Wave Motion

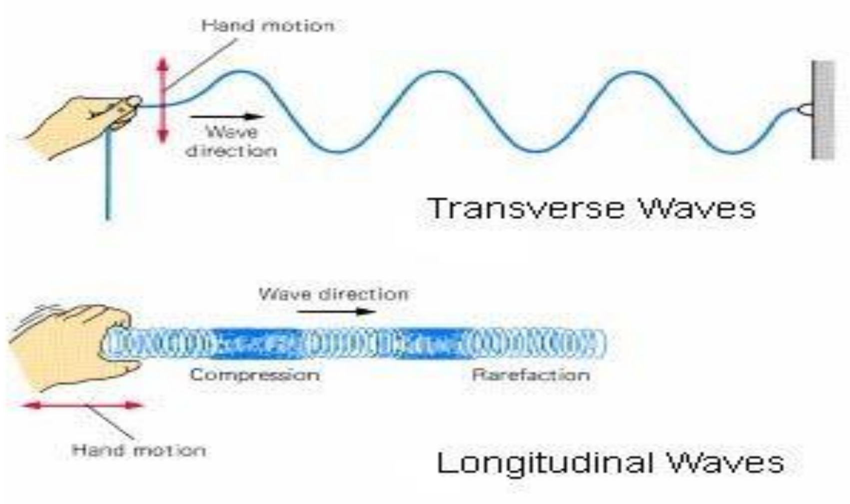
Wave motion is a form of disturbance which travels through a medium due to the repeated periodic motion of the particles of the medium about their mean position.

Characteristics of wave motion:-

- Wave motion is the disturbance produced in the medium due to repeated periodic motion of the particle.
- Wave travels in forward direction while the particle of the medium vibrates about their mean position.
- In each vibration, a particle handover some of its energy to the next particles, which it receive from the previous particle.
- The velocity of wave is different from the velocity of the particle.
- Wave velocity is uniform while particle velocity is different at different position. It is maximum at mean position and minimum at extreme position.

Types of wave:-

On the basis of modes of vibration of the particle of the medium, there are two types of waves;



Transverse Wave:-

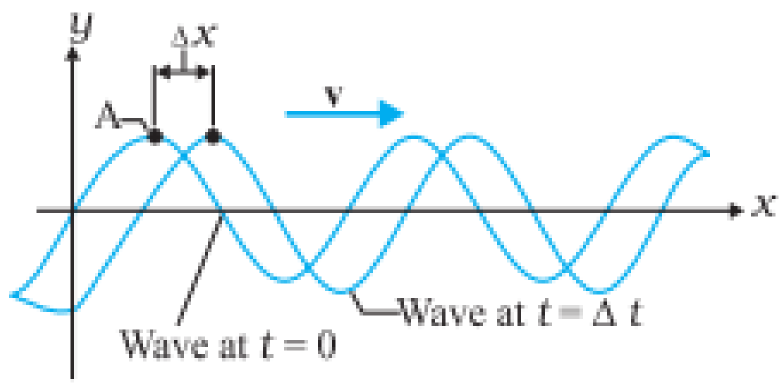
When the particle of the medium vibrates about their mean position in a direction perpendicular to the direction of propagation of wave, the wave is called transverse wave. Eg. Transverse wave produced in water, when stone is thrown to the water.

Longitudinal wave:-

If the particles of the medium propagating the wave motion vibrate in the direction of propagation then the wave is called longitudinal wave. Eg. Sound waves.

Equation of propagation of wave (travelling waves):-

A wave travelling from one region to another region of medium through the successive vibration of the particle of the medium is called progressive wave.



Suppose a wave originate at ‘O’ the displacement of the particle at that instant is;

$$y = a\sin wt \dots\dots\dots (1)$$

Also consider another particle ‘P’ at a distance ‘x’ from ‘O’ to its right side.

