Chapter - 1

VECTOR ANALYSIS AND COORDINATE SYSTEM

1.1 INTRODUCTION

A coordinate system provides us a means to locate ourselves in any point in space with a particular reference. A solution to a particular problem is independent of what coordinate system we use but it is true that if we take the advantage of the particular situation in particular problem and use the appropriate coordinate system, our computational efforts will be minimized and the problem gets simplified. Thus, depending upon the particular geometry of the problem, one of the available coordinate systems might lead us to solving the problem more efficiently.

We have been studying right from our secondary level school course that the position of Nepal lies in the latitude and longitude of approximately 28°N and 84°E respectively. What does this mean? Why is the position of Nepal in global map not mentioned in terms of (x, y, z), the coordinate system we have been studying till date? The answer is obvious, the problem of locating Nepal in global map is simplified by mentioning it in angles rather than in distance of x, y, and z. Thus, study of single coordinate system does not suffice our requirements and hence, in this chapter, we will be dealing with newer coordinate systems like spherical coordinate system and cylindrical coordinate system in addition to the rectangular coordinate system.

During the course of electromagnetics, we find ourselves in different kinds of symmetrical situations like spherical symmetry and cylindrical symmetry. As demanded by the situation, we would like to switch between different coordinate systems. Now, question arises: what is cylindrical symmetry? or what is spherical symmetry? The answer is simple. The situation which demands the use of spherical coordinate system is the spherically symmetrical situation and so on.

Before we really begin with explanation of coordinate systems, let's have some insights on scalars, vectors, and the related topics.

1.2 SCALAR AND VECTOR FIELDS

The term "scalar" refers to a quantity whose value may be represented by a single (positive or negative) real number. In other words, the quantities having only magnitude are the scalar quantities. Examples of scalar quantities include mass, density, pressure, volume, temperature, etc. Vector quantities refer to those which have both a magnitude and a direction in space. Examples are force, velocity, acceleration, magnetic flux density, current density, etc.

A field (scalar or vector) may be defined mathematically as some function of that vector which connects an arbitrary origin to a general point in space. It is usually possible to associate some physical effect with a field, such as the force on a compass needle in the earth's magnetic field, or the movement of smoke particles in the field defined by the vector velocity of air in some region of space. Both scalar fields and vector fields exist. The temperature throughout the bowl of soup and the density at any point in the earth are examples of scalar fields. The gravitational and magnetic fields of the earth, the voltage gradient in a cable and the temperature gradient in a soldering-iron tip are examples of vector fields. The value of a field varies in general with both position and time.

1.3 ALGEBRAIC OPERATIONS INVOLVING VECTORS

(i) Vector addition and subtraction

- Addition of two vectors consists simply of adding the three pairs of like components of the vectors.

$$\vec{A} + \vec{B} = (A_x\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z) + (B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z)$$
$$= (A_x + B_x)\,\hat{a}_x + (A_y + B_y)\,\hat{a}_y + (A_z + B_z)\,\hat{a}_z$$

Vector subtraction is a special case of addition. Thus,

$$\vec{B} - \vec{C} = \vec{B} + (-\vec{C})$$
$$= (B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z) + (-C_x\,\hat{a}_x - C_y\,\hat{a}_y - C_z\,\hat{a}_z)$$
$$= (B_x - C_x)\,\hat{a}_x + (B_y - C_y)\,\hat{a}_y + (B_z - C_z)\,\hat{a}_z$$

(ii) Multiplication and division by a scalar

- Multiplication of a vector $\vec{A}$ by a scalar m is the same as repeated addition of the vector.

$$m\vec{A} = m\,(A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z)$$
$$= mA_x\,\hat{a}_x + mA_y\,\hat{a}_y + mA_z\,\hat{a}_z$$

Division by a scalar is a special case of multiplication by a scalar. Thus,

$$\frac{\vec{B}}{n} = \frac{1}{n}(\vec{B}) = \frac{1}{n}(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z)$$

$$= \frac{B_x}{n}\hat{a}_x + \frac{B_y}{n}\hat{a}_y + \frac{B_z}{n}\hat{a}_z$$

(iii) **Magnitude of a vector**

- The magnitude of a vector $\vec{A}$ is given as

$$|\vec{A}| = |A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

(iv) **Unit vector**

- The unit vector along A, $\hat{a}_A$ has a magnitude equal to unity and has the same direction as that of $\vec{A}$.

$$\hat{a}_A = \frac{\vec{A}}{|\vec{A}|} = \frac{1}{|\vec{A}|}(A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z$$

$$= \frac{1}{\sqrt{A_x^2 + A_y^2 + A_z^2}}(A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z)$$

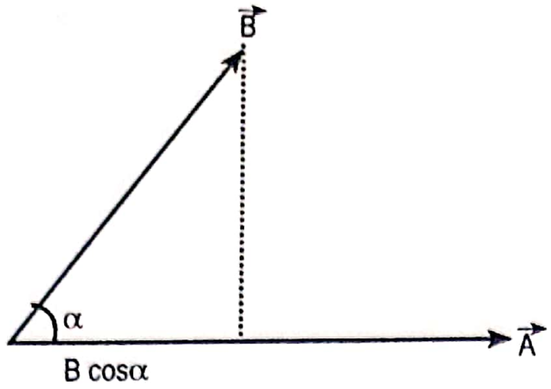
(v) **Scalar or dot product of two vectors.**

- The scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$ is a scalar quantity equal to the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of the angle between $\vec{A}$ and $\vec{B}$. Mathematically,

$$\vec{A}.\vec{B} = |\vec{A}|\;|\vec{B}|\cos\alpha = AB\cos\alpha$$

where α = the angle between $\vec{A}$ and $\vec{B}$.

By noting that $\vec{A}.\vec{B} = A\,(B\cos\alpha) = B\,(A\cos\alpha)$, we observe that the dot product operation consists of multiplying the magnitude of one vector by the scalar obtained by projecting the second vector onto the first vector.



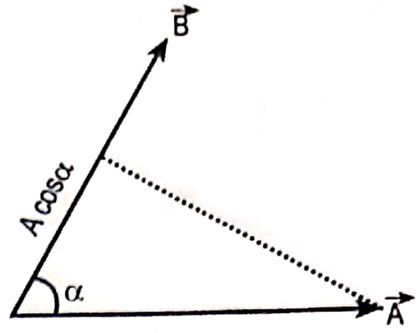


Figure 1.1 *To show that the dot product of two vectors $\vec{A}$ and $\vec{B}$ is the product of the magnitude of one vector and the projection of the second vector onto the first vector.*

(vi) Vector or cross product of two vectors

- The vector or cross product of two vectors $\vec{A}$ and $\vec{B}$ is a vector quantity whose magnitude is equal to the product of the magnitude of $\vec{A}$ and $\vec{B}$ and the sine of the angle α between $\vec{A}$ and $\vec{B}$ and whose direction is normal to the plane containing $\vec{A}$ and $\vec{B}$ and toward the side of advance of a right-handed screw as it is turned from $\vec{A}$ to $\vec{B}$ through the angle α, as shown in the figure below. Mathematically,

$$\vec{A} \times \vec{B} = |\vec{A}|\ |\vec{B}| \sin\alpha\ \hat{a}_N = AB \sin\alpha\ \hat{a}_N$$

where $\hat{a}_N$ = the unit vector in the direction of advance of the right-handed screw.

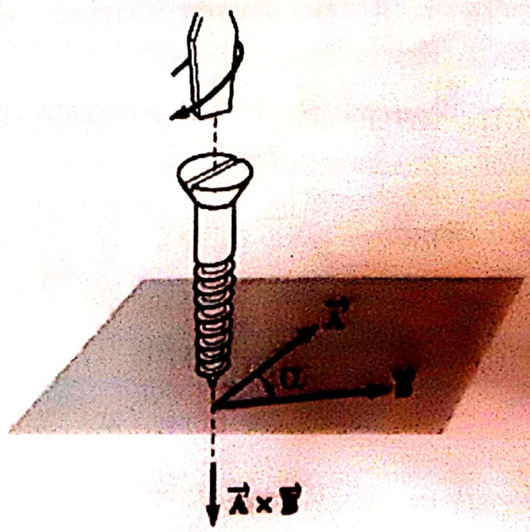


Figure 1.2 *The direction of $\vec{A} \times \vec{B}$ is in the direction of advance of a right-handed screw as $\vec{A}$ is turned into $\vec{B}$.*

(vii) Vector Triple Product

- The vector triple product is defined as the cross product of one vector with the cross product of the other two. The vector triple product of three vectors $\vec{A}, \vec{B}, \vec{C}$ denoted by $\vec{A} \times (\vec{B} \times \vec{C})$ is a vector coplanar with $\vec{B} \times \vec{C}$. The following relationships hold:

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$$

$$(\vec{A} \times \vec{B}) \times \vec{C} = - \vec{C} \times (\vec{A} \times \vec{B}) = - (\vec{C}.\vec{B})\vec{A} + (\vec{C}.\vec{A})\vec{B}$$

(vii) Scalar triple product

- The scalar triple product involves three vectors in a dot product operation and a cross product operation as, for example, $\vec{A}.\ \vec{B} \times \vec{C}$. It is not necessary to include parentheses since this quantity can be evaluated in only one manner, that is, by evaluating $\vec{B} \times \vec{C}$ first and then dotting the resulting vector with $\vec{A}$.

$$\vec{A}.\ \vec{B} \times \vec{C} = (A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z) . \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$\text{or, } \vec{A}.\ \vec{B} \times \vec{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Since the value of the determinant on the right side of equation (i) remains unchanged if the rows are interchanged in a cyclical manner, we can write

$$\vec{A}.\ \vec{B} \times \vec{C} = \vec{B}.\ \vec{C} \times \vec{A} = \vec{C}.\ \vec{A} \times \vec{B}$$

1.4 COORDINATE SYSTEM

(i) Rectangular or Cartesian Coordinate System

- In this system, we set up three coordinate axes mutually at right angles to each other namely x, y, and z axes.
- Any point P is specified as P (x, y, z) and the point is the intersection of three mutually perpendicular planes namely x = constant, y = constant, and z = constant.

