

6th CLASS

Ex-23 Q. No. 13 $\frac{dy}{dx} + y = \cos x$

Sol: Given diff. eqⁿ is $\frac{dy}{dx} + y = \cos x \quad \text{---(1)}$

which is linear diff. eqⁿ on y

So comparing it with $\frac{dy}{dx} + py = q$

$$p = 1, q = \cos x$$

$$\text{Now, I.F.} = e^{\int p dx} = e^{\int 1 dx} = e^x$$

Multiplying eqⁿ(1) by I.F. = e^x on both sides we get;

$$y \times \text{I.F.} = \int (q \times \text{I.F.}) dx + C$$

$$\text{or, } y \times e^x = \int \underline{e^x \cos x dx} + C$$

* Using $\int e^{ax} \cos bx dx = \frac{e^{ax} (\cos bx + b \sin bx)}{a^2 + b^2}$

** $\int e^{ax} \sin bx dx = \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2}$

$$y \cdot e^x = \frac{e^x (1 \cdot \cos x + 1 \cdot \sin x)}{1^2 + 1^2} + C$$

$$\text{or, } y e^x = \frac{e^x (\cos x + \sin x)}{2} + C$$

$$\Rightarrow y = \cos x + \sin x + 2C e^{-x}$$

$$\text{or, } y = \cos x + \sin x + K e^{-x}$$

where $K = 2C$ which is
the required general soln

Exact differential eqⁿ

Exact differential eq

Exact diff. eqn: - A diff. eqn
is said to be exact if there exist
a function $f(x,y)$ such that
 $M(x,y) dx + N(x,y) dy = \underline{df(x,y)}$

$$\underline{M(x,y) dx + N(x,y) dy} = \underline{df(x,y)}$$

$$\text{e.g. } \underline{x dy} + \underline{y dx} = \underline{d(xy)x}$$

i.e. the given diff. eqn is exact if
 $M(x,y) dx + N(x,y) dy$ is exact or
perfect differential.

Note:- The diff. eqn

$$M(x,y) dx + N(x,y) dy = 0$$

will be exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \leftarrow$$

where $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x}$ denotes the
partial derivative.

$$U = ax^2 + bxy + by^2$$

~~diff. partial w.r.t. x~~

$$\frac{\partial U}{\partial x} = 2ax + by + 0$$

diff. partial w.r.t. y

$$\frac{\partial U}{\partial y} = 0 + bx + 2by$$

Note:-

Every diff. eqn

$$M(x,y) dx + N(x,y) dy = 0$$

===== Every $\frac{M(x,y)}{N(x,y)} \rightarrow M(x,y)dx + N(x,y)dy = 0$
is not exact.

for ex. $x^2 dy + 2xy dx = 0$ is exact
because

$$x^2 dy + 2xy dx = \underline{d(x^2 y)}$$

Some formulae

$$\textcircled{1} \quad x dy + y dx = d(xy)$$

$$\textcircled{2} \quad \frac{x dy - y dx}{x^2} = d\left(\frac{y}{x}\right)$$

$$\textcircled{3} \quad \frac{y dx - x dy}{y^2} = d\left(\frac{x}{y}\right)$$

and so on. — page 329

Ex-25

Solve the following diff. eq" by
reducing to exact form

e.g. $x dy + y dx = 0$

Sol: Given diff. eq" is

$$x dy + y dx = 0$$

or, $\int d(xy) = 0$ Integrating

$xy = C$ which is the required
general soln ~~**~~

or $x dy + y dx = 0$

$$\underline{\underline{OK}} \quad x dy + y dx = 0$$

$$\text{or, } x dy = -y dx$$

$$\text{or, } \int y dy = - \int \frac{1}{x} dx \quad \text{Int.}$$

$$\log y = -\log x + \log C$$

$$\text{or, } \log y = \log \left(\frac{C}{x} \right) \Rightarrow y = \frac{C}{x}$$

$$\therefore xy = C$$

~~Solve~~ ~~Ex-21, 23, 24~~ which is the required general soln of the given diff. eq "

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\text{e.g. } \frac{dy}{dx} = \frac{2x-y+1}{x+2y-3} \quad \text{Ex-22}$$

$$(x+2y-3)dy = (2x-y+1)dx$$

$$\text{or, } x dy + 2y dy - 3 dy = dx - y dx + dx$$

$$\text{or, } \cancel{x dy} + \cancel{2y dy} - \cancel{3 dy} = dx - \cancel{y dx} + \cancel{dx}$$

$$\text{or, } \int (2y) dy + \int (-3) dy = \int dx + \int dx$$

Integrating

$$xy + \frac{y^2}{2} - 3y = 2x + x + C$$

$$\text{or, } xy + y^2 - 3y - x^2 - x - C = 0$$

$$x^2 + y^2 - 3y - x^2 - x - c = 0$$

$$\cancel{x^2} - 25x \rightarrow x \cancel{x}$$

First order but not first degree differential equation

An eqⁿ of the form $f(x, y, p) = 0$ where $p = \frac{dy}{dx}$ is called first order but first degree diff. eqⁿ. The solution of such type of diff. eqⁿ contains only one constant.