Let the wave travelling from left to right with a velocity v and displacement of particle at point ‘P’ will be;

$$y = a\sin(wt - \emptyset) \dots\dots\dots (2)$$

Also, $\text{phse difference}(\emptyset) = \frac{2\pi}{\lambda} \times \text{path difference}$

$$\therefore \emptyset = \frac{2\pi}{\lambda} \times x$$

And, also $w = \frac{2\pi}{T}$ $but\ v = \frac{\lambda}{T}$ $i.e.\ T = \frac{\lambda}{v}$

$$\therefore w = \frac{2\pi v}{\lambda}$$

Therefore equation (2) becomes;

$$y = \mathrm{asin}\frac{2\pi}{\lambda}(vt - x)$$

Above equation can be expressed in terms of wave number $k = \frac{2\pi}{\lambda}$ and angular frequency w as;

$$\mathrm{y = asin(wt - kx)} \quad \text{left to right} \quad \text{and}$$

$$y = \mathrm{asin(wt + kx)} \quad \text{right to left}$$

Differential equation of wave motion:-

We have general wave equation travelling from left to right is

$$y = \mathrm{asin}\frac{2\pi}{\lambda}(vt - x) \ldots\ldots\ldots (1)$$

Differentiating equation (1) with respect to time

$$\frac{dy}{dt} = \frac{2\pi}{\lambda} va\ cos\frac{2\pi}{\lambda}(vt - x) \ldots\ldots\ldots (2)$$

Now, differentiating equation (1) with respect to x

$$\frac{dy}{dx} = \frac{-2\pi}{\lambda} a\ cos\frac{2\pi}{\lambda}(vt - x) \ldots\ldots\ldots (3)$$

Again, differentiating equation (2) with respect to time

$$\frac{d^2y}{dt^2} = \frac{-4\pi^2v^2}{\lambda^2}\text{a}\sin\frac{2\pi}{\lambda}(vt - x) \ldots\ldots\ldots (4)$$

And again differentiating equation (3) with respect to x

$$\frac{d^2y}{dx^2} = \frac{-4\pi^2}{\lambda^2}\text{asin}\frac{2\pi}{\lambda}(vt - x) \ldots\ldots\ldots (5)$$

From equation (2) and (3)

$$\frac{dy}{dt} = -v\frac{dy}{dx}$$

∴ Velocity of particles = (-)ve of velocity of wave and product of slope of curve.

And from equation (4) and (5)

$$\frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2}$$

∴ Particle acceleration at a point = (wave velocity)^2 × curvature of the displacement curve at that point.

Wave velocity and particle velocity:-

We have *v* is the velocity of wave and y is the displacement of the particle

$$i.e.\ y = \text{asin}\frac{2\pi}{\lambda}(vt - x)$$

$$\therefore v_{particle} = \frac{dy}{dt} = \frac{2\pi}{\lambda} va\, cos \frac{2\pi}{\lambda}(vt - x)$$

$$\therefore (v_{particle}) max = \frac{2\pi}{\lambda} va = wa$$

And, Acceleration of the particle (a_p) is

$$\frac{d^2y}{dt^2} = \frac{-4\pi^2 v^2}{\lambda^2} \text{a sin} \frac{2\pi}{\lambda}(vt - x)$$

$$(a_p) \text{ max} = \frac{-4\pi^2 v^2}{\lambda^2} a = -w^2 a$$

We have, $\frac{dy}{dx} = \frac{-2\pi}{\lambda} a\, cos \frac{2\pi}{\lambda}(vt - x)$

$$\therefore v_{particle} = -v \frac{dy}{dx}$$

Where, $\frac{dy}{dx}$ represent the slop of the displacement curve.

Energy, power and intensity of progressive wave:-

Total energy is the sum of kinetic energy and potential energy. Therefore, potential energy in the progressive wave is;

$$U = \frac{1}{2} ky^2$$

Where, y is the displacement of the particle

$$i.e. y = \mathrm{a}\sin\frac{2\pi}{\lambda}(vt - x)$$

$$\therefore U = \frac{1}{2}ka^2sin^2\frac{2\pi}{\lambda}(vt - x)$$

$$= \frac{1}{2}mw^2a^2sin^2\frac{2\pi}{\lambda}(vt - x)$$

Now, potential energy per unit volume is

$$U = \frac{1}{2}\rho w^2a^2sin^2\frac{2\pi}{\lambda}(vt - x)$$

And, Kinetic energy per unit volume is

$$K.E. = \frac{1}{2}\rho v^2$$

$$= \frac{1}{2}\rho(\frac{dy}{dt})^2$$

$$= \frac{1}{2}\rho\left(\frac{2\pi}{\lambda}va\ cos\frac{2\pi}{\lambda}(vt - x)\right)^2$$

$$= \frac{1}{2}\rho\frac{4\pi^2v^2a^2}{\lambda^2}\ \cos^2\frac{2\pi}{\lambda}(vt - x)$$

$$= \frac{1}{2}\rho w^2a^2\ \cos^2\frac{2\pi}{\lambda}(vt - x)$$

So, Total energy per unit volume is sum of kinetic energy and potential energy.