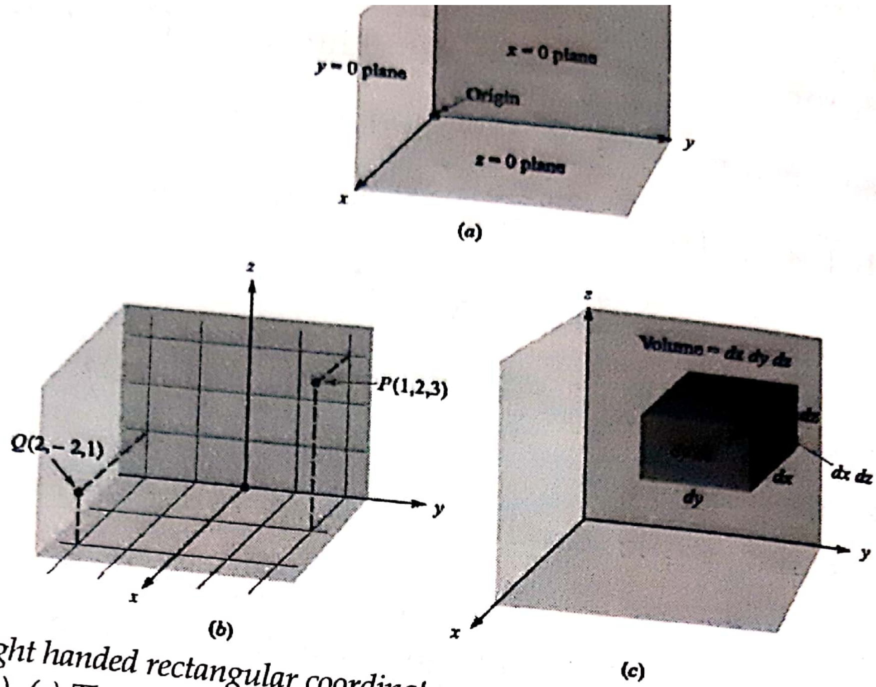


Figure 1.3 (a) A right handed rectangular coordinate system. (b) The location of points P(1,2,3) and Q(2,-2,1). (c) The differential volume element in rectangular coordinates; dx, dy and dz are, in general, independent differentials.

- The three unit vectors of the rectangular coordinate system are $\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$ towards positive x-axis, y-axis, and z-axis respectively.
- **Differential length, area, and volume**
  - Differential length is given by

$$\boxed{d\vec{l} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z}$$

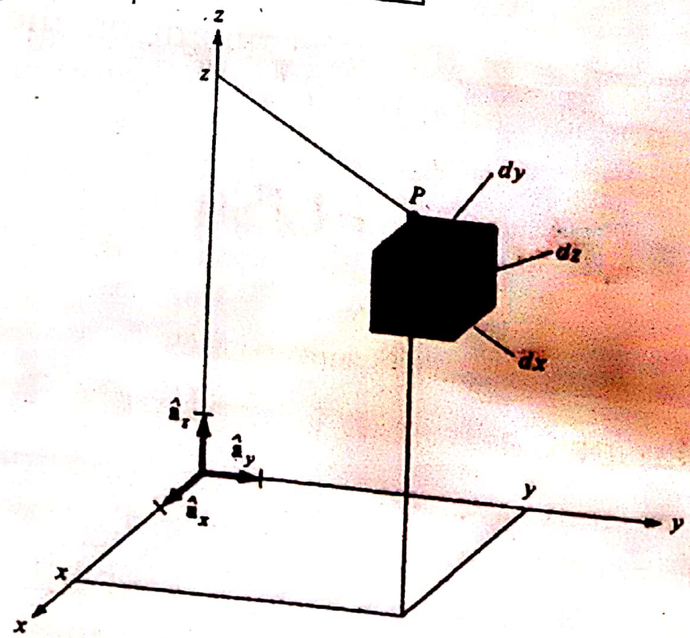


Figure 1.4 Differential elements in the right-handed Cartesian coordinate system

- Differential normal area is given by

$$\boxed{\begin{aligned} d\vec{S} &= dydz\ \hat{a}_x \\ &\quad dxdz\ \hat{a}_y \\ &\quad dxdy\ \hat{a}_z \end{aligned}}$$

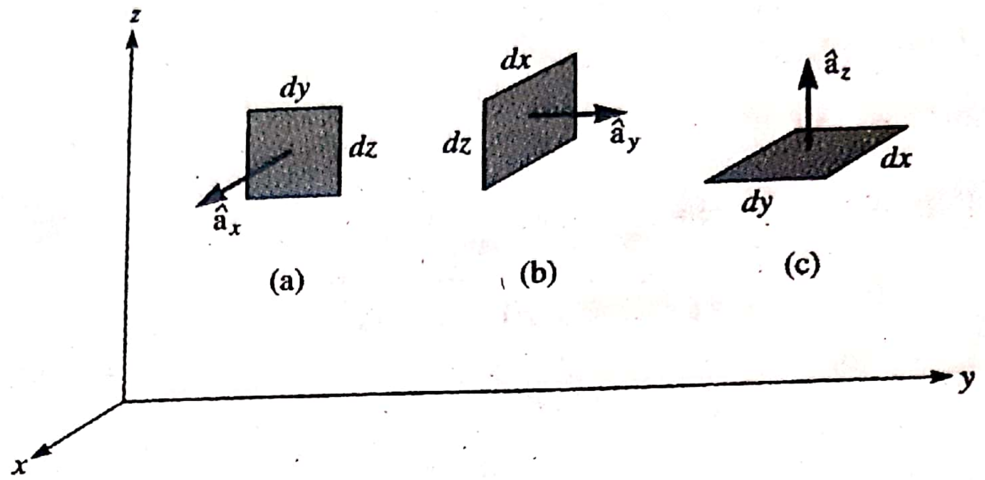


Figure 1.5 Differential normal areas in the Cartesian coordinate system.

- Differential volume is given by $\boxed{dv = dxdydz}$

- Any vector $\vec{A}$ is written as $\boxed{\vec{A} = A_x\ \hat{a}_x + A_y\ \hat{a}_y + A_z\hat{a}_z}$

(ii) Circular Cylindrical Coordinate System

- It is the three dimensional version of the polar coordinates of analytic geometry. The circular cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry.
- Any point P is specified as P (ρ, ϕ, z) and the point is the intersection of three mutually perpendicular surfaces namely a circular cylinder (ρ = constant), a plane (ϕ = constant), and another plane (z = constant).

where ρ = radius of the circular cylinder

ϕ = angle between the x-axis and the projection of the line joining origin and point P in the z = 0 plane.

z = height of the cylinder.

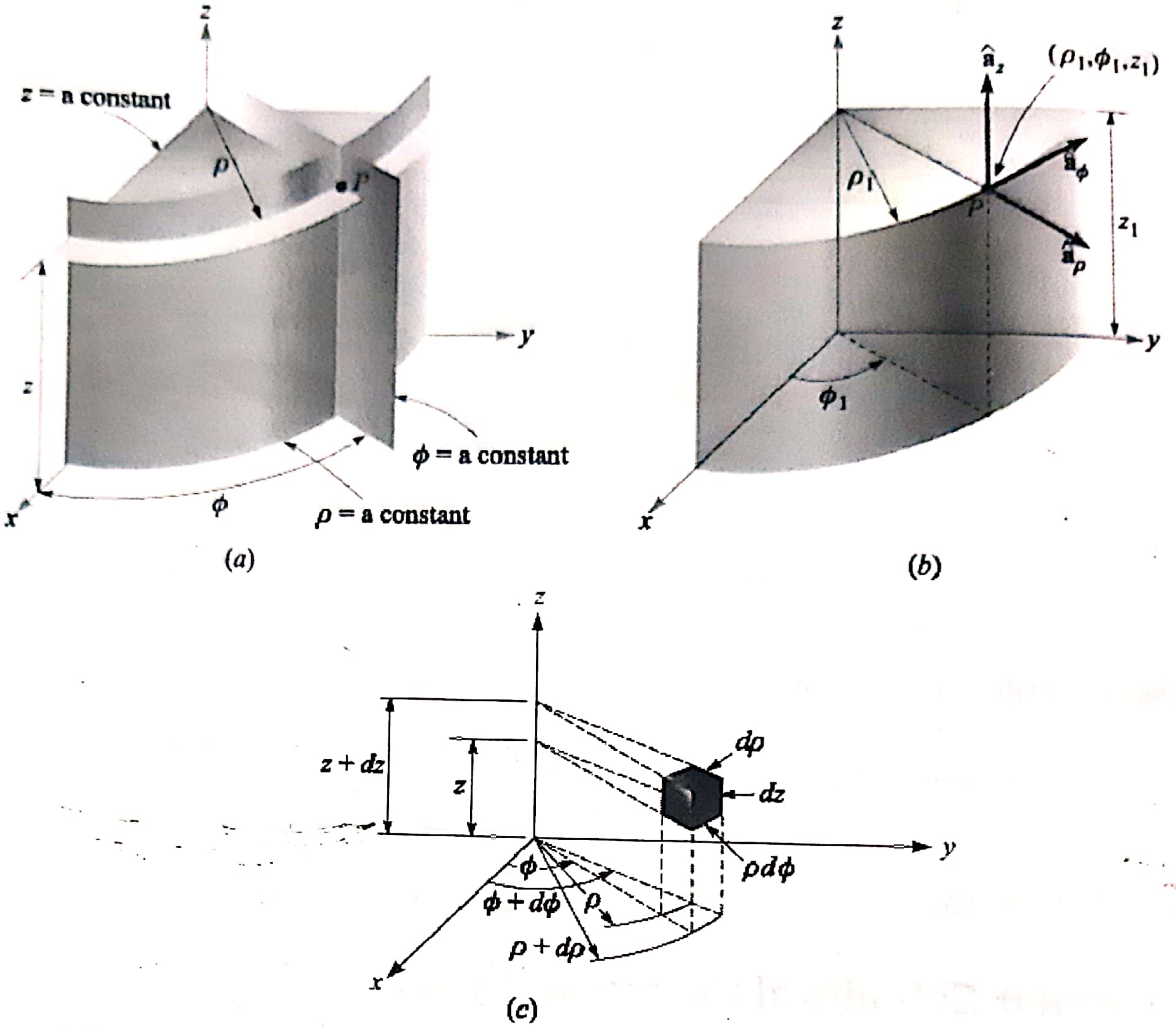


Figure 1.6 (a) The three mutually perpendicular surfaces of the circular cylindrical coordinate system. (b) The three unit vectors of the circular cylindrical coordinate system. (c) The differential volume unit in the circular cylindrical coordinate system; dρ, ρdφ and dz are all elements of length.

- The three unit vectors of the cylindrical coordinate system are $\hat{a}_\rho$, $\hat{a}_\phi$ and $\hat{a}_z$.
  - The unit vector $\hat{a}_\rho$ is directed radially outward, normal to the cylindrical surface $\rho = \rho_1$.
  - The unit vector $\hat{a}_\phi$ is normal to the plane $\phi = \phi_1$, points in the direction of increasing ϕ, lies in the plane $z = z_1$, and is tangent to the cylindrical surface $\rho = \rho_1$.
  - The unit vector $\hat{a}_z$ gives the direction towards positive z-axis.

