Particular Integral

An equation is of the form $(D^2 + P_1D + P_2)\,y = Q$

i.e. $f(D)\,y = Q$ is called second order linear differential equation with constant coefficient.

Where, P_1, P_2 are constant and Q is function of x or constant.

Clearly, this equations is satisfied by,

$$y = \frac{1}{f(D)} \,.\, Q$$

So that $\frac{1}{f(D)} \,.\, Q$ is the particular Integral (P. I.) of the given differential equation.

$$\therefore \text{ P. I.} = \frac{1}{f(D)} \,.\, Q$$

The general solution of $(D^2 + P_1D + P_2)\,y = 0$ is called the complementary function (C. F.)

Theorem : II

If $y = f(x)$ is the complete solution of $f(D)\,y = 0$ and $y = g(x)$ is a particular solution of the equation $f(D)y = Q$ then the complete solution of the equation $f(D)\,y = Q$ is $y = f(x) + g(x)$.

Theorem : III

If Q be function of x then $\frac{1}{D}$ Q operates the integration of Q with respect to x.

i.e. $\frac{1}{D} Q = \int Q\, dx$.

Theorem : IV

$\frac{1}{(D-a)} Q = e^{ax} \int Q.e^{-ax}\, dx$ where a is any constant.

Working Rules for Finding Particular Integral (P.I.)

For P. I. $= \frac{1}{f(D)} \,.\, Q$

1. When $Q = e^{ax}$ where a is any constant.

   P. I. $= \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$; if $f(a) \neq 0$

Note : (i) If $f(a) = 0$ then,

$$\text{P.I.} = x.\frac{1}{f'(D)}e^{ax} = x\,.\frac{1}{f'(a)}e^{ax};\ \text{if } f'(a) \neq 0$$

(ii) If $f'(a) = 0$ then,

$$\text{P.I.} = x^2\frac{1}{f''(D)}e^{ax} = x^2\,.\frac{1}{f''(a)}e^{ax};\ \text{if } f''(a) \neq 0 \text{ and so on.}$$

2. When Q = sin ax . or cos ax

$$\text{P. I.} = \frac{1}{f(D^2)}\ \sin ax = \frac{1}{f\left(-a^2\right)}\sin ax \text{ if } f(-a^2) \neq 0.$$

Note : (i) If $f(-a^2) = 0$ then,

$$\text{P. I.} = x\,.\frac{1}{f'(D^2)}\ \sin ax = x\,.\frac{1}{f'(a^2)}\sin ax \text{ if } f'(-a^2) \neq 0$$

(ii) If $f'(-a^2) = 0$ then,

$$\text{P. I.} = x^2\frac{1}{f''(D^2)}\sin ax \text{ if } f''(-a^2) \neq 0 \text{ and so on.}$$

Similarly, $\text{P. I.} = \dfrac{1}{f\left(D^2\right)}\ \cos ax = \dfrac{1}{f\left(-a^2\right)}\cos ax$ if $f(-a^2) \neq 0$

Note : (i) If $f(-a^2) = 0$; then,

$$\text{P. I.} = x\frac{1}{f'(D^2)}\ \cos ax = x\frac{1}{f'(-a^2)}\ \cos ax \text{ if } f'(-a^2) \neq 0$$

(ii) $f'(-a^2) = 0$ then,

$$\text{P. I.} = x^2\,.\frac{1}{f''\left(D^2\right)}\ \cos ax = x^2\frac{1}{f''\left(-a^2\right)}\cos ax;$$

if $f''(-a^2) \neq 0$ and so on.

3. If $Q = x^m$, where m is positive integer.

Then, $\text{P. I.} = \dfrac{1}{f(D)}Q = \dfrac{1}{f(D)}\ x^m$

$$= \frac{1}{\left(D^2 + P_1D + P_2\right)}x^m = \frac{1}{p^2}\left(1 + \frac{D^2 + P_1D}{P_2}\right)^{-1}x^m$$

$[f(D)]^{-1}$ can be expanded in ascending power of D and then operate on x^m with each term of the expansions.

It can be expanded by the Binomial Theorem as follows:

(i) $(1 + D)^{-1} = 1 - D + D^2 - D^3 + \ldots\ldots\ldots$

(ii) $(1 - D)^{-1} = 1 + D + D^2 + D^3 + \ldots\ldots\ldots$

(iii) $(1 + D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \ldots\ldots\ldots$

(iv) $(1 - D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \ldots\ldots\ldots..$

4. If $Q = e^{ax}$ V where V is the function of x, then

$$\text{P. I.} = \frac{1}{f(D)}Q = \frac{1}{f(D)}\ e^{ax}\,V = e^{ax}\frac{1}{f(D+a)}\,.\,V$$

5. If Q = x V where V is function of x. Then

$$\text{P. I.} = \frac{1}{f(D)}\ Q = \frac{1}{f(D)}x\,V = x\frac{1}{f(D)}V - \frac{f'(D)}{\left[F(D)\right]^2}V$$

6. If $Q = x^m \cos ax$ and $Q = x^m \sin ax$, Then,

$$\text{P.I.} = \frac{1}{f(D)}Q = \frac{1}{f(D)}x^m\cos ax = \frac{1}{f(D)}\ [\text{Real part of } x^m e^{iax}]$$

and

$$\text{P. I.} = \frac{1}{f(D)}\ Q = \frac{1}{f(D)}\ x^m \sin ax$$

$$= \frac{1}{f(D)}\ [\text{Imaginary part of } x^m e^{iax}]$$

Note : General solution $(D^2y + P_1D + P_2)\,y = Q$ is y = C. F. + P. I.

Exercise - 30

Solve the following equation.

1. $(D^2 - 1)\,y = 5e^{2x}$

Sol^n. Given differential equation is,

$(D^2 - 1)\,y = 5e^{2x}$

So, It's A. E. is,

$m^2 - 1 = 0 \Rightarrow m = \pm 1$

So, C. F. $= C_1 e^x + C_2 e^{-x}$

and $\text{P. I.} = \dfrac{1}{\left(D^2 - 1\right)}\ 5e^{2x}$

$$= \frac{1}{\left(2^2 - 1\right)}\ 5e^{2x} = \frac{5}{3}e^{2x}$$

Thus, y = C. F. + P. I.

or, $y = C_1e^x + C_2 e^{-x} + \dfrac{5}{3}e^{2x}$ is the required general solution.

2. $(D - 2)^2y = e^{4x}$

Sol^n. Given differential equation is,

$(D - 2)^2\,y = e^{4x}$

So, it's A. E. is, $(m - 2)^2 = 0 \Rightarrow m = 2, 2$

So, C. F. $= (C_1 + C_2x)\,e^{2x}$

and $\text{P. I.} = \dfrac{1}{\left(D-2\right)^2}e^{4x} = \dfrac{1}{\left(4-2\right)^2}e^{4x} = \dfrac{1}{4}\ e^{4x}$

Thus, y = C. F. + P. I.

or, $y = (C_1 + C_2x)\, e^{2x} + \dfrac{e^{4x}}{4}$ is the required general solution.

3. $\mathbf{\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 2e^{3x}}$

Sol^n. Given differential equation is,

$\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 2e^{3x}$

or, $(D^2 - 4D + 3)\, y = 2e^{3x}$

So, It's A. E. is,

$m^2 - 4m + 3 = 0$

or, $m^2 - 3m - m + 3 = 0$

or, $m (m - 3) - 1 (m - 3) = 0$

or, $(m - 1) (m - 3) = 0$

$\Rightarrow m = 1, 3$

So, C. F. = $C_1e^x + C_2e^{3x}$

and P. I. = $\dfrac{1}{\left(D^2 - 4D + 3\right)} 2 . e^{3x}$

$= x . \dfrac{1}{(2D - 4)} 2 . e^{3x} = x . \dfrac{1}{(6-4)} . 2e^{3x} = x . \dfrac{1}{2} . 2\, e^{3x} = x\, e^{3x}$

Thus, y = C. P + F. I.

$y = C_1e^x + C_2\, e^{3x} + x\, e^{3x}$ is the required general solution.

4. $\mathbf{\dfrac{d^2y}{dx^2} + 4y = \sin 2x}$

Sol^n. Given differential equation is,

$\dfrac{d^2y}{dx^2} + 4y = \sin 2x$

or, $(D^2 + 4)\, y = \sin 2x$

So, it's A. E. is; $m^2 + 4 = 0 \Rightarrow m = \underline{+}\, 2i$

So, C. F. = A cos2x + B sin 2x

and P. I. = $\dfrac{1}{\left(D^2 + 4\right)} . \sin 2x$

$= x . \dfrac{1}{2D} \sin 2x = \dfrac{x}{2} \int \sin 2x\, dx = - \dfrac{x}{4} \cos 2x$

Thus, y = C. F. + P. I.

or, $y = A \cos 2x + B \sin 2x - \dfrac{x}{4} \cos 2x$ is the required general solution.

