Differential Equations

An Equation which contains or involving the derivatives of the function is called the differential equations.

If the function, whose derivatives are used in the equation is of single independent variable then it is called an ordinary differential equation (ODE).

For example:

- $\frac{dy}{dx} + y = x$
- $y'' + 6y' + 5y = 6x\ etc$

When the equation contains the derivative of the function with respective to two or more than two independent variables it is called a partially differential equation (PDE).

For example:

- $x\frac{\partial u}{\partial t} = k^2\frac{\partial^2 u}{\partial x^2}$
- $x\frac{\partial f}{\partial t} + y\frac{\partial f}{\partial t} = \lambda f$

Order and Degree of Differential equation

Order ordinary differential equation (ODE)

The highest derivative of y with respective to x occurring in the differential equation is called the order of **ODE.**

For example:

- $\frac{dy}{dx} + y = x$ is an **ODE of order $\mathbf{1}$**.
- $y'' + 6y' + 5y = 6x$ is an **ODE of order $\mathbf{2}$**.

Degree of ordinary differential equation (ODE)

The degree of the highest derivative of the function present in the differential equation after making free from the fractions and the radical is called the degree of **ODE.**

For example:

- $\left(\frac{dy}{dx}\right)^1 + y = x$ is an **ODE of degrre $\mathbf{1}$**.
- $(y'')^1 + 6(y')^3 + 5y = 6x$ is an **ODE of degree $\mathbf{1}$**.
- $\left(\frac{dy}{dx}\right)^2 - 5\left(\frac{d^2y}{dx^2}\right)^3 + 4 = 0$ is an **ODE of degree $\mathbf{3}$**.

Differential Equations of First Order and First Degree

A differential equation of the form

$M(x, y)dx + N(x, y)dy = 0$

or, $Mdx + Ndy = 0$

e.g. $xdy + ydx = 0$

Where $M\ and\ N$ are function of x and y or constant is called First Order and First Degree Differential Equations.

Note:

- The general solution of First Order and First Degree Differential Equations contains only one constant.
- We cannot solve all the Differential Equations.

We will discuss the following form of ordinary differential equation of First Order and First Degree

- Differential Equations with Separable Variables,
- Homogeneous differential equation of first order
- First order differential equation Reducible Homogeneous form
- First order Linear differential equations

- Bernoulli,s equations and
- Exact Differential Equations

Differential Equations with Separable Variables

In the differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

Where $M \; and \; N$ are function of x and y or constants,

The variables are separated in such way that the coefficient of dx must be functions of x only or constant and the coefficient of dy must be functions of y only or constant, and after integration we get the general solution of differential equation.

Note:

If we give particular value to the constant c then the solution is called particular solution.

Exercise

- **Solve the following differential equation using the separation of variable.**

a). $\boldsymbol{xdx - ydy = 0}$

Solution:

Given differential equation is

$xdx - ydy = 0$

Or, $xdx = ydy$ integrating on both sides we get;

Or, $\int xdx = \int ydy$

$$Using \int x^n dx = \frac{x^{n+1}}{n+1} + c \ \ as\, n \neq -1$$

Or, $\frac{x^2}{2} = \frac{y^2}{2} + c$

Or, $x^2 = y^2 + 2c$

Or, $x^2 - y^2 = C$ where $C = 2c$

$\therefore x^2 - y^2 = C$ Ans.

b). $\boldsymbol{\frac{dy}{dx} = \frac{y}{x}}$

Solution:

Given differential equation is

$$\frac{dy}{dx} = \frac{y}{x}$$

Or, $\frac{dy}{y} = \frac{dx}{x}$ integrating on both sides we get;

Or, $\int \frac{1}{y} dy = \int \frac{1}{x} dx$

Or, $logy = logx + logc$

Or, $logy = logxc$

Or, $y = xc$

$\therefore y = xc$ Ans.

c). $\frac{dy}{dx} = \frac{x^3+1}{y^3+1}$

Solution:

Given differential equation is

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{x^3+1}{y^3+1}$$

Or, $(y^3 + 1)dy = (x^3 + 1)dx$ integrating on both sides we get;

Or, $\int (y^3 + 1)dy = \int (x^3 + 1)dx$

Or, $\int y^3 dy + \int 1. dy = \int x^3 dx + \int 1. dx +$

Or, $\frac{y^4}{4} + y = \frac{x^4}{4} + x + c$

$\therefore \frac{y^4}{4} + y - \frac{x^4}{4} - x = c$ Ans.

d). $(1 + x^2)y' = 1$

Soluton:

Given differential equation is

$$(1 + x^2)y' = 1$$

Or, $(1 + x^2)\frac{dy}{dx} = 1$

Or, $dy = \frac{1}{(1+x^2)} dx$ integrating on both sides we get;

Or, $\int dy = \int \frac{1}{(1+x^2)} dx$

Using $\int \frac{1}{(a^2+x^2)} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$

