BCT- B First year

Bernoulli's Equation

6th class

Ex-2.4

Q.N. 5 $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$.

Soln: Given diff. eqn is $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

dividing both sides by $\cos^2 y$.

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{x\cdot 2\sin y\cos y}{\cos^2 y} = x^3$$

or $\sec^2 y \frac{dy}{dx} + 2x\tan y = x^3$ ——— (1)

put $\tan y = v$

diff. w.r.t. 'x' we get,

$$\sec^2 y\frac{dy}{dx} = \frac{dv}{dx}$$

Now eqn (1) becomes;

$$\frac{dv}{dx} + 2xv = x^3 \text{ ——— (2)}$$

which is linear diff. eqn in v

So it's I.F. $= e^{\int 2x\,dx} = e^{\frac{2x^2}{2}} = e^{x^2}$

Multiplying eqn (2) by I.F. $= e^{x^2}$ on both sides we get,

$$v \times e^{x^2} = \int x^3 \cdot e^{x^2} dx + C$$

$* \int e^{4x}dx = \frac{e^{4x}}{4}$

put $x^2 = t$

$\therefore 2x\,dx = dt \Rightarrow x\,dx = \frac{1}{2}dt$.

$$= \int x^2 \cdot x\, e^{x^2} dx + C$$

$$= \int t\, e^t \frac{dt}{2} + C$$

$$= \frac{1}{2}\int t\, e^t\, dt + C$$

$$= \frac{1}{2}\left(t e^t - \int 1 \cdot e^t dt\right) + C$$

$$= \frac{1}{2}\left(t e^t - e^t\right) + C$$

Since $t = x^2$

$$v \cdot e^{x^2} = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C$$

Since $v = \tan y$

$$\tan y\, e^{x^2} = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C$$

or, $2\tan y = x^2 - 1 + 2Ce^{-x^2}$

(7) $e^y - p^3 - p = 0$

Soln:

$$\frac{dv}{dx} - \frac{1}{x}v = -\frac{\log x}{x} \text{ ——— (2)}$$

I.F. $= \frac{1}{x}$

multiplying eqn (2) I.F. $= \frac{1}{x}$

$$\frac{1}{x}\left(\frac{dv}{dx} - \frac{1}{x}v\right) = -\frac{\log x}{x}\cdot\frac{1}{x}$$

$$\frac{d}{dx}\left(v \times \frac{1}{x}\right) = -\frac{\log x}{x^2}$$

$$\int d\left(v \times \frac{1}{x}\right) = -\int \frac{\log x}{x^2}dx. \text{ Integrating}$$

$$v \times \frac{1}{x} = -\int \frac{\log x}{x^2}dx + C$$

linear function $\times$ I.F. $= \int (Q \times I.F.)dx + C$

$v \times$ I.F.

or, $2\tan y = x^2 - 1 + 2ce^{-}$

$\therefore 2\tan y = x^2 - 1 + Ke^{-x^2}$

where $K = 2c$

Which is the required general soln of given diff. eqn. #

Exact diff. eqns

Exact diff. eqn: A diff. eqn

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be exact if there exists a function $f(x,y)$ such that

$$\boxed{M(x,y)\,dx + N(x,y)\,dy} = d\,f(x,y)$$

i.e. the given diff. eqn is exact if $M(x,y)\,dx + N(x,y)\,dy$ is exact or perfect diff eqn.

Note: The diff. eqn

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

will be exact if and only if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ denotes the partial derivative.

e.g. $u = ax^2 + 2hxy + by^2$

diff. p. w.r.t. 'x'

$$\frac{\partial u}{\partial x} = 2ax + 2hy + 0$$

diff. p. w.r.t 'y'

$$\frac{\partial u}{\partial y} = 0 + 2hx + 2by.$$

Note: Every diff. eqn $M(x,y)dx + N(x,y)dy = 0$ is not exact.