5. $\mathbf{(D^2 + 16) = \cos 4x}$ ***(B. E. 2068)***

Sol^n. Given differential equation is,

$(D^2 + 16)\, y = \cos 4x$

So, Its, A. E. is; $m^2 + 16 = 0 \Rightarrow m = \underline{+}\, 4i$

So, C. F. = A cos 4x – B sin 4x

and P. I. = $\dfrac{1}{\left(D^2 + 16\right)} . \cos 4x$

or, P. I. = $x . \dfrac{1}{2D} \cos 4x = \dfrac{x}{2} \int \cos 4x\, dx$

or, P.I. = $\dfrac{x}{8} \sin 4x$

Thus, y = C. F. + P. I.

or, $y = A \cos 4x + B \sin 4x + \dfrac{x}{8} \sin 4x$ is the required general solution.

6. $\mathbf{\dfrac{d^2y}{dx^2} + y = \cos^2 x}$

Sol^n. Given differential equation is,

$\dfrac{d^2y}{dx^2} + y = \cos^2 x$

or, $(D^2 + 1)\, y = \cos^2 x$

or, It's A. E. is; $m^2 + 1 = 0 \Rightarrow m = \underline{+}\, i$

So, C. F. = A cosx + B sinx

and P. I. = $\dfrac{1}{\left(D^2 + 1\right)} \cos^2 x = \dfrac{1}{\left(D^2 + 1\right)} \dfrac{\left(1 + \cos 2x\right)}{2}$

$= \dfrac{1}{2}\left[\dfrac{1}{\left(D^2 + 1\right)} e^{0.x} + \dfrac{1}{\left(D^2 + 1\right)} \cos 2x\right]$

$= \dfrac{1}{2}\left[1 + \dfrac{1}{\left(-2^2 + 1\right)} \int \cos 2x\right] = \dfrac{1}{2}\left(1 - \dfrac{\cos 2x}{3}\right)$

or, P. I. = $\dfrac{1}{2} - \dfrac{1}{6} \cos 2x$

Thus, y = C. F. + P. I.

or, $y = A \cos x + B \sin x + \dfrac{1}{2} - \dfrac{1}{6} \cos 2x$ is the required general solution.

7. $\mathbf{(D^2 - D - 2)\ y = \sin 2x + e^x}$ ***(B. E. 2072)***

Sol^n. Given differential equation is,

$(D^2 - D - 2)y = \sin 2x + e^x$

So, it's A. E. is $m^2 - m - 2 = 0$

or, $m^2 - 2m + m - 2 = 0$

or, $m (m - 2) + 1 (m - 2) = 0$

or, $(m + 1) (m - 2) = 0$

$\therefore\ m = -1, 2$

So, C. F. $= C_1 e^{-x} + C_2 e^{2x}$

and P. I. $= \dfrac{1}{\left(D^2 - D - 2\right)}$ $(\sin 2x + e^x)$

$$= \frac{1}{\left(D^2 - D - 2\right)} \sin 2x + \frac{1}{\left(D^2 - D - 2\right)} e^x$$

$$= \frac{1}{\left(-2^2 - D - 2\right)} \sin 2x + \frac{1}{1 - 1 - 2} e^x$$

$$= \frac{1}{-\left(D + 6\right)} \sin 2x - \frac{1}{2} e^x$$

$$= \frac{-\left(D - 6\right)}{\left(D^2 - 36\right)} \sin 2x - \frac{1}{2} e^x$$

$$= \frac{-\left(D - 6\right)}{\left(-2^2 - 36\right)} \sin 2x - \frac{1}{2} e^x$$

$$= \frac{2}{40} \cos 2x + \frac{6}{40} \sin 2x - \frac{1}{2} e^x$$

$$= \frac{1}{20} \cos 2x + \frac{3}{20} \sin 2x - \frac{1}{2} e^x$$

$$= \frac{1}{20} [\cos 2x + 3 \sin 2x] - \frac{1}{2} e^x$$

Thus, y = C. F. + P. I.

$y = C_1 e^{-x} + C_2 e^{2x} - \dfrac{1}{2} e^x + \dfrac{1}{20} (\cos 2x + 3 \sin 2x)$ is the required general solution.

