Bernoulli's equation

An equation of the form $\frac{dy}{dx} + Py = Qy^n$, where $P \; and \; Q$ are function of x or constant is called Bernoulli's equation. It can be solved by reducing it to linear form.

For this

$$\frac{dy}{dx} + Py = Qy^n$$

Dividing both sides by y^n

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^n}Py = Q$$

$\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}}P = Q \qquad (1)$

Put $\qquad \frac{1}{y^{n-1}} = v$

$$y^{-n+1} = v,$$

Diff. w. r. t. $'x'$ we get;

$$(-n+1)y^{-n+1-1}\frac{dy}{dx} = \frac{dv}{dx}$$

Or, $(-n+1)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$

Or, $(-n+1)\frac{1}{y^n}\frac{dy}{dx} = \frac{dv}{dx}$

Or, $\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(-n+1)}\frac{dv}{dx}$

Now equation (1) becomes;

$$\frac{1}{(-n+1)}\frac{dv}{dx} + Pv = Q$$

Or, $\frac{1}{(-n+1)}\frac{dv}{dx} + Pv = Q$

Or, $\frac{dv}{dx} + P(-n+1)v = (-n+1)Q$

Which is the linear differential equation in v,

So its integrating factor (I.F.)= $e^{\int P(-n+1)dx}$

And its general solution is

$$v \times I.F. = \int (-n+1)Q \times I.F \ \ dx + c$$

Exercise – 24

Solve the following differential equation.

1. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

Solution:

Given differential equation is

$$\frac{dy}{dx}+\frac{y}{x}=\frac{y^2}{x^2}$$

Which is of the $\frac{dy}{dx}+Py=Qy^n$

Dividing both sides by y^2

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y^2}\times\frac{y}{x}=\frac{1}{x^2}$$

Or, $\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y}.\frac{1}{x}=\frac{1}{x^2}$ (1)

Put $\frac{1}{y}=v$

Diff. w. r. t. $'x'$ we get;

$$\frac{d}{dx}\left(\frac{1}{y}\right)=\frac{dv}{dx}$$

$$\frac{dy^{-1}}{dx}=\frac{dv}{dx}$$

Or, $-1y^{-2}\frac{dy}{dx}=\frac{dv}{dx}$

Or, $\frac{1}{y^2}\frac{dy}{dx}=-\frac{dv}{dx}$

Now equation (1) becomes;

$$-\frac{dv}{dx}+v\frac{1}{x}=\frac{1}{x^2}$$

Or, $\frac{dv}{dx}-\frac{1}{x}v=-\frac{1}{x^2}$ $\quad(2)$

Which is linear differential equation on v

So its $I.F.=e^{\int -\frac{1}{x}dx}$

$$=e^{-\int\frac{1}{x}dx}$$

$$=e^{-logx}$$

$$=e^{logx^{-1}}$$

$$=x^{-1}$$

$$=\frac{1}{x}$$

Multiplying equation (2) by $I.F.=\frac{1}{x}$ on both sides we get

$$v\times I.F.=\int(Q\times I.F.)\,dx+c$$

$$v\times\frac{1}{x}=\int\left(-\frac{1}{x^2}\times\frac{1}{x}\right)dx+c$$

$$=-\int\frac{1}{x^3}\,dx+c$$

$$= \frac{1}{2x^2} + c$$

$$v \times \frac{1}{x} = \frac{1}{2x^2} + c$$

Since $v = \frac{1}{y}$

Or $\frac{1}{y} \times \frac{1}{x} = \frac{1}{2x^2} + c$

Or $2x = y + c.2x^2y$

Or $2x - y = Cx^2y$ where $C = 2c$

$$y = Cx^2y - 2x$$

Which is the required general solution.