We will discuss the following first order but not first degree diff. eqⁿ

- Solvable for p
- Solvable for y
- Solvable for x
- Clairaut's Equation

Solvable for P

An eqⁿ of the form $f(x, y, p) = 0$ where $p = \frac{dy}{dx}$ can be factorized into

linear factors such as

$$(P - f_1(x, y)) \cdot (P - f_2(x, y)) \cdots (P - f_n(x, y)) = 0$$

Such type of first order but not first degree diff. eqⁿ is called solvable for P .

"Solvble for p."

It's general soln is

$$\begin{aligned} \text{L. } & \overline{P^2 + P - 6} = 0 \\ \text{Soln. } & \text{ Given diff. eqn is } \frac{dy}{dx} = P \\ & \underline{P^2 + P - 6 = 0} \end{aligned}$$

or, $(\frac{dy}{dx})^2 + \frac{dy}{dx} - 6 = 0$
whose $\frac{dy}{dx} = P$

$$\text{or, } P^2 + 3P - 2P - 6 = 0$$

$$\text{or, } P(P+3) - 2(P+3) = 0$$

$$\text{or, } (P+3)(P-2) = 0$$

$$\text{Either } P+3 = 0$$

$$\text{or } P-2 = 0$$

$$\text{or, } \frac{dy}{dx} + 3 = 0$$

$$\frac{dy}{dx} - 2 = 0$$

$$\text{or, } \int dy = -3 dx \text{ Int.}$$

$$\text{or, } \int dy = \int 2 dx \text{ Int.}$$

$$\Rightarrow y = -3x + C$$

$$\Rightarrow y = 2x + C$$

$$\text{or, } y + 3x - C = 0$$

$$y - 2x - C = 0$$

$$0 = w \cdot y - 2u - c = 0$$

\therefore The required general soln is

$$(y + 2u - c)(y - 2u - c) = 0$$

$$\textcircled{2} \quad p^2 + 2px - 3x^2 = 0$$

Soln.: Given diff. eqns

$$\textcircled{3} \quad p^2 - p(e^x + e^{-x}) + 1 = 0 \quad \left\{ \begin{array}{l} f(x, y, p) = 0 \\ p = f_1(x, y) \end{array} \right.$$

Soln.: Given diff. eqn

$$p^2 - p(e^x + e^{-x}) + 1 = 0$$

$$\text{or}, \quad p^2 - pe^x - pe^{-x} + 1 = 0$$

$$\text{or}, \quad p(p - e^x) - e^{-x}(p - \frac{1}{e^{-x}}) = 0$$

$$\text{or}, \quad p(p - e^x) - e^{-x}(p - e^x) = 0$$

$$\text{or}, \quad (p - e^x)(p - e^{-x}) = 0$$

Either

$$p - e^x = 0$$

$$\text{or} \quad p - e^{-x} = 0$$

$$\frac{dy}{dx} = e^x$$

$$dy = e^x dx \text{ Int. } \int dy = \int e^x dx$$

$$y = e^x + C$$

$$y - e^x - C = 0$$

$$dy = e^{-x} dx \text{ Int. } \int dy = \int e^{-x} dx$$

$$y = -e^{-x} + C$$

$$\Rightarrow y + e^{-x} - C = 0$$

\therefore The required general soln is

$$(y - e^x - C)(y + e^{-x} - C) = 0$$

$$\textcircled{8} \quad p^3 - p(x^2 + xy + y^2) + x^2y + xy^2 = 0$$

Soln.: Given diff. eqn

$\text{Q. } \begin{matrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{matrix}$
Sol: Given diff. eqn is
 $P^3 - Px^2 - Pxy - Py^2 + x^2y + xy^2 = 0$
or, $P^3 - Px^2 - Pxy + x^2y - Py^2 + xy^2 = 0$
or, $P(P^2 - x^2) - xy(P - x) - y^2(P - x) = 0$
or, $(P - x)\{P(P + x) - xy - y^2\} = 0$
or, $(P - x)(P^2 + Px - xy - y^2) = 0$
or, $(P - x)(P^2 - y^2 + Px - xy) = 0$
or, $(P - x)\{(P - y)(P + y) + x(P - y)\} = 0$
or, $(P - x)(P - y)(P + y + x) = 0$

