Chapter - 14

Ordinary Differential Equations

14.1 Introduction

The equation, which contains dependent variables, independent variable and differential coefficient of the dependent variable with respect to independent variable with constant, is called *Ordinary Differential Equation.*

For example,

i. $\frac{dy}{dx} = 0$ ii. $xy\frac{dy}{dx} + y^2 = 3$

iii. $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 6 = x^2$ iv. $x^4\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 6 = x^2$

v. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2}\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^3 = 0$ vi. $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = c\frac{d^2y}{dx^2}$

vii. $y = x\frac{dy}{dx} + \frac{c}{(dy/dx)}$

A differential equation is also used as mathematical model of the problems. For example, if we drop a stone then its acceleration at time t, $\frac{d^2y}{dt^2}$ is equal to the acceleration of gravity g, which is constant. The model of this problem is

$$\frac{d^2y}{dt^2} = g \text{ (neglecting the air resistance)}$$

After integration, we get the velocity.

$$\frac{dy}{dt} = gt + v_0$$

Where v_0 is the initial velocity from which the motion is started.

Again after integration,

$$y = \frac{g}{2}t^2 + v_0t + y_0$$

where y_0 is the distance from O at the beginning

It can be concluded that the model

$\frac{d^2y}{dt^2} = g$ is the differential equation and

$y = \frac{g}{2}t^2 + v_0t + y_0$ is the general solution of the equation.

14.2 Order of Differential Equation

The highest derivative of y with respect to x occurring in the differential equation is called *order*.

For example,

i. $\frac{d^2y}{dx^2} = \cos 2x + \sin 2x$ is second order.

ii. $y = px + \sqrt{a^2p + b^2}$ where $p = \frac{dy}{dx}$ is first order.

14.3 Degree of Differential Equation

The degree of a differential equation is the power of the highest order derivative which occurs in it provided the equation has been made free from radicals and fractions as far as the derivatives are concerned.

For example,

1. $x\frac{dy}{dx} = y$

2. $x^2y\frac{d^2y}{dx^2} - \frac{dy}{dx} + y^2 = 0$

3. $y = x\frac{\left(\frac{dy}{dx}\right)^2 + c}{\frac{dy}{dx}}$

4. $\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2} = c\frac{d^2y}{dx^2}$

The equations (1), (2) are of first degree differential equations where as the equations (3) and (4) are of the second degree because (3) can be written as

$$y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + c$$

and (4) can be written as $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = c^2\left(\frac{d^2y}{dx^2}\right)^2$

14.4 Solution of Differential Equation

The relation between the dependent and independent variable involving arbitrary constants and if it satisfies the differential equation is called a *Solution of a Differential Equation.* It should be noted that a solution of differential equation does not involve the derivatives of the dependent variable with respect to independent variable. Thus the function involving arbitrary constants is called a *General Solution* of the differential equation.

For example,

$y = ce^{-x}$ is general solution of $\frac{dy}{dx} + y = 0$

Because the values of y satisfies in this equation.

i.e. $\frac{d}{dx}(ce^{-x}) + ce^{-x} = 0$

or $-ce^{-x} + ce^{-x} = 0 \qquad \therefore \quad 0 = 0$

If we choose a specific c, i.e. c = 0, 5, - 2 etc. then what we obtain is called a *Particular Solution* of the differential equation.

14.5 Formation of Differential Equation

If the equation of curve contains only one arbitrary constant it will be the general solution of first order differential equation, so we differentiate it once and the first order differential equation is formed by eliminating one arbitrary constant.

If the equation of curve contains two arbitrary constants then it will be the general solution of second order differential equation, so we differentiate the equation twice and the differential equation is formed by eliminating two arbitrary constants.

Similarly, the elimination of n arbitrary constants lead to n^{th} derivatives and hence we obtain a differential equation of n^{th} order.

For example,

The equation of curve is

$$y = cx + c^2 \qquad \text{.............(1)}$$

This equation contains only one arbitrary constant.

Differentiating it with respect to x once

$$\frac{dy}{dx} = c \qquad \text{.............(2)}$$

Eliminating c from (1) and (2)

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 \qquad \therefore \quad x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 - y = 0$$

is the required differential equation of the first order.

Let the equation of a curve is

$$y = e^x (A\cos x + B\sin x) \qquad \text{.........(1)}$$

This equation contains two arbitrary constants.

Differentiating it with respect to x twice

$$\frac{dy}{dx} = e^x (-A\sin x + B\cos x) + e^x (A\sin x + B\cos x)$$

or $$\frac{dy}{dx} = e^x (-A\sin x + B\cos x) + y \qquad \text{.............(2)}$$

and $$\frac{d^2y}{dx^2} = e^x (-A\cos x - B\sin x) + e^x(-A\sin x + B\cos x) + \frac{dy}{dx} \qquad \text{.....(3)}$$

Eliminating the constants 'A' and 'B' from (1), (2) and (3).

We have,

$$\frac{d^2y}{dx^2} = -e^x (A\cos x - B\sin x) + e^x (-A\sin x + B\cos x) + \frac{dy}{dx}$$

or $\quad \frac{d^2y}{dx^2} = -y + \frac{dy}{dx} - y + \frac{dy}{dx}$

$\therefore \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ is the required differential equation of second order.

14.6 Different Forms of Solution

As we see that the equation of curves

$y = A\cos x - B\sin x$ ....(1)

and $y = \alpha\cos(x - \delta)$ ....(2)

both contain two arbitrary constants.

So. differentiating (1) with respect to x twice

$\frac{dy}{dx} = -A\sin x - B\cos x$ ....(2)

and $\frac{d^2y}{dx^2} = -A\cos x + B\sin x$

$\therefore \quad \frac{d^2y}{dx^2} + y = 0$ is the required differential equation of second order.

Also. differentiating (2) with respect to x twice

$\frac{dy}{dx} = -\alpha\sin(x - \delta)$ ....(5)

and $\frac{d^2y}{dx^2} = -\alpha\cos(x - \delta)$ ....(6)

Eliminating α and δ from (2) and (6).

We get. $\frac{d^2y}{dx^2} + y = 0$ is the required differential equation of second order.

It shows that the general solution of a differential equation may be different forms.

Worked Out Examples

Ex.1: Form the differential equation of the family of curve

$y = A e^{-2x} + B e^{-3x}$

Solution:

Here. the family of curve is

$y = A e^{-2x} + B e^{-3x}$ ....(1)

Since it contains two arbitrary constants. differentiating it with respect to x twice

$\frac{dy}{dx} = -2A e^{-2x} - 3B e^{-3x}$ ....(2)

$$\frac{d^2y}{dx^2} = 4A\, e^{-2x} + 9B\, e^{-3x} \qquad \text{...............(3)}$$

From (2) and (3),

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3B\, e^{-3x}$$

$$\therefore \quad B = \frac{1}{3}\, e^{3x}\left(\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right)$$

From (2),

$$\frac{dy}{dx} = -2A\, e^{-2x} - \frac{d^2y}{dx^2} - 2\frac{dy}{dx}$$

$$\text{or} \quad \frac{d^2y}{dx^2} + 3\frac{dy}{dx} = -2A\, e^{-2x}$$

$$\therefore \quad A = -\frac{1}{2}\, e^{2x}\left(\frac{d^2y}{dx^2} + 3\frac{dy}{dx}\right)$$

Putting the value of A and B in (1),

$$y = -\frac{1}{2}\left(\frac{d^2y}{dx^2} + 3\frac{dy}{dx}\right) + \frac{1}{3}\left(\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right)$$

$$\text{or} \quad y = -\frac{1}{6}\frac{d^2y}{dx^2} - \frac{5}{6}\frac{dy}{dx}$$

$\therefore \quad \frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$ is the required differential equation of second order.

Ex. 2: Obtain the differential equation of all circles touching the axis of y at the origin and centers on the axis of x.

Solution:

Let the equation of all circles touching the axis of y at the origin and centres on the axis of x is

$$(x - h)^2 + (y - 0)^2 = h^2$$

$$\text{or} \quad x^2 - 2hx + h^2 + y^2 = h^2$$

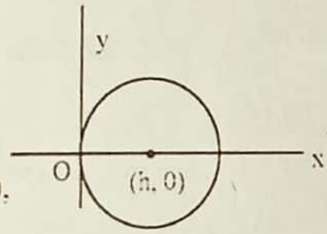
$$\text{or} \quad x^2 + y^2 - 2hx = 0 \qquad \text{............(1)}$$

Where h is the only one arbitrary constant.

Differentiating (1) with respect to x once

$$2x + 2y\frac{dy}{dx} - 2h = 0$$

$$h = x + y\frac{dy}{dx} \qquad \text{.......(2)}$$



Eliminating 'h' between (1) and (2),

$$x^2 - y^2 - 2x\left(x + y\frac{dy}{dx}\right) = 0$$

$\therefore \quad 2xy\frac{dy}{dx} - x^2 - y^2 = 0$ is the required differential equation.

Exercise - 18

1. Determine the order and degree of each of the following differential equations.
   i. $(x + 3y - 2)\,dx + (2x - 3y + 5)\,dy = 0$
   ii. $y = x\dfrac{d^2y}{dx^2} + \dfrac{k}{d^2y/dx^2}$
   iii. $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{-3/2} = k\dfrac{d^2y}{dx^2}$
   iv. $x^2\dfrac{d^2y}{dx^2} + 2x\,y\dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^3 = 0$
2. Form the differential equations from the following equations
   i. $y = a\log x + b$
   ii. $xy = a + bx$
   iii. $xy = Ae^x + Be^{-x}$
   iv. $y = ax^3 + bx^2$
   v. $a\cos(\log x) + b\sin(\log x)$
3. Obtain the differential equation of all circles of radius a and centre (h, k).
4. Form a differential equation of simple harmonic motion given by $x = A\cos(nt + \alpha)$

Answers

1. i. Fist order, first degree. ii. Second order second degree
   iii. Second order, second degree. iv. Second order first degree
2. i. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$ ii. $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 0$
   iii. $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} - y = 0$ iv. $x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = 0$
   v. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$ 3. $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{-3} = a^2\left(\dfrac{d^2y}{dx^2}\right)^2$ 4. $\dfrac{d^2x}{dt^2} + n^2x = 0$

Chapter - 15

First Order and First Degree Differential Equation

15.1 Introduction

The differential of the form $M(x, y)\,dx + N(x, y)\,dy = 0$ where M and N are functions of x and y or constant is called *First Order and First Degree Differential Equation.*

15.2 Separation of the Variables

In the differential equation $M\,dx + N\,dy = 0$, the variables are separated in such way that the coefficient of dx must be function x only or constant and the coefficient of dy must be function of y only or constant.

Worked out Examples

Ex. 1 Solve: $\sqrt{1 - x^2}\,dy + \sqrt{1 - y^2}\,dx = 0$

Solution:

Here, the equation is

$$\sqrt{1 - x^2}\,dy + \sqrt{1 - y^2}\,dx = 0.$$

Separating the variables,

$$\frac{dy}{\sqrt{1 - y^2}} + \frac{dx}{\sqrt{1 - x^2}} = 0$$

Integrating,

$$\frac{dy}{\sqrt{1 - y^2}} + \frac{dx}{\sqrt{1 - x^2}} = 0$$

$\therefore$ $\sin^{-1} y + \sin^{-1} x = c$ is the required solution.

