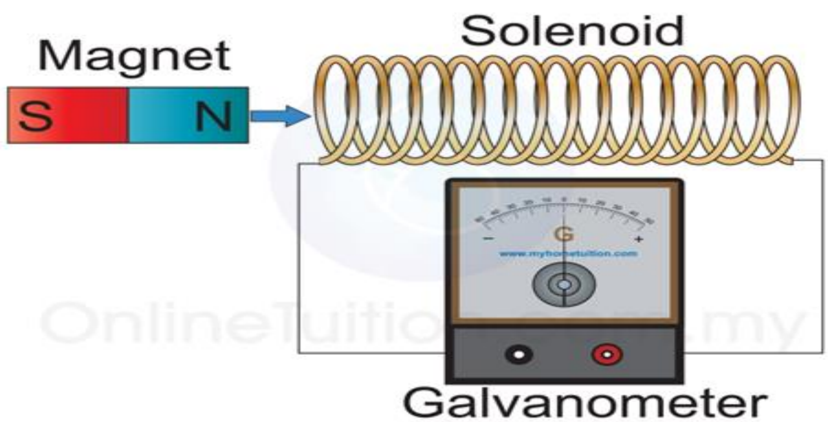
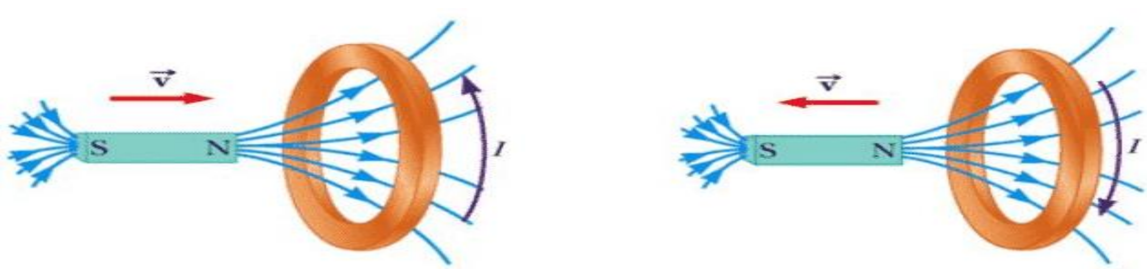
Electromagnetic Induction

The phenomenon of generation of induced emf in a coil due to change in magnetic flux is called electromagnetic induction. For example, an electric generator produces a current because of electromagnetic induction.



Faraday's Law of Induction:-



1. First law:-

Whenever the amount of magnetic flux linked with a coil changes an induced emf is developed in the circuit. The induced emf last as long as the change in magnetic flux continues.

2. Second law:-

The magnitude of emf induced in a conducting loop is equal to the rate of change of magnetic flux through that loop with time.

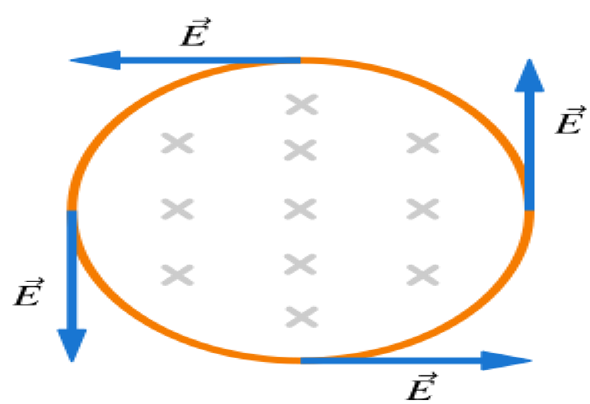
$$i.e. induced\ emf\,(\epsilon) = -\frac{d\phi_B}{dt}$$

$$\therefore \int \vec{E}\,.\,\overrightarrow{dl} = \frac{-d\phi_B}{dt}$$

Induced Electric Field:-

Consider a cupper ring of radius 'r' is placed in uniform magnetic field 'B'. If we increase the strength of this field at steady state rate, then by Faraday's law an induced current and an induced emf appear in the ring.

$$i.e.\,\text{induced emf}(\epsilon) = -\frac{d\phi_B}{dt} \dots\dots\dots (i)$$



If there is a current in the ring, an electric field must be present along the ring. This electric field is called induced electric field. This is real as that produced by a static charge. A changing magnetic field would still create an electric field in the free space in the absence of conducting loop.

