9. $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \dots$

Given series is, $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \dots$

It's general term after first term is,

$$u_n = \frac{n^n}{(n+1)^{n+1}}$$

Taking series $\Sigma v_n = \Sigma \frac{1}{n}$

and its general term is $v_n = \frac{1}{n}$

Now,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^n}{n^{n+1}(1+1/n)^{n+1}} \times n$$

$$= \lim_{n\to\infty} \frac{1}{(1+1/n)^n \cdot (1+1/n)^1}$$

$$= \lim_{n\to\infty} \frac{1}{e} \cdot \frac{1}{(1+1/n)}$$

$$= \frac{1}{e} \cdot \frac{1}{1}$$

$= e^{-1}$ which is finite and non zero

$\Sigma v_n = \Sigma \frac{1}{n}$ is divergent by p test

$\therefore$ $\Sigma u_n = \Sigma \frac{n^n}{(n+1)^{n+1}}$ is divergent by limit comparision test.

Ex-28

4 $\sum \left(\frac{nx}{1+n}\right)^n$

Given series,

$$\sum \left(\frac{nx}{1+n}\right)^n$$

The general term of the given series is,

$$u_n = \left(\frac{nx}{1+n}\right)^n$$

By root test,

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} (u_n)^{1/n}$$

$$= \lim_{n\to\infty} \frac{nx}{1+n}$$

$$= \lim_{n\to\infty} \frac{nx}{n(1+1/n)}$$

$$= \lim_{n\to\infty} \frac{x}{(1+1/n)}$$

$$= x$$

By root test, given series is convergent for $x<1$

divergent for $x>1$

when $x=1$, the test fails,

when $x=1$,

$$\sum \left(\frac{n\cdot 1}{1+n}\right)^n$$

It's general term $u_n = \left(\frac{n}{1+n}\right)^n$

$$= \frac{n^n}{(1+n)^n}$$

Thus the given series is convergent for $x < 1$
divergent for $x > 1$
test fail for $x = 1$

when $x = 1$,
The series becomes,

$$\sum \frac{1^n}{n(n+1)} = \sum \frac{1}{n(n+1)}$$

The, general term is $\frac{1}{n(n+1)}$

Let's suppose a series $\sum v_n = \sum \frac{1}{n^2}$, [convergent by p test]

Now,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n(n+1)} \times n^2$$

$$= \lim_{n\to\infty} \frac{1}{n^2(1+1/n)} \times n^2$$

$$= \lim_{n\to\infty} \frac{1}{1+1/n}$$

$$= \frac{1}{1}$$

$= 1$ which is finite and non-zero,

$\therefore$ The series is convergent for $x = 1$.

So, The series is convergent for $x \leq 1$
divergent for $x > 1$.

8) $1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \cdots\cdots$, $x > 0$

= Here, the series is,

$$1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \cdots$$

Now,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n^n}{(1+n)^n}$$

$$= \lim_{n\to\infty} \frac{1}{(1+1/n)^n}$$

$$= \lim_{n\to\infty} \frac{1}{(1+1/n)^n}$$

$$= \cancel{e}\,\frac{1}{e}$$

$\therefore \lim_{n\to\infty} u_n = \frac{1}{e} \neq 0$

So, the given series is divergent for $x=1$

Thus, given series is convergent for $x<1$

divergent for $x \geq 1$.

(7) $\frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \dots\dots, x>0$

Soln: Given series is,

$$\frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \dots + \frac{x^n}{n(n+1)}$$

The ~~given~~ general term is,

$$u_n = \frac{x^n}{n\cdot(n+1)}$$

Now

$$u_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$$

Now,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(\cancel{n+1})(n+2)} \times \frac{n(\cancel{n+1})}{x^n}$$

$$= \lim_{n\to\infty} \frac{x \quad n}{n(1+2/n)}$$

$$= \lim_{n\to\infty} \frac{x}{1+2/n} = x$$

The general term after first term is,

$$u_n = \frac{x^n}{(n+1)^n}$$

Now,
Taking root test

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \left(\frac{x^n}{(n+1)^n} \right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{x}{n+1}$$

$$= \frac{x}{\infty}$$

$$= 0 < 1$$

The series is convergent for all values of x.