Solvable for y

If an equation of the form y = f(x, p) then it can be solved by the method of solvable for y.

Let y = f (x, y) …… (i)

Differentiating with respect to x,

$$\frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right)$$

$$\text{or, } p = \phi\left(x, p, \frac{dp}{dx}\right) \text{ ……… (ii)}$$

Which is differential equation in the two variables x and p. Suppose its solution is f(x, p, c) = 0 …… (iii)

Eliminating p from (i) and (iii) gives the required solution. It p can not be eliminated from given two equations (i) and (iii), then the equations (i) and (iii) together give required solution.

Solvable for x

If an equation of the form x = f(y, p) then it can be solved by the method of solvable for x.

Let x = f(y, p) ……. (i)

Differentiating with respect to y,

$$\frac{dy}{dx} = \phi\left(y, p, \frac{dp}{dy}\right)$$

$$\text{or, } \frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right) \text{ …….. (ii)}$$

Which is a differential equation of two variables y and p. Suppose its solution be f(y, p, c)

Eliminating p from (i) and (iii) gives the required solution.

If p can not be eliminated from the given two equations (i) and (iii), then the equation (i) and (iii) together given required solution.

Exercise - 27

Solve the following equations

1. $\mathbf{p^3x - p^2y - 1 = 0}$

Sol^n. Given differential equation is,

$p^3x - p^2y - 1 = 0$

or, $p^2y = p^3x - 1$

or, $y = \frac{p^3x-1}{p^2} = px - \frac{1}{p}$ ……… (i)

Differential equation (i) w. r. t. 'x'

$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{1}{p^2}\frac{dp}{dx}$

or, $p = p + x\frac{dp}{dx} + \frac{1}{p^2}\frac{dp}{dx}$

or, $\left(x + \frac{1}{p^2}\right)\frac{dp}{dx} = 0$

Either $\frac{dp}{dx} = 0$

or, $\int dp = \int 0 \, . \, dx$; Integrating

$p = C$ ……. (ii)

From (i) and (ii)

$C^3x - C^2y - 1 = 0$ is the required solution.

2. $\mathbf{y = 2px + p^3y^2}$

Sol^n. Given differential equation is,

$y = 2px + p^3y^2$

or, $2px = y - p^3y^2$

or, $2x = \frac{y}{p} - p^2y^2$ …….. (i)

Differential equation (i) w. r. t. 'y'

or, $2\frac{dx}{dy} = \frac{p.1 - y\frac{dp}{dx}}{p^2} - 2p^2y - 2y^2p\frac{dp}{dy}$

or, $\frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2p^2y - 2y^2p\frac{dp}{dy}$

or, $\frac{2}{p} - \frac{1}{p} + 2p^2y = -\frac{dp}{dy}\left(\frac{y}{p^2} + 2y^2p\right)$

or, $\left(\frac{2 - 1 + 2yp^3}{p}\right) = \frac{-\left(y + 2y^2p^3\right)}{p^2}\frac{dp}{dy}$

or, $\frac{\left(1 + 2yp^3\right)}{p} = \frac{-y\left(1 + 2yp^3\right)}{p^2}\frac{dp}{dy}$

or, $1 = \frac{-y}{p}\frac{dp}{dy}$

or, $\int\frac{1}{y}dy = -\int\frac{1}{p}dp$; Integrating

$\log y = -\log p + \log C$

or, $\log yp = \log C \Rightarrow yp = C$

or, $p = \frac{C}{y}$ ……. (ii)

From (i) and (ii)

$y = 2 \, . \, \frac{C}{y}\,x + \left(\frac{C}{y}\right)^3 y^2$

$y^2 = 2Cx + C^3$ is the required solution.

3. $\mathbf{x = 4p + 4p^3}$

Sol^n. Given differential equation is,

$x = 4p + 4p^3$ ……. (i)

Differential equation (i) w. r. t. 'y'

$\frac{dx}{dy} = 4\frac{dp}{dy} = 4 \, . \, 3p^2\frac{dp}{dy}$

or, $\frac{1}{p} = (4 + 12p^2)\frac{dp}{dy}$

or, $\int dy = \int (4p + 12p^3)\, dp$ Integrating

$y = 2p^2 + 3p^4 + C$ …….(ii)

From (i) and (ii)

$x = 4p + 4p^3$ and $y = 2p^2 + 3p^4 + C$ is the required solution.