Ex. 2 Solve: $(\sin x + \cos x)\,dy = (\cos x - \sin x)\,dx$

Solution:

Here, the equation is

$$(\sin x + \cos x)\,dy = (\cos x - \sin x)\,dx.$$

Separating the variables,

$$dy = \frac{(\cos x - \sin x)}{(\sin x + \cos x)}\,dx$$

Integrating,

$$y = \log(\sin x + \cos x) + \log c$$

or $\quad y = \log c\,(\sin x + \cos x)$

$\therefore \quad e^{y} = c(\sin x + \cos x)$ is the required solution.

Ex. 3: Solve: $(1 + x)\, y\, dx + (1 + y)\, x\, dy = 0$

Solution:

Here. the equation is

$$(1 + x)\, y\, dx + (1 + y)\, x\, dy = 0$$

Separating the variables.

$$\frac{1 + x}{x} dx + \frac{1 + y}{y} dy = 0$$

or $$\frac{(1 + x)}{x} dx + \frac{(1 + y)}{y} dy = 0.$$

or $$\left(\frac{1}{x} + 1\right) dx + \left(\frac{1}{y} + 1\right) dy = 0$$

Integrating

$$\log x + x + \log y + y = c$$

$\therefore \quad \log (xy) + x + y = c$ is the required solution.

Ex. 4: Solve: $\log \left(\frac{dy}{dx}\right) = ax + by$

Solution:

Here, the equation is

$$\log \left(\frac{dy}{dx}\right) = ax + by$$

or $$\frac{dy}{dx} = e^{ax+by}$$

Separating the variables

$$\frac{dy}{e^{by}} = e^{ax}\, dx.$$

or $$e^{-by}\, dy = e^{ax}\, dx$$

or $$e^{ax}\, dx - e^{-by}\, dy = 0$$

Integrating.

$$\frac{e^{ax}}{a} + \frac{e^{-by}}{b} + c = 0$$

$\therefore \quad be^{ax} + ae^{-by} + abc = 0$ is the required solution.

Ex. 5:Solve:$(1 - x^2)\,(1 - y)\, dx = xy\,(1 + y)\, dy$

Solution:

Here. the equation is

$$(1 - x^2)\,(1 - y)\, dx = xy\,(1 + y)\, dy.$$

Separating the variables.

$$\frac{1-x^2}{x}\,dx = y\,\frac{(1+y)}{1-y}\,dy$$

or $$\frac{1-x^2}{x}\,dx = \frac{y(1+y)}{1-y}\,dy$$

or $$\frac{dx}{x} - x\,dx = \left(-y + \frac{2}{1-y} - 2\right) dy$$

Integrating

$$\log x - \frac{x^2}{2} = -\frac{y^2}{2} - 2\log(1-y) - 2y + c$$

$\therefore$ $\log x - \frac{x^2}{2} + \frac{y^2}{2} + 2y + 2\log(1-y) = c$ is the required solution.

Ex. 6:Solve: $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$

Solution:

Here, the equation is

$$\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0.$$

Separating the variables.

$$\frac{\sec^2 x}{\tan x}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0$$

or $$\frac{\sec^2 x}{\tan x}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0$$

Integrating,

$$\log(\tan x) + \log(\tan y) = \log c$$

or $\log(\tan x \tan y) = \log c$

$\therefore$ $\tan x \tan y = c$ is the required solution.

Ex. 7: Solve: $\frac{dy}{dx} + \frac{1+\cos 2y}{1-\cos 2x} = 0$

Solution:

Here, the equation is

$$\frac{dy}{dx} + \frac{1+\cos 2y}{1-\cos 2x} = 0.$$

Separating the variables.

$$\frac{dy}{1+\cos 2y} + \frac{dx}{1-\cos 2x} = 0$$

or $$\frac{dy}{2\cos^2 y} + \frac{dx}{2\sin^2 x} = 0$$

or $\sec^2 y.\,dy + \operatorname{cosec}^2 x\,dx = 0$

Integrating

$\tan y - \cot x = c$ is the required solution.

Ex. 8: Solve: $\frac{dy}{dx} = \frac{x(2 \log x + 1)}{\sin y + y \cos y}$

Solution:

Here, the equation is

$$\frac{dy}{dx} = \frac{x(2 \log x + 1)}{\sin y + y \cos y}$$

Separating the variables.

$$(\sin y + y \cos y)\, dy = (2x \log x + x)\, dx$$

or $\sin y\, dy + y \cos y\, dy = 2x \log x\, dx + x dx$

Integrating.

$$- \cos y + y \sin y - \sin y\, dy = x^2 \log x - \frac{1}{x}\, x^2\, dx + \frac{x^2}{2}$$

or $\quad - \cos y + y \sin y + \cos y = x^2 \log x - \frac{x^2}{2} + \frac{x^2}{2} + c$

$\therefore \quad y \sin y = x^2 \log x + c$ is the required solution.

Ex.9: Solve: $\cos y\, dx + (1 + 2e^{-x}) \sin y\, dy = 0$ given $y = \frac{\pi}{4}$ when $x = 0$

Solution:

Here, the equation is

$$\cos y\, dx + (1 + 2e^{-x}) \sin y\, dy = 0$$

Separating the variables,

$$\frac{dx}{(1 + 2e^{-x})} + \frac{\sin y}{\cos y}\, dy = 0$$

or $\quad \frac{e^x}{e^x + 2}\, dx + \frac{\sin y}{\cos y}\, dy = 0$

Integrating.

$$\log (e^x + 2) + \log \sec y = \log c$$

or $\quad (e^x + 2) \sec y = c \qquad$ ....(1)

Using the given condition, $y = \frac{\pi}{4}$ when $x = 0$.

Then (1) becomes,

$3\sqrt{2} = c. \qquad \therefore \quad c = 3\sqrt{2}$

$\therefore \quad (e^x + 2) \sec y = 3\sqrt{2}$ is the required solution.

Ex.10: Find the particular solution of the following differential equation $\log \left(\frac{dy}{dx}\right) = 3x + 4y$ given that $y=0$, when $x=0$

Solution:

Here, the equation is

$$\log \left(\frac{dy}{dx}\right) = 3x - 4y.$$

or $\frac{dy}{dx} = e^{3x+4y}$

Separating the variables,

$e^{-4y}\, dy = e^{3x}\, dx$

or $e^{3x}dx - e^{-4y}\, dy = 0$

Integrating

$\frac{e^{3x}}{3} + \frac{e^{-4y}}{4} = c$ ....(1)

Given that $y = 0$ when $x = 0$ then (1) becomes,

$\frac{1}{3} + \frac{1}{4} = c$

$\therefore c = \frac{7}{12}$.

Thus the general solution is $4e^{3x} + 3e^{-4y} = 7$.

Exercise-19

Solve the following differential equations

1. $x\sqrt{1+y^2}\, dx + y\sqrt{1+x^2}\, dy = 0$
2. $(x^2+1)\frac{dy}{dx} = 1$
3. $y\, dx = (e^x + 1)\, dy$
4. $(xy^2 + x)\, dx + (yx^2 + y)\, dy = 0$
5. $\tan y\, dx + \tan x\, dy = 0$
6. $\left(y - x\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$
7. $(1+x)(1+y^2)\, dx + (1+y)(1+x^2)\, dy = 0$
8. $(e^x + 1)\, y\, dy = (y+1)\, e^x\, dx$
9. $3e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$
10. $e^{x-y}\, dx + e^{y-x}\, dy = 0$
11. $(a^2 + y^2)\, x\, dx + y(x^2 - a^2)\, dy = 0$
12. $(e^y - 1)\cos x\, dx - e^y \sin x\, dy = 0$

13. Find the particular solution of $2xy' = 3y$ given that $y = 4$ when $x = 1$

14. Find the particular solution of $y' = \sec y$, given that $y = 0$ when $x = 0$

15. Find the particular solution of $\frac{dy}{dx} = e^{x+y}$ and it is given that for $x=1$, $y = 1$ find y as $x= -1$

16. Find the equation of the curve which passes through the point (1, 2) and has at every point, $\frac{dy}{dx} = \frac{-2xy}{x^2+1}$

17. Find the particular solution of $y(1-x^2)\frac{dy}{dx} + x(1-y^2) = 0$ given that $y = 1$ when $x = 0$

18. Find the equation of the curve represented by $(y - yx)\,dx + (x + xy)dy = 0$ and passes the point (1, 1)

Answers

1. $\sqrt{1+x^2} + \sqrt{1+y^2} = c$
2. $y = \tan^{-1}x + c$
3. $y(1 + e^{-x}) = c$
4. $(1+x^2)(1+y^2) = c$
5. $\sin x \sin y = c$
6. $(a + x)(1 - ay) = cy$
7. $2(\tan^{-1} x + \tan^{-1} y) + \log(1 + x^2)(1 + y^2) = c$
8. $y - c = \log[(1 + y)(e^x + 1)]$
9. $\tan y = c(1 - e^x)^3$ 10. $e^{2x} + e^{2y} = c$
11. $(x^2 - a^2)(a^2 + y^2) = c$ 12. $\sin x\,(e^y + 1) = c$
13. $y = 4x^{3/2}$ 14. $\sin y = x$
15. $-e^{-y} = e^x - e - e^{-1}$. $y = -1$ 16. $y(x^2 + 1) = 4$
17. $(1-y^2)(1-x^2) = 0$ 18. $\log xy = x - y$

15.3 Change a Variable

If the first order differential equation $M(x, y)dx + N(x, y)\,dy = 0$ of the form $\frac{dy}{dx} = f(ax + by + c)$.

Then it can be solved by putting

$$ax + by + c = v.$$

or $$a + b\frac{dy}{dx} = \frac{dv}{dx}.$$

$$\frac{dy}{dx} = \frac{1}{b}\frac{dv}{dx} - \frac{a}{b}.$$

So, the differential equation can be written as

$$\frac{1}{b}\frac{dv}{dx} - \frac{a}{b} = f(v).$$

Separating the variables and integrating we get the required solution.

Worked Out Examples

Ex. 1: Solve $\frac{dy}{dx} = \cos(x + y)$

Solution:

Here, the equation is

$$\frac{dy}{dx} = \cos(x + y)$$

Put $x + y = v$,

or $$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

or $$\frac{dy}{dx} = \frac{dv}{dx} - 1$$

Thus, the equation reduces to

$$\frac{dv}{dx} - 1 = \cos v.$$

or $$\frac{dv}{dx} = 1 + \cos v$$

Separating the variables,

$$\frac{dv}{1 + \cos v} = dx.$$

or $$\frac{dv}{2\cos^2\frac{v}{2}} = dx$$

or $$\frac{1}{2}\sec^2\frac{v}{2}\,dv = dx$$

Integrating,

$$\tan\left(\frac{v}{2}\right) = x + c$$

$\therefore \quad \tan\left(\frac{x+y}{2}\right) = x + c$ is the required solution.