8. $\mathbf{\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{2x} \sin x}$ ***(B. E. 2062, 071, 073- Shrawan)***

Sol^n. Given differential equation is,

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{2x} \sin x$$

or, $(D^2 + 3D + 2)\ y = e^{2x} \sin x$

So, It,s A. E. is;

$m^2 + 3m + 2 = 0$

or, $m^2 + 2m + m + 2 = 0$

or, $m (m + 2) + 1 (m + 2) = 0$

or, $(m + 2) (m + 1) = 0$

$\therefore\ m = -1, -2$

So, C. F. $= C_1 e^{-x} + C_2 e^{-2x}$

and P. I. $= \dfrac{1}{\left(D^2 + 3D + 2\right)} e^{2x} \sin x$

$$= e^{2x} \frac{1}{\left[\left(D + 2\right)^2 + 3\left(D + 2\right) + 2\right]} \sin x$$

$$= e^{2x} \frac{1}{\left(D^2 + 4D + 4 + 3D + 6 + 2\right)} \sin x$$

$$= e^{2x} \frac{1}{\left(D^2 + 7D + 12\right)} \sin x$$

$$= e^{2x} \frac{1}{\left(-1^2 + 7D + 12\right)} \sin x$$

$$= e^{2x} \frac{1}{\left(7D + 1\right)} \sin x$$

$$= e^{2x} \frac{\left(7D - 11\right)}{\left(49D^2 - 121\right)} \sin x$$

$$= e^{2x} \frac{\left(7D - 11\right)}{\left(-49 - 121\right)} \sin x$$

$$= e^{2x} \left(\frac{7}{-170} \cos x + \frac{11}{70} \sin x\right)$$

$$= \frac{e^{2x}}{170} (11 \sin x - 7 \cos x)$$

Thus y = C. F. + P. I.

or, $y = C_1e^{-x} + C_2e^{-2x} + \dfrac{e^{2x}}{170}$ $(11 \sin x - 7 \cos x)$ is the required general solution.

9. $\mathbf{\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2 e^{3x}}$

Sol^n. Given differential equation is,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2 e^{3x}$$
or, $(D^2 - 2D + 1)\, y = x^2 e^{3x}$
So, it's A. E.;
$$m^2 - 2m + 1 = 0$$
or, $(m - 1)^2 = 0 \Rightarrow m = 1, 1$
So, C. F. $= (C_1 + C_2x)\, e^x$

and P. I. $= \dfrac{1}{(D^2 - 2D + 1)}\, x^2 e^{3x}$

$$= e^{3x} \frac{1}{\left[(D+3)^2 - 2(D+3) + 1\right]} x^2$$

$$= e^{3x} \frac{1}{\left[D^2 + 6D + 9 - 2D - 6 + 1\right]} x^2$$

$$= e^{3x} \frac{1}{(D^2 + 4D + 4)} x^2$$

$$= \frac{e^{3x}}{4} \frac{1}{\left[1 + \left(\frac{D^2 + 4D}{4}\right)\right]} x^2$$

$$= \frac{e^{3x}}{4} \left[1 + \left(\frac{D^2 + 4D}{4}\right)\right]^{-1} . x^2$$

$$= \frac{e^{3x}}{4} \left[1 - \left(\frac{D^2 + 4D}{4}\right) + \left(\frac{D^2 + 4D}{4}\right)^2 - \ldots\right] x^2$$

$$= \frac{e^{3x}}{4} \left[x^2 - \left(\frac{2 + 8x}{4}\right) + \left(\frac{2}{4}\right)^2\right]$$

$$= \frac{e^{3x}}{4} \left[x^2 - \frac{1}{2} - 2x + \frac{1}{4}\right]$$

$$= \frac{e^{3x}}{4} \left(x^2 - 2x + \frac{3}{2}\right)$$

$$= \frac{e^{3x}}{8} (2x^2 - 4x + 3)$$

Thus, y = C. E. + P. I.

or, $y = (C_1 + C_2x)\, e^x + \dfrac{e^{3x}}{8}\,(2x^2 - 4x + 3)$ is the required general solution.

