LAW OF COULOMB AND ELECTRIC FIELD INTENSITY

2.1 INTRODUCTION

Electrostatics is a fascinating subject that has grown up in diverse areas of application. Electric power transmission, X-ray machines, and lightning protection are associated with strong electric fields and will require a knowledge of electrostatics to understand and design suitable equipment. The operation of devices used in solid-state electronics need good understanding of electrostatics. Application of electrostatics in industry ranges from paint spraying, electrochemical machining to separation of fine particles. Sorting of seeds, measurement of moisture content in crops, etc. are some of the usefulness of electrostatics in agriculture.

In electrostatics, electric charges (the sources) are at rest, and electric fields do not change with time. There are no magnetic fields; hence, we deal with a relatively simple situation. Actually, nearly all real electric fields vary to some extent with time, but for many problems, the time variation is slow and the field may be considered stationary in time (static) over the interval of interest. After we have studied the behavior of static electric fields and mastered the techniques for solving electrostatic boundary-value problems, we will go on to the subject of magnetic fields, and time-varying electromagnetic fields.

2.2 COULOMB'S LAW

It states that the force between two charges in a vacuum or free space separated by a distance which is large compared to their size is directly proportional to the product of the two charges and inversely proportional to the square of the distance between them.

If Q_1 and Q_2 are the positive or negative charges separated by a distance R, then mathematically Coulomb's law is illustrated as

$$F \propto Q_1 Q_2$$

$$F \propto \frac{1}{R^2}$$

Combining both,

$$F \propto \frac{Q_1 Q_2}{R^2}$$

$$\text{or,} \quad F = K\frac{Q_1 Q_2}{R^2} \quad ; \text{where } K = \text{proportionality constant}$$

If we adopt the International System of Units (SI); Q in Coulombs (C), R in meters and the force in Newtons (N), then the value of K should be as

$$K = \frac{1}{4\pi\varepsilon_o}; \quad \varepsilon_o = \text{permittivity of free space} = 8.854 \times 10^{-12}\ \text{F/m}$$

For any other medium,

$$K = \frac{1}{4\pi\varepsilon} \; ; \; \varepsilon = \text{permittivity of the medium}$$

The final scalar expression is therefore

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2} \text{ (for vacuum or free space)}$$

Vector expression

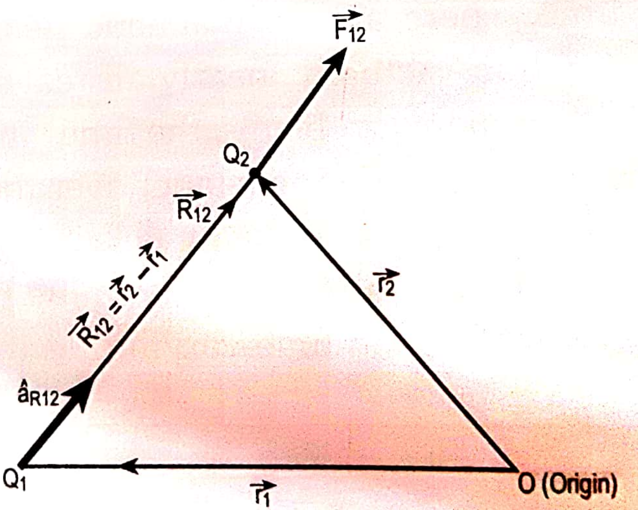


Figure 2.1 Illustration of force between two like charges.

Consider as shown above, two like charges Q_1 & Q_2 identified by position vectors $\vec{r}_1$ and $\vec{r}_2$ respectively, reference being the origin. The vector $\vec{R}_{12} = \vec{r}_2 - \vec{r}_1$ represents the directed line segment from Q_1 to Q_2. Since the two charges are alike the force $\vec{F}$ is repulsive and can also be thought as the force on Q_2.

Now, Coulomb's law has the form

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R_{12}^2} \hat{a}_{R12}$$

where $\hat{a}_{R12}$ = unit vector that gives the direction of $\vec{F}$

$$= \frac{\vec{R}_{12}}{|\vec{R}_{12}|} \left(\text{i.e., } \frac{\vec{R}_{12}}{R_{12}}\right)$$

$$\text{or, } \vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R_{12}^2} \frac{\vec{R}_{12}}{R_{12}} = \frac{Q_1 Q_2}{4\pi\varepsilon_o} \frac{\vec{R}_{12}}{(R_{12})^3} = \frac{Q_1 Q_2}{4\pi\varepsilon_o} \frac{\vec{R}_{12}}{|\vec{R}_{12}|^3}$$

$$\boxed{\therefore \vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_o} \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}}$$

2.3 ELECTRIC FIELD INTENSITY

Consider a charge Q_1 which is fixed and Q_t be the test charge as shown Figure 2.2.

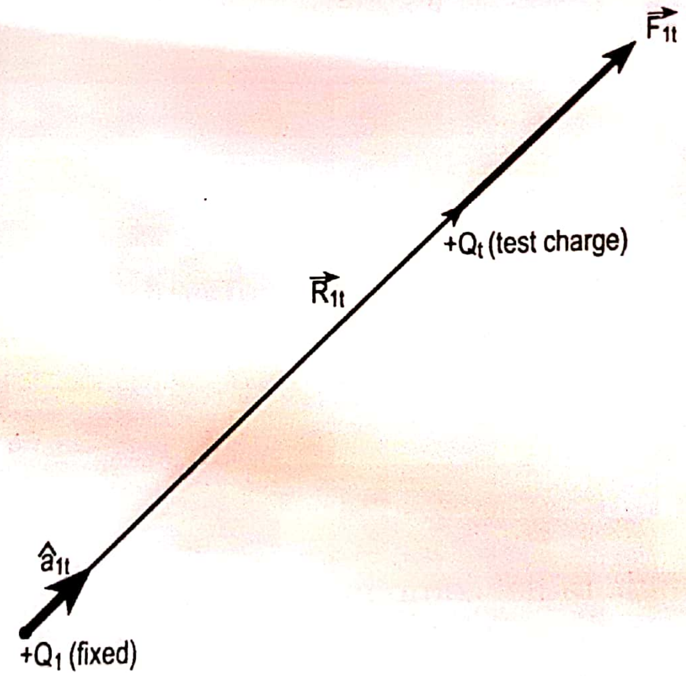


Figure 2.2 For defining electric field intensity.

Let Q_1 and Q_t both are positive charges, then the direction of force on Q_t due to Q_1 is given by $\hat{a}_{1t}$ (directed from Q_1 to Q_t).

From Coulomb's law,

$$\vec{F}_{1t} = \frac{Q_1 Q_t}{4\pi\varepsilon_o} \frac{\hat{a}_{1t}}{R_{1t}^2}$$

Electric field intensity is the force per unit test charge.

$$\vec{E}_{1t} = \frac{\vec{F}_{1t}}{Q_t} = \frac{\frac{Q_1 Q_t}{4\pi\varepsilon_o} \frac{\hat{a}_{1t}}{R_{1t}^2}}{Q_t} = \frac{Q_1}{4\pi\varepsilon_o R_{1t}^2} \hat{a}_{1t}$$

where $\hat{a}_{1t} = \frac{\vec{R}_{1t}}{R_{1t}}$

In general,

$$\vec{E} = \frac{Q}{4\pi\varepsilon_o R^2} \hat{a}_R$$

where $\hat{a}_R \left(= \frac{\vec{R}}{R} \right)$ is the unit vector in the $\vec{R}$ direction. $\vec{R}$ is the directed line segment from Q to a point where $\vec{E}$ is desired.