Ex. 2: Solve $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

Solution:

Here, the equation is

$$\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$$

Put $x + y = v$,

$$1 + \frac{dy}{dx} = \frac{dv}{dx} \qquad \text{or} \quad \frac{dy}{dx} = \frac{dv}{dx} - 1$$

So the given equation reduces to

$$\frac{dv}{dx} - 1 = \sin v + \cos v$$

or $$\frac{dv}{dx} = 1 + \sin v + \cos v,$$

or $$\frac{dv}{dx} = 2\cos^2\frac{v}{2} + 2\sin\frac{v}{2}\cos\frac{v}{2}$$

Separating the variables

$$\frac{dv}{2\cos^2\frac{v}{2} + 2\sin\frac{v}{2}\cos\frac{v}{2}} = dx$$

or $$\frac{\sec^2\left(\frac{v}{2}\right)dv}{2 + 2\tan\frac{v}{2}} = dx$$

Integrating

$$\log\left(1 + \tan\frac{v}{2}\right) = x + c$$

$\therefore \quad \log\left(1 + \tan\frac{x+y}{2}\right) = x + c$ is the required solution .

Ex. 3: Solve: $\left(\frac{x+y-a}{x+y-b}\right)\frac{dy}{dx} = \frac{x+y+a}{x+y+b}$

Solution:

Here, the equation is

$$\frac{dy}{dx} = \frac{(x+y+a)(x+y-b)}{(x+y-a)(x+y+b)}$$

Put $x + y = v$

$$1 + \frac{dy}{dx} = \frac{dv}{dx} \qquad \text{or} \quad \frac{dy}{dx} = \frac{dv}{dx} - 1$$

So, the equation changes into.

$$\frac{dv}{dx} - 1 = \frac{(v + a)(v - b)}{(v - a)(v + b)}$$

or $$\frac{dv}{dx} = \frac{v^2 + (a - b) v - ab}{v^2 + (b - a) v - ab} + 1$$

or $$\frac{dv}{dx} = \frac{v^2 + (a - b) v - ab + v^2 - (a - b) v - ab}{v^2 - (a - b) v - ab}$$

or $$\frac{dv}{dx} = \frac{2v^2 - 2ab}{v^2 - (a-b) v - ab}$$

Separating the variables

$$\frac{v^2 - (a - b) v - ab}{(v^2 - ab)} dv = 2dx$$

or $$\left[1 - \frac{(a - b)v}{(v^2 - ab)}\right] dv = 2dx$$

or $$\left(2 - \frac{(a - b) 2v}{v^2 - ab}\right) dv = 4 dx$$

Integrating

$$2v - (a - b) \log(v^2 - ab) = 4x + c$$

or $$2x + 2y + (b - a) \log\{(x + y)^2 - ab\} = 4x + c$$

$\therefore$ $$(b - a) \log \{(x + y)^2 - ab\} = 2(x - y + c)$$

is the required solution.

Ex. 4: Solve: $$\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1 - x^2 - y^2}{x^2 + y^2}}$$

Solution:

Here, the equation is

$$\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1 - x^2 - y^2}{x^2 + y^2}}$$

or $$\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1 - (x^2 + y^2)}{x^2 + y^2}}$$

Put $x = r\cos\theta$, $\quad y = r\sin\theta$

$x^2 + y^2 = r^2$, $\quad \tan\theta = \frac{y}{x}$

or $x\,dx + y\,dy = r\,dr$, $\quad \sec^2\theta\, d\theta = \frac{x\,dy - y\,dx}{x^2}$

So, the equation changes into

$$\frac{r\,dr}{x^2 \sec^2\theta\, d\theta} = \sqrt{\frac{1 - r^2}{r^2}}$$

or $$\frac{r\,dr}{r^2 \cos^2\theta \sec^2\theta\, d\theta} = \frac{\sqrt{1 - r^2}}{r}$$

or $\dfrac{dr}{\sqrt{1-r^2}} = d\theta$

Integrating.

$$\sin^{-1} r = \theta + c$$

or $\sin^{-1}\sqrt{x^2+y^2} = \tan^{-1}\left(\dfrac{y}{x}\right) + c$

or $\sqrt{x^2+y^2} = \sin\left[\tan^{-1}\left(\dfrac{y}{x}\right)\right] + c$

$\therefore$ $x^2+y^2 = \sin^2\left[\tan^{-1}\left(\dfrac{y}{x}\right)\right] + c$ is the required solution.

Exercise-20

Solve the following differential equations

1. $(x+y)^2 \dfrac{dy}{dx} = a^2$
2. $\cos(x+y)\, dy = dx$
3. $\sin^{-1}\left(\dfrac{dy}{dx}\right) = x+y$
4. $\dfrac{dy}{dx} + 1 = e^{x+y}$
5. $\dfrac{dy}{dx} + 1 = e^{x-y}$
6. $\dfrac{dy}{dx} - x\tan(y-x) = 1$
7. $\dfrac{dy}{dx} = (4x+y+1)^2$
8. $(x^2+y^2+2xy+1)\, dy = (x+y)\, dx$
9. $(x+y+1)\dfrac{dy}{dx} = 1$
10. $\dfrac{dy}{dx} = \sqrt{y-x}$
11. $x^2(xdx + y\,dy) + 2y(x\,dy - ydx) = 0$

Answers

1. $x+y = a\tan^{-1}\dfrac{(y-c)}{a}$
2. $\tan\dfrac{x+y}{2} = y-c$
3. $\tan(x+y) - \sec(x+y) = x - c$

4. $(x+c)e^{x} + e^{-y} = 0$

5. $e^{y} = \frac{1}{2}e^{x} + ce^{-x}$

6. $\sin(y-x) = ce^{x^2/2}$

7. $(4x+y+1) = 2\tan(2x+c)$

8. $y - \frac{1}{2}\log\{(x+y)^2 + x + y + 1\}$

$+ \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+2y+1}{\sqrt{3}} = c$

9. $y - \log(x+y+2) = c$

10. $\sqrt{y-x} + \log\left(\sqrt{y-x} - 1\right) = \frac{x}{2} + c$

11. $\sqrt{x^2+y^2}\left(1+\frac{2}{x}\right) = c$

15.4 Homogeneous Differential Equation of First Order

If the first order differential equation is of the form $\frac{dy}{dx} = \frac{\psi(x, y)}{\phi(x, y)}$

where $\psi(x, y)$ and $\phi(x, y)$ are homogeneous function of same degree n. then it can be solved by putting $y = vx$.

Differentiating with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So. the differential equation reduces to

$$v + x\frac{dv}{dx} = f(v).$$

Separating the variables and integrating we get the required solution.

Worked Out Examples

Ex. 1: Solve $\frac{dy}{dx} = \frac{x^2y}{x^3+y^3}$

Solution:

Here. $\frac{dy}{dx} = \frac{x^2y}{x^3-y^3}$

This is homogenous. Put $y = vx$. $\frac{dy}{dx} = v + x\frac{dv}{dx}$

We get. $v + \frac{xdv}{dx} = \frac{x^2.vx}{x^3 + v^3x^3}$

$$\frac{xdv}{dx} = \frac{v}{1+v^3} - v = \frac{v - v - v^4}{1+v^3}$$

or $x\frac{dv}{dx} = \frac{-v^4}{1+v^3}$

Separating the variables

$$\frac{dx}{x} + \frac{(1+v^3)}{v^4}dv = 0$$

or $\frac{dx}{x} + \frac{1}{v^4}dv + \frac{1}{v}dv = 0$

Integrating

$$\log x - \frac{1}{3v^3} + \log v + \frac{c}{3} = 0$$

or $\log(xv) + \frac{c}{3} = \frac{1}{3v^3}$

or $3\log y + c = \frac{x^3}{y^3}$ is the required solution.

Ex. 2: Solve $(x^2 + y^2)\,dx + 2xy\,dy = 0$

Solution:

Here, $(x^2 + y^2)\,dx + 2xy\,dy = 0$.

or $\frac{dy}{dx} = -\frac{x^2+y^2}{2xy}$

This is homogenous. Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

We get,

$$v + \frac{xdv}{dx} = -\frac{x^2 + v^2.x^2}{2x.vx},$$

or $\frac{xdv}{dx} = -v - \frac{1+v^2}{2v}$

or $x\frac{dv}{dx} = -\frac{1+3v^2}{2v}$

or $\frac{dx}{x} + \frac{2v\,dv}{1+3v^2} = 0$

or $3\frac{dx}{x} + \frac{6v\,dv}{1+3v^2} = 0$

Integrating

$3 \log x + \log (1 + 3v^2) = \log c$. or $x^3 (1 + 3v^2) = c$

$x(x^2 + 3y^2) = c$ is the required solution.

Ex. 3: Solve $y^2\, dx + (xy + x^2)\, dy = 0$

Solution:

Here, $y^2\, dx + (xy + x^2)\, dy = 0$.

$$\frac{dy}{dx} = -\frac{y^2}{xy + x^2}$$

This is homogenous. Put $y = vx$,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

We get $v + x\dfrac{dv}{dx} = -\dfrac{v^2x^2}{x.vx + x^2}$

$$\frac{xdv}{dx} = -\frac{v^2}{v+1} - v = \frac{-v^2 - v^2 - v}{v+1}$$

or $x\dfrac{dv}{dx} + \dfrac{2v^2 + v}{v+1} = 0.$

Separating the variables

$$\frac{dx}{x} + \frac{v+1}{2v^2 + v}\, dv = 0$$

or $2\dfrac{dx}{x} + 2\left(\dfrac{1}{v} - \dfrac{1}{2v+1}\right) dv = 0$

Integrating

$$2 \log x + 2 \log v - \log (2v + 1) = \log c$$

or $\log x^2v^2 = \log c(2v + 1)$

or $xy^2 = c(2y + x)$ is the required solution.

Ex. 4: Solve $(x - y)\, dx + (y - x)\, dy = 0$

Solution:

Here, $(x - y)\, dx + (y - x)\, dy = 0$.

or $\dfrac{dy}{dx} = -\dfrac{x + y}{y - x}$

This is homogenous. Put, $y = vx$, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

We get $v + x\dfrac{dv}{dx} = -\dfrac{x + vx}{vx - x}$.

or $\dfrac{xdv}{dx} = -\dfrac{1 + v}{v - 1} - v$

or $\dfrac{xdv}{dx} = \dfrac{-1 - v - v^2 + v}{v - 1}$

or $x\dfrac{dv}{dx} = -\dfrac{v^2+1}{v-1}$.

Separating the variables

$$\frac{dx}{x} + \frac{v-1}{v^2+1}\,dv = 0$$

or $2\dfrac{dx}{x} + \dfrac{2v}{v^2+1}\,dv - \dfrac{2}{v^2+1}\,dv = 0$

Integrating

$$2\log x + \log(v^2+1) - 2\tan^{-1} v = c$$

or $-2\tan^{-1}\dfrac{y}{x} + \log x^2\left(\dfrac{y^2+x^2}{x^2}\right) = c$

$\therefore$ $\tan^{-1}\left(\dfrac{y}{x}\right) - \dfrac{1}{2}\log(y^2+x^2) = c$ is the required solution.

Ex. 5: Solve: $\dfrac{dy}{dx} = -\dfrac{x-2y}{2x-y}$

Solution:

Here, $\dfrac{dy}{dx} = -\dfrac{x-2y}{2x-y}$

This is homogenous. Put $y = vx$.