For example:

$x^2 dy + 2xy\,dx = 0$ is exact

because $x^2 dy + 2xy\,dx = d(x^2 y)$

Some Formulae:

1. $x\,dy + y\,dx = d(xy)$

2. $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$

3. $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$

4. $x\,dx + y\,dy = d\left(\dfrac{x^2}{2}\right) + d\left(\dfrac{y^2}{2}\right)$

$= d\left(\dfrac{x^2+y^2}{2}\right)$ and so on.

Ex-25

Solve the following diff. eq^n by reducing to exact form.

e.g. $x\,dy + y\,dx = 0$

Sol^n: Given diff. eq^n is

$x\,dy + y\,dx = 0$

or, $d(xy) = 0$

Integrating on both sides

$\int d(xy) = \int 0$

or, $xy = C$

which is the required general sol^n #

or

$x\,dy + y\,dx = 0$

or, $x\,dy = -y\,dx$

or, $\int \dfrac{1}{y}\,dy = -\int \dfrac{1}{x}\,dx$ Int.

or, $\log y = -\log x + \log$

$\therefore \log y = \log\left(\dfrac{c}{x}\right)$

or, $y = \dfrac{c}{x} \Rightarrow \boxed{xy = c}$

e.g. $\dfrac{dy}{dx} = \dfrac{2x - y + 1}{x + 2y - 3}$

Here $\dfrac{2}{1} \neq \dfrac{-1}{2}$

put $x = X + h$

$y = Y + K$.

$(x + 2y - 3)\,dy = (2x - y + 1)\,dx$

or $x\,dy + 2y\,dy - 3\,dy = 2x\,dx - y\,dx + dx$

or, $x\,dy + y\,dx + 2y\,dy - 3\,dy - 2x\,dx = dx$

or $d(xy) + 2y\,dy - 3\,dy - 2x\,dx = dx$

Int. on both sides

or $d(xy) + 2y\,dy - 3dy - 2x\,dx = dx$

Intd. on both sides

or, $\int d(xy) + 2\int y\,dy - 3\int dy - 2\int x\,dx = \int dx$

or, $xy + 2\frac{y^2}{2} - 3y - 2\cdot\frac{x^2}{2} = x + c$

$\Rightarrow xy + y^2 - 3y - x^2 - x = C$

which is required general solⁿ ##

$\boxed{\text{Ex-21, 23, 24}}$

{Ex-25} {Ex-22} [Ex-19, 18, 20]

(1) Q.N. 5 marks

(1) **First order but not first degree diff. eqns.**

An eqⁿ of the form $f(x, y, P) = 0$ where $P = \frac{dy}{dx}$ is called first order but not first degree diff. eqⁿ. The solution of such equations containts only one arbitary constants.

We will discuss the following first order but not first degree diff. eqⁿ

→ Solvable for p
→ Solvable for y
→ Solvable for x

→ Clairaut's Eqⁿ

Solvable for p

$p_x^2 + 5p_{xy} + 6y^2 = 0$

An eqⁿ of the form $f(x, y, p) = 0$ where $p = \frac{dy}{dx}$ can be factorized into linear factor such

$\{P - f_1(x,y)\}\{P - f_2(x,y)\}\cdots\{P - f_n(x,y)\}$

Such type of first order but not first degree diff. eqⁿ is called solvable for p. ##

Each factor equated to zero, we get the solⁿ of the form

$F_1(x, y, c_1) = 0$, $F_2(x, y, c_2) = 0$, ... $F_n(x, y, c_n) = 0$

It's general solⁿ is,

$F_1(x, y, c) \cdot F_2(x, y, c) \cdots F_n(x, y, c) = 0$

$F_1(x,y,c) \cdot F_2(x,y,c) \cdots F_n(x,y,c) = 0$

Ex-26

Solve the following diff. eqⁿs

1. $p^2 + p - 6 = 0$ where, $p = \frac{dy}{dx}$

or, $\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} - 6 = 0$

Solⁿ: Given diff. eqⁿ is

$f(x,y,p)=0 \leftarrow p^2 + p - 6 = 0$

Solvable for p

linear factor

or, $p^2 + 3p - 2p - 6 = 0$

or, $p(p+3) - 2(p+3) = 0$

or, $(p+3)(p-2) = 0$

Either $p+3=0$ or $p-2=0$

$\frac{dy}{dx} + 3 = 0$

or, $\int dy = -\int 3\,dx$ Int.