4. **siny cos px – cos y sin px = p**

Sol^n. Given differential equation is,

$\sin y \cos px - \cos y \sin px = p$

or, $\sin(y - px) = p$

or, $y - px = \sin^{-1}(p)$

or, $y = px + \sin^{-1}(p)$ …….. (i)

Differential equation (i) w. r. t. 'x'

$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$

or, $p = p + x\frac{dp}{dx} + \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$

or, $\frac{dp}{dx}\left(x + \frac{1}{\sqrt{1-p^2}}\right) = 0$

Either $\frac{dp}{dx} = 0$

or, $\int dp = \int 0 \,.\, dx$; Integrating

$p = C$ …….. (ii)

From (i) and (ii)

$y = Cx + \sin^{-1}C$

$y - C\,x = \sin^{-1}C$

or, $\sin(y - Cx) = C$ is the required solution.

5. $\mathbf{y + px = x^4p^2}$

Sol^n. Given differential equation is,

$y = x^4p^2 - px$ …….. (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = 2x^4p\,\frac{dp}{dx} + 4x^3p^2 - p - x\,\frac{dp}{dx}$$

or, $p - 4x^3p^2 + p = (2x^4\,p - x)\,\frac{dp}{dx}$

or, $(2p - 4x^3\,p^2) = (2x^4p - x)\,\frac{dp}{dx}$

or, $2p\,(1 - 2x^3p) = x\,(2x^3\,p - 1)\,\frac{dp}{dx}$

or, $2p\,(1 - 2x^3p) = -\,x\,(1 - 2x^3P)\,\frac{dp}{dx}$

or, $2p = -\,x\,\frac{dp}{dx}$

or, $2\int\frac{dx}{x} = -\int\frac{1}{p}\,dp$; Integrating

$2\log x = -\log p + \log C$

$\log x^2 + \log p = \log C$

or, $\log x^2p + \log C \Rightarrow x^2p = C$

or, $p = \frac{C}{x^2}$ ……… (ii)

From (i) and (ii)

$y = x^4,\ \frac{C^2}{x^4} - \frac{C}{x^2}\,.\,x$

or, $y = C^2 - \frac{C}{x}$

or, $xy = C^2\,x - C \Rightarrow xy + C = C^2\,x$ is the required solution.

6. $\mathbf{p^2 - py + x = 0}$

Sol^n. Given differential equations,

$p^2 - py + x = 0$

or, $yp = x + p^2$

or,. $y = \frac{x}{p} + p$ ……… (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = \frac{1}{p} - \frac{x}{p^2}\,\frac{dp}{dx} + \frac{dp}{dx}$$

or, $p - \frac{1}{p} = \frac{dp}{dx}\left(1 - \frac{x}{p^2}\right)$

or, $\frac{\left(p^2-1\right)}{p} = \frac{dp}{dx}\,\frac{\left(p^2-x\right)}{p^2}$

or, $(p^2 - 1) = \frac{\left(p^2-x\right)}{p}\,\frac{dp}{dx}$

$$\frac{dx}{dp} = \frac{\left(p^2-x\right)}{p\left(p^2-1\right)} = \frac{p}{p^2-1} - \frac{x}{p\left(p^2-1\right)}$$

or, $\frac{dx}{dp} + \frac{1}{p\left(p^2-1\right)}\,x = \frac{p}{p^2-1}$ ……. (ii)

Equation (ii) is linear in x so,

$$\text{I. F.} = e^{\int\frac{1}{p(p^2-1)}dp} = e^{\int\left(\frac{-1}{p}-\frac{p}{p^2-1}\right)dp} - e^{-\int\log p - \frac{1}{2}\log\left(p^2-1\right)}$$

$$= e^{-\left(\log p + \log\sqrt{p^2-1}\right)} = e^{-\log\frac{p}{\sqrt{p^2-1}}} = e^{\log\frac{\sqrt{p^2-1}}{p}}$$

I. F. $= \frac{\sqrt{p^2-1}}{p}$

Multiplying (ii) by I. F. we get,

$$x\,.\,\frac{\sqrt{p^2-1}}{p} = \int\frac{p}{p^2-1}\,.\,\frac{\sqrt{p^2-1}}{p}\,dp + C$$

$$= \int\frac{1}{\sqrt{\left(p^2-1\right)}}\,dp + C$$

or, $\frac{x\sqrt{p^2-1}}{p} = \log\left(p + \sqrt{p^2-1}\right) + C$

$$\text{or, } x = \frac{p\left\{\log\left(p+\sqrt{p^2-1}\right)+C\right\}}{\sqrt{p^2-1}} \text{ …….. (iii)}$$