10. $(D^2 + 2)\, y = \sin \sqrt{2}\, x$

Sol^n. Given differential equation is,

$(D^2 + 2)\, y = \sin\left(\sqrt{2}\, x\right)$

So, it's A. E. is, $m^2 + 2 = 0 \Rightarrow m = \pm \sqrt{2}\, i$

So, C. F. $= A \cos \sqrt{2}\, x + B \sin \sqrt{2}\, x$

and P. I. $= \dfrac{1}{(D^2 + 2)} \sin(\sqrt{2}\, x)$

$$= x . \frac{1}{2D} \sin(\sqrt{2}\, x)$$

$$= \frac{x}{2} . \int \sin(\sqrt{2}\, x)\, dx$$

$$= -\frac{x}{2\sqrt{2}} \cos \sqrt{2}\, x$$

Thus, y = C. F. + P. I.

or, $y = A\cos(\sqrt{2}\, x) + B \sin(\sqrt{2}\, x) - \dfrac{x}{2\sqrt{2}} \cos(\sqrt{2}\, x)$ is the required general solution.

11. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 4y = e^x \cos x$

Sol^n. Given differential equation is,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x \cos x$$

or, $(D^2 - 2D + 4)\, y = e^x \cos x$
or, It's A. E. is, $m^2 - 2m + 4 = 0$

or, $m = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4.1.4}}{2.1} = \dfrac{2 \pm i\sqrt{12}}{2} = 1 \pm \sqrt{3}\, i$

So, C. E. $= e^x (A \cos \sqrt{3}\, x + B \sin \sqrt{3}\, x)$

and P. I. $= \dfrac{1}{(D^2 - 2D + 4)}\, e^x \cos x$

$$= e^x \frac{1}{\left[(D+1)^2 - 2(D+1) + 4\right]} \cos x$$

$$= e^x \frac{1}{\left(D^2 + 2D + 1 - 2D - 2 + 4\right)} \cos x$$

$$= e^x \frac{1}{\left(D^2 + 3\right)} \cos x$$

$$= e^x \frac{1}{\left(-1 + 3\right)} \cos x$$

$$= \frac{e^x \cos x}{2}$$

Thus, y = C. F. + P. I.

or, $y = e^x \; A \cos\sqrt{3}\, x + B\sin\sqrt{3}\, x + \frac{e^x \cos x}{2}$ is the required general solution.

12. $\mathbf{\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x}$

Sol^n. Given differential equation is,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$$

or, $(D^2 + D - 2)\, y = x + \sin x$

So, It's A. E. is;

$m^2 + m - 2 = 0$

or, $m^2 + 2m - m - 2 = 0$

or, $m(m + 2) - 1(m + 1) = 0$

or, $(m - 1)(m + 2) = 0$

or, $m = 1, 2$

So, C. F. = $C_1 e^x + C_2 e^{2x}$

and P. I. = $\frac{1}{\left(D^2 + D - 2\right)} \cdot (x + \sin x)$

$$= \frac{1}{\left(D^2 + D - 2\right)} x + \frac{1}{\left(D^2 + D - 2\right)} \sin x$$

$$= \frac{-1}{2} \left[1 - \left(\frac{D^2 + D}{2}\right)\right]^{-1} x + \frac{1}{\left(-1 + D - 2\right)} \sin x$$

$$= \frac{-1}{2} \left[1 + \left(\frac{D^2 + D}{2}\right) + ...\right] x + \frac{1}{\left(D - 3\right)} \sin x$$

$$= \frac{-1}{2} \left(x + \frac{1}{2}\right) + \frac{\left(D + 3\right)}{\left(D^2 - 9\right)} \sin x$$

$$= -\frac{x}{2} - \frac{1}{4} + \frac{\left(D + 3\right)}{\left(-1 - 9\right)} \sin x$$

$$= -\frac{x}{2} - \frac{1}{4} - \frac{\cos x}{10} - \frac{3}{10} \sin x$$

or, P. I. $= -\frac{x}{2} - \frac{1}{4} - \frac{1}{10} (\cos x + 3 \sin x)$

Thus, y = C. F. + P. I.

or, $y = C_1 e^x + C_2 e^{2x} - \frac{x}{2} - \frac{1}{4} - \frac{1}{10} (\cos x + 3 \sin x)$

or, $y = C_1 e^x + C_2 e^{2x} - \frac{1}{10} (\cos x + 3\sin x) - \frac{1}{4} (2x + 1)$

is the required general solution.

