Linear differential equations

Linear differential equation:

An equation of the form $\frac{dy}{dx} + Py = Q$ where P and Q are function of x or constant is called linear differential equation in y of first order and first degree.

And its solution is

$$\boldsymbol{y \times I.F. = \int (Q \times I.F.)\, dx + c}$$

Where $I.F = integrating\ factor = e^{\int Pdx}$

Also

The first order differential equation is of the form

$$\frac{dx}{dy} + Px = Q$$

where P and Q are function of y or constant is called linear differential equation in $\boldsymbol{x}$ of first order and first degree.

And its solution is

$$x \times I.F. = \int (Q \times I.F.)\, dy + c$$

Where $I.F = integrating\ factor = e^{\int Pdy}$

Exercise

Solve the following differential equations.

1. $\frac{dy}{dx} + y = 1$

Solution:

Given differential equation is

$$\frac{dy}{dx} + y = 1 \qquad (1)$$

Which is the linear differential equation in y

So comparing (1) with $\frac{dy}{dx} + Py = Q$ we get

$$P = 1 \ and\ Q = 1$$

Now $\qquad I.F. = e^{\int Pdx}$

$$= e^{\int dx}$$

$$I.F. = e^{x}$$

Multiplying equation (1) by
$I.F. = e^{x}$ on both sides we get,

$$\mathbf{y \times I.F. = \int (Q \times I.F.)\, dx + c}$$

$$y \times e^{x} = \int (1 \times e^{x})\, dx + c$$

$$= \int e^{x}\, dx + c$$

$$= e^{x} + c$$

$$\therefore y \times e^{x} = e^{x} + c$$

$y = 1 + ce^{-x}$ which is required general solution.

2. $\frac{dy}{dx} - y = e^{x}$

Solution:

Given differential equation is

$$\frac{dy}{dx} - y = e^{x} \qquad (1)$$

Which is the linear differential equation in y

So comparing (1) with $\frac{dy}{dx}+Py=Q$ we get

$$P=-1 \ \ and \ Q=e^{x}$$

Now $\quad I.F.=e^{\int Pdx}$

$$=e^{\int -dx}$$

$$I.F.=e^{-x}$$

Multiplying equation (1) by
$I.F.=e^{-x}$ on both sides we get,

$$y\times I.F.=\int (Q\times I.F.)\,dx+c$$

$$y\times e^{-x}=\int (e^{x}\times e^{-x})\,dx+c$$

$$=\int dx+c$$

$$=x+c$$

$$\therefore y\times e^{-x}=x+c$$

$y = e^x(x + c)$ which is required general solution.

3. $\frac{dy}{dx} + \frac{y}{x} = x$

Solution:

Given differential equation is

$$\frac{dy}{dx} + \frac{y}{x} = x \qquad (1)$$

Which is the linear differential equation in y

So comparing (1) with $\frac{dy}{dx} + Py = Q$ we get

$$P = \frac{1}{x} \ \ and\ Q = x$$

Now $\qquad I.F. = e^{\int P dx}$

$$= e^{\int \frac{1}{x} dx}$$

$$= e^{log x}$$

$$I.F. = x$$

Multiplying equation (1) by
$I.F. = x$ on both sides we get,

$$y \times I.F. = \int (Q \times I.F.)\, dx + c$$

$$y \times x = \int (x \times x)\, dx + c$$

$$= \int x^2 dx + c$$

$$= \frac{x^3}{3} + c$$

$$\therefore y \times x = \frac{x^3}{3} + c$$

$xy = \frac{x^3}{3} + c$ which is required general solution.

4. $x\frac{dy}{dx} + y = x^4$

Solution:

Given differential equation is

$$x\frac{dy}{dx} + y = x^4$$

Dividing both sides by x we get;

$$\frac{dy}{dx} + \frac{y}{x} = x^3 \qquad (1)$$

Which is the linear differential equation in y

So comparing (1) with $\frac{dy}{dx} + Py = Q$ we get

$$P = \frac{1}{x} \ and \ Q = x^3$$

Now $\qquad I.F. = e^{\int P dx}$

$$= e^{\int \frac{1}{x} dx}$$

$$= e^{log x}$$

$$I.F. = x$$

Multiplying equation (1) by

$I.F. = x$ on both sides we get,

$$y \times I.F. = \int (Q \times I.F.)\, dx + c$$

$$y \times x = \int (x^3 \times x)\, dx + c$$

$$= \int x^4 dx + c$$

$$= \frac{x^5}{5} + c$$

$$\therefore y \times x = \frac{x^5}{5} + c$$

$xy = \frac{x^5}{5} + c$ which is required general solution.

