Clairaut's Equation

An equation is of the form y = px + f(p) is called clairaut's equation.
If can be solved by the following method.
Let y = px + f(p) ……. (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\text{or, } p = p + [x + f'(p)]\frac{dp}{dx}$$

$$\text{or, } (x + f'(p))\frac{dp}{dx} = 0$$

Either $\frac{dp}{dx} = 0 \Rightarrow \int dp = \int 0 \,.\, dx$, Integrating

P = C ……… (ii)
From (i) and (ii)
y = Cx + f(c) is the required general solution.
or, [x + f ' (p)] = 0 ⇒ f '(p) = – x ……. (iii)

Eliminating p from (i) and (iii) gives the required singular solution.

Exercise - 28

Solve the following equations

1. $y = px + p - p^2$

Sol^n. Given differential equation is,

$y = px + p - p^2$ …….. (i)

Differential equation (i) w. r. t. 'x'.

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$P = P + x\frac{dp}{dx} + \frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\text{or, } (x + 1 - 2p)\frac{dp}{dx} = 0$$

$$\text{Either } \frac{dp}{dx} = 0$$

⇒ p = C …… (ii)
From (i) and (ii)
$y = Cx + C - C^2$ is the required general solution.
or, x + 1 – 2p = 0
or, 2p = x + 1

or, $p = \frac{1}{2}(x+1)$ …….. (iii)

From (i) and (iii)

$$y = \left(\frac{x+1}{2}\right).x + \left(\frac{x+1}{2}\right) - \left(\frac{x+1}{2}\right)^2$$

$$= \left(\frac{x+1}{2}\right)\left\{x+1-\frac{x+1}{2}\right\}$$

$$= \left(\frac{x+1}{2}\right)\left(\frac{2x+2-x-1}{2}\right)$$

$$= \left(\frac{x+1}{2}\right)\left(\frac{x+1}{2}\right) = \left(\frac{x+1}{2}\right)^2$$

$4y = (x+1)^2$ is the singular solution.

2. $y = px + \frac{a}{p}$

Sol^n. Given differential equation is,

$y = px + \frac{a}{p}$ ………. (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$$

or, $p = p + \left(x - \frac{a}{p^2}\right)\frac{dp}{dx}$

or, $\left(x - \frac{a}{p^2}\right)\frac{dp}{dx} = 0$

Either $\frac{dp}{dx} = 0 \Rightarrow p = C$ ……. (ii)

From (i) and (ii)

$y = Cx + \frac{a}{C}$ is the required general solution.

or, $x - \frac{a}{p^2} = 0$

or, $xp^2 = a$

or, $p^2 = \frac{a}{x} \Rightarrow p = \sqrt{\frac{a}{x}}$ ……… (iii)

From (i) and (iii)

$$y = x.\sqrt{\frac{a}{x}} + \frac{a}{\sqrt{\frac{a}{x}}}$$

$y = \sqrt{ax} - \sqrt{ax}$

or, $y = 2\sqrt{ax}$ squaring on both sides

$y^2 = 4ax$ is the required singular solution.

3. $py = p^2(x-b) + a$

Sol^n. Given differential equation is,

$py = p^2(x-b) + a$

or, $y = px + \frac{a}{p} - bp$ ……… (i)

Differential equation w. r.t . 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{a}{p}\frac{dp}{dx} - b\frac{dp}{dx}$$

or, $p = p + \left(x - \frac{a}{p^2} - b\right)\frac{dp}{dx}$

or, $\left(x - \frac{a}{p^2} - b\right)\frac{dp}{dx} = 0$

Either $\frac{dp}{dx} = 0 \Rightarrow p = c$ …… (ii)

From (i) and (ii)

$yC = C^2(x-b) + a$ is the required general solution.

or, $x - \frac{a}{p^2} - b = 0$

or, $\frac{a}{p^2} = (x-b)$

$p = \sqrt{\frac{a}{x-b}}$ ……… (ii)

From (i) and (iii)

$$\sqrt{\frac{a}{x-b}}.y = \frac{a}{(x-b)}(x-b) + a$$

or, $y\frac{\sqrt{a}}{\sqrt{x-b}} = a + a = 2a$

Squaring on both sides,

$y^2a = 4a^2(x-b)$

or, $y^2 = 4a(x-b)$ is the required singular solution.