13. $\mathbf{(D^2 - 4D + 4)y = x^2 + e^{2x}}$

Sol^n. Given differential equation is,

$(D^2 - 4D + 4)\, y = x^2 + e^{2x}$

So, its, A. E. is,

$m^2 - 4m + 4 = 0$

or, $(m - 2) = 0$

or, $m = 2, 2$

So, C. F. = $(C_1 + C_2 x)\, e^{2x}$

and P. I. = $\frac{1}{\left(D^2 - 4D + 4\right)} \cdot (x^2 + e^{2x})$

$$= \frac{1}{\left(2 - D\right)^2} x^2 + \frac{1}{\left(D^2 - 4D + 4\right)} e^{2x}$$

$$= \frac{1}{4} \left(1 - \frac{D}{2}\right)^{-2} x^2 + x \cdot \frac{1}{\left(2D - 4\right)} e^{2x}$$

$$= \frac{1}{4} \left[1 + 2D + 3D^2 + ...\right] x^2 + x \cdot \frac{1}{\left(2D - 4\right)} e^{2x}$$

$$= \frac{1}{4} \left(x^2 + \frac{4x}{2} + \frac{6}{4}\right) + x^2 \frac{1}{2} e^{2x}$$

$$= \frac{1}{4} \left(x^2 + 2x + \frac{3}{2}\right) + \frac{x^2 e^{2x}}{2}$$

Thus, y = C. F. + P. I.

$$y = (C_1 + C_2 x)\, e^{2x} + \frac{1}{4} \left(x^2 + 2x + \frac{3}{2}\right) + \frac{x^2 e^{2x}}{2}$$

is the required general solution.

14. $(D-2)^2 y = x^2 e^{2x}$ ***(B. E. 2060)***

Sol^n. Given differential equation is,

$(D-2)^2 y = x^2 e^{2x}$

So, It's A. E. is, $(m-2)^2 = 0 \Rightarrow m = 2.\ 2$

So, C. E. $= (C_1 + C_2x)\, e^{2x}$

and P. I. $= \dfrac{1}{(D-2)^2} x^2 e^{2x}$

$$= e^{2x} \frac{1}{\{(D+2)-2\}^2} x^2$$

$$= e^{2x} \frac{1}{D^2} x^2 = e^{2x} \frac{1}{D} \int x^2\, dx = e^{2x} \frac{1}{D} \frac{x^3}{3} = \frac{e^{2x}}{3} \int x^3\, dx$$

$$= \frac{e^{2x}}{3} \cdot \frac{x^4}{4} = \frac{e^{2x}x^4}{12}$$

Thus, y = C. F. + P. I.

or, $y = (C_1 + C_2x)\, e^{2x} + \dfrac{e^{2x}x^4}{12}$ is the required general solution.

15. $(D^2 - 3D + 2)\, y = \cosh x$

Sol^n. Given differential equation is,

$(D^2 - 3D + 2)\, y = \cosh x$

So, A. E. is;

$m^2 - 3m + 2 = 0$

or, $m^2 - 2m - m + 2 = 0$

or, $m(m-2) - 1(m-2) = 0$

or, $(m-2)(m-1) = 0$

$\Rightarrow m = 1, 2$

So, C. F. $= C_1e^x + C_2e^{2x}$

and P. I. $= \dfrac{1}{(D^2 - 3D + 2)} \cosh x = \dfrac{1}{(D^2 - 3D + 2)} \dfrac{(e^x + e^{-x})}{2}$

$$= \frac{1}{2}\left[\frac{1}{(D^2 - 3D + 2)} e^x + \frac{1}{(D^2 - 3D + 2)} e^{-x}\right]$$

$$= \frac{1}{2}\left[x.\frac{1}{(2D-3)} e^x + \frac{1}{(-1)^2 - 3(-1) + 2} e^{-x}\right]$$

$$= \frac{1}{2}\left(x \cdot \frac{1}{2.1-3} e^x + \frac{1}{6} e^{-x}\right)$$

$$= -\frac{1}{2} xe^x + \frac{1}{12} e^{-x}$$

Thus, y = C. F. + P. I.

or, $y = C_1e^x + C_2e^{2x} + \dfrac{1}{12} e^{-x} - \dfrac{1}{2} x\, e^x$ is the required general solution.