If we locate Q_1 at the center of a spherical coordinate system, then at the surface of a sphere,

$$\vec{E} = \frac{Q_1}{4\pi\varepsilon_o r^2} \hat{a}_r$$

where $\hat{a}_r$ = radial unit vector $= \frac{\vec{r}}{r}$, r = radius of the sphere

Since $\vec{E} = E_r \hat{a}_r$,

$$E_r = \frac{Q_1}{4\pi\varepsilon_o r^2}$$

The field has a single radial component, and its inverse-square-law relationship is quite obvious.

(i) Electric Field Intensity due to a Point Charge

Consider a point charge Q is located at (x', y', z') and (x, y, z) be the point where electric field intensity is to be determined.

By a "point charge" we mean a charge that is located on a body whose dimensions are much smaller than other relevant dimensions. For example, a collection of electric charges on a pinhead may be regarded as a point charge.

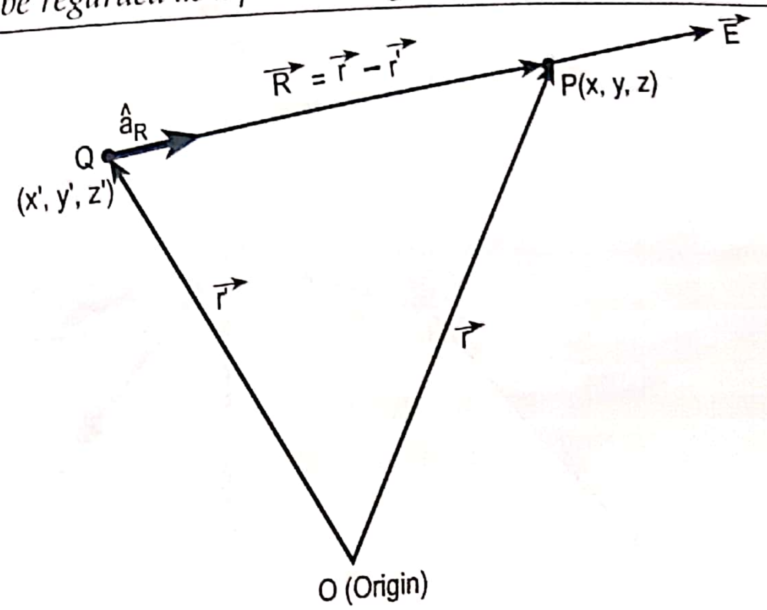


Figure 2.3 The vector $\vec{r}'$ locates the point charge Q and the vector $\vec{r}$ identifies the point in space P(x, y, z) where electric field intensity has to be determined.

$$\vec{E} = \frac{Q}{4\pi\varepsilon_o R^2}\,\hat{a}_R = \frac{Q}{4\pi\varepsilon_o R^2}\frac{\vec{R}}{R} = \frac{Q}{4\pi\varepsilon_o}\frac{\vec{R}}{|\vec{R}|^3}$$

$$\text{or, } \vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_o}\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

$$\vec{r}' = x'\hat{a}_x + y'\,\hat{a}_y + z'\,\hat{a}_z,\quad \vec{r} = x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z$$

$$\vec{r} - \vec{r}' = (x - x')\,\hat{a}_x + (y - y')\,\hat{a}_y + (z - z')\,\hat{a}_z$$

$$|\vec{r} - \vec{r}'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

The expression for electric field intensity is then reduced to

$$\boxed{\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_o}\frac{\left[(x - x')\,\hat{a}_x + (y - y')\,\hat{a}_y + (z - z')\,\hat{a}_z\right]}{\left[\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}\right]^3}}$$

(ii) Electric Field Intensity due to n Point Charges

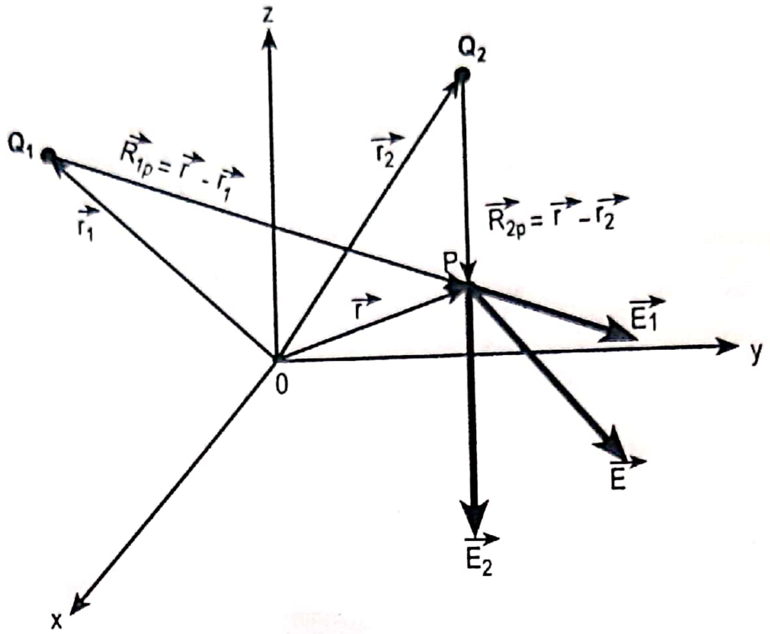


Figure 2.4 *Illustration of the vector addition of the electric field intensities due to Q_1 and Q_2.*

For simplicity, we have shown two point charges in the figure. First, we derive electric field intensity at point P due to two point charges Q_1 and Q_2, and then we generalize the expression for n number of point charges.

$$\vec{E}_1 = \frac{Q_1}{4\pi\varepsilon_o R_{1p}^2}\hat{a}_{1p}\ ,\ \hat{a}_{1p} = \frac{\vec{R}_{1p}}{R_{1p}}$$

$$= \frac{Q_1}{4\pi\varepsilon_o|\vec{r} - \vec{r}_1|^2}\hat{a}_{1p}$$

$$\vec{E}_2 = \frac{Q_2}{4\pi\varepsilon_o R^2_{2p}}\hat{a}_{2p} = \frac{Q_2}{4\pi\varepsilon_o|\vec{r} - \vec{r}_2|^2}\hat{a}_{2p}$$

Resultant electric field intensity due to two point charges is

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{Q_1}{4\pi\varepsilon_o|\vec{r} - \vec{r}_1|^2}\hat{a}_{1p} + \frac{Q_2}{4\pi\varepsilon_o|\vec{r} - \vec{r}_2|^2}\hat{a}_{2p}$$

If there are n point charges, then

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \dots\dots\dots\dots + \vec{E}_n$$

$$= \frac{Q_1}{4\pi\varepsilon_o |\vec{r}-\vec{r}_1|^2}\hat{a}_{1p} + \frac{Q_2}{4\pi\varepsilon_o |\vec{r}-\vec{r}_2|^2}\hat{a}_{2p} + \dots\dots\dots + \frac{Q_n}{4\pi\varepsilon_o |\vec{r}-\vec{r}_n|^2}\hat{a}_{np}$$

$$\boxed{\therefore \vec{E} = \sum_{m=1}^{n} \frac{Q_m}{4\pi\varepsilon_o |\vec{r}-\vec{r}_m|^2}\hat{a}_{mp}}$$

2. CHARGE DISTRIBUTIONS

Volume Charge

When charge is distributed throughout a specified volume, each charge element contributes to the electric field at an external point. A summation or integration is then required to obtain the total electric field. Even though electric charge in its smallest division is found to be an electron or proton, it is useful to consider continuous (in fact, differentiable) charge distributions and to define a volume charge density by

$$\rho_v = \frac{dQ}{dv} \quad (C/m^3)$$

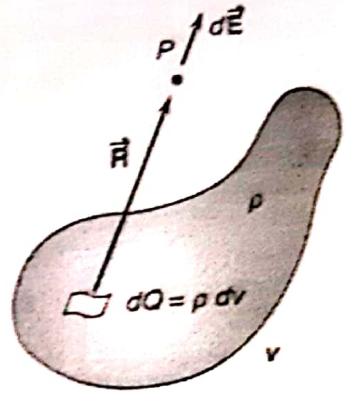


Figure 2.5 For defining volume charge density

Surface Charge

Charge may also be distributed over a surface or a sheet. Then, each differential charge dQ on the sheet results in a differential electric field.

