

Capacitor

A capacitor is a device that stores electric potential energy or electric charge. Two conducting plates enclosing an insulating material form a capacitor.

Capacitance:-

The ratio of the charge on each conducting plate to the potential difference is a constant quantity called the capacitance.

$$\text{i. e. Capacitance}(C) = \frac{q}{V} = \frac{\text{charge}}{\text{potential difference}}$$

It is also defined as the ability of a capacitor to store electric potential energy.

General method for calculating the capacitance:-

- Assume a charge 'q' on the plate under consideration.
- Calculate the electric field 'E' between the plates in terms of charge using Gauss law;

$$\text{i. e. } \oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

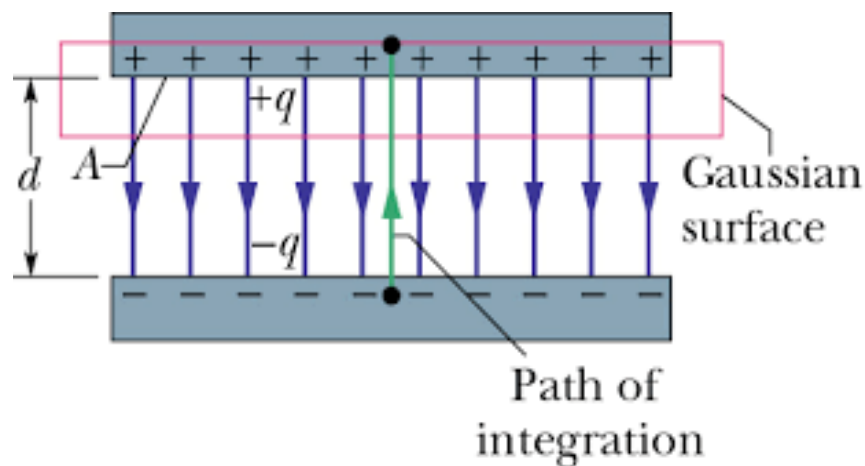
- Knowing 'E' calculate the potential difference between the plates. Now, potential difference between the plates of capacitor is given by;

$$V = \oint_{\infty}^r \vec{E} \cdot d\vec{r}$$

- Now capacitance (C) = $\frac{q}{V}$

Types of capacitor:-

1. Parallel plate capacitor:-



The arrangement of two parallel conducting plates with area 'A' separated by a small distance 'd' form a parallel plate capacitor. We draw a Gaussian surface that encloses the charge 'q' on the positive plate.

∴ From Gauss law;

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}, \Rightarrow EA = \frac{q}{\epsilon_0}, \Rightarrow q = \epsilon_0 EA, \therefore E = \frac{q}{\epsilon_0 A}$$

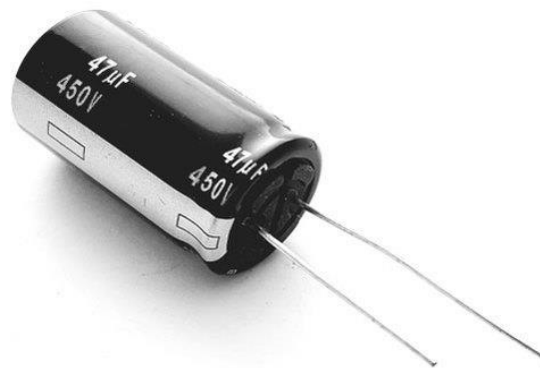
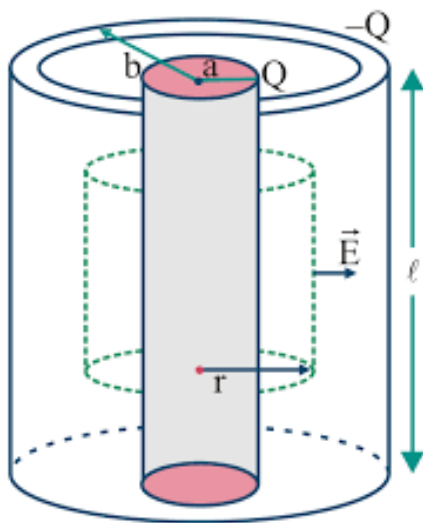
$$\text{Now, } V = \int_0^d \vec{E} \cdot d\vec{r}, \Rightarrow V = E \cdot d, \quad \therefore V = \frac{q}{\epsilon_0 A} \cdot d$$

$$\text{Now, Capacitance (C)} = \frac{q}{V}$$

$$\therefore C = \frac{\epsilon_0 A}{d}$$

\therefore Capacitance depends only on the geometry of the capacitor.
i. e. area of the plates and distance between them.