- **Differential length, area, and volume**

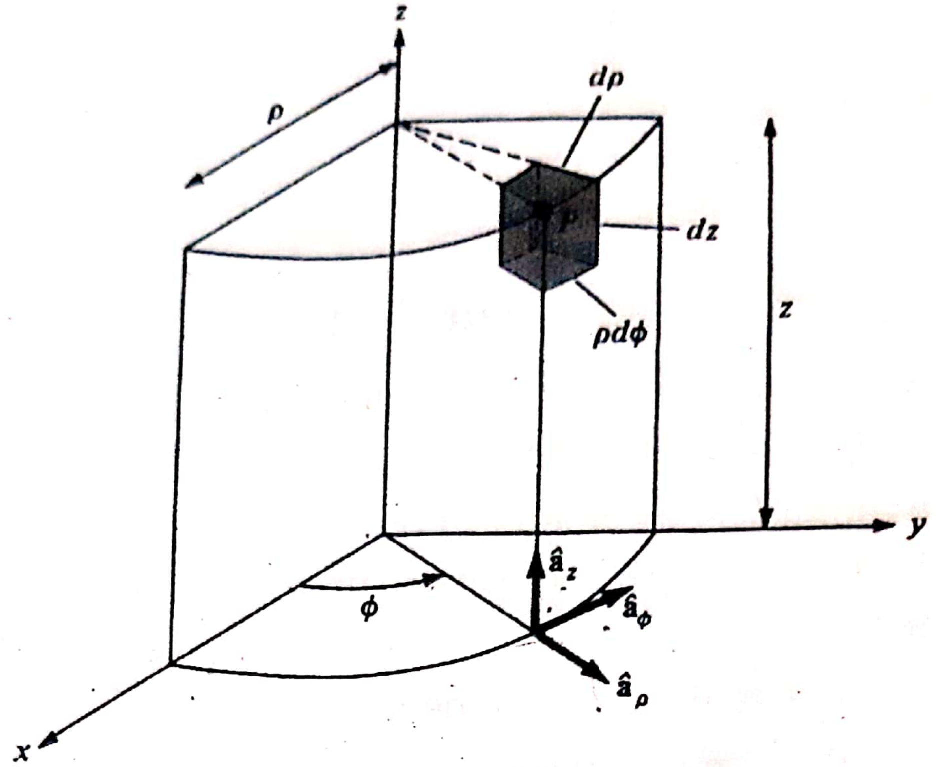


Figure 1.7 Differential elements in the cylindrical coordinate system.

- Differential length is given by: $\boxed{d\vec{l} = d\rho\, \hat{a}_\rho + \rho\, d\phi\, \hat{a}_\phi + dz\, \hat{a}_z}$

- Differential normal area is given by:

$$d\vec{S} = \begin{cases} \rho\, d\phi\, dz\, \hat{a}_\rho \\ d\rho\, dz\, \hat{a}_\phi \\ \rho\, d\phi\, d\rho\, \hat{a}_z \end{cases}$$

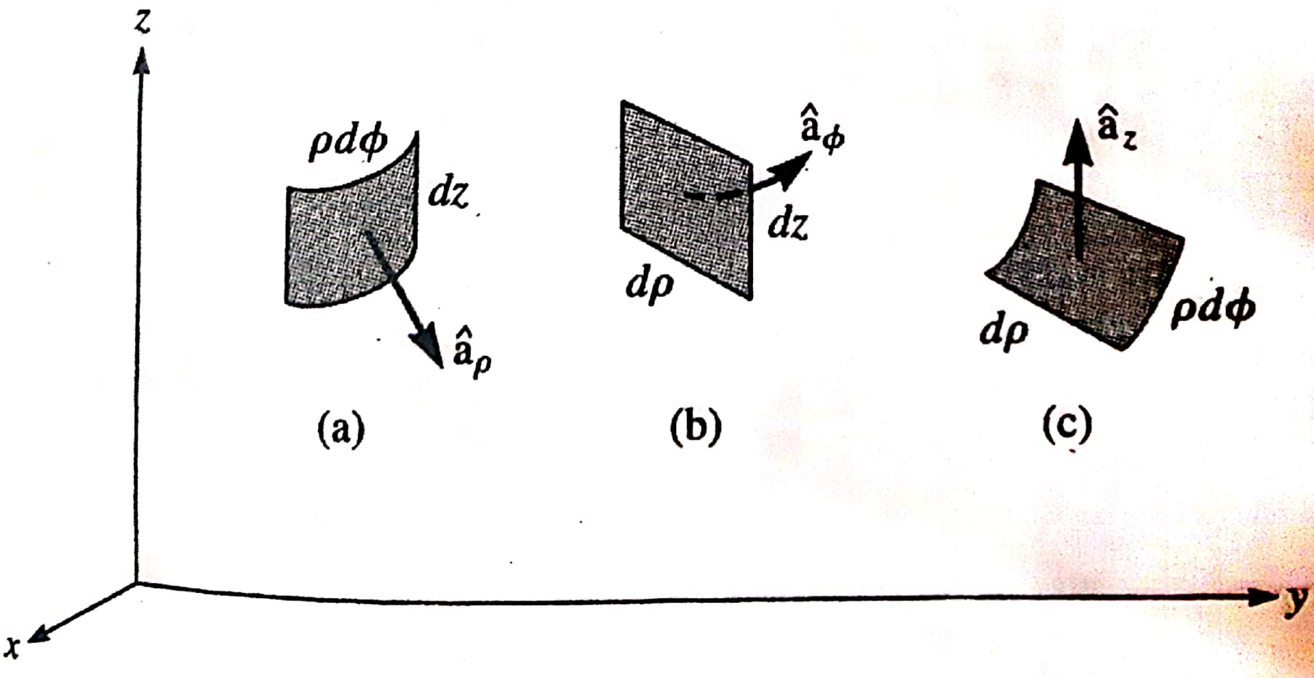


Figure 1.8 Diff...

- Differential volume is given by

$$\boxed{dv = \rho\, d\rho\, d\phi\, dz}$$

- Any vector $\vec{A}$ is written as

$$\boxed{\vec{A} = A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z}$$

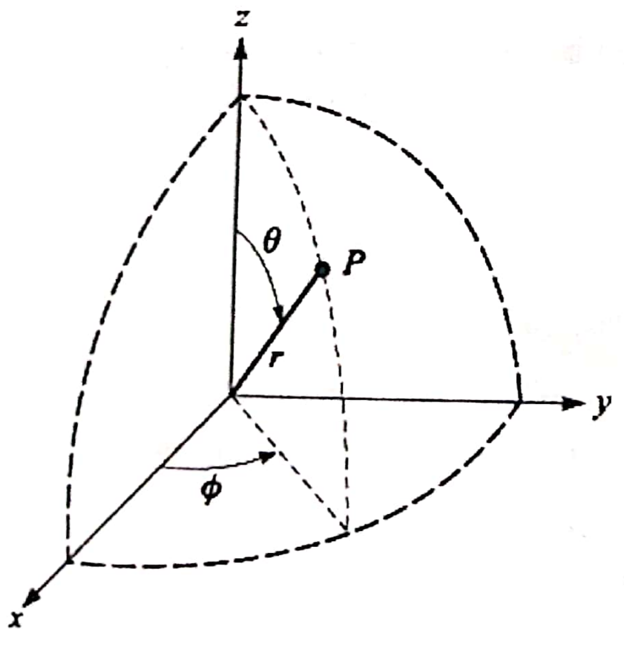
(iii) The Spherical Coordinate System

- The spherical coordinate system is most appropriate when one is dealing with problems having a degree of spherical symmetry.
- Any point P is specified as P(r, θ, ϕ) and the point is the intersection of three mutually perpendicular surfaces namely a sphere, a cone, and a plane.

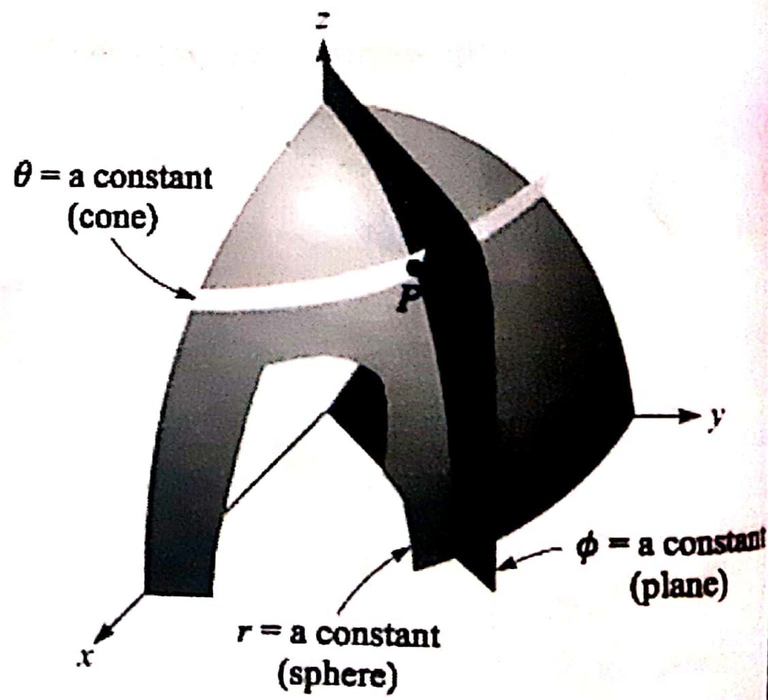
where r = radius of the sphere

θ = angle between the line joining origin and point P, and the z-axis.

ϕ = angle between the x-axis and the projection of line joining origin and point P in the z = 0 plane.



(a)



(b)

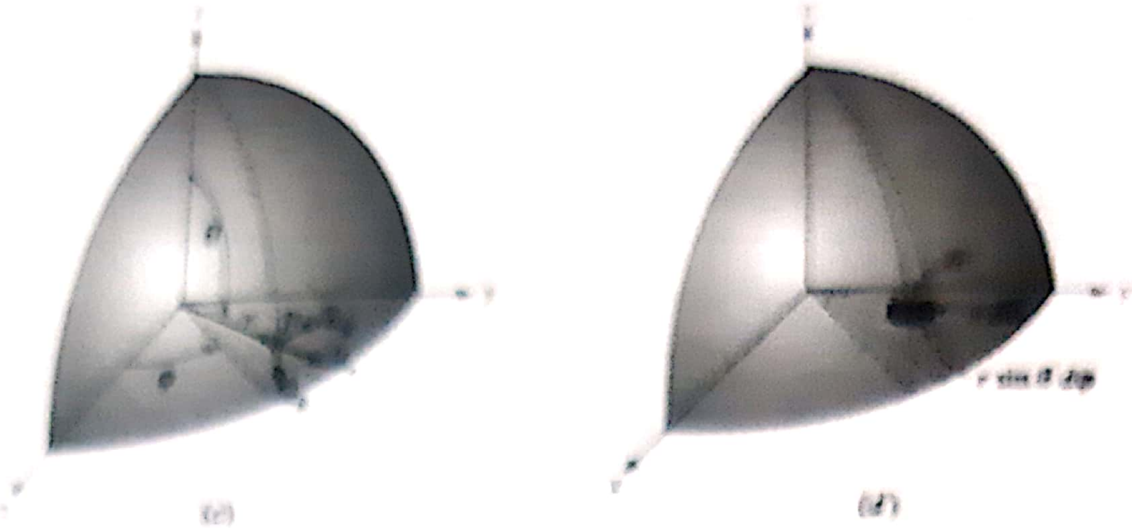


Figure 1.9 (a) The three spherical coordinates. (b) The three mutually perpendicular surfaces of the spherical coordinate system. (c) The three unit vectors of spherical coordinates. $\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi$. (d) The differential volume element in the spherical coordinate system.