Or, $y = \tan^{-1} x + c$

$\therefore y = \tan^{-1} x + c$ Ans.

e). $\boldsymbol{ydx - xdy = xydy}$

Solution:

Given differential equation is

$$\boldsymbol{ydx - xdy = xydy}$$

Dividing both sides by $\boldsymbol{xy}$

Or, $\boldsymbol{\frac{ydx-xdy}{xy} = \frac{xy}{xy}dy}$

Or, $\boldsymbol{\frac{dx}{x} - \frac{dy}{y} = dy}$ integrating on both sides we get;

Or, $\int \left(\boldsymbol{\frac{dx}{x} - \frac{dy}{y}}\right) = \int \boldsymbol{dy}$

Or, $\int \boldsymbol{\frac{dx}{x}} - \int \boldsymbol{\frac{dy}{y}} = \int \boldsymbol{dy}$

Or, $logx - logy = y + c$

$\therefore logx - logy = y + c$ Ans.

f). $(xy^2 + x)dx + (yx^2 + y)dy = 0$

g). $\frac{dy}{dx} = \frac{e^x+1}{y}$

h). $\frac{dy}{dx} = e^{x-y} + e^y$

i). $e^{x-y}dx + e^{y-x}dy = 0$

j). $sec^2xtanydx + sec^2ytanxdy = 0$

k). $\frac{dy}{dx} + \frac{1+cos2y}{1-cos2x}$ =0

l).$x\sqrt{1+y^2}\,dx + y\sqrt{1+x^2}dy = 0$

Solution:

Given differential equation is

$$x\sqrt{1+y^2}\,dx + y\sqrt{1+x^2}dy = 0$$

Or, $\frac{x}{\sqrt{1+x^2}}\,dx = -\frac{y}{\sqrt{1+y^2}}dy$ integrating on both sides we get;

Or, $\int \frac{x}{\sqrt{1+x^2}}\,dx = -\int \frac{y}{\sqrt{1+y^2}}dy$

Or, $\frac{1}{2}\int \frac{\mathbf{2x}}{\sqrt{\mathbf{1+x^2}}}\, \boldsymbol{dx} = -\frac{1}{2}\int \frac{\mathbf{2y}}{\sqrt{\mathbf{1+y^2}}}\boldsymbol{dy}$

Using $\int \frac{f\prime(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$

$$\int \frac{f'(x)}{f(x)} dx = logf(x) + c$$

Or, $\frac{1}{2}.2\sqrt{\mathbf{1}+\boldsymbol{x}^{\mathbf{2}}} = -\frac{1}{2}.2\sqrt{\mathbf{1}+\boldsymbol{y}^{\mathbf{2}}} + c$

Or, $\sqrt{\mathbf{1}+\boldsymbol{x}^{\mathbf{2}}} = -\sqrt{\mathbf{1}+\boldsymbol{y}^{\mathbf{2}}} + c$

$\therefore \sqrt{\mathbf{1}+\boldsymbol{x}^{\mathbf{2}}} + \sqrt{\mathbf{1}+\boldsymbol{y}^{\mathbf{2}}} = c$ Ans.

3. Solve the initial value problem.

a). $\boldsymbol{\frac{dy}{dx} = 2x - 7, y(2) = 0}$

Solution:

Given differential equation is

$$\boldsymbol{\frac{dy}{dx} = 2x - 7, y(2) = 0}$$

Or, $dy = (\mathbf{2}\boldsymbol{x} - \mathbf{7})dx$ integrating on both sides we get;

Or, $\int dy = \int(\mathbf{2x-7})dx$

Or, $\int dy = 2\int \boldsymbol{x}dx - 7\int dx$

Or, $y = 2\frac{x^{1+1}}{1+1} - 7x + c$

Or, $y = 2\frac{x^2}{2} - 7x + c$

Or, $y = x^2 - 7x + c$

$\therefore y = x^2 - 7x + c \qquad (1)$

which is the general solution .

By given $\boldsymbol{y(2) = 0}$

$$\boldsymbol{i.e. x = 2, y = 0}$$

Then from (1) we have,

Or, $\mathbf{0} = 2^2 - 7 \times 2 + c$

0r, $\mathbf{0} = 4 - 14 + c$

0r, $\mathbf{0} = -10 + c$

0r, $\boldsymbol{c} = 10$

Substituting the value $\boldsymbol{c} = 10$ in(1), we get.

$y = x^2 - 7x + 10$

This is the required particular solution.

b). $\frac{dy}{dx} = 10 - x, y(0) = -1$

c). $\frac{dy}{dx} = 9x^2 - 4x + 5, y(-1) = 0$

- Find the order and degree of the following differential equations.

a). $\frac{dy}{dx} = 2$

b). $\frac{d^2y}{dx^2} = sinx$

c). $x\frac{d^3y}{dx^3} + y + \left(\frac{dy}{dx}\right)^4 = 0$

d). $(y'')^3 + 4y' = e^x$

e). $\frac{d^2y}{dx^2} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3}$

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$$