4. $y = px + ap - ap^2$

Sol^n. Given differential equation is,

$y = px + ap - ap^2$ ……… (i)

Differential equation w. r. t. 'x' we get,

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + a\frac{dp}{dx} - 2ap\frac{dp}{dx}$$

$$\text{or, } p = p + x\frac{dp}{dx} + a\frac{dp}{dx} - 2ap\frac{dp}{dx}$$

$$\text{or, } 0 = (x + a - 2ap)\frac{dp}{dx}$$

$$\text{or, } (x + a - 2ap)\frac{dp}{dx} = 0$$

Either $\frac{dp}{dx} = 0 \Rightarrow p = C$ ……. (ii)

From (i) and (ii)

$y = Cx + aC - aC^2$ is the required general solution.

or, $x + a - 2ap = 0$

or, $2ap = x + a$

$p = \frac{x+a}{2a}$ ……… (iii)

From (i) and (iii)

$$y = x\,.\,\frac{(x+a)}{2a} + a\frac{(x+a)}{2a} - a\frac{(x+a)^2}{4a^2}$$

$$\text{or, } y = \frac{(x+a)}{2a}\left[x + a - \frac{a(x+a)}{2a}\right]$$

$$= \frac{(x+a)}{2a}\frac{(2ax + 2a^2 - ax - a^2)}{2a}$$

$$= \frac{(x+a)}{2a}\frac{(ax + a^2)}{2a}$$

$$= \frac{(x+a)}{2a}\frac{(x+a)}{2}$$

$4ay = (x + a)^2$ is the required singular solution.

5. $(y - px)^2 (1 + p^2) = a^2p^2$

Sol^n. Given differential equation is,

$(y - px)^2 (1 + p^2) = a^2p^2$

$$\text{or, } (y - px)^2 = \frac{a^2p^2}{(1+p^2)}$$

$$\Rightarrow y - px = \frac{ap}{\sqrt{1+p^2}}$$

$$\text{or, } y = px + \frac{ap}{\sqrt{1+p^2}} \text{ ……… (i)}$$

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \left[\frac{\sqrt{1+p^2}\,.\,a\frac{dp}{dx} - aP\,.\,\frac{2p}{2\sqrt{1+p^2}}\,.\,\frac{dp}{dt}}{(1+p^2)}\right]$$

$$p = p + x\frac{dp}{dx} + \frac{dp}{dx}\left[\frac{a\sqrt{1+p^2}}{(1+p^2)} - \frac{ap^2}{(1+p^2)\sqrt{1+p^2}}\right]$$

$$\text{or, } 0 = \frac{dp}{dx}\left[x + a\left\{\frac{1+p^2-p^2}{(1+p^2)}\right\}\right]$$

$$\text{or, } \frac{dp}{dx}\left[x + \frac{a}{(1+p^2)\left(\sqrt{1+p^2}\right)}\right] = 0$$

Either $\frac{dp}{dx} = 0 \Rightarrow p = C$ ……. (ii)

From (i) and (2)

$(y - Cx)^2 (1 + C^2) = a^2C^2$ is the required general solution.

$$\text{or, } x + \frac{a}{\left(1+p^2\right)\sqrt{1+p^2}} = 0$$

$$\text{or, } x = -\frac{a}{\left(1+p^2\right)^{\frac{3}{2}}}$$

$$\text{or, } x^{\frac{2}{3}} = \frac{a^{\frac{2}{3}}}{(1+p^2)} \text{ ……… (iii)}$$

Also, from $y = px + \frac{ap}{\sqrt{1+p^2}}$

$$y = -\frac{pa}{\left(1+p^2\right)^{\frac{3}{2}}} + \frac{ap}{\sqrt{1+p^2}} = \frac{ap[-1+1+p^2]}{\left(1+p^2\right)^{\frac{3}{2}}}$$

or, $y = \dfrac{ap^3}{\left(1+p^2\right)^{\frac{3}{2}}}$

or, $y^{\frac{2}{3}} = \dfrac{a^{\frac{2}{3}}\, p^2}{\left(1+p^2\right)}$ ……… (iv)

From (iii) and (iv)

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = \frac{a^{\frac{2}{3}}}{\left(1+p^2\right)} + \frac{a^{\frac{2}{3}}\,.\,p^2}{1+p^2} = \frac{a^{\frac{2}{3}}\,(1+p^2)}{\left(1+p^2\right)}$$

$\therefore\ x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is the required singular solution.