Surface charge density is defined as

$$\rho_S = \frac{dQ}{dS} \quad (C/m^2)$$

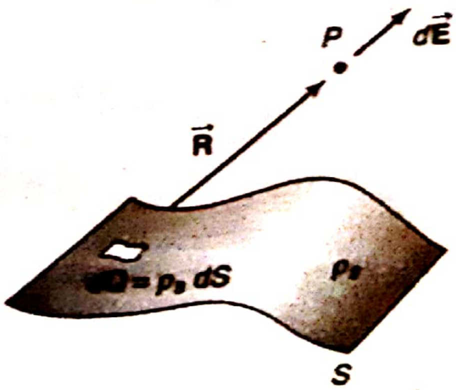


Figure 2.6 For defining surface charge density

Line Charge

If charge is distributed over a (curved) line, each differential charge dQ along the line produces a differential electric field. Line charge density is defined as

$$\rho_L = \frac{dQ}{d\ell} \quad (C/m)$$

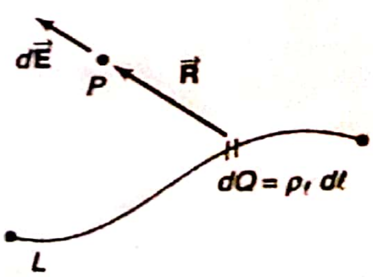
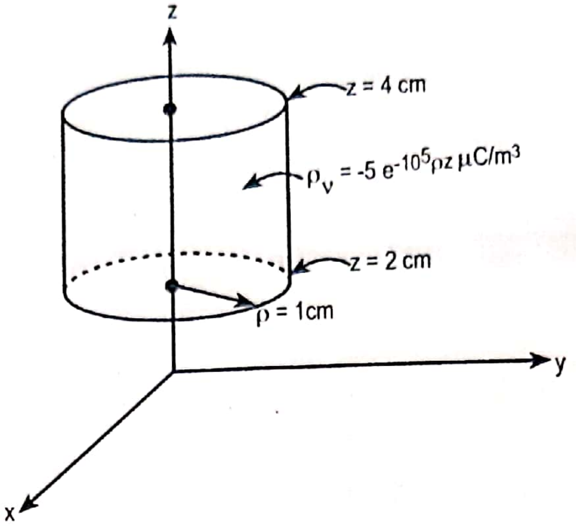


Figure 2.7 For defining line charge density

1. **Find the total charge contained in a 2 cm length of the electron beam having volume charge density, $\rho_v = -5 \times 10^{-6}\, e^{-10^5 \rho z}$ C/m³.**



Solution:

The total charge is given as

$$Q = \int_{vol} \rho_v \, dv$$

dv (in cylindrical coordinate system) = ρ dρ dϕ dz

$$\therefore \quad Q = \int_{vol} (-5 \times 10^{-6}\, e^{-10^5 \rho z})\,(\rho d\rho\, d\phi\, dz)$$

$$= \int_{z=0.02}^{0.04} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{0.01} -5 \times 10^{-6}\, e^{-10^5 \rho z}\, \rho\, d\rho\, d\phi\, dz$$

$$= \int_{z=0.02}^{0.04} \int_{\rho=0}^{0.01} -5 \times 10^{-6}\, e^{-10^5 \rho z}\, \rho\, d\rho\, dz\, [\phi]_0^{2\pi}$$

$$= \int_{z=0.02}^{0.04} \int_{\rho=0}^{0.01} -10^{-5}\pi\, e^{-10^5 \rho z}\, \rho\, d\rho\, dz$$

$$= \int_{\rho=0}^{0.01} -10^{-5}\pi\rho\, d\rho \left[\frac{e^{-10^5 \rho z}}{-10^5 \rho}\right]_{0.02}^{0.04}$$

$$= 10^{-10}\pi \int_{\rho=0}^{0.01} [e^{-4000\rho} - e^{-2000\rho}]\, d\rho$$

$$= 10^{-10}\pi \left[\frac{e^{-4000\rho}}{-4000} - \frac{e^{-2000\rho}}{-2000}\right]_0^{0.01} = 0.0785 \text{ pC}$$

(iii) Electric Field Intensity due to a Line Charge

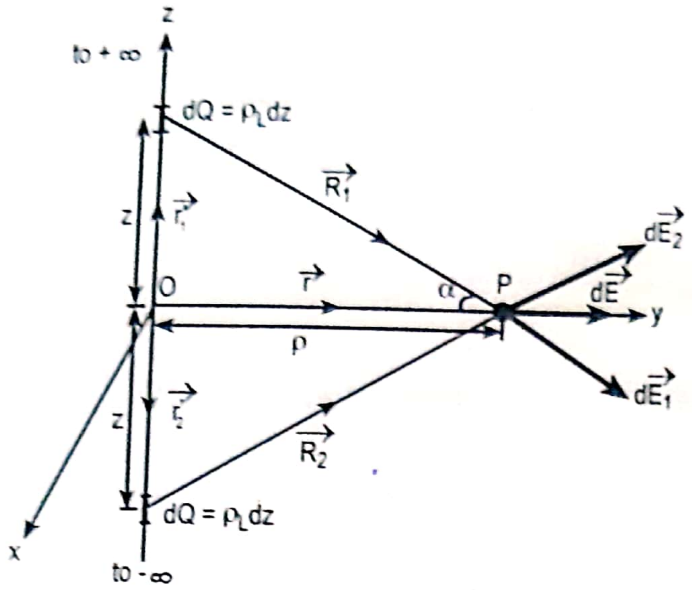


Figure 2.8 A uniform infinite line charge extending along the entire z axis.

Consider a uniform infinite line charge is placed on z axis having line charge density ρ_L. Since the problem has a cylindrical symmetry, it would be most convenient to work with cylindrical coordinates. As we move around the line charge by varying ϕ while keeping ρ and z constant, electric field does not vary. This is azimuthal symmetry. If we maintain ρ and ϕ constant while moving up and down the line charge by changing z, field remains unchanged again. This is axial symmetry. But, if we maintain ϕ and z constant and vary ρ, there is variation in strength of the field. This helps us to accept the fact that the field varies only with ρ.

Let P(0, y, 0) be the point on the y axis where the value of electric field intensity has to be calculated. This is a perfectly general point knowing that electric field does not vary with ϕ and z.

Firstly, we determine the small value of electric field intensity at point P due to elemental charge dQ and then integrate suitably to get electric field intensity due to the entire infinite line charge.