16. $(D^2 - 1)\, y = \sinh x$

Sol^n. Given differential equation is,

$(D^2 - 1)\, y = \sin hx$

or, It's A. E. is $m^2 - 1 = 0 \Rightarrow m = 1.\ 1$

So, C. F. $= (C_1 + C_2x)\, e^x$

and P. I. $= \dfrac{1}{(D^2 - 1)} \sin hx$

$$= \frac{1}{(D^2-1)} \frac{(e^x - e^{-x})}{2}$$

$$= \frac{1}{2}\left[\frac{1}{(D^2-1)} e^x - \frac{1}{(D^2-1)} e^{-x}\right]$$

$$= \frac{1}{2}\left[x \frac{1}{2D} e^x - x \frac{1}{2D} e^{-x}\right]$$

$$= \frac{1}{4} x \left[\int e^x\, dx - \int e^{-x}\, dx\right]$$

$$= \frac{1}{4} x\, (e^x + e^{-x})$$

$$= \frac{x}{2}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{x}{2} \cosh x$$

Thus, y = C. F. + P. I.

or, $y = (C_1 + C_2x)\, e^x + \dfrac{x}{2} \cosh x$ is the required general solution.

17. $(D^2 + 4)\, y = x \sin^2 x$

Sol^n. Given differential equation is,

$(D^2 + 4)\, y = x \sin^2 x$

So, it's A. E. is, $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

So, C. F. $= A\cos 2x + B \sin 2x$

and P. I. $= \frac{1}{(D^2+4)} x \sin^2 x$

$$= \frac{1}{(D^2+4)} x \left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{(D^2+4)} x - \frac{1}{(D^2+4)} x\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1} x - \frac{1}{(D^2+4)} . \text{Real Part of } xe^{i2x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}\left(1-\frac{D^2}{4}+...\right)x - \text{Real part of}\left\{e^{i2x}\frac{1}{\left[(D+2i)^2+4\right]} . x\right\}\right]$$

$$= \frac{1}{8}x - \frac{1}{2}\text{ Real part of }\left[e^{i2x}\frac{1}{(D^2+4Di-4+4)} . x\right]$$

$$= \frac{1}{8}x - \frac{1}{2}\text{ Real part of }\left\{e^{i2x}\frac{1}{(D^2+4Di)} . x\right\}$$

$$= \frac{1}{8}x - \frac{1}{2}\text{ Real part of}\left\{e^{i2x}\frac{1}{4Di}\left(1+\frac{D}{4i}\right)^{-1} x\right\}$$

$$= \frac{1}{8}x - \frac{1}{2}\text{ Real part of }\left\{e^{i2x}\frac{1}{4Di}\left(1-\frac{D}{4i}+\left(\frac{D}{4i}\right)^2-...\right)x\right\}$$

$$= \frac{1}{8}x - \frac{1}{2}\text{ Real part of }\left\{e^{i2x}.\frac{1}{4Di}\left(x-\frac{1}{4i}\right)\right\}$$

$$= \frac{1}{8}x - \frac{1}{2}\text{ Real part of }\left[e^{i2x}\left(\frac{x^2}{8i}+\frac{x}{16}\right)\right]$$

$$= \frac{1}{8}x - \frac{1}{2}\text{ Real part of }\left\{\cos 2x + i\sin 2x\left[-\frac{ix^2}{8}+\frac{x}{16}\right]\right\}$$

$$= \frac{x}{8}+\frac{1}{32}\text{ real part of }\{\cos 2x + i\sin 2x\,(2ix^2 - x)\}$$

$$= \frac{x}{8}+\frac{1}{32}(-2x^2\sin 2x - x\cos 2x)$$

$\therefore$ P. I. $= \frac{x}{8} - \frac{x}{32}\cos 2x - \frac{1}{16}x^2\sin 2x$

Thus, y = C. F. + P. I.

$y = A\cos 2x + B\sin 2x + \frac{x}{8} - \frac{x}{32}\cos 2x - \frac{1}{16}x^2\sin 2x$ is the required solution.

18. $(D^2 - 4)\, y = x \sin hx$

Sol^n. Given differential equation is,

$(D^2 - 4)\, y = x \sin hx$

So, It's A. E. is $m^2 - 4 = 0 \Rightarrow m = \pm 2$

So, C. F. $= C_1e^{2x} + C_2e^{-2x}$

and P. I. $= \frac{1}{(D^2-4)} x \sin hx$

$$= \frac{1}{(D^2-4)} x . \left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{(D^2-4)} x e^x - \frac{1}{(D^2-4)} xe^{-x}\right]$$

$$= \frac{1}{2}\left\{e^x\frac{1}{\left[(D+1)^2-4\right]}x - e^{-x}\frac{1}{\left[(D-1)^2-4\right]}x\right\}$$