- The three unit vectors of the spherical coordinate system are $\hat{a}_r$, $\hat{a}_\theta$, and $\hat{a}_\phi$.
  - The unit vector $\hat{a}_r$ is directed radially outward, normal to the sphere r = constant, and lies in the cone θ = constant and the plane ϕ = constant.
  - The unit vector $\hat{a}_\theta$ is normal to the conical surface, lies in the plane, and is tangent to the sphere.
  - The unit vector $\hat{a}_\phi$ is normal to the plane and tangent to both the cone and sphere.
- **Differential length, area, and volume**

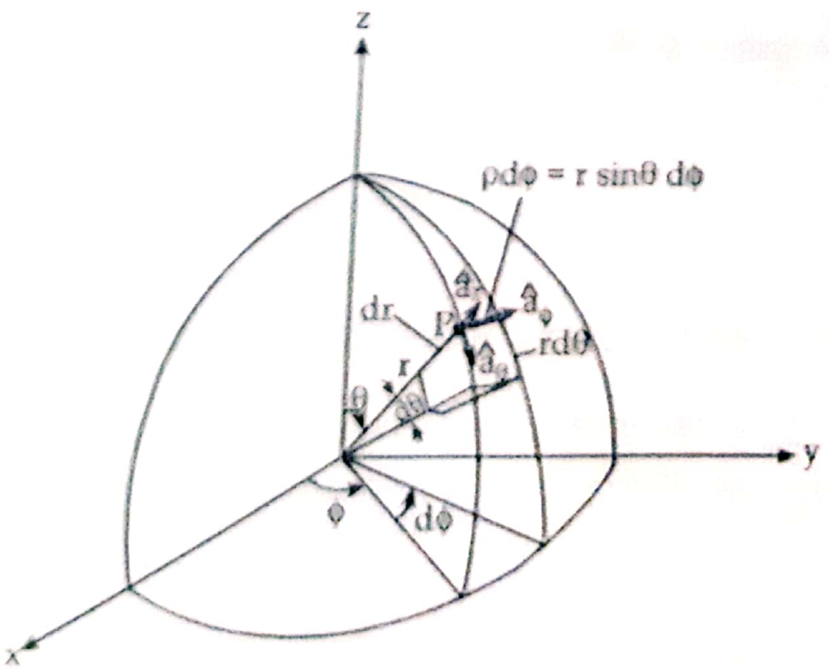


Figure 1.10 Differential elements in the spherical coordinate system.

- Differential length is given by

$$\boxed{d\vec{l} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi}$$

- Differential normal area is given by

$$\boxed{\begin{aligned} d\vec{S} &= r^2\sin\theta\,d\theta\,d\phi\,\hat{a}_r \\ &\quad r\sin\theta\,dr\,d\phi\,\hat{a}_\theta \\ &\quad r\,dr\,d\theta\,\hat{a}_\phi \end{aligned}}$$

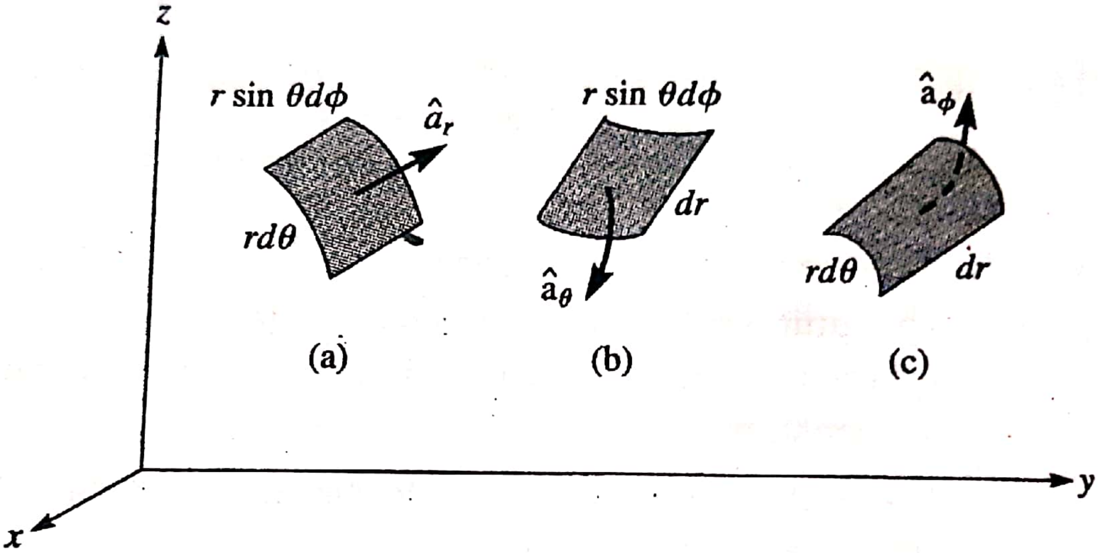


Figure 1.11 Differential normal areas in the spherical coordinate system.

- Differential volume is given by $\boxed{dv = r^2\sin\theta\,dr\,d\theta\,d\phi}$

- Any vector $\vec{A}$ is written as $\boxed{\vec{A} = A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_\phi\hat{a}_\phi}$

Note that, in order to construct a sphere of any radius r, we have to integrate θ from 0 to π radians (not from 0 to 2π radians) and φ from 0 to 2π. A little reflection with stress of geometry will convince us that a half-circle (not a full-circle) rotated about the z-axis through 2π radians (φ from 0 to 2π) generates a sphere.

A picture is worth a thousand words if we just know what picture to draw. Figure 1.12 will favour this popular saying.

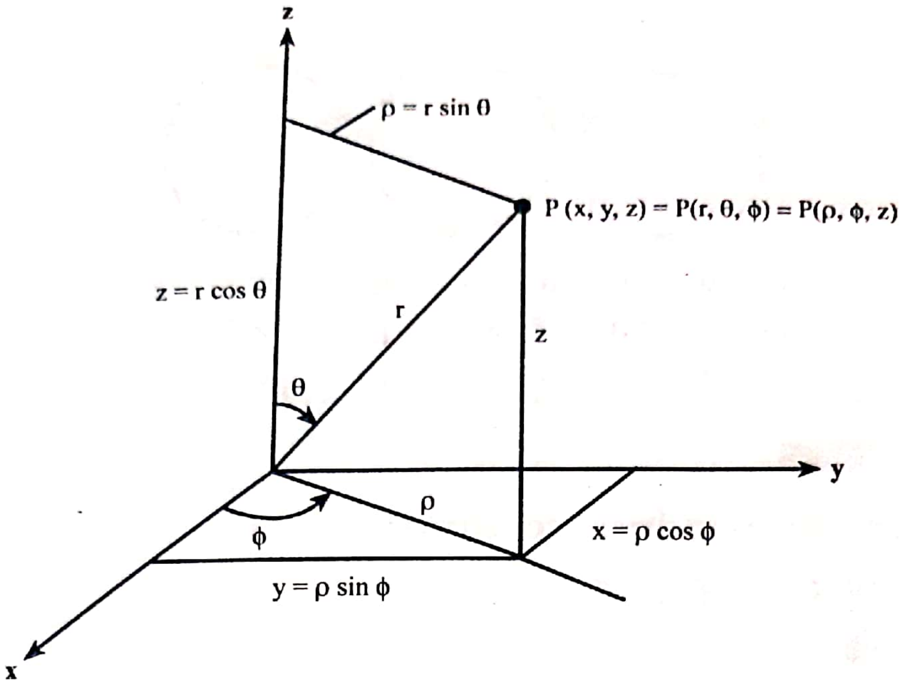


Figure 1.12 Relationships between (x, y, z), (r, θ, ϕ), and (ρ, ϕ, z).

IMPORTANT RESULTS

- **Dot products of unit vectors in spherical and rectangular coordinate system**

	$\hat{a}_r$	$\hat{a}_\theta$	$\hat{a}_\phi$
$\hat{a}_x \cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\hat{a}_y \cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\hat{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0

- **Dot products of unit vectors in cylindrical and rectangular coordinate system**

	$\hat{a}_\rho$	$\hat{a}_\phi$	$\hat{a}_z$
$\hat{a}_x \cdot$	$\cos\phi$	$-\sin\phi$	0
$\hat{a}_y \cdot$	$\sin\phi$	$\cos\phi$	0
$\hat{a}_z \cdot$	0	0	1

- **Dot products of unit vectors in cylindrical and spherical coordinate system**

	$\hat{a}_\rho$	$\hat{a}_\phi$	$\hat{a}_z$
$\hat{a}_r \cdot$	$\sin\theta$	0	$\cos\theta$
$\hat{a}_\theta \cdot$	$\cos\theta$	0	$-\sin\theta$
$\hat{a}_\phi \cdot$	0	1	0

- **Cross products**

$\hat{a}_x \times \hat{a}_y = \hat{a}_z$ $\quad$ $\hat{a}_\rho \times \hat{a}_\phi = \hat{a}_z$ $\quad$ $\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi$

$\hat{a}_y \times \hat{a}_z = \hat{a}_x$ $\quad$ $\hat{a}_\phi \times \hat{a}_z = \hat{a}_\rho$ $\quad$ $\hat{a}_\theta \times \hat{a}_\phi = \hat{a}_r$

$\hat{a}_z \times \hat{a}_x = \hat{a}_y$ $\quad$ $\hat{a}_z \times \hat{a}_\rho = \hat{a}_\phi$ $\quad$ $\hat{a}_\phi \times \hat{a}_r = \hat{a}_\theta$

TRANSFORMATIONS

(1) Rectangular and cylindrical coordinate system

(i) Cylindrical to rectangular

$$x = \rho\cos\phi$$

$$y = \rho\sin\phi$$

$$z = z$$

(ii) Rectangular to cylindrical

$$\rho = \sqrt{x^2 + y^2}\ (\rho \geq 0)$$

$$\phi = \tan^{-1}\frac{y}{x}$$

$$z = z$$

The proper value of ϕ is determined by inspecting the signs of x and y and then locating the quadrant.

