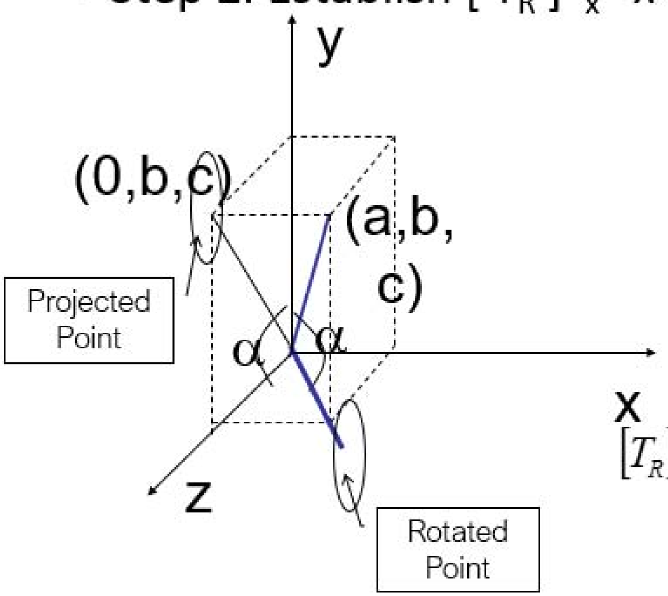
- **Rotation about any arbitrary axis in 3D Space**
  - Step 1. Translation



$$T_{TR} = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Three Dimensional Transformation

- **Rotation about any arbitrary axis in 3D Space**
  - Step 2. Establish $[\,T_R\,]^{\alpha}_{x}$ x axis



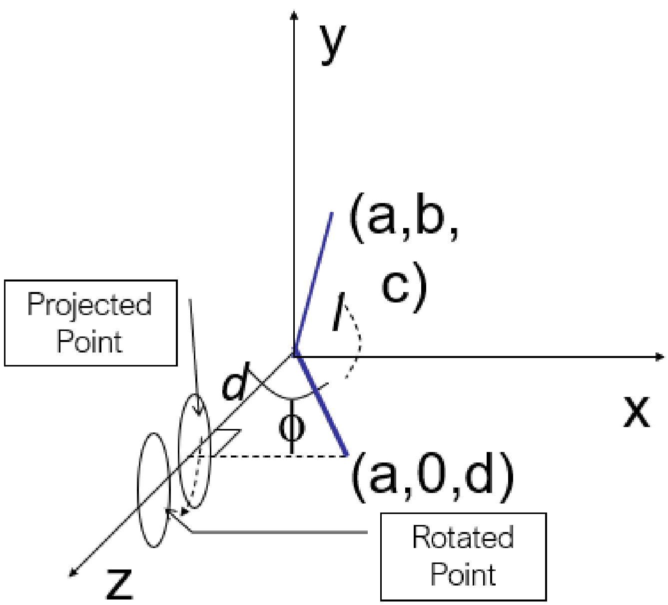
$$\sin\alpha = \frac{b}{\sqrt{b^2+c^2}} = \frac{b}{d}$$

$$\cos\alpha = \frac{c}{\sqrt{b^2+c^2}} = \frac{c}{d}$$

$$[T_R]_x^{\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Three Dimensional Transformation

- **Rotation about any arbitrary axis in 3D Space**
  - Step 3. Rotate about *y* axis by ϕ



$$\sin\phi = \frac{a}{l}, \qquad \cos\phi = \frac{d}{l}$$

$$l^2 = a^2 + b^2 + c^2 = a^2 + d^2$$

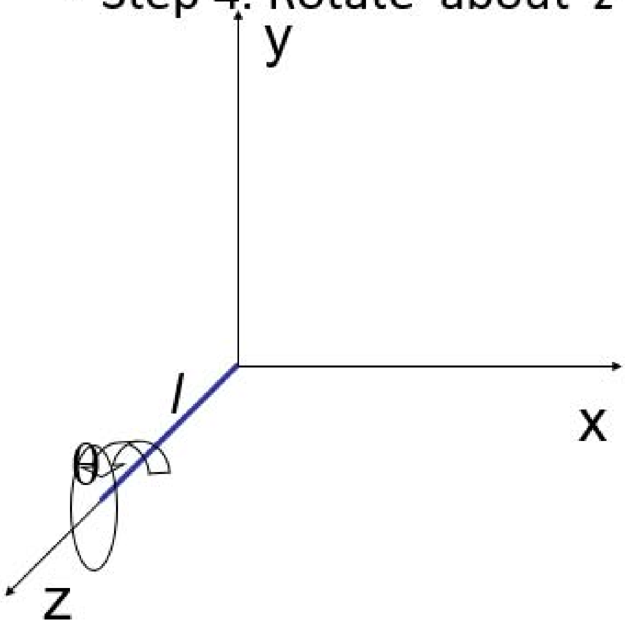
$$d = \sqrt{b^2 + c^2}$$

$$[T_R]_y^\phi = \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d/l & 0 & -a/l & 0 \\ 0 & 1 & 0 & 0 \\ a/l & 0 & d/l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Three Dimensional Transformation