5. $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

Solution:

Given differential equation is

$$(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$$

Dividing both sides by $(1 + x^2)$ we get;

$$\frac{dy}{dx}+\frac{2xy}{(1+x^2)}=\frac{4x^2}{(1+x^2)} \qquad (1)$$

Which is the linear differential equation in y

So comparing (1) with $\frac{dy}{dx}+Py=Q$ we get

$$P=\frac{2x}{(1+x^2)} \quad and\ Q=\frac{4x^2}{(1+x^2)}$$

Now $\qquad I.F.=e^{\int P dx}$

$$=e^{\int \frac{2x}{(1+x^2)}dx}$$

$$=e^{log(x^2+1)}$$

$$I.F.=(x^2+1)$$

Multiplying equation (1) by
$I.F.=(x^2+1)$ on both sides we get,

$$y\times I.F.=\int (Q\times I.F.)\,dx+c$$

$$y \times (1 + x^2) = \int \left(\frac{4x^2}{(1 + x^2)} \times (x^2 + 1) \right) dx + c$$

$$= \int 4x^2 dx + c$$

$$= \frac{4x^3}{3} + c$$

$$\therefore y \times (1 + x^2) = \frac{4x^3}{3} + c$$

$$y(1 + x^2) = \frac{4x^3}{3} + c$$

which is required general solution.

6. $\frac{dy}{dx} + 2ytanx = sinx$

Solution:

Given differential equation is

$$\frac{dy}{dx} + 2ytanx = sinx \qquad (1)$$

Which is the linear differential equation in y

So comparing (1) with $\frac{dy}{dx}+Py=Q$ we get

$$P=2tanx \;\; and \; Q=sinx$$

Now $\qquad I.F.=e^{\int Pdx}$

$$=e^{\int 2tanxdx}$$

$$=e^{2logsecx}$$

$$=e^{logsec^2x}$$

$$I.F.=sec^2x$$

Multiplying equation (1) by
$I.F.=sec^2x$ on both sides we get,

$$y\times I.F.=\int(Q\times I.F.)\,dx+c$$

$$y\times sec^2x=\int(sinx\times sec^2x)\,dx+c$$

$$=\int secxtanxdx+c$$

$$= secx + c$$

$$\therefore y \times sec^2x = secx + c$$

$ysec^2x = secx + c$ which is required general solution.

7. $sinx\frac{dy}{dx} + ycosx = xsinx$

Solution:

Given differential equation is

$$sinx\frac{dy}{dx} + ycosx = xsinx$$

$$\frac{dy}{dx} + \frac{ycosx}{sinx} = \frac{xsinx}{sinx}$$

$$\frac{dy}{dx} + ycotx = x \qquad (1)$$

Which is the linear differential equation in y

So comparing (1) with $\frac{dy}{dx} + Py = Q$ we get

$$P = cotx \ \ and \ Q = x$$

Now $\qquad I.F. = e^{\int Pdx}$

$$= e^{\int cotxdx}$$

$$= e^{logsinx}$$

$$I.F. = sinx$$

Multiplying equation (1) by
$I.F. = sinx$ on both sides we get,

$$y \times I.F. = \int (Q \times I.F.)\, dx + c$$

$$y \times sinx = \int (x \times sinx)\, dx + c$$

$$= \int xsinxdx + c$$

$$= x \int sinxdx - \int \left\{ \frac{dx}{dx} \int sinx \right\} dx + c$$

$$= x \times (-cosx) - \int 1 \times (-cosx) dx + c$$

$$= -xcosx + \int cosx\, dx + c$$

$$= -xcosx + sinx + c$$

$$\therefore y \times sinx = -xcosx + sinx + c$$

$ysinx = sinx - xcosx + c$ which is required general solution.

8. $cos^2x\frac{dy}{dx} + y = 1$

Solution:

Given differential equation is

$$cos^2x\frac{dy}{dx} + y = 1$$

$$\frac{dy}{dx} + \frac{y}{cos^2x} = \frac{1}{cos^2x}$$

$$\frac{dy}{dx} + ysec^2x = sec^2x \qquad (1)$$

Which is the linear differential equation in y

So comparing (1) with $\frac{dy}{dx} + Py = Q$ we get

$$P = sec^2x \ \ and\ Q = sec^2x$$

Now $\qquad I.F. = e^{\int Pdx}$

$$= e^{\int sec^2xdx}$$

$$= e^{tanx}$$

$$I.F. = e^{tanx}$$

Multiplying equation (1) by
$I.F. = e^{tanx}$on both sides we get,

$$y \times I.F. = \int (Q \times e^{tanx}.)\, dx + c$$

$$y \times e^{tanx} = \int (sec^2x \times e^{tanx})\, dx + c$$

$$= \int (sec^2xe^{tanx})\, dx + c$$

$$put\ tanx = t$$

Diff. w.r.t. x

$$secx^2dx = dt$$

$$y \times e^{tanx} = \int e^t\, dt + c$$

$$= e^t + c$$

$$= e^{tanx} + c$$

$$\therefore y \times e^{tanx} = e^{tanx} + c$$

$y = 1 + ce^{-tanx}$ which is required general solution.

8. $(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}$

Solution:

Given differential equation is

$(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}$

$$\frac{dy}{dx} + \frac{y}{(1+x^2)} = \frac{e^{\tan^{-1} x}}{(1+x^2)} \qquad (1)$$

Which is the linear differential equation in y

So comparing (1) with $\frac{dy}{dx} + Py = Q$ we get

$$P = \frac{1}{(1 + x^2)} \quad and\ Q = \frac{e^{\tan^{-1} x}}{(1 + x^2)}$$

Now $\qquad I.F. = e^{\int Pdx}$

$$= e^{\int \frac{1}{(1+x^2)} dx}$$

$$= e^{\tan^{-1} x}$$

$$I.F. = e^{\tan^{-1} x}$$

Multiplying equation (1) by

$I.F. = e^{\tan^{-1} x}$ on both sides we get,

$$y \times I.F. = \int (Q \times I.F.)\, dx + c$$

$$y \times e^{\tan^{-1} x} = \int \left(\frac{e^{\tan^{-1} x}}{(1 + x^2)} \times e^{\tan^{-1} x} \right) dx + c$$

$$= \int \left(\frac{e^{2 \tan^{-1} x}}{(1 + x^2)} \right) dx + c$$

$$put\ \tan^{-1} x = t$$

Diff. w.r.t. x

$$\frac{1}{1 + x^2} dx = dt$$

$$y \times e^{\tan^{-1}x} = \int e^{2t}\,dt + c$$

$$= \frac{e^{2t}}{2} + c$$

$$= \frac{e^{2\tan^{-1}x}}{2} + c$$

$$\therefore y \times e^{\tan^{-1}x} = \frac{e^{2\tan^{-1}x}}{2} + c$$

$y = \frac{e^{\tan^{-1}x}}{2} + ce^{-\tan^{-1}x}$ which is required general solution.

Exercise: - 23

Solve the following differential equations.

1. $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2}$

Solution:

Given differential equations is

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2} \qquad (1)$$

$$(1+x^2)\left(\frac{dy}{dx}+\frac{2x}{1+x^2}y\right)=\frac{1}{(1+x^2)^2}(1+x^2)$$

(1)

$$\frac{d\{y\times(1+x^2)\}}{dx}=\frac{1}{(1+x^2)^2}(1+x^2)$$

$$\int d\{y\times(1+x^2)\}=\int\frac{1}{(1+x^2)^2}(1+x^2)\mathrm{dx}$$

$$y\times(1+x^2)=\int\frac{1}{(1+x^2)^2}(1+x^2)\mathrm{dx}+\mathrm{c}$$

Which is the linear differential equation in y

So comparing (1) with $\frac{dy}{dx}+Py=Q$ we get

$$P=\frac{2x}{1+x^2}\ \ and\ Q=\frac{1}{(1+x^2)^2}$$

Now $\qquad I.F.=e^{\int Pdx}$

$$=e^{\int\frac{2x}{1+x^2}dx}$$

$$=e^{log(1+x^2)}$$

$$I.F. = 1 + x^2$$

Multiplying equation (1) by
$I.F. = 1 + x^2$ on both sides we get,

$$y \times I.F. = \int (Q \times I.F.)\, dx + c$$

$$y \times 1 + x^2 = \int \left\{ \frac{1}{(1+x^2)^2} \times (1+x^2) \right\} dx + c$$

$$= \int \frac{1}{1+x^2}\, dx + c$$

$$y \times (1 + x^2) = \tan^{-1} x + c$$

$$\therefore y(1 + x^2) = \tan^{-1} x + c$$

which is required general solution.

