Second Order Linear Differential Equation with Constant Coefficient

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An equation of the form $\frac{d^2y}{dx^2} + p_1 \frac{dy}{dx} + p_2y = Q$

Where p_1 and p_2 are constants and Q is function of x or constant is called second order linearm differential equation with constant coefficient.

Note : If we write $\frac{d}{dx} = D$ and $\frac{d^2}{dx^2} = D^2$

Then second order linear differential equation with constant coefficient can be written as,

$(D^2 + p_1D + p_2)\, y = 0$ and

$(D^2 + p_1D + p_2)\, y = Q$

Theorem : 1

If $y = f_1(x)$ and $y = f_2(x)$ are two independent solutions of the equation

$(D^2 + p_1D + p_2)\, y = 0$

i.e. $f(D)\, y = 0$ then $y = C_1 f_1(x) + C_2 f_2(x)$

Will be the general solution of the equation.

Solution of $\frac{d^2y}{dx^2} + p_1 \frac{dy}{dx} + p_2y = 0$

or, $(D^2 + p_1D + p_2)\, y = 0$ ……… (i)

Where, $\frac{d}{dx} = D$ and $\frac{d^2}{dx^2} = D^2$

Let $y = e^{mx}$ be a solution of (i) then (i) becomes,

$m^2e^{mx} + p_1me^{mx} + p_2 e^{mx} = 0$

or, $e^{mx} (m^2 + p_1m + p_2) = 0$

Since, $e^{mx} \neq 0$,

So, $m^2 + p_1m + p_2 = 0$

Which is called Auxillary equation (A. E.) of the given differential equation and it is quadratic equation in m. So m has two roots.

Case - 1

If m has two different real roots, say $m = \alpha$ and $m = \beta$ then the general solution of (i) is

$y = C_1 e^{\alpha x} + C_2 e^{\beta x}$

Case - 2

If m has two real and equal roots, say $m = \alpha, \alpha$ then the general solution (i) is $y = (C_1 + C_2x)\, e^{\alpha x}$

Case - 3

If m has two imaginary roots say $m = \alpha + i\beta$ and $\alpha - i\beta$ then the general solution of (i) is,

$$y = e^{\alpha x} (A \cos\beta x + B \sin\beta x)$$

Exercise - 29

Solve the following equations.

1. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 0$

Sol^n. Given differential equation is,

$\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 0$

or, $(D^2 + 3D + 2)\, y = 0$

So, It's A. E. is,

$m^2 + 3m + 2 = 0$

or, $m^2 + 2m + m + 2 = 0$

or, $m\,(m + 2) + 1\,(m + 2) = 0$

or, $(m + 2)\,(m + 1) = 0$

$\therefore\ m = -1, -2$

Thus, $y = C_1 e^{-x} + C_2 e^{-2x}$ is the required general solution.

2. $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 4y = 0$

Sol^n. Given differential equation is,

$\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 4y = 0$

or, $(D^2 + 5D + 4)\, y = 0$

So, It's A. E. is,

$m^2 + 5m + 4 = 0$

or, $m^2 + 4m + m + 4 = 0$

or, $m\,(m + 4) + 1\,(m + 4) = 0$

or, $(m + 4)\,(m + 1) = 0$

$\therefore\ m = -1, -4$

Thus, $y = C_1 e^{-x} + C_2 e^{-4x}$ is the required general solution.

3. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$

Sol^n. Given differential equation is,

$\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$

or, $(D^2 + 2D + 1)\, y = 0$

So, It's A. E. is,

$m^2 + 2m + 1 = 0$

or, $(m + 1)^2 = 0$

$\Rightarrow m = -1, -1$

This, $y = (C_1 + C_2x)\, e^{-x}$ is the required general solution.

4. $(D^2 + D)\, y = 0$

Sol^n. Given differential equation is,

$(D^2 + D)\, y = 0$

So, It's A. E. is;

$m^2 + m = 0$

or, $m\,(m + 1) = 0$

$\Rightarrow m = 0, m = -1$

Thus, $y = C_1 + C_2 e^{-x}$ is the required general solution.

5. $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 25y = 0$

Sol^n. Given differential equation is,

$\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 25y = 0$

or, $(D^2 + 6D + 25)\, y = 0$

So, It's A. E. is the $m^2 + 6m + 25 = 0$

or, $m = \dfrac{-6 \pm \sqrt{(6)^2 - 4\,.\,1\,.\,25}}{2.1}$

$= \dfrac{-6 \pm \sqrt{36 - 100}}{2} = \dfrac{-6 \pm \sqrt{-64}}{2} = \dfrac{-6 \pm 8i}{2} = -3 \pm 4i$

Thus, $y = e^{-3x}$ (A cos 4x + B sin 4x) is the required general solution.)

6. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 13\, y = 0$

Sol^n. Given differential equation is,

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+13y=0$$

or, $(D^2 + 4D + 13)\, y = 0$

So, it's A. E. is,

$$m^2 + 4m + 13 = 0$$

$$\text{or, } m = \frac{-4 \pm \sqrt{4^2 - 4.1.13}}{2.1} = \frac{-4 \pm 6i}{2}$$

$\therefore$ $m = -2 \pm 3i$

Thus, $y = e^{-2x}$ (A cos 3x + B sin 3x) is the required general solution.