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

We get, $v + \dfrac{xdv}{dx} = -\dfrac{x-2vx}{2x-vx}$

$$\frac{xdv}{dx} = -\frac{1-2v}{2-v} - v = \frac{-1+2v-2v+v^2}{2-v}$$

or $x\dfrac{dv}{dx} = -\dfrac{1-v^2}{2-v}$

Separating the variables

$$\frac{dx}{x} + \frac{2-v}{1-v^2}\,dv = 0.$$

or $\dfrac{dx}{x} + \dfrac{1+1-v}{1-v^2}\,dv = 0$

or $\dfrac{dx}{x} + \dfrac{1}{1-v^2}\,dv + \dfrac{1}{1+v}\,dv = 0$

Integrating

$$\log x + \frac{1}{2}\log\frac{1+v}{1-v} + \log(1+v) = \log c$$

or $\left\{\log x\dfrac{(1+v)^{1/2}}{(1-v)^{1/2}}(1+v)\right\} = \log c$

$\therefore$ $(x+y)^3 = c(y-x)$ is the required solution.

Ex. 6: Solve $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

2069 Chaitra B. E.

Solution:

Here, $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

This is homogenous, so put $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = \frac{vx}{x} + \tan\frac{vx}{x} = v + \tan v$$

or $x\frac{dv}{dx} = \tan v.$

Separating the variables, we get

$$\cot v\, dv = \frac{dx}{x}.$$

Integrating it we get

$$\log \sin v = \log x + \log c = \log cx$$

or $\sin v = cx$

$\therefore$ $\sin\frac{y}{x} = c$ is the required solution.

Ex. 7: Solve: $\frac{dy}{dx} + \frac{3xy + y^2}{x^2 + xy} = 0$

Solution:

Here, $\frac{dy}{dx} + \frac{3xy + y^2}{x^2 + xy} = 0$

or $\frac{dy}{dx} = -\frac{3xy + y^2}{x^2 + xy}$

This is homogenous. Put $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

We get

$$v + x\frac{dv}{dx} = -\frac{3x.vx + v^2x^2}{x^2 + x.vx}$$

or $x\frac{dv}{dx} = -\frac{3v + v^2}{1 + v} - v = \frac{-3v - v^2 - v - v^2}{1 + v}$

or $x\frac{dv}{dx} = -\frac{2v^2 + 4v}{1 + v}.$

Separating the variables

$$\frac{dx}{x} - \frac{1 + v}{2v(v + 2)}dv = 0$$

or $\quad \dfrac{dx}{x} + \dfrac{1}{2}\left(\dfrac{1}{2v} + \dfrac{1}{2(v+2)}\right) dv = 0$

Integrating

$$\log x + \frac{1}{4}\log v + \frac{1}{4}\log(v+2) = \log c$$

or $\quad \log\left\{x\, v^{1/4}(v+2)^{1/4}\right\} = \log c$

or $\quad \left\{x\, v^{1/4}(v+2)^{1/4}\right\} = c$

or $\quad x^4 v(v+2) = c^4$

$\therefore \quad x^2 y(y+2x) = c$ is the required solution.

Ex. 8: Solve : $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$ 2061/062 B.E.

Solution:

Here, $\quad x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

or $\quad \dfrac{dy}{dx} = \dfrac{\sqrt{x^2+y^2} + y}{x}$

This is homogenous, so put $y = vx$, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$; we get

$$v + x\frac{dv}{dx} = \frac{\sqrt{x^2+v^2x^2} + vx}{x} = \sqrt{1+v^2} + v$$

or $\quad x\dfrac{dv}{dx} = \sqrt{1+v^2}$.

Separating the variables, we get

$$\frac{1}{\sqrt{1+v^2}}\,dv = \frac{dx}{x}.$$

Integrating it we get

$$\log\left(v + \sqrt{1+v^2}\right) = \log x + \log c = \log cx$$

or $\quad \dfrac{y}{x} + \sqrt{1 + \dfrac{y^2}{x^2}} = cx$

$\therefore \quad y + \sqrt{x^2+y^2} = c\,x^2$ is the general solution.

Exercise-21

Solve the following differential equations

1. $x + y\dfrac{dy}{dx} = 2y$
2. $(x^2 - y^2)\,dx - 2xy\,dy = 0$

3. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$
4. $(x^2 + y^2)\, dx = (x^2 + xy)\, dy$
5. $x(x - y)\, dy = y\,(x + y)\, dx$
6. $(x^2 + y^2)\, dy = xy\, dx$
7. $x^2 dy + y(x + y)\, dx = 0$
8. $x\frac{dy}{dx} = y - \sqrt{x^2 + y^2}$
9. $x \sin\frac{y}{x}\, dy = \left(y \sin\frac{y}{x} - x\right) dx$
10. $(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$
11. $\frac{dy}{dx} = \frac{3xy + y^2}{3x^2}$
12. $(x^2 + 2xy^2)\, dx + (2x^2y + y^2)\, dy = 0$

Answers

1. $\log(y - x) = c + \frac{x}{y - x}$
2. $y^2 + x^2 - cx = 0$
3. $2x - y = cx^2 y$
4. $(x - y)^2 = c\, xe^{-y/x}$
5. $xy = e^{-x/y}$
6. $y = c\, e^{x^2/2y^2}$
7. $x^2y = c(y + 2x)$
8. $y + \sqrt{x^2 + y^2} = c$
9. $\log x = \cos\left(\frac{y}{x}\right) + c$
10. $x + ye^{x/y} = c$
11. $3x + y \log x = cy$
12. $x^3 + y^3 + 3x^2y^2 = c$

15.5 First Order Differential Equations Reducible to Homogenous form .

If the first order differential equation is of the form

$\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$, then it can be solved by the following ways:

I. If the first order differential equation is of the form

$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$ and $\frac{a}{A} = \frac{b}{B} = \frac{1}{l}$, then the equations can easily be reduced to the form

$$\frac{dy}{dx} = \frac{ax - by - c}{l(ax - by) - C} \text{ and}$$

Put $\quad ax + by = v.$

$$\therefore \quad a + b\frac{dy}{dx} = \frac{dv}{dx}.$$

Separating the variables and integrating we get the required solution.

II. If $\frac{a}{A} \neq \frac{b}{B}$ then the equation can be reduced to the homogeneous form by putting $x = X+h,\ y = Y+k$,where h, k are constants which will be determined by solving the two equations in terms of h and k. the equations are to be chosen in such a way that the differential equation should be homogeneous.

Worked Out Examples

Ex. 1: Solve $\frac{dy}{dx} = \frac{x - y + 3}{2x - 2y + 5}$

Solution:

Here, $\frac{dy}{dx} = \frac{x - y + 3}{2x - 2y + 5} = \frac{x - y + 3}{2(x - y) + 5}$

Put $x - y = v$.

$$1 - \frac{dy}{dx} = \frac{dv}{dx}$$

The equation becomes

$$1 - \frac{dv}{dx} = \frac{v + 3}{2v + 5}$$

$$\frac{dv}{dx} = 1 - \frac{v + 3}{2v + 5} = \frac{2v + 5 - v - 3}{2v + 5} = \frac{v + 2}{2v + 5}$$

Separating the variables

$$dx = \frac{(2v + 5)\,dv}{v + 2} = \frac{2(v + 2) + 1}{v + 2}\,dv$$

$$\text{or } dx = \left(2 + \frac{1}{v + 2}\right) dv$$

Integrating

$$x + c = 2v + \log(v + 2)$$

$$\text{or} \quad x + c = 2x - 2y + \log(x - y + 2)$$

$$x - 2y - \log(x - y + 2) = c \text{ is the required solution.}$$

Ex. 2: Solve: $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0$

Solution:

Here, $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0$

$$\frac{dy}{dx} = -\frac{(2x+y)+1}{2(2x+y)-1}$$

Put $2x + y = v$.

$$2 + \frac{dy}{dx} = \frac{dv}{dx}.$$

or $$\frac{dy}{dx} = \frac{dv}{dx} - 2$$

So, the equation becomes

$$\frac{dv}{dx} - 2 = -\frac{v+1}{2v-1} = -\frac{v+1}{2v-1} + 2$$

$$= \frac{-v+1+4v-2}{2v-1} = \frac{3v-3}{2v-1}$$

or $$\frac{dv}{dx} = \frac{3(v-1)}{(2v-1)}$$

Separating the variables

$$\frac{2v-1}{v-1}\,dv = 3\,dx$$

or $$\left(2 + \frac{1}{v-1}\right) dv = 3\,dx$$

Integrating,

$$2v + \log(v-1) = 3x + c$$

or $$4x + 2y + \log(2x + y - 1) = 3x + c$$

$\therefore x + 2y + \log(2x + y - 1) = c$ is the required solution.

Ex. 3: Solve: $(3y - 7x + 7)\,dx + (7y - 3x + 3)\,dy = 0$

Solution:

Here, $(3y - 7x + 7)\,dx + (7y - 3x + 3)\,dy = 0$

$$\frac{dy}{dx} = -\frac{3y - 7x + 7}{7y - 3x + 3}$$

Put $x = X + h$, $y = Y + k$ where h, k are constants.

$$\frac{dY}{dX} = -\frac{3Y - 7X + (3k - 7h + 7)}{7Y - 3X + (7k - 3h + 3)}$$

Choose h, k such that $3k - 7h + 7 = 0$

and $7k - 3h - 3 = 0$

Solving these we get $h = 1$, $k = 0$

So. $$\frac{dY}{dX} = -\frac{3Y - 7X}{7Y - 3X}$$

This is homogenous. Put $Y = vX$.

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

The equation becomes

$$v + X\frac{dv}{dX} = -\frac{3vX - 7X}{7vX - 3X}$$

or $$X\frac{dv}{dX} = -\frac{3v - 7}{7v - 3} - v$$

or $$X\frac{dv}{dX} = -\frac{(7v^2 - 7)}{7v - 3}$$

Separating the variables

$$\frac{dX}{X} + \frac{(7v - 3)\,dv}{7(v^2 - 1)} = 0$$

or $$2\frac{dX}{X} + \frac{2v\,dv}{v^2 - 1} - \frac{6}{7}\frac{1}{v^2 - 1}\,dv$$

Integrating

$$2\log X + \log(v^2 - 1) - \frac{6}{7}\cdot\frac{1}{2}\log\frac{v - 1}{v + 1} = \log c$$

or $$\log X^2\frac{(v^2 - 1)}{(v - 1)^{3/7}}(v + 1)^{3/7} = \log c$$

or $$X^2\frac{(v + 1)(v - 1)(v + 1)^{3/7}}{(v - 1)^{3/7}} = c$$

or $$X^2(v - 1)^{4/7}(v + 1)^{10/7} = c$$

or $$(Y - X)^4(Y + X)^{10} = c^7$$

or $$(y - 0 - x + 1)^4(y - 0 + x - 1)^{10} = c^7$$

or $$(y - x + 1)^2(x + y - 1)^5 = c^{7/2}$$

$\therefore$ $(y - x + 1)^2(x + y - 1)^5 = c$ is the required solution.

Ex. 4: Solve: $(2x + 3y - 5)\,dy + (3x + 2y - 5)\,dx = 0$

Solution:

Here, $(2x + 3y - 5)\,dy + (3x + 2y - 5)\,dx = 0$

$$\frac{dy}{dx} = -\frac{(3x + 2y - 5)}{2x + 3y - 5}$$

Put $x = X + h$, $y = Y + k$ where h and k are constants.

$$\frac{dy}{dx} = \frac{dY}{dX}.$$

The given equation becomes

$$\frac{dY}{dX} = -\frac{3X + 2Y + (3h + 2k - 5)}{2X + 3Y + (2h + 3k - 5)}$$

Choose h, k such that $3h + 2k - 5 = 0$

and $2h + 3k - 5 = 0$.