4. $\frac{dy}{dx} + \frac{y}{x} logy = \frac{y}{x^2}(logy)^2$

Solution:

Given differential equation

$$\frac{dy}{dx} + \frac{y}{x} logy = \frac{y}{x^2}(logy)^2$$

Which is of the $\frac{dy}{dx} + Py = Qy^n$

Dividing both sides by $y(logy)^2$

$$\frac{1}{y(logy)^2}\frac{dy}{dx}+\frac{1}{y(logy)^2}\times\frac{y}{x}logy=\frac{1}{x^2}$$

$$\text{Or, }\frac{1}{y(logy)^2}\frac{dy}{dx}+\frac{1}{logy}\times\frac{1}{x}=\frac{1}{x^2} \qquad (1)$$

Put $\frac{1}{logy}=v$

Diff. w. r. t. $'x'$ we get;

$$\frac{d}{dx}\left(\frac{1}{logy}\right)=\frac{dv}{dx}$$

$$\frac{d(logy)^{-1}}{dx}=\frac{dv}{dx}$$

Or, $\frac{d(logy)^{-1}}{d(logy)}.\frac{d(logy)}{dy}.\frac{dy}{dx}=\frac{dv}{dx}$

Or, $-(logy)^{-2}\frac{1}{y}\frac{dy}{dx}=\frac{dv}{dx}$

Or, $-\frac{1}{(logy)^2}\frac{1}{y}\frac{dy}{dx}=\frac{dv}{dx}$

Or, $\frac{1}{(logy)^2}\frac{1}{y}\frac{dy}{dx}=-\frac{dv}{dx}$

Now equation (1) becomes;

$$-\frac{dv}{dx}+v\frac{1}{x}=\frac{1}{x^2}$$

Or, $\qquad \frac{dv}{dx}-\frac{1}{x}v=-\frac{1}{x^2} \qquad (2)$

Which is linear differential equation on v

So its $I.F.=e^{\int -\frac{1}{x}dx}$

$$=e^{-\int\frac{1}{x}dx}$$

$$=e^{-logx}$$

$$=e^{logx^{-1}}$$

$$=x^{-1}$$

$$=\frac{1}{x}$$

Multiplying equation (2) by $I.F.=\frac{1}{x}$ on both sides we get

$$v\times I.F.=\int(Q\times I.F.)\,dx+c$$

$$v \times \frac{1}{x} = \int \left(-\frac{1}{x^2} \times \frac{1}{x}\right) dx + c$$

$$= -\int \frac{1}{x^3} dx + c$$

$$= \frac{1}{2x^2} + c$$

$$v \times \frac{1}{x} = \frac{1}{2x^2} + c$$

Since $v = \frac{1}{logy}$

Or $\frac{1}{logy} \times \frac{1}{x} = \frac{1}{2x^2} + c$

Or $2x = logy + c.2x^2 logy$

Or $2x = logy + 2cx^2 logy$

Or $2x = 2cx^2 logy + logy$

Or $2x = logy(2cx^2 + 1)$

Or $x = logy\left(\frac{2cx^2+1}{2}\right)$

Or $x = \left(cx^2 + \frac{1}{2}\right) logy$

Which is the required general solution

9. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

Solution:

Given differential equation is

$$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$$

Which is of the $\frac{dy}{dx} + Py = Qy^n$

Dividing both sides by $\sec y$

$$\frac{1}{\sec y}\frac{dy}{dx} - \frac{1}{\sec y} \times \frac{\tan y}{1+x} = (1+x)e^x$$

Or, $\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x \qquad (1)$

Put $\quad sin\ y = v$

Diff. w. r. t. $'x'$ we get;

$$\cos y \frac{dy}{dx} = \frac{dv}{dx}$$

Now equation (1) becomes;

$$\frac{dv}{dx} - \frac{v}{1+x} = (1+x)e^{x}$$

Or, $\frac{dv}{dx} - \frac{1}{1+x}v = (1+x)e^{x}$ (2)