2. Cylindrical capacitor:-



Two cylinders of radii 'a' and 'b' placed co-axially each of length 'l' separated by small distance form a cylindrical capacitor. As a Gaussian surface we chose a cylinder of length l

and radius 'r' that enclose a charge 'q' on the positive inner cylinder. From Gauss law;

$$\oint \vec{E} \cdot \vec{dA} = \frac{q}{\epsilon_0}, \Rightarrow EA = \frac{q}{\epsilon_0}, \Rightarrow q = \epsilon_0 EA, \Rightarrow q = \epsilon_0 E \cdot 2\pi r l$$

$$\therefore E = \frac{q}{\epsilon_0 2\pi r l}$$

$$\text{Now, } V = \int_a^b E \cdot dr, \Rightarrow V = \int_a^b \frac{q}{\epsilon_0 2\pi r l} \cdot dr$$

$$\text{or, } V = \frac{q}{\epsilon_0 2\pi l} [\ln r]_a^b$$

$$= \frac{q}{\epsilon_0 2\pi l} [\ln b - \ln a]$$

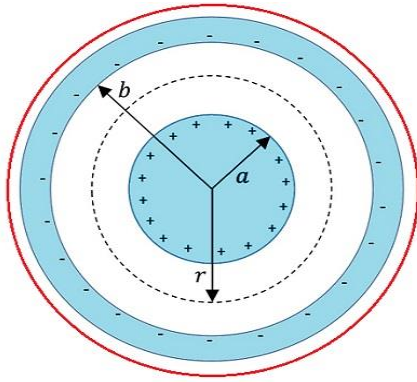
$$\therefore V = \frac{q}{\epsilon_0 2\pi l} \ln\left(\frac{b}{a}\right)$$

$$\text{Now, capacitance } (C) = \frac{q}{V}$$

$$\therefore C = \frac{2\pi\epsilon_0 l}{\ln\left(\frac{b}{a}\right)}$$

Therefore, capacitance depends only upon geometrical factor i. e. l , b and a .

3. Spherical capacitor:-



Two concentric spheres of radii 'a' and 'b', such that $b > a$, insulated from each other form a spherical capacitor. The inner conductor is solid sphere of radius 'a' and the outer conductor is hollow sphere of radius 'b'. As a Gaussian surface we draw a sphere of radius 'r' concentric with two spheres, such that $a < r < b$.

From Gauss law;

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}, \Rightarrow EA = \frac{q}{\epsilon_0}, \Rightarrow q = \epsilon_0 EA, \quad \Rightarrow E = \frac{q}{\epsilon_0 A}$$

$$\therefore E = \frac{q}{4\pi\epsilon_0 r^2}$$

$$\text{Now, } V = \int_a^b E \cdot dr, \Rightarrow V = \frac{q}{4\pi\epsilon_0} \int_a^b \frac{dr}{r^2}, \Rightarrow V = \frac{q}{4\pi\epsilon_0} \left[-\frac{1}{r} \right]_a^b$$

$$\text{or, } V = \frac{q}{4\pi\epsilon_0} \left[-\frac{1}{b} + \frac{1}{a} \right], \Rightarrow V = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{a} - \frac{1}{b} \right]$$

$$\therefore V = \frac{q}{4\pi\epsilon_0} \left(\frac{b-a}{ab} \right)$$

$$\therefore \text{Capacitance (C)} = \frac{q}{V} = 4\pi\epsilon_0 \left(\frac{ab}{b-a} \right)$$

Hence, the capacitance of spherical capacitor depends only on geometrical factor i. e. a and b.

Energy Stored in an electric field:-

The energy stored in the electric field between the plates of a capacitor can be visualized as the total amount of the work done to charge a capacitor. Suppose at any instant a charge dq is transferred from one plate to another. The applied potential difference at that instant be V . Then the amount of work done required is;

$$dw = Vdq$$

The total work done to store charge is;

$$W = \int_0^q Vdq, \Rightarrow W = \int_0^q \frac{q}{C} dq, \Rightarrow W = \frac{1}{C} \left[\frac{q^2}{2} \right]_0^q$$

$$\therefore W = \frac{q^2}{2C}$$

$$\text{or, } W = \frac{C^2 V^2}{2C} = \frac{1}{2} CV^2$$

$$\therefore \text{Electric potential energy (U)} = W = \frac{1}{2} CV^2$$

Energy Density:-

In a parallel plate capacitor the electric field has nearly the same value at all the points between the plates.