6. p = log (px – y)

Sol^n. Given differential equation is,

p = log (px – y)

$\Rightarrow px - y = e^p$

or, $y = px - e^p$ ……… (i)

Differential equation (i) w. r. t. 'x'

$\dfrac{dy}{dx} = p + x\dfrac{dp}{dx} - e^p\,\dfrac{dp}{dx}$

$p = p + (x - e^p)\,\dfrac{dp}{dx}$

or, $\dfrac{dp}{dx}\ (x - e^p)\ \dfrac{dp}{dx}$

or, $\dfrac{dp}{dx}\ (x - e^p) = 0$

Either $\dfrac{dp}{dx} = 0 \Rightarrow p = C$ ……. (ii)

From (i) and (ii)

$y = Cx - e^C$ is the required general solution.

or, $x - e^p = 0$

or, $x = e^p$

$\Rightarrow p = \log x$ ……. (iii) from equation (i) and (iii)

$y = x \log x - e^{\log x}$

or, y = x log x – x is the required singular solution.

7. (y + 1) p – xp^2 + 2 = 0

Sol^n. Given differential equation is,

$(y + 1)\,p - xp^2 + 2 = 0$

or, $yp + p = xp^2 - 2$

or, $yp = xp^2 - p - 2$

or, $y = px - 1 - \dfrac{2}{p}$ ……. (i)

Differential equation (i) w. r. t. 'x'

$\dfrac{dy}{dx} = p + x\,\dfrac{dp}{dx} + \dfrac{2}{p^2}\,\dfrac{dp}{dx}$

or, $p = p + \left(x + \dfrac{2}{p^2}\right)\dfrac{dp}{dx}$

or, $\dfrac{dp}{dx}\left(x + \dfrac{2}{p^2}\right) = 0$

Either $\dfrac{dp}{dx} = 0 \Rightarrow p = C$ …… (ii)

From equation (i) and (ii)

$(y + 1)\,C - xC^2 + 2 = 0$ is the required general solution.

or, $x + \dfrac{2}{p^2} = 0$

or, $p^2 = -\dfrac{2}{x}$

Using $p^2 = -\dfrac{2}{x}$ in $(y + 1)\,p = xp^2 - 2$

or, $(y + 1)^2\, p^2 = x^2\,(p^2 - 2)^2$

or, $(y + 1)^2 \left(-\dfrac{2}{x}\right) = x^2 \left[\left(-\dfrac{2}{x}\right)(-2)\right]^2$

or, $-\dfrac{2}{x}\ (y + 1)^2 = x^2 \left(\dfrac{4}{x}\right)^2$

or, $-\dfrac{2}{x}\ (y + 1) = 16$

or, $(y + 1)^2 = -\dfrac{16x}{2}$

or, $(y + 1)^2 + 8x = 0$ is the required singular solution.

8. (xp – y)^2 = p^2 – 1

Sol^n. Given differential equation is,

$(xp - y)^2 = p^2 - 1$

or, $xp - y = \sqrt{p^2 - 1}$

$y = xp - \sqrt{p^2 - 1}$ …….. (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{1}{2}\cdot\frac{2p}{\sqrt{p^2-1}}\frac{dp}{dx}$$

$$\text{or, } p = p + \left(x - \frac{p}{\sqrt{p^2-1}}\right)\frac{dp}{dx}$$

$$\text{or, } \frac{dp}{dx}\left(x - \frac{p}{\sqrt{p^2-1}}\right) = 0$$

Either $\frac{dp}{dx} = 0 \Rightarrow p = C$ ……. (ii)

From (i) and (ii)

$(xC - y)^2 = C^2 - 1$ is the required general solution.

$$\text{or, } x - \frac{p}{\sqrt{p^2-1}} = 0$$

$$\text{or, } x = \frac{p}{\sqrt{p^2-1}} \quad \text{……. (iii)}$$

From (i) and (iii)

$$y = \frac{p}{\sqrt{p^2-1}}\,p + \sqrt{p^2-1} = \frac{p^2+p^2-1}{\sqrt{p^2-1}} = \frac{-1}{\sqrt{p^2-1}}$$

$$\text{or, } y = -\frac{1}{\sqrt{p^2-1}} \text{ squaring on both sides}$$

$$y^2 = \frac{1}{\sqrt{p^2-1}}$$

$$\text{or, } p^2 - 1 = \frac{1}{y^2}$$

$$\text{or, } p^2 = 1 + \frac{1}{y^2} = \frac{y^2+1}{y^2} \quad \text{……… (iv)}$$

From (iii) and (iv)

$$x^2 = \frac{p^2}{1-p^2} = \frac{\frac{y^2+1}{y^2}}{1-\frac{y^2+1}{y^2}} = \frac{y^2+1}{y^2}\cdot\frac{y^2}{\left(y^2-y^2+1\right)}$$

or, $x^2 = y^2 + 1$

or, $x^2 - y^2 = 1$ is the required singular solution.