CURRENT AND CONDUCTORS

5.1 INTRODUCTION

In this chapter, we will define current and current density, and then develop continuity equation. Having finished this, we will study Ohm's law and proceed further to find out the conditions which must be met at conductor boundaries. But before we do that, we will have a glance at classification of materials.

5.2 CONDUCTORS, SEMICONDUCTORS, AND INSULATORS

Materials are broadly classified in terms of their electrical properties as conductors and nonconductors. Nonconducting materials are usually referred to as insulators or dielectrics. Thus, materials may be described in terms of their conductivity σ, in mho per meter (℧/m) or siemens per meter (S/m), as conductors and nonconductors, or technically as metals and insulators (or dielectrics). The conductivity of a material usually depends on temperature and frequency. A material with high conductivity ($\sigma >> 1$) is referred to as a metal whereas one with low conductivity ($\sigma << 1$) is referred to as an insulator. A material whose conductivity lies somewhere between those of metals and insulators is called a semiconductor.

The conductivity of metals generally increases with decrease in temperature. At temperatures near absolute zero (T = 0°K), some conductors exhibit infinite conductivity and are called superconductors. Lead and aluminum are typical examples of such metals. The conductivity of lead at 4°K is of the order of 10^{20} mhos/m.

Microscopically, the major difference between a metal and an insulator lies in the amount of electrons available for conductor of current. Dielectric materials have few electrons available for conduction of current in contrast to metals which have an abundance of the electrons.

A semiconductor is a substance which has resistivity (10^{-4} to 0.5 Ωm) in between conductors and insulators e.g., germanium, silicon, carbon etc. In semiconductors, two types of current carriers are present, which are called electrons and holes. Both the carriers move in an electric field and they move in opposite directions and hence, the total current is the sum of currents due to electrons and holes. The conductivity of a semiconductor is expressed as

$$\sigma = \sigma_e + \sigma_h \quad = - \rho_e \mu_e + \rho_h \mu_h$$

where ρ_e and μ_e are electron charge density (a negative number) and electron mobility respectively whereas ρ_h and μ_h are the corresponding values for holes. For pure or intrinsic silicon, the electron and hole mobilities are 0.12 and 0.025 respectively while for germanium, the mobilities are respectively, 0.36 and 0.17. The mobilities listed above are given for a temperature of 300°K.

5.3 CURRENT AND CURRENT DENSITY

Current: Electric charges in motion constitute a current. The total amount of positive charge passing through a reference plane normal to the direction of motion of the charge in one second is referred to the current, and is denoted by I. The unit of current is the ampere (A).

$$I = \frac{dQ}{dt}$$

Current is defined as the motion of positive charges, even though conduction in metals takes place through the motion of electrons.

Current Density ($\vec{J}$): The current density is defined as the current passing through a unit surface area. It is a vector quantity, and its direction is the direction of current flow and is denoted by $\vec{J}$. The unit is amperes per square meter (A/m²).

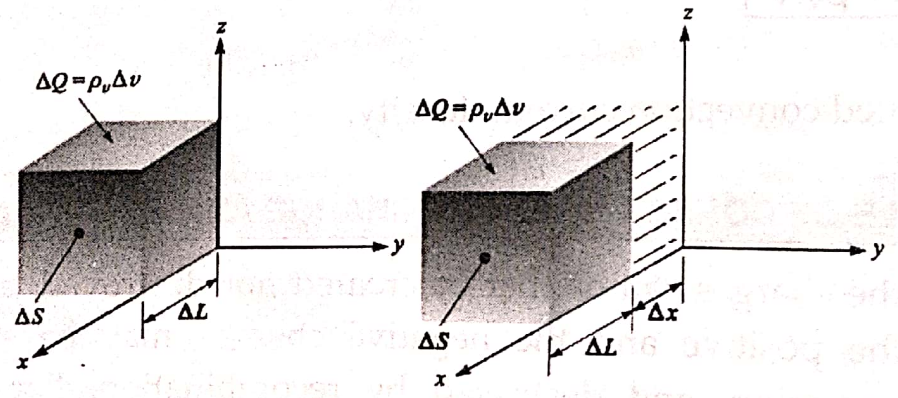

Figure 5.1 An increment of charge, $\Delta Q = \rho_v \Delta S \Delta l$, which moves a distance Δx in a time Δt, produces a component of current density in the limit of $J_x = \rho_v v_x$.

