Homogeneous differential equations

An equation of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} \qquad (1)$$

Where $f(x, y)$and $g(x, y)$ are the homogeneous function of x and y are the **same degree** is called the homogeneous differential equations.

Solving method of homogeneous differential equations.

Put $y = vx$ in (1)

Diff. (1) w.r.t. x we get,

$$\frac{dy}{dx} = v\frac{dx}{dx} + x\frac{dv}{dx}$$

$$\frac{dy}{dx} = v.1 + x\frac{dv}{dx}$$

Now equation (1) becomes

$$v + x\frac{dv}{dx} = \phi(v)$$

Or $x\frac{dv}{dx} = \phi(v) - v$

Or $\frac{dv}{\phi(v)-v} = \frac{dx}{x}$

Which is the variable separable form and we solve it by integrating both sides we get the required solution.

Exercise

Solve the following differential equations.

1. $\frac{dy}{dx} = \frac{x+y}{x}$

Solution:

Given differential equations is

$$\frac{dy}{dx} = \frac{x+y}{x} \qquad (1)$$

This is homogeneous differential equations.

So put $y = vx$ in (1)

Diff. (1) w.r.t. x we get,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now equation (1) becomes

$$v + x\frac{dv}{dx} = \frac{x+vx}{x}$$

Or, $v + x\frac{dv}{dx} = \frac{x(1+v)}{x}$

Or, $v + x\frac{dv}{dx} = 1 + v$

Or, $x\frac{dv}{dx} = 1$

Or, $dv = \frac{1}{x}dx$ integrating on both sides we get;

Or, $\int dv = \int \frac{1}{x}dx$

Or, $v = logx + c$

Since $y = vx$

$\Longrightarrow v = \frac{y}{x}$

$$\therefore \frac{y}{x} = logx + c$$

$y = x(logx + c)$ which is the required solution.

2. $\frac{dy}{dx} = \frac{2x+y}{x}$

Solution:

Given differential equations is

$$\frac{dy}{dx} = \frac{2x + y}{x} (1)$$

This is homogeneous differential equations.

So put $y = vx$ in (1)

Diff. (1) w.r.t. x we get,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now equation (1) becomes

$$v + x\frac{dv}{dx} = \frac{2x + vx}{x}$$

Or, $v + x\frac{dv}{dx} = \frac{x(2+v)}{x}$

Or, $v + x\frac{dv}{dx} = 2 + v$

Or, $x\frac{dv}{dx} = 2$

Or, $dv = \frac{2}{x}dx$ integrating on both sides we get;

Or, $\int dv = 2\int \frac{1}{x}dx$

Or, $v = 2logx + c$

Or, $v = logx^2 + c$

Since $y = vx$

$\Longrightarrow v = \frac{y}{x}$

$$\therefore \frac{y}{x} = logx^2 + c$$

$y = x(logx^2 + c)$ which is the required solution.

3. $\frac{dy}{dx} = \frac{2y-x}{x}$

Solution:

Given differential equations is

$\frac{dy}{dx} = \frac{2y-x}{x}$ (1)

This is homogeneous differential equations.

So put $y = vx$ in (1)

Diff. (1) w.r.t. x we get,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now equation (1) becomes

$$v + x\frac{dv}{dx} = \frac{2vx - x}{x}$$

Or, $v + x\frac{dv}{dx} = \frac{x(2v-1)}{x}$

Or, $v + x\frac{dv}{dx} = 2v - 1$

Or, $x\frac{dv}{dx} = 2v - v - 1$

Or, $x\frac{dv}{dx} = v - 1$

Or, $\frac{dv}{v-1} = \frac{1}{x}dx$ integrating on both sides we get;

Or, $\int \frac{dv}{v-1} = \int \frac{1}{x}dx$

Or, $log(v - 1) = logx + logc$

Or, $log(v - 1) = logxc$

$$v - 1 = xc$$

Since $y = vx$

$\Longrightarrow v = \frac{y}{x}$

$$\therefore \frac{y}{x} - 1 = xc$$

$$y - x = x^2c$$

$$y - x = x^2c$$

which is the required solution.

4. $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$

Solution:

Given differential equations is

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2} \qquad (1)$$

This is homogeneous differential equations.