7. $16\dfrac{d^2y}{dx^2}+24\dfrac{dy}{dx}+9y=0$

Sol^n. Given differential equation is,

$$16\frac{d^2y}{dx^2}+24\frac{dy}{dx}+9y=0$$

or, $(16D^2 + 24D + 9)\, y = 0$

So, it's A. E. is,

$16m^2 + 24m + 9 = 0$

or, $16m^2 + 12m + 12m + 9 = 0$

or, $4m(4m + 3) + 3(4m + 3) = 0$

or, $(4m + 3)(4m + 3) = 0$

$\Rightarrow m = -\dfrac{3}{4}, -\dfrac{3}{4}$

Thus, $y = (C_1 + C_2 x)\, .\, e^{-\frac{3}{4}x}$ is the required general solution.

8. $(D + 3)^2\, y = 0$

Sol^n. Given differential equation is,

$(D + 3)^2\, y = 0$

So, it's, A. E. is;

$(m + 3)^2 = 0$

$\Rightarrow m = -3, -3$

Thus, $y = (C_1 + C_2x)\, e^{-3x}$ is the required general solution.

9. $(D^2 + 3aD - 4a^2)\, y = 0$

Sol^n. Given differential equation is,

$(D^2 + 3aD - 4a^2)\, y = 0$

So, it's, A. E. is,

$m^2 + 3am - 4a^2 = 0$

or, $m^2 + 4am - am - 4a^2 = 0$

or, $m(m + 4a) - a(m + 4a) = 0$

or, $(m + 4a)(m - a) = 0$

$\Rightarrow m = a, -4a$

Thus, $y = C_1 e^{ax} + C_2 e^{-4ax}$ is the required general solution.

10. **Solve** $\dfrac{d^2x}{dt^2} + \mu x = 0, \mu > 0$ **given that** $x = a$ **and** $\dfrac{dx}{dt} = 0$ **when** $t = \dfrac{\pi}{2\sqrt{\mu}}$.

Sol^n. Given differential equation is,

$$\frac{d^2x}{dt^2} + \mu x = 0$$

or, $(D^2 + \mu)\, x = 0$ where $D = \dfrac{d}{dt}$ and $D^2 = \dfrac{d^2}{dt^2}$

So, it's A. E. is;

$m^2 + \mu = 0$

or, $m = \pm i\sqrt{\mu}$

Thus, $x = A \cos\sqrt{\mu}\, t + B \sin\sqrt{\mu}\, t$ ……. (i)

Differential equation (i) w. r. t. 't'

$\dfrac{dx}{dt} = -A\sqrt{\mu}\, \sin\sqrt{\mu}\, t + B\sqrt{\mu}\, \cos\sqrt{\mu}\, t$ …… (ii)

Using, when $t = \dfrac{\pi}{2\sqrt{\mu}}$ given,

$x = a$ and $\dfrac{dx}{dt} = 0$ in (i) and (ii) respectively.

We get,

$a = 0 + B \Rightarrow B = a$

and $0 = -A\sqrt{\mu} \Rightarrow \mu > 0$

So, A = 0

Hence, (i) becomes,

$x = a \sin\sqrt{\mu}\, t$ is the required particular solution.

11. $\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} + 2x = 0$ **given that** $x = 1$ **when** $t = 0$ **and** $\dfrac{dx}{dt} = 0$ **when** $t = 0$.

Sol^n. Given differential equation is,

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$$

or, $(D^2 - 3D + 2)\, x = 0$ where, $D = \frac{d}{dt}$ and $D^2 = \frac{d^2}{dt^2}$

So, it's A. E. is;

$m^2 - 3m + 2 = 0$

or, $m^2 - 2m - m + 2 = 0$

or, $m\,(m - 2) - 1\,(m - 2) = 0$

or, $(m - 1)\,(m - 2) = 0$

$\Rightarrow m = 1, m = 2$

Thus, $x = C_1 e^t + C_2 e^{2t}$ …….. (i)

Differential equation (i) w. r. t. 't'

$\frac{dx}{dt} = C_1 e^t + 2C_2 e^{2t}$ ……. (ii)

Using, when $t = 0$ given that $x = 1$ and $\frac{dx}{dt} = 0$ in (i) and (ii) respectively we get,

$1 = C_1 + C_2$ and $0 = C_1 + 2C_2$

or, $C_1 = -2C_2$

$\therefore 1 = -2C_2 + C_2$

$\Rightarrow C_2 = -1$ and $C_1 = 2$

Hence, (i) becomes,

$x = 2e^t - e^{2t}$ is the required particular solution.

12. $\mathbf{\frac{d^2x}{dt^2} + y = 0}$ **given that y = 4 when x = 0 and** $\mathbf{\frac{dy}{dx} = 0}$ **when x = 0**

Sol^n. Given differential equation is,

$\frac{d^2x}{dt^2} + y = 0$

or, $(D^2 + 1)\, y = 0$

So, it's A. E. is;

$m^2 + 1 = 0 \Rightarrow m = \pm i$

Thus, $y = A \cos x + B \sin x$ …….. (i)

Differential equation (i) w. r. t. 'x'

$\frac{dy}{dx} = -A \sin x + B \cos x$ ……. (ii)

Using when $x = 0$ given $y = 4$ and $\frac{dy}{dx} = 0$ in (i) and (ii) respectively we get,

$4 = A$ and $0 = B$

Hence, (i) becomes,

$y = 4 \cos x$ is the required particular solution.