The incremental current ΔI crossing an incremental surface ΔS normal to the current density is $\Delta I = J_N \, \Delta S$. If the current density is not perpendicular to the surface, then

$$\Delta I = \vec{J} \cdot \Delta \vec{S}$$

Total current is

$$I = \int_S \vec{J} \cdot d\vec{S}$$

Consider the element of charge $\Delta Q = \rho_v \, \Delta_v = \rho_v \, \Delta S \, \Delta l$ as shown in Figure 5.1 and let the charge element be oriented such that it possesses only an x-component of velocity.

Now, we assume that some element of charge moves a distance Δx in time Δt. This element of charge has the value, $\Delta Q = \rho_v \, \Delta S \, \Delta x$ and the corresponding current is

$$\Delta I = \frac{\Delta Q}{\Delta t} = \frac{\rho_v \, \Delta S \, \Delta x}{\Delta t}$$

or, $\Delta I = \rho_v \, \Delta S \dfrac{\Delta x}{\Delta t}$

or, $\Delta I = \rho_v \, \Delta S \, v_x$; v_x = x component of the velocity

or, $\dfrac{\Delta I}{\Delta S} = \rho_v \, v_x$

$\therefore \quad J_x = \rho_v \, v_x$

In general, $\boxed{\vec{J} = \rho_v \, \vec{v}}$

where $\vec{J}$ is called convection current density.

5.4 PRINCIPLE OF CONSERVATION OF CHARGE AND CONTINUITY EQUATION

It states that the charges can neither be created nor destroyed, although equal amounts of the positive and the negative charges may be simultaneously created by separation, and destroyed by recombination. For example, the positive charges on the glass rod, and the negative charges on a piece of silk are simultaneously created when they are rubbed together.

Continuity of Current (Continuity Equation)

The continuity equation is based on the principle of conservation of charge.

Consider a region bounded by a closed surface. The current through the closed surface is

$$I = \oint_S \vec{J} \cdot d\vec{S}$$

This outward flow of positive charge owing to current should lead to a decrease of positive charge (or perhaps an increase of negative charge) within the closed surface by the same amount.

Let the charge inside the closed surface be Q_i, then the rate of decrease is $-\frac{dQ_i}{dt}$.

Now, we can write

$$I = \oint_S \vec{J} \cdot d\vec{S} = -\frac{dQ_i}{dt} \quad \text{............. (i)}$$

where –ve sign is interpreted as an outward flowing current.

Equation (i) is the integral form of the continuity equation.

Using the divergence theorem,

$$\oint_S \vec{J} \cdot d\vec{S} = \int_{vol} (\nabla \cdot \vec{J})\, dv$$

∴ Equation (i) becomes

$$\int_{vol} (\nabla \cdot \vec{J})\, dv = -\frac{dQ_i}{dt}$$

$$\text{or,} \quad \int_{vol} (\nabla \cdot \vec{J})\, dv = -\frac{d}{dt} \int_{vol} \rho_v \, dv$$

In moving the time derivative of ρ_v inside the volume integral, it is necessary to use partial differentiation because ρ_v may be a function of time as well as of space coordinates.

$$\int_{vol} (\nabla \cdot \vec{J})\, dv = \int_{vol} -\frac{\partial \rho_v}{\partial t}\, dv$$

Since the expression is true for any volume, however small, it is true for an incremental volume. This results

$$(\nabla.\vec{J})\,\Delta v = -\frac{\partial \rho_v}{\partial t}\Delta v$$

$$\boxed{\therefore\ \nabla.\vec{J} = -\frac{\partial \rho_v}{\partial t}}$$

which is the expression of the continuity equation in **differential** or **point** form.

For steady currents, charge density does not vary with time, $\frac{\partial \rho_v}{\partial t} = 0$ and thus,

$\nabla.\vec{J} = 0$, Thus, steady currents are divergenceless or solenoidal.

Since $\nabla.\vec{J} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{J}.d\vec{S}}{\Delta v}$, the above equation explains that the current, or charge per second, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point. Since, the sign of divergence is negative, each point within Δv enclosed by surface is the sink of positive charges for these charges are decreasing within the volume Δv.

