Ex-18

Prove the following Legendre's function

(1) $\frac{2}{5} P_3(x) + \frac{3}{5} P_1(x) = x^3$

Sol^n: we know the Legendre's polynomials are:

$$P_1(x) = x^1$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Now

$$L.H.S. = \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x)$$

$$= \frac{2}{5} \cdot \frac{1}{2}(5x^3 - 3x) + \frac{3}{5} x$$

$$= x^3 - \frac{3}{5}x + \frac{3}{5}x$$

$$= x^3 \quad R.H.S.$$

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$f(x) = 5x^3 + x$

we know,

$P_1(x) = x$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$2P_3(x) = 5x^3 - 3x$

$2P_3(x) = 5x^3 - 3P_1(x)$

$5x^3 = 2P_3(x) + 3P_1(x)$

Now,

$f(x) = 2P_3(x) + 3P_1(x) + P_1(x)$

$f(x) = 2P_3(x) + 4P_1(x)$

$f(x) = x^3 - 5x^2 + 6x + 1$

we know,

$P_0(x) = 1$

$P_1(x) = x$

$6P_1(x) = 6x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$2P_2(x) = 3x^2 - P_0(x)$

$2P_2(x) + P_0(x) = 3x^2$

$\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x) = x^2$

$5x^2 = \frac{10}{3}P_2(x) + \frac{5}{3}P_0(x)$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

or, $$2P_3(x) = \frac{5}{\cancel{2}} x^3 - 3x$$

or, $$2P_3(x) = 5x^3 - 3P_1(x)$$

or $$5x^3 = 2P_3(x) + 3P_1(x)$$

$$x^3 = 2/5\, P_3(x) + 3/5\, P_1(x)$$

Now,

$$f(x) = \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x) - \frac{10}{3} P_2(x) - \frac{5}{3} P_0(x) + 6P_1(x) + P_0(x)$$

or, $$f(x) = \frac{2}{5} P_3(x) - \frac{10}{3} P_2(x) + \frac{33}{5} P_1(x) - \frac{2}{3} P_0(x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Ex-18

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

we know,

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m}$$

Put $n = 1/2$

$$J_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\frac{1}{2}+1)} \left(\frac{x}{2}\right)^{\frac{1}{2}+2m}$$

$$J_{\frac{1}{2}}(x) = \left(\frac{x}{2}\right)^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+\frac{3}{2})} \left(\frac{x}{2}\right)^{2m}$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{x}{2}} \left[\frac{(-1)^0}{0!\,\Gamma(0+3/2)} \cdot \left(\frac{x}{2}\right)^{2 \cdot 0} + \frac{(-1)^1}{1!\,\Gamma(1+\frac{3}{2})} \left(\frac{x}{2}\right)^{2 \cdot 1} + \cdots \right]$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{x}{2}} \left[\frac{1}{\frac{1}{2}\sqrt{\pi}} - \frac{1}{\frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}} \cdot \frac{x^2}{4} + \frac{1}{2 \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}} \frac{x^4}{16} - \cdots \right]$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{x}{2}} \times \frac{1}{\frac{\sqrt{\pi}}{2}} \left[1 - \frac{x^2}{3 \cdot 2 \cdot 1} + \frac{x^4}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} - \cdots \right]$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{x}{2}} \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{1}{x} \left[x - \frac{x^3}{3 \cdot 2 \cdot 1} + \frac{x^5}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} - \cdots \right]$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