Thus, the energy density *i.e.* the potential energy per unit volume between the plates should be uniform.

$$\therefore \text{Energy density } (\mu) = \frac{\text{Energy(U)}}{\text{Volume}}$$

$$\text{or, } \mu = \frac{U}{A \times d}$$

$$\text{or, } \mu = \frac{CV^2}{2Ad}$$

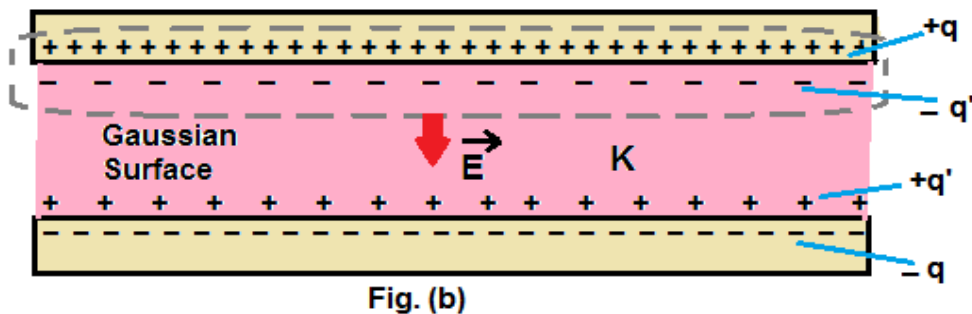
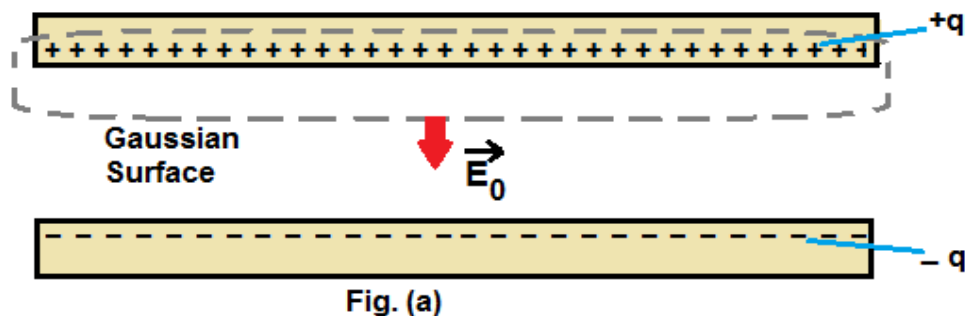
$$\text{or, } \mu = \frac{\epsilon_0 A}{d} \cdot \frac{V^2}{2Ad}$$

$$\text{or, } \mu = \frac{1\epsilon_0}{2} \left(\frac{V^2}{d^2} \right)$$

$$\therefore \mu = \frac{1}{2} \epsilon_0 E^2$$

Although, we derived this result for the special case of the parallel plate capacitor, it holds good for every type of capacitor, whatever may be the source of electric field.

Dielectric and Gauss law:-



Consider a parallel plate capacitor of plate area 'A' having charge 'q' on each plate. Now, draw a Gaussian surface that just encloses the charge 'q' on the positive plate. Let E_0 be the magnitude of the field and ϵ_0 be the permittivity of air between two plates then from Gauss law;

$$\oint \vec{E}_0 \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

$$E_0 A = \frac{q}{\epsilon_0}$$

$$\therefore E_0 = \frac{q}{A\epsilon_0} \dots \dots \dots (1)$$

When the dielectric is inserted as in figure (b), the induced charge ‘q’ on the top plate is ‘ q^I ’ which is distributed on the top face of the dielectric. Take the same Gaussian surface, the net charge enclosed by it is $q - q^I$. Now, using Gauss law;

$$\oint \vec{E} \cdot d\vec{A} = \frac{q - q^I}{\epsilon_0}$$

$$\therefore E = \frac{q - q^I}{A\epsilon_0} \dots \dots \dots (2)$$

Where, E is the electric field across capacitor with dielectric. Since, the effect of dielectric in capacitor is to decrease electric field by a factor of $\frac{1}{K}$. So ;

$$E = \frac{E_0}{K}$$

$$\frac{q - q^I}{A\epsilon_0} = \frac{q}{A\epsilon_0} \cdot \frac{1}{K}$$

$$\therefore q - q^I = \frac{q}{K}$$

Where, K is dielectric constant, now we can write the Gaussian law as;

$$\oint \vec{E} \cdot d\vec{A} = \frac{q - q^I}{\epsilon_0} = \frac{1}{\epsilon_0} \cdot \frac{q}{K}$$

$$\therefore \oint K\vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} \dots \dots \dots (3)$$

This is the Gauss law with dielectric of dielectric constant 'K'.

