# First Order but not First Degree Differential Equations

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An equation is of the form f (x, y, p) = 0 where $p = \frac{dy}{dx}$ is called the first order but not a first degree differential equation. The solution of such equations contrains only one arbitary constant.

**Solvable for p**

If the differential equation f (x, y, p) = 0 can be factorized into linear factor seen as $\{p - f_1 (x, y)\}\ \{p - f_2 (x, y)\} \ldots.$ $\{p - f_n (x, y)\}$, then it is called solvable for p.

Each factor equated to zero, we get the solution of the form.

$$F_1 (x, y, c) = 0, F_2 (x, y, c) = 0, \ \ldots\ldots.\ F_n (x, y, c_n) = 0$$

It general solution is,

$$F_1 (x, y, c), F_2 (x, y, c)\ \ldots..\ F_n (x, y, c) = 0$$

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**Solve the following equations**

**1. $p^2 + p - 6 = 0$**

**Sol$^n$.** Given differential equation is,

$p^2 + p - 6 = 0$

or, $p^2 + 3p - 2p - 6 = 0$

or, $p (p + 3) - 2 (p + 3) = 0$

or, $(p + 3) (p - 2) = 0$

Either $p + 3 = 0$

$\Rightarrow \frac{dy}{dx} = -3$

$\int dy = -\int 3dx$ Integrating

$y = -3x + C$

$\Rightarrow y + 3x + C = 0$

or, $p - 2 = 0$

$\Rightarrow \frac{dy}{dx} = 2$

$\int dy = \int 2dx$; Integrating

$y = 2x + C$

$\Rightarrow y - 2x + C = 0$

Hence the general solution is,

$(y + 3x + C) (y - 2x + C) = 0$

**2.** $\mathbf{p^2 + 2px - 3x^2 = 0}$

**Sol$^n$.** Given differential equation is,

$p^2 + 2px - 3x^2 = 0$

or, $p^2 + 3px - px - 3x^2 = 0$

or, $p(p + 3x) - x(p + 3x) = 0$

or, $(p + 3x)(p - x) = 0$

either $p + 3x = 0$

$\Rightarrow \dfrac{dy}{dx} = -3x$

$\int dy = -\int 3x\, dx$; Integration

or, $y = \dfrac{3x^2}{2} + C$

or, $\left(y - \dfrac{3x^2}{2} + C\right) = 0$

or, $(2y - 3x^2 - 2C) = 0$

or, $(2y - 3x^2 - K) = 0$

Where $K = 2C$

or, $p - x = 0$

$\Rightarrow \dfrac{dy}{dx} = x$

$\int dy = \int x\, dx$; Integration

or, $y = \dfrac{x^2}{2} + C$

or, $2y - x^2 = 2C$

or, $2y - x^2 - 2C = 0$

$(2y - x^2 - K) = 0$

where $K = 2C$

Hence the general solution is,

$(2y - 3x^2 - K)(2y - x^2 - K) = 0$

**3.** $\mathbf{p^2 - p(e^x + e^{-x}) + 1 = 0}$

**Sol$^n$.** Given differential equation is,

$p^2 - p(e^x + e^{-x}) + 1 = 0$

or, $p^2 - pe^x - pe^{-x} + 1 = 0$

or, $p(p - e^x) - e^{-x}(p - e^x) = 0$

or, $(p - e^x)(p - e^{-x}) = 0$

either $p - e^x = 0$

or, $\dfrac{dy}{dx} = e^x$

or, $\int dy = \int e^x dx$; Integration

or, $y = e^x + C$

or, $(y - e^x - C) = 0$

or, $p - e^{-x} = 0$

or, $\dfrac{dy}{dx} = e^{-x}$

or, $\int dy = \int e^{-x}\, dx$; Integration

or, $y = -e^{-x} + C$

or, $y + e^{-x} - C = 0$

Hence the general solution is,

$(y - e^x + C)(y + e^{-x} + C) = 0$

**4.** $\mathbf{p(p^2 + xy) = p^2(x + y)}$

**Sol$^n$.** Given differential equation is,

$p(p^2 + xy) = p^2(x + y)$

or, $p^3 + pxy - p^2x - p^2y = 0$

or, $p(p^2 + xy - px - py) = 0$

or, $p(p^2 - px + xy - py) = 0$

or, $p\{p(p - x) + y(x - p)\} = 0$



or, $p\{p(p - x) - y(p - x)\} = 0$

or, $p(p - x)(p - y) = 0$

either $p = 0$

or, $\dfrac{dy}{dx} = 0$

or, $\int dy = \int dx$; Int.

or, $y = x + C$

$\Rightarrow y - x - C = 0$

or, $p - x = 0$

or, $\dfrac{dy}{dx} = x$

or, $\int dy = \int x\, dx$; int.

or, $y = \dfrac{x^2}{2} + \dfrac{C}{2}$

or, $2y = x^2 + C$

$\Rightarrow (2y - x^2 - C) = 0$

or, $p - y = 0$

or, $\dfrac{dy}{dx} = y$

or, $\int \dfrac{1}{y}\, dy = \int dx$; int.

or, $\log y = x + \log C$

or, $\log \dfrac{y}{C} = x$

$\Rightarrow y - Ce^x = 0$

Hence, general solution is,

$(y - x - C)(2y - x^2 - C)(y - Ce^x) = 0$

**5.** $\mathbf{yp^2 + (x - y)p - x = 0}$

**Sol$^n$.** Given differential equation is,

$yp^2 + (x - y)p - x = 0$

or, $yp^2 + xp - yp - x = 0$

or, $yp^2 - yp + xp - x = 0$

or, $yp(p - x) + x(p - x) = 0$

or, $(p - x)(yp + x) = 0$

either, $p - x = 0$

or, $\dfrac{dy}{dx} = x$

or, $\int dy = \int x\, dx$; integration

or, $y = \dfrac{x^2}{2} + \dfrac{C}{2}$

or, $2y - x^2 - C = 0$

or, $yp + x = 0$

or, $y\dfrac{dy}{dx} = -x$

or, $\int y\, dy = -\int x\, dx$; Integration

or, $\dfrac{y^2}{2} = -\dfrac{x^2}{2} + \dfrac{C}{2}$

or, $y^2 = -x^2 + C$

or, $y^2 + x^2 - C = 0$

Hence, the general solution is,

$(2y - x^2 - C)(y^2 + x^2 - C) = 0$

**6.** $\mathbf{p^2 + 2px + py + 2xy = 0}$

**Sol$^n$.** Given differential equation is,

$p^2 + 2px + py + 2xy = 0$

or, $p(p + 2x) + y(p + 2x) = 0$

or, $(p + 2x)(p + y) = 0$

either $p + 2x = 0$

or, $\dfrac{dy}{dx} = -2x$

or, $\int dy = -\int 2x\, dx$; Integration

or, $p + y = 0$

or, $\dfrac{dy}{dx} = -y$

or, $\int \dfrac{1}{y}\, dy = -\int dx$ Integration or, $y = -x^2 + C$

or, $y + x^2 - C = 0$

or, $\log y = -x + \log C$

or, $\log \frac{y}{C} = -x$

or, $y = Ce^{-x}$

or, $y - Ce^{-x} = 0$

Hence, the general solution is,

$(y + x^2 - C)(y - Ce^{-x}) = 0$

**7.** $\mathbf{p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0}$

**Sol$^n$.** Given, Differential equation is,

$p^3 + 2xp^2 - y^2p^2 - 2x\,y^2p = 0$

or, $p^2(p + 2x) - y^2p(p + 2x) = 0$

or, $(p + 2x)(p^2 - y^2p) = 0$

or, $p(p + 2x)(p - y^2) = 0$

either $p = 0$

$\frac{dy}{dx} = 0$

or, $\int dy = \int 0.\,dx$; Int.

or, $y = C$

or, $p + 2x = 0$

or, $\frac{dy}{dx} = -2x$

or, $\int dy = \int -2x\,dx$; Int.

or, $y = -x^2 + C$

or, $p - y^2 = 0$

or, $\frac{dy}{dx} = y^2$

$\int \frac{1}{y^2}\,dy = \int dx$; Int.

or, $-\frac{1}{y} = x + C$

or, $-1 = xy + Cy$

or, $xy + 1 + Cy = 0$

Hence, the general solution.

$(y - C)(y + x^2 - C)(xy + 1 + Cy) = 0$

**8.** $\mathbf{p^3 - p(x^2 + xy + y^2) + x^2y + xy^2 = 0}$

**Sol$^n$.** Given differential equation is,

$p^3 - p(x^2 + xy + y^2) + x^2y + xy^2 = 0$

or, $p^3 - px^2 - pxy - py^2 + x^2y + xy^2 = 0$

or, $p^3 - px^2 - pxy + x^2y - py^2 + xy^2 = 0$

or, $p(p^2 - x^2) - xy(p - x) - y^2(p - x) = 0$

or, $p(p + x)(p - x) - xy(p - x) - y^2(p - x) = 0$

or, $(p - x)\{p(p + x) - xy - y^2\} = 0$

or, $(p - x)(p^2 + px - xy - y^2) = 0$

or, $(p - x)(p^2 - y^2 + px - xy) = 0$

or, $(p - x)\{1(p^2 - y^2) + x(p - y)\} = 0$

or, $(p - x)\{(p + y)(p - y) + x(p - y)\} = 0$

or, $(p - x)(p - y)(p + y + x) = 0$

either $p - x = 0$

or, $\frac{dy}{dx} = x$

or, $\int dy = \int x\,dx$; Int.

or, $y = \frac{x^2}{2} + \frac{C}{2}$

or, $2y = x^2 + C$

or, $2y - x^2 - C = 0$

or, $p - y = 0$

or, $\frac{dy}{dx} = y$

or, $\int \frac{1}{y}\,dy = \int dx$; Int.

