Bernoulli's equation

An equation is of the form $\frac{dy}{dx}+py=Qy^n$ Where P and Q are function of x or constant is called Bernoulli's equation.
It can be solved by reducing it to linear equation.

For this $\frac{dy}{dx}+py=Qy^n$ ....(1)

Dividing both sides by y^n

$$\frac{1}{y^n}\frac{dy}{dx}+\frac{y}{y^n}P=Q$$

or, $\frac{1}{y^n}\frac{dy}{dx}+\frac{1}{y^{n-1}}p=Q$

Put $\frac{1}{y^{n-1}}=v$

$y^{-n+1}=v$ differentiating it w.r. to x

$(-n+1)\,y^{-n}\frac{dy}{dx}=\frac{dv}{dx}$

or, $\frac{1}{y^n}\frac{dy}{dx}=\frac{1}{(-n+1)}\frac{dv}{dx}$

So the eq^n (1) reduces to

$$\frac{1}{(-n+1)}\frac{dv}{dx}+pv=v$$

$$\frac{dv}{dx}+P(-n+1)\,v=(-n+1)Q\ ....(2)$$

Thus eq^n (2) is linear differential eq^n of first order in v so its integrating factor (I. F.) = $e^{\int p(-n+1)dx}$
and its general solution is
$v\times I.F.=\int(-n+1)\,Q\times(I.F)\,dx+c$

Exercise - 24

Solve the following differential equations

1. $\frac{dy}{dx}+\frac{y}{x}=\frac{y^2}{x^2}$

Sol^n. Given differential equation is

$$\frac{dy}{dx}+\frac{y}{x}=\frac{y^2}{x^2}$$

Dividing both sides by y^2

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y}.\frac{1}{x}=\frac{1}{x^2} \text{ ....(1)}$$

Put $\frac{1}{y}=v$ Then $-\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$

or, $\frac{1}{y^2}\frac{dy}{dx}=-\frac{dv}{dx}$

Now eq^n (1) becomes

$$-\frac{dv}{dx}+\frac{1}{x}v=\frac{1}{x^2}$$

or, $\frac{dv}{dx}-\frac{1}{x}v=-\frac{1}{x^2}$ ....(2)

eq^n (2) is linear differential eq^n. form

$P=-\frac{1}{x}, Q=-\frac{1}{x^2}$

I.F. $= e^{\int pdx}=e^{-\int\frac{1}{x}dx}=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$

Multiplying eq^n (2) by I.F. we get

v. $\frac{1}{x}=-\int\frac{1}{x^2}.\frac{1}{x}dx+c$

$=-\int\frac{1}{x^3}dx+c$

$\frac{v}{x}=\frac{1}{2x^2}+c$

Restoring the value of $v=\frac{1}{y}$ we get

$\frac{1}{xy}=\frac{1}{2x^2}+c$

or, $2x = y + 2x^2y . c$

or, $2x = y + 2cx^2y$ is the required solution.

2. $(1-x^2)\frac{dy}{dx}+xy=xy^2$ ***(B. E. 2067)***

Sol^n. Given differential equation is

$(1-x^2)\frac{dy}{dx}+xy=xy^2$

or, $\frac{dy}{dx}+\frac{x}{1-x^2}.y=\frac{x}{1-x^2}.y^2$

Dividing both sides by y^2

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{x}{1-x^2}.\frac{1}{y}=\frac{x}{1-x^2} \text{ ....(1)}$$

Put $\frac{1}{y}=v$Then $-\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$

Now eq^n (1) becomes

$-\frac{dv}{dx}+\frac{x}{1-x^2}v=\frac{x}{1-x^2}$ ....(2)

or, $\frac{dv}{dx}-\frac{x}{1-x^2}v=-\frac{x}{1-x^2}$ ....(2)

eq^n (2) is lineal differential equation form.

Here, $P=-\frac{x}{1-x^2}, Q=-\frac{x}{1-x^2}$

$\therefore$ I . F. $=e^{\int pdx}=e^{\int -\frac{x}{1-x^2}dx}=e^{\frac{1}{2}\log(1-x^2)}$

$=e^{\log\left(\sqrt{1-x^2}\right)}=\sqrt{1-x^2}$

Multiplying eq^n (2) by I.F. we get,

v. $\sqrt{1-x^2}=-\int.\frac{x}{(1-x^2)}\sqrt{1-x^2}dx+c$

$-\int\frac{x}{\sqrt{1-x^2}}dx+c$

$v\sqrt{1-x^2}=\sqrt{1-x^2}+c$

Restoring the value of $v=\frac{1}{y}$ we get

$\frac{1}{y}\sqrt{1-x^2}=\sqrt{1-x^2}+c$

$\sqrt{1-x^2}=y\sqrt{1-x^2}-cy$

$\left(\sqrt{1-x^2}\right)(1-y)=cy$ is the required solution.