Either

$$dx + y + x = 0$$

or, $\cancel{\frac{dy}{x}} + y = -x$ $\quad \text{--- (1)}$
which is linear diff. eqn on y

So it's I.F. $= e^{\int 1 dx} = e^x$
Multiplying eqn (1) by I.F. $= e^x$
on both sides we get

$$y \times \text{I.F.} = \int (y \times \text{I.F.}) dx + C$$

$$y \cdot e^x = - \int x e^x dx + C$$

$x \sim 1, -1, 0, 1, -1, \dots$

$$\int e^{-x} - \int e^x + C$$

$$= -(xe^x - \int 1 \cdot e^x dx) + C$$

$$y \cdot e^x = -xe^x + e^x + C$$

$$\text{a. } y = -x + 1 + Ce^{-x}$$

$$\text{a. } y + x - 1 - Ce^{-x} = 0$$

\therefore the required general soln is

$$() () (y + x - 1 - Ce^{-x}) = 0$$

#

$$Ax^2 + Bx + C = 0$$

$$\Rightarrow x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

~~$$p^3 + 3xp^2 - y^3 p^2 - 3xy^3 p = 0$$~~

$$\times \int \frac{6 - 5v}{(5v^2 - 7v + 2)} dv$$

$$\int \frac{\frac{1}{2}(10v - 7) - \frac{7}{2} + 6}{(5v^2 - 7v + 2)} dv$$

$$= \frac{1}{2} \int \frac{10v - 7}{5v^2 - 7v + 2} dv + \frac{5}{2} \int \frac{1}{(5v^2 - 7v + 2)} dv$$

$$= -\frac{1}{2} \log(5v^2 - 7v + 2) + \frac{5}{2} \int \frac{1}{(v^2 - \frac{7}{5}v + \frac{2}{5})} dv$$

$$= " + \frac{1}{2} \int \frac{1}{v^2 - \frac{7}{5}v + \frac{2}{5}} dv$$

$$= " + \frac{1}{2} \int \frac{1}{\left(9 - \frac{x}{10}\right)^2 - \left(\frac{3}{10}\right)^2} dx$$

$\frac{+2}{5}$

$$* \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right) + C$$

BCT-A First year ~~7th class~~ II
do 78-02-26

First order but not first degree diff. eq"

→ Solvable for y An eqn of the form $y = f(x, p)$
where $p = \frac{dy}{dx}$ Such type of first order but
not first degree diff. eq's is called solvable for
y.

The method of solvable for y

Let $y = f(x, p)$ — (1)

diff. (1) w.r.t. 'x'

$$\frac{dy}{dx} = \phi(x, p, \frac{dp}{dx})$$

or $P = \phi(x, p, \frac{dp}{dx})$ — (2)

or Which is the diff. eqn in two variables

x and P

Suppose its soln is

— C. n. —

$$\begin{aligned} P &= \frac{dy}{dx} \\ &= x^2 + p^2 = f(x, p) \\ &= 2x + p \frac{dp}{dx} \\ &= \phi(x, p, \frac{dp}{dx}) \end{aligned}$$

$$\begin{cases} \frac{dy}{dx} = \sin x \\ \int dy = \int \sin x dx \\ y = -\cos x + C \\ \therefore y + \cos x + C = 0 \end{cases}$$

- 11 - or ...

$$F(x, p, c) = 0 \quad (3)$$

$$\begin{cases} y + px + c = 0 \\ F(x, y, c) = 0 \end{cases}$$

Eliminating p from (1) and (3)

We get two required general soln.

If P can not be eliminated from two given eqn (1) and (3), Then the eqn (1) and (3) together give the required general soln.

Ex-27.

Find the general soln of the following diff. eqn.

$$(5) \quad y + px = x^4 p^2 \quad \text{where } P = \frac{dy}{dx}$$

Soln. Given diff. eqn $y + px = x^4 p^2$.

$$\text{or, } y = \underline{x^4 p^2} - px \quad (1)$$

which is of the form $y = f(x, p)$

So it can be solved by the method for solvable for y .