$$= \frac{1}{2}\left\{e^x\frac{1}{(D^2+2D+1-4)}x - e^{-x}\frac{1}{(D^2-2D+1-4)}x\right\}$$

$$= \frac{1}{2}\left\{e^x\frac{1}{(D^2+2D-3)}x - e^{-x}\frac{1}{(D^2-2D-3)}x\right\}$$

$$= \frac{1}{2}\left\{\frac{e^x}{-3}\left[1-\frac{(D^2+2D)}{3}\right]^{-1} x + \frac{e^{-x}}{3}\left[1-\frac{D^2-2D}{3}\right]^{-1} x\right\}$$

$$= \frac{1}{2}\left[\frac{e^x}{-3}\left\{1+\frac{D^2+2D}{3}+...\right\}x + \frac{e^{-x}}{3}\left\{1+\frac{D^2-2D}{3}+....\right\}x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{-3}\left(x+\frac{2}{3}\right)+\frac{e^{-x}}{3}\left(x-\frac{2}{3}\right)\right]$$

$$= -\frac{x}{3}\frac{\left(e^x - e^{-x}\right)}{2} - \frac{2}{18}\times 2\frac{\left(e^x + e^{-x}\right)}{2}$$

$$\text{P. I.} = -\frac{x}{3}\ \sinh x - \frac{2}{9}\ \cosh x$$

This, y = C. F. + P. I.

or, $y = C_1e^{2x} + C_2e^{-2x} - \frac{x}{3}\ \sinh x - \frac{2}{9}\ \cosh x$ is the required general solution.

19. $\mathbf{\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x}$

Sol^n. Given differential equation is,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\, e^x \sin x$$

or, $(D^2 - 2D + 1)\, y = x\, e^x \sin x$

So, its, A. E. is

$$m^2 - 2m + 1 = 0$$

or, $(m - 1)^2 = 0 \Rightarrow m = 1, 1$

So, C. F. = $(C_1 + C_2x)\, e^x$

and P. I. = $\frac{1}{\left(D^2 - 2D + 1\right)}\, x\, e^x \sin x$

$$= e^x\, \frac{1}{\left[\left(D+1\right)^2 - 2\left(D+1\right) + 1\right]}\, x \sin x$$

$$= e^x\, \frac{1}{\left(D^2 + 2D + 1 - 2D - 2 + 1\right)}\ x \sin x$$

$$= e^x\, \frac{1}{D^2}\ x \sin x$$

$$= e^x\left[x.\frac{1}{D^2}\sin x - \frac{2D}{\left(D^2\right)^2}\sin x\right]$$

$$= e^x\, [x\,(-\sin x) - 2\cos x]$$

$$= -\, e^x\, (x \sin x + 2 \cos x)$$

Thus, y = C. E. + P. I.

or, $y = (C_1 + C_2x)\, e^x - e^x\, (x \sin x + 2\cos x)$ is the required general solution.

20. $\mathbf{(D^2 + 2D + 1)\ y = x \cos x}$

Sol^n. Given differential equation is,

$(D^2 + 2D + 1)\ y = x \cos x$

So, it's A. E. is

$$m^2 + 2m + 1 = 0$$

or, $(m + 1)^2 = 0 \Rightarrow m = -1, -1$

So, C. F. = $(C_1 + C_2x)\, e^{-x}$

And P. I. = $\frac{1}{\left(D^2 + 2D + 1\right)}\ (x \cos x)$

$$= x\ \frac{1}{\left(D^2 + 2D + 1\right)} \cos x - \frac{2D + 2}{\left(D^2 + 2D + 1\right)^2}\ \cos x$$

$$= x\ \frac{1}{\left(-1 + 2D + 1\right)} \cos x - \frac{\left(2D + 2\right)}{\left(-1 + 2D + 1\right)^2}\ \cos x$$

$$= \frac{x}{2}.\frac{1}{D}\cos x - \frac{2(D+1)}{4D^2}\ \cos x$$

$$= \frac{x}{2}\sin x - \frac{1}{2D^2}\ (D + 1) \cos x$$

$$= \frac{x}{2}\ \sin x - \frac{1}{2D^2}\ (\sin x + \cos x)$$

$$= \frac{x}{2}\ \sin x - \frac{1}{2}\ (\sin x - \cos x)$$

Thus, y = C. F. + P . I.

or, $y = (C_1 + C_2x)\, e^{-x} + \frac{x}{2}\ \sin x - \frac{1}{2}\ (\sin x - \cos x)$ is the required general solution.