14. $(1 + y^2)dx = (\tan^{-1} y - x)dy$

Solution:

Given differential equation is

$$(1+y^2)dx = (\tan^{-1} y - x)dy$$

$$\frac{dy}{dx} = \frac{1+y^2}{\tan^{-1} y - x}$$

$$\therefore \frac{dx}{dy} = \frac{\tan^{-1} y - x}{1+y^2}$$

Or, $\frac{dx}{dy} = \frac{\tan^{-1} y}{1+y^2} - \frac{x}{1+y^2}$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2} \qquad (1)$$

Which is the linear differential equation in x

So comparing (1) with $\frac{dx}{dy} + Px = Q$ we get

$$P = \frac{1}{1+y^2} \quad and\ Q = \frac{\tan^{-1} y}{1+y^2}$$

Now $\qquad I.F. = e^{\int P dy}$

$$= e^{\int \frac{1}{1+y^2} dy}$$

$$= e^{\tan^{-1} y}$$

$$I.F. = e^{\tan^{-1} y}$$

Multiplying equation (1) by
$I.F. = e^{\tan^{-1} y}$ on both sides we get,

$$x \times I.F. = \int (Q \times I.F.)\, dy + c$$

$$x \times e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1 + y^2} \times e^{\tan^{-1} y}\, dy + c$$

Put $\tan^{-1} y = t$

Diff. w.r.t. $'y'$ we get

$$\frac{1}{1 + y^2} dy = dt$$

$$\therefore x \times e^{\tan^{-1} y} = \int t e^t\, dt + c$$

Note: $\int uvdx = u \int v\, dx - \int \left\{ \frac{du}{dx} \int vdx \right\} dx$

$$x \times e^{\tan^{-1} y} = te^t - \int 1. e^t \, dt + c$$

$$= te^t - e^t + c$$

Since $\quad t = \tan^{-1} y$

$$\therefore x \times e^{\tan^{-1} y} = \tan^{-1} y \, e^{\tan^{-1} y} - e^{\tan^{-1} y} + c$$

$$x = \tan^{-1} y - 1 + \frac{c}{e^{\tan^{-1} y}}$$

$$x = (\tan^{-1} y - 1) + ce^{-\tan^{-1} y}$$

which is required general solution.

Exact Differential Equations

Exact differential equations

A differential equation

$M(x, y)dx + N(x, y)dy = 0$

Is said to be exact if there exists a function $f(x, y)$such that

$$M(x, y)dx + N(x, y)dy = df(x, y)$$

$i.e.$ the given differential equations is exact if

$M(x, y)dx + N(x, y)dy$ is exact or perfect differential.

Note:

The differential equation

$M(x, y)dx + N(x, y)dy = 0$ will be exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Where $\frac{\partial}{\partial x}$ denotes the partial derivative.

Note:

Every differential equation

$M(x, y)dx + N(x, y)dy = 0$ is not exact.

Some formula

1. $xdy + ydx = d(xy)$

2. $\frac{xdy-ydx}{x^2} = d\left(\frac{y}{x}\right)$

3. $\frac{ydx-xdy}{y^2} = d\left(\frac{x}{y}\right)$

4. $xdx + ydy = d\left(\frac{x^2}{2}\right) + d\left(\frac{y^2}{2}\right) = d\left(\frac{x^2+y^2}{2}\right)$

5. $\frac{xdy-ydx}{xy} = \frac{dy}{y} - \frac{dx}{x} = d(logy) - d(logx) =$
$d(logy - logx) = d\left(log\frac{y}{x}\right)$

$$i.e. \quad \frac{xdy - ydx}{xy} = d\left(log\frac{y}{x}\right)$$

6. $\frac{2xydx-x^2dy}{y^2} = d\left(\frac{x^2}{y}\right)$

7. $\frac{2xydy-y^2dx}{x^2} = d\left(\frac{y^2}{x}\right)$

8. $\frac{ydx-xdy}{x^2+y^2} = \frac{\frac{ydx-xdy}{x^2}}{1+\left(\frac{x}{y}\right)^2} = d\left(\tan^{-1}\frac{x}{y}\right)$

Exercise

Solve the following differential equation by reducing to exact form.

1. $xdy + ydx = 0$

Solution:

Given differential equation is

$$xdy + ydx = 0$$

Or, $d(xy) = 0$

Integrating on both sides we get;

$$xy = c$$

Or

Given differential equation is

$$xdy + ydx = 0$$

Or,

$$xdy = -ydx$$

Dividing both sides by xy

$$\frac{xdy}{xy} = \frac{-ydx}{xy}$$

$\frac{dy}{y} = \frac{-dx}{x}$ Integrating on both sides we get;

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$logy + logc$$

$$= -logx$$

$$logyc = logx^{-1}$$

$$yc = x^{-1}$$

$$yc = \frac{1}{x}$$

2. $2xydy + y^2dx = 0$
3. $ydx - xdy = 0$
4. $2xydx - x^2dy$
5. $ydx + (x + y)dy = 0$
6. $(2xy + y^2)dy + (y^2 + x)dx = 0$
7. $\frac{dy}{dx} = \frac{y-x+1}{y-x+5}$
8. $(x^2 + 5xy^2)dx + (5x^2y + y^2)dy = 0$
9. $sinxcosxdx + sinycosydy = 0$