Relation Between \vec{D} , \vec{E} , and \vec{P} :-

From equation (2);

$$E = \frac{q - q^I}{A\epsilon_0} = \frac{1}{\epsilon_0} \left(\frac{q}{A} - \frac{q^I}{A} \right)$$

The quantity $\frac{q}{A}$ is called electric displacement vector \vec{D} , which is also equivalent to free surface charge density and the quantity $\frac{q^I}{A}$ is called polarization vector \vec{P} . It represents the capacity of dipole formation due to applied field. It is equivalent to induced surface charge density.

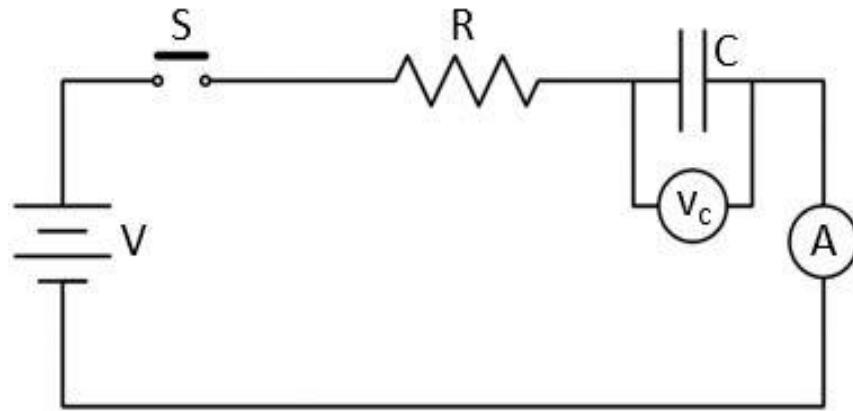
$$\therefore \vec{E} = \frac{1}{\epsilon_0} (\vec{D} - \vec{P})$$

$$\text{or,} \quad \vec{D} - \vec{P} = \epsilon_0 \vec{E}$$

$$\therefore \vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

Which is required Relation between \vec{D} , \vec{E} , and \vec{P} .

Charging and discharging of capacitor:-



Let a capacitor having capacitance C is connected in an electric circuit with resistance R , Ammeter A , battery with potential V and switch S . When the switch is on, the positive and negative charge appears on the plates. As the charge accumulate the potential difference between the plate increases and the charging current falls to zero. Now, applying Kirchhoff's voltage law;

$$V = V_c + V_R$$

$$\text{or, } \frac{q}{C} + IR = V$$

$$\text{or, } \frac{q}{C} + \frac{dq}{dt}R = V$$

$$\text{or, } R \frac{dq}{dt} = V - \frac{q}{C} = \frac{VC - q}{C} = \frac{q_0 - q}{C}$$

Where, $q_0 = VC$ is the maximum charge stored in the capacitor.

$$\text{or, } \frac{dq}{q_0 - q} = \frac{dt}{RC}$$

Now, integrating both sides;

$$\int_0^q \frac{dq}{q_0 - q} = \frac{1}{RC} \int_0^t dt$$

$$\text{or, } -[\ln(q_0 - q)]_0^q = \frac{t}{RC}$$

$$\text{or, } \ln(q_0 - q) - \ln q_0 = -\frac{t}{RC}$$

$$\text{or, } \ln\left(\frac{q_0 - q}{q_0}\right) = -\frac{t}{RC}$$

$$\text{or, } q_0 - q = q_0 e^{-t/RC}$$

$$\therefore q = q_0(1 - e^{-t/RC}) \dots \dots \dots (1)$$

Which is charging equation for capacitor in terms of charge.

Differentiating this equation w.r.to t, the current during the charging is;

$$I = \frac{dq}{dt} = -q_0 \left(-\frac{1}{RC}\right) e^{-t/RC}$$

$$\text{or, } I = \frac{q_0}{RC} e^{-t/RC}$$

$$\text{or, } I = \frac{VC}{RC} e^{-t/RC}$$

$$\therefore I = I_0 e^{-t/RC} \dots \dots \dots (2)$$

Where, $I_0 = V/R$ is the maximum current, this is the charging equation in terms of current. The term RC in equation (1) and (2) is called capacitive time constant of circuit. It has dimension of time.