- **Rotation about any arbitrary axis in 3D Space**

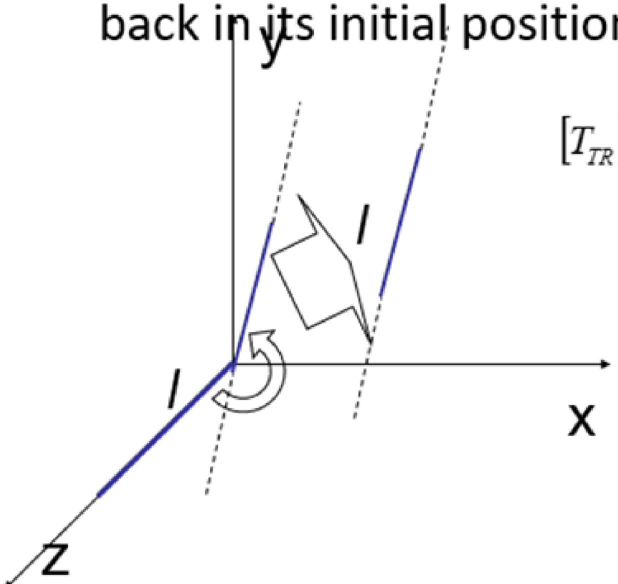
  - Step 4. Rotate about *z* axis by the desired angle θ



$$[T_R]_z^\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Three Dimensional Transformation

- **Rotation about any arbitrary axis in 3D Space**
  - Step 5. Apply the reverse transformation to place the axis back in its initial position

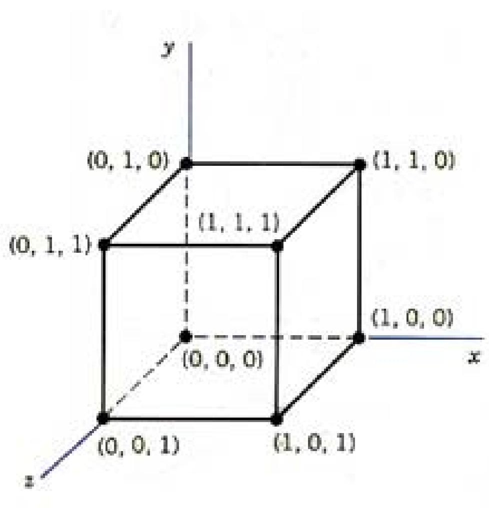


$$[T_{TR}]^{-1}[T_R]_x^{-\alpha}[T_R]_y^{-\phi} = \begin{bmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T_R]_{ARB} = [T_{TR}]^{-1}[T_R]_x^{-\alpha}[T_R]_y^{-\phi}[T_R]_z^{\theta}[T_R]_y^{\phi}[T_R]_x^{\alpha}[T_{TR}]$$

Example

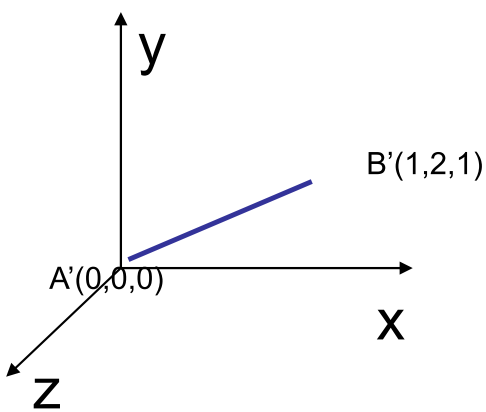
Find the new coordinates of a unit cube 90º-rotated about an axis defined by its endpoints A(2,1,0) and B(3,3,1).



A Unit Cube

Example

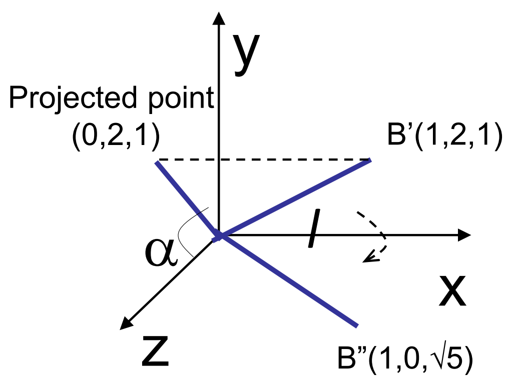
- Step1. Translate point A to the origin



$$[T_{TR}] = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

- Step 2. Rotate axis *A'B'* about the x axis by an angle α, until it lies on the *xz* plane.



$$\sin\alpha = \frac{2}{\sqrt{2^2+1^2}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\cos\alpha = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\sin\alpha = \frac{b}{\sqrt{b^2+c^2}} = \frac{b}{d}$$

$$\cos\alpha = \frac{c}{\sqrt{b^2+c^2}} = \frac{c}{d}$$

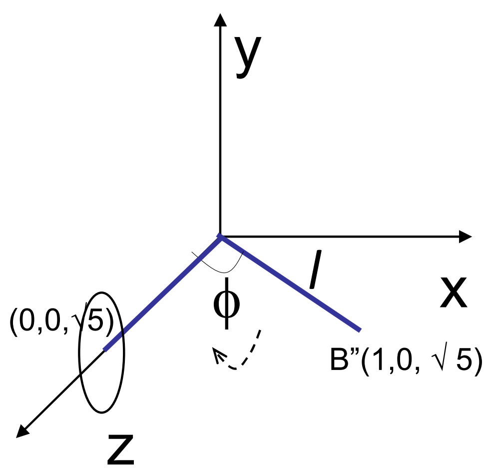
$$l = \sqrt{1^2+2^2+1^2} = \sqrt{6}$$

$$[T_R]_x^\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} & 0 \\ 0 & \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

- Step 3. Rotate axis *A'B''* about the y axis by and angle ϕ, until it coincides with the z axis.