Electric field intensity at point P due to dQ is

$$\vec{dE_1} = \frac{dQ}{4\pi\varepsilon_o R_1^2}\hat{a}_{R1} = \frac{dQ}{4\pi\varepsilon_o R_1^2}\frac{\vec{R_1}}{R_1} = \frac{dQ}{4\pi\varepsilon_o R_1^3}\vec{R_1}$$

$$\vec{R_1} = \vec{r} - \vec{r_1} = \rho\,\hat{a}_\rho - z\,\hat{a}_z$$

> *An example of line charge is a very fine, sharp beam in a cathode-ray tube or a charged conductor of very small radius. In the case of the electron beam, the charges are in motion and it is true that we do not have an electrostatic problem. However, if the electron motion is steady and uniform (a dc beam) and if we ignore for the moment the magnetic field which is produced, the electron beam may be considered to be composed of stationary electrons, for snapshots taken at any time will show the same charge distribution.*

$$R_1 = |\vec{R}_1| = \sqrt{(\rho)^2 + (-z)^2} = \sqrt{\rho^2 + z^2}$$

$$\text{So, } d\vec{E}_1 = \frac{dQ}{4\pi\varepsilon_o}\frac{(\rho\,\hat{a}_\rho - z\,\hat{a}_z)}{\left(\sqrt{\rho^2 + z^2}\right)^3} = \frac{dQ\,(\rho\,\hat{a}_\rho - z\,\hat{a}_z)}{4\pi\varepsilon_o(\rho^2 + z^2)^{\frac{3}{2}}}$$

$$d\vec{E}_2 = \frac{dQ}{4\pi\varepsilon_o R_2{}^2}\,\hat{a}_{R2} = \frac{dQ}{4\pi\varepsilon_o R_2{}^2}\frac{\vec{R}_2}{R_2} = \frac{dQ}{4\pi\varepsilon_o R_2{}^3}\vec{R}_2$$

$$\vec{R}_2 = \vec{r} - \vec{r}_2 = \rho\,\hat{a}_\rho - [\,z\,(-\hat{a}_z)] = \rho\,\hat{a}_\rho + z\,\hat{a}_z$$

$$R_2 = |\vec{R}_2| = \sqrt{(\rho)^2 + (z)^2} = \sqrt{\rho^2 + z^2}$$

$$\text{So, } d\vec{E}_2 = \frac{dQ}{4\pi\varepsilon_o R_2{}^2}\frac{\vec{R}_2}{R_2} = \frac{dQ\,(\rho\,\hat{a}_\rho + z\,\hat{a}_z)}{4\pi\varepsilon_o(\rho^2 + z^2)^{\frac{3}{2}}}$$

$$d\vec{E} = d\vec{E}_1 + d\vec{E}_2$$

$$= \frac{dQ}{4\pi\varepsilon_o}\frac{(\rho\,\hat{a}_\rho - z\,\hat{a}_z)}{(\rho^2 + z^2)^{\frac{3}{2}}} + \frac{dQ}{4\pi\varepsilon_o}\frac{(\rho\,\hat{a}_\rho + z\,\hat{a}_z)}{(\rho^2 + z^2)^{\frac{3}{2}}}$$

$$= \frac{dQ}{4\pi\varepsilon_o(\rho^2 + z^2)^{\frac{3}{2}}}[\rho\hat{a}_\rho - z\,\hat{a}_z + \rho\,\hat{a}_\rho + z\,\hat{a}_z]$$

$$= \frac{2\rho\,dQ}{4\pi\varepsilon_o(\rho^2 + z^2)^{\frac{3}{2}}}\,\hat{a}_\rho$$

Finally, electric field intensity at point P due to infinitely long line charge can be obtained by integrating $d\vec{E}$ from 0 to ∞ (since we have calculated $d\vec{E}$ as the result of two dQ).

$$\vec{E} = \int_0^\infty d\vec{E} = \int_0^\infty \frac{2\rho\,dQ}{4\pi\varepsilon_o(\rho^2 + z^2)^{\frac{3}{2}}}\,\hat{a}_\rho = \int_0^\infty \frac{2\rho\,\rho_L\,dz}{4\pi\varepsilon_o(\rho^2 + z^2)^{\frac{3}{2}}}\,\hat{a}_\rho$$

$$= \frac{2\,\rho\,\rho_L}{4\pi\varepsilon_o}\,\hat{a}_\rho\int_0^\infty \frac{dz}{(\rho^2 + z^2)^{\frac{3}{2}}} \qquad \text{............ (i)}$$

Let $I = \int_0^{\infty} \frac{dz}{(\rho^2 + z^2)^{\frac{3}{2}}}$

Put $z = \rho \tan\alpha$ $\left(\because \tan\alpha = \frac{z}{\rho}\right)$

$\therefore dz = \rho \sec^2\alpha \, d\alpha$

$z = 0 \Rightarrow \alpha = 0$ and $z = \infty \Rightarrow \alpha = \frac{\pi}{2}$

$$\therefore I = \int_0^{\pi/2} \frac{\rho \sec^2\alpha \, d\alpha}{\rho^3 \sec^3\alpha} = \frac{1}{\rho^2} \int_0^{\pi/2} \cos\alpha \, d\alpha = \frac{1}{\rho^2} [\sin\alpha]_0^{\pi/2} = \frac{1}{\rho^2} \left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{1}{\rho^2}$$

From equation (i),

$$\vec{E} = \frac{2\rho_L \hat{a}_\rho}{4\pi \varepsilon_o} \frac{1}{\rho} \qquad \boxed{\therefore \vec{E} = \frac{\rho_L}{2\pi\varepsilon_o\rho} \hat{a}_\rho}$$

It should be kept in mind that above equation is obtained for an infinite line charge along the z-axis so that ρ and $\hat{a}_\rho$ have their usual meaning. If the line is not along the z-axis, ρ is the perpendicular distance from the line to the point of interest, and $\hat{a}_\rho$ is a unit vector along that distance directed from the line charge to the field point.

(iv) Electric Field Intensity due to Infinite Sheet of Charge

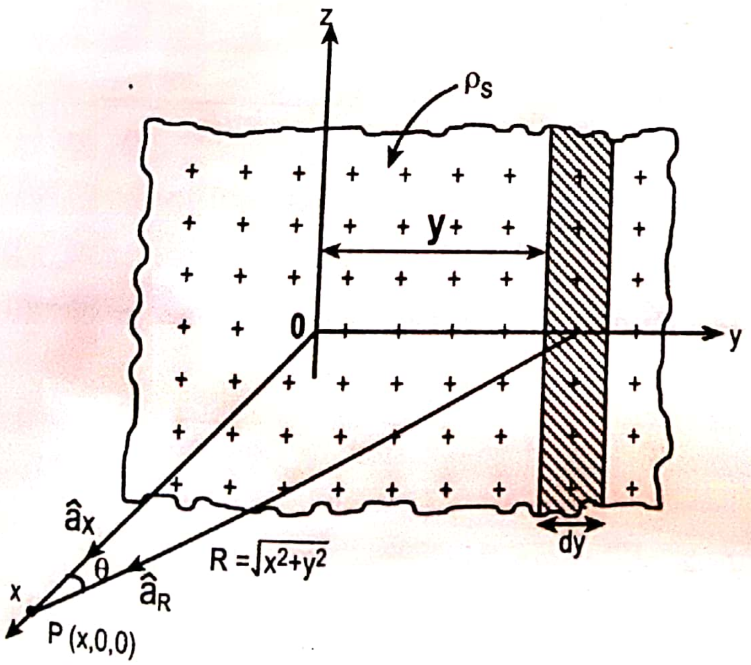


Figure 2.9 An infinite sheet of charge in the yz plane.

Consider an infinite sheet of charge having a uniform charge density ρ_s (C/m²) placed in the yz plane i.e., x = 0 plane. Symmetry should be considered this time again. Field cannot vary with y or with z, but only along x-axis. This is due to the fact that the y and z components arising from differential elements of charge symmetrically located with respect to the point at which we evaluate the field will cancel. Hence, only E_x is present.

Let P(x, 0, 0) be the point on x-axis where the value of electric field intensity is to be determined. Firstly, we divide the sheet into differential-width strips each having width dy and evaluate field at point P due to a strip and then we integrate to find the electric field intensity due to the infinite sheet of charge.