Solving these, we get $h = 1, k = 1$.

So, $\frac{dY}{dX} = -\frac{3X + 2Y}{2X + 3Y}$

This is homogenous. Put $Y = vX$.

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

So the equation reduces to

$$v + X\frac{dv}{dX} = -\frac{3X + 2vX}{2X + 3vX} = -\frac{3 + 2v}{2 + 3v}$$

or $X\frac{dv}{dX} = -\frac{3v^2 + 4v + 3}{2 + 3v}$

Separating the variables

$$\frac{dX}{X} + \frac{(2 + 3v)\, dv}{(3v^2 + 4v + 3)} = 0$$

or $2\frac{dX}{X} + \frac{(4 + 6v)}{3v^2 + 4v + 3} = 0$

Integrating

$$2\log X + \log(3v^2 + 4v + 3) = \log c$$

or $\log X^2 + \log(3c^2 + 4v + 3) = \log c$

or $X^2(3v^2 + 4v + 3) = c$

or $X^2\left(3\frac{Y^2}{X^2} + \frac{4Y}{X} + 3\right) = c$

or $(3Y^2 + 4XY + 3X^2) = c$

or $3(y - 1)^2 + 4(x - 1)(y - 1) + 3(x - 1)^2 = c$

or $3y^2 + 4xy + 3x^2 - 10x - 10y = c$

$\therefore$ $3x^2 + 3y^2 + 4xy - 10x - 10y = c$ is the required solution.

Ex. 5: Solve: $\frac{dy}{dx} + \frac{2x - y + 1}{2y - x - 1} = 0$

Solution:

Here, $\frac{dy}{dx} + \frac{2x - y + 1}{2y - x - 1} = 0$ ....(1)

Put $x = X + h,\ y = Y + k$ where h and k are constants.

$$\frac{dy}{dx} = \frac{dY}{dX}$$

The given equation becomes

$$\frac{dY}{dx} = -\frac{2X - Y + (2h - k + 1)}{2Y - X + (2k - h - 1)}$$

Choose h, k such that $2h - k + 1 = 0$

and $2k - h - 1 = 0$

Solving these, we get $h = -\frac{1}{3}, k = \frac{1}{3}$

The equation (1) reduces to

$$\frac{dY}{dX} = -\frac{2X - Y}{2Y - X} \quad \text{.........(2)}$$

This is homogenous. Put $Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dx}$$

Thus the equation (2) becomes

$$v + X\frac{dv}{dX} = \frac{2X - vX}{2vX - X}$$

or $$X\frac{dv}{dX} = -\frac{2 - v}{2v - 1} - v$$

or $$X\frac{dv}{dx} = -\frac{2v^2 - 2v + 2}{2v - 1}$$

Separating the variables

$$2\frac{dX}{X} + \frac{(2v - 1)\,dv}{(v^2 - v + 1)} = 0$$

or $2\log X + \log(v^2 - v + 1) = \log c$

or $X^2(v^2 - v + 1) = c$

or $X^2\left\{\left(\frac{Y}{X}\right)^2 - \frac{Y}{X} + 1\right\} = c$ or $X^2\frac{(Y^2 - XY + X^2)}{X^2} = c$

or $\left\{\left(y - \frac{1}{3}\right)^2 - \left(x + \frac{1}{3}\right)\left(y - \frac{1}{3}\right) + \left(x + \frac{1}{3}\right)^2\right\} = c$

$\therefore$ $3x^2 + 3y^2 - 3xy - 3y + 3x = c$ is the required solution.

Exercise - 22

Solve the differential equations

1. $\frac{dy}{dx} = \frac{x + y + 1}{x + y - 1}$ $\boxed{\text{2058 B.E.}}$
2. $(x + y + 1)\,dx - (2x + 2y + 1)\,dy = 0$
3. $(4x + 6y + 5)\,dy = (3y + 2x + 4)\,dx$
4. $(2x + 2y + 3)\,dy - (x + y + 1)\,dx = 0$ $\boxed{\text{2057/2060 B.E.}}$

5. $\frac{dy}{dx} = \frac{x + y}{x + y - 2}$

6. $(6x - 5y + 4)\, dy + (y - 2x - 1)\, dx = 0$

7. $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$

8. $\frac{dy}{dx} + \frac{2x + 3y + 1}{3x + 5y - 1}$

9. $(x - 3y + 4)\, dy + (7y - 5x)\, dx = 0$

10. $(x - y)\, dy - (x + y + 1)\, dx = 0$

Answers

1. $y - x - c = \log (x + y)$
2. $6y - 3x = \log (3x + 3y + 2) + c$
3. $7(6y - 3x) - 9 \log (14x + 21y + 22) = c$
4. $6y - 3x + \log (3x + 3y + 4) = c$
5. $y - x - \log (x + y - 1) = c$
6. $(5y - 2x - 3)^4 = c(4y - 4x - 3)$ 7. $c^2(x - y)^3 = (x + y - 2)$
8. $5y^2 + 2x^2 + 6xy - 2y + 2x = c$ 9. $(3y - 4x + 10)^2 = c(y - x + 1)$
10. $\tan^{-1}\left(\frac{2y + 1}{2x + 1}\right) = \frac{1}{2} \log\left[c\left\{\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2\right\}\right]$

15.6 First Order Linear Differential Equation

If the first order, differential equation is of the form

$\frac{dy}{dx} + Py = Q$ where P and Q are function of x or constant, is called *First Order Linear Differential Equation.*

To solve these types of differential equations, we multiply $e^{\int Pdx}$ on both sides of the given equation.

So the equation becomes

$$e^{\int Pdx} \frac{dy}{dx} + Py\, e^{\int Pdx} = Q\, e^{\int Pdx}$$

or $$\frac{d}{dx}\left(y \times e^{\int Pdx}\right) = Q\, e^{\int Pdx}$$

Integrating

$$y \times e^{\int P dx} = \int \left(Q\, e^{\int P dx}\, dx\right) + c \text{ is the required solution.}$$

So its formula is remembered as

$$y \times e^{\int P dx} = \left(Q\, e^{\int P dx}\, dx\right) + c.$$

The factor $e^{\int P dx}$ is called *Integrating Factor* and is sometimes shortly written as $(\text{I.F.}) = e^{\int P dx}$.

So, $y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx + c$

Similarly

The first order differential equation is of the form

$\frac{dx}{dy} + Px = Q$ where P and Q are function of y or constants is called linear in x and its $\text{I.F.} = e^{\int P dy}$ then its solution is

$$x \times \text{I.F.} = \int (Q \times \text{I.F.})\, dy + c$$

Worked Out Examples

Ex.1 Solve: $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

Solution:

Here, $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

$$\frac{dy}{dx} + \frac{2x}{1 + x^2}\, y = \frac{4x^2}{1 + x^2}$$

This is linear form, $P = \frac{2x}{1 + x^2}$, $Q = \frac{4x^2}{1 + x^2}$

So, $\text{I.F.} = e^{\int \frac{2x}{1 + x^2} dx} = e^{\log(1 + x^2)} = 1 + x^2$

Its general solution is

$$y \times (\text{I.F.}) = \int Q \times (\text{I.F.})\, dx$$

or $y(1 + x^2) = \int \frac{4x^2}{(1 + x^2)} (1 + x^2)\, dx + c$

$\therefore\ y(1 + x^2) = \frac{4x^3}{3} + c$ is the required solution.

Ex. 2 Solve: $x(x - 1)\frac{dy}{dx} - y = \frac{x^2}{(x - 1)^2}$

Solution:

Here, $x(x-1)\dfrac{dy}{dx} - y = \dfrac{x^2}{(x-1)^2}$

or $\dfrac{dy}{dx} - \dfrac{1}{x(x-1)}y = \dfrac{x}{(x-1)^3}$

This is linear form, $P = -\dfrac{1}{x(x-1)}$, $Q = \dfrac{x}{(x-1)^3}$

So, $\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{1}{x(x-1)}dx} = e^{\int\left(\frac{1}{x} - \frac{1}{x-1}\right)dx}$

$= e^{\log x - \log(x-1)} = e^{\log\frac{x}{x-1}} = \dfrac{x}{x-1}$

Its general solution is

$y \times (\text{I.F}) = \int Q \times (\text{I.F.})\,dx$

or $y \times \dfrac{x}{x-1} = \int \dfrac{x}{(x-1)^3}\,\dfrac{x}{(x-1)}\,dx + c$

or $y\dfrac{x}{(x-1)} = \int \dfrac{x^2}{(x-1)^4}\,dx + c$

Put $x - 1 = t$, $dx = dt$

$= \int \dfrac{(t+1)^2}{t^4}\,dt + c = \int \dfrac{t^2+2t+1}{t^4}\,dt + c$

or $\dfrac{yx}{(x-1)} = \int\left(\dfrac{1}{t^2} + \dfrac{2}{t^3} + \dfrac{1}{t^4}\right)dt = -\dfrac{1}{t} - \dfrac{1}{t^2} - \dfrac{1}{3t^3} + c$

or $\dfrac{yx}{(x-1)} = -\dfrac{1}{x-1} - \dfrac{1}{(x-1)^2} - \dfrac{1}{3}\dfrac{1}{(x-1)^3} + c$

or $3xy = -3 - \dfrac{3}{(x-1)} - \dfrac{1}{(x-1)^2} + 3c(x-1)$

$\therefore\ 3xy = -\dfrac{3x}{x-1} - \dfrac{1}{(x-1)^2} + 3c\,(x-1)$ is the required solution.

Ex. 3 Solve: $\cos x\dfrac{dy}{dx} + y\sin x = \sec^2 x$

Solution:

Here, $\cos x\dfrac{dy}{dx} + y\sin x = \sec^2 x$

or $\dfrac{dy}{dx} + \tan x\, y = \sec^3 x$

This is linear form, $P = \tan x$, $Q = \sec^3 x$

So, $\text{I.F.} = e^{\int P dx} = e^{\int \tan x\,dx} = e^{\log\sec x} = \sec x$

Its general solution is

$y \times \text{I.F.} = \int Q \times (\text{I.F.})\,dx$

or $\quad y \sec x = \int \sec^2 x + c$

or $\quad y \sec x = \int (1 + \tan^2 x) \sec^2 x \, dx + c$

Put $\tan x = t, \quad \sec^2 x dx = dt$

$$= \int (1 + t^2) \, dt + c = t + \frac{t^3}{3} + c$$

or $\quad y \sec x = \tan x + \frac{\tan^3 x}{3} + c$

$\therefore \quad 3y \sec x = c + 3 \tan x + \tan^3 x$ is the required solution.