It must be kept in mind that the continuity equation is derived from the principle of conservation of charge and essentially states that there can be no accumulation of charge at any point.

5.5 POINT FORM OF OHM'S LAW

The resistivity of a material is defined as

$$S = \frac{RA}{l}\ (\Omega\,m)$$

The conductivity σ of the material is the reciprocal of resistivity, that is

$$\sigma = \frac{1}{S} = \frac{l}{RA}\ (\text{mho m}^{-1}) \quad \text{.......... (i)}$$

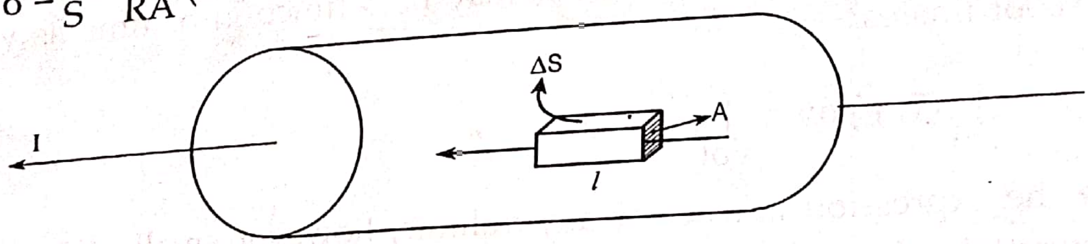

Figure 5.2 Current emerging out of a conductor.

The potential difference between the sides of an infinitely small volume dv enclosed by the surface ΔS is given by

$$V = E\,l \quad \text{.................. (ii)}$$

The total current passing through the area A is

$$I = J\,A \quad \text{.................. (iii)}$$

Ohm's law is

$$V = I\,R$$

Using equation (i) and (ii),

$$E\,l = J\,A\,R$$

$$\text{or,} \quad J = \frac{E\,l}{A\,R}$$

$$\text{or,} \quad J = \frac{l}{AR}E$$

$$\text{or,} \quad J = \sigma E \quad \text{[from equation (i)]}$$

In vector form, $\boxed{\vec{J} = \sigma \vec{E}}$ .... (iv)

Since the volume is assumed infinitely small, a point, equation (iv) is interpreted as the **point form of Ohm's law.**

5.6 RELAXATION TIME CONSTANT (RTC)

Any charge placed inside the conductor moves towards the surface and eventually appears on the surface of the conductor as shown in the figure below.

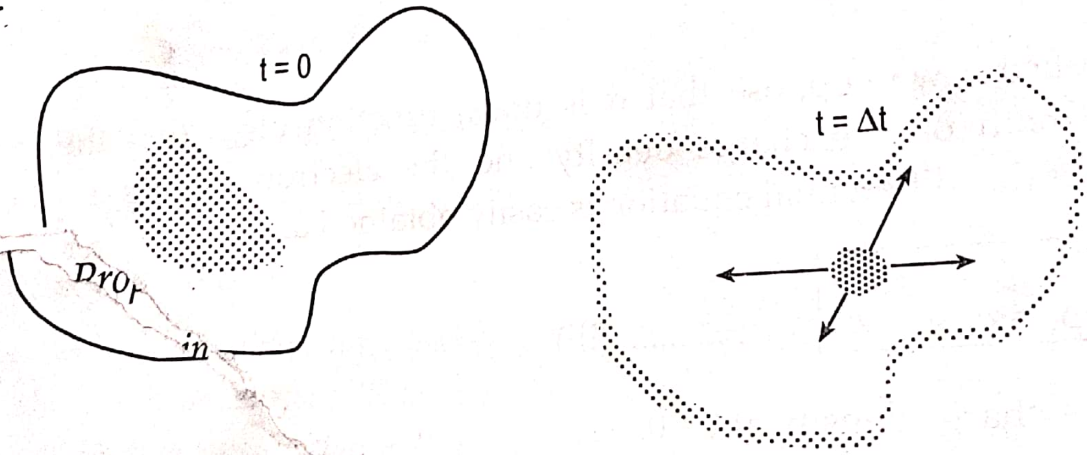

Figure 5.3 The charges decaying at a point within the conductor, and reappearing on the surface.

