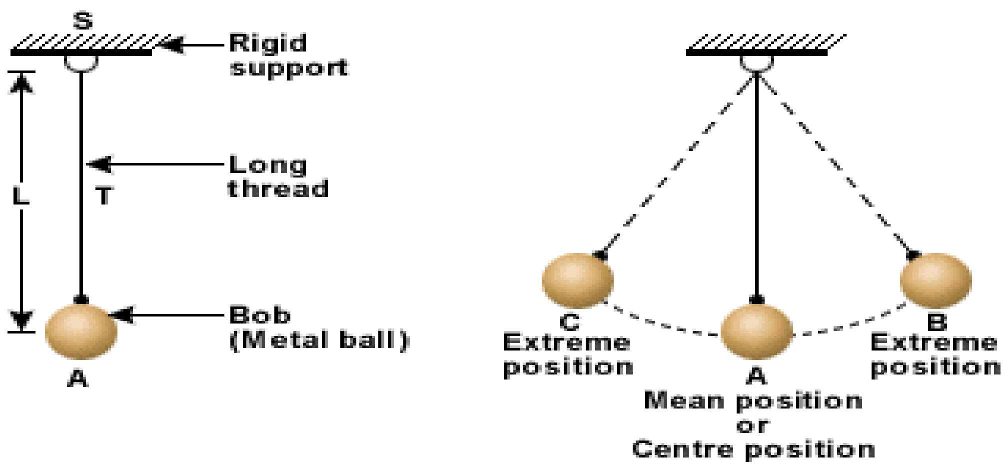
Mechanical Oscