Assignment-2

1) $\int_0^\infty \frac{\log(1+x^2)}{1+x^2} dx = \pi \log 2$ [BE 2070]

$\Rightarrow$ Sol^n.

let $I = \int_0^\infty \frac{\log(1+x^2)}{1+x^2} dx$

put $x = \tan\theta$, $dx = \sec^2\theta \, d\theta$

when $x = 0$, $\theta = 0$ when $x = \infty$, $\theta = \frac{\pi}{2}$

so, $I = \int_0^{\frac{\pi}{2}} \frac{\log(1+\tan^2\theta)}{1+\tan^2\theta} \sec^2\theta \, d\theta$

$= \int_0^{\pi/2} \log(\sec^2\theta) \, d\theta = -2\int_0^{\pi/2} \log(\cos\theta) \, d\theta$ -(1)

Using properties of definite integral

$I = -2\int_0^{\pi/2} \log\cos\left(\frac{\pi}{2} - \theta\right) d\theta$

$= -2\int_0^{\pi/2} \log\sin\theta \, d\theta$ -(2)

Adding (1) & (2), we get

$2I = -2\int_0^{\pi/2} (\log\sin\theta + \log\cos\theta) \, d\theta$

$I = -\int_0^{\pi/2} \log\left(\frac{\sin 2\theta}{2}\right) d\theta$

$= -\int_0^{\pi/2} \log\sin 2\theta + \log 2 \int_0^{\pi/2} d\theta$

put $2\theta = t$

$d\theta = \frac{dt}{2}$

$$I = -\int_0^{\pi} \log \sin t \frac{dt}{2} + \log 2 [\theta]_0^{\pi/2}$$

$$= -\frac{1}{2}\int_0^{\pi} \log \sin t \, dt + \log 2 \left(\frac{\pi}{2} - 0\right)$$

$$= -\frac{1}{2}\int_0^{2(\pi/2)} \log \sin t \, dt + \frac{\pi}{2}\log 2$$

$$= -\frac{1}{2} \times 2 \int_0^{\pi/2} \log \sin t \, dt + \frac{\pi}{2}\log 2 \quad \left[\sin\left(\frac{2\pi}{2}\right.\right.$$

$$= -\frac{1}{2} \times 2 \int_0^{\pi/2} \log \sin \theta \, d\theta + \frac{\pi}{2}\log 2$$

$$I = \frac{1}{2} + \frac{\pi}{2}\log 2$$

$$\frac{I}{\cancel{2}} = \frac{\pi}{\cancel{2}}\log 2$$

$$I = \pi \log 2$$ #

Q. (2) $\int_0^{\pi} \frac{x \tan x}{\sec x + \cos x} dx = \frac{\pi^2}{4}$ [BE. 2068]

$\Rightarrow$ Sol^n

Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \cos x} dx$

by using properties of definite integral

$$I = \int_0^{\pi} \frac{(\pi - x) \tan x (\pi - x)}{\sec(\pi - x) + \cos(\pi - x)} dx$$

$= \sin t$

$$= \int_0^{\pi} \frac{-(\pi - x) \tan x \, dx}{-\sec x - \cos x}$$

$$= \int_0^{\pi} \frac{+(\pi - x) \tan x \, dx}{+(\sec x + \cos x)}$$

$$= \int_0^{\pi} \frac{\pi \tan x \, dx}{\sec x + \cos x} - \int_0^{\pi} \frac{x \tan x \, dx}{\sec x + \cos x}$$

$$= \pi \int_0^{\pi} \frac{\tan x \, dx}{\sec x + \cos x} - I$$

$$2I = \pi \int_0^{2 \times (\pi/2)} \frac{\tan x \, dx}{\sec x + \cos x}$$

$$= \pi \int_0^{2 \times (\pi/2)} \frac{\sin x}{1 + \cos^2 x} dx$$

$$= 2\pi \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx \quad \left[f\left(2\frac{\pi}{2} - x\right) = f(x) \right]$$

put $\cos x = t$

$-\sin x \, dx = dt$

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when $x = 0,\ t = 1$
when $x = \pi/2 ;\ t = 0$

$$I = -\pi \int_1^0 \frac{dt}{1+t^2}$$

$$= -\pi \left[\tan^{-1} t\right]_1^0$$

$$I = \frac{\pi^2}{4} \quad \#$$

Q.3) $\displaystyle\int_0^{\pi/2} \frac{\sin^2 x \, dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1).$

$\Rightarrow$ soln.

let $I = \displaystyle\int_0^{\pi/2} \frac{\sin^2 x \, dx}{\sin x + \cos x} dx$ —(1)

by using properties of definite integral

$$I = \int_0^{\pi/2} \frac{\left[\sin\left(\frac{\pi}{2} - x\right)\right]^2}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx \quad -(2)$$

Adding (1) & (2) we get

or, $2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x}\, dx$

$= \int_0^{\pi/2} \frac{1}{\cos x + \sin x}\, dx$

$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{1}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}\, dx$

$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{1}{\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}}\, dx$

$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{1}{\sin\left(x + \frac{\pi}{4}\right)}\, dx$

$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \operatorname{cosec}\left(x + \frac{\pi}{4}\right) dx$

$= \frac{1}{\sqrt{2}} \log \left[\operatorname{cosec}\left(x+\frac{\pi}{4}\right) - \cot\left(x+\frac{\pi}{4}\right)\right]_0^{\pi/2}$

$= \frac{1}{\sqrt{2}} \log \left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$

$= \frac{1}{\sqrt{2}} \log (\sqrt{2}+1)^2$

$2I = \frac{1}{\sqrt{2}} \times 2 \log (\sqrt{2}+1)$

$I = \frac{1}{\sqrt{2}} \log (\sqrt{2}+1)$. #

Q, 14) $\int_0^{\pi/2} \sin^5\theta \cos^3\theta \, d\theta$

let

$$I = \int_0^{\pi/2} \sin^5\theta \cos^3\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^5\theta \, (1-\sin^2\theta) \cos\theta \, d\theta$$

put $\sin\theta = t$, $\cos\theta \, d\theta = dt$

when $\theta = 0$, $t = 0$, when $\theta = \pi/2$

$t = 1$

$$= \int_0^1 t^5(1-t^2)\, dt = \int_0^1 t^5 dt - \int_0^1 t^7 dt$$

$$= \left[\frac{t^6}{6}\right]_0^1 - \left[\frac{t^8}{8}\right]_0^1$$

$$= \frac{1}{6} - \frac{1}{8}$$

$$I = \frac{2}{48} \#$$

$$= \frac{1}{24} \#$$

(b) $\int_0^{\pi/2} \frac{dx}{4+5\sin x}$

let $I = \int_0^{\frac{\pi}{2}} \frac{dx}{4+5\sin x} = \int_0^{\pi/2} \frac{dx}{4+10\sin\frac{x}{2}\cdot\cos\frac{x}{2}}$

$= \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}\,dx}{4\sec^2\frac{x}{2}+10\tan\frac{x}{2}} = \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}\,dx}{4+4\tan^2\frac{x}{2}+10\tan\frac{x}{2}}$

put $\tan\frac{x}{2} = t$, $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

when $x=0$, $t=0$

$x = \pi/2$, $t=1$

$I = \int_0^1 \frac{2\,dt}{4t^2+10t+4} = \frac{2}{4}\int_0^1 \frac{dt}{t^2+\frac{10t}{4}+1}$

$= \frac{2}{4}\int_0^1 \frac{dt}{t^2+2\cdot t\cdot\frac{5}{4}+\left(\frac{5}{4}\right)^2+1-\frac{25}{16}}$

$= \frac{2}{4}\int_0^1 \frac{dt}{\left(t+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}$