3. $\frac{dy}{dx}+\frac{1}{x}=\frac{e^y}{x^2}$

Sol^n. Given differential equation is

$\frac{dy}{dx}+\frac{1}{x}=\frac{e^y}{x^2}$

Dividing both sides by e^y

$$\frac{1}{e^y}\frac{dy}{dx}+\frac{1}{e^y}\cdot\frac{1}{x}=\frac{1}{x^2}\ ....(1)$$

Put $\frac{1}{e^y}=v$

$e^{-y}=v$ Then $-\ e^{-y}\frac{dy}{dx}=\frac{dv}{dx}$

or, $\frac{1}{e^y}\frac{dy}{dx}=\frac{-dv}{dx}$

Now (1) becomes

$$-\frac{dv}{dx}+\frac{1}{x}v=\frac{1}{x^2}$$

or, $\frac{dv}{dx}-\frac{1}{x}v=\frac{-1}{x^2}$ ....(2)

eq^n (2) is linear differential equation form

$P=-\frac{1}{x},\ Q=-\frac{1}{x^2}$

$\therefore$ I. F $= e^{\int pdx}=e^{\int -\frac{1}{x}dx}=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$

Multiplying eq^n (2) by I.F. we get,

$$v.\ \frac{1}{x}=\int -\frac{1}{x^2}\cdot\frac{1}{x}dx+c$$

$$=-\int\frac{1}{x^3}dx+c$$

or, $\frac{v}{x}=\frac{1}{2x^2}+c$

Restoring the value of $v=\frac{1}{e^y}$ we get

$$\frac{1}{xe^y}=\frac{1}{2x^2}+c$$

or, $2x = (1 + 2cx^2)\ e^y$

or, $2x = (1 + kx^2)\ e^y$ where K = 2c is the required solution.

4. $\mathbf{\frac{dy}{dx}+\frac{y}{x}logy=\frac{y}{x^2}(logy)^2}$ ***(B. E. 2067)***

Sol^n. Given differential equation is

$$\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^2}(\log y)^2$$

dividing both sides by y $(logy)^2$

$$\frac{1}{y(\log y)^2}\frac{dy}{dx}+\frac{1}{(\log y)}\cdot\frac{1}{x}=\frac{1}{x^2}\ ....(1)$$

Put $\frac{1}{\log y}=Q$

or, $\frac{-1}{(\log y)^2}\cdot\frac{1}{y}\frac{dy}{dx}=\frac{dv}{dx}$

or, $\frac{1}{y(\log y)^2}\frac{dy}{dx}=-\frac{dv}{dx}$

Now eq^n (1) becomes\

$$\frac{-dv}{dx}+\frac{1}{x}v=\frac{1}{x^2}$$

or, $\frac{dv}{dx}-\frac{1}{x}v=\frac{-1}{x^2}$ ....(2)

eq^n (2) is linear differential equation form ;

Here, $P=\frac{-1}{x},Q=-\frac{1}{x^2}$

$\therefore$ I. F. $= e^{\int pdx}=e^{-\int\frac{1}{x}dx}=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$

multiplying eq^n (2) by I.F. = we get

$$v.\ \frac{1}{x}=\int -\frac{1}{x^2}\cdot\frac{1}{x}dx+c$$

$$=-\int\frac{1}{x^3}dx+c$$

$$\frac{v}{x}=\frac{1}{2x^2}+c$$

Restoring the value of $v=\frac{1}{(\log y)}$ we get

$$\frac{1}{x\log y}=\frac{1}{2x^2}+c$$

or, x = logy ($\left(\frac{1}{2}+cx^2\right)$ is the required solution.