\therefore diff. (1) w.r.t. x

$$\frac{dy}{dx} = x^4 \cancel{dP} \frac{dp}{dx} + p^2 \cdot 4x^3 - p \cdot 1 - x \frac{dp}{dx}$$

$$\text{or, } \cancel{\frac{P}{x}} = \cancel{dP} x^4 \frac{dp}{dx} + \cancel{4p^2 x^3} - p - x \frac{dp}{dx}$$

$$P - 4p^2 x^3 + p = \cancel{2P} x^4 \frac{dp}{dx} - x \frac{dp}{dx}$$

$$\text{or, } (P - 4p^2 x^3) = (\cancel{2P} x^4 - x) \frac{dp}{dx}$$

$$\text{or, } (P - 4P^2x^3) = (2Px^4 - x) \frac{dp}{dx}$$

$$\text{or, } \frac{(dp - 4P^2x^3)}{(2Px^4 - x)} = \frac{dp}{dx}$$

$$\text{or, } \frac{2P(1 - 2Px^3)}{-x(1 - 2Px^3)} = \frac{dp}{dx}$$

$$\text{or, } \frac{2P}{-x} = \frac{dp}{dx}$$

$$\text{or, } \int \frac{dx}{x} = -\int \frac{1}{P} dp \quad \text{Integrating}$$

$$\therefore d \log x = -\log P + \log C$$

$$\therefore \log x^2 = \log \frac{C}{P}$$

$$\Rightarrow x^2 = \frac{C}{P}$$

$$\text{or, } P = \frac{C}{x^2} \quad \text{--- (2)}$$

$$y = x^4 P^2 - Px \quad \text{--- (1)}$$

Eliminating P from (1) and (2),

$$\therefore y = x^4 \cdot \left(\frac{C}{x^2}\right)^2 - \frac{C}{x^2} \cdot x$$

$$\therefore y = C^2 - \frac{C}{x}$$

$\Rightarrow xy = C^2x - C$ which is the required
general soln of the
given diff. eq $\underline{\underline{y' = x}}$

Ans.

Solvable for x

An eqⁿ of the form $x = f(y, p)$
where $p = \frac{dy}{dx}$ such type of first order
but not first degree diff. eqⁿ is called
solvable for x .

The method of solvable for x

Let $\underline{x} = f(y, p) \quad \text{--- (1)}$

diff. (1) wrt. 'y'

$$\frac{dx}{dy} = \phi(y, p, \frac{dp}{dy})$$

$$\therefore \frac{1}{p} = \phi(y, p, \frac{dp}{dy}) \quad \text{--- (2)}$$

which is the diff. eqⁿ in two variable
 y and p .

Suppose its solution is

$$F(y, p, \underline{\underline{)}} = 0 \quad \text{--- (3)}$$

Eliminating p from (1) and (3) gives the
required general solⁿ.

If p can not be eliminated from two
given eqⁿ (1) and (3) together give
required general solⁿ of the given diff. eq^s

Ex-27.

Ex-27.

Find the general soln. of the following diff. eq.

(2) $y = 2px + p^3y^2$

Soln.: Given diff. eq " $y = 2px + p^3y^2$

or, $2px = y - p^3y^2$

or, $dx = \frac{y - p^3y^2}{p} = \frac{y - p^2y^2}{p}$

or, $dx = \frac{y}{p} - p^2y^2 \quad \text{--- (1)}$

which is of the form $x = f(y, p)$

so it can be solved by the method for
solvable for x

∴ diff. (1) w.r.t. 'y'

$$2 \frac{dx}{dy} = \frac{p \frac{dy}{dy} - y \frac{dp}{dy}}{p^2} - p^2 \cdot 2y - y^2 \cdot 2p \frac{dp}{dy}$$

$$\text{or, } 2 \cdot \frac{1}{p} = \frac{1}{p} - \frac{y \frac{dp}{dy}}{p^2} - 2p^2y - 2py^2 \frac{dp}{dy}$$

$$\text{or, } \frac{1}{p} - \frac{1}{p} + 2p^2y = -\left(\frac{y}{p^2} + 2py^2\right) \frac{dp}{dy}$$

$$\text{or, } \frac{(2-1+2p^3y)}{p} = -\left(\frac{y+2p^3y^2}{p^2}\right) \frac{dp}{dy}$$

$$\text{or, } \frac{(1+2p^3y)}{p} = -y \frac{(1+2p^3y)}{p^2} \frac{dp}{dy}$$

$$\text{or, } 1 = -\frac{y}{p} \frac{dp}{dy}$$

$$\therefore 1 = - \frac{y}{p} \frac{dp}{dy}$$

$$\text{or, } \int y dy = - \int \frac{1}{p} dp \quad \text{Int.}$$

$$\log y = -\log p + \log C$$

$$\text{or } \log y = \log \left(\frac{C}{p} \right)$$

$$\text{or } y = \frac{C}{p}$$

$$\text{or, } p = \frac{C}{y} \quad \text{--- (2)}$$

Eliminating p from (1) and (3) we get

$$2x = \frac{y}{\left(\frac{C}{y}\right)} - \left(\frac{C}{y}\right)^2 \cdot y^2$$

$$\Rightarrow 2x = \frac{y}{C} - C^2$$

$$\Rightarrow 2Cx = y^2 - C^3$$

which is the required

general solⁿ of the given diff. eqⁿ.