So put $y = vx$ in (1)

Diff. (1) w.r.t. x we get,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now equation (1) becomes

$$v + x\frac{dv}{dx} = \frac{xvx}{x^2 + v^2x^2}$$

Or, $v + x\frac{dv}{dx} = \frac{x^2v}{x^2(1+v^2)}$

Or, $v + x\frac{dv}{dx} = \frac{v}{(1+v^2)}$

Or, $x\frac{dv}{dx} = \frac{v}{(1+v^2)} - v$

Or, $x\frac{dv}{dx} = \frac{v-v-v^3}{(1+v^2)}$

Or, $x\frac{dv}{dx} = \frac{-v^3}{(1+v^2)}$

Or, $\frac{1+v^2}{v^3}dv = -\frac{1}{x}dx$ integrating on both sides we get;

Or, $\int \frac{1+v^2}{v^3}dv = -\int \frac{1}{x}dx$

Or, $\int \frac{1}{v^3}dv + \int \frac{v^2}{v^3}dx = -\int \frac{1}{x}dx$

Or, $\int \frac{1}{v^3} dv + \int \frac{1}{v} dx = -\int \frac{1}{x} dx$

Or, $\int v^{-3} dv + \int \frac{1}{v} dx = -\int \frac{1}{x} dx$

Or, $\frac{v^{-3+1}}{-3+1} + logv = -logx + c$

Or, $\frac{v^{-2}}{-2} + logv = -logx + c$

Or, $\frac{1}{-2v^2} + logv = -logx + c$

Since $y = vx$

$\Rightarrow v = \frac{y}{x}$

$$\therefore \frac{y}{x} = logx^2 + c$$

$y = x(logx^2 + c)$ which is the required solution.

5. $\frac{dy}{dx} = \frac{2x+y}{x}$

Solution:

Given differential equations is

$$\frac{dy}{dx} = \frac{2x + y}{x} (1)$$

This is homogeneous differential equations.

So put $y = vx$ in (1)

Diff. (1) w.r.t. x we get,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now equation (1) becomes

$$v + x\frac{dv}{dx} = \frac{2x + vx}{x}$$

Or, $v + x\frac{dv}{dx} = \frac{x(2+v)}{x}$

Or, $v + x\frac{dv}{dx} = 2 + v$

Or, $x\frac{dv}{dx} = 2$

Or, $dv = \frac{2}{x}dx$ integrating on both sides we get;

Or, $\int dv = 2\int \frac{1}{x} dx$

Or, $v = 2logx + c$

Or, $v = logx^2 + c$

Since $y = vx$

$\Longrightarrow v = \frac{y}{x}$

$$\therefore \frac{y}{x} = logx^2 + c$$

$y = x(logx^2 + c)$ which is the required solution.

6. $2xy\frac{dy}{dx} = x^2 + y^2$

Solution:

Given differential equations is

$$2xy\frac{dy}{dx} = x^2 + y^2$$

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \qquad (1)$$

This is homogeneous differential equations.

So put $y = vx$ in (1)

Diff. (1) w.r.t. x we get,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now equation (1) becomes

Or, $v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2xvx}$

Or, $v + x\frac{dv}{dx} = \frac{x^2(1+v^2)}{2xvx}$

Or, $v + x\frac{dv}{dx} = \frac{(1+v^2)}{2v}$

Or, $x\frac{dv}{dx} = \frac{(1+v^2)}{2v} - v$

Or, $x\frac{dv}{dx} = \frac{1+v^2-2v^2}{2v}$

Or, $x\frac{dv}{dx} = \frac{1-v^2}{2v}$

Or, $\frac{2v}{1-v^2}dv = \frac{1}{x}dx$ integrating on both sides we get;

Or, $\int \frac{2v}{1-v^2}dv = \int \frac{1}{x}dx$

put$1 - v^2 = t$

Diff. w.t.x $-2vdv = dt$

$$2vdv = -dt$$

Now$\int \frac{2v}{1-v^2}dv = -\int \frac{1}{t}dt = -logt = -log(1 - v^2)$

Or, $-log(1 - v^2) = logx + logc$

Or, $log(1 - v^2)^{-1} = logxc$

Or, $(1 - v^2)^{-1} = xc$

Or, $\frac{1}{1-v^2} = xc$

Since $y = vx$

$\Longrightarrow v = \frac{y}{x}$

Or, $\frac{1}{1-\left(\frac{y}{x}\right)^2} = xc$

Or, $\frac{x^2}{x^2-y^2} = xc$

Or, $x^2 = x(x^2 - y^2)c$

Or, $x = (x^2 - y^2)c$

which is the required solution.

7. $\frac{dy}{dx} = \frac{x+y}{x-y}$
8. $\frac{dy}{dx} = \frac{y}{x} - sin^2 \frac{y}{x}$

Solution:

Given differential equations is

$$\frac{dy}{dx} = \frac{2x + y}{x} (1)$$

This is homogeneous differential equations.