From equation (i) and (iii)

$$\left.\begin{aligned} x &= \frac{p\left\{\log p+\sqrt{p^2-1}+C\right\}}{\sqrt{p^2-1}} \\ &\text{and } x = py - p^2 \end{aligned}\right\} \text{ is the required solution.}$$

7. $e^y - p^3 - p = 0$

Sol^n. Given differential equation is,

$e^y - p^3 - p = 0$

or, $e^y = p^3 + p$

or, $y = \log(p^3 + p)$ ……. (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = \frac{1}{\left(p^3+p\right)}(3p^2+1)\frac{dp}{dx}$$

$$\text{or, } p = \frac{\left(3p^2+1\right)}{p\left(p^2+1\right)}\frac{dp}{dx}$$

$$\text{or, } \int dx = \int \frac{3p^2+1}{p^2\left(p^2+1\right)} dp; \text{ Integrating}$$

$$\text{or, } \int dx = \int\left(\frac{2}{1+p^2}+\frac{1}{p^2}\right) dp$$

$$\text{or, } x = 2\tan^{-1} p - \frac{1}{p} + C \text{ …….. (ii)}$$

From equation (i) and (ii)

$x = 2\tan^{-1} p - \frac{1}{p} + C$ and $y = \log(p^3 + p)$ is the required solution.

8. $4(xp^2 + yp) = y^4$

Sol^n. Given differential equation is,

$4(xp^2 + yp) = y^4$

or, $4xp^2 + 4yp = y^4$

$4xp^2 = y^4 - 4yp$

$$\text{or, } 4x = \frac{y^4}{p^2} - \frac{4y}{p} \text{ ……. (i)}$$

Differential equation (i) w. r. t. 'y'

$$4\frac{dx}{dy} = \frac{p^2.4y^3 - y^4.2p\frac{dp}{dy}}{p^4} - \frac{4p-4y\frac{dp}{dy}}{p^2}$$

$$\text{or, } \frac{4}{p}+\frac{4}{p}-\frac{4y^3}{p^2} = \left(\frac{4y}{p^2}-\frac{2y^4}{p^3}\right)\frac{dp}{dy}$$

$$\text{or, } \left(\frac{8p-4y^3}{p^2}\right) = \frac{\left(4py-2y^4\right)}{p^3}\frac{dp}{dy}$$

$$\text{or, } \frac{4\left(2p-y^3\right)}{p^2} = \frac{2y\left(2p-y^3\right)}{p^3}\frac{dp}{dy}$$

$$\text{or, } 2 = \frac{y}{p}\frac{dp}{dy}$$

$$\text{or, } \int\frac{2}{y}dy = \int\frac{1}{p}dp; \text{ Integrating}$$

$2\log y = \log p = \log C$

$$\text{or, } y^2 = pC \Rightarrow p = \frac{y^2}{C} \text{ …….. (ii)}$$

From (i) and (ii)

$$4x = \frac{y^4}{y^4}\times C^2 - \frac{4y}{y^2}.C$$

$$\text{or, } 4x = C^2 - \frac{4}{y}C \text{ is the required solution.}$$

9. $x + \dfrac{p}{\sqrt{1-p^2}} = a$

Sol^n. Given differential equation is,

$$x + \frac{p}{\sqrt{1-p^2}} = a$$

$$\text{or, } x = a - \frac{p}{\sqrt{1-p^2}} \text{ ……. (i)}$$

Differential equation (i) w. r. t. 'y' we get,

$$\frac{dx}{dy} = -\frac{1}{\sqrt{1-p^2}}\frac{dp}{dy} - p.\frac{(-1)}{\left(1-p^2\right)^{-\frac{3}{2}}}.\frac{1}{2}.(-2p)\frac{dp}{dy}$$

$$\frac{1}{p} = \left[\frac{-1}{\sqrt{1-p^2}} - \frac{p^2}{\left(1-p^2\right)^{-\frac{3}{2}}}\right] \frac{dp}{dy}$$

$$\text{or, } \frac{1}{p} = -\left[\frac{1-p^2+p^2}{\left(1-p^2\right)^{-\frac{3}{2}}}\right] \frac{dp}{dy}$$

$$\text{or, } \frac{1}{p} = -\frac{1}{\left(1-p^2\right)^{-\frac{3}{2}}} \frac{dp}{dy}$$

$$\text{or, } \int dy = -\int \frac{p}{\left(1-p^2\right)^{-\frac{3}{2}}} dp\text{; Integrating}$$