Ex. 4: $\mathbf{x \frac{dy}{dx} + 2y = x^2 \log x}$

Solution:

Here, $x \frac{dy}{dx} + 2y = x^2 \log x$

or $\quad \frac{dy}{dx} + \frac{2}{x} y = x \log x$

This is linear form, $P = \frac{2}{x}$, $\qquad Q = x \log x$

So, I.F. $= e^{\int P dx} = e^{\int \frac{2}{x} dx} = e^{2 \log x} = x^2$

Its general solution is

$$y \times \text{I.F.} = \int Q \times (\text{I.F.}) \, dx$$

or $\quad yx^2 = \int x^3 \log x \, dx + c$

or $\quad yx^2 = \log x \frac{x^4}{4} - \int \frac{1}{x} \frac{x^4}{4} \, dx + c$

or $\quad yx^2 = \frac{x^4}{4} \log x - \frac{x^4}{16} + c$

$\therefore \quad 16x^2 y = 4x^4 \log x - x^4 + c$ is the required solution.

Ex. 5: Solve: $\mathbf{(x^3 + 1) \frac{dy}{dx} + 3x^2 y = \sin^2 x}$ $\boxed{2057 \text{ B.E.}}$

Solution:

Here, $(x^3 + 1) \frac{dy}{dx} + 3x^2 y = \sin^2 x$

or $\quad \frac{dy}{dx} + \frac{3x^2}{x^3 + 1} y = \frac{\sin^2 x}{x^3 + 1}$

This is linear form, $P = \frac{3x^2}{x^3 + 1}$, $Q = \frac{\sin^2 x}{x^3 + 1}$

So. I.F. $= e^{\int Pdx} = e^{\int \frac{3x^2}{x^3-1}dx} = e^{\log(x^3-1)} = x^3 - 1$

Its general solution is

$$y \times \text{I.F.} = \int Q \times (\text{I.F.})\, dx$$

or $y(x^3+1) = \int \frac{\sin^2 x}{x^3+1}(x^3+1)\, dx$

or $y(x^3+1) = \frac{x}{2} - \frac{\sin 2x}{4} + c$

$\therefore$ $4y(x^3+1) = 2x - \sin 2x + c$ is the required solution.

Ex. 6: $(x + y + 1)\frac{dy}{dx} = 1$

Solution:

Here, $(x + y + 1)\frac{dy}{dx} = 1$

or $\frac{dy}{dx} = \frac{1}{x + y + 1}$.

or $\frac{dx}{dy} = x + y + 1$

or $\frac{dx}{dy} - x = y + 1$

This is linear form. $P = -1$, $Q = y + 1$

I.F. $= e^{\int Pdy} = e^{-1\int dy} = e^{-y}$

Its general solution is

$$x \times \text{I. F.} = \int Q \times (\text{I. F.})\, dy$$

or $x e^{-y} = \int (e^{-y}y + e^{-y})\, dy + c$

or $x e^{-y} = -y e^{-y} + \int e^{-y}\, dy + \int e^{-y}\, dy = -ye^{-y} - 2e^{-y} + c$

$\therefore$ $(x + y + 2) = ce^{y}$ is the required solution.

Ex. 7: $(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$ $\boxed{\textit{2070 Chaitra B. E.}}$

Solution:

Here, $(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$.

or $\frac{dy}{dx} = -\frac{1 + y^2}{x - e^{\tan^{-1} y}}$.

or $\dfrac{dx}{dy} = -\dfrac{x - e^{\tan^{-1} y}}{1+y^2}$,

or $\dfrac{dx}{dy} + \dfrac{1}{1+y^2}x = \dfrac{e^{\tan^{-1} y}}{1+y^2}$.

This is linear form. $P = \dfrac{1}{1+y^2}$, $Q = \dfrac{e^{\tan^{-1} y}}{1+y^2}$.

$$\text{I. F.} = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Its general solution is

$$x \times \text{I. F.} = \int Q \times (\text{I. F.})\, dy,$$

or $x\, e^{\tan^{-1} y} = \int \dfrac{e^{\tan^{-1} y}}{1+y^2} \times e^{\tan^{-1} y}\, dy + c,$

Put $\tan^{-1} y = t$, $\dfrac{1}{1+y^2}\, dy = dt$,

or $x\, e^{\tan^{-1} y} = \int e^{2t}\, dt + c = \dfrac{e^{2t}}{2} + \dfrac{c}{2}$,

$\therefore\ 2x\, e^{\tan^{-1} y} = e^{2\tan^{-1} y} + c$ is the required solution.

Exercise - 23

Solve the following differential equations

1. $\dfrac{dy}{dx} + \dfrac{2x}{1+x^2}\, y = \dfrac{1}{(1+x^2)^2}$

2. $\dfrac{dy}{dx} + \dfrac{y}{x^2} = \dfrac{1}{x^2}$

3. $(1 - x^2)\dfrac{dy}{dx} - xy = 1$

4. $\dfrac{dy}{dx} + 2y = 4x$

5. $(1 + x)\dfrac{dy}{dx} - xy = 1 - x$

6. $(1 + x^2)\dfrac{dy}{dx} + y = e^{\tan^{-1} x}$

7. $x \cos x \dfrac{dy}{dx} + y\,(x \sin x + \cos x) = 1$

8. $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

9. $\sin x \dfrac{dy}{dx} + y \cos x = x \sin x$

10. $\dfrac{dy}{dx} + y \cot x = 2 \cos x$ 2068 Baishakh B. E.

11. $\dfrac{dy}{dx} + y \tan x = \sec x$

12. $x \log x \dfrac{dy}{dx} + y = 2 \log x$ 2061 B.E.

13. $\dfrac{dy}{dx} + y = \cos x$

14. $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$ 2070 Ashad B. E.

15. Solve: $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$ if $y = 1$ when $x = 1$

Answers

1. $y(1 + x^2) = \tan^{-1} x + c$	2. $y = 1 + ce^{\frac{1}{x}}$
3. $y\sqrt{1 - x^2} = \sin^{-1} x + c$	4. $y = 2x - 1 + ce^{-2x}$
5. $y(1 + x) = x + ce^x$	6. $2y\, e^{\tan^{-1} x} = e^{2\tan^{-1} x} + c$
7. $yx \sec x = \tan x + c$	8. $(y - \tan x + 1)\, e^{\tan x} = c$
9. $y \sin x + x \cos x - \sin x = c$	10. $2y \sin x = c - \cos 2x$
11. $y = \sin x + c \cos x$	12. $y \log x = (\log x)^2 + c$
13. $2y = \cos x + \sin x + c\, e^{-x}$	14. $x = (\tan^{-1} y - 1) + c\, e^{-\tan^{-1} y}$
15. $2x\, e^{\tan^{-1} y} + c = e^{2\tan^{-1} y}$	15. $4xy = x^4 + 3$

15.7 Bernoulli's Equation

The first order differential equation of the form

$\dfrac{dy}{dx} + P y = Q y^n$, where P and Q are functions of x alone is called *Bernoulli's Equation*. It can be solved by reducing it to linear equation.

For this

$$\frac{dy}{dx} + Py = Qy^n$$

Dividing both sides by y^n,

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q$$

Put $\quad \frac{1}{y^{n-1}} = v$,

$$y^{-n+1} = v,$$

Differentiating it with respect to x

$$(-n+1)\, y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$$

So the equation reduces to

$$\frac{1}{(-n+1)}\frac{dv}{dx} + Pv = Q$$

or $\quad \frac{dv}{dx} + P(-n+1)\, v = (-n+1)\, Q$

This is linear in v, so its integrating factor (I.F.) $= e^{\int P(-n+1)\,dx}$ and its general solution is

$$v \times \text{I.F.} = \int(-n+1)\, Q \times (\text{I.F.})\, dx + c$$

Worked Out Examples

Ex. 1: Solve: $\frac{dy}{dx} + \frac{2y}{x} = \frac{y^3}{x^3}$

Solution:

Here, $\frac{dy}{dx} + \frac{2y}{x} = \frac{y^3}{x^3}$

Dividing by y^3,

$$y^{-3}\frac{dy}{dx} - \frac{2}{x}\, y^{-2} = \frac{1}{x^3}$$

Put $y^{-2} = v$, $\quad -2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$

So the equation reduces to

$$-\frac{1}{2}\frac{dv}{dx} - \frac{2}{x}v = \frac{1}{x^3}$$

or $\quad \frac{dv}{dx} - \frac{4}{x}v = \frac{-2}{x^3}$

This is linear form. $P = \frac{-4}{x}$, $\quad Q = -\frac{2}{x^3}$

So, I.F. $= e^{\int P dx} = e^{-2\log x} = \dfrac{1}{x^2}$

Its general solution is.

$$v \times \text{I.F.} = \int Q \times (\text{I.F.})\, dx$$

or $\quad v\dfrac{1}{x^2} = -\displaystyle\int \dfrac{2}{x^7}\, dx + c$

or $\quad \dfrac{1}{y^2}\dfrac{1}{x^2} = \dfrac{1}{3x^6} + c.$

$\therefore \quad 3x^2 = y^2 + cx^6y^2$ is the required solution.

Ex. 2: Solve: $\dfrac{dy}{dx} + \dfrac{x}{1-x^2}\, y = x\sqrt{y}$

Solution:

Here, $\dfrac{dy}{dx} + \dfrac{x}{1-x^2}\, y = x\sqrt{y}$

Dividing by $\sqrt{y}$, $\quad y^{-1/2}\dfrac{dy}{dx} + \dfrac{x}{1-x^2}\, y^{-1/2} = x$

Put $y^{1/2} = v$, $\quad \dfrac{1}{2}\, y^{-1/2}\dfrac{dy}{dx} = \dfrac{dv}{dx}$

So the equation reduces to

$$2\frac{dv}{dx} + \frac{x}{1-x^2}\, v = x$$

or $\quad \dfrac{dv}{dx} + \dfrac{x}{2(1-x^2)}\, v = \dfrac{x}{2}$

This is linear form. $P = \dfrac{x}{2(1-x^2)}$, $Q = \dfrac{x}{2}$

So I.F. $= e^{\int P dx} = e^{\int \frac{x}{2(1-x^2)} dx}$

$$= e^{\int -\frac{(-2x)}{4(1-x^2)} dx} = e^{-\frac{1}{4}\log(1-x^2)} = \frac{1}{(1-x^2)^{1/4}}$$

Its general solution is

$$v \times \text{I.F.} = \int Q \times (\text{I.F.})\, dx$$

or $\quad \dfrac{v}{(1-x^2)^{1/4}} = \displaystyle\int \dfrac{x}{2(1-x^2)^{1/4}}\, dx + c$

Put $1 - x^2 = t$, $-2x\, dx = dt$, $x dx = \dfrac{-dt}{2}$

$$= -\int \frac{dt}{4t^{1/4}} = -\frac{t^{3/4}}{3}$$

or $\quad \dfrac{y^{1/2}}{(1-x^2)^{1/4}} = -\dfrac{1}{3}(1-x^2)^{3/4} - c$

$\therefore \quad \sqrt{y} = -\frac{1}{3}(1-x^2) + c(1-x^2)^{1/4}$ is the required solution.

Ex. 3: Solve: $x\frac{dy}{dx} + y \log y = xye^x$ 2062/069/072 Kartik B. E.

