First Order and First Degree DIfferential Equation

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Exercise - 19

Solve the following differential equations

1. $\mathbf{x\sqrt{1+y^2}\,dx + y\sqrt{1+x^2}\,dy = 0}$

Sol^n. Given differential equation is,

$$x\sqrt{1+y^2}\;dx + y\;\sqrt{1+x^2}\;dy = 0$$

Separating the variables,

$$x\frac{dx}{\sqrt{1+x^2}} + y\frac{dy}{\sqrt{1+y^2}} = 0$$

$$\text{or, } 2x\frac{dx}{\sqrt{1+x^2}} + \frac{2y\,dy}{\sqrt{1+y^2}} = 0$$

Integrating we get,

$$\sqrt{1+x^2} + \sqrt{1+y^2} = C \text{ Ans.}$$

2. $\mathbf{(x^2 + 1)\,\frac{dy}{dx} = 1}$

Sol^n. Given differential equation is,

$$(x^2 + 1)\,\frac{dy}{dx} = 1$$

Separating the variables,

$$dy = \frac{dx}{x^2+1}$$

Integrating we get,

$y = \tan^{-1}x + C$ Ans.

3. $\mathbf{y\,dx = (e^x + 1)\,dy}$

Sol^n. Given differential equation is,

$y\,dx = (e^x + 1)\,dy$

Separating the variables,

$$\frac{dx}{e^x+1} = \frac{dy}{y}$$

$$\text{or, } -\frac{e^{-x}}{1+e^{-x}} = -\frac{dy}{y}$$

Integrating we get,

$\log(1 + e^{-x}) = -\log y - \log C$

or, $\log y + \log(1 + e^{-x}) + \log C = 0$

$\therefore\; y(1 + e^{-x})\,C = 0$ Ans.

4. $\mathbf{(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0}$

Sol^n. Given differential equation is,

$(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0$

or, $x(y^2 + 1)\,dx + y(x^2 + 1)\,dy = 0$

Separating the variables,

$$x\frac{dx}{(x^2+1)} + y\frac{dy}{(y^2+1)} = 0$$

$$\text{or, } 2x\frac{dx}{\left(x^2+1\right)} + \frac{2y\,dy}{\left(y^2+1\right)} = 0$$

Integrating we get,

$\log(x^2 + 1) + \log(y^2 + 1) = \log C$

$\therefore\; (x^2 + 1)(y^2 + 1) = C$ Ans.

5. $\mathbf{\tan y\,dx + \tan x\,dy = 0}$

Sol^n. Given differential equation is,

$\tan y\,dx + \tan x\,dy = 0$

Separating the variables,

$$\frac{dx}{\tan x} + \frac{dy}{\tan y} = 0$$

or, $\cot x\,dx + \cot y\,dy = 0$

$$\text{or, } \frac{\cos x}{\sin x}dx + \frac{\cos y}{\sin y}dy = 0$$

Integrating we get,

$\log(\sin x) + \log(\sin y) = \log C$

$\therefore\; \sin x \sin y = C$ Ans.

6. $\mathbf{\left(y - x\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)}$

Sol^n. Given differential equation is,

$$\left(y - x\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$$

or, $y - x\frac{dy}{dx} = ay^2 + a\frac{dy}{dx}$

or, $y(1 - ay) = (a + x)\frac{dy}{dx}$

Separating the variables,

$$\frac{dx}{(a+x)} = \frac{dy}{y(1-ay)}$$

or, $\frac{dx}{(a+x)} = \frac{1}{y}\,dy - \frac{-a}{1-ay}\,dy$

Integrating we get,

$\log(a + x) = \log y - \log(1 - ay) + \log C$

or, $\log(a + x) + \log(1 - ay) = \log yC$

$\therefore (a + x)(1 - ay) = yC$ Ans.

7. $(1 + x)(1 + y^2)\,dx + (1 + y)(1 + x^2)\,dy = 0$

Sol^n. Given differential equation is,

$(1 + x)(1 + y^2)\,dx + (1 + y)(1 + x^2)\,dy = 0$

Separating the variables,

$$\frac{(1+x)\,dx}{(1+x^2)} + \frac{(1+y)\,dy}{(1+y^2)} = 0$$

or, $\frac{dx}{1+x^2} + x\frac{dx}{1+x^2} + \frac{dy}{1+y^2} + y\frac{dy}{1+y^2} = 0$

or, $\frac{dx}{1+x^2} + \frac{1}{2}\cdot\frac{2x\,dx}{1+x^2} + \frac{dy}{1+y^2} + \frac{1}{2}\frac{2y\,dy}{1+y^2} = 0$

Integrating we get,

$\tan^{-1}x + \frac{1}{2}\log(1 + x^2) + \tan^{-1}y + \frac{1}{2}\log(1 + y^2) = \log C$

or, $\tan^{-1}x + \tan^{-1}y + \log\left(1+x^2\right)^{\frac{1}{2}} + \log\left(1+y^2\right)^{\frac{1}{2}} = \log C$

$\therefore \tan^{-1}x + \tan^{-1}y + \log\sqrt{1+x^2}\,\sqrt{1+y^2} = \log C$ Ans.