When, $t = RC$, from equation (1);

$$q = q_0(1 - e^{-1}) = q_0(1 - 0.37) = 0.63q_0$$

$$\Rightarrow q = 63\% \text{ of } q_0$$

Hence, the time constant τ of a charging circuit is defined as the time in which capacitor charges by about 63% of its maximum charge.

The potential across the capacitor is;

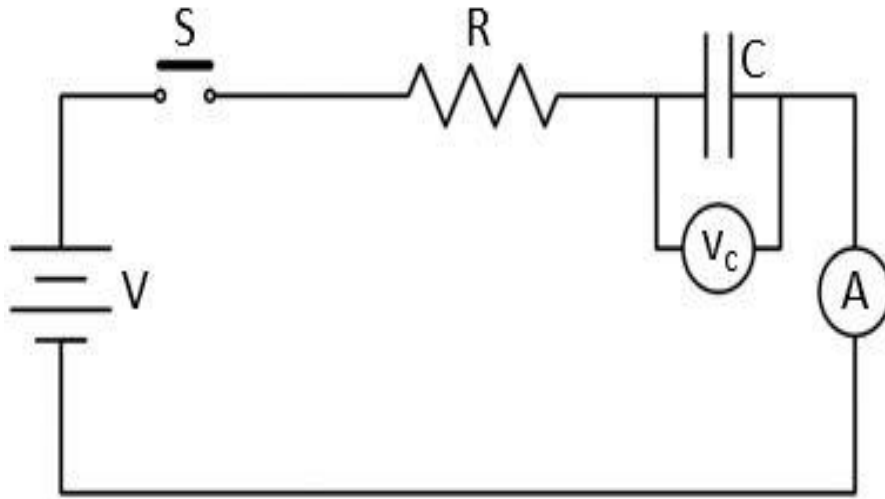
$$V_C = \frac{q}{C} = \frac{q_0}{C} (1 - e^{-t/RC})$$

$$\therefore V_C = V(1 - e^{-t/RC}) \dots \dots \dots (3)$$

And potential across resistor is;

$$V_R = IR = V e^{-t/RC} \dots \dots \dots (4)$$

Discharging of Capacitor:-



When capacitor is fully charged and switch is off. Discharging occurs in the capacitor through resistor. Now, using Kirchhoff's voltage law;

$$0 = V_c + V_R$$

$$\text{or, } \frac{q}{C} + IR = 0$$

$$\text{or, } \frac{q}{C} + \frac{dq}{dt}R = 0$$

$$\text{or, } R \frac{dq}{dt} = -\frac{q}{C}$$

$$\text{or, } \frac{dq}{q} = -\frac{dt}{RC}$$

Now, integrating both sides;

$$\int_{q_0}^q \frac{dq}{q} = -\frac{1}{RC} \int_0^t dt$$

$$\text{or, } [\ln q]_{q_0}^q = -\frac{t}{RC}$$

$$\text{or, } \ln q - \ln q_0 = -\frac{t}{RC}$$

$$\text{or, } \ln \left(\frac{q}{q_0} \right) = -\frac{t}{RC}$$

$$\text{or, } \frac{q}{q_0} = e^{-t/RC}$$

$$\therefore q = q_0(e^{-t/RC}) \dots \dots \dots (5)$$

Which is discharging equation for capacitor in terms of charge.

Differentiating this equation w.r.to t, the current during the charging is;

$$I = \frac{dq}{dt} = q_0 \left(-\frac{1}{RC} \right) e^{-t/RC}$$

$$\text{or, } I = \frac{-q_0}{RC} e^{-t/RC}$$

$$\text{or, } I = \frac{-VC}{RC} e^{-t/RC}$$

$$\therefore I = -I_0 e^{-t/RC} \dots \dots \dots (6)$$

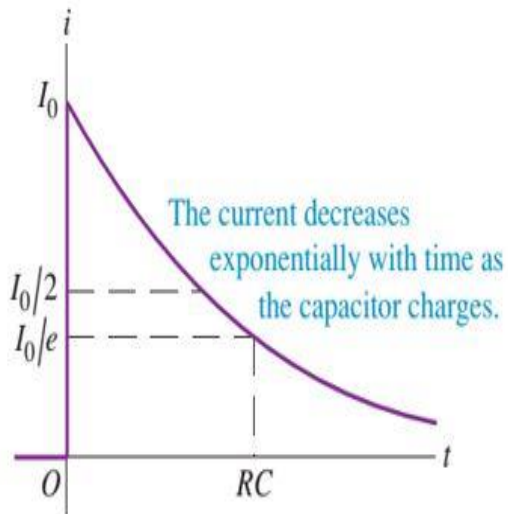
Equation (5) and (6) are called discharging equation in terms of charge and current. Here, -ve sign indicates discharging current. The term RC in equation (5) and (6) is called capacitive time constant of circuit. It has dimension of time.