A time constant that shows how fast the charges decay at a point within the conductor, and reappear on the surface is termed as the relaxation time constant (RTC) for the conductor.

The continuity equation is

$$\nabla . \vec{J} = -\frac{\partial \rho_v}{\partial t} \quad \text{............ (i)}$$

The point form of Ohm's law is

$$\vec{J} = \sigma \vec{E} \quad \text{............. (ii)}$$

From equations (i) & (ii),

$$\nabla . (\sigma \vec{E}) = -\frac{\partial \rho_v}{\partial t}$$

$$\text{or, } \nabla . \left(\frac{\sigma}{\varepsilon} \vec{D} \right) = -\frac{\partial \rho_v}{\partial t}$$

Assuming a homogeneous medium where σ and ε do not vary with position,

$$\nabla . \vec{D} = -\frac{\varepsilon}{\sigma} \frac{\partial \rho_v}{\partial t}$$

Maxwell's first equation is $\nabla . \vec{D} = \rho_v$,

$$\rho_v = -\frac{\varepsilon}{\sigma} \frac{\partial \rho_v}{\partial t}$$

$$\text{or, } \frac{\varepsilon}{\sigma} \frac{\partial \rho_v}{\partial t} + \rho_v = 0$$

$$\text{or, } \frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon} \rho_v = 0$$

For simplicity, we suppose that σ is not a function of ρ_v (but the fa depends both on the charge density and the electron mobility). Then, solution of this differential equation is easily obtained as

$$\boxed{\rho_v = \rho_o \, e^{-\left(\frac{\sigma}{\varepsilon}\right) t}} \quad \text{............. (iii)}$$

where ρ_o = charge density at t = 0.

Equation (iii) shows an exponential decay of charge density at every point with a time constant of $\frac{\varepsilon}{\sigma}$.

$$\rho_v = \rho_o e^{\frac{-t}{T}} \text{ where } T = \frac{\varepsilon}{\sigma}$$

When $t = T$, $\rho_v = \rho_o e^{-1} = 0.37\ \rho_o$

Descriptively, when time equals T, the charges within the conductor decays to 37% of the initial charges ρ_o, and simultaneously reappear on the conductor surface. This time is called the relaxation time constant for the conductor of interest.