8. $(e^x + 1)\,y\,dy = (y + 1)\,e^x\,dx$

Sol^n. Given differential equation is,

$(e^x + 1)\,y\,dy = (y + 1)\,e^x\,dx$

Separating the variables,

$$\frac{y\,dy}{(1+y)} = \frac{e^x dx}{\left(e^x+1\right)}$$

or, $\left(1 - \frac{1}{1+y}\right)dy = \frac{e^x}{e^x+1}\,dx$

Integrating we get,

$y - \log(1 + y) = \log(e^x + 1) + \log C$

or, $y = \log(1 + y) + \log(e^x + 1) + \log C$

$\therefore y = \log(1 + y)(e^x + 1)\,C$ Ans.

9. $3e^x \tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0$

Sol^n. Given differential equation is,

$3e^x \tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0$

Separating the variables,

$$\frac{3e^x dx}{1-e^x} + \frac{\sec^2 y\,dy}{\tan y} = 0$$

or, $-3\left[\frac{e^x dx}{1-e^x}\right] + \frac{\sec^2 y\,dy}{\tan y} = 0$

Integrating we get,

$-3\log(1 - e^x) + \log\tan y = \log C$

or, $\log\tan y = \log C + \log(1 - e^x)^3$

$\therefore \tan y = C(1 - e^x)^3$ Ans.

10. $e^{x-y}\,dx + e^{y-x}\,dy = 0$

Sol^n. Given differential equation is,

$e^{x-y}\,dx + e^{y-x}\,dy = 0$

or, $e^x \cdot e^{-y}\,dx + e^y e^{-x}\,dy = 0$

Separating the variables,

or, $\frac{e^x}{e^{-x}}\,dx + \frac{e^y dy}{e^{-y}} = 0$

or, $e^{2x}dx + e^{2y}dy = 0$

Integrating we get,

$$\frac{e^{2x}}{2} + \frac{e^{2y}}{2} = \frac{C}{2}$$

$\therefore e^{2x} + e^{2y} = C$ Ans.

11. $(a^2 + y^2)\,x\,dx + y\,(x^2 - a^2)\,dy = 0$

Sol^n. Given differential equation is,

$(a^2 + y^2)\,x\,dx + y\,(x^2 - a^2)\,dy = 0$

Separating the variables,

$$\frac{x\,dx}{\left(x^2-a^2\right)} + \frac{y\,dy}{\left(a^2+y^2\right)} = 0$$

or, $\dfrac{2x\,dx}{\left(x^2-a^2\right)}+\dfrac{2y\,dy}{\left(a^2+y^2\right)}=0$

Integrating we get,

$\log(x^2-a^2)+\log(a^2+y^2)=\log C$

$\therefore (x^2-a^2)(a^2+y^2)=C$ Ans.

12. $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

Sol^n. Given differential equation is,

$(e^y+1)\cos x\,dx + e^y\sin x\,dy = 0$

Separating the variables,

$\dfrac{\cos x\,dx}{\sin x}+\dfrac{e^y dy}{\left(e^y+1\right)}=0$

Integrating we get,

$\log(\sin x)+\log(e^y+1)=\log C$

$\therefore \sin x\,(e^y+1)=C$ Ans.

13. Find the particular solution of 2xy' = 3y given that y = 4 when x = 1

Sol^n. Given differential equation is,

$2xy^1 = 3y$

i.e. $2x\dfrac{dy}{dx}=3y$

Separating the variables,

$2\dfrac{dy}{dx}=3\dfrac{dx}{x}$

Integrating both sides we get,

$2\log y = 3\log x + \log C$

or, $\log y^2 = \log x^3 + \log C$

or, $\dfrac{y^2}{x^3}=C$

When y = 4, x = 1 then,

$\dfrac{16}{1}=C \Rightarrow C=16$

$\therefore \dfrac{y^2}{x^3}=16$

or, $y^2=16x^3$

$\therefore y = 4x^{\frac{3}{2}}$ Ans.