$$i.e. E = K.E + U$$

$$\therefore E = \frac{1}{2}\rho w^2 a^2$$

$$or, E = \frac{2\pi^2 \rho v^2 a^2}{\lambda^2}$$

Since, $v = f\lambda$,

$$\therefore \; E = \frac{2\pi^2 \rho f^2 \lambda^2 a^2}{\lambda^2}$$

$$\therefore \; E = 2\pi^2 \rho f^2 a^2$$

Now, total energy for volume V is

$$E = 2\pi^2 \rho f^2 a^2 V$$

Since, $V = Al$

But, $l = vt$

$$\therefore V = Avt$$

$$\therefore \; E = 2\pi^2 \rho f^2 a^2 Avt$$

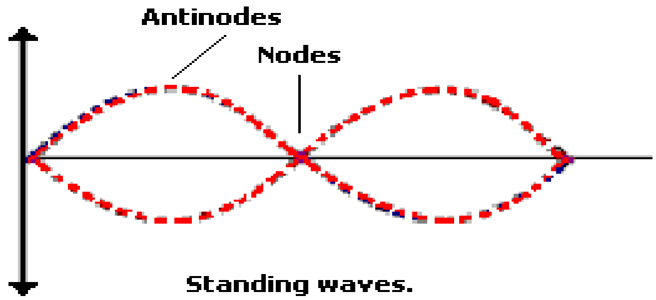
Now, power $(P) = \frac{E}{t}$

$$\therefore P = 2\pi^2 \rho f^2 a^2 Av$$

$\therefore$ Intensity $(I) = \frac{P}{A}$

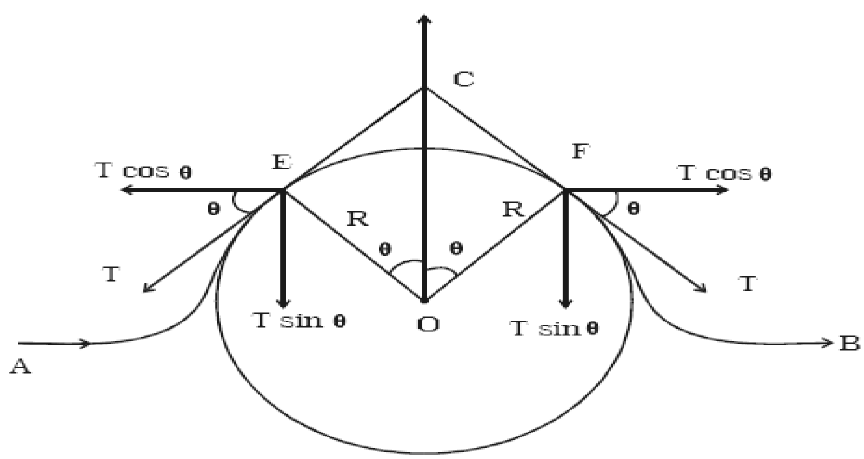
$$\therefore I = 2\pi^2 \rho f^2 a^2 v$$

Standing or Stationary wave:-



When two progressive waves of the same amplitude and frequency travelling through a medium with the same speed but in opposite direction superimpose in each other, stationary or standing wave is formed. The wave is so called stationary because it does not travel in both direction and all the particles of medium are permanently at rest. There is not flow of energy along the wave.

Velocity of transverse wave in a stretched string:-



Transverse vibration of a string

Consider a portion of AEFB of a stretched string with a transverse wave is travelling from left to right with velocity *v*. Consider a small string element EF of length Δl forming an arc of a circle of radius R and subtending an angle 2θ at the centre of that circle. A force with a magnitude is equal to the tension on the string T, pulls tangentially on this element at each end. The horizontal components are equal and opposite so they canceled each other and the vertical components gets added since they are along same direction to give resultant tension.

$$i.e. F = 2(T sin\theta)$$

As θ is small $sin\,\theta \approx \theta,$

$$\therefore \text{Resultant tension}(F) = T(2\theta) = T\frac{\Delta l}{R} \quad where, EF = \Delta l$$