When, $t = RC$, from equation (5);

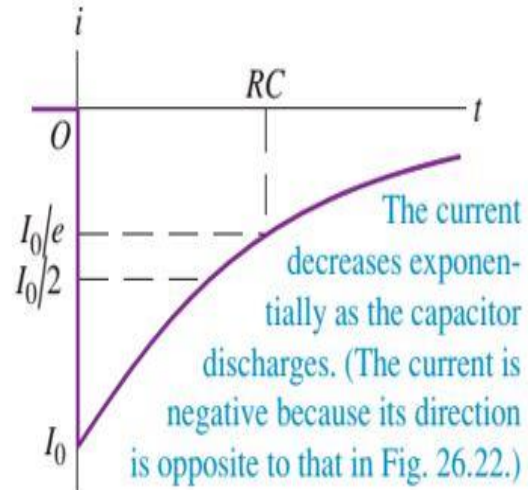
$$q = q_0(e^{-1}) = q_0(0.37) = 0.37q_0$$
$$\Rightarrow q = 37\% \text{ of } q_0$$

Hence, the time constant τ of a discharging circuit is defined as the time in which capacitor discharges by about 37% of its initial value.

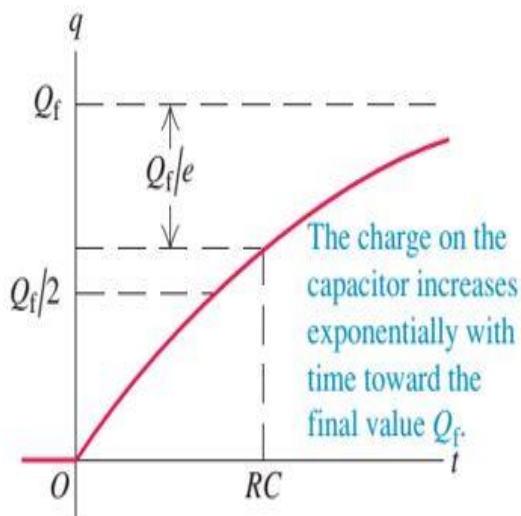
(a) Graph of current versus time for a charging capacitor



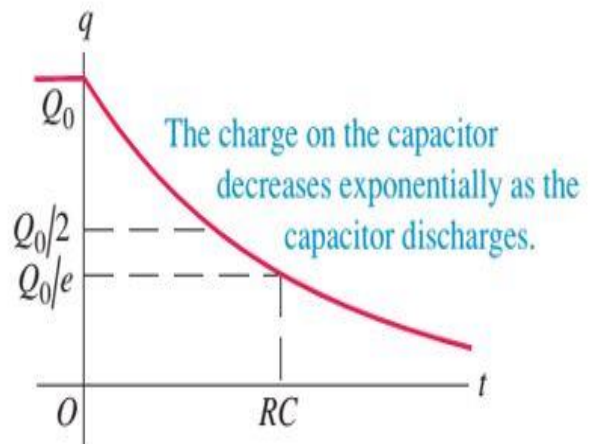
(a) Graph of current versus time for a discharging capacitor



(b) Graph of capacitor charge versus time for a charging capacitor



(b) Graph of capacitor charge versus time for a discharging capacitor



Numerical Examples:-

- 1. Two capacitors $2\ \mu\text{F}$ and $4\ \mu\text{F}$ are connected in parallel across $300\ \text{V}$ potential difference. Calculate the total energy in the system.**

Solution:-

$$\begin{aligned}\text{Total capacitance (C)} &= C_1 + C_2 = 2\ \mu\text{F} + 4\ \mu\text{F} = 6\ \mu\text{F} \\ &= 6 \times 10^{-6}\ \text{F}\end{aligned}$$

$$\text{Potential difference (V)} = 300\ \text{V}$$

$$\begin{aligned}\therefore \text{Energy stored in capacitor (U)} &= \frac{1}{2} CV^2 \\ &= \frac{1}{2} \times 6 \times 10^{-6} \times 300^2 \\ &= 0.27\ \text{Joule}\end{aligned}$$

- 2. An air filled parallel plate capacitor has a capacitance of $10^{-12}\ \text{F}$. The separation between the plate is double and wax is inserted between them which increases the capacitance to $2 \times 10^{-12}\ \text{F}$. Calculate the dielectric constant of wax.**