14. Find the particular solution of y' = sec y, given that y = 0 when x = 0.

Sol^n. Given differential equation is,

$y' = \sec y$

i.e. $\dfrac{dy}{dx}=\sec y$

Separating the variables,

$\dfrac{dy}{\sec y}=dx \Rightarrow \cos y\,dy = dx$

Integrating both sides we get,

$\sin y = x + C$

when y = 0, x = 0 then

$\sin 0 = 0 + C$

$\Rightarrow C = 0$

$\therefore \sin y = x$ Ans.

15. Find the particular solution of $\dfrac{dy}{dx}=e^{x+y}$ and it is given that for x = 1, y = 1 find y as x = – 1

Sol^n. Given differential equation is,

$\dfrac{dy}{dx}=e^{x+y}$

or, $\dfrac{dy}{dx}=e^x\,.\,e^y$

or, $\dfrac{dy}{e^y}=e^x\,dx$

or, $\int e^{-y}\,dy=\int e^x\,dx$ Integrating

$-e^{-y}=e^x+C$

$-e^x-e^{-y}=C$ ……. (i)

Using x = 1, y = 1 in (i)

$-e-e^{-1}=C$ ……. (ii)

Again using x = – 1 in (i)

$-e^{-1}-e^{-y}=C$ ….. (iii)

From (ii) and (iii)

$-e-e^{-1}=-e^{-1}-e^{-y}$

$\Rightarrow e=e^{-y}$

Taking log on both sides,

$\log e=\log e^{-y}$

$-y=1$

$\Rightarrow y=-1$ Ans.

Also, from (i) and (ii)

$-e^x-e^{-y}=-e-e^{-1}$ is the particular solution of the given differential equation.

16. Find the equation of the curve which passes through the point (1, 2) and has at every point, $\frac{dy}{dx} = \frac{-2xy}{x^2+1}$

Sol^n. Given differential equation is,

$$\frac{dy}{dx} = \frac{-2xy}{x^2+1}$$

$$\int \frac{dy}{y} = -\int \frac{2x}{x^2+1} \text{; Integrating}$$

$\log y = -\log (x^2 + 1) + \log C$

$$\log y = \log \frac{c}{\left(x^2+1\right)} \Rightarrow y = \frac{C}{\left(x^2+1\right)}$$

or, $y (x^2 + 1) = C$ …… (i)

Since equation (i) passes through the point (1, 2)

So, $2 (1^2 + 1) = C \Rightarrow C = 4$

Hence (i) gives,

$y (x^2 + 1) = 4$ Ans.

17. Find the particular solution of $y (1 - x^2) \frac{dy}{dx} + x (1 - y^2) = 0$ given that $y = 1$ when $x = 0$

Sol^n. Given differential equation is,

$$y (1 - x^2) \frac{dy}{dx} + x (1 - y^2) = 0$$

$$\text{or, } \int \frac{y}{1-y^2} dy = -\int \frac{x}{1-x^2} dx\text{; Integrating}$$

$-\log (1 - y^2) = \log (1 - x^2) \log C$

$$\text{or, } \log (1 - y^2)^{-1} = \log \frac{\left(1-x^2\right)}{C}$$

$$\Rightarrow \frac{1}{\left(1-y^2\right)} = \frac{\left(1-x^2\right)}{C}$$

or, $(1 - x^2) (1 - y^2) = C$ …… (i)

Using $x = 0$ and $y = 1$, in (i) we get

$C = 0$

Hence (i) gives,

$(1 - x^2) (1 - y^2) = 0$ Ans.

18. Find the equation of the curve represented by $(y - yx) dx + (x + xy) dy = 0$ and passes the point (1, 1)

Sol^n. Given differential equation is,

$(y - yx) dx + (x + xy) dy = 0$

or, $y (1 - x) dx + x (1 + y) dy = 0$

$$\text{or, } \frac{(1-x)}{x} dx = -\frac{\left(1+y\right)}{y} dy$$

$$\text{or, } \int \left(\frac{1}{x} - 1\right) dx = \int \left(\frac{1}{y} + 1\right) dy \text{ Integrating}$$

$\log x - x = -\log y - y + C$

or, $\log x + \log y = x - y + C$

$\log xy = x - y + C$ …… (i)

Hence equation (i) passes through the point (1, 1)

$\log 1 = 1 - 1 + C \Rightarrow C = 0$

Hence (i) givens,

$\log xy = x - y$ Ans.