Put $1 - p^2 = t$

$- 2p\, dp = dt$

$- p\, dp = \frac{1}{2}\, dt$

$$\text{or, } \int dp = \frac{1}{2} \int \frac{1}{t^{-\frac{3}{2}}}\, dt$$

$$\text{or, } y = -\frac{1}{t^{\frac{1}{2}}} - C$$

$$\text{or, } y + C = \frac{-1}{\sqrt{1-p^2}} \quad \text{Squaring on both side}$$

$$(y + C)^2 = \frac{1}{\left(1-p^2\right)}$$

$$1 - p^2 = \frac{1}{\left(y+C\right)^2} \quad \text{........ (ii)}$$

$$p^2 = 1 - \frac{1}{\left(y+C\right)^2} \quad \text{....... (iii)}$$

Also form equation (i) $(x - a)^2 = \dfrac{p^2}{1-p^2}$ .... (iv)

From (ii) and (iii) putting the value of p^2 and $1 - p^2$ in (iv) we get,

$$(x - a)^2 = \frac{1-\dfrac{1}{\left(y+C\right)^2}}{\dfrac{1}{\left(y+C\right)^2}}$$

or, $(x - a)^2 = (y + C)^2 - 1$

or, $(y + C)^2 - (x - a)^2 = 1$ is the required solution.

10. $\mathbf{xp^3 = a + bp}$

Sol^n. Given differential equation is,

$xp^3 = a + bp$

$$\text{or, } x = \frac{a}{p^3} + \frac{b}{p^2} \quad \text{......... (i)}$$

Differential equation (i) w. r. t. 'y'

$$\frac{dx}{dy} = -\frac{3a}{p^4}\frac{dp}{dy} - \frac{2b}{p^3}\frac{dp}{dy}$$

$$\text{or, } \frac{1}{p} = -\left(\frac{3a}{p^4} + \frac{2b}{p^3}\right)\frac{dp}{dy}$$

$$\text{or, } \frac{1}{p} = -\frac{\left(3a+2bp\right)}{p^4}\frac{dp}{dy}$$

$$\text{or, } 1 = -\left(\frac{3a}{p^3} + \frac{2b}{p^2}\right)\frac{dp}{dy}$$

$$\text{or, } \int dy = -\int\left(\frac{3a}{p^3} + \frac{2b}{p^2}\right) dp\text{; Integrating}$$

$$\text{or, } y = -\left\{\frac{3a}{-2p^2} + \frac{2b}{-p}\right\}$$

$$\text{or, } y = \frac{3a}{2p^2} + \frac{2b}{p} \quad \text{......... (ii)}$$

From (i) and (ii)

$y = \dfrac{3a}{2p^2} + \dfrac{2b}{p}$ and $x = \dfrac{a}{p^3} + \dfrac{b}{p^2}$ is the required solution.

11. $\mathbf{y = 2px + p^2}$

Sol^n. Given differential equation is,

$y = 2px + p^2$ .... (i)

differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

or, $p - 2p = (2x + 2p)\ \dfrac{dp}{dx}$

or, $-p = 2\ (x + p)\ \dfrac{dp}{dx}$

or, $\dfrac{dp}{dx} = \dfrac{2(x+p)}{-p}$

or, $\dfrac{dp}{dx} + \dfrac{2}{p}\ .\ x = -2$ …….. (ii)

Which is linear in p so,

I. F. $= e^{\int \frac{2}{p}dp} = e^{2\log p} = e^{\log p^2} = p^2$

Multiplying (ii) by I. F. we get,

or, $xp^2 = -2\int p^2 dp + C$

or, $xp^2 = -\dfrac{2}{3}\ p^3 + C$

$x = -\dfrac{2}{3}\ p + \dfrac{C}{p^2}$ …….. (iii)

From equation (i) and (iii)

$y = 2px + p^2$ and $x = -\dfrac{2}{3}p + \dfrac{C}{p^2}$ is the required solution.