$$\sin\phi = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

$$\cos\phi = \frac{\sqrt{5}}{\sqrt{6}} = \frac{\sqrt{30}}{6}$$

$$\sin\phi = \frac{a}{l}, \qquad \cos\phi = \frac{d}{l}$$

$$l^2 = a^2 + b^2 + c^2 = a^2 + d^2$$

$$d = \sqrt{b^2 + c^2}$$

$$[T_R]_y^\phi = \begin{bmatrix} \frac{\sqrt{30}}{6} & 0 & -\frac{\sqrt{6}}{6} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{30}}{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

- Step 4. Rotate the cube 90° about the z axis

$$[T_R]_z^{90^\circ} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, the concatenated rotation matrix about the arbitrary axis AB becomes,

$$[T_R]_{ARB} = [T_{TR}]^{-1}[T_R]_x^{-\alpha}[T_R]_y^{-\phi}[T_R]_z^{90^\circ}[T_R]_y^{\phi}[T_R]_x^{\alpha}[T_{TR}]$$

Example

$$[T_R]_{ARB} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} & 0 \\ 0 & -\frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{30}}{6} & 0 & \frac{\sqrt{6}}{6} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{30}}{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{30}}{6} & 0 & -\frac{\sqrt{6}}{6} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{30}}{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} & 0 \\ 0 & \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.166 & -0.075 & 0.983 & 1.742 \\ 0.742 & 0.667 & 0.075 & -1.151 \\ -0.650 & 0.741 & 0.167 & 0.560 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

- Multiplying $[T_R]_{AB}$ by the point matrix of the original cube

$$[P^*] = [T_R]_{ARB} \cdot [P]$$

$$[P^*] = \begin{bmatrix} 0.166 & -0.075 & 0.983 & 1.742 \\ 0.742 & 0.667 & 0.075 & -1.151 \\ -0.650 & 0.741 & 0.167 & 0.560 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.650 & 1.667 & 1.834 & 2.816 & 2.725 & 1.742 & 1.909 & 2.891 \\ -0.558 & -0.484 & 0.258 & 0.184 & -1.225 & -1.151 & -0.409 & -0.483 \\ 1.467 & 1.301 & 0.650 & 0.817 & 0.726 & 0.560 & -0.091 & 0.076 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Three Dimensional Transformation

- **Reflection**

➢ A three-dimensional reflection can be performed relative to a selected reflection axis or with respect to a selected reflection plane.

➢ The matrix representation for the reflection of a point relative to XY-plane is given by

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\boxed{\begin{gathered} x' = x \\ y' = y \\ z' = -z \end{gathered}}$$

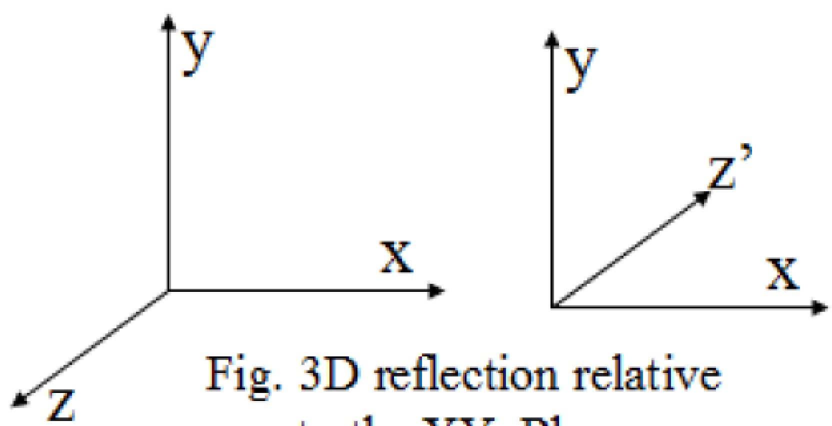


Fig. 3D reflection relative to the XY- Plane

Three Dimensional Transformation

- **Reflection**

➢ The matrix representation for the reflection of a point relative to YZ-plane is given by

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

x' = -x
y' = y
z' = z

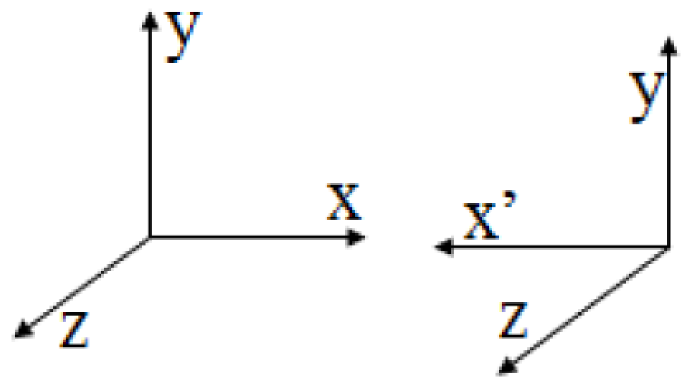


Fig. 3D reflection relative to the YZ- Plane

Three Dimensional Transformation

- **Reflection**

➢ The matrix representation for the reflection of a point relative to ZX-plane is given by

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

x’ = x
y’ = -y
z’ = z

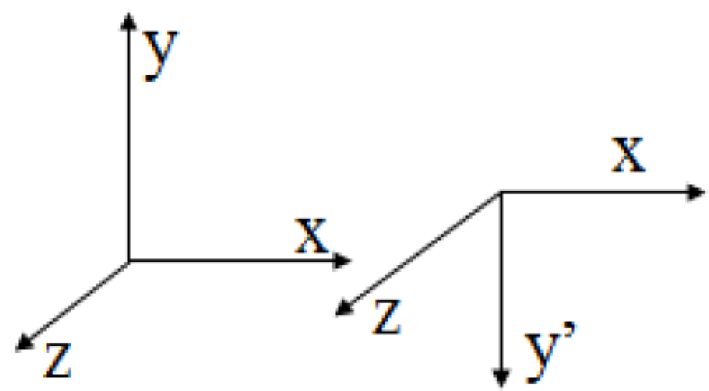


Fig. 3D reflection relative to the ZX- Plane

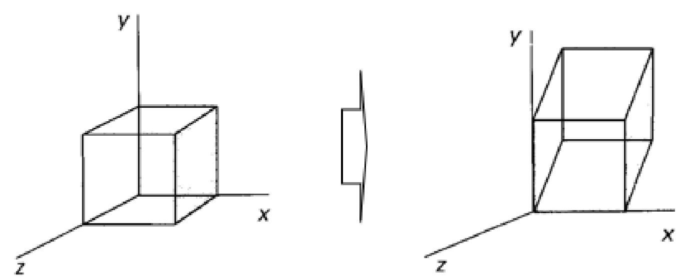
Three Dimensional Transformation

- **Shear**

➢ Shearing transformations are used to modify object shapes.

➢ shearing relative to the z axis:

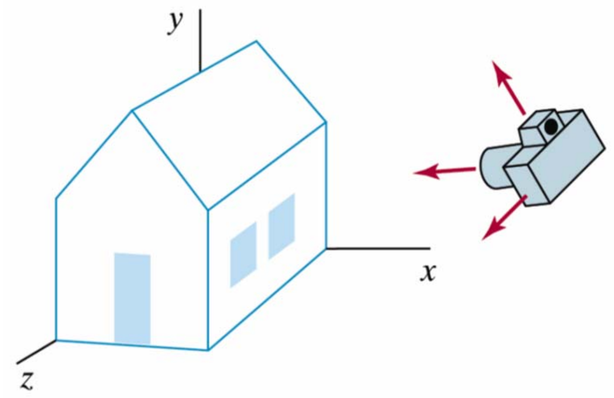
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



x' = x + az	a=shear factor for x
y' = y + bz	b=shear factor for y
z' = z	

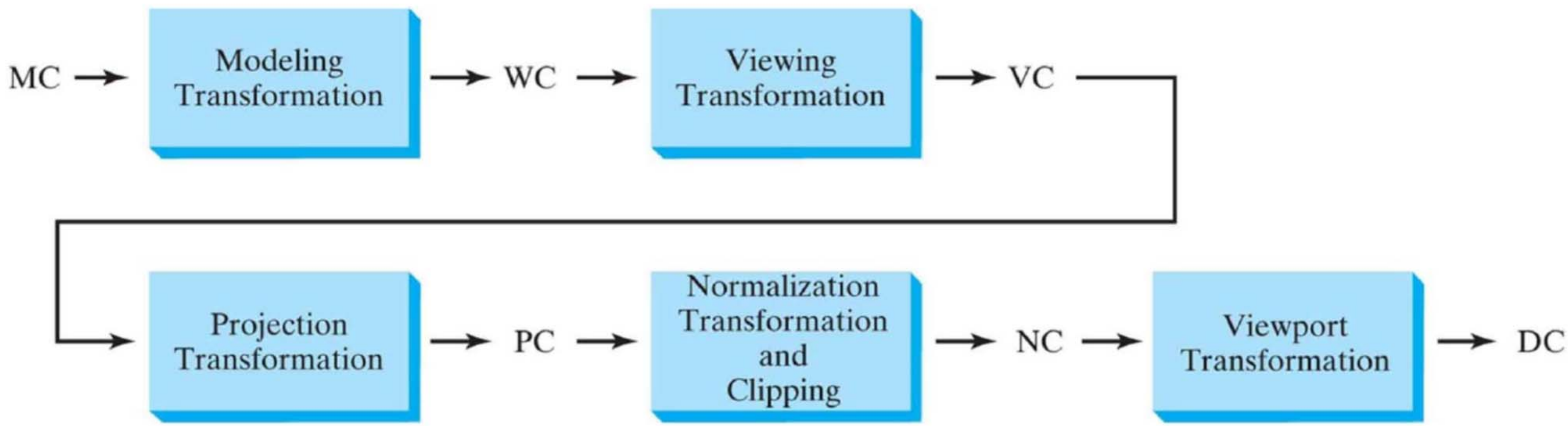
Three Dimensional Viewing Pipeline

- Three dimensional viewing process is inherently more complex than two dimensional viewing process because of the added dimension and the fact that even though object is three dimensional, the display device are only two dimensional
- The mismatch between 3D object and 2D display is compensated by introducing projection. The projection transform 3D objects into a 2D projection plane.
- The steps for computer generation of a view of a three dimensional scene are somewhat analogous to the processes involved in taking a photograph.
- It is the general processing steps for modeling and converting a world coordinate description of a scene to device coordinates