Solution:-

$$C_1 = 10^{-12}\ \text{F} \quad d_1 = d, \quad A_1 = A, \text{ and } \epsilon_1 = \epsilon_0$$

$$C_1 = \frac{\epsilon_1 A_1}{d_1} = \frac{\epsilon_0 A}{d} \dots \dots \dots (i)$$

When separation is double and wax is inserted then;

$$C_2 = 2 \times 10^{-12} \text{F} \quad d_2 = 2d, \quad A_2 = A, \text{ and } \epsilon_2 = \epsilon$$

$$C_2 = \frac{\epsilon_2 A_2}{d_2} = \frac{\epsilon A}{2d} = \frac{k\epsilon_0 A}{2d} \dots \dots \dots (ii)$$

$$\text{Then,} \quad \frac{C_2}{C_1} = \frac{k}{2}$$

$$\therefore k = \frac{2 \times 2 \times 10^{-12}}{10^{-12}} = 4$$

3. A parallel plate capacitor has capacitance of $100 \mu\mu\text{F}$. A plate area of 100 cm^2 and mica as dielectric material, with 50 V potential difference. Calculate (i) Electric field in mica. (ii) The free charge on the plates and (iii) The induced surface charge (q^I). Given $k = 5.4$ for mica.

Solution:-

$$C = 100 \mu\mu\text{F} = 100 \times 10^{-6} \times 10^{-6} \text{ F}$$

$$V = 50 \text{ V}$$

$$(ii) q = CV = 100 \times 10^{-12} \times 50 = 5 \times 10^{-9} \text{ C}$$

$$(iii) \text{The induced surface charge } (q^I) = q(1 - \frac{1}{k})$$

$$= 5 \times 10^{-9} (1 - \frac{1}{5.4})$$

$$= 4.1 \times 10^{-9} \text{ C}$$

$$(i) \text{ Electric field in mica } (E) = \frac{q}{k\epsilon_0 A}$$

$$= \frac{5 \times 10^{-9}}{5.4 \times 8.85 \times 10^{-12} \times 10^{-2}}$$

$$= 1.05 \times 10^4 \text{ V/m}$$

4. A capacitor of capacitance C is discharge through a resistor of resistance R. After how many time constant is the stored energy becomes 1/4th of initial value?

Solution:-

$$\text{We have, energy stored by capacitor } (U) = \frac{1}{2} CV^2 = \frac{q^2}{2C}$$

$$\text{or, } U = \frac{q_0^2 e^{-2t/RC}}{2C} = U_0 e^{-2t/RC}$$

$$\text{According to question; } U = \frac{U_0}{4}$$

$$\therefore \frac{U_0}{4} = U_0 e^{-2t/RC}$$

$$\text{or, } e^{2t/RC} = 4$$

$$\text{or, } \frac{2t}{RC} = \ln 4$$

$$\text{or, } t = \frac{\ln 4}{2} \times RC$$

$$\therefore t = 0.693 \tau$$

5. A capacitor of capacitance C is charged through a resistor R. Calculate the time at which potential across the resistor is equal to the potential across the capacitor.

Solution:-

According to the question; $V_R = V_C$

$$\text{or, } IR = \frac{q}{C}$$

$$\text{or, } R \cdot I_0 e^{-t/RC} = \frac{q_0}{C} (1 - e^{-t/RC})$$

$$\text{or, } e^{-t/RC} = 1 - e^{-t/RC}$$

$$\text{or, } 2e^{-t/RC} = 1$$

$$\text{or, } e^{t/RC} = 2$$

$$\text{or, } t = \ln(2) RC$$

$$\therefore t = 0.693 RC$$

6. If n drops each of capacitance C combine to form a single big drop. Find the capacitance of big drop.

Solution:-

$$\text{The mass of the small drop } (M_1) = \frac{4}{3}\pi r^3 \times \rho$$

$$\text{The mass of the large drop} = \frac{4}{3}\pi R^3 \times \rho$$

$$\text{Since, } M_2 = nM_1$$

$$\text{or, } \frac{4}{3}\pi r^3 \times \rho = n \frac{4}{3}\pi R^3 \times \rho$$

$$\therefore R = n^{1/3}r$$

$$\therefore \text{Capacitance of big drop} = 4\pi\epsilon_0 R$$

$$= 4\pi\epsilon_0 n^{1/3}r$$

$$= C \cdot n^{1/3}$$

7. A parallel plate capacitor has circular plate of 8.2 cm radius and 1.3 mm separation in air. (i) Calculate the capacitance and (ii) What charge will appear on the plates, if potential difference of 120 V is applied.