(2) Rectangular and spherical coordinate system

(i) Spherical to rectangular

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

(ii) Rectangular to spherical

$$r = \sqrt{x^2 + y^2 + z^2}\ \ (r \geq 0)$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}}\ (0° \leq \theta \leq 180°)$$

$$\phi = \tan^{-1}\frac{y}{x}$$

The proper values of θ, ϕ are determined by inspecting the signs of x, y and z and then locating the quadrant.

(3) Cylindrical and spherical coordinate system

(i) Cylindrical to spherical

$$\phi = \phi$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2}$$

$$\theta = \cos^{-1}\frac{z}{r} = \cos^{-1}\frac{z}{\sqrt{\rho^2 + z^2}} \Rightarrow \cos\theta = \frac{z}{\sqrt{\rho^2 + z^2}}$$

(ii) Spherical to cylindrical

$$\phi = \phi$$

$$\rho = \sqrt{x^2 + y^2} = \sqrt{(r \sin\theta \cos\phi)^2 + (r \sin\theta \sin\phi)^2} = r \sin\theta$$

$$z = r \cos\theta$$

PROBLEMS SOLVED AND SCRAMBLED

1. If $\vec{A} = 3\,\hat{a}_r + 2\,\hat{a}_\theta - 6\,\hat{a}_\phi$ and $\vec{B} = 4\,\hat{a}_r + 3\,\hat{a}_\phi$, determine

a. $\vec{A} . \vec{B}$

b. $|\vec{A} \times \vec{B}|$

c. The vector component of $\vec{A}$ along $\hat{a}_z$ at $(1, \frac{\pi}{3}, \frac{5\pi}{4})$.

Solution:

a. $\vec{A} . \vec{B} = (3\,\hat{a}_r + 2\,\hat{a}_\theta - 6\,\hat{a}_\phi).(4\,\hat{a}_r + 3\,\hat{a}_\phi) = 12 - 18 = -6$

b. $$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_r & \hat{a}_\theta & \hat{a}_\phi \\ 3 & 2 & -6 \\ 4 & 0 & 3 \end{vmatrix} = +\,\hat{a}_r(6-0) - \hat{a}_\theta(9+24) + \hat{a}_\phi(0-8) = 6\,\hat{a}_r - 33\,\hat{a}_\theta - 8\,\hat{a}_\phi$$

$$\therefore |\vec{A} \times \vec{B}| = \sqrt{(6)^2 + (-33)^2 + (-8)^2} = 34.48$$

c. The vector component of $\vec{A}$ along $\hat{a}_z$ is $(\vec{A} . \hat{a}_z)\,\hat{a}_z$

$$\vec{A} . \hat{a}_z = (3\,\hat{a}_r + 2\,\hat{a}_\theta - 6\,\hat{a}_\phi).\,\hat{a}_z = 3\,\hat{a}_r . \hat{a}_z + 2\,\hat{a}_\theta . \hat{a}_z - 6\,\hat{a}_\phi . \hat{a}_z$$

$$= 3\cos\theta + 2(-\sin\theta) - 0 = 3\cos\theta - 2\sin\theta$$

Now, we express $\hat{a}_z$ in terms of $\hat{a}_r$, $\hat{a}_\theta$, $\hat{a}_\phi$ as

Let $\vec{F} = \hat{a}_z$

In spherical coordinate system,

$$\vec{F} = F_r\hat{a}_r + F_\theta\hat{a}_\theta + F_\phi\hat{a}_\phi$$

where $F_r = \vec{F}.\hat{a}_r = \hat{a}_z.\hat{a}_r = \cos\theta$

$F_\theta = \vec{F}.\hat{a}_\theta = \hat{a}_z.\hat{a}_\theta = -\sin\theta$

$F_\phi = \vec{F}.\hat{a}_\phi = \hat{a}_z.\hat{a}_\phi = 0$

$\therefore\ \vec{F} = \cos\theta\ \hat{a}_r + (-\sin\theta)\ \hat{a}_\theta + 0$

Thus, $\hat{a}_z = \cos\theta\ \hat{a}_r - \sin\theta\ \hat{a}_\theta$

The vector component $= (\vec{A}.\hat{a}_z)\ \hat{a}_z = (3\cos\theta - 2\sin\theta)(\cos\theta\ \hat{a}_r - \sin\theta\ \hat{a}_\theta)$

At $(1, \frac{\pi}{3}, \frac{5\pi}{4})$,

The vector component $= (3\cos 60^\circ - 2\sin 60^\circ)(\cos 60^\circ\ \hat{a}_r - \sin 60^\circ\ \hat{a}_\theta)$

$= -0.323\ (0.5\ \hat{a}_r - 0.866\ \hat{a}_\theta) = -0.116\ \hat{a}_r + 0.2009\ \hat{a}_\theta$

2. Express the unit vector $\hat{a}_x$ in spherical components at $\rho = 2.5, \phi = 0.7$ rad, $z = 1.5$.

Solution:

Let $\vec{F} = \hat{a}_x$

In spherical coordinate system, we have

$$\vec{F} = F_r\ \hat{a}_r + F_\theta\hat{a}_\theta + F_\phi\hat{a}_\phi$$

where $F_r = \vec{F}.\hat{a}_r = \hat{a}_x.\hat{a}_r = \sin\theta\cos\phi$

$F_\theta = \vec{F}.\hat{a}_\theta = \hat{a}_x.\hat{a}_\theta = \cos\theta\cos\phi$

$F_\phi = \vec{F}.\hat{a}_\phi = \hat{a}_x.\hat{a}_\phi = -\sin\phi$

For $(\rho = 2.5, \phi = 0.7$ rad, $z = 1.5)$,

$\phi = 0.7$ rad $= 40.107^\circ$

$$\theta = \cos^{-1}\frac{z}{\sqrt{\rho^2+z^2}} = \cos^{-1}\frac{1.5}{\sqrt{(2.5)^2+(1.5)^2}} = 59.036^\circ$$

$\therefore F_r = \sin\theta \cos\phi = \sin 59.036^\circ \cos 40.107^\circ = 0.6558$

$F_\theta = \cos\theta \cos\phi = \text{coas } 59.036^\circ \cos 40.107^\circ = 0.3935$

$F_\phi = -\sin\phi = -\sin 40.107^\circ = -0.6442$

$\therefore \vec{F}_{sph} = 0.6558\, \hat{a}_r + 0.3935\, \hat{a}_\theta - 0.6442\, \hat{a}_\phi$

Hence, $\hat{a}_x = 0.6558\, \hat{a}_r + 0.3935\, \hat{a}_\theta - 0.6442\, \hat{a}_\phi$

3. **Transform the vector (a) $\vec{A} = y\, \hat{a}_x + (x+y)\, \hat{a}_z$ at P(−2, 6, 3) in cylindrical system (b) $\vec{A} = 3\hat{a}_\rho - \hat{a}_\phi$ at Q(3, 4, −1) in Cartesian coordinate system.** *[2068 Jestha]*

Solution:

a. $\rho = \sqrt{(-2)^2 + (6)^2} = \sqrt{4 + 36} = \sqrt{40}$

$\phi = \tan^{-1}\left(\frac{6}{-2}\right) = 180^\circ - \tan^{-1}\left(\frac{6}{2}\right) = 108.435^\circ$

At point (−2, 6, 3), $\vec{A} = +6\hat{a}_x + 4\hat{a}_z$

In cylindrical coordinate system,

$\vec{A} = A_\rho\, \hat{a}_\rho + A_\phi\, \hat{a}_\phi + A_z\, \hat{a}_z$

where $A_\rho = \vec{A}\, .\, \hat{a}_\rho = 6\hat{a}_x\, .\, \hat{a}_\rho + 4\, \hat{a}_z.\, \hat{a}_\rho = 6\cos\phi + 4\times 0 = 6\cos(108.435^\circ) = -1.897$

$A_\phi = \vec{A}\, .\, \hat{a}_\phi = 6\, \hat{a}_x\, .\, \hat{a}_\phi + 4\, \hat{a}_z\, .\, \hat{a}_\phi = -6\sin\phi + 4\times 0 = -6\sin(108.435^\circ) = -5.69$

$A_z = 4$

$\therefore \vec{A}(\rho, \phi, z) = -1.897\, \hat{a}_\rho - 5.69\, \hat{a}_\phi + 4\, \hat{a}_z$

b. $\vec{A} = 3\hat{a}_\rho - \hat{a}_\phi$ at Q(3, 4, −1)

$\rho = \sqrt{(3)^2 + (4)^2} = \sqrt{9+16} = 5,\ \phi = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ,\ z = -1$

In Cartesian coordinate system,

$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$

where $A_x = \vec{A}\, .\, \hat{a}_x = 3\, \hat{a}_\rho.\, \hat{a}_x - \hat{a}_\phi\, .\, \hat{a}_x = 3\cos\phi - (-\sin\phi)$

$= 3\cos 53.13^\circ + \sin 53.13^\circ = 2.6$

$A_y = \vec{A}\, .\, \hat{a}_y = 3\, \hat{a}_\rho.\, \hat{a}_y - \hat{a}_\phi\, .\, \hat{a}_y = 3\sin\phi - \cos\phi$

$= 3\sin 53.13^\circ - \cos 53.13^\circ = 1.799$

$A_z = 0$

$\therefore \vec{A}(x, y, z) = 2.6\hat{a}_x + 1.799\, \hat{a}_y$

4. **Transform the vector field $\vec{B} = y\hat{a}_x - x\hat{a}_y + z\hat{a}_z$ into cylindrical coordinates.**

Solution:

For cylindrical coordinates, $\vec{B} = B_\rho\,\hat{a}_\rho + B_\phi\,\hat{a}_\phi + B_z\,\hat{a}_z$

where $B_\rho = \vec{B}.\hat{a}_\rho = y\,\hat{a}_x.\,\hat{a}_\rho - x\,\hat{a}_y.\,\hat{a}_\rho + z\,\hat{a}_z.\,\hat{a}_\rho = y\cos\phi - x\sin\phi$

$= \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0$

$B_\phi = \vec{B}.\hat{a}_\phi = y\,(\hat{a}_x.\,\hat{a}_\phi) - x\,(\hat{a}_y.\,\hat{a}_\phi) + z\,(\hat{a}_z.\,\hat{a}_\phi) = -y\sin\phi - x\cos\phi$

$= -\rho(\sin^2\phi + \cos^2\phi) = -\rho$

$B_z = z$

$\therefore\quad \vec{B} = -\rho\,\hat{a}_\phi + z\,\hat{a}_z$

5. **(a) Give the Cartesian coordinates of the point C ($\rho = 4.4$, $\phi = -115°$, $z = 2$)**
   (b) Give the cylindrical coordinates of point D ($x = -3.1$, $y = 2.6$, $z = -3$)
   (c) Specify the distance from C to D.