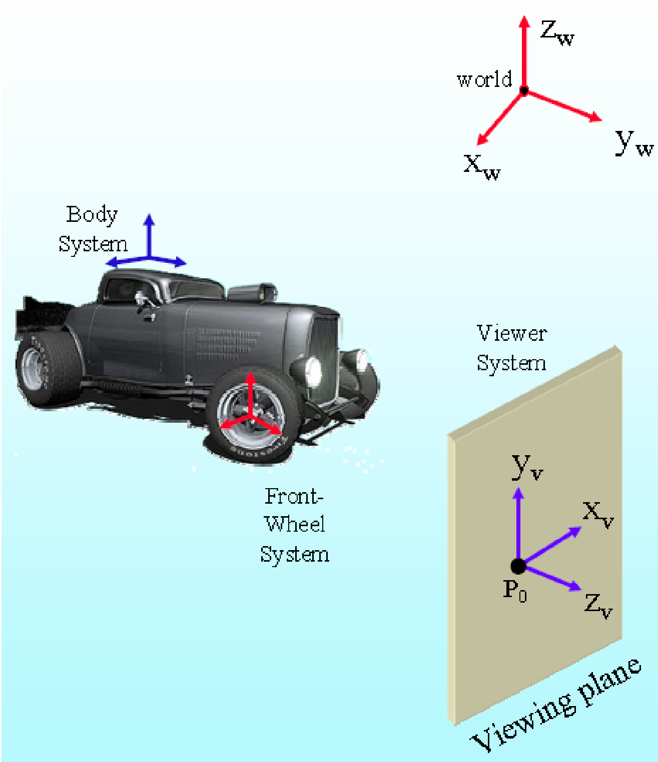
Three Dimensional Viewing Pipeline

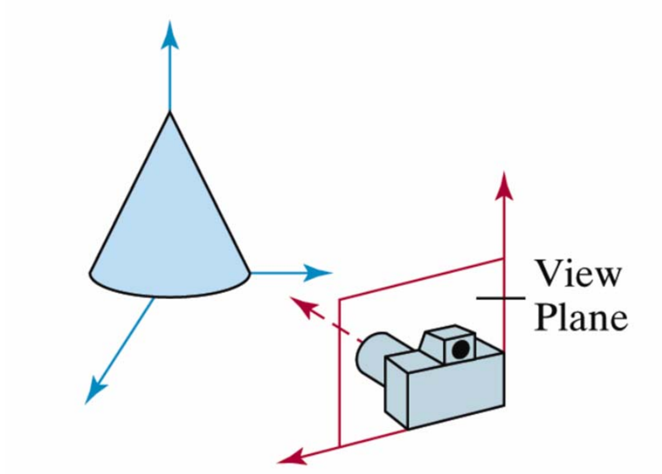
- Steps
  - Construct the shape of individual objects in a scene within modeling coordinate, and place the objects into appropriate positions within the scene (world coordinate).
  - World coordinate positions are converted to viewing coordinates.
  - Convert the viewing coordinate description of the scene to coordinate positions on the projection plane.
  - Positions on the projection plane, will then mapped to the Normalized coordinate and output device.



Viewing Coordinate

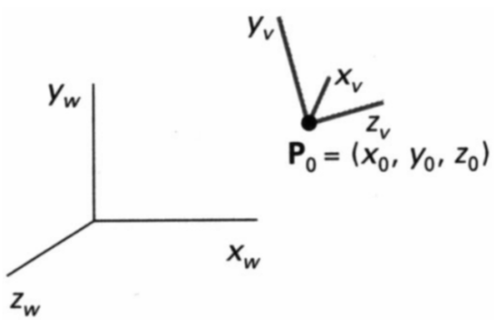
- Viewing coordinates system describe 3D objects with respect to a viewer.
- A Viewing (Projector) plane is set up perpendicular to z_v and aligned with (x_v, y_v).

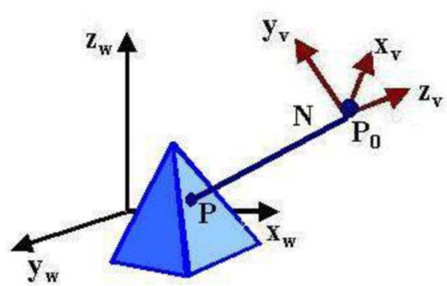


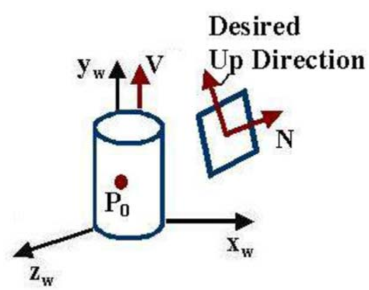


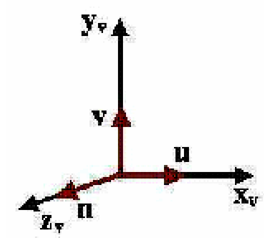
Viewing Coordinate

- We first pick a world coordinate position called **view reference point** (origin of our viewing coordinate system).
- View reference point (P_0) is a point where a camera or eye is located.
- Next, we select the positive direction for the viewing $\mathbf{z_v}$ axis, by specifying the **view plane normal vector**, **N**.
- The direction of **N**, is from the **look at point** (P) to the view reference point (P_0).
- Finally, we choose the ***up direction*** for the view by specifying a vector ***V***, called the ***view up vector***.
- This vector is used to establish the positive direction for the $\mathbf{y_v}$ axis.
- ***V*** is projected into a plane that is perpendicular to the normal vector.
- Using vectors **N** and **V**, the graphics package computer can compute a third vector **U**, perpendicular to both **N** and **V**, to define the direction for the $\mathbf{x_v}$ axis.