Small value of electric field due to the strip is

$$d\vec{E} = \frac{\rho_L}{2\pi\varepsilon_o R}\,\hat{a}_R = \frac{\rho_L}{2\pi\varepsilon_o\sqrt{x^2+y^2}}\,\hat{a}_R$$

As only x–component is present,

$$dE_x = d\vec{E}\,.\,\hat{a}_x = \frac{\rho_L}{2\pi\varepsilon_o\sqrt{x^2+y^2}}\,\hat{a}_R.\hat{a}_x$$

$$\text{We have, }\cos\theta = \frac{\vec{a}\,.\,\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{\hat{a}_x.\,\hat{a}_R}{|\,\hat{a}_x\,|\,|\,\hat{a}_R\,|} = \hat{a}_x.\hat{a}_R$$

$$\text{So, } dE_x = \frac{\rho_L}{2\pi\varepsilon_o\sqrt{x^2+y^2}}\cos\theta = \frac{\rho_L}{2\pi\varepsilon_o\sqrt{x^2+y^2}}\,\frac{x}{\sqrt{x^2+y^2}}$$

$$= \frac{\rho_s\,dy}{2\pi\varepsilon_o\sqrt{x^2+y^2}}\,\frac{x}{\sqrt{x^2+y^2}} = \frac{x\,\rho_s\,dy}{2\pi\varepsilon_o\,(x^2+y^2)}$$

Total electric field due to entire infinite sheet can be calculated as

$$E_x = \int_{-\infty}^{\infty} dE_x = \int_{y=-\infty}^{\infty}\frac{x\rho_s\,dy}{2\pi\varepsilon_o\,(x^2+y^2)} = \frac{\rho_s x}{2\pi\varepsilon_o}\int_{-\infty}^{\infty}\frac{dy}{x^2+y^2} \quad \text{.......... (i)}$$

$$\text{Let } I = \int_{-\infty}^{\infty}\frac{dy}{(x^2+y^2)}$$

Put $y = x\tan\theta$

$\therefore\ dy = x\sec^2\theta\,d\theta$

$$y = -\infty \Rightarrow \theta = -\frac{\pi}{2} \text{ and } y = \infty \Rightarrow \theta = \frac{\pi}{2}$$

$$\therefore I = \int_{-\pi/2}^{\pi/2} \frac{x \sec^2\theta \, d\theta}{x^2 + x^2 \tan^2\theta} = \int_{-\pi/2}^{\pi/2} \frac{x \sec^2\theta d\theta}{x^2(1 + \tan^2\theta)} = \frac{1}{x} \int_{-\pi/2}^{\pi/2} d\theta = \frac{1}{x} [\theta]_{-\pi/2}^{\pi/2} = \frac{\pi}{x}$$

From equation (i),

$$E_x = \frac{\rho_s}{2\pi\varepsilon_o} \pi = \frac{\rho_s}{2\varepsilon_o}$$

In vector form,

$$\vec{E}_x = \frac{\rho_s}{2\varepsilon_o} \hat{a}_x$$

If P is at (−x, 0, 0),

$\vec{E}_x = \frac{\rho_s}{2\varepsilon_o} (-\hat{a}_x)$ for the field is always directed away from the positive charge.

In general,

$$\vec{E}_s = \frac{\rho_s}{2\varepsilon_o} \hat{a}_N$$

where $\hat{a}_N$ = unit vector normal to sheet and directed away from it.

This equation promises that the field is just as strong a thousand kilometers away from the sheet as it is right off the surface.

The lightning scheme of an office or a classroom may consist of incandescent bulbs, long fluorescent tubes, or ceiling panel lights. These are analogous to point sources, line sources, and planar sources respectively. We can now make an estimation that light intensity will fall off rapidly as the square of the distance from the source in the case of incandescent bulbs, less rapidly as the first power of the distance for long fluorescent tubes, and not at all for ceiling panel lights. Note that, holding the book closer to ceiling panel lights will do no good if we want greater illumination.

(v) Electric Field due to Parallel Plate Capacitor

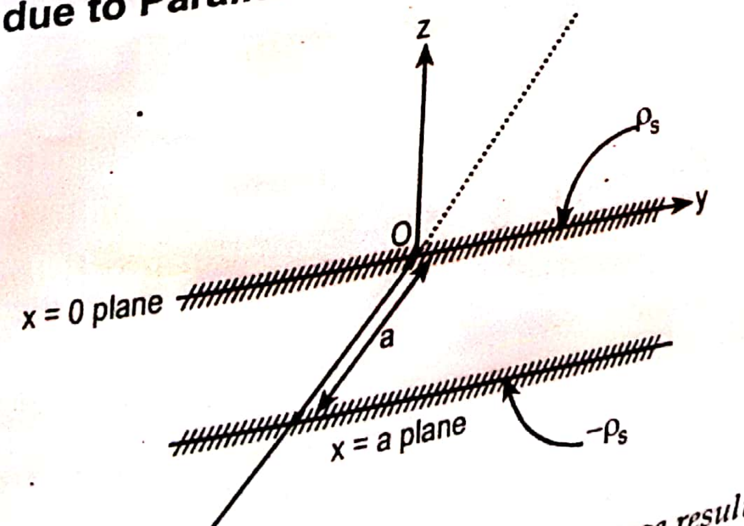


Figure 2.10 Arrangement of two infinite sheets of charge resulting a capacitor.

Consider two infinite sheets of charge placed on x = 0 and x = a plane as shown. Let $+\rho_s$ and $-\rho_s$ be the surface charge density on these sheets. This arrangement can be thought as a parallel plate capacitor with air in between. Let $\vec{E}_+$ and $\vec{E}_-$ be the electric field due to infinite sheets having positive and negative charges respectively.

In the region $x < 0$,

$$\vec{E}_+ = \frac{+\rho_s}{2\varepsilon_o}(-\hat{a}_x)\ ,\ \vec{E}_- = \frac{-\rho_s}{2\varepsilon_o}(-\hat{a}_x)$$

$$\therefore\ \vec{E} = \vec{E}_+ + \vec{E}_- = \frac{-\rho_s}{2\varepsilon_o}\hat{a}_x + \frac{\rho_s}{2\varepsilon_o}\hat{a}_x = 0$$

In the region $x > a$,

$$\vec{E}_+ = \frac{+\rho_s}{2\varepsilon_o}(\hat{a}_x)\ ,\ \vec{E}_- = \frac{-\rho_s}{2\varepsilon_o}(\hat{a}_x)$$

$$\therefore\ \vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho_s}{2\varepsilon_o}\hat{a}_x - \frac{\rho_s}{2\varepsilon_o}\hat{a}_x = 0$$

In the region $0 < x < a$,

$$\vec{E}_+ = \frac{+\rho_s}{2\varepsilon_o}(+\hat{a}_x)\ ,\ \vec{E}_- = \frac{-\rho_s}{2\varepsilon_o}(-\hat{a}_x)$$

$$\therefore\ \vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho_s}{2\varepsilon_o}\hat{a}_x + \frac{\rho_s}{2\varepsilon_o}\hat{a}_x = \frac{\rho_s}{\varepsilon_o}\hat{a}_x$$

Hence, the electric field exists only in between the capacitor.

2.5 STREAMLINES AND SKETCHES OF FIELDS

It is difficult to visualize the nature of electric fields due to various charge distributions from the mathematical equations. A pictorial representation of electric fields, on the other hand, gives clear understanding of the nature of the fields.

The continuous lines which are used to represent the electric field around a charge are called streamlines. These are also called flux lines or direction lines. The streamlines have basically two properties:

i. These are the continuous lines from the charge which only show the direction of $\vec{E}$ and are everywhere tangent to $\vec{E}$.

ii. The magnitude of the field can be shown to be inversely proportional to the spacing of the streamlines.

The streamlines are associated with the arrows which are used to show the direction of $\vec{E}$. If a small positive test charge is placed at any point in the field and is free to move, then the direction in which it will accelerate is indicated by the streamline at that point.