(4) Find the general solⁿ of following diff. eqⁿ

Solⁿ: Given diff. eqⁿ is

$$\sin y \cos px - \cos y \sin px = p$$

$$\text{or, } \sin(y - px) = p$$

$$\text{or, } \underline{\underline{y - px}} = \sin^{-1} p$$

$$\underline{\underline{y}} = px + \sin^{-1}(p) \quad \text{--- (1)}$$

Ans. $\therefore \dots \dots \dots$ (Ans)

$$y = P'' + \sin^{-1} P - C$$

Diff. w.r.t. (x)

$$\frac{dy}{dx} = p + x \frac{dp}{dx} + \frac{1}{\sqrt{1-p^2}} \frac{dp}{dx}$$

$$\therefore \cancel{p} = \cancel{p} + x \frac{dp}{dx} + \frac{1}{\sqrt{1-p^2}} \frac{dp}{dx}$$

$$\therefore 0 = \left[x + \frac{1}{\sqrt{1-p^2}} \right] \frac{dp}{dx}$$

$$\text{or, } \left[x + \frac{1}{\sqrt{1-p^2}} \right] \frac{dp}{dx} = 0$$

P x - 28

$$\text{Either } \frac{dp}{dx} = 0$$

$$\text{or, } \int dp = \int 0 \cdot dx \quad \text{Int.}$$

$$\Rightarrow p = c$$

putting two values of $p = c$ in ①

$$y = cx + \sin^{-1} c$$

which is the required general solⁿ
of the given diff. eqn.

$$⑥ \quad P^2 - py + x = 0$$

Soln. Given diff. eqn $P^2 - py + x = 0$

$$\therefore py = P^2 + x$$

$$\therefore y = P + \frac{x}{P} \quad \text{--- ①}$$

diff. ① w.r.t. x

$$\frac{dy}{dx} = \frac{dp}{dx} + \frac{P \cdot 1 - x \cdot \frac{dp}{dx}}{P^2}$$

$\therefore \frac{dy}{dx} = 1$

$$\frac{dx}{dt} = \frac{1}{\sqrt{x}} - \frac{1}{t}$$

$$\therefore P = \frac{\frac{dp}{dx}}{\sqrt{x}} + \frac{1}{P} - \frac{x}{P^2} \frac{dp}{dx}$$

$$\text{or, } P - \frac{1}{P} = \left(1 - \frac{x}{P^2}\right) \frac{dp}{dx}$$

$$\therefore \frac{(P^2 - 1)}{P} = \left(\frac{P^2 - x}{P^2}\right) \frac{dp}{dx}$$

$$\text{or, } \frac{dp}{dx} = \frac{P(P^2 - 1) - x}{(P^2 - x)}$$

$$\text{or, } \frac{dx}{dp} = \frac{P^2 - x}{P(P^2 - 1)}$$

$$\text{or, } \frac{dx}{dp} = \frac{P^2}{P(P^2 - 1)} - \frac{x}{P(P^2 - 1)}$$

$$\text{or, } \frac{dx}{dp} + \frac{1}{P(P^2 - 1)} \cdot x = \frac{P}{(P^2 - 1)} \quad \text{--- (2)}$$

which is linear diff. eqn in x

$$\text{So it's I.F.} = e^{\int \frac{1}{P(P^2 - 1)} dp}$$

$$= e^{\int \left[-\frac{1}{P} + \frac{P}{P^2 - 1} \right] dp}$$

$$= e^{-\log P + \frac{1}{2} \log(P^2 - 1)}$$

$$= e^{-\log P + \log(P^2 - 1)^{1/2}}$$

$$= e^{\log \frac{\sqrt{P^2 - 1}}{P}}$$

$$= \frac{\sqrt{P^2 - 1}}{P}$$

∴

$$= \frac{p^{e-1}}{P}$$

Multiplying eqn ② by I.F. $= \frac{\sqrt{p^2-1}}{P}$ on both sides we get,

$$\begin{aligned} x \times \frac{\sqrt{p^2-1}}{P} &= \int \frac{P}{(p^2-1)} \cdot \frac{\sqrt{p^2-1}}{P} dp + C \\ &= \int \frac{1}{\sqrt{p^2-1}} dp + C \end{aligned}$$

using $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log \{x + \sqrt{x^2-a^2}\}$

$$\therefore x \frac{\sqrt{p^2-1}}{P} = \log(p + \sqrt{p^2-1}) + C$$

$$\Rightarrow x = \frac{p \log(p + \sqrt{p^2-1})}{\sqrt{p^2-1}} + \frac{CP}{\sqrt{p^2-1}} - ③$$