$= \cancel{\frac{2}{4}\int_0^1 \frac{dt}{\left(t+\frac{5}{4}\right)^2}}$

$= \frac{2}{4}\cdot\frac{4}{3}\left[\tan^{-1}\frac{t+5/4}{3/4}\right]_0^1$

$$= \frac{2}{3}\left[\tan^{-1}\frac{1+5/4}{3/4} - \tan^{-1}\frac{5/4}{3/4}\right]$$

$$= \frac{2}{3}\left\{\tan^{-1}3 - \tan^{-1}\frac{5}{3}\right\}$$

$$= \frac{2}{3}\tan^{-1}\frac{3-\frac{5}{3}}{1+3*\frac{5}{3}}$$

$$= \frac{2}{3}\tan^{-1}\frac{2}{9}$$

$$I = \frac{2}{3}\tan^{-1}\frac{2}{9} \quad \#$$

Q 5) (a) $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} = \frac{\pi}{4}$

=> Sol^n

$$= \int_a^a \frac{dx}{x+\sqrt{a^2-x^2}}$$

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

when $x=a$ $\quad a = a\sin\theta$

$\sin^{-1}(1) = 0 \quad \theta = \frac{\pi}{2}$

when $x=0 \quad 0 = a\sin\theta$

$\sin\theta = 0, \theta = \sin^{-1}(0)$

$$= \int_0^{\pi/2} \frac{a\cos\theta\, dx}{a\sin\theta + \sqrt{a^2 - a^2\sin^2\theta}}$$

$$= \int_0^{\pi/2} \frac{a\cos\theta d\theta}{a\sin\theta + \sqrt{a^2(1-\sin^2\theta)}}$$

$\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$$= \int_0^{\pi/2} \frac{a\cos\theta d\theta}{a\sin\theta + \sqrt{a^2\cos^2\theta}}$$

$$= \int_0^{\pi/2} \frac{a\cos\theta \, d\theta}{a\sin\theta + \sqrt{(a\cos\theta)^2}}$$

$$= \int_0^{\pi/2} \frac{a\cos\theta \, d\theta}{a\sin\theta + a\cos\theta}$$

$$= \int_0^{\pi/2} \frac{\cancel{a}\cos\theta \, d\theta}{\cancel{a}(\sin\theta + \cos\theta)}$$

$$= \int_0^{\pi/2} \frac{\cos\theta \, d\theta}{\sin\theta + \cos\theta} \quad -(1) \qquad \left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right) + \cos\left(\frac{\pi}{2}-\theta\right)} d\theta$$

$$= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta + \sin\theta} d\theta \quad -(2)$$

Adding (1) & (2)

$$2I = \int_0^{\pi/2} \frac{\sin\theta}{\sin\theta + \cos\theta} d\theta + \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta + \cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cancel{\cos\theta + \sin\theta}}{\cancel{\cos\theta + \sin\theta}} d\theta$$

$$2I = \int_0^{\pi/2} 1 \cdot d\theta = [\theta]_0^{\pi/2}$$

$$2I = \left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi}{2 \times 2} = \frac{\pi}{4} \#$$

(b) $\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2}+1)$.

=> Sol^n,

let $I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$

by using properties of definite integral.

$$I = \int_0^{\pi/2} \frac{\pi/2 - x}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\pi/2 - x}{\cos x + \sin x} dx$$

$$= \int_0^{\pi/2} \frac{1}{\cos x + \sin x} dx - \int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \frac{1}{\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x} dx - I$$

$$2I = \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \frac{1}{\cos\frac{\pi}{4} \sin x + \sin\frac{\pi}{4} \cos x} dx$$

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$$= \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \frac{1}{\sin\left(x + \frac{\pi}{4}\right)} dx = \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \operatorname{cosec}\left(x + \frac{\pi}{4}\right) dx$$

$$= \frac{\pi}{2\sqrt{2}} \log \left[\operatorname{cosec}\left(x + \frac{\pi}{4}\right) - \cot\left(x + \frac{\pi}{4}\right) \right]_0^{\pi/2}$$

$$= \frac{\pi}{2\sqrt{2}} \log \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right)$$

$$= \frac{\pi}{2\sqrt{2}} \log (\sqrt{2}+1)^2$$

$$2I = \frac{\pi}{\cancel{2}\sqrt{2}} \times \cancel{2} \log(\sqrt{2}+1)$$

$$2I = \frac{\pi}{\sqrt{2}} \log(\sqrt{2}+1)$$

$$I = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2}+1)$$

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