So put $y = vx$ in (1)

Diff. (1) w.r.t. x we get,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now equation (1) becomes

$$v + x\frac{dv}{dx} = \frac{2x + vx}{x}$$

Or, $v + x\frac{dv}{dx} = \frac{x(2+v)}{x}$

Or, $v + x\frac{dv}{dx} = 2 + v$

Or, $x\frac{dv}{dx} = 2$

Or, $dv = \frac{2}{x}dx$ integrating on both sides we get;

Or, $\int dv = 2\int \frac{1}{x}dx$

Or, $v = 2logx + c$

Or, $v = logx^2 + c$

Since $y = vx$

$\Longrightarrow v = \frac{y}{x}$

$$\therefore \frac{y}{x} = logx^2 + c$$

$y = x(logx^2 + c)$ which is the required solution.

9. $\frac{dy}{dx} = \frac{x+y}{x-y}$

10. $\frac{dy}{dx} = \frac{2x+y}{x}$

Solution:

Given differential equations is

$$\frac{dy}{dx} = \frac{2x + y}{x} (1)$$

This is homogeneous differential equations.

So put $y = vx$ in (1)

Diff. (1) w.r.t. x we get,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now equation (1) becomes

$$v + x\frac{dv}{dx} = \frac{2x + vx}{x}$$

Or, $v + x\frac{dv}{dx} = \frac{x(2+v)}{x}$

Or, $v + x\frac{dv}{dx} = 2 + v$

Or, $x\frac{dv}{dx} = 2$

Or, $dv = \frac{2}{x}dx$ integrating on both sides we get;

Or, $\int dv = 2\int \frac{1}{x}dx$

Or, $v = 2logx + c$

Or, $v = logx^2 + c$

Since $y = vx$

$\Longrightarrow v = \frac{y}{x}$

$$\therefore \frac{y}{x} = logx^2 + c$$

$y = x(logx^2 + c)$ which is the required solution.

11. $\frac{dy}{dx} = \frac{y}{x} - sin^2 \frac{y}{x}$

12. $\frac{dy}{dx} = \frac{2x+y}{x}$

Solution:

Given differential equations is

$$\frac{dy}{dx} = \frac{2x + y}{x} (1)$$

This is homogeneous differential equations.

So put $y = vx$ in (1)

Diff. (1) w.r.t. x we get,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now equation (1) becomes

$$v + x\frac{dv}{dx} = \frac{2x + vx}{x}$$

Or, $v + x\frac{dv}{dx} = \frac{x(2+v)}{x}$

Or, $v + x\frac{dv}{dx} = 2 + v$

Or, $x\frac{dv}{dx} = 2$

Or, $dv = \frac{2}{x}dx$ integrating on both sides we get;

Or, $\int dv = 2\int \frac{1}{x}dx$

Or, $v = 2logx + c$

Or, $v = logx^2 + c$

Since $y = vx$

$\Longrightarrow v = \frac{y}{x}$

$$\therefore \frac{y}{x} = logx^2 + c$$

$y = x(logx^2 + c)$ which is the required solution.

Exercise -21

Solve the following differential equations:

1. $\boldsymbol{x + y\frac{dy}{dx} = 2y}$

Solution:

Given differential equations is

$$x + y\frac{dy}{dx} = 2y$$

Or, $y\frac{dy}{dx} = 2y - x$

Or, $\frac{dy}{dx} = \frac{2y-x}{y}$ $\qquad\qquad\qquad\qquad (1)$

This is homogeneous differential equations.

So put $y = vx$ in (1)

Diff. (1) w.r.t. x we get,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now equation (1) becomes

$$v + x\frac{dv}{dx} = \frac{2vx - x}{vx}$$

Or,

$$v + x\frac{dv}{dx} = \frac{x(2v - 1)}{vx}$$

Or, $v+x\frac{dv}{dx}=\frac{2v-1}{v}$

Or $\quad x\frac{dv}{dx}=\frac{2v-1}{v}-v$

Or $\quad x\frac{dv}{dx}=\frac{2v-1-v^2}{v}$

Or $\quad x\frac{dv}{dx}=\frac{-(v^2-2v+1)}{v}$

Or $\quad x\frac{dv}{dx}=\frac{-(v^2-2v+1)}{v}$

Or, $\frac{v}{v^2-2v+1}dv=-\frac{1}{x}dx$ integrating on both sides we get;

Or, $\int\frac{v}{v^2-2v+1}dv=-\int\frac{1}{x}dx$

Or, $\int\frac{\frac{1}{2}(2v-2)+1}{v^2-2v+1}dv=-\int\frac{1}{x}dx$

Or, $\int\frac{\frac{1}{2}(2v-2)}{v^2-2v+1}dv+\int\frac{1}{v^2-2v+1}dv=-\int\frac{1}{x}dx$