5. $\mathbf{\frac{dy}{dx}+xsin^2y=x^3cos^2y}$ ***(B. E. 2061)***

Sol^n. Given differential equation is

$$\frac{dy}{dx}+x\sin^2 y=x^3\cos^2 y$$

Dividing both sides by cos^2y we get,

$$\frac{1}{\cos^2 y}\frac{dy}{dx}+2x.\tan y = x^3 \text{ ....(1)}$$

Put tan y = v

Then $\sec^2 y \frac{dy}{dx}=\frac{dv}{dx}$

or, $\frac{1}{\cos^2 y}\frac{dy}{dx}=\frac{dv}{dx}$

Now eq^n (1) becomes

$$\frac{dv}{dx}+2xv = x^3 \text{........(2)}$$

eq^n (2) is linear differential equation form;

Here, p = 2x, Q = x^3

$\therefore$ I.F. = $e^{\int p dx} = e^{\int 2x dx} = e^{x^2}$

Multiplying eq^n (2) by I.F. we get,

$$v.\ e^{x^2} = \int x^3 e^{x^2} dx. + c$$

$$= \int x.\ x^2 e^{x^2} dx + c$$

put x^2 = t then 2xdx = dt or, xdx = $\frac{1}{2}$dt

$$\therefore\ ve^{x^2} = \frac{1}{2}\int te^t dt$$

$$= \frac{1}{2}\left[te^t - e^t\right] + c$$

$$ve^{x^2} = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + c$$

Restoring the value of v = tany we get

$$\tan y\ e^{x^2} = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + c$$

or, $e^{x^2}\tan y = \frac{1}{2}e^{x^2}(x^2-1)+c$ is the required solution.

6. $\mathbf{\cos x \frac{dy}{dx} = y(\sin x - y)}$

Sol^n. Given differential equation is

$$\text{Cosx}\frac{dy}{dx} = y(\sin x - y)$$

or, $\frac{dy}{dx} = \frac{(y\sin x - y)}{\cos x} = y\frac{\sin x}{\cos x} - \frac{y^2}{\cos x}$

or, $\frac{dy}{dx} = y\tan x - \frac{y^2}{\cos x}$

or, $\frac{dy}{dx} - y\tan x = \frac{-y^2}{\cos x}$ ....(1)

dividing both sides by – y^2

or, $\frac{-1}{y^2}\frac{dy}{dx} + \frac{1}{y}\tan x = \frac{1}{\cos x}$

Put $\frac{1}{y}$ = v Then $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$

Now eq^n (1) becomes

$$\frac{dv}{dx} + \tan v = \frac{1}{\cos x}\text{........(2)}$$

eq^n (2) is linear differential eq^n form p = tanx, Q = $\frac{1}{\cos x}$

$\therefore$ I.F. = $e^{\int p dx} = e^{\int \tan x dx} = e^{\log \sec x}$

= secx

Multiplying eq^n (2) by I.F. we get

v. secx = $\int \frac{1}{\cos x}.\sec x dx + c$

= $\int \sec^2 x dx$ + c

or, v. secx = tanx + c

Restaring the value of v = $\frac{1}{y}$ we get

$$\frac{1}{y}\sec x = \tan x + c$$

secx = y tanx + cy is the required solution.

7. $\mathbf{x\frac{dy}{dx} + y = y^2 \log x}$

Sol^n. Given differential equation is

$$x\frac{dy}{dx} + y = y^2 \log x$$

Dividing both sides by xy^2 we get,

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}.\frac{1}{y} = \frac{\log x}{x} \text{ ....(1)}$$

put $\frac{1}{y}$ = v Then $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{1}{y^2}\frac{dy}{dx} = -\frac{dv}{dx}$

Now eq^n (1) becomes

$$-\frac{dv}{dx}+\frac{1}{x}.v=\frac{\log x}{x}$$

or, $\frac{dv}{dx}-\frac{1}{x}v=\frac{-\log x}{x}$ ....(2)

eq^n (2) is linear differential equation form

$p=\frac{-1}{x}$, $Q=\frac{\log x}{x}$

$\therefore$ I. F $= e^{\int p dx}=e^{-\int \frac{1}{x}dx}=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$

Multiplying eq^n (2) by I.F. we get

$$v.\frac{1}{x}=-\int \frac{\log x}{x}.\frac{1}{x}dx+c$$

or, $\frac{v}{x}=-\int \frac{1}{x^2}\log x dx+c$

$$=-\left\{\log x\left(\frac{-1}{x}\right)-\int \frac{1}{x}.\left(\frac{-1}{x}\right)dx\right\}+c$$

$$=\frac{\log x}{x}-\int \frac{1}{x^2}dx+c$$

$$\frac{v}{x}=\frac{\log x}{x}+\frac{1}{x}+c$$

Restoring the value of $v=\frac{1}{y}$ we get

$$\frac{1}{xy}=\frac{\log x}{x}+\frac{1}{x}+c$$

or, 1 = y(logx + 1) + cxy

or, cxy + y(logx + 1) = 1 is the required solution.

8. $\mathbf{\frac{dy}{dx}=ytanx-y^2secx}$ ***[B.E. 2060]***

Sol^n. Given differential equation is

$$\frac{dy}{dx}=y\tan x-y^2\sec x$$

or, $\frac{dy}{dx}-y\tan x=-y^2\sec x$

Dividing both sides by – y^2 we get

$$\frac{-1}{y^2}\frac{dy}{dx}+\tan x.\frac{1}{y}=\sec x \text{ ....(1)}$$

Put $\frac{1}{y}=v$ then

$$\frac{-1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$$

Now eq^n (1) becomes,

$$\frac{dv}{dx}+\tan x.v=\sec x \text{ ....(2)}$$

eq^n (2) is limear differential equation form p = tanx, Q = secx

$\therefore$ I. F. $= e^{\int p dx}=e^{\int \tan x dx}=e^{\log \sec x}=\sec x$

Multiplying eq^n (2) by I.F. we get,

v . secx = $\int \sec x.\sec x dx+c$

$= \int \sec^2 x dx+c$

v. secx = tanx + c

Restoring the value of v = $\frac{1}{y}$ we get,

$$\frac{\sec x}{y}=\tan x+c$$

or, secx = y tanx + cy is the required solution.

9. $\mathbf{\frac{dy}{dx}-\frac{tany}{1+x}=(1+x)\,e^x\,secy.}$

Sol^n. Given differential equation is

$$\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)\,e^x \sec y.$$

dividing both sides by secy

or, $\frac{1}{\sec y}\frac{dy}{dx}-\frac{1}{1+x}\sin y=(1+x)e^x$ ....(1)

Put siny = v then cosy $\frac{dy}{dx}=\frac{dv}{dx}$

or, $\frac{1}{\sec y}\frac{dy}{dx}=\frac{dv}{dx}$

Now eq^n (1) becomes

$$\frac{dv}{dx}-\frac{1}{1+x}v=(1+x)e^x \text{ ....(2)}$$

eq^n (2) is linear differential equation form

$p=\frac{-1}{1+x}$, $Q=(1+x)e^x$

$\therefore$ I. F. $= e^{\int p dx}=e^{\int -\frac{1}{1+x}dx}=e^{-\log(1+x)}$

$= e^{\log(1+x)^{-1}}=(1+x)^{-1}=\frac{1}{(1+x)}$

Multiplying eq^n (2) by I. F. we get

$$v.\frac{1}{(1+x)} = \int e^x(1+x).\frac{1}{(1+x)}dx + c = \int e^x dx + c$$

$$\frac{v}{(1+x)} = e^x + c$$

Restoring the value of v = siny we get

$$\frac{\sin y}{(1+x)} = e^x + c$$

or, siny = (1 + x) (e^x + c) is the required solution

10. $\mathbf{\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^2\cos^2 y}$

Sol^n. Given differential equation is

$$\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3\cos^2 y$$

dividing both sides by $\cos^2 y$

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{2}{x}\tan y = x^3 \text{ .... (1)}$$

Put tany = v Then $\sec^2 y \; \frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{1}{\cos^2 y}\frac{dy}{dx} = \frac{dv}{dx}$

Now eq^n (1) becomes

$$\frac{dv}{dx} + \frac{2}{x}v = x^3 \text{ ....(2)}$$

Equation (2) is linear differential equation form

$p = \frac{2}{x}$ and $Q = x^3$

$$\therefore \text{ I. F.} = e^{\int p dx} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$$

Multiplying eq^n (2) by I.F. we get,

$$v.x^2 = \int x^3.x^2 dx + c$$

$$= \int x^5 dx + c$$

$$vx^2 = \frac{x^6}{6} + c$$

Restoring the value of v = tany we get

$$\tan y.\ x^2 = \frac{x^6}{6} + c$$

or, $6x^2$ tany = x^6 + 6c

or, $6x^2$ tany = x^6 + K where K = 6c is the required solution.