12. $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$

Sol: we have,

$$2n\, J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right] \quad -①$$

Put $n = 3$,

$$6\, J_3(x) = x\left[J_2(x) + J_4(x)\right]$$

or, $6J_3(x) = xJ_2(x) + xJ_4(x)$

or, $xJ_4(x) = 6J_3(x) - xJ_2(x)$

or, $J_4(x) = \frac{6}{x} J_3(x) - J_2(x) \quad -②$

Put, $n = 2$ in ①

$$4J_2(x) = x\left[J_1(x) + J_3(x)\right]$$

or, $4J_2(x) = xJ_1(x) + xJ_3(x)$

$\therefore J_3(x) = \frac{4}{x} J_2(x) - J_1(x) \quad -③$

From ② & ③

$$J_4(x) = \frac{6}{x}\left[\frac{4}{x} J_2(x) - J_1(x)\right] - J_2(x)$$

$$J_4(x) = \frac{24}{x^2} J_2(x) - J_2(x) - \frac{6}{x} J_1(x)$$

$$J_4(x) = \left(\frac{24}{x^2} - 1\right) J_2(x) - \frac{6}{x} J_1(x) \quad -④$$

Put $n = 1$ in (1)

$$2J_1(x) = x J_0(x) + x J_2(x)$$

$$\Rightarrow J_2(x) = \frac{2}{x} J_1(x) - J_0(x) \quad \text{--- (5)}$$

using (5) in (4)

$$J_4(x) = \left(\frac{24}{x^2} - 1\right)\left(\frac{2}{x} J_1(x) - J_0(x)\right) - \frac{6}{x} J_1(x)$$

or, $$J_4(x) = \left(\frac{48}{x^3} - \frac{2}{x} - \frac{6}{x}\right) J_1(x) - \left(\frac{24}{x^2} - 1\right) J_0(x)$$

$$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

$\cancel{4J_n''(x) =}$

$$4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$$

we know,

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \quad -(1)$$

diff (1) w.r.t. x,

$$2J_n''(x) = J_{n-1}'(x) - J_{n+1}'(x)$$

Multiplying both sides by 2,

$$4J_n''(x) = 2J_{n-1}'(x) - 2J_{n+1}'(x) \quad -(2)$$

Put, n as n-1 in (1) and n+1 in 1

$$2J_{n-1}'(x) = J_{n-2}(x) - J_n(x) \quad -(3)$$

$$2J_{n+1}'(x) = J_n(x) - J_{n+2}(x) \quad -(4)$$

using (3) & (4) in (2)

$$4J_n'(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$$ proved

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19. $J_{-\frac{1}{2}}(x) = J_{1/2}(x) \cot x$

Solⁿ: we know,

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{——} (i)$$

and,

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x \quad \text{——} (ii)$$

Dividing (ii) by (i)

$$\frac{J_{-\frac{1}{2}}(x)}{J_{\frac{1}{2}}(x)} = \frac{\sin x}{\cos x}$$

$\therefore$ $J_{-1/2}(x) = J_{1/2}(x) \cot x$ //

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20. $\left[J_{1/2}(x)\right]^2 + \left[J_{-1/2}(x)\right]^2 = \frac{2}{\pi x}$

soln: we know,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{—(i)}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x \quad \text{—(ii)}$$

Squaring and adding (i) and (ii)

$$\left[J_{1/2}(x)\right]^2 + \left[J_{-1/2}(x)\right]^2 = \frac{2}{\pi x}\sin^2 x + \frac{2}{\pi x}\cos^2 x$$

or, $\left[J_{1/2}(x)\right]^2 + \left[J_{-1/2}(x)\right]^2 = \frac{2}{\pi x}\left[\sin^2 x + \cos^2 x\right]$

$\therefore \left[J_{1/2}(x)\right]^2 + \left[J_{-1/2}(x)\right]^2 = \frac{2}{\pi x}$ proved.

(17) $J_2(x) - J_0(x) = 2J_0''(x)$

solⁿ From Q.N. 14,

$$4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$$

Put $n = 0$,

$$4J_0''(x) = J_{-2}(x) - 2J_0(x) + J_2(x)$$

since,

$$J_{-n}(x) = (-1)^n J_n(x)$$

$$4J_0''(x) = (-1)^2 J_2(x) - 2J_0(x) + J_2(x)$$

$$4J_0''(x) = 2J_2(x) - 2J_0(x)$$

$$\therefore J_2(x) - J_0(x) = 2J_0''(x)$$