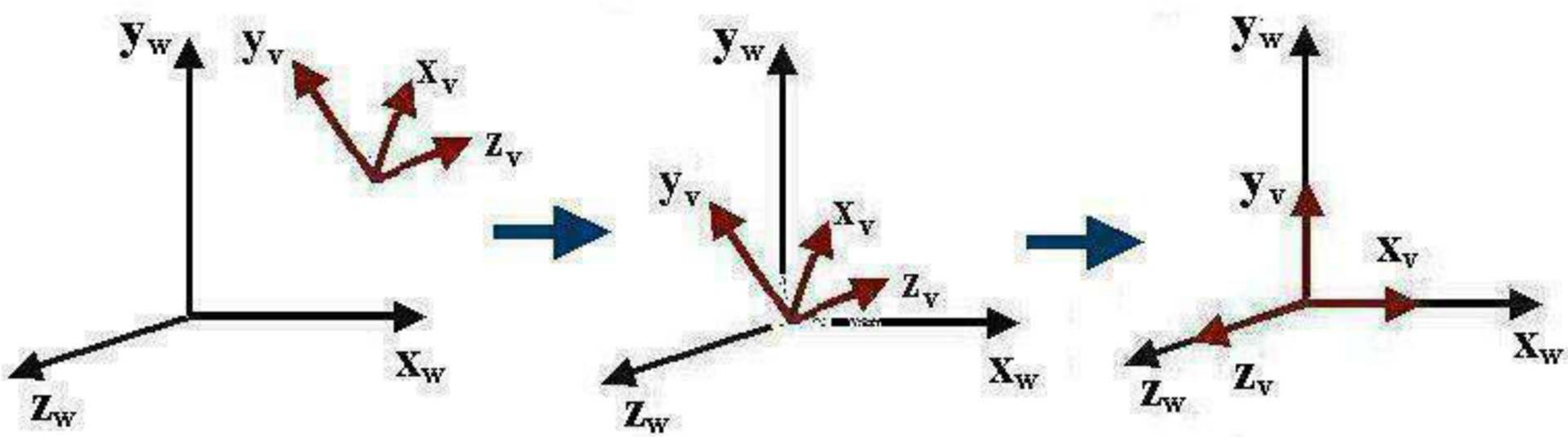


Transforming World to Viewing Coordinate

- Transforming world to viewing coordinate include following sequences:

1. Translate the view reference point to the origin of the world coordinate system
2. Apply the rotation to align the x_v, y_v and z_v axes with the world x_w, y_w and z_w axes respectively.

$$T = \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Transforming World to Viewing Coordinate

- Background

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{i} = -\mathbf{k}$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

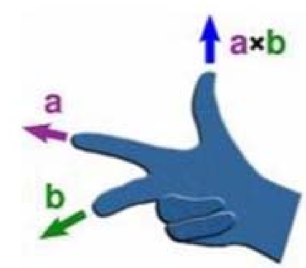
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Given vectors N and V, calculating the unit vector

$$\mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|} = (n_x, n_y, n_z)$$
$$\mathbf{u} = \frac{\mathbf{V} \times \mathbf{N}}{|\mathbf{V} \times \mathbf{N}|} = (u_x, u_y, u_z)$$
$$\mathbf{v} = \mathbf{n} \times \mathbf{u} = (v_x, v_y, v_z)$$

Rotation Matrix

$$R = \begin{bmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ n_x & n_y & n_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



- The complete world to viewing coordinate transformation matrix is

$$M_{wc\text{-}vc} = R \cdot T$$

Projection

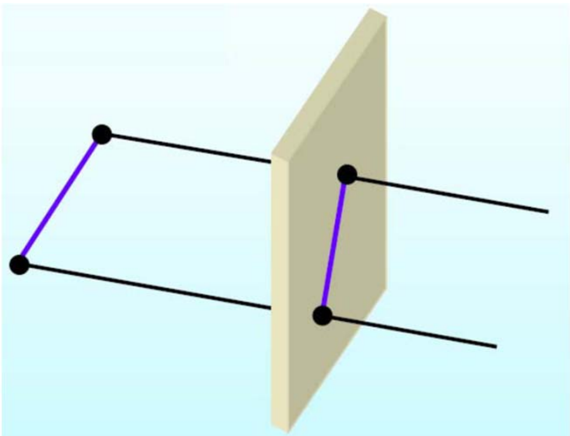
- Projection is the process of representing a three dimensional object or scene into two dimensional medium
- Types of projection

  1. **Parallel Projection**

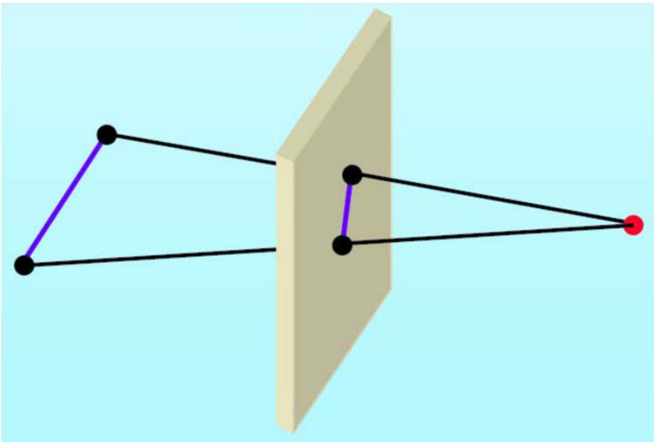
  ➢ Coordinate position are transformed to the view plane along **parallel lines**.

  2. **Perspective Projection**

  ➢ Object positions are transformed to the view plane along lines that converge to the **projection reference (center) point**.