Since p cannot be easily eliminated from ① and ③ so ① and ③ together gives the required general solution.

i.e. $x = \frac{p \log(p^2 + \sqrt{p^2-1})}{\sqrt{p^2-1}} + \frac{CP}{\sqrt{p^2-1}}$ } u

$$y = p + \frac{x}{p}$$

Aus.

Solve:- $Dy = y p^2 + 2p^2 n$

$$② \quad y - 2p^2 n + 2yp^2 = 0$$

$$\textcircled{2} \quad y - 2px + ayp^2 = 0$$

$$\textcircled{3} \quad p^3 - 4ayp + 8y^2 = 0$$

$$\textcircled{5} \quad xp^2 - 2yp + a = 0$$

Ex-27

\textcircled{1}

\textcircled{2}

\textcircled{4}

\textcircled{5}

old question

BCT-A gth Class

$$\text{Ex-26} \quad \text{Q-H.10} \quad \left(\frac{dy}{dx}\right)^2 + 2y \cot x \frac{dy}{dx} = y^2$$

$$\text{Soln, Given diff. eqn is } \left(\frac{dy}{dx}\right)^2 + 2y \cot x \frac{dy}{dx} = y^2$$

$$\text{or, } \underline{\underline{p^2 + 2y \cot x p - y^2 = 0}} \quad \text{where } p = \frac{dy}{dx}$$

which ~ quadratic in p so,

$$\text{or, } p = \frac{-2y \cot x \pm \sqrt{(2y \cot x)^2 - 4 \cdot 1 \cdot (-y^2)}}{2 \cdot 1}$$

$$\begin{aligned} & A u^2 + B u + C = 0 \\ & u = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \\ & \underline{\underline{p = f(u, y)}}$$

$$\therefore p = \frac{-2y \cot x \pm \sqrt{4y^2 \cot^2 x + 4y^2}}{2}$$

$$= \frac{-2y \cot x \pm 2y \sqrt{\cot^2 x + 1}}{2}$$

$$= \frac{-2y \cot x \pm 2y \cosec x}{2}$$

$$= y(-\cot x \pm \cosec x)$$

$$\begin{aligned}
 &= y \left(-\cos n \pm i \sin n \right) \\
 &= y \left(-\frac{\cos n}{\sin n} \pm \frac{i}{\sin n} \right) = y \left(\frac{\cos n \pm i}{\sin n} \right)
 \end{aligned}$$

$$\therefore P = y \left(-\frac{\cos n \pm i}{\sin n} \right)$$

Taking (+ve) sign

$$P = y \left(-\frac{\cos n + i}{\sin n} \right)$$

$$\therefore \frac{dy}{dx} = y \left(\frac{-\cos n + i}{\sin n} \right)$$

$$\text{or, } \int \frac{1}{y} dy = \int \left(\frac{1 - \cos n}{\sin n} \right) dx \quad \text{Integrating}$$

$$\begin{aligned}
 \log y &= \int \frac{(1 - \cos n)(1 + \cos n)}{\sin n (1 + \cos n)} dx \\
 &= \int \frac{1 - \cos^2 n}{\sin n (1 + \cos n)} dx \\
 &= \int \frac{\sin^2 n}{\sin n (1 + \cos n)} dx \\
 &= \int \frac{\sin n}{1 + \cos n} dx = -\log(1 + \cos n) + \log C
 \end{aligned}$$

$$\therefore \log y = \log \frac{C}{(1 + \cos n)}$$

$$\Rightarrow y = \frac{C}{(1 + \cos n)}$$

$$\Rightarrow \delta = \frac{1}{(1 + \cos n)}$$

or, $y(1 + \cos n) - c = 0$

taking $(-ve)$ sign,

$$p = \frac{y(-\cos n - 1)}{\sin n} x$$

Clairaut's equation

An eqⁿ of the form $y = px + f(p)$

where $p = \frac{dy}{dx}$ such type of

first order but not first degree
diff eqⁿ is called Clairaut's eqⁿ.