or, $\log \frac{y}{C} = x$

or, $y = C\,e^x$

or, $y - C\,e^x = 0$

or, $p + y + x = 0$

or, $\frac{dy}{dx} + y = -x$

which is linear differential form

$\therefore$ I.F. $= e^{\int p dx} = e^{\int dx} = e^x$

Multiplying it by I.F. we get

$y.e^x = \int -x\,e^x\,dx + C$

or, $ye^x = -x\,e^x + e^x + C$

or, $y = -x + 1 + C\,e^{-x} = 0$

or, $y + x - 1 - C\,e^{-x} = 0$

Hence, $(2y - x^2 - C)(y - C\,e^x)(y + x - 1 - C\,e^{-x}) = 0$



**9.** $\mathbf{x^2\left(\frac{dy}{dx}\right)^2 + xy\,\frac{dy}{dx} - 6y^2 = 0}$

**Sol$^n$.** Given differential equation is,

$x^2\left(\frac{dy}{dx}\right)^2 + xy\,\frac{dy}{dx} - 6y^2 = 0$

or, $x^2p^2 + xyp - 6y^2 = 0$ where $p = \frac{dy}{dx}$

or, $x^2p^2 + 3xyp - 2xyp - 6y^2 = 0$

or, $xp(xp + 3y) - 2y(xp + 3y) = 0$

or, $(xp + 3y)(xp - 2y) = 0$

either $xp + 3y = 0$

or, $x\frac{dy}{dx} = -3y$

or, $\int \frac{1}{y}\,dy = -\int \frac{3}{x}\,dx$; Integration

or, $\log y = -3\log x + \log C$

or, $\log y + \log x^3 = \log C$

or, $\log yx^3 = \log C$

or, $yx^3 = C$

or, $yx^3 - C = 0$

or, $xp - 2y = 0$

or, $x\frac{dy}{dx} = 2y$

or, $\int \frac{1}{y}\,dy = \int \frac{2}{x}\,dx$; Integration

or, $\log y = 2\log x + \log C$

or, $\log y = \log x^2 + \log C$

or, $\log y = \log x^2C$

or, $y = C\,x^2$

or, $y - C\,x^2 = 0$

Hence the general solution is,

$(yx^3 - C)(y - C\,x^2) = 0$

**10.** $\mathbf{\left(\frac{dy}{dx}\right)^2 + 2y\cot x\,\frac{dy}{dx} = y^2}$

**Sol$^n$.** Given differential equation is,

$\left(\frac{dy}{dx}\right)^2 + 2y\cot x\,\frac{dy}{dx} = y^2$

or, $p^2 + 2y\cot x\,p - y^2 = 0$ where $\frac{dy}{dx} = p$.

Which is quadratic in p So,

$$p = \frac{-2y\cot x \pm \sqrt{(2y\cot x)^2 - 4.1.y^2}}{2.1}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x - 4y^2}}{2}$$

$$= \frac{-2y\cot x \pm 2y\sqrt{\cot^2 x - 1}}{2}$$

$$= y\,(-\cot x \pm \operatorname{cosec} x) = y\left(-\frac{\cos x}{\sin x} \pm \frac{1}{\sin x}\right)$$

$$= y\,\frac{(-\cos x \pm 1)}{\sin x}$$

Taking (+ ve) sign,

$$p = y\,\frac{(-\cos x + 1)}{\sin x} \quad \text{or,} \quad \frac{dy}{dx} = y\,\frac{(-\cos x + 1)}{\sin x}$$

$$\int \frac{1}{y}\,dy = \int \frac{(1-\cos x)(1+\cos x)}{\sin x\,(1+\cos x)}\,dx; \text{ Integrating}$$

$$\text{or, } \log y = \int \frac{\sin^2 x}{\sin x\,(1+\cos x)}\,dx + \log C$$

$$\text{or, } \log y = -\int \frac{\sin x}{1+\cos x}\,dx + \log C$$

or, log y = – log (1 + cosx) + log C
or, log y + log (1 + cosx) = log C
or, log y (1 + cosx) = log C
or, y (1 + cosx) = C
or, (y + y cosx – C) = 0
Taking (– ve) sign,

$$p = y\,\frac{(-\cos x - 1)}{\sin x}$$

$$\text{or, } \frac{dy}{dx} = -y\,\frac{(1+\cos x)\,(1-\cos x)}{\sin x\,(1-\cos x)} \quad \text{or, } \frac{dy}{dx} = -y\,\frac{\sin^2 x}{\sin x(1-\cos x)}$$

$$\text{or, } \frac{dy}{dx} = -y\,\frac{\sin x}{(1-\cos x)} \quad \text{or, } \int \frac{1}{y}\,dy = -\int \frac{\sin x}{(1-\cos x)}\,dx; \text{ Integrating}$$

or, log y = – log (1 – cosx) + log C
or, log y = log (1 – cosx) = log C
or, log y (1 – cosx) = log C
or, y (1 – cosx) = C
or, y – y cosx – C = 0
Hence, the general solution is
(y + y cosx – C) (y – y cosx – C) = 0