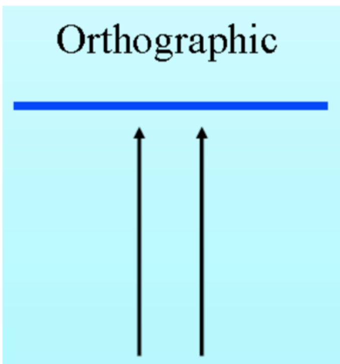
Parallel Projection

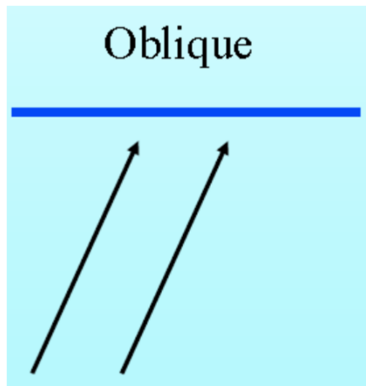


Perspective Projection

Projection

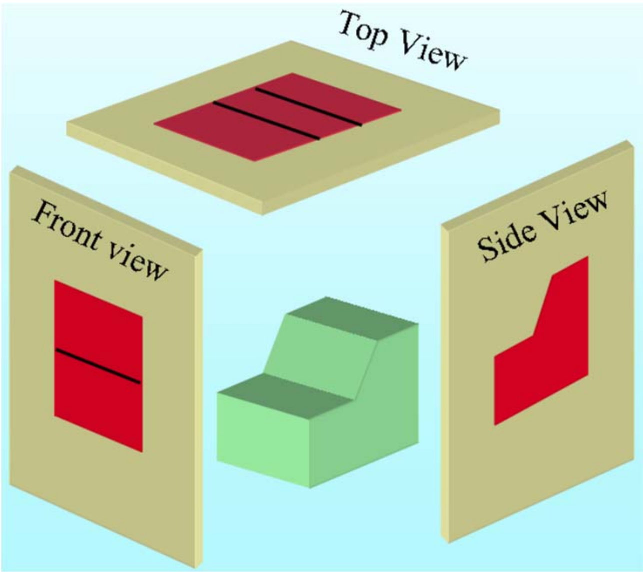
- **Parallel Projection Types**
- **Orthographic-** when the projection is perpendicular to the view plane. Used to produce Front, Side and Top view of an object. Most commonly used projection
- **Oblique** – when the projection is not perpendicular to the view plane. Not commonly used





Projection

- **Orthographic Projection**
  - when the projection is perpendicular to the view plane.
  - Most often used to produce front, side and top view of an object
  - Orthographic projection can display more than one face of an object. Such views are called axonometric orthographic projection
  - The most commonly used axonometric projection is isometric projection

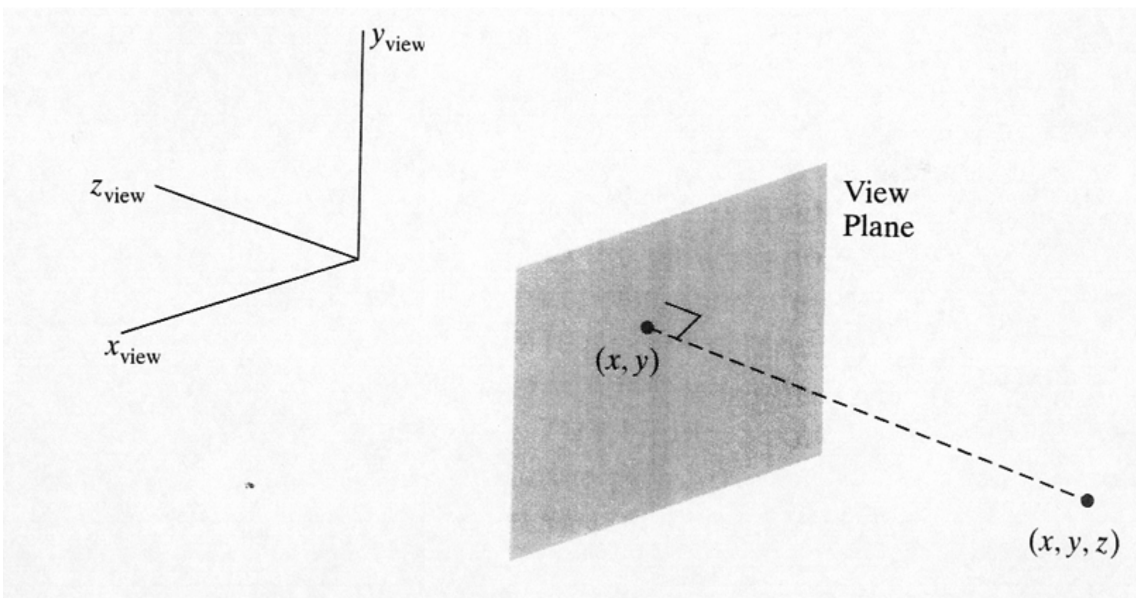


Projection

- **Orthographic Projection**
- If the view plane is placed at position z_{vp} along the z_v axis. Then any point (x,y,z) in viewing coordinates is transformed to projection coordinates as:

  $x_p = x, \; y_p = y$

  where the original z-coordinate is kept for depth information

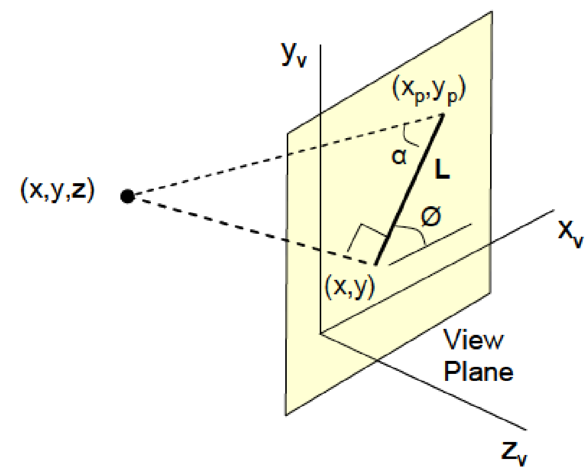


Projection

- **Oblique Projection**
  - when the projection is not perpendicular to the view plane
  - A vector direction is defining the projection lines
  - Can improve the view of an object
  - Point (x,y,z) is projected to position (x_p,y_p) on the view plane.
  - Projector (oblique) from (x,y,z) to (x_p,y_p) makes an angle α with the line (L) on the projection plane that joins (x_p,y_p) and (x,y).
  - Line L is at an angle ϕ with the horizontal direction in the projection plane.
  - Expressing projection coordinates in terms of x, y, L and ϕ:

$$x_p = x + L.\cos\varnothing$$
$$y_p = y + L.\sin\varnothing$$

  - Length L depends on the angle α and z-coordinate of the line to be projected



$$\tan\alpha = \frac{Z}{L} \qquad , L = \frac{Z}{\tan\alpha}, L = ZL_1 \qquad \text{where,} \quad L_1 = \frac{1}{\tan\alpha}$$

$$x_p = x + z(L_1\cos\varnothing)$$
$$y_p = y + z(L_1\sin\varnothing)$$

Projection

- **Oblique Projection**
- The transformation matrix for parallel projection onto $x_v y_v$-plane can be written as,

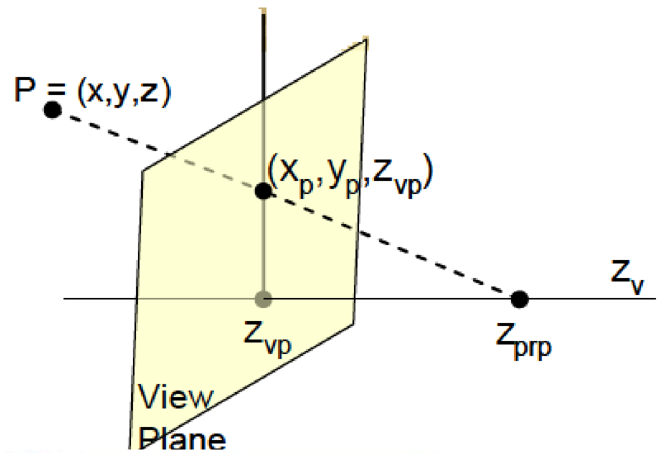
$$M_{parallel} = \begin{bmatrix} 1 & 0 & L_1 \cos\phi & 0 \\ 0 & 1 & L_1 \sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- If $L_1 = 0$, orthographic projection is obtained

- Common choices for angle ϕ are 30 degree and 45 degree
- Two commonly used values for α are those for which tanα = 1 and tanα = 2
- If tanα = 1, α=45degree, the views obtained are called cavalier projections
- If tanα = 2, α=63.4 degree (approx.), the views is obtained are called cabinet projection
- Cabinet projection appear more realistic than cavalier projection

Projection

- **Perspective Projection**
- For a perspective projection, object positions are transformed to the view plane along lines that converge to a point called projection reference point or center of projection.
- Suppose, we set the projection reference point at position z_{prp} along z_v axis, and we place the view plane at z_{vp}



- Parametric equation of perspective projection line to describe coordinate positions

$$x' = x - xu$$
$$y' = y - yu$$
$$z' = z - (z - z_{prp})u$$

u takes value between 0 and 1

P'(x',y',z') represents any point along the projection line

- On view plane

$$z' = z_{vp}; \text{ therefore}$$
$$u = (z_{vp} - z)/(z_{prp} - z)$$

Projection

- **Perspective Projection**

So,

$$\left.\begin{aligned} x_p &= x\left(\frac{z_{prp} - z_{vp}}{z_{prp} - z}\right) = x\left(\frac{d_p}{z_{prp} - z}\right) \\ &\text{and} \\ y_p &= y\left(\frac{z_{prp} - z_{vp}}{z_{prp} - z}\right) = y\left(\frac{d_p}{z_{prp} - z}\right) \end{aligned}\right\} \quad \text{.................. (1)}$$

In a 3D homogeneous coordinate system representation

$$x_p = x_h/h \quad \text{and} \quad y_p = y_h/h \quad \text{.................. (2)}$$

Now comparing eqn 1 and eqn 2, we get

$$x_h = x \quad \text{and} \quad y_h = y$$

and

$$h = (z_{prp} - z)/d_p$$

Projection

- **Perspective Projection**

We know that,

$$z_p = z_{vp}$$

So,

$$z_h = z_p \text{ x } h$$

$$= z_{vp} \text{ x } (z_{prp} - z)/d_p$$

$$z_h = - z.z_{vp}/d_p + z_{vp}.z_{prp}/d_p$$

Also,

$$h = - z/d_p + z_{prp}/d_p$$

Now, the perspective projection transformation matrix in homogeneous coordinate representation is

$$\begin{bmatrix} x_h \\ y_h \\ z_h \\ h \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -z_{vp}/d_p & z_{vp}(z_{prp}/d_p) \\ 0 & 0 & -1/d_p & z_{prp}/d_p \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$