If μ is the linear mass density of the string, and Δm is the mass of the small element,

$$\Delta m = \mu\,\Delta l$$

The element has a centripetal acceleration: $a = \frac{v^2}{R}$

From Newton's second law; $F = ma$

$$\text{Therefore,} \quad T\frac{\Delta l}{R} = (\mu\,\Delta l)\frac{v^2}{R}$$

$$v = \sqrt{\frac{T}{\mu}}$$

Numerical Examples:-

1. **A stretched string has linear density $525\ \mathrm{gm/m}$ and is under tension of $45\ \mathrm{N}$ and sinusoidal wave with frequency $120\ \mathrm{Hz}$ and amplitude $8.5\ \mathrm{mm}$ is stretched along a string from one end. At what average rate does the wave transport energy?**

Solution:-

Linear density$(\mu) = 525\ \mathrm{gm/m} = 0.525\ \mathrm{kg/m}$

Tension $(\mathrm{T}) = 45\ \mathrm{N}$

Frequency $(\mathrm{f}) = 120\ \mathrm{Hz}$

Amplitude $(\mathrm{a}) = 8.5\ \mathrm{mm} = 0.0085\ \mathrm{m}$

We have, velocity of transverse wave in stretched string is;

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{45}{0.525}} = 9.258\ \mathrm{m/sec}$$

Now rate of energy is;

$$P = 2\pi^2 \rho f^2 a^2 A v$$

$$\text{or, } P = 2\pi^2 \frac{\mu}{A} f^2 a^2 A v = 2\pi^2 \mu f^2 a^2 v$$

$$\text{or, } \mathrm{P} = 2 \times (3.14)^2 \times 0.525 \times (120)^2 \times (0.0085)^2 \times 9.258$$

$$\therefore \mathrm{P} = 99.7\ \mathrm{J/sec}$$

2. **A source of sound has a frequency of $\mathbf{512}$ Hz and amplitude of $\mathbf{0.25}$ cm. What is the flow of energy across a square cm per second? If the velocity of sound in air is $\mathbf{340\ m/sec}$. And density of air is $\mathbf{0.00129\ g/cm^3}$.**

Solution:-

Frequency (f) $= 512$ Hz

Amplitude (a) $= 0.25 \text{ cm} = 2.5 \times 10^{-3}$ m

Velocity $(v) = 340$ m/sec

Density (ρ) $= 0.00129 \text{ gm/cm}^3 = 1.29 \text{ Kg/m}^3$

Now we know that;

$$\text{Intensity } (I) = 2\pi^2\rho f^2 a^2 v$$

$$= 2 \times (3.14)^2 \times 1.29 \times (512)^2 \times (2.5 \times 10^{-3})^2 \times 340$$

$$\therefore \text{I} = 14170.26 \text{ Watt/m}^2$$

3. **The equation of transverse wave travelling in a rope is given by $\mathbf{y = 10\ sin\pi\ (0.01x - 2t)cm}$. Find the amplitude, frequency, velocity and wave length of the wave.**

Solution:-

We have;

$$y = 10\sin\pi\,(0.01x - 2t) \ldots\ldots\ldots \text{(i)}$$

Comparing this equation with;

$$y = a\sin(kx - wt)$$

$$a = 10, \qquad k = 0.01\pi, \qquad w = 2\pi$$

$$\text{now,} \qquad w = 2\pi f \quad \therefore f = 1\ \text{Hz}$$

$$\text{Also,} \qquad k = \frac{2\pi}{\lambda}, \qquad \text{or, } 0.01\pi = \frac{2\pi}{\lambda} \qquad \therefore\ \lambda = 200\ \text{cm}$$

$$\text{And,} \qquad v = f\lambda = 200\ \text{cm/sec}$$

4. **In simple harmonic motion when the displacement is one half the amplitude. What fraction of total energy is the kinetic energy and what fraction is potential energy. At what displacement is the energy half kinetic and half potential energy?**