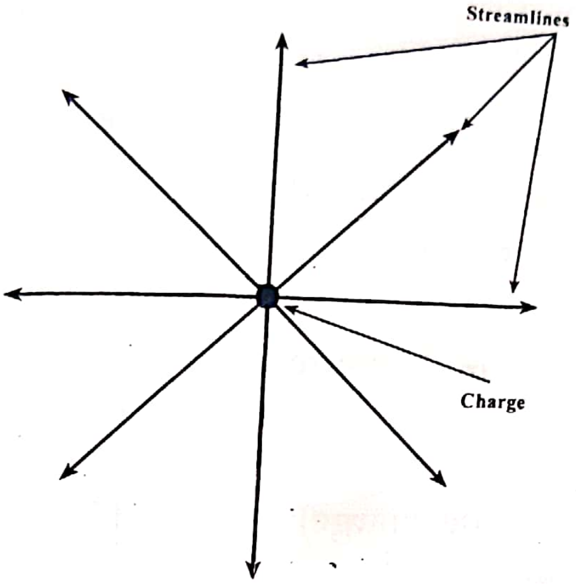


Figure 2.11 The streamlines for a line charge (the cross-section of the line charge is shown).

The three dimensional sketching of streamlines is very difficult. Hence, in practice, only two dimensional sketching is used. For such sketching, the z-component of $\vec{E}$ is assumed to be zero.

Equation of streamlines

Consider the streamlines shown in Figure 2.12.

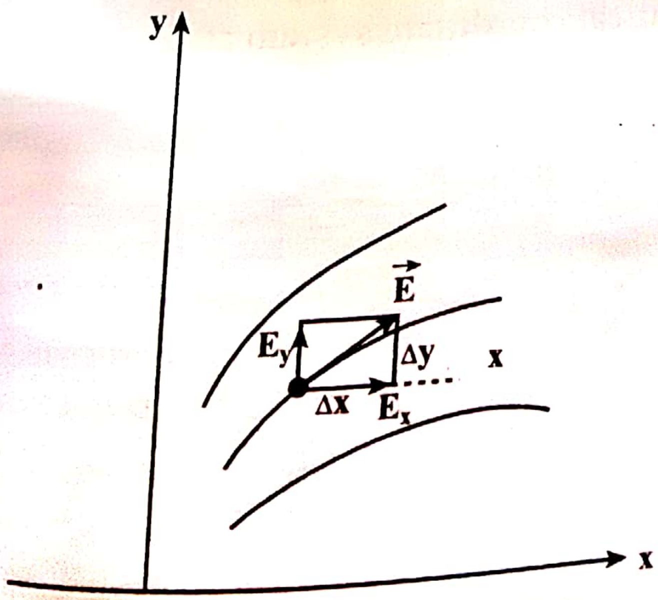


Figure 2.12 The equation of a streamline is obtained by solving the differential equation $E_y/E_x = dy/dx$.

Consider a point P on any of the streamlines. The $\vec{E}$ is tangential to the streamline at that point P. The $\vec{E}$ can be resolved into two components E_x in x direction and E_y in y direction.

It can be seen that E_y is proportional to a small component Δy in y direction while E_x is proportional to the small component Δx in x direction. Thus, we can write

$$\frac{E_y}{E_x} = \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

SOME IMPORTANT FORMULAE

$$\vec{E}_P = \frac{Q}{4\pi\varepsilon_o R^2}\hat{a}_R \qquad \text{(point charge)}$$

$$\vec{E}_L = \frac{\rho_L}{2\pi\varepsilon_o R}\hat{a}_R \qquad \text{(line charge)}$$

$$\vec{E}_S = \frac{\rho_s}{2\varepsilon_o}\hat{a}_N \qquad \text{(surface charge)}$$

PROBLEMS SOLVED AND SCRAMBLED

1. **An infinite line charge ρ_L = 2 nC/m lies along the x-axis in free space while point charges of 8 nC each are located at (0, 0, 1) & (0, 0, – 1). Find the electric field intensity in cylindrical coordinate system at (2, 3, –4).** *[2063 Kartik]*

Solution:

Total electric field intensity is given as

$$\vec{E} = \vec{E}_P + \vec{E}_L$$

Electric field intensity at point P due to two point charges is given by

$$\vec{E}_P = \vec{E}_{P1} + \vec{E}_{P2} = \frac{Q_1}{4\pi\varepsilon_o}\frac{\vec{R}_1}{R_1^3} + \frac{Q_2}{4\pi\varepsilon_o}\frac{\vec{R}_2}{R_2^3}$$

where $\vec{R}_1$ = (field point) – (source point) = (2, 3, – 4) – (0, 0, 1)=$2\hat{a}_x+3\hat{a}_y-5\hat{a}_z$

$$R_1 = \sqrt{4+9+25} = \sqrt{38}$$

$\vec{R_2}$ = (field point) – (source point) = (2,3,–4)–(0,0,–1)=$2\hat{a}_x+3\hat{a}_y-3\hat{a}_z$

$R_2 = \sqrt{4+9+9} = \sqrt{22}$

$$\therefore \vec{E_P} = 8 \times 10^{-9}(9 \times 10^9)\left[\frac{2\hat{a}_x + 3\hat{a}_y - 5\hat{a}_z}{(38)^{\frac{3}{2}}} + \frac{2\hat{a}_x + 3\hat{a}_y - 3\hat{a}_z}{(22)^{\frac{3}{2}}}\right]$$

$= 72\ [8.5379 \times 10^{-3}\hat{a}_x + 0.0128\ \hat{a}_y - 0.2134\ \hat{a}_z + 0.01938\ \hat{a}_x + 0.029\ \hat{a}_y - 0.029\ \hat{a}_z]$

$= 2.01\ \hat{a}_x + 3.009\ \hat{a}_y - 3.628\ \hat{a}_z$ V/m

$$\vec{E_L} = \frac{\rho_L}{2\pi\varepsilon_o}\frac{\vec{R}}{R^2}$$

where $\vec{R}$ = (field point) – (source point) =(2, 3, – 4) – (2, 0, 0) = $3\hat{a}_y - 4\hat{a}_z$

$R = \sqrt{9+16} = 5$

$$\therefore \vec{E_L} = \frac{2 \times 10^{-9} \times (18 \times 10^9)\ (3\hat{a}_y - 4\hat{a}_z)}{25} = 4.32\ \hat{a}_y - 5.76\ \hat{a}_z \text{ V/m}$$

Total Electric Field $(\vec{E}) = \vec{E_P} + \vec{E_L}$

$= (2.01\hat{a}_x + 3.009\hat{a}_y - 3.628\hat{a}_z + 4.32\hat{a}_y - 5.76\hat{a}_z)$

$= 2.01\ \hat{a}_x + 7.329\ \hat{a}_y - 9.388\ \hat{a}_z$ V/m

Expressing $\vec{E}$ in cylindrical coordinate system at point (2, 3, – 4)

$\rho = \sqrt{x^2+y^2} = \sqrt{4+9} = \sqrt{13}$

$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{3}{2}\right) = 56.3099^o$

$z = -4$

$\vec{E}$ in cylindrical coordinate system is calculated as

$\vec{E} = E_\rho\ \hat{a}_\rho + E_\phi\hat{a}_\phi + E_z\hat{a}_z$

where $E_\rho = \vec{E}.\hat{a}_\rho = E_x\ \hat{a}_x.\ \hat{a}_\rho + E_y\ \hat{a}_y.\ \hat{a}_\rho + E_z\ \hat{a}_z.\ \hat{a}_\rho$

$= 2.01 \times \cos(56.3099^o) + 7.32 \sin(56.3099^o) + 0$

$= 1.1149 + 6.098 = 7.212$

$E_\phi = \vec{E}.\hat{a}_\phi = E_x\ \hat{a}_x.\ \hat{a}_\phi + E_y\ \hat{a}_y.\ \hat{a}_\phi + E_z\ \hat{a}_z.\hat{a}_\phi$