12. $\mathbf{p^3 - p\ (y + 3) + x = 0}$

Sol^n. Given differential equation is,

$p^3 - p\ (y + 3) + x = 0$

or, $x = p\ (y + 3) - p^3$ ……. (i)

Differential equation (i) w. r. t. 'y'

$\dfrac{dx}{dy} = p + (y+3)\ \dfrac{dp}{dx} - 3p^2\ \dfrac{dp}{dy}$

or, $\dfrac{1}{p} - p = (y + 3 - 3p^2)\ \dfrac{dp}{dy}$

or, $\dfrac{\left(1-p^2\right)}{p} = (y + 3 - 3p^2)\ \dfrac{dp}{dy}$

or, $\dfrac{dp}{dy} = \dfrac{\left(y+3-3p^3\right)}{\left(1-p^2\right)}\ p$

or, $\dfrac{dp}{dy} - \dfrac{p}{1-p^2}\ .\ y = \dfrac{\left(3-3p^2\right)p}{\left(1-p^2\right)}$ …….. (ii)

Equation (ii) is linear in y so,

I. F. $= e^{\int -\frac{p}{1-p^2}dp} = e^{\frac{1}{2}\log(1-p^2)} = e^{\log\sqrt{1-p^2}} = \sqrt{1-p^2}$

Multiplying equation (ii) by I. F. we get,

$y\ .\ \sqrt{1-p^2} = \displaystyle\int \frac{3p\left(1-p^2\right)}{\left(1-p^2\right)}\ .\ \sqrt{1-p^2}\ dp + C$

$y\sqrt{1-p^2} = 3\int p\ \sqrt{1-p^2}\ dp + C$

Put $1 - p^2 = t$

$-2p\ dp = dt \Rightarrow p\ dp = -\dfrac{1}{2}\ dt$

or, $y\ \sqrt{1-p^2} = -\dfrac{3}{2}\displaystyle\int t^{\frac{1}{2}}\ dt + C$

$= -\dfrac{3}{2}\cdot\dfrac{2}{3}\ t^{\frac{3}{2}} + C$

$y\sqrt{1-p^2} = -\left(1-p^2\right)^{\frac{3}{2}} + C$

or, $y\sqrt{1-p^2} + \left(1-p^2\right)^{\frac{3}{2}} = C$ …….. (iii)

From (i) and (iii)

$y\sqrt{1-p^2} + \left(1-p^2\right)^{\frac{3}{2}} = C$ and $x = p\ (y + 3) - p^3$ is the required solution.

13. $\mathbf{x + yp = ap^2}$

Sol^n. Given differential equation is,

$x + yp = ap^2$

$yp = -x + ap^2$

or, $y = \dfrac{-x}{p} + ap$ …….. (i)

Differential equation (i) w. r. t. 'x' we get,

$\dfrac{dy}{dx} = -\left[\dfrac{1}{p} + x\left(\dfrac{-1}{p^2}\right)\dfrac{dp}{dx}\right] + \dfrac{adp}{dx}$

or, $p + \dfrac{1}{p} = \left(\dfrac{x}{p^2} + a\right)\dfrac{dp}{dx}$

or, $\dfrac{\left(p^2+1\right)}{p} = \dfrac{\left(x+ap^2\right)}{p^2}\ \dfrac{dp}{dx}$

$$\frac{dx}{dp}\left(\frac{p^2+1}{p}\right)=\frac{\left(x+ap^2\right)}{p^2}$$

$$\text{or, } \frac{dx}{dp}=\frac{\left(x+ap^2\right)}{p^2}\cdot\frac{p}{\left(p^2+1\right)}$$

$$=\frac{\left(x+ap^2\right)}{p\left(p^2+1\right)}=\frac{1}{p\left(p^2+1\right)}\ x\ .+\frac{ap}{\left(p^2+1\right)}$$

$$\text{or, } \frac{dx}{dp}-\frac{1}{p\left(p^2+1\right)}x=\frac{ap}{p^21}\ \text{........ (ii)}$$

Which is linear in p so,

$$\text{I.F.} = e^{-\int\frac{1}{p\left(p^2+1\right)}dp}=e^{\int\left[-\frac{1}{p}+\frac{p}{p^2+1}\right]dp}$$

$$=e^{-\log p+\frac{1}{2}\log\left(p^2+1\right)}=e^{-\log p+\log\sqrt{p^2+1}}=e^{\log\sqrt{\left(\frac{p^2+1}{p}\right)}}=\frac{\sqrt{p^2+1}}{p}$$