Consider a moving charge q_o around a circular path of figure as shown. The work done on it by induced emf is.

$$W = q_o V = q_o \epsilon \text{ (}\epsilon\text{ is induced emf)}$$

Also,

$$W = \int \vec{F} . \overrightarrow{dl} = \int q_o \vec{E} . \overrightarrow{dl}$$

$$\therefore W = q_o \int \vec{E} . \overrightarrow{dl}$$

$$\therefore q_o \epsilon = q_o \int \vec{E} . \overrightarrow{dl}, \qquad \overrightarrow{E} \text{ is induced electric field}$$

$$So, \qquad \epsilon = \int \vec{E} . \overrightarrow{dl} \ldots \ldots \ldots (2)$$

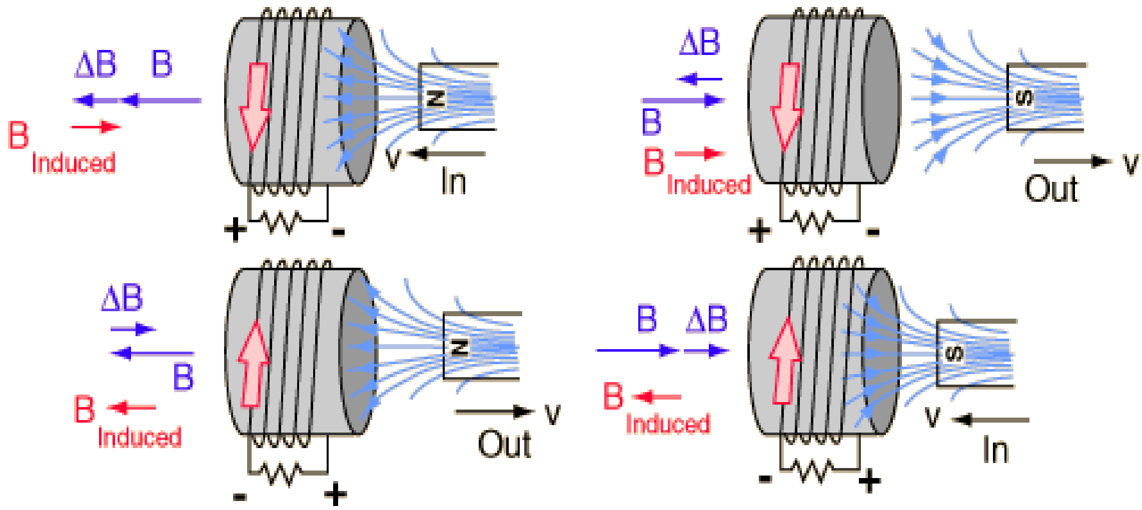
From equation (1) and (2) we can Rewrite Faraday’s law as

$$\int \vec{E} . \overrightarrow{dl} = \frac{-d\emptyset_B}{dt}$$

This equation says that changing magnetic field induces an electric field.

Lenz's Law:-

Lenz law is used to determine the direction of induced current in a loop, which states that "the induced current flow always in such a direction that it opposes the change in magnetic field".



When magnet north pole approaches the loop, it increases the magnetic flux in loop. The induced current flows in the loop, such that the magnetic field produced by it tend to decrease the applied field 'B'. Therefore the induced current is in anticlockwise direction.

Similarly the movement of magnet is in opposite direction. It decreases the magnetic flux through the loop. So the magnetic field produced by induced current must have same direction as that of applied field. Therefore, the induced current is in clockwise direction.

Self induction:-

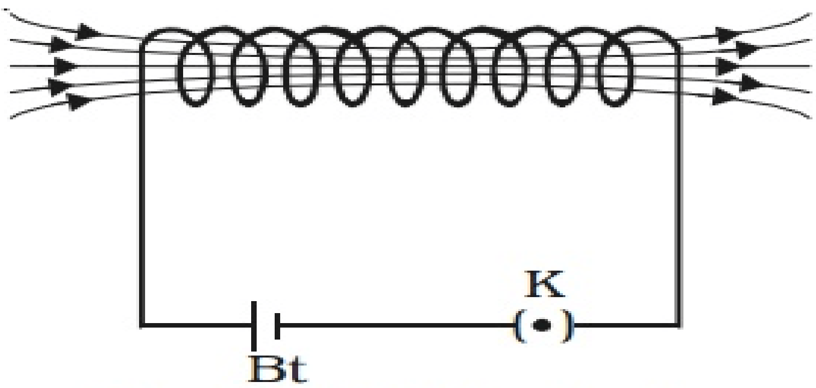


Fig *Self Induction*

The phenomenon in which the induced emf is produced as a result of change in current flowing through the same coil is known as self induction and thus produced emf is called self induced emf.

Let a coil consists of 'N' turns with flux in each turn ($\emptyset_B$). Then According to Henry the total flux through coil ($N\emptyset_B$) is directly proportional to current flowing through it.

$$i.e.\ N\emptyset_B \propto I$$

$$or\ \ N\emptyset_B = LI$$

$$\therefore\ L = \frac{N\emptyset_B}{I}$$

Where, L is a constant, called self inductance of coil.

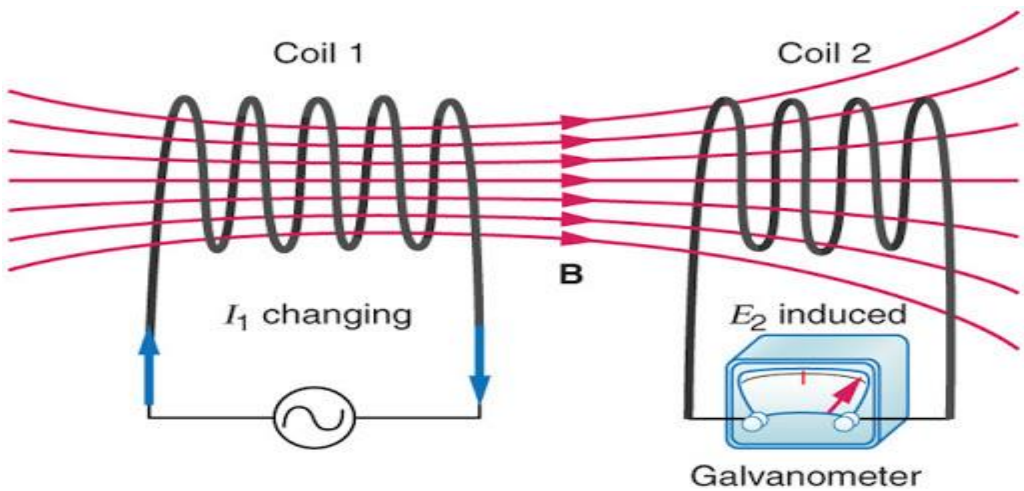
According to Faraday's law of induction;

Induced $emf\ (\epsilon) = -\frac{d\emptyset_B}{dt} = \frac{-d(LI)}{dt} = -L\frac{dI}{dt}$

$$\therefore\ \epsilon = -L\frac{dI}{dt}$$

Thus in any inductor a self induced emf appear when the current changes with time.

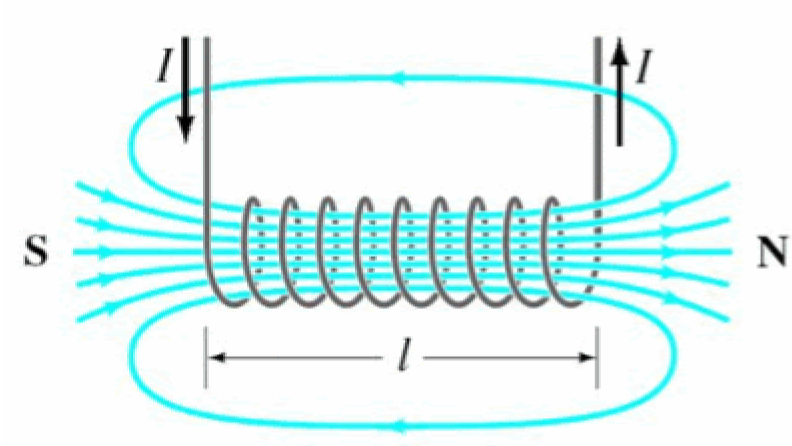
Mutual Induction:-



The phenomenon in which the induced emf is produced as a result of change in current flowing through another coil is known as mutual induction. Thus produced emf is called mutual induced emf.

Inductance of a Solenoid:-

Consider a solenoid of length '*l*' cross section area 'A' and having 'n' turns per unit length.



Therefore, total number of turns $(N) = nl$

Therefore, the flux linkage in length 'l' of solenoid is;

$$\therefore N\phi_B = NBA = nlBA$$

The magnetic field due to solenoid is;

$$B = \mu_0 nI$$

$$\therefore N\phi_B = nlA\mu_0 nI$$

$$= \mu_0 n^2 AlI$$

$$\therefore \text{Inductance of solenoid } (L) = \frac{N\phi_B}{I} = \frac{\mu_0}{I} n^2 AlI$$

$$\therefore L = \mu_0 n^2 Al$$

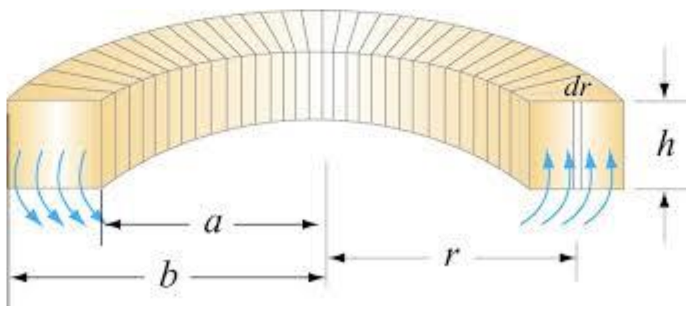
Inductance per unit length $\left(\frac{L}{l}\right) = \mu_0 n^2 A$

Hence, self inductance depends upon the length, area, and number of turns. i.e. geometry of the solenoid.

Self inductance of toroid:-

A toroid is a solenoid bent in to the shape of a hollow doughnut. Consider a toroid having rectangular cross sectional area A carrying current I . 'a' and 'b' are the internal and external radii of toroid. 'h' be the width of cross section of toroid.

Let us take a small strip of length 'dr' at a distance 'r' from centre. Since 'h' be the width of the strip, therefore its area is $h.dr$.



The magnetic flux through toroid is;

$$\phi_B = \int \vec{B}.\overrightarrow{dA} = \int_a^b B.hdr$$

The magnetic field due to toroid is, $B = \frac{\mu_0 NI}{2\pi r}$, where N is the total number of turns in toroid.

$$\therefore \ \phi_B = \frac{\mu_0 NIh}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 NIh}{2\pi}\ln b/a$$

$$\therefore \text{Inductance of toroid, } L = \frac{N\emptyset_B}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln b/a$$

L-R Circuit:-

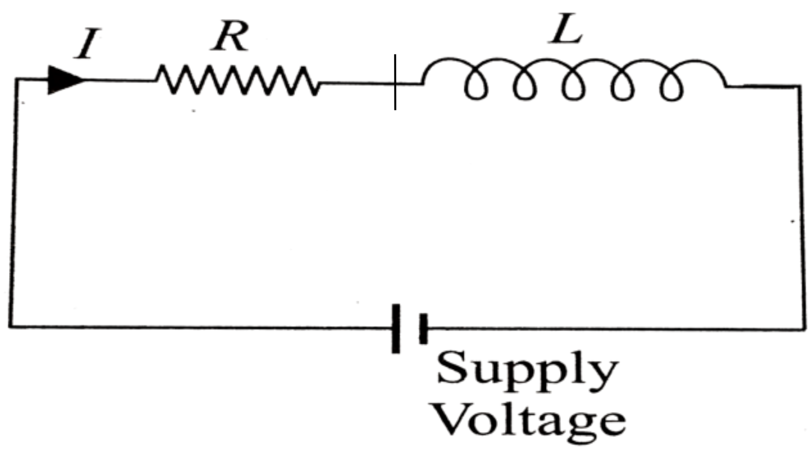


Figure:- LR circuit

Let a resistor 'R' inductor 'L' are connected in series with a battery of emf 'V' as shown in figure. When the switch is closed, the current in resistor start to raise. If the inductor were not present it would rise to steady value V/R. Because of inductor, a self induced emf appear in circuit, so current in the circuit will be less than V/R.

Now close the switch, the current flow in anticlockwise direction in above circuit. Now, applying Kirchhoff's voltage law;

$$V = L\frac{dI}{dt} + IR$$

$$or, \qquad \frac{V}{R} = \frac{L}{R}\frac{dI}{dt} + I$$

Here, $\frac{V}{R} = I_0$ is maximum current and

$$\frac{L}{R} = \tau \text{ is inductive time constant.}$$

$$\therefore I_0 = \tau\frac{dI}{dt} + I$$

$$or, \qquad \tau\frac{dI}{dt} = I_0 - I$$

$$or, \qquad \frac{dI}{I_0 - I} = \frac{dt}{\tau}$$

Now, integrating both sides we get;

$$\int_0^I \frac{dI}{I_0 - I} = \int_0^t \frac{dt}{\tau}$$

$$or, \qquad -[\ln(I_0 - I)]_0^I = t/\tau$$

$$or, \qquad \ln(I_0 - I) - \ln(I_0) = -t/\tau$$

$$or, \qquad ln\frac{I_0 - I}{I_0} = -t/\tau$$

$$or, \qquad \frac{I_0 - I}{I_0} = e^{-t/\tau}$$

$$or, \qquad I_0 - I = I_0 e^{-t/\tau}$$

$$\therefore I = I_0\left(1 - e^{-\frac{t}{\tau}}\right) \ldots\ldots\ldots (1)$$

This shows that the current growth is exponential in L-R circuit.

When, $t = \tau$ then;

$$I = I_0\,(1 - e^{-1}) = 0.63 I_0$$

Thus the time constant τ is the time required to current to reach about 63% of its maximum value.

Now the decay of current starts due to absence of battery. Now using Kirchhoff's voltage law;

$$0 = L\frac{dI}{dt} + IR$$

$$or, \qquad IR = -L\frac{dI}{dt}$$

$$or, \qquad I = -\frac{L}{R}\frac{dI}{dt}$$

$$or, \qquad \frac{dI}{I} = -\frac{dt}{\tau}$$

Now, integrating both sides we get;

$$\int_{I_0}^{I} \frac{dI}{I} = \int_{0}^{t} \frac{-dt}{\tau}$$

$$or, \qquad \ln\left(\frac{I}{I_0}\right) = -t/\tau$$

$$or, \qquad \frac{I}{I_0} = e^{-t/\tau}$$

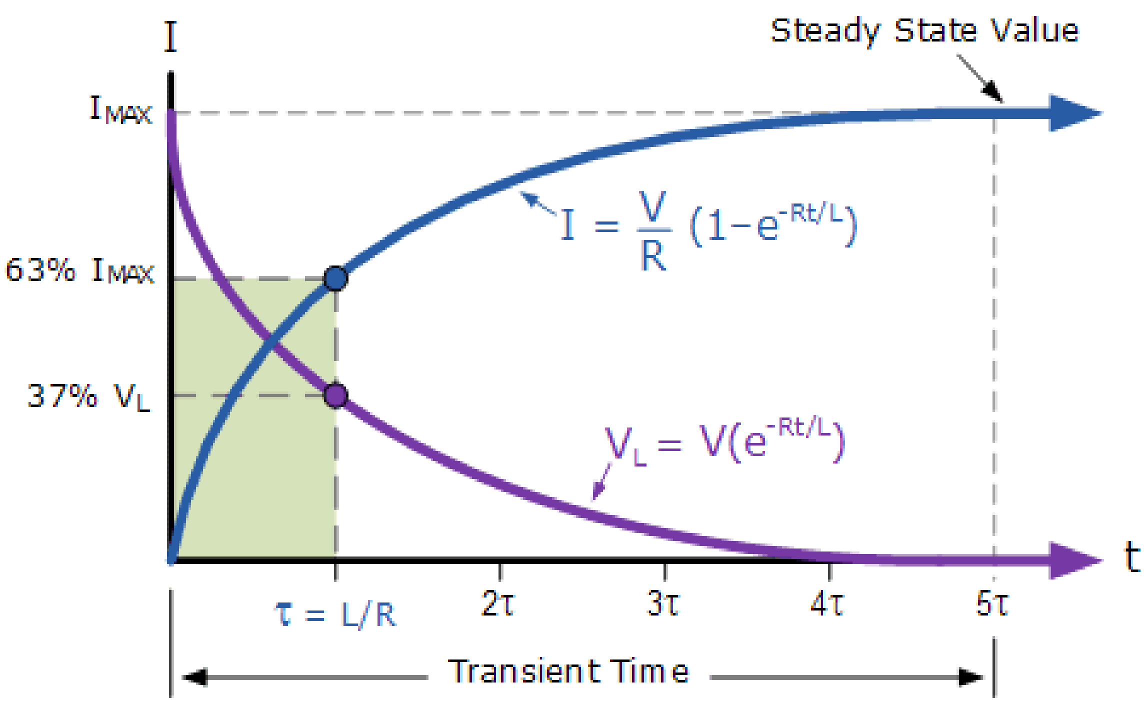
$$\therefore I = I_0\left(e^{-\frac{t}{\tau}}\right) \dots\dots\dots (2)$$

This shows that the decay of current L-R circuit is also exponential in L-R circuit.

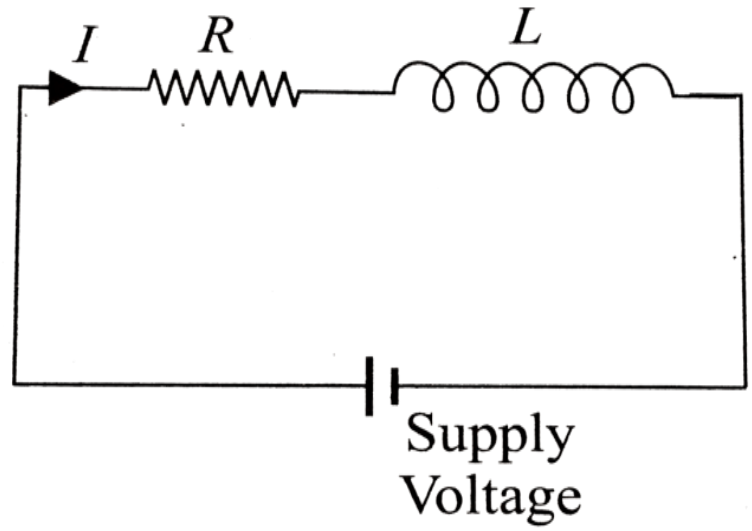
When, $t = \tau$ then;

$$I = I_0\,(e^{-1}) = 0.37 I_0$$

Thus the time constant τ is the time for which current decrease to 37% of its initial value.



Energy stored in Magnetic field:-



Consider a resistor ‘R’ inductor ‘L’ is connected in series with a battery of emf ‘V’ as shown in figure.

Then, applying Kirchhoff’s voltage law;

$$V = L\frac{dI}{dt} + IR$$

Multiplying both sides by I;

$$VI = LI\frac{dI}{dt} + I^2R \ldots\ldots\ldots (1)$$

The term I^2R represents the rate at which energy appears as thermal energy in the resistor. The term VI represents rate of energy supplied by battery. Therefore the term $LI\frac{dI}{dt}$ must represent the rate at which energy is stored in magnetic field or in inductor.

Thus, $\frac{dU_B}{dt} = LI\frac{dI}{dt}$

$$or, \qquad dU_B = LIdI$$

Integrating;

$$\int_0^{U_B} dU_B = \int_0^{I} LIdI$$

$\therefore\ U_B = \frac{LI^2}{2}$

Which represents the energy stored by an inductor of inductance 'L', carrying current 'I'.

Energy Density of magnetic field:-

Consider a solenoid having length '*l*' and cross-section area 'A' carrying current 'I'. The volume associated with this length is equal to *Al*.

Thus, the magnetic energy stored per unit volume of this field is;

$$\mu_B = \frac{U_B}{Al} = \frac{LI^2}{2Al}$$

Thus, the inductance 'L' of solenoid is given by;

$$L = \mu_0 n^2 A l$$

$$\therefore \mu_B = \frac{\mu_0 n^2 A l I^2}{2 A l} = \frac{1}{2}\mu_0 n^2 I^2 = \frac{1}{2\mu_0}\mu_0^2 n^2 I^2$$

$$\therefore \mu_B = \frac{B^2}{2\mu_0} \quad where\ B = \mu_0 n I$$

This is the expression for the magnetic energy density. Though it is derived for solenoid. It holds good for all magnetic field, no matter how they are generated. Above result shows that magnetic energy density $\mu_B \propto B^2$, i.e. square of magnetic field.

Numerical Examples:-

1. **An inductor of self inductance $100\ mH$, and resistor of resistance $50\ \Omega$ are connected to a $2\ V$ battery. Calculate the time required for the current to fall to half of its steady state value.**

Solution:-

Inductance (L) $= 100\ \text{mH} = 100 \times 10^{-3}\ \text{H}$

Resistance (R) $= 50\ \Omega$

Potential (V) $= 2\ \text{V}$

Now, $\quad I_0 = \frac{V}{R} = \frac{2}{50} = 0.04\ \text{A}$

$$\tau = \frac{L}{R} = \frac{100 \times 10^{-3}}{50} = 2 \times 10^{-3}\, sec$$

$$\therefore \text{fall current (I)} = I_0\, e^{-t/\tau}$$

$$or, \qquad \frac{I_0}{2} = I_0\, e^{-t/\tau} \qquad (\because I = \frac{I_0}{2})$$

$$or, \qquad \frac{1}{2} = e^{-t/2\times 10^{-3}}$$

$$or, \qquad \ln\left(\frac{1}{2}\right) = \frac{t}{2 \times 10^{-3}}$$

$$\therefore t = 1.38 \times 10^{-3}\, sec$$

2. A solenoid of inductance L and resistor R is connected to a battery. After how many time constant the magnetic energy fall to 1/4th of its maximum value?

Solution:-

$$\text{Given; } \mathrm{U_B} = \frac{\mathrm{U_0}}{4}$$

$$\text{For the condition of decay current (I)} = \mathrm{I_0\, e^{-t/\tau}}$$

$$\text{We have; } \mathrm{U_B} = \frac{LI^2}{2}$$

$$or, \qquad \frac{1}{4}U_0 = \frac{L(\mathrm{I_0\, e^{-t/\tau}})^2}{2}$$

$$or, \quad \frac{1}{4}U_0 = U_0 e^{-2t/\tau}$$

$$or, \quad \ln\frac{1}{4} = \frac{-2t}{\tau}$$

$\therefore$ t = 0.693 τ Sec

3. A 45 V of potential difference is suddenly applied to a coil with $L = 50\ mH$ and $R = 180\ \Omega$. At what rate is the current increasing after 1.2 mili second.

Solution:-

Inductance (L) = 50 mH = 50×10^{-3} H

Resistance (R) = 180 Ω

Potential (V) = 45 V

Time (t) = 1.2 mili second = $1.2 \times 10^{-3}\ sec$

We have, for growth current $(I) = I_0(1 - e^{-\frac{t}{\tau}})$

$$or, \frac{dI}{dt} = I_0(0 + \frac{1}{\tau}\ e^{-\frac{t}{\tau}})$$

$$or, \frac{dI}{dt} = \frac{I_0}{\tau}\ e^{-t/\tau} = \frac{V}{R}\frac{R}{L}\ e^{-tR/L}$$

$$= \frac{45}{50 \times 10^{-3}} . e^{\frac{1.2\times10^{-3}\times180}{50\times10^{-3}}}$$

$$\therefore \frac{dI}{dt} = 11.97\ A/sec$$

4. **An ideal inductor of self inductance $5\,H$ and a resistance of $100\ \Omega$ is suddenly connected in series to a battery of $6\,V$. Calculate (i) The steady current in the circuit. (ii) The maximum rate of increase of current. (iii) The time constant.**

Solution:-

Inductance $(\mathrm{L}) = 5\ \mathrm{H}$

Resistance $(\mathrm{R}) = 100\ \Omega$

Potential $(\mathrm{V}) = 6\ \mathrm{V}$

(i) $\mathrm{I_0} = \frac{\mathrm{V}}{\mathrm{R}} = \frac{6}{100} = 0.06\ \mathrm{A}$

(ii)We have, $\mathrm{V} = \mathrm{L}\frac{\mathrm{dI}}{\mathrm{dt}}$

$$\therefore \frac{\mathrm{dI}}{\mathrm{dt}} = \frac{V}{L} = \frac{6}{5} = 1.2\ A/sec$$

(iii) Also, $\tau = \frac{\mathrm{L}}{\mathrm{R}} = \frac{5}{100} = 0.05\ \mathrm{sec}$

5. **A Circular loop of wire 5 cm in radius carries a current of 5 A. What is the energy density at the centre of the loop?**

Solution:

We have, $B = \frac{\mu_0 N I a^2}{2(x^2+R^2)^{3/2}}$

Here $N = 1 \ and \ x = 0$

$$\therefore \ B = \frac{\mu_0 I}{2R}$$

$\therefore$ Energy density, $\mu_B = \frac{B^2}{2\mu_0} = \frac{1}{2\mu_0} . \frac{\mu_0^2 I^2}{4R^2}$

$$= \frac{\mu_0^2 I^2}{8R^2} = \frac{4\pi \times 10^{-7} \times 5^2}{8 \times (0.05)^2} = 1.58 \times 10^{-3} \ joule/m^3$$

6. **What is the magnetic energy density at the centre of a circulating electron in the hydrogen atom. Assume that the electron circulate around the nucleus in a path of radius $\boldsymbol{5.1 \times 10^{-11} \ metre}$ at a frequency of $\boldsymbol{6.8 \times 10^{15} \ revs/sec.}$**

Solution:

Here, $q = e = 1.6 \times 10^{-19} \ C$, $R = 5.1 \times 10^{-11} \ metre$, $f = 6.8 \times 10^{15} \ revs/sec.$

We have,

$$B = \frac{\mu_0 N I a^2}{2(x^2 + R^2)^{3/2}}$$

Here $N = 1\ and\ x = 0$

$$B = \frac{\mu_0 I}{2R} = \frac{\mu_0 (\frac{q}{T})}{2R} = \frac{\mu_0 e f}{2R}$$

Magnetic energy density,

$$\mu_B = \frac{B^2}{2\mu_0}$$

$$= \frac{1}{2\mu_0} . \frac{\mu_0^2 e^2 f^2}{4R^2} = \frac{\mu_0^2 e^2 f^2}{8R^2}$$

$$= \frac{4\pi \times 10^{-7} \times (1.6 \times 10^{-19})^2 \times (6.8 \times 10^{15})^2}{8 \times (5.1 \times 10^{-11})^2}$$

$$\therefore \mu_B = 7.15 \times 10^7\ J/m^3$$

7. Flux ϕ (in weber) in a closed circuit of resistance 10 Ω varies with time t (in second) According to the equation $\emptyset = 6t^2 - 5t + 2$. Calculate the magnitude of induced current in the circuit at t = 0.25 sec.

Solution:-

Here, $\emptyset = 6t^2 - 5t + 2$

$$\frac{d\emptyset}{dt} = 12t - 5$$

$$\text{Induced emf}(\epsilon) = -\frac{d\emptyset_B}{dt} = -12t + 5$$

At $t = 0.25\ sec, \epsilon = -12 \times 0.25 + 5 = 2V$

Induced current, $I = \frac{\epsilon}{R} = \frac{2}{10} = 0.2\ A$

Exercise:-

1. Derive inductance of solenoid and toroid. Then show that inductance is the property of the coil.
2. Determine the energy stored in an inductor. Also, determine the energy density in magnetic field.
3. Differentiate between electromagnetic induction and self induction. Develop an expression for self induction for toroid.
4. Explain meaning of self induction. Calculate inductance for a solenoid and toroid.
5. Derive the relation for rise and fall of current in LR circuit. Plot a graph between current and time and explain the figure.
6. Determine the energy stored in an inductor. Hence prove that the energy density in magnetic field is directly proportional to square of magnetic field.
7. What is self inductance? Define inductance of a coil. Show by calculation inductance of a coil depends on permeability of a medium and the geometry of the coil.
8. Show that the energy per unit volume in electric field and magnetic field are proportional to the square of their fields.
9. A toroid has number of turns 1250, internal radius 52 mm, external radius 95 mm and thickness of the ring 13mm calculate the inductance.
10. A solenoid having an inductance of 6.3 μH is connected in series with a 1.2 kΩ resistance. (a) If a 14 V

battery is connected across the pair, how long it will take for the current through the resistor to reach 80 % of its initial value? (b) What is the current through the resistor at time $t = \tau_L$.

11. An inductance L is connected to a battery of emf E through a resistance. Show that the potential difference across the inductance after time t is $V_L = \varepsilon\, e^{\frac{-Rt}{L}}$. At what time is the potential difference across the inductance equal to that across the resistance such that $i = \frac{i_0}{2}$.

12. A solenoid is 1.3 m long and 2.6 cm in diameter carries a current of 18 A. The magnetic field inside the solenoid is 23 mT. Find the length of the wire forming the solenoid. Also calculate the inductance of solenoid.

13. What must be the magnitude of a uniform electric field have the same energy density that passed by a 0.50 T magnetic field?