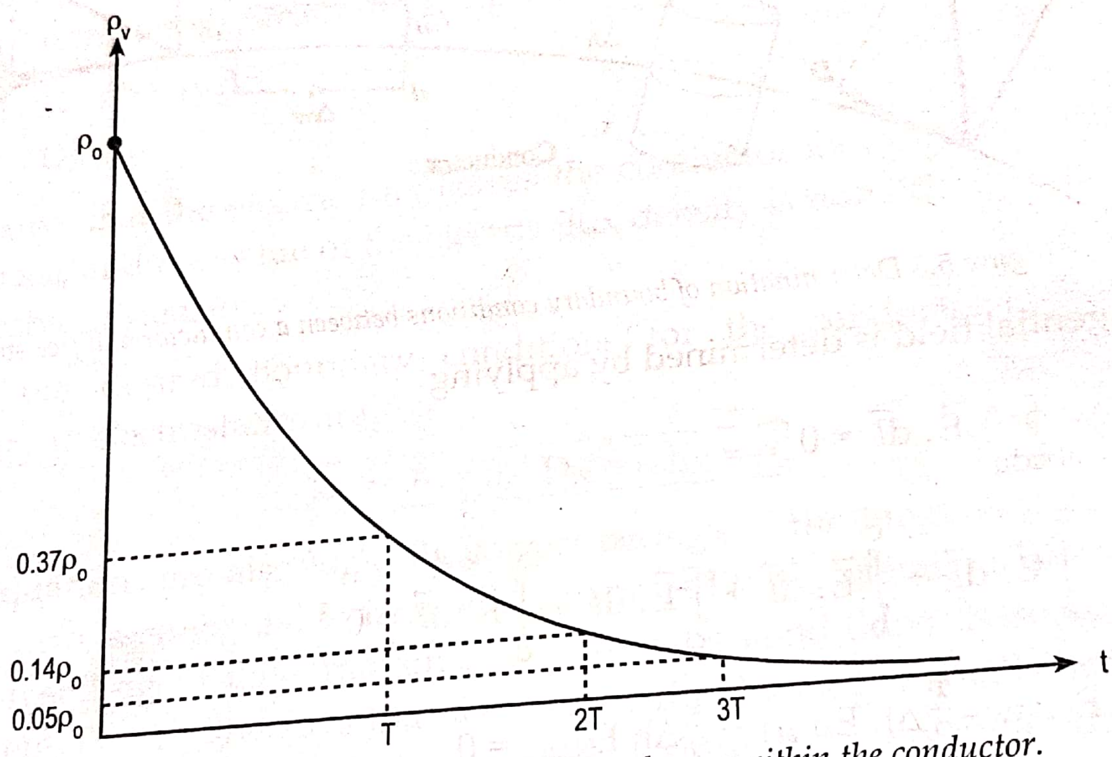

Figure 5.4 Profile of decaying of the charges within the conductor.

5.7 BOUNDARY CONDITION BETWEEN THE CONDUCTOR AND FREE SPACE

Before starting with the derivation, we shall understand conductor properties.

Conductor Properties

No charge may remain within the conductor. If it did, the resulting electric field would force the charges to the surface. Therefore, the final result within a conductor is zero charge density, and a surface charge density resides on the exterior surface. This is one of the two characteristics of a good conductor. The other characteristic for static conditions is that the electric field intensity within the conductor is zero.

Hence, for electrostatics, no charge and no electrical field may exist at any point within a conductor.

Derivation

Consider an arrangement which depicts a boundary between a conductor and free space showing the tangential and normal components of $\vec{D}$ and $\vec{E}$ on the free-space side of the boundary. Both fields are zero in the conductor.

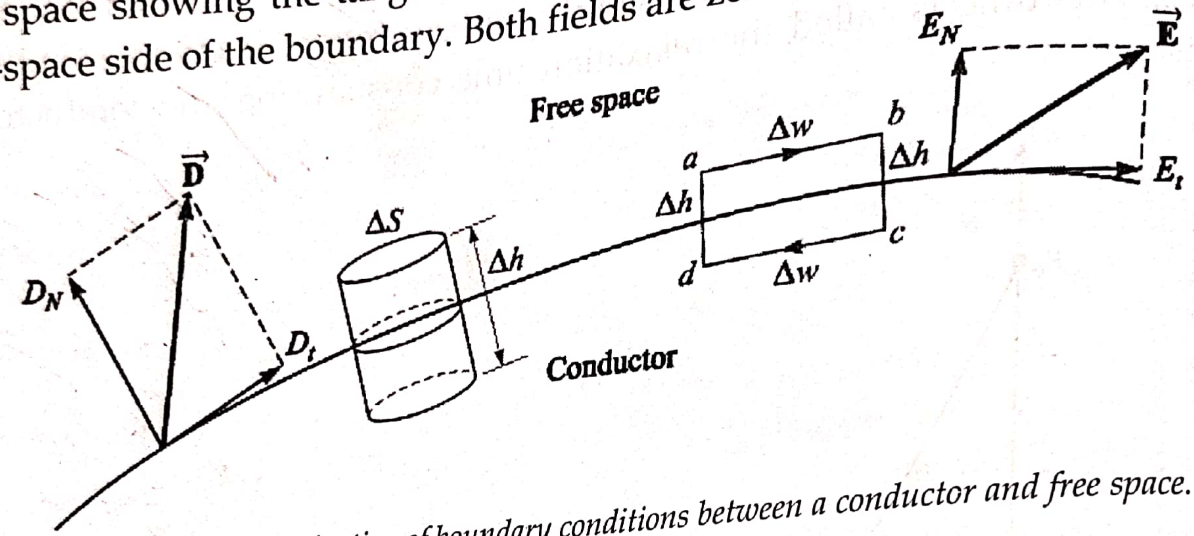

Figure 5.5 Determination of boundary conditions between a conductor and free space.

