6th Class

Ex-13 Q.N. 13 $\frac{dy}{dx} + y = \cos x$.

Soln: Given diff. eqn is $\frac{dy}{dx} + y = \cos x$ —(1)

Which is linear diff. eqn on y

So comparing it with $\frac{dy}{dx} + py = Q$

$p = 1$, $Q = \cos x$

Now, I.F. $= e^{\int p\,dx} = e^{\int 1.dx} = e^{x}$

Multiplying eqn (1) by I.F. $= e^{x}$ on both sides we get;

$$y \times I.F. = \int (Q \times I.F.)\,dx + C$$

or, $y \times e^{x} = \int e^{x} \cos x \, dx + C$

* using $\int e^{ax} \cos bx \, dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

** $\int e^{ax} \sin bx \, dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

$$y.e^{x} = \frac{e^{x}(1.\cos x + 1.\sin x)}{1^2+1^2} + C$$

or, $y e^{x} = \frac{e^{x}(\cos x + \sin x)}{2} + C$

$\Rightarrow 2y = \cos x + \sin x + 2Ce^{-x}$

$$\Rightarrow 2y = \cos x + \sin x + 2ce^{-x}$$

or, $2y = \cos x + \sin x + Ke^{-x}$

where $K = 2c$ which is the required general soln

Exact differential eqn

Exact diff. eqn:- A diff. eqn

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be exact if there exist a function $f(x,y)$ such that

$$M(x,y)\,dx + N(x,y)\,dy = d\,f(x,y)$$

e.g. $x\,dy + y\,dx = d(xy)$

i.e. the given diff. eqn is exact if $M(x,y)\,dx + N(x,y)\,dy$ is exact or perfect differential.

Note:- The diff. eqn

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

will be exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

where ∂ and ∂ denotes the where $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x}$ denotes the partial derivative.

$$\underline{U} = \underline{ax^2} + 2hxy + by^2$$

diff. partial w.r.t. 'x'

$$\frac{\partial U}{\partial x} = 2ax + 2hy + 0$$

diff. partial w.r.t. y

$$\frac{\partial U}{\partial y} = 0 + 2hx + 2by$$

Note:: Every diff. eqn

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is not exact.

for e.g. $x^2 dy + 2xy\,dy = 0$ is exact because

$$x^2 dy + 2xy\,dy = \underline{d(x^2 y)}$$

Some formula

(1) $x\,dy + y\,dx = d(xy)$

(2) $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$

(2) $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$

(3) $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$

and so on. — page 329

Ex-25

Solve the following diff. eqⁿ by reducing to exact form

e.g. $x\,dy + y\,dx = 0$

Solⁿ: Given diff. eqⁿ is

$x\,dy + y\,dx = 0$

or, $\int d(xy) = \int 0$ Integrating

$xy = c$ which is the required general solⁿ #

OR $x\,dy + y\,dx = 0$

or, $x\,dy = -y\,dx$

or, $\int \frac{1}{y}\,dy = -\int \frac{1}{x}\,dx$ Int.

$\log y = -\log x + \log c$

$$\log y = -\log x + \log c$$

or, $\log y = \log\left(\frac{c}{x}\right) \Rightarrow y = \frac{c}{x}$

$\therefore xy = c$

Which is the required general soln of the given diff. eqn

Solve: Ex-21, 23, 24

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

~~e.g.~~ $\frac{dy}{dx} = \frac{2x - y + 1}{x + 2y - 3}$ Ex-22 (4)

$$(x+2y-3)dy = (2x-y+1)dx$$

or, $x\,dy + 2y\,dy - 3\,dy = 2x\,dx - y\,dx + dx$

or, $x\,dy + y\,dx + 2y\,dy - 3\,dy = 2x\,dx + dx$

or, $\int d(xy) + 2\int y\,dy - 3\int dy = 2\int x\,dx + \int dx$

Integrating

$$xy + 2\cdot\frac{y^2}{2} - 3y = 2\frac{x^2}{2} + x + c$$

or, $xy + y^2 - 3y - x^2 - x - c = 0$

$xy + y^2 - 3y - x^2 - x - c = 0$

Ex-25

First order but not first degree differential equation

An eqⁿ of the form $f(x, y, p) = 0$ where $p = \frac{dy}{dx}$ is called first order but first degree diff. eqⁿ. The solution of such type of diff. eqⁿ contains only one constant.

We will discuss the following first order but not first degree diff. eqⁿ

→ Solvable for p
→ Solvable for y
→ Solvable for x
→ Clairaut's Equation

Solvable for p

An eqⁿ of the form $f(x, y, p) = 0$ where $p = \frac{dy}{dx}$ can be factorized into linear factors such as

$$\{P - f_1(x,y)\}\{P - f_2(x,y)\} \cdots \cdots \{P - f_n(x,y)\} = 0$$

Such type of first order but not first degree diff. eqⁿ is called solvable for p.

It's general solⁿ is

1. $p^2 + p - 6 = 0$ or, $\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} - 6 = 0$ where $\frac{dy}{dx} = p$

Solⁿ: Given diff. eqⁿ is

$$p^2 + p - 6 = 0$$

or, $p^2 + 3p - 2p - 6 = 0$

or, $p(p+3)-2(p+3)=0$

or, $(p+3)(p-2)=0$

Either $p+3=0$ | or $p-2=0$

or, $\frac{dy}{dx}+3=0$ | $\frac{dy}{dx}-2=0$

or, $\int dy = -\int 3\,dx$ Int. | or, $\int dy = \int 2\,dx$ Int.

$\Rightarrow y = -3x + c$ | $\Rightarrow y = 2x + c$

or, $y + 3x - c = 0$ | or, $y - 2x - c = 0$

$\therefore$ The required general solⁿ is

$$(y+3x-c)(y-2x-c)=0$$

(2) $p^2 + 2px - 3x^2 = 0$

Solⁿ: Given diff. eqⁿs

(3) $p^2 - p(e^x + e^{-x}) + 1 = 0$ $\left\{ f(x,y,p)=0 \quad S\{p-f_1(x,y)\} \right.$

Solⁿ: Given diff. eqⁿ

$$p^2 - p(e^x + e^{-x}) + 1 = 0$$

or, $p^2 - pe^x - pe^{-x} + 1 = 0$

or, $p(p-e^{x}) - e^{-x}\left(p - \frac{1}{e^{-x}}\right) = 0$

or, $p(p-e^{x}) - e^{-x}(p-e^{x}) = 0$

or, $(p-e^{x})(p-e^{-x}) = 0$

Either $p - e^{x} = 0$ or $p - e^{-x} = 0$

$\frac{dy}{dx} = e^{x}$ | $\frac{dy}{dx} = e^{-x}$

$\int dy = \int e^{x} dx$ Int. | $\int dy = \int e^{-x} dx$ Int.

$y = e^{x} + c$ | $y = -e^{-x} + c$

$y - e^{x} - c = 0$ | $\Rightarrow y + e^{-x} - c = 0$

$\therefore$ The required general solⁿ is,

$(y - e^{x} - c)(y + e^{-x} - c) = 0$ #

(8) $p^3 - p(x^2 + xy + y^2) + x^2 y + xy^2 = 0$

Solⁿ: Given diff. eqⁿ is

$p^3 - px^2 - pxy - py^2 + x^2 y + xy^2 = 0$

or, $p^3 - px^2 - pxy + x^2 y - py^2 + xy^2 = 0$

or, $p(p^2 - x^2) - xy(p-x) - y^2(p-x) = 0$

$\quad (p^2 - x^2) = (p-x)(p+x)$

or, $(p-x)\{p(p+x)-xy-y^2\}=0$

or, $(p-x)(p^2+px-xy-y^2)=0$

or, $(p-x)(p^2-y^2+px-xy)=0$

or, $(p-x)\{(p-y)(p+y)+x(p-y)\}=0$

or, $(p-x)(p-y)(p+y+x)=0$

Either

$p-x=0$ or $p-y=0$ or $p+y+x=0$

$''$ $''$ $\frac{dy}{dx}+y+x=0$

or, $\frac{dy}{dx}+y=-x$ ⟶ ①

Which is linear diff. eqn on y

So it's I.F. $= e^{\int 1\,dx} = e^x$

Multiplying eqn ① by I.F. $= e^x$ on both sides we get

$y \times I.F. = \int(Q \times I.F.)\,dx + C$

$y \cdot e^x = -\int x e^x\,dx + C$

$= -\left(xe^x - \int 1 \cdot e^x\,dx\right) + C$

$$= -\left(xe^{x} - \int 1 \cdot e^{x}\, dx\right) + C$$

$$y \cdot e^{x} = -xe^{x} + e^{x} + C$$

or, $y = -x + 1 + Ce^{-x}$

or, $y + x - 1 - Ce^{-x} = 0$

∴ The required general solⁿ is,

$$(\quad)(\quad)(y + x - 1 - Ce^{-x}) = 0$$

#

$$Ax^2 + Bx + C = 0$$

$$\Rightarrow x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2 \cdot A}$$

e.g. $p^3 + 3xp^2 - y^3 p^2 - 3xy^3 p = 0$

$$\int \frac{6 - 5v^{2}}{(5v^2 - 7v + 2)}\, dv$$

$$\int \frac{-\frac{1}{2}(10v - 7) - \frac{7}{2} + 6}{(5v^2 - 7v + 2)}\, dv$$

$$= -\frac{1}{2} \int \frac{10v - 7}{5v^2 - 7v + 2}\, dv + \frac{5}{2} \int \frac{1}{(5v^2 - 7v + 2)}\, dv$$

$$= -\frac{1}{2} \log(5v^2 - 7v + 2) + \frac{5}{2} \cdot \frac{1}{5} \int \frac{1}{}$$

$$= \frac{1}{2}\log(5\theta^2-7\theta+2) + \frac{\cancel{5}}{2}\cdot\frac{1}{\cancel{5}}\int \frac{1}{\theta^2-\frac{7}{5}\theta+\frac{2}{5}}\,d\theta$$

$$= \text{''} + \frac{1}{2}\int \frac{1}{\theta^2 - 2\cdot\theta\cdot\frac{7}{10} + \left(\frac{7}{10}\right)^2 - \left(\frac{7}{10}\right)^2 + \frac{2}{5}}\,d\theta$$

$$= \text{''} + \frac{1}{2}\int \frac{1}{\left(\theta-\frac{7}{10}\right)^2 - \left(\frac{3}{10}\right)^2}\,d\theta$$

$$* \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$$