$= 2.01\,[-\sin(56.3099^\circ)] + 7.32\cos(56.2099^\circ) + 0$

$= -1.6744 + 4.6654 = 2.39$

$E_z = -9.388$

$\therefore \vec{E} = 7.212\,\hat{a}_\rho + 2.39\,\hat{a}_\phi - 9.38\,\hat{a}_z \text{ V/m}$

2. **Find the electric field intensity at origin if the following charge distributions are present at free space, point charge 12 nC at P(2, 0, 6), uniform line charge density 3 nC/m at x = 2, y = 3, uniform surface charge density 0.2 nC/m² at x = 2.**

[2065 Chaitra]

Solution:

For point charge $\vec{E}_P = \dfrac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R = \dfrac{Q}{4\pi\varepsilon_0}\dfrac{\vec{R}}{R^3}$

where $\vec{R} = (0, 0, 0) - (2, 0, 6) = -2\hat{a}_x - 6\hat{a}_z$; $R = \sqrt{40}$

$$\therefore \quad \vec{E}_P = \frac{9 \times 10^9 \times 12 \times 10^{-9} \times (-2\hat{a}_x - 6\hat{a}_z)}{(40)^{\frac{3}{2}}} = -0.8538\,\hat{a}_x - 2.561\,\hat{a}_z \text{ V/m}$$

For line charge,

$$\vec{E}_L = \frac{\rho_L}{2\pi\varepsilon_o R}\hat{a}_R = \frac{\rho_L}{2\pi\varepsilon_0}\frac{\vec{R}}{R^2}$$

where $\vec{R} = (0,0,0) - (2,3,0) = -2\,\hat{a}_x - 3\,\hat{a}_y$; $R = \sqrt{4+9} = \sqrt{13}$

$$\therefore \quad \vec{E}_L = \frac{2 \times 9 \times 10^9 \times 3 \times 10^{-9} \times (-2\,\hat{a}_x - 3\hat{a}_y)}{13} = -8.307\,\hat{a}_x - 12.461\,\hat{a}_y \text{ V/m}$$

For surface charge,

$$\vec{E}_S = \frac{\rho_S}{2\varepsilon_o}\hat{a}_N = \frac{0.2 \times 10^{-9}}{2 \times 8.85 \times 10^{-12}}(-\hat{a}_x) = -11.294\,\hat{a}_x \text{ V/m}$$

$\therefore$ Total electric field is

$$\vec{E} = \vec{E}_P + \vec{E}_L + \vec{E}_S = -20.455\hat{a}_x - 12.461\hat{a}_y - 2.561\hat{a}_z \text{ V/m}$$

3. **Planes x = 2 and y = −3 respectively, carry charges 10 nC/m² and 15 nC/m². If the line x = 0, z = 2 carries charge 10π nC/m, calculate $\vec{E}$ at (1, 1, −1) due to three charge distributions.**

Solution:

Total electric field is given as

$$\vec{E} = \vec{E}_S + \vec{E}_L \quad \text{- (i)}$$

$$\vec{E}_S = \vec{E}_{S1} + \vec{E}_{S2} = \frac{\rho_{S1}}{2\varepsilon_o}\hat{a}_{N1} + \frac{\rho_{S2}}{2\varepsilon_o}\hat{a}_{N2} = \frac{10\times 10^{-9}}{2\times 8.854\times 10^{-12}}(-\hat{a}_x) + \frac{15\times 10^{-9}}{2\times 8.854\times 10^{-12}}(\hat{a}_y)$$

$$= -564.9717\,\hat{a}_x + 847.4576\,\hat{a}_y \text{ V/m}$$

$$\vec{E}_L = \frac{\rho_L}{2\pi\varepsilon_o R}\hat{a}_R = \frac{\rho_L}{2\pi\varepsilon_o R^2}\vec{R}$$

where $\vec{R}$ = F.P.– S.P.= $(1, 1, -1) - (0, 1, 2) = \hat{a}_x - 3\hat{a}_z$; $R = \sqrt{(1)^2 + (-3)^2} = \sqrt{10}$

$$\vec{E}_L = \frac{10\pi\times 10^{-9}}{2\pi\times 8.85\times 10^{-12}\sqrt{10^2}}(\hat{a}_x - 3\hat{a}_z) = 56.4971\,\hat{a}_x - 169.4915\,\hat{a}_z \text{ V/m}$$

From equation (i),

$$\vec{E} = \vec{E}_S + \vec{E}_L = (-564.9717\,\hat{a}_x + 847.4576\,\hat{a}_y) + (56.4971\,\hat{a}_x - 169.4915\,\hat{a}_z)$$

$$= -508.4746\,\hat{a}_x + 847.4576\,\hat{a}_y - 169.4915\,\hat{a}_z \text{ V/m}$$

4. A point charge of 30 nC is located at the origin while plane y = 3 carries charge 10 nC/m². Find $\vec{E}$ at (0, 4, 3).

Solution:

$$\vec{E} = \vec{E}_P + \vec{E}_S \quad \text{.......................... (i)}$$

$$\vec{E}_P = \frac{Q}{4\pi\varepsilon_o}\frac{\vec{R}}{R^3}$$

where $\vec{R}$ = F. P. – S. P. = $(0, 4, 2) - (0, 0, 0) = 4\,\hat{a}_y + 3\,\hat{a}_z$; $R = \sqrt{(4)^2 + (3)^2} = 5$

$$\therefore \quad \vec{E}_P = \frac{30\times 10^{-9}\times 9\times 10^{9}}{(5)^3}(4\,\hat{a}_y + 3\,\hat{a}_z) = 8.64\,\hat{a}_y + 6.48\,\hat{a}_z \text{ V/m}$$

$$\vec{E}_S = \frac{\rho_S}{2\varepsilon_o}\hat{a}_N = \frac{10\times 10^{-9}}{2\times 8.85\times 10^{-12}}\hat{a}_y = 564.971\hat{a}_y \text{ V/m}$$

From equation (i),

$$\vec{E} = \vec{E}_P + \vec{E}_S$$

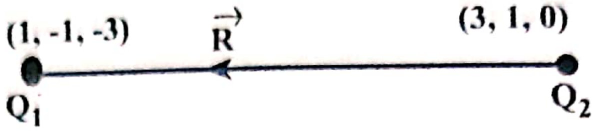
$$= (8.64\,\hat{a}_y + 6.48\,\hat{a}_z) + 564.971\hat{a}_y = 573.611\,\hat{a}_y + 6.48\,\hat{a}_z \text{ V/m}$$

5. **Two point charges, $Q_1 = 50\ \mu C$ and $Q_2 = 10\ \mu C$, are located at (-1, 1, -3) m and (3, 1, 0) m, respectively. Find the force on Q_1.**

Solution:

Force on Q_1 due to Q_2 is

$$\vec{F}_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2}\,\hat{a}_R = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^3}\,\vec{R}$$

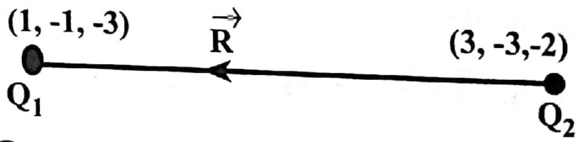


$$\vec{R} = (-1, 1, -3) - (3, 1, 0) = -4\hat{a}_x - 3\hat{a}_z\ ;\ R = \sqrt{(-4)^2 + (-3)^2} = 5$$

$$\therefore\ \vec{F}_{21} = \frac{(50\times10^{-6})\times(10\times10^{-6})}{4\pi\times8.854\times10^{-12}\times(5)^3}(-4\hat{a}_x - 3\hat{a}_z) = 0.0359\,(-4\hat{a}_x - 3\hat{a}_z)\ \text{N}$$