Solution:-

We have in SHM;

$$\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2}mw^2A^2\cos^2(wt + \emptyset)$$

$$\text{P.E.} = \frac{1}{2}kx^2 = \frac{1}{2}mw^2x^2 = \frac{1}{2}mw^2A^2\sin^2(wt + \emptyset)$$

$$\therefore \text{T.E.} = \frac{1}{2}mw^2A^2$$

From question;

$$x = \frac{A}{2}$$

$$A\sin(wt + \emptyset) = \frac{A}{2}$$

$$\therefore\ \sin(wt + \emptyset) = \frac{1}{2}$$

$$\therefore \cos(wt + \emptyset) = \sqrt{1 - (\frac{1}{2})^2} = \frac{\sqrt{3}}{2}$$

$$\text{So, P. E.} = \frac{1}{2} mA^2w^2 . \frac{1}{4}$$

$$\therefore\ \text{P. E.} = \frac{1}{4}\ \text{T. E.}$$

$$\text{And,} \qquad \text{K. E.} = \frac{1}{2} mA^2w^2 . \frac{3}{4}$$

$$\therefore\ \text{K. E.} = \frac{3}{4}\ \text{T. E.}$$

$$\text{Now given that,} \quad \text{P. E.} = \frac{1}{2}\text{T. E}$$

$$\text{or,} \qquad \frac{1}{2} mx^2w^2 = \frac{1}{2}[\frac{1}{2} mA^2w^2]$$

$$\text{or,} \qquad x^2 = \frac{A^2}{2}$$

$$\therefore x = \frac{A}{\sqrt{2}}$$

5. **The equation of transverse wave on a string is $\mathbf{y = (2.0\ mm)\sin[(20\ m^{-1})x - (600\ s^{-1})t]}$. The tension on the string is $\mathbf{15}$ N. What is the wave speed? Find the linear density of the string in gm/m.**

Solution:-

$$y = (2.0\ \text{mm})\sin[(20\ \text{m}^{-1})x - (600\ \text{s}^{-1})t]$$

Now comparing this equation with;

$$y = a\sin(kx - wt)$$

$$a = 2\ \text{mm} = 2 \times 10^{-3}\ \text{m}$$

$$k = 20\ \text{m}^{-1}\ \text{ or,} \frac{2\pi}{\lambda} = 20 \quad \therefore \lambda = 0.314\ \text{m}$$

$$w = 600\ \text{s}^{-1}\ \text{ or,} 2\pi f = 600 \quad \therefore f = 95.5\ \text{Hz}$$

$$\text{we have,} \qquad w^2 = \frac{k}{m}$$

$$\therefore m = 5.55 \times 10^{-5}\ \text{kg}$$

$$\text{And,} v = f\lambda = 95.5 \times 0.314 = 30\ \text{m/sec}$$

$$\text{Also we have} \quad v = \sqrt{\frac{T}{\mu}}$$

$$\therefore \text{linear density } (\mu) = \frac{T}{v^2} = \frac{15}{30^2} = 0.0167 \text{ kg/m}$$

$$\therefore \mu = 16.67 \text{ gm/m}$$

Exercise:-

1. Define transverse wave. Develop a differential equation of the wave in a stretched string and then find the velocity of transverse wave.
2. Prove that if a transverse wave is travelling along a string, then the slope at any point of the string is numerically equal to the ratio of the particle speed to the wave speed at that point.
3. In the progressive wave, show that the potential energy and kinetic energy of every particle will change with time but the average KE per unit volume and PE per unit volume remains constant.
4. Show that the wave equation of a transverse wave in string is $\frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2}$, where $v = \sqrt{\frac{F}{\mu}}$, where μ is mass per unit length.
5. Write a plane progressive wave equation for a wave propagating along the +ve x-axis. Prove that (i) particle

velocity at a point = -(wave velocity) × (slope of the displacement curve at that point) and (ii) particle acceleration at a point = -(wave velocity)^2 × (curvature of the displacement curve at that point).

6. Derive a relation for speed of transverse wave in a stretched string and show that the average rate of energy transfer is $\frac{1}{2}\mu v w^2 A^2$, where the symbols are having usual meaning.
7. The speed of transverse wave on a string is 170 m/s when the string tension is 120 N. To what value must the tension be changed to raise the wave speed to 180 m/s?
8. A rod vibrating at 12 Hz generates harmonic waves with amplitude of 1.5 mm in a string of linear mass density 2 gm/m. If the tension on the string is 15 N, What is the average power supplied by the source?
9. A string has linear density of 625 gm/m and is stretched with a tension 50 N. A wave, whose frequency and amplitude are 160 Hz and 10 mm respectively, is travelling along the string. At what average rate is the wave transporting energy along the string?
10. The elastic limit of steel forming a piece of wire is equal to $2.7 \times 10^8 Pa$. what is the maximum speed at which transverse wave pulses can propagate along this wire without exceeding this stress? (Density of steel = $7.89 \times 10^3\ kg/m^3$).