The tangential field is determined by applying

$$\oint_{abcda} \vec{E} . d\vec{l} = 0$$

$$\text{or, } \int_a^b \vec{E} . d\vec{l} + \int_b^c \vec{E} . d\vec{l} + \int_c^d \vec{E} . d\vec{l} + \int_d^a \vec{E} . d\vec{l} = 0$$

$$\text{or, } E_t \Delta w - \frac{1}{2}\Delta h\, E_{N,\text{ at b}} + \frac{1}{2}\Delta h\, E_{N,\text{ at a}} = 0$$

Here, $\frac{1}{2}\Delta h$ (instead of Δh) has been written for both $\int_b^c dl$ and $\int_d^a dl$. This is because E is zero within conductor. Also, note that in each integral either normal or tangential component makes 90° with dl and the other makes 0° or 180° with dl. Let $\Delta h \to 0$ and keep Δw small and finite.

$$\text{or, } E_t \Delta w - 0 + 0 = 0$$

or, $E_t\, \Delta w = 0$

Since $\Delta w \neq 0$,

$$E_t = 0$$

The normal field is determined by applying

$$\oint_S \vec{D} \cdot d\vec{S} = \Delta Q$$

or, $\int_{top} \vec{D} \cdot d\vec{S} + \int_{bottom} \vec{D} \cdot d\vec{S} + \int_{sides} \vec{D} \cdot d\vec{S} = \Delta Q$

or, $D_N\, \Delta S \cos 0° + 0 + 0 = \Delta Q$

The second term is zero because no electric field exists within a conductor, and the third term is zero because the height of the cylinder Δh is infinitely small.

or, $D_N\, \Delta S = \rho_s\, \Delta S$

$\therefore\ D_N = \rho_s$

This shows that the electric flux leaves the conductor in a direction normal to the surface, and the value of the electric flux density is numerically equal to the surface charge density.

Hence, the desired boundary conditions for the conductor-to-free space boundary in electrostatics are

$$\boxed{D_t = \varepsilon_o E_t = 0, \quad D_N = \varepsilon_o E_N = \rho_s}$$

These equations are also logically correct because if the tangential component was not zero, a tangential force would be applied to the elements of the surface charge, resulting in their motion and non static conditions. Also for normal component, Gauss' law says that the electric flux leaving a small increment of surface must be equal to the charge residing on that incremental surface. The flux cannot penetrate into the conductor, for the total field there is zero. It must then leave the surface normally.

Lastly, owing to zero tangential electric field intensity, we conclude that a conductor surface is an **equipotential surface.**

Similarly, if we calculate boundary conditions between the conductor and dielectric, the results are

$$D_t = \varepsilon_0 \varepsilon_r E_t = 0, \quad D_N = \varepsilon_0 \varepsilon_r E_N = \rho_s$$

1. Given the vector current density $\vec{J} = 10\rho^2 z\ \hat{a}_\rho - 4\rho \cos^2\phi\ \hat{a}_\phi\ \frac{mA}{m^2}$, find the current flowing outward through the circular band $\rho = 3$, $0<\phi<2\pi$, $2<z<2.8$. [2074 Chaitra]

Solution:

$$I = \int_S \vec{J} \cdot d\vec{S}$$

$$= \int_S \left[(10\rho^2 z\ \hat{a}_\rho - 4\rho\cos^2\phi\ \hat{a}_\phi)\times 10^{-3}\right] \cdot (\rho d\phi dz\ \hat{a}_\rho)$$

$$= \int_S 10^{-2}\rho^3 z\ d\phi dz = 10^{-2}\times 3^3 \int_{\phi=0}^{0.3\pi}\int_{z=0}^{2.8} z\ d\phi dz = 3.257\ A$$

2. A current density in certain region is given as $\vec{J} = 20\sin\theta\cos\phi\ \hat{a}_r + \frac{1}{r}\hat{a}_\phi\ A/m^2$. Find : (i) the average value of J_r over the surface $r = 1$, $0 < \theta < \frac{\pi}{2}$, $0 < \phi < \frac{\pi}{2}$ (ii) $\frac{\partial \rho_v}{\partial t}$. [2073 Shrawan]

Solution:

$$\vec{J} = 20\sin\theta\cos\phi\ \hat{a}_r + \frac{1}{r}\hat{a}_\phi\ A/m^2$$

i. $$I = \oint_S \vec{J}\cdot d\vec{S} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} (20\sin\theta\cos\phi\ \hat{a}_r + \frac{1}{r}\hat{a}_\phi)\cdot(r^2\sin\theta\ d\theta\ d\phi\ \hat{a}_r)$$

$$= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} 20\sin^2\theta\cos\phi\ d\theta\ d\phi \qquad \text{for } r=1$$

$$= 5\pi\ A$$

Now, using

$$I = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} (J_r)(r^2\sin\theta\ d\theta\ d\phi)$$

$$\text{or, } 5\pi = J_r[-0+1]\frac{\pi}{2} \quad \text{for } r = 1$$

$$\therefore J_r = 10\ A/m^2$$

ii. $\frac{\partial \rho_v}{\partial t} = -[\nabla \cdot \vec{J}]$

$$= -\left[\frac{1}{r^2}\frac{\partial(r^2 J_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(J_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(J_\phi)}{\partial\phi}\right]$$

$$= \frac{-40\sin\theta\cos\phi}{r}$$

3. The potential field, $V = 2x^2 + 4y - 2z^2$ V, exists in the free space surrounding a perfectly conducting surface. Point P(4, 3, 2) lies on the surface. (a) Give the equation of the surface. (b) Find the unit vector outward normal to the surface at P, assuming the origin is inside the surface.

a. $V|_{(4,3,2)} = 2 \times (4)^2 + 4 \times 3 - 2 \times (2)^2 = 36$ V

The equation of the surface is calculated as

$$V = 2x^2 + 4y - 2z^2$$

or, $36 = 2x^2 + 4y - 2z^2$

$\therefore$ $2x^2 + 4y - 2z^2 - 36 = 0$

b. Assuming that origin is inside the surface, the unit vector outward normal to the surface is given as

$$\hat{a}_N = -\frac{\vec{E}}{|\vec{E}|}$$

Here,

$$\vec{E} = -\nabla V$$

$$= -\left[\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right]$$

$$= -\left[\frac{\partial(2x^2+4y-2z^2)}{\partial x}\hat{a}_x + \frac{\partial(2x^2+4y-2z^2)}{\partial y}\hat{a}_y + \frac{\partial(2x^2+4y-2z^2)}{\partial z}\hat{a}_z\right]$$

$$= -16\,\hat{a}_x - 4\,\hat{a}_y + 8\,\hat{a}_z \text{ V/m}$$

$$|\vec{E}| = \sqrt{(-16)^2 + (-4)^2 + (8)^2} = 18.33 \text{ V/m}$$

Thus,

$$\hat{a}_N = \frac{-\vec{E}}{|\vec{E}|} = 0.8729\,\hat{a}_x + 0.2182\,\hat{a}_y - 0.4364\,\hat{a}_z$$

Hammered Problems

1. If $\vec{J} = \frac{1}{r^3}(2\cos\theta\hat{a}_r + \sin\theta\,\hat{a}_\theta)\frac{A}{m^2}$, calculate the current passing through

(a) A hemispherical shell of radius 20 cm, $0 < \theta < \frac{\pi}{2}, 0 < \phi < 2\pi$

Answer: (a) 31.4 A (b) 0

(b) A spherical shell of radius 10 cm.

2. For the current density $\vec{J} = 10.2\ \sin^2\phi\ \hat{a}_\rho \frac{A}{m^2}$, find the current through the cylindrical surface $\rho = 2, 1 \le z \le 5$m.

Answer: 753.963 A

3. The current density in certain region is approximated by $\vec{J} = \left(\frac{0.1}{r}\right) e^{-10^6 t}\hat{a}_r \frac{A}{m^2}$ in spherical coordinates.

(a) How much current is crossing the surface r = 50 cm at t = 1 μs?

[2072 Kartik]

(b) Find $\rho_v(r, t)$ assuming that $\rho_v \to 0$ as $t \to \infty$.

Answer: (a) 0.116 A (b) $-\left(\frac{0.1}{r}\right) e^{-10^6 t} \frac{\mu C}{m^2}$

4. It is found that $\vec{E} = 60\hat{a}_x + 20\hat{a}_y - 30\hat{a}_z$ mV/m at a particular point on the interface between air and a conducting surface. Find $\vec{D}$ and ρ_s at that point.

Answer: $0.531\hat{a}_x + 0.177\hat{a}_y - 0.265\hat{a}_z$ pC/m², 0.619 pC/m²