Multiplying (i) by I. F,. we get,

$$x\ .\ \frac{\sqrt{p^2+1}}{p}=\int\frac{ap}{\left(p^2+1\right)}\cdot\frac{\sqrt{p^2+1}}{p}dp+C$$

$$=a\int\frac{1}{\sqrt{p^2+1}}\ dp+C$$

$$\text{or, } \frac{x\sqrt{p^2+1}}{p}=a\log\left(p+\sqrt{p^2+1}\right)+C$$

$$\text{or, } x\left(p^2+1\right)^{\frac{1}{2}}=P\left[a\log\left(p+\sqrt{p^2+1}\right)\right]+C\ \text{........ (iii)}$$

From (i) and (iii)

$x + yp = ap^2$ and $x\left(p^2+1\right)^{\frac{1}{2}}=p\left[a\log\left(p+\sqrt{p^2+1}\right)\right]+C$ is the required solution.

14. y = sinp – pcosp

Sol^n. Given differential equation is,

y = sin p – p cos p. .... (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx}=\cos p\frac{dp}{dx}+p\sin p\frac{dp}{dx}-\cos p\frac{dp}{dx}$$

$$\text{or, } p=(\cos p+p\sin p-\cos p)\ \frac{dp}{dx}$$

$$\text{or, } p=p\sin p\frac{dp}{dx}$$

$$\text{or, } dx=\frac{p\sin p}{p}dp$$

or, $\int dx = \int \sin p\ d\ p$; Integrating

x = – cosp + C

or, x = C – cosp .... (ii)

from (i) and (ii)

y = sin p – p cosp and x = c –cos p is the required solution.

15. $p^2 – 2px + 1 = 0$

Sol^n. Given differential equation is,

$p^2 – 2px + 1 = 0$

or, $2px = p^2 + 1$

or, $2x = p + \frac{1}{p}$ .... (i)

Differential (i) w. r. t. ' y'

$$2\frac{dx}{dy}=\frac{dp}{dy}-\frac{1}{p^2}\ \frac{dp}{dy}$$

$$\text{or, } \frac{2}{p}=\left(1-\frac{1}{p^2}\right)\frac{dp}{dy}$$

$$\text{or, } \frac{2}{p}=\left(\frac{p^2-1}{p^2}\right)\frac{dp}{dy}$$

$$\text{or, } 2=\frac{\left(p^2-1\right)}{p}\ \frac{dp}{dy}$$

or, $\int dy=\frac{1}{2}\int\left(p-\frac{1}{p}\right)dp$; Integrating

$$y=\frac{1}{2}\left(\frac{p^2}{2}-\log p\right)+C$$

$$\text{or, } y=\frac{p^2}{4}-\frac{1}{2}\log p+C\ \text{........ (ii)}$$

From (i) and (ii)

$x = \frac{1}{2}\left(p + \frac{1}{2}\right)$ and $y = \frac{p^2}{4} - \frac{1}{2}$ log p + C is the required solution.

16. $y = (1 + p)\, x + ap^2$

Sol^n. Given differential equation is,

$y = (1 + p)\, x + ap^2$ …….. (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = (1 + p) + x\frac{dp}{dx} + a \,.\, 2p\frac{dp}{dx}$$

or, $p = 1 + p + (x + 2ap)\,\frac{dp}{dx}$

or, $-1 = (x + 2ap)\,\frac{dp}{dx}$

or, $\frac{dx}{dp} = -(x + 2ap)$

or, $\frac{dx}{dp} + x = -2ap$ …….. (ii)

Equation (ii) is linear differential equation

So, I.F. = $e^{\int 1.dp} = e^p$

Multiplying (ii) by I. F. we get,

$x \,.\, e^p = -2a \int pe^p\, dp + C$

$= -2a\,(pe^p - e^p) + C$

$xe^p = -2ae^p\,(p - 1) + C$

or, $x = -2a\,(p - 1) + ce^{-p}$

or, $x = 2a\,(1 - p) + ce^{-p}$ ……… (iii)

From (i) and (iii)

$x = 2a\,(1 - p) + C\, e^{-p}$ and $y = (1 + p)\, x + ap^2$ is the required solution.