Solution:

Here. $x\frac{dy}{dx} + y \log y = xye^x$

Dividing by xy,

$$y^{-1}\frac{dy}{dx} + \frac{\log y}{x} = e^x$$

Put $\log y = v$, $\quad y^{-1}\frac{dy}{dx} = \frac{dv}{dx}$

So the equation reduces to,

$$\frac{dv}{dx} + \frac{v}{x} = e^x$$

This is linear form, $P = \frac{1}{x}$, $\quad Q = e^x$

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Its general solution is,

$$v \times \text{I.F.} = \int Q \times (\text{I.F.})\, dx$$

or $\quad v\,x = \int xe^x\, dx + c$

$\therefore \quad x.\log y = xe^x - e^x + c$ is the required solution.

Ex. 4: Solve: $\frac{dy}{dx} - 2y \tan x = y^2 \tan^2 x$ 2068 Chaitra B. E.

Solution:

Here. $\frac{dy}{dx} - 2y \tan x = y^2 \tan^2 x.$

Dividing by y^2

$$y^{-2}\frac{dy}{dx} - 2y^{-1}\tan x = \tan^2 x$$

Put $y^{-1} = v$, $\quad -y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$

So the equation reduces to

$$-\frac{dv}{dx} - 2v \tan x = \tan^2 x$$

or $\frac{dv}{dx} + 2\tan x. v = -\tan^2 x$

This is linear form, $P = 2\tan x$, $Q = -\tan^2 x$.

So. $I.F. = e^{\int P dx} = e^{2\int \tan x\, dx} = e^{2\log \sec x} = \sec^2 x$

Its general solution is

$$v \times I.F. = \int Q \times (I.F.)\, dx$$

or $v \sec^2 x = -\int \tan^2 x \sec^2 x\, dx + c$

Put $\tan x = t$, $\sec^2 x\, dx = dt$

or $v \sec^2 x = -\int t^2\, dt = -\frac{t^3}{3} - c = -\frac{\tan^3 x}{3} - c$

or $v \sec^2 x = -\frac{\tan^3 x}{3} + c$

$\therefore 3\sec^2 x + y\tan^3 x + cy = 0$ is the required solution.

Ex. 5: Solve: $\frac{dy}{dx} + \frac{1}{x}\tan y = \frac{1}{x^2}\tan y \sin y$

Solution:

Here, $\frac{dy}{dx} + \frac{1}{x}\tan y = \frac{1}{x^2}\tan y \sin y$.

Dividing tany siny,

$$\operatorname{cosec} y \cot y \frac{dy}{dx} + \frac{\operatorname{cosec} y}{x} = \frac{1}{x^2}$$

Put cosecy = v,

$$-\operatorname{cosec} y \cot y \frac{dy}{dx} = \frac{dv}{dx}$$

The equation reduces to

$$-\frac{dv}{dx} + \frac{1}{x} v = \frac{1}{x^2}, \quad \text{or} \quad \frac{dv}{dx} - \frac{v}{x} = -\frac{1}{x^2}$$

This is linear form, $P = \frac{-1}{x}$, $Q = -\frac{1}{x^2}$

So. $I.F. = e^{\int P dx} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$

Its general solution is,

$$v \times I.F. = \int Q \times (I.F.)\, dx$$

or $v\frac{1}{x} = -\int \frac{1}{x^3}\, dx - c$

or $\frac{\text{cosecy}}{x} = \frac{1}{2x^2} + c$

$\therefore$ $2x = (1 + 2cx^2)$ siny is the required solution.

Ex. 6: Solve: $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

Solution:

Here, $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

Dividing by secy,

$$\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)\, e^x$$

Put siny = v, $\quad \cos y \frac{dy}{dx} = \frac{dv}{dx}$

So, the equation reduces to,

$$\frac{dv}{dx} - \frac{v}{1+x} = (1+x)\, e^x$$

This islinear form, $P = -\frac{1}{1+x}$ $Q = (1+x)\, e^x$

So, I.F. $= e^{\int P dx} = e^{-\int \frac{1}{1+x} dx} = e^{-\log(1+x)} = \frac{1}{1+x}$

Its general solution is,

$$v \times \text{I.F.} = \int Q \times (\text{I.F.})\, dx$$

or $\quad v.\frac{1}{1+x} = \int e^x\, dx + c.$

or $\quad \frac{\sin y}{1+x} = e^x + c$

or $\quad \sin y = (1+x)(e^x + c)$ is the required solution.

Exercise - 24

Solve the following differential equations

1. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

2. $(1 - x^2)\frac{dy}{dx} + xy = xy^2$ $\quad$ *2067 Ashwin B.E.*

3. $\frac{dy}{dx} - \frac{1}{x} = \frac{e^y}{x^2}$

4. $\dfrac{dy}{dx} + \dfrac{y}{x} \log y = \dfrac{y}{x^2} (\log y)^2$ 2067 Ashwin B.E.

2 5. $\dfrac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$ 2061 Ashwin B.E.

6. $\cos x \dfrac{dy}{dx} = y(\sin x - y)$

7. $x\dfrac{dy}{dx} + y = y^2 \log x$

8. $\dfrac{dy}{dx} = y \tan x - y^2 \sec x$ 2060 B.E.

9. $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)\, e^x \sec y$

10. $\dfrac{dy}{dx} + \dfrac{1}{x} \sin 2y = x^3 \cos^2 y$

Answers

1. $2x = cx^2y + y$
2. $\sqrt{1 - x^2}\ (1 - y) = cy$
3. $2x = (cx^2 + 1)\, e^y$
4. $x = \log y \left(cx^2 + \dfrac{1}{2}\right)$
5. $e^{x^2} \tan y = \dfrac{1}{2} e^{x^2} (x^2 - 1) + c$
6. $\sec x = y(\tan x + c)$
7. $c\, xy + y(\log x + 1) = 1$
8. $\sec x = y \tan x + cy$
9. $\sin y = (1 + x)(e^x + c)$
10. $6x^2 \tan y = x^6 + c$

15.8 Exact Differential Equation

The first order differential equation $Mdx + Ndy = 0$ where M and N are functions of x, and y is said to be *exact* if there exist a function $U(x, y)$ such that $Mdx + Ndy = d\big(U(x, y)\big)$

i.e. if $Mdx + Ndy$ is perfect differential.

An exact differential equation can always be obtained by differentiating of its primitive.

For example.

$x^2\, dy + 2xy\, dx = 0$ is *exact* because

$x^2dy + 2xy\, dx = d(x^2y)$ and $U = x^2y$ is the *Primitive* of the given differential equation.

A necessary and sufficient condition for differential equation $Mdx + Ndy = 0$ *to be exact is* $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Let $Mdx + Ndy = 0$ is exact, we have to show that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

For this.

Let U(x, y) = c be its primitive.

$$\frac{\partial U}{\partial x}\, dx + \frac{\partial U}{\partial y}\, dy = 0$$

By definition of exactness, it should be equal to given equation.

Thus $M = \frac{\partial U}{\partial x}$ and $N = \frac{\partial U}{\partial y}$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial y} = \frac{\partial^2 U}{\partial x \partial y}$$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Hence the necessary condition.

To prove the sufficient condition.

Let $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, we have to show that Mdx + Ndy = 0 is exact.

For this,

Let $\quad M = \frac{\partial U}{\partial x}$,

or $\quad \frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y \partial x}$

or $\quad \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

or $\quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)$.

Integrating

$$N = \frac{\partial U}{\partial y} + f(y)$$

Thus

$$\begin{aligned} Mdx + Ndy &= \frac{\partial U}{\partial x}\, dx + \left(\frac{\partial U}{\partial y} + f(y)\right) dy \\ &= \frac{\partial U}{\partial x}\, dx + \frac{\partial U}{\partial y}\, dy + f(y)\, dy \\ &= d\left[U + \int f(y)\, dy\right] = d[U + F(y)]. \end{aligned}$$

It shows that the equation Mdx + Ndy = 0 is exact.

15.9 Rules for Solving Exact Differential Equation

The rules for solving the first order exact differential equation $M\,dx + N\,dy = 0$ are as follows.

I. Integrate M with respect to x treating y as constant.

II. Take those terms of N which are free from x and integrate them with respect to y.

III. Add the above results and equate the sum to some constant, gives the required solution.

15.10 Integrating Factor

If the first order differential equation $Mdx + Ndy = 0$ is linear but not exact then it becomes exact if we multiply it by suitable factor called *Integrating Factor* (I.F.). In many cases, the integrating factors are found by inspection as given below:.

Equation	*I.F.*	*Exact equation*
1. $x\,dy + y\,dx$	1	$x\,dy + y\,dx = d(xy)$
$x\,dy + y\,dx$	$\dfrac{1}{xy}$	$\dfrac{x\,dy + y\,dx}{xy} = d\,(\log xy)$.
$xdy + ydx$	$(xy)^n$	$(xy)^n\,(x\,dy + y\,dx) = \dfrac{1}{n+1}\,d(xy)^{n+1}$
2. $x\,dy - y\,dx$	$\dfrac{1}{x^2}$	$\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$
$x\,dy - y\,dx$	$\dfrac{1}{y^2}$	$\dfrac{x\,dy - y\,dx}{y^2} = -\,d\left(\dfrac{x}{y}\right)$
$x\,dy - y\,dx$	$\dfrac{1}{xy}$	$\dfrac{x\,dy - y\,dx}{xy} = d\left(\log\dfrac{y}{x}\right)$
$x\,dy - y\,dx$	$\dfrac{1}{x^2+y^2}$	$\dfrac{x\,dy - y\,dx}{x^2+y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$
$x\,dy - y\,dx$	$\dfrac{1}{x^2+y^2}$	$-\dfrac{y\,dx - x\,dy}{x^2+y^2} = -\,d\left(\tan^{-1}\dfrac{x}{y}\right)$
3. $x\,dx + y\,dy$	$\dfrac{1}{2}$	$\dfrac{x\,dx + y\,dy}{2} = \dfrac{1}{2}\,d(x^2+y^2)$
$x\,dx + y\,dy$	$\dfrac{1}{x^2+y^2}$	$\dfrac{x\,dx + y\,dy}{x^2+y^2} = \dfrac{1}{2}\,d\left(\log\,(x^2+y^2)\right)$
4. $dx + dy$	$\dfrac{1}{x+y}$	$\dfrac{dx+dy}{x+y} = d\,[\log(x+y)]$
5. $2xy\,dx - x^2dy$	$\dfrac{1}{y^2}$	$\dfrac{2xydx - x^2dy}{y^2} = d\left(\dfrac{x^2}{y}\right)$
6. $2xy\,dy - y^2dx$	$\dfrac{1}{x^2}$	$\dfrac{2xy\,dy - y^2dx}{x^2} = d\left(\dfrac{y^2}{x}\right)$

Worked Out Examples

Ex. 1: Solve: $(x^2 + 2xy^2)\, dx + (2x^2y + y^2)\, dy = 0$

Solution:

Here, $(x^2 + 2xy^2)\, dx + (2x^2y + y^2)\, dy = 0$

$$M = x^2 + 2xy^2, \quad N = 2x^2y + y^2$$

$$\frac{\partial M}{\partial y} = 4xy, \quad \frac{\partial N}{\partial x} = 4xy.$$

So, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

Now, $\int M dx$, keeping y is constant

$$= \int (x^2 + 2xy^2)\, dx = \frac{x^3}{3} + x^2y^2$$

Taking the term free from x in N, i.e., y^2

So, $\int y^2 dy = \dfrac{y^3}{3}$

Hence the required solution is,

$$\frac{x^3}{3} + \frac{y^3}{3} + x^2y^2 = k$$

or $\quad x^3 + y^3 + 3x^2y^2 = c.$

Ex. 2: Solve: $x\, dy + (x + 1)\, dx = 0$

Solution:

Here, $xdy + (x + 1)\, dx = 0$

$$M = (x + 1), \quad N = x$$

$$\frac{\partial M}{\partial y} = 0, \quad \frac{\partial N}{\partial x} = 1.$$

So, $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, the equation is not exact.