Which is linear differential equation on v

So its $I.F. = e^{\int -\frac{1}{1+x}dx}$

$$= e^{-\int \frac{1}{1+x}dx}$$

$$= e^{-log(1+x)}$$

$$= e^{log(1+x)^{-1}}$$

$$= (1+x)^{-1}$$

$$= \frac{1}{(1+x)}$$

Multiplying equation (2) by $I.F. = \frac{1}{(1+x)}$ on both sides we get

$$v \times I.F. = \int (Q \times I.F.)\, dx + c$$

$$v \times \frac{1}{(1+x)} = \int \left((1+x)e^{x} \quad \times \frac{1}{(1+x)} \right) dx + c$$

$$= \int e^{x}\, dx + c$$

$$= e^{x} + c$$

$$v \times \frac{1}{(1+x)} = e^{x} + c$$

Since $v = \sin y$

Or $\quad \sin y \times \frac{1}{(1+x)} = e^{x} + c$

Or $\quad \sin y = (1+x)(e^{x} + c)$

Which is the required general solution

<u>Exact Differential Equations</u>

Exact differential equations

A differential equation

$M(x,y)dx + N(x,y)dy = 0$

Is said to be exact if there exists a function $f(x,y)$such that

$$M(x,y)dx + N(x,y)dy = df(x,y)$$

$i.e.$ the given differential equations is exact if

$M(x,y)dx + N(x,y)dy$ is exact or perfect differential.

Note:

The differential equation

$M(x,y)dx + N(x,y)dy = 0$ will be exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Where $\frac{\partial}{\partial x}$ denotes the partial derivative.

Note:

Every differential equation

$M(x,y)dx + N(x,y)dy = 0$ is not exact.

For example;

$x^2dy + 2xydx = 0$ is exact because

$$x^2dy + 2xydx = d(x^2y)$$

Some formula

1. $xdy + ydx = d(xy)$

2. $\frac{xdy-ydx}{x^2} = d\left(\frac{y}{x}\right)$

3. $\frac{ydx-xdy}{y^2} = d\left(\frac{x}{y}\right)$

4. $xdx + ydy = d\left(\frac{x^2}{2}\right) + d\left(\frac{y^2}{2}\right) = d\left(\frac{x^2+y^2}{2}\right)$

5. $\frac{xdy-ydx}{xy} = \frac{dy}{y} - \frac{dx}{x} = d(logy) - d(logx) =$
$d(logy - logx) = d\left(log\frac{y}{x}\right)$

$$i.e.\quad \frac{xdy - ydx}{xy} = d\left(log\frac{y}{x}\right)$$

6. $\frac{2xydx-x^2dy}{y^2} = d\left(\frac{x^2}{y}\right)$

7. $\frac{2xydy-y^2dx}{x^2} = d\left(\frac{y^2}{x}\right)$

8. $\frac{ydx-xdy}{x^2+y^2} = \frac{\frac{ydx-xdy}{x^2}}{1+\left(\frac{x}{y}\right)^2} = d\left(\tan^{-1}\frac{x}{y}\right)$

Exercise

Solve the following differential equation by reducing to exact form.

1. $xdy + ydx = 0$

Solution:

Given differential equation is

$$xdy + ydx = 0$$

Or, $d(xy) = 0$

Integrating on both sides we get;

$$xy = c$$

Or

Given differential equation is

$$xdy + ydx = 0$$

Or,

$$xdy = -ydx$$

Dividing both sides by xy

$$\frac{xdy}{xy} = \frac{-ydx}{xy}$$

$\frac{dy}{y} = \frac{-dx}{x}$ Integrating on both sides we get;

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$logy + logc$$

$$= -logx$$

$$logyc = logx^{-1}$$

$$yc = x^{-1}$$

$$yc = \frac{1}{x}$$

2. $2xydy + y^2dx = 0$
3. $ydx - xdy = 0$
4. $2xydx - x^2dy$

5. $ydx + (x + y)dy = 0$
6. $(2xy + y^2)dy + (y^2 + x)dx = 0$
7. $\frac{dy}{dx} = \frac{y-x+1}{y-x+5}$
8. $(x^2 + 5xy^2)dx + (5x^2y + y^2)dy = 0$
9. $sinxcosxdx + sinycosydy = 0$