Solution:

a. For Cartesian coordinates, $x = \rho\cos\phi = 4.4\cos(-115°) = -1.86$

$y = \rho\sin\phi = 4.4\sin(-115°) = -3.987$

$z = 2$

$\therefore$ C ($x = -1.86$, $y = -3.987$, $z = 2$)

b. For cylindrical coordinates,

$\rho = \sqrt{x^2 + y^2} = \sqrt{(-3.1)^2 + (2.6)^2} = 4.045$

$\phi = 180° - \tan^{-1}\left(\frac{2.6}{3.1}\right)$ [$\because$ the point (x, y) lies in the second quadrant]

$= 140.01°$

$z = -3$

$\therefore$ D ($\rho = 4.045$, $\phi = 140.01°$, $z = -3$)

c. Distance from C to D

$d_{CD} = \sqrt{(x_D - x_C)^2 + (y_D - y_C)^2 + (z_D - z_C)^2}$

$= \sqrt{(-3.1 + 1.86)^2 + (2.6 + 3.987)^2 + (-3 - 2)^2} = 8.36$

6. **(a) Transform to cylindrical coordinates**

   (i) $\vec{F} = 10\hat{a}_x - 8\hat{a}_y + 6\,\hat{a}_z$ at point P (10, –8, 6)

   (ii) $\vec{G} = (2x + y)\,\hat{a}_x - (y - 4x)\,\hat{a}_y$ at point Q (ρ, ϕ, z)

(b) Give the ... vector $H = 20\hat{a}_\rho - 10\hat{a}_\phi + 3\hat{a}_z$ at point P (x = 5, y = 2, z = −1)

Solution:

(a)

(i) In cylindrical coordinate system,

$$\vec{F} = F_\rho\hat{a}_\rho + F_\phi\hat{a}_\phi + F_z\hat{a}_z \quad \text{.......................} \quad (i)$$

At point P (10, −8, 6),

$$\phi = 360° - \tan^{-1}\left(\frac{8}{10}\right) \quad [\because \text{ the point lies in the } 4^{th} \text{ quadrant}]$$

$$= 321.34°$$

$$F_\rho = \vec{F}.\hat{a}_\rho = (10\hat{a}_x - 8\hat{a}_y + 6\hat{a}_z).\ \hat{a}_\rho = 10(\hat{a}_x.\hat{a}_\rho) - 8(\hat{a}_y.\hat{a}_\rho) + 6(\hat{a}_z.\hat{a}_\rho)$$

$$= 10\cos\phi - 8\sin\phi + 0$$

$$= 10\cos 321.34° - 8\sin 321.34° = 12.8$$

$$F_\phi = \vec{F}.\hat{a}_\phi = (10\hat{a}_x - 8\hat{a}_y + 6\hat{a}_z).\ \hat{a}_\phi = 10(\hat{a}_x.\hat{a}_\phi) - 8(\hat{a}_y.\hat{a}_\phi) + 6(\hat{a}_z.\hat{a}_\phi)$$

$$= -10\sin\phi - 8\cos\phi + 0$$

$$= -10\sin 321.34° - 8\cos 321.34° \cong 0$$

$$F_z = \vec{F}.\hat{a}_z = (10\hat{a}_x - 8\hat{a}_y + 6\hat{a}_z).\ \hat{a}_z = 10(\hat{a}_x.\hat{a}_z) - 8(\hat{a}_y.\hat{a}_z) + 6(\hat{a}_z.\hat{a}_z)$$

$$= 0 - 0 + 6 = 6$$

From equation (i),

$$\vec{F} = 12.8\ \hat{a}_\rho + 6\ \hat{a}_z$$

(ii) In cylindrical coordinate system,

$$\vec{G} = G_\rho\hat{a}_\rho + G_\phi\hat{a}_\phi + G_z\hat{a}_z \quad \text{.......................} \quad (i)$$

$$G_\rho = \vec{G}.\ \hat{a}_\rho = [(2x + y)\ \hat{a}_x - (y - 4x)\ \hat{a}_y].\ \hat{a}_\rho$$

$$= (2x + y)\ (\hat{a}_x.\hat{a}_\rho) - (y - 4x)\ (\hat{a}_y.\hat{a}_\rho)$$

$$= (2\rho\cos\phi + \rho\sin\phi)\cos\phi - (\rho\sin\phi - 4\rho\cos\phi)\sin\phi$$

$$= 2\rho\cos^2\phi + \rho\sin\phi\cos\phi - \rho\sin^2\phi + 4\rho\cos\phi\sin\phi$$

$$= 2\rho\cos^2\phi + 5\rho\sin\phi\cos\phi - \rho\sin^2\phi$$

$$G_\phi = \vec{G}.\hat{a}_\phi = [(2x + y)\hat{a}_x - (y - 4x)\hat{a}_y].\hat{a}_\phi$$
$$= (2\rho\cos\phi + \rho\sin\phi)(\hat{a}_x.\hat{a}_\phi) - (\rho\sin\phi - 4\rho\cos\phi)(\hat{a}_y.\hat{a}_\phi)$$
$$= (2\rho\cos\phi + \rho\sin\phi)(-\sin\phi) - (\rho\sin\phi - 4\rho\cos\phi)\cos\phi$$
$$= -2\rho\sin\phi\cos\phi - \rho\sin^2\phi - \rho\sin\phi\cos\phi + 4\rho\cos^2\phi$$
$$= 4\rho\cos^2\phi - 3\rho\sin\phi\cos\phi - \rho\sin^2\phi$$

$$G_z = \vec{G}.\hat{a}_z = [(2x + y)\hat{a}_x - (y - 4x)\hat{a}_y].\hat{a}_z = 0$$

From equation (i),

$$\vec{G} = (2\rho\cos^2\phi - \rho\sin^2\phi + 5\rho\sin\phi\cos\phi)\hat{a}_\rho + (4\rho\cos^2\phi - \rho\sin^2\phi - 3\rho\sin\phi\cos\phi)\hat{a}_\phi$$

(b) In Cartesian coordinate system,

$$\vec{H} = H_x\hat{a}_x + H_y\hat{a}_y + H_z\hat{a}_z \qquad \text{.......................} \text{ (i)}$$

$$\phi = \tan^{-1}\left(\frac{2}{5}\right) = 21.80^\circ$$

$$H_x = \vec{H}.\hat{a}_x = (20\hat{a}_\rho - 10\hat{a}_\phi + 3\hat{a}_z).\hat{a}_x = 20(\hat{a}_\rho.\hat{a}_x) - 10(\hat{a}_\phi.\hat{a}_x) + 3(\hat{a}_z.\hat{a}_x)$$
$$= 20\cos 21.80^\circ - 10(-\sin 21.80^\circ) = 22.28$$

$$H_y = \vec{H}.\hat{a}_y = (20\hat{a}_\rho - 10\hat{a}_\phi + 3\hat{a}_z).\hat{a}_y = 20(\hat{a}_\rho.\hat{a}_y) - 10(\hat{a}_\phi.\hat{a}_y) + 3(\hat{a}_z.\hat{a}_y)$$
$$= 20\sin 21.80^\circ - 10\cos 21.80^\circ = -1.857$$

$$H_z = \vec{H}.\hat{a}_z = (20\hat{a}_\rho - 10\hat{a}_\phi + 3\hat{a}_z).\hat{a}_z = 20(\hat{a}_\rho.\hat{a}_z) - 10(\hat{a}_\phi.\hat{a}_z) + 3(\hat{a}_z.\hat{a}_z)$$
$$= 3$$

From equation (i),

$$\vec{H} = 22.28\hat{a}_x - 1.857\hat{a}_y + 3\hat{a}_z$$

7. Transform the vector field $\vec{G} = \left(\frac{xz}{y}\right)\hat{a}_x$ into spherical components of variables.

Solution:

In spherical coordinate system,

$$\vec{G} = G_r\hat{a}_r + G_\theta\hat{a}_\theta + G_\phi\hat{a}_\phi \qquad \text{................} \text{ (i)}$$

where $G_r = \vec{G}.\hat{a}_r = \frac{xz}{y}(\hat{a}_x.\hat{a}_r) = \frac{xz}{y}\sin\theta\cos\phi = r\sin\theta\cos\theta\frac{\cos^2\phi}{\sin\phi}$

$$G_\theta = \vec{G}.\hat{a}_\theta = \frac{xz}{y}(\hat{a}_x.\,\hat{a}_\theta) = \frac{xz}{y}\cos\theta\cos\phi = r\cos^2\theta\frac{\cos^2\phi}{\sin\phi}$$

$$G_\phi = \vec{G}.\hat{a}_\phi = \frac{xz}{y}(\hat{a}_x.\,\hat{a}_\phi) = \frac{xz}{y}(-\sin\phi) = -r\cos\theta\cos\phi$$

From equation (i),

$$\vec{G} = r\cos\theta\cos\phi\,(\sin\theta\cot\phi\,\hat{a}_r + \cos\theta\cot\phi\,\hat{a}_\theta - \hat{a}_\phi)$$

8. **Given the two points C (–3, 2, 1) & D (r = 5, θ = 20°, φ = – 70°) find**
(a) the spherical coordinates of C
(b) Cartesian coordinates of D
(d) the distance from C to D

Solution:

(a) C (x = – 3, y = 2, z = 1)

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{9 + 4 + 1} = \sqrt{14} = 3.7416$$

$$\theta = \cos^{-1}\frac{z}{r} = \cos^{-1}\frac{1}{3.7416} = 74.498^\circ$$

$$\phi = 180^\circ - \tan^{-1}\left(\frac{2}{3}\right) = 146.30^\circ$$ [∵ the point (x, y) lies in the second quadrant]

∴ C (r = 3.873, θ = 74.498°, φ = 146.30°)

(b) D (r = 5, θ = 20°, φ = – 70°)

$x = r\sin\theta\cos\phi = 5\sin 20^\circ\cos(-70^\circ) = 0.585$

$y = r\sin\theta\sin\phi = 5\sin 20^\circ\sin(-70^\circ) = -1.607$

$z = r\cos\theta = 5\cos 20^\circ = 4.698$

∴ D (x = 0.585, y = – 1.607, z = 4.698)

(c) Distance from C to D

$$d_{CD} = \sqrt{(x_d - x_c)^2 + (y_d - y_c)^2 + (z_d - z_c)^2}$$

$$= \sqrt{(0.585 + 3)^2 + (-1.607 - 2)^2 + (4.698 - 1)^2} = 6.287$$

9. **Express vector $\vec{B} = \frac{10}{r}\,\hat{a}_r + r\cos\theta\,\hat{a}_\theta + \hat{a}_\phi$ in Cartesian and cylindrical coordinates. Find $\vec{B}$ (–3, 4, 0) and $\vec{B}$ $(5, \frac{\pi}{2}, -2)$.**