So, the given differential equation can be written as

$$xdy + (x + 1)\, ydx = 0.$$

Separating the variables

$$\frac{dy}{y} + \frac{x + 1}{x}\, dx$$

Integrating

$$\log y + x + \log x = c$$

$\log(xy) + x = c.$ is the required solution.

Ex. 3: Solve: $(x + y)(dx - dy) = dx + dy$

Solution:

Here. $(x + y)(dx - dy) = dx + dy$.

The equation can be written as

$$dx - dy = \frac{dx + dy}{x + y}$$

or $\int dx - \int dy = \int d(\log(x + y))$

or $c + x - y = \log(x + y)$

$\therefore$ $\log(x + y) = c + x - y$ is the required solution.

Ex. 4: Solve: $y \sin 2x\, dx - (y^2 + \cos^2 x)\, dy = 0$

Solution:

Here, $y \sin 2x\, dx - (y^2 + \cos^2 x)\, dy = 0$

or $(\cos^2 x\, dy - 2y \sin x \cos x\, dx) + y^2\, dy = 0$

or $\int d(\cos^2 x\, y) + \int y^2\, dy = 0$ or $\cos^2 x\, y + \frac{y^3}{3} = c$

or $y \cos^2 x + y^3 = c$ or $\frac{3y(1 + \cos 2x)}{2} + y^3 = c$,

$\therefore$ $3y \cos 2x + 2y^3 + 3y = c$ is the required solution.

Ex. 5: Solve: $(x^3y^2 - y)\, dx - (x^2y^3 + x)\, dy = 0$

Solution:

Here. $(x^3y^2 - y)\, dx - (x^2y^3 + x)\, dy = 0$

The equation can be written as

$$x^2y^2 (x\, dy - y\, dy) = (y\, dx + x\, dy)$$

or $(x\, dx - y\, dy) - \frac{(x\, dy + y\, dx)}{x^2y^2} = 0$

or $\int x\, dx - \int y\, dy + \int d\left(\frac{1}{xy}\right) = 0$

or $\frac{x^2}{2} - \frac{y^2}{2} + \frac{1}{xy} = c$

$\therefore$ $x^3y - xy^3 + 2 = cxy$ is the required solution.

Ex. 6: Solve: $x\, dy - y\, dx = \sqrt{x^2 - y^2}\, dx$ 2062 B.E.

Solution:

Here. $x\, dy - y\, dx = \sqrt{x^2 - y^2}\, dx$

$$\frac{x dy - y dx}{x^2 \sqrt{1 - y^2/x^2}} = \frac{dx}{x}$$

or $\int d\left(\sin^{-1}\frac{y}{x}\right) = \int \frac{dx}{x}$

or $\sin^{-1}\frac{y}{x} = \log x + \log c,$

or $\sin^{-1}\frac{y}{x} = \log cx$

$\therefore$ y = x sin (log cx) is the required solution.

Ex. 7: Solve: $y(axy + e^x)\,dx - x\,dy = 0$

Solution:

Here, $y(axy + e^x)\,dx - x\,dy = 0$

or $ax\,dx + \frac{y\,e^x\,dx - e^x dy}{y^2} = 0$

or $\int ax\,dx + \int d\left(\frac{e^x}{y}\right) = 0$

or $\frac{ax^2}{2} + \frac{e^x}{y} = c$

$\therefore$ $ax^2y + 2e^x = cy$ is the required solution.

Ex. 7: Solve: $(x^2 - y^2)dx + 2xy\,dy = 0$ 2068 Baishakh B. E.

Solution:

Here, $(x^2 - y^2)dx + 2xy\,dy = 0$

or $x^2dx + 2xy\,dy - y^2\,dx = 0,$

or $dx + \frac{2xy\,dy - y^2\,dx}{x^2} = 0,$

or $dx + d\left(\frac{y^2}{x}\right) = 0.$

or $\int dx + \int d\left(\frac{y^2}{x}\right) = 0.$

or $x + \left(\frac{y^2}{x}\right) = c.$

$\therefore$ $x^2 - y^2 = cx$ is the required solution.

Exercise - 25

Solve the following equations

1. $(2ax + by)\, y\, dx + (ax + 2by)\, x\, dy = 0$
2. $(x^2 - ay)\, dx - (ax - y^2)\, dy = 0$
3. $\dfrac{dy}{dx} = \dfrac{2x - y + 1}{x + 2y - 3}$
4. $(x + y)\, dy + (y - x)\, dx = 0$
5. $2xy\, dx - (x^2 - y^2)\, dy = 0$
6. $(x^2 + y^2 + 2x)\, dx + xy\, dy = 0$
7. $x\dfrac{dy}{dx} + y = y^2 \log x$
8. $x dy - y dx + a(x^2 + y^2)\, dx = 0$ 2058 B.E.
9. $x dx + y dy + (x^2 + y^2)\, dy = 0$
10. $(1 + xy)\, y\, dx + (1 - xy)\, x\, dy = 0$
11. $\sin x\, dy - y \cos x\, dx + y^2\, dx = 0$
12. $x\dfrac{dy}{dx} = y + x^2 \log x$
13. $x \cos\left(\dfrac{y}{x}\right)(y\, dx + x\, dy) = y \sin\left(\dfrac{y}{x}\right)(x\, dy - y\, dx)$
14. $\cos x (\cos x - \sin y)\, dx + \cos y (\cos y - \sin x)\, dy = 0$
15. $(x + 2y^3)\, dy = y\, dx$
16. $x^2y^3\, dx + 3x^2y\, dy + 2y\, dx = 0$
17. $\dfrac{x\, dx + y\, dy}{x\, dy - y\, dx} = \dfrac{\sqrt{a^2 - x^2 - y^2}}{x^2 + y^2}$
18. $(x^2 + y^2 + 2x)\, dx + 2y\, dy = 0$
19. $x^2\, dy + xy\, dx + 2\sqrt{1 - x^2y^2}\, dx = 0$
20. $x\, dx + y\, dy = a^2 \dfrac{(x\, dy - y\, dx)}{x^2 + y^2}$
21. $(x^2 + y^2)\, dx - 2xy\, dy = 0$

Answers

1. $ax^2y + bxy^2 = c$
2. $x^3 + y^3 - 3axy = c$
3. $y^2 - x^2 + xy - 3y - x = c$
4. $x^2 - y^2 - 2xy = c$
5. $x^2 + y^2 = cy$
6. $3x^4 + 8x^3 + 6x^2y^2 = c$
7. $cxy + 1 = y(1 + \log x)$
8. $\tan^{-1}\left(\frac{y}{x}\right) + ax = c$
9. $x^2 + y^2 = c\,e^{-2y}$
10. $\log\left(\frac{x}{y}\right) = c + \frac{1}{xy}$
11. $\sin x = y(x + c)$
12. $y = x^2 \log x - x^2 + cx$
13. $xy\cos\left(\frac{y}{x}\right) = c$
14. $2x + 2y + \sin 2x + \sin 2y - 4\sin y\sin x = c$
15. $y^3 + cy = x$
16. $x(xy - 2)^3 = c(xy - 1)^3$
17. $\sqrt{x^2 + y^2} = a\sin\left(\tan^{-1}\left(\frac{y}{x}\right) + c\right)$
18. $x + \log(x^2 + y^2) = c$
19. $\sin^{-1}(xy) + 2\log x = c$
20. $x^2 + y^2 = 2a^2\tan^{-1}\left(\frac{y}{x}\right) + c$
21. $y^2 = x(x + c)$

7.2 Fundamental Integrals

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1}\ (n \neq -1)$

2. $\int \frac{1}{x}\,dx = \log x$, for $x > 0$.

3. $\int e^x\,dx = e^x$

4. $\int a^x\,dx = \frac{a^x}{\log a}$

5. $\int \sin x\,dx = -\cos x$

6. $\int \cos x\,dx = \sin x$

7. $\int \sec^2 x\,dx = \tan x$

8. $\int \operatorname{cosec}^2 x\,dx = -\cot x$

9. $\int \sec x \tan x\,dx = \sec x$

10. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x$

11. $\int \sinh x\,dx = \cosh x$

12. $\int \cosh x\,dx = \sinh x$

7.3 Standard Integrals

13. $\int \frac{f'(x)}{f(x)}\,dx = \log f(x)$

14. $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)}$

15. $\int \tan x\,dx = \log \sec x$

16. $\int \cot x\,dx = \log \sin x$

17. $\int \sec x\,dx = \log(\sec x + \tan x) = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

18. $\int \operatorname{cosec}\,dx = \log(\operatorname{cosec} x - \cot x) = \log \tan \frac{x}{2}$

19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$

20. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x}$

21. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a}$

22. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$

23. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \log\left(x + \sqrt{a^2 + x^2}\right) = \sinh^{-1} \frac{x}{a}$

23. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left(x + \sqrt{x^2 - a^2}\right) = \cosh^{-1} \frac{x}{a}$

24. $\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx} \cdot \int v\,dx\right] dx$ (Integration by parts)

25. $\int \sqrt{a^2 + x^2}\,dx = \frac{x\sqrt{a^2 + x^2}}{2} + \frac{a^2}{2} \log\left(x + \sqrt{a^2 + x^2}\right)$

26. $\int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$

27. $\int \sqrt{x^2 - a^2}\, dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \log \left(x + \sqrt{x^2 - a^2}\right)$

28. $\int e^{ax} \sin bx\, dx = \frac{e^{ax}\,(a \sin bx - b \cos bx)}{a^2 + b^2}$

29. $\int e^{ax} \cos bx\, dx = \frac{e^{ax}\,(a \cos bx + b \sin bx)}{a^2 + b^2}$

30. $\int_{-a}^{a} f(x)\, dx = 2 \int_{0}^{a} f(x)\, dx,$ if f(x) is an even function.

$=0,$ if f(x) is odd function.

7.4 Some Standard Forms

Standard Form I: $\int \frac{1}{px^2 + qx + r}\, dx$

Let $I = \int \frac{1}{px^2 + qx + r}\, dx$

In this form, if $px^2 + qx + r$ can be factorized into two linear factors, then it can be written as

$$I = \int \frac{1}{(ax + b)\,(cx + d)}\, dx$$

$$= \frac{1}{(ad - bc)} \int \left(\frac{a}{ax + b} - \frac{c}{cx + d}\right) dx$$

If $px^2 + ax + r$ can not be factorized as two linear factors then it can be written as

$$I = \int \frac{1}{px^2 + qx + r}\, dx = \frac{1}{p} \int \frac{1}{x^2 + \frac{q}{p}x + \frac{r}{p}}\, dx$$

$$= \frac{1}{p} \int \frac{1}{(x)^2 + 2.x.\frac{q}{2p} + \left(\frac{q}{2p}\right)^2 + \frac{r}{p} - \left(\frac{q}{2p}\right)^2}\, dx$$