It can be solved by two following method

$$\text{Let } y = px + f(p) \quad (1)$$

diff (1) wrt 'n'

$$\frac{dy}{dx} = p + x \frac{dp}{dx} + f'(p) \frac{dp}{dx}$$

$$\text{or, } p = p + x \frac{dp}{dx} + f'(p) \frac{dp}{dx}$$

$$\text{or, } 0 = [x + f'(p)] \frac{dp}{dx}$$

$$\text{or, } [x + f'(p)] \frac{dp}{dx}$$

$$f(y, p) = 0$$

↳ linear factor

$$y = f(x, p)$$

$$x = f(y, p)$$

$$\text{or, } dp = \int_0^x dx \quad \text{Integrating}$$

$$\therefore \frac{dp}{dx} = f_0 - dx \text{ Integrating}$$

$$\Rightarrow p = C \quad \text{--- (2)}$$

Eliminating p from (1) and (2) we get
 $y = Cx + f(C)$
 which is the required

$$\boxed{\text{general soln}} \stackrel{\text{find eq}}{=} \checkmark$$

$$\text{or, } x + f'(p) = 0$$

$$\therefore x = -f'(p) \quad \text{--- (3)}$$

Eliminating p from (1) and (3) we get
 $\boxed{\text{Singular soln}}$

Ex-28

Solve the following diff. eq"

$$(1) \quad y = px + p - p^2 \quad \text{where } p = \frac{dy}{dx}$$

Soln:- Given diff. eq" is,

$$y = px + p - p^2 \quad (1) \text{ D.E}$$

which is of the form $y = px + f(p)$

So it can be solved by the Clairaut's eq" method

\therefore diff. (1) w.r.t. "x"

$$\frac{dy}{dx} = p + x \frac{dp}{dx} + \frac{df}{dx} - p \frac{dp}{dx}$$

$$\text{or, } p = p + x \frac{dp}{dx} + \frac{df}{dx} - p \frac{dp}{dx}$$

$$\text{or, } 0 = f_x + (1-p) \frac{dp}{dx}$$

$$\text{or, } O = (x+1-2p) \frac{dp}{dx}$$

$$\frac{dp}{dx} = 0$$

$$\therefore dp = 0$$

$$p = c \quad \text{--- (2)}$$

Eliminating p from (1) & (2)

$$\boxed{y = cx + c - c^2}$$

which is the required general solⁿ
of the given diff-eq (1)

$$\text{or } x+1-2p=0$$

$$\therefore dp = x+1$$

$$\Rightarrow p = \frac{x+1}{2} \quad \text{--- (3)}$$

eliminating p from (1) and (3)
we get,

$$\begin{aligned} \text{or, } y &= \frac{x+1}{2} \cdot x + \frac{x+1}{2} - \left(\frac{x+1}{2}\right)^2 \\ &= \frac{x^2+x}{2} + \frac{x+1}{2} - \frac{x^2+2x+1}{4} \\ &= \underline{\underline{2x^2+2x+2-x^2-2x-1}} \end{aligned}$$

$$= \frac{4n^2 + 2x + 2n + 2 - n^2 - 2x - 1}{4}$$

$$= \frac{x^2 + 2x + 1}{4} = \frac{(n+1)^2}{4}$$

" $4y = (n+1)^2$
 which is required singular soln

$$\textcircled{6} \quad p = \log(pn-y)$$

Soln: Given diff. eqn is $p = \log(pn-y)$

$$\Rightarrow e^p = pn - y$$

$$\text{or, } y = pn - e^p \quad \text{---(i)}$$

diff. $\textcircled{1}$ w.r.t x

$$\frac{dy}{dx} = p + n \frac{dp}{dx} - e^p \frac{dp}{dx}$$

$$\text{" } p' = p + (n - e^p) \frac{dp}{dx}$$

$$\text{or, } (n - e^p) \frac{dp}{dx} = 0$$

$$\text{or, } \frac{dp}{dx} = 0 \quad \text{or, } \int dp = \int 0 \quad \Rightarrow p = C$$