Solution:-

$$\begin{aligned} \text{(i) Capacitance (C)} &= \frac{\epsilon_0 A}{d} \\ &= \frac{8.85 \times 10^{-12} \times \pi \times (0.082)^2}{1.3 \times 10^{-3}} \\ &= 1.497 \times 10^{-10} \text{ F} \\ \text{(ii) Charge (q)} &= CV = 1.497 \times 10^{-10} \times 120 \\ &= 1.725 \times 10^{-8} \text{ C} \end{aligned}$$

Exercise:-

1. Explain how electric energy is stored in a capacitor and derive an expression for energy density of electric field.
2. Define the three electric vectors, E, P, D and develop the relation between them.
3. A cylindrical capacitor has radii 'a' and 'b'. Show that half the energy stored lies within the cylinder whose radius is $r = \sqrt{ab}$.
4. Derive the relation for rise and fall of current in charging and discharging of capacitor through resistor. Plot graphs between current and time and explain the figure.
5. Write the general methods to calculate the capacitance of a capacitor and hence determine the capacitance of the cylindrical capacitor of inner and outer radii 'a' and 'b' respectively.
6. Write the general methods to calculate the capacitance of a capacitor and hence determine the capacitance of the spherical capacitor.
7. A capacitor of capacitance C is discharging through a resistor of resistance R. After how many times constant is the stored energy $1/8$ of its initial value?
8. A capacitor of capacitance C is discharging through a resistor of resistance R. After how many time constant is the stored energy becomes one fourth of its initial value?
9. Prove that the capacitance per unit length of a capacitor varies inversely with logarithm of ratio of external and

internal radii. Obtain an expression for energy stored per unit volume in a parallel plate capacitor.

10. If a parallel plate capacitor is to be designed to operate in an environment of fluctuating temperature, prove that the rate of change of capacitance with temperature T is given by $\frac{dC}{dT} = C \left[\frac{1}{A} \frac{dA}{dT} - \frac{1}{x} \frac{dx}{dT} \right]$, where symbols carry its usual meaning.
11. Prove that the capacitance of a concentric spherical capacitor of radii 'a' and 'b' is $C = 4\pi\epsilon_0 \left(\frac{b^2}{b-a} \right)$. If outer plate is positively charged and inner plate is earthed.
12. The space between two concentric conducting spherical shells of radii $b = 1.70$ cm and $a = 1.20$ cm is filled with a substance of dielectric constant $k = 23.5$. A potential difference $V = 73$ V is applied across the inner and outer shells. Determine (a) The capacitance of the device (b) the free charge q on the inner shell.
13. A parallel plate capacitor whose capacitance C is 13.5 pF is charged by a battery to a potential difference $V = 12.5$ V between its plates. The charging battery disconnected and a porcelain slab ($k = 6.50$) is supplied between the plates. (a) What is the potential energy of the capacitor before the slab is inserted (b) what is the potential energy of the capacitor-slab device after the slab is inserted?
14. The parallel plates in a capacitor, with a plate area 8.5 cm² and air filled separation of 3 mm are charged by 6 V batteries. They are then disconnected from the battery and

- pulled apart to a separation of 8 mm. Neglecting fringing; find (a) the potential difference between the plates (b) the initial energy stored and (c) final energy stored.
15. A capacitor discharge through a resistor R. (a) after how many time constants does its charge fall to one half of its original value? (b) After how many time constants does the stored energy drops to half of its initial value?
 16. A spherical drop of water carrying a charge of 30 pC has a potential of 500 V at its surface. (a) What is the radius of the drop? (b) If two such drops of the same charge and radius combine to form a single spherical drop, what is the potential at the surface of the new drop?
 17. What is the force per unit area with which plates of parallel plate capacitor attract each other if they are separated by 1 mm and maintain at 100 V potential difference and electric constant of the medium in unity.
 18. A long cylindrical conductor 1 m and is surround by a co-axial cylindrical conducting shell with inner radius double that of long cylindrical conductor. Calculate the capacitance of this capacitor assuming that there is vacuum in space between cylinders.
 19. A parallel plate capacitor each of area 100 cm^2 has a potential difference of 500 V and capacitance of $100 \times 10^{-6} \mu F$. If a mica of dielectric constant 5.4 is inserted between plates find the magnitude of (a) Electric field in mica (b) Displacement vector (c) mobility of electrons.