Solution:

In Cartesian coordinates,

$$\vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$$

where $B_x = \vec{B} \cdot \hat{a}_x = \left(\frac{10}{r}\hat{a}_r + r\cos\theta\,\hat{a}_\theta + \hat{a}_\phi\right).\hat{a}_x$

$$= \frac{10}{r}(\hat{a}_r.\hat{a}_x) + r\cos\theta\ (\hat{a}_\theta.\hat{a}_x) + (\hat{a}_\phi.\hat{a}_x)$$

$$= \frac{10}{r}(\sin\theta\cos\phi) + r\cos\theta\,(\cos\theta\cos\phi) + (-\sin\phi)$$

$$= \frac{10}{r}\sin\theta\cos\phi + r\cos^2\theta\cos\phi - \sin\phi$$

$$B_y = \vec{B} \cdot \hat{a}_y = \left(\frac{10}{r}\hat{a}_r + r\cos\theta\,\hat{a}_\theta + \hat{a}_\phi\right).\hat{a}_y$$

$$= \frac{10}{r}(\hat{a}_r.\hat{a}_y) + r\cos\theta\ (\hat{a}_\theta.\hat{a}_y) + (\hat{a}_\phi.\hat{a}_y)$$

$$= \frac{10}{r}(\sin\theta\sin\phi) + r\cos\theta\,(\cos\theta\sin\phi) + \cos\phi$$

$$= \frac{10}{r}\sin\theta\sin\phi + r\cos^2\theta\ \sin\phi + \cos\phi$$

$$B_z = \vec{B} \cdot \hat{a}_z = \left(\frac{10}{r}\hat{a}_r + r\cos\theta\,\hat{a}_\theta + \hat{a}_\phi\right).\hat{a}_z$$

$$= \frac{10}{r}(\hat{a}_r.\hat{a}_z) + r\cos\theta\ (\hat{a}_\theta.\hat{a}_z) + (\hat{a}_\phi.\hat{a}_z)$$

$$= \frac{10}{r}(\cos\theta) + r\cos\theta\,(-\sin\theta) + 0 \quad = \frac{10}{r}\cos\theta - r\cos\theta\sin\theta$$

But $r = \sqrt{x^2+y^2+z^2}$, $\cos\theta = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$, $\tan\phi = \dfrac{y}{x}$

$$\sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}, \quad \sin\phi = \frac{y}{\sqrt{x^2+y^2}}$$

$$\cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

$$\therefore B_x = \left(\frac{10}{\sqrt{x^2+y^2+z^2}}\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\frac{x}{\sqrt{x^2+y^2}}\right) + \left(\sqrt{x^2+y^2+z^2}\,\frac{z^2}{x^2+y^2+z^2}\frac{x}{\sqrt{x^2+y^2}}\right) - \frac{y}{\sqrt{x^2+y^2}}$$

$$= \frac{10x}{x^2+y^2+z^2} + \frac{xz^2}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} - \frac{y}{\sqrt{x^2+y^2}}$$

$$B_y = \frac{10y}{x^2+y^2+z^2} + \frac{yz^2}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} + \frac{x}{\sqrt{x^2+y^2}}$$

$$B_z = \frac{10z}{x^2+y^2+z^2} - \frac{z\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

Hence,

$$\vec{B} = \left(\frac{10x}{x^2+y^2+z^2} + \frac{xz^2}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} - \frac{y}{\sqrt{x^2+y^2}}\right)\hat{a}_x$$

$$+\left(\frac{10y}{x^2+y^2+z^2} + \frac{yz^2}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} + \frac{x}{\sqrt{x^2+y^2}}\right)\hat{a}_y$$

$$+\left(\frac{10z}{x^2+y^2+z^2} - \frac{z\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\right)\hat{a}_z$$

In cylindrical coordinates,

$$\vec{B} = B_\rho\,\hat{a}_\rho + B_\phi\,\hat{a}_\phi + B_z\,\hat{a}_z$$

where $B_\rho = \vec{B}\cdot\hat{a}_\rho = \left(\frac{10}{r}\hat{a}_r + r\cos\theta\,\hat{a}_\theta + \hat{a}_\phi\right).\hat{a}_\rho$

$$= \frac{10}{r}(\hat{a}_r.\hat{a}_\rho) + r\cos\theta\,(\hat{a}_\theta.\hat{a}_\rho) + (\hat{a}_\phi.\hat{a}_\rho)$$

$$= \frac{10}{r}(\sin\theta) + r\cos\theta\,(\cos\theta) + 0 \quad = \quad \frac{10}{r}\sin\theta + r\cos^2\theta$$

$$B_\phi = \vec{B}\cdot\hat{a}_\phi = \left(\frac{10}{r}\hat{a}_r + r\cos\theta\,\hat{a}_\theta + \hat{a}_\phi\right).\hat{a}_\phi$$

$$= \frac{10}{r}(\hat{a}_r.\hat{a}_\phi) + r\cos\theta\,(\hat{a}_\theta.\hat{a}_\phi) + (\hat{a}_\phi.\hat{a}_\phi) = 0 + 0 + 1 = 1$$

$$B_z = \vec{B}\cdot\hat{a}_z = \left(\frac{10}{r}\hat{a}_r + r\cos\theta\,\hat{a}_\theta + \hat{a}_\phi\right).\hat{a}_z$$

$$= \frac{10}{r}(\hat{a}_r.\hat{a}_z) + r\cos\theta\,(\hat{a}_\theta.\hat{a}_z) + (\hat{a}_\phi.\hat{a}_z)$$

$$= \frac{10}{r}(\cos\theta) + r\cos\theta\,(-\sin\theta) + 0 \quad = \quad \frac{10}{r}\cos\theta - r\cos\theta\sin\theta$$

But $r = \sqrt{\rho^2+z^2}$, $\quad \cos\theta = \frac{z}{\sqrt{\rho^2+z^2}}$

$$\tan\theta = \frac{\rho}{z}$$

$$\therefore\ B_\rho = \frac{10}{\sqrt{\rho^2+z^2}}\frac{\rho}{\sqrt{\rho^2+z^2}} + \sqrt{\rho^2+z^2}\frac{z^2}{\rho^2+z^2} = \frac{10\rho}{\rho^2+z^2} + \frac{z^2}{\sqrt{\rho^2+z^2}}$$

$$B_\phi = 1$$

$$B_z = \frac{10}{\sqrt{\rho^2+z^2}}\frac{z}{\sqrt{\rho^2+z^2}} - \sqrt{\rho^2+z^2}\frac{z}{\sqrt{\rho^2+z^2}}\frac{\rho}{\sqrt{\rho^2+z^2}} = \frac{10z}{\rho^2+z^2} - \frac{\rho z}{\sqrt{\rho^2+z^2}}$$

Hence,

$$\vec{B} = \left(\frac{10\rho}{\rho^2+z^2} + \frac{z^2}{\sqrt{\rho^2+z^2}}\right)\hat{a}_\rho + \hat{a}_\phi + \left(\frac{10z}{\rho^2+z^2} - \frac{\rho z}{\sqrt{\rho^2+z^2}}\right)\hat{a}_z$$

Thus,

$$\vec{B}(-3, 4, 0) = \left(\frac{-30}{25} + 0 - \frac{4}{5}\right)\hat{a}_x + \left(\frac{40}{25} + 0 - \frac{3}{5}\right)\hat{a}_y + (0-0)\,\hat{a}_z$$

$$= -2\hat{a}_x + \hat{a}_y$$

$$\vec{B}\left(5, \frac{\pi}{2}, -2\right) = \left(\frac{50}{29} + \frac{4}{\sqrt{29}}\right)\hat{a}_\rho + \hat{a}_\phi + \left(\frac{-20}{29} + \frac{10}{\sqrt{29}}\right)\hat{a}_z$$

$$= 2.467\,\hat{a}_\rho + \hat{a}_\phi + 1.167\,\hat{a}_z$$

10. Given a vector field $\vec{D} = r\sin\phi\,\hat{a}_r - \frac{1}{r}\sin\theta\cos\phi\,\hat{a}_\theta + r^2\hat{a}_\phi$, determine

(a) $\vec{D}$ at P (10, 150°, 330°)

(b) The component of $\vec{D}$ tangential to the spherical surface r = 10 at P.

(c) A unit vector at P perpendicular to $\vec{D}$ and tangential to the cone $\theta = 150°$

Solution:

(a) At P($r = 10$, $\theta = 150°$, $\phi = 330°$),

$$\vec{D} = 10\sin 330°\,\hat{a}_r - \frac{1}{10}\sin 150°\cos 330°\,\hat{a}_\theta + 100\,\hat{a}_\rho$$

$$= -5\,\hat{a}_r - 0.043\,\hat{a}_\theta + 100\,\hat{a}_\phi$$

(b) $\vec{D} = \vec{D}_t + \vec{D}_N$

In our case, $\hat{a}_r$ is normal to the surface r = 10.

$$\vec{D}_N = r\sin\phi\,\hat{a}_r = -5\hat{a}_r$$

$$\therefore \vec{D}_t = \vec{D} - \vec{D}_N = -0.043\,\hat{a}_\theta + 100\,\hat{a}_\phi$$

(c) A vector at P perpendicular to $\vec{D}$ and tangential to the cone $\theta = 150°$ is the same as the vector as the vector perpendicular to both $\vec{D}$ and $\hat{a}_\theta$. Hence,

$$\vec{D} \times \hat{a}_\theta = \begin{vmatrix} \hat{a}_r & \hat{a}_\theta & \hat{a}_\phi \\ -5 & -0.043 & 100 \\ 0 & 1 & 0 \end{vmatrix} = -100\,\hat{a}_r - 5\,\hat{a}_\phi$$

A unit vector along this is

$$\hat{a} = \frac{-100\,\hat{a}_r - 5\,\hat{a}_\theta}{\sqrt{(-100)^2 + (-5)^2}} = -0.9988\,\hat{a}_r - 0.0499\,\hat{a}_\phi$$

11. **Transform $\vec{A}_c = x\,\hat{a}_x + xy\,\hat{a}_z$ at point (1, 2, 3) in Cartesian coordinate system to $\vec{A}_{cy}$ in cylindrical coordinate system.** *[2067 Mangsir]*

Solution:

Here, $\vec{A}_c = x\,\hat{a}_x + xy\,\hat{a}_z$

At (1, 2, 3), $\vec{A}_c = \hat{a}_x + 2\hat{a}_z$

In cylindrical coordinate system,

$$\vec{A}_{cy} = A_\rho\,\hat{a}_\rho + A_\phi\,\hat{a}_\phi + A_z\,\hat{a}_z \quad \text{......................} \quad (i)$$

where $A_\rho = \vec{A}_c \cdot \hat{a}_\rho = (\hat{a}_x + 2\hat{a}_z) \cdot \hat{a}_\rho$

$= \hat{a}_x \cdot \hat{a}_\rho + 2\,\hat{a}_z \cdot \hat{a}_\rho = \cos\phi + 2 \times 0 = \cos\phi$

$\phi = \tan^{-1}\frac{2}{1} = 63.43°$

$\therefore A_\rho = \cos 63.43° = 0.447$

$A_\phi = \vec{A}_c \cdot \hat{a}_\phi = (\hat{a}_x + 2\hat{a}_z) \cdot \hat{a}_\phi$

$= \hat{a}_x \cdot \hat{a}_\phi + 2\,\hat{a}_z \cdot \hat{a}_\phi$

$= -\sin\phi + 2 \times 0 = -\sin\phi = -\sin 63.43° = -0.894$

$A_z = \vec{A}_c \cdot \hat{a}_z = (\hat{a}_x + 2\hat{a}_z) \cdot \hat{a}_z$

$= \hat{a}_x \cdot \hat{a}_z + 2\hat{a}_z \cdot \hat{a}_z$

$= 0 + 2 = 2$

From equation (i),

$\vec{A}_{cy} = \cos 63.43°\,\hat{a}_\rho - \sin 63.43°\,\hat{a}_\phi + 2\hat{a}_z = 0.447\,\hat{a}_\rho - 0.894\,\hat{a}_\phi + 2\hat{a}_z$

12. Express the vector field $\vec{W} = (x - y)\,\hat{a}_y$ in cylindrical and spherical coordinates. [2068 Baishakh]

Solution:

a. For cylindrical coordinate system, $\vec{W}_{cy} = W_\rho\,\hat{a}_\rho + W_\phi\,\hat{a}_\phi + W_z\,\hat{a}_z$

where $W_\rho = \vec{W}\cdot\hat{a}_\rho = (x - y)\,\hat{a}_y\cdot\hat{a}_\rho = (\rho\cos\phi - \rho\sin\phi)\sin\phi$

$W_\phi = \vec{W}\cdot\hat{a}_\phi = (x - y)\,\hat{a}_y\cdot\hat{a}_\phi = (\rho\cos\phi - \rho\sin\phi)\cos\phi$

$Wz = \vec{W}\cdot\hat{a}_z = (x - y)\,\hat{a}_y\cdot\hat{a}_z = 0$

$\therefore\ \vec{W}_{cy} = \rho(\cos\phi - \sin\phi)\sin\phi\,\hat{a}_\rho + \rho(\cos\phi - \sin\phi)\cos\phi\,\hat{a}_\phi$

b. For spherical coordinate system,

$\vec{W}_{sp} = W_r\,\hat{a}_r + W_\theta\,\hat{a}_\theta + W_\phi\,\hat{a}_\phi$

where $W_r = \vec{W}\cdot\hat{a}_r = (x - y)\,\hat{a}_y\cdot\hat{a}_r$

$= (r\sin\theta\cos\phi - r\sin\theta\sin\phi)\sin\theta\sin\phi$

$W_\theta = \vec{W}\cdot\hat{a}_\theta = (x - y)\,\hat{a}_y\cdot\hat{a}_\theta$

$= (r\sin\theta\cos\phi - r\sin\theta\sin\phi)\cos\theta\sin\phi$

$W_\phi = \vec{W}\cdot\hat{a}_\phi = (x - y)\,\hat{a}_y\cdot\hat{a}_\phi$

$= (r\sin\theta\cos\phi - r\sin\theta\sin\phi)\cos\phi$

$\therefore\ \vec{W}_{sp} = r\sin\theta(\cos\phi - \sin\phi)\sin\theta\sin\phi\,\hat{a}_r + r\sin\theta(\cos\phi - \sin\phi)\cos\theta\sin\phi\,\hat{a}_\theta + r\sin\theta(\cos\phi - \sin\phi)\cos\phi\,\hat{a}_\phi$

13. Transform vector $\vec{A} = (r\cos\phi\,\hat{a}_r + z\hat{a}_z)$ at point $(1, 30^\circ, 2)$ in cylindrical co-ordinate system to a vector in spherical co-ordinate system. [2068 Chaitra]

Solution:

$P(1, 30^\circ, 2) \leftrightarrow P(r, \phi, z)$

In spherical coordinate system,

$\vec{A} = A_R\,\hat{a}_R + A_\theta\,\hat{a}_\theta + A_\phi\,\hat{a}_\phi$

where $A_R = \vec{A}\cdot\hat{a}_R = (r\cos\phi\,\hat{a}_r + z\,\hat{a}_z)\cdot\hat{a}_R$

$= r\cos\phi\,\hat{a}_r\cdot\hat{a}_R + z\,\hat{a}_z\cdot\hat{a}_R$

$= r\cos\phi\sin\theta + z\cos\theta$

$A_\theta = \vec{A} \cdot \hat{a}_\theta = (r\cos\phi\, \hat{a}_r + z\,\hat{a}_z) \cdot \hat{a}_\theta$

$= r\cos\phi\, \hat{a}_r \cdot \hat{a}_\theta + z\,\hat{a}_z \cdot \hat{a}_\theta$

$= r\cos\phi\cos\theta + z(-\sin\theta) = r\cos\phi\cos\theta - z\sin\theta$

$A_\phi = \vec{A} \cdot \hat{a}_\phi = (r\cos\phi\, \hat{a}_r + z\,\hat{a}_z) \cdot \hat{a}_\phi$

$= r\cos\phi\, \hat{a}_r \cdot \hat{a}_\phi + z\,\hat{a}_z \cdot \hat{a}_\phi = 0 + 0 = 0$

For point (1, 30°, 2), we can write

$\phi = \phi = 30°,$

$R = \sqrt{r^2 + z^2} = \sqrt{1^2 + 2^2} = \sqrt{5}$, $\theta = \cos^{-1}\frac{z}{R} = \cos^{-1}\frac{2}{\sqrt{5}} = 26.565°$

$A_R = 2.17614$, $A_\theta = -0.11982$, $A_\phi = 0$

$\therefore \vec{A} = 2.17614\, \hat{a}_R - 0.11982\, \hat{a}_\theta$

14. **Express a scalar potential field $V = x^2 + 2y^2 + 3z^2$ in spherical coordinates. Find the value of V at a point P(2, 60°, 90°).** *[2073 Chaitra]*

Solution:

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$V = (r\sin\theta\cos\phi)^2 + 2(r\sin\theta\sin\phi)^2 + 3(r\cos\theta)^2$

$= r^2\sin^2\theta\cos^2\phi + 2r^2\sin^2\theta\sin^2\phi + 3r^2\cos^2\theta$

$V(2, 60°, 90°) = (2)^2\sin^2 60°\cos^2 90° + 2(2)^2\sin^2 60°\sin^2 90° + 3(2)^2\cos^2 60° = 9$ V.

15. **Express in Cartesian components: (a) the vector at A($\rho = 4$, $\phi = 40°$, $z = -2$) that extends to B($\rho = 5$, $\phi = -110°$, $z = 2$) (b) a unit vector at B directed towards A.** *[2075 Ashwin]*

Solution:

a. The Cartesian coordinates for A($\rho = 4$, $\phi = 40°$, $z = -2$) is calculated as

$x = \rho\cos\phi = 4\cos 40° = 3.064$

$y = \rho\sin\phi = 4\sin 40° = 2.571$

$\therefore$ A($x = 3.064$, $y = 2.571$, $z = -2$)

The Cartesian coordinates for B($\rho = 5$, $\phi = -110°$, $z = 2$) will be

$x = 5\cos(-110°) = -1.71$

$y = 5\sin(-110°) = -4.698$

$\therefore$ B($x = -1.71$, $y = -4.698$, $z = 2$)

$\overrightarrow{AB} = (-1.71, -4.698, 2) - (3.064, 2.571, -2) = -4.774\hat{a}_x - 7.269\hat{a}_y + 4\hat{a}_z$

b. $\overrightarrow{BA} = (3.064, 2.571, -2) - (-1.71, -4.698, 2) = 4.774\hat{a}_x + 7.269\hat{a}_y - 4\hat{a}_z$

$$\hat{a}_{BA} = \frac{\overrightarrow{BA}}{|\overrightarrow{BA}|} = \frac{4.774\hat{a}_x + 7.269\hat{a}_y - 4\hat{a}_z}{\sqrt{(4.774)^2 + (7.269)^2 + (-4)^2}} = 0.498\hat{a}_x + 0.759\hat{a}_y - 0.417\hat{a}_z$$

Hammered Problems

1. **Given a point P(-3, 4, 5), express the vector that extends from P to Q(2, 0, -1) in (a) rectangular coordinates (b) spherical coordinates (c) cylindrical coordinates.** *[2069 Chaitra]*

Answer: (a) $\vec{A} = 5\hat{a}_x - 4\hat{a}_y - 6\hat{a}_z$ (b) $\vec{A} = (5\sin\theta\cos\phi - 4\sin\theta\sin\phi - 6\cos\theta)\,\hat{a}_r + (5\cos\theta\cos\phi - 4\cos\theta\sin\phi + 6\sin\theta)\,\hat{a}_\theta + (-5\sin\phi - 4\cos\phi)\,\hat{a}_\phi$ (c) $\vec{A} = (5\cos\phi - 4\sin\phi)\hat{a}_\rho + (-5\sin\phi - 4\cos\phi)\hat{a}_\phi - 6\hat{a}_z$

2. **Express the vector field $\vec{G} = (x^2 + y^2)^{-1}(x\,\hat{a}_x + y\,\hat{a}_y)$ in cylindrical components and cylindrical variables.** *[2071 Shrawan]*

Answer: $\vec{G} = \frac{1}{\rho}\hat{a}_\rho$

3. **Convert the vector $\vec{A} = A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_\phi\hat{a}_\phi$ into Cartesian coordinates.**

Answer:

$$\vec{A} = \left[\frac{A_r x}{\sqrt{x^2+y^2+z^2}} + \frac{A_\theta xz}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} - \frac{A_\phi y}{\sqrt{x^2+y^2}}\right]\hat{a}_x + \left[\frac{A_r y}{\sqrt{x^2+y^2+z^2}} + \frac{A_\theta yz}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} + \frac{A_\phi x}{\sqrt{x^2+y^2}}\right]\hat{a}_y + \left[\frac{A_r z}{\sqrt{x^2+y^2+z^2}} - \frac{A_\theta\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\right]\hat{a}_z$$