$y = -3x + c$

or, $y + 3x - c = 0$

or, $\frac{dy}{dx} - 2 = 0$

$\int dy = \int 2\,dx$ Int.

$y = 2x + c$

or, $y - 2x - c = 0$

∴ The required general solⁿ is,

$(y+3x-c)\cdot(y-2x-c) = 0$ Ans.

(2) $p^2 + 2px - 3x^2 = 0$

(3) $p^2 - p(e^x + e^{-x}) + 1 = 0$

Solⁿ: Given diff. eqⁿ is

$p^2 - p(e^x + e^{-x}) + 1 = 0$

or $p^2 - pe^x - pe^{-x} + 1 = 0$

or, $p(p - e^x) - e^{-x}\left(p - \frac{1}{e^{-x}}\right) = 0$

or, $p(p - e^x) - e^{-x}(p - e^x) = 0$

or, $(p - e^x)(p - e^{-x}) = 0$

Either $(p - e^x) = 0$

$\frac{dy}{dx} = e^{-x}$

$\int dy = \int e^{-x}\, dx$ Ind.

$\Rightarrow y = -e^{-x} + C$

or, $y + e^{-x} - C = 0$

or, $\frac{dy}{dx} = e^{x}$

$\int dy = \int e^{x}\, dx$ Ind.

$\Rightarrow y = e^{x} + C$

or, $y - e^{x} - C = 0$

$\therefore$ The required general soln is,

$(y + e^{-x} - c)(y - e^{x} - c) = 0$ Ans

(8) $p^3 - p(x^2 + xy + y^2) + x^2y + xy^2 = 0$

Soln: Given diff. eqn is

$p^3 - p(x^2 + xy + y^2) + x^2y + xy^2 = 0$

or, $p^3 - px^2 - pxy - py^2 + x^2y + xy^2 = 0$

or, $p^3 - px^2 - pxy + x^2y - py^2 + xy^2 = 0$

or, $p(\underbrace{p^2 - x^2}_{(p+x)(p-x)}) - xy(p-x) - y^2(p-x) = 0$

or, $(p-x)\{p(p+x) - xy - y^2\} = 0$

or, $(p-x)(p^2 + px - xy - y^2) = 0$

or, $(p-x)(p^2 - y^2 + px - xy) = 0$

or, $(p-x)\{(p+y)(p-y) + x(p-y)\} = 0$

or, $(p-x)(p-y)(p+y+x) = 0$

Either $p - x = 0$ | or $p - y = 0$

or, $p + y + x = 0$

$\frac{dy}{dx} + y + x = 0$

or $\frac{dy}{dx} + y = -x$ (1) first order first degree

Which is linear diff. eqn in y

So Its $I.F = e^{\int 1 \cdot dx} = e^{x}$

Multiplying eq^n ① by I.F. $= e^x$ on both sides we get

$$y \cdot e^x = -\int x e^x \, dx + C$$

$$= -\left(x e^x - \int 1 \cdot e^x \, dx\right) + C$$

$$e^x \cdot y = -x e^x + e^x + C$$

$$\Rightarrow y + x - 1 - Ce^{-x}$$

∴ The required general sol^n of the given diff. eq^n is

$$(\ldots) \quad (\ldots) \quad (y + x - 1 - Ce^{-x}) = 0$$

$$Ax^2 + Bx + C = 0$$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2 \cdot A}$$