Or, $\frac{1}{2}\int\frac{2v-2}{v^2-2v+1}dv+\int\frac{1}{(v-1)^2}dv=-\int\frac{1}{x}dx$

Or, $\frac{1}{2}log(v^2-2v+1)-\frac{1}{v-1}=-logx+c$

Or, $log(v-1)^{2.\frac{1}{2}} + logx = \frac{1}{v-1} + c$

Or, $log(v-1) + logx = \frac{1}{v-1} + c$

Or, $log\{(v-1).x\} = \frac{1}{v-1} + c$

Since $y = vx$

$\Longrightarrow v = \frac{y}{x}$

Or, $log\left\{\left(\frac{y}{x} - 1\right).x\right\} = \frac{1}{\frac{y}{x}-1} + c$

Or, $log\left\{\left(\frac{y-x}{x}\right).x\right\} = \frac{1}{\frac{y-x}{x}} + c$

$\therefore log(y-x) = \frac{x}{y-x} + c$

which is the required general solution.

2. $(x^2 - y^2)dx + 2xydy = 0$

Solution:

Given differential equations is

$$(x^2 - y^2)dx + 2xydy = 0$$

Or, $\frac{dy}{dx} = -\frac{x^2-y^2}{2xy}$ (1)

This is homogeneous differential equations.

So put $y = vx$ in (1)

Diff. (1) w.r.t. x we get,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now equation (1) becomes

$$v + x\frac{dv}{dx} = -\frac{x^2 - x^2v^2}{2x.vx}$$

Or, $v + x\frac{dv}{dx} = -\frac{x^2(1-v^2)}{2v.x^2}$

Or, $v + x\frac{dv}{dx} = -\frac{(1-v^2)}{2v}$

Or $x\frac{dv}{dx} = -\frac{(1-v^2)}{2v} - v$

Or $x\frac{dv}{dx} = \frac{-1+v^2-2v^2}{2v}$

Or $x\frac{dv}{dx} = \frac{-1-v^2}{2v}$

Or $x\frac{dv}{dx} = \frac{-(1+v^2)}{2v}$

Or $\frac{2v}{1+v^2}dv = -\frac{1}{x}dx$ integrating on both sides we get;

Or, $\int \frac{2v}{1+v^2}dv = -\int \frac{1}{x}dx$

Or, $log(1 + v^2) = -logx + logc$

Or, $log(1 + v^2) = log\frac{c}{x}$o

Or, $1 + v^2 = \frac{c}{x}$

Since $y = vx$

$\Rightarrow v = \frac{y}{x}$

Or, $1 + \frac{y^2}{x^2} = \frac{c}{x}$

Or, $\frac{x^2+y^2}{x^2} = \frac{c}{x}$

$\therefore x^2 + y^2 = cx$

which is the required general solution.

10. $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

Solution:

Given differential equations is

$$\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$$

Or, $\frac{dx}{dy}=-\frac{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{y}}}$ $\qquad (1)$

This is homogeneous differential equations.

So put $x=vy$ in (1)

Diff. (1) w.r.t. y we get,

$$\frac{dx}{dy}=v+y\frac{dv}{dy}$$

Now equation (1) becomes

$$v+y\frac{dv}{dy}=-\frac{e^{\frac{vy}{y}}\left(1-\frac{vy}{y}\right)}{1+e^{\frac{vy}{y}}}$$

0r,

$$v+y\frac{dv}{dy}=-\frac{e^{v}(1-v)}{1+e^{v}}$$

Or $\quad y\frac{dv}{dy}=-\frac{e^{v}(1-v)}{1+e^{v}}-v$

Or $y\frac{dv}{dy} = \frac{-e^v+ve^v-v-ve^v}{1+e^v}$

Or $y\frac{dv}{dy} = \frac{-e^v-v}{1+e^v}$

Or $y\frac{dv}{dy} = \frac{-(e^v+v)}{1+e^v}$

Or $\frac{1+e^v}{e^v+v}dv = -\frac{1}{y}dy$ integrating on both sides we get;

Or, $\int \frac{1+e^v}{e^v+v}dv = -\int \frac{1}{y}dy$

Or, $log(e^v + v) = -logy + logc$

Or, $log(e^v + v) = log\frac{c}{y}$

Or, $e^v + v = \frac{c}{y}$

Since $x = vy$

$\Longrightarrow v = \frac{x}{y}$

Or, $e^{\frac{x}{y}} + \frac{x}{y} = \frac{c}{y}$

Or, $ye^{\frac{x}{y}} + x = c$

$\therefore x + ye^{\frac{x}{y}} = c$

which is the required general solution.