6. **Point charge $Q_1 = 300\ \mu C$, located at (1, -1, -3) m, experiences a force $\vec{F}_1 = 8\hat{a}_x - 8\hat{a}_y + 4\hat{a}_z$ N due to point charge Q_2 at (3, -3, -2) m. Determine Q_2.**

Solution:



Force on Q_1 due to Q_2 is

$$\vec{F}_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2}\,\hat{a}_R = \frac{Q_1 Q_2}{4\pi\varepsilon_o}\frac{\vec{R}}{R^3}$$

$$\vec{R} = (1, -1, -3) - (3, -3, -2) = -2\,\hat{a}_x + 2\,\hat{a}_y - \hat{a}_z\ ;\ R = \sqrt{(-2)^2 + (2)^2 + (-1)^2} = 3$$

$$\text{So,}\quad 8\hat{a}_x - 8\,\hat{a}_y + 4\hat{a}_z = \frac{300\times10^{-6}\times Q_2}{4\pi\times8.854\times10^{-12}\times(3)^3}(-2\,\hat{a}_x + 2\,\hat{a}_y - \hat{a}_z)$$

$$\therefore\ Q_2 = -40\times10^{-6} = -40\ \mu C$$

7. **A uniform sheet of charge with $\rho_S = \left(\frac{1}{3\pi}\right)$ nC/m² is located at z = 5 m and a uniform line charge with $\rho_L = \left(\frac{-25}{9}\right)$ nC/m at z = -3 m, y = 3 m. Find $\vec{E}$ at (x, -1, 0) m.**

Solution:

$$\vec{E} = \vec{E}_L + \vec{E}_S \ \ldots\ldots\text{(i)}$$

$$\vec{E}_L = \frac{\rho_L}{2\pi\varepsilon_o}\frac{\vec{R}}{R^2}$$

where $\vec{R} = (x, -1, 0) - (x, 3, -3) = -4\,\hat{a}_y + 3\hat{a}_z$; $R = \sqrt{(-4)^2+(3)^2} = 5$

$$\therefore \vec{E_L} = \frac{\frac{-25}{9}\times 10^{-9}}{2\pi\times 8.854\times^{-12}\times(5)^2}(-4\,\hat{a}_y + 3\hat{a}_z) = 8\,\hat{a}_y - 6\hat{a}_z \text{ V/m}$$

$$\vec{E_S} = \frac{\rho_S}{2\varepsilon_o}\hat{a}_N = \frac{\frac{1}{3\pi}\times 10^{-9}}{2\times 8.85\times 10^{-12}}(-\hat{a}_z) = -6\hat{a}_z \text{ V/m}$$

From equation (i),

$$\vec{E} = \vec{E_L} + \vec{E_S} = (8\hat{a}_y - 6\hat{a}_z) + (-6\hat{a}_z) = 8\,\hat{a}_y - 12\hat{a}_z \text{ V/m}$$

8. **Two point charges 1 mC and –2 mC are located at (3, 2, –1) and (–1, –1, 4) respectively. Calculate the electric force on 10 nC located at (0, 3, 1) and electric field at that point.**

Solution:

$$\vec{F} = \vec{F_1} + \vec{F_2} = \frac{Q_1Q_2}{4\pi\varepsilon_0}\frac{\vec{R_1}}{R_1^3} + \frac{Q_1Q_2}{4\pi\varepsilon_0}\frac{\vec{R_2}}{R_2^3}$$

where $\vec{R_1} = (0, 3, 1) - (3, 2, -1) = -3\,\hat{a}_x + \hat{a}_y + 2\,\hat{a}_z$; $R_1 = \sqrt{14}$

$\vec{R_2} = (0, 3, 1) - (-1, -1, 4) = \hat{a}_x + 4\,\hat{a}_y - 3\hat{a}_z$; $R_2 = \sqrt{26}$

$$\therefore \vec{F} = 10\times 10^{-9}\times 9\times 10^9\left[\frac{1\times 10^{-3}(-3\hat{a}_x + \hat{a}_y + 2\hat{a}_z)}{(14)^{3/2}} + \frac{(-2\times 10^{-3})(\hat{a}_x + 4\,\hat{a}_y - 3\,\hat{a}_z)}{(26)^{3/2}}\right]$$

$$= -6.512\times 10^{-3}\,\hat{a}_x - 3.7128\times 10^{-3}\,\hat{a}_y + 7.5093\times 10^{-3}\hat{a}_z \text{ N}$$

Electric field is

$$\vec{E} = \frac{\vec{F}}{Q} = \frac{-6.512\times 10^{-3}\,\hat{a}_x - 3.7128\times 10^{-3}\,\hat{a}_y + 7.5093\times 10^{-3}\hat{a}_z}{10\times 10^{-9}}$$

$$= -651.2\hat{a}_x - 371.28\hat{a}_y + 750.93\hat{a}_z \text{ kV/m}$$

9. **Plane x + 2y = 5 carries charge $\rho_s = 6$ nC/m². Determine $\vec{E}$ at (–1, 0, 1).**

Solution:

Let $f(x, y) = x + 2y - 5$

$$\nabla f = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)(x + 2y - 5) = \hat{a}_x + 2\hat{a}_y$$

$$\therefore \hat{a}_N = \pm\frac{\nabla f}{|\nabla f|} = \pm\left(\frac{\hat{a}_x + 2\hat{a}_y}{\sqrt{5}}\right)$$

Since field point (– 1, 0, 1) lies below the plane, $\hat{a}_N = -\dfrac{\hat{a}_x + 2\hat{a}_y}{\sqrt{5}}$

$$\therefore \vec{E} = \frac{\rho_s}{2\varepsilon_o}\hat{a}_N = \frac{6\times10^{-9}}{2\times8.85\times10^{-12}}\{-\frac{\hat{a}_x + 2\hat{a}_y}{\sqrt{5}}\} = -151.7\,\hat{a}_x - 303.5\hat{a}_y \text{V/m}$$

Hammered Problems

1. **Assuming that a cloud of electrons confined in a region between two spheres of radii 2 and 5 cm has a charge density of $\dfrac{-3 \times 10^{-8}}{r^4}\cos^2\phi$ C/m^3, find the total charge contained in the region.**

   Answer: $-1.8\pi\,\mu$C

2. **Cosider $Q_1 = 3\times10^{-4}$ C at M(1, 2, 3) and $Q_2 = -10^{-4}$ C at N(2, 0, 5) in a vacuum. Calculate the force exerted on Q_2 by Q_1.**

   Answer: $-10\hat{a}_x + 20\hat{a}_y - 20\hat{a}_z$ **N**

3. **Three infinite uniform sheets of charge are located in free space as follows: 3 nC/m^2 at $z = -4$, 6 nC/m^2 at $z = 1$, and -8 nC/m^2 at $z = 4$. Find $\vec{E}$ at the point: (a) $P_A(2, 5, -5)$ (b) $P_B(4, 2, -3)$ (c) $P_C(-1, -5, 2)$ (d) $P_D(-2, 4, 5)$.**

   Answer: (a) $-56.5\hat{a}_z$ V/m (b) $283\hat{a}_z$ V/m (c) $961\hat{a}_z$ V/m (d) $56.5\hat{a}_z$ V/m

4. **Infinite uniform line charges of 5 nC/m lie along the (positive and negative) x and y axes in free space. Find $\vec{E}$ at: (a) $P_A(0, 0, 4)$ (b) $P_B(0, 3, 4)$.**

   Answer: (a) $45\hat{a}_z$ V/m (b) $10.8\hat{a}_y + 36.9\hat{a}_z$ V/m