Putting the value of $p = C$ in $\textcircled{1}$

$$y = Cx - e^C$$

which is the required general soln

$$\text{or, } n - e^p = 0$$

$$\text{or, } n - e^p = 0$$

$$\Rightarrow x = e^p$$

$$\Rightarrow \log n = p$$

Putting the values of $p = \log n$ in ①

$$y = x \cdot \log n - e^{\log n}$$

$$y = n \log n - n$$

$$\therefore y = n(\log n - 1)$$

which is the required \Rightarrow Singular Soln

$$\textcircled{P} (y+1)p - np^2 + 2 = 0$$

Soln: Given diff. eqn

$$(y+1)p - np^2 + 2 = 0$$

$$\therefore (y+1)p = np^2 - 2$$

$$\therefore y+1 = \frac{np^2 - 2}{p}$$

$$\therefore y = p(n - \frac{2}{p}) - 1 \quad \text{--- ①}$$

diff. ① w.r.t. n

$$\frac{dy}{dx} = p + n \frac{dp}{dx} - 2 \left(\frac{1}{-p^2} \right) \frac{dp}{dx}$$

$$\therefore p = p + n \frac{dp}{dx} + \frac{2}{p^2} \frac{dp}{dx}$$

$$\therefore \left(n + \frac{2}{p^2} \right) \frac{dp}{dx} = 0$$

$$\text{either, } \frac{dp}{dx} = 0 \quad \text{or, } n + \frac{2}{p^2} = 0$$

or, $\frac{y}{x} = p - 1$
 $\Rightarrow p = \frac{y+1}{x}$

or, $x + \frac{2}{p^2} = 0$

$\therefore x = -\frac{2}{p^2}$ $\left[y = p \cdot -\frac{2}{p^2} + 1 \right] \quad \text{①}$
 $\therefore p^2 = -\frac{2}{x}$

we have, $(y+1)p - np^2 + 2 = 0$

$\therefore (y+1)p = (np^2 - 2)$

Sq. on both sides,

$$(y+1)^2 p^2 = (np^2 - 2)^2$$

putting the value of $p^2 = -\frac{2}{x}$

$$(y+1)^2 \left(-\frac{2}{x}\right) = [1 \cdot \left(\frac{-2}{x}\right) - 2]^2$$

$$-\frac{2}{x}(y+1)^2 = 16$$

$$\Rightarrow (y+1)^2 = -\frac{16x}{2}$$

or, $(y+1)^2 + 8x = 0$ which is the
 reg no d
 singular soln

⑧ $(xp-y)^2 = p^2 - 1$

sim. since diff. eqn is,

Soln. Given diff. eqn is
 $(xp - y)^2 = p^2 - 1$

$$xp - y = \sqrt{p^2 - 1}$$

$$\therefore y = px - \sqrt{p^2 - 1} \quad \text{---(1)}$$

diff. (1) wrt 'n'

$$\frac{dy}{dx} = p + n \frac{dp}{dn} - \frac{1}{2} \frac{1}{\sqrt{p^2 - 1}} \frac{dp}{dn}$$

$$\text{or, } p' = p + n \frac{dp}{dn} - \frac{p}{\sqrt{p^2 - 1}} \frac{dp}{dn}$$

$$\text{or, } \left[x - \frac{p}{\sqrt{p^2 - 1}} \right] \frac{dp}{dn} = 0$$

either $\frac{dp}{dn} = 0$

$$\int dp = 0 \rightarrow \text{Int.}$$

$$\Rightarrow p = C$$

putting the value of $p = C$ in (1) we get

$$y = Cx - \sqrt{C^2 - 1}$$

which is the required general soln

$$\text{or } x - \frac{p}{\sqrt{p^2 - 1}} = 0$$

$$\text{or, } x = \frac{p}{\sqrt{p^2 - 1}} \quad \text{---(2)}$$

putting value of $x = \frac{p}{\sqrt{p^2 - 1}}$ in (1)

$$y = pn - \sqrt{p^2 - 1}$$

↳ (1)

putting value of $x = \frac{p}{\sqrt{p^2-1}}$

$$y = p \cdot \frac{p}{\sqrt{p^2-1}} - \frac{\sqrt{p^2-1}}{\sqrt{p^2-1}}$$

$$y = \frac{p^2}{\sqrt{p^2-1}} - \frac{\sqrt{p^2-1}}{\sqrt{p^2-1}}$$

$$y^2 = \frac{1}{p^2-1}$$

$$p^2-1 = \frac{1}{y^2}$$

$$x = \frac{p}{\sqrt{p^2-1}}$$

②

Now from ①

$$\text{Eq. } n^2 = \frac{p^2}{p^2-1}$$

putting the value of $p^2 = \frac{1+y^2}{y^2}$

$$n^2 = \frac{\frac{1+y^2}{y^2}}{\frac{1+y^2}{y^2} - 1} = \frac{1+y^2}{1+y^2-y^2}$$

$$\text{or, } x^2 = 1 + y^2$$
$$\Rightarrow x^2 - y^2 - 1 = 0$$

which is the required singular

~~So, if~~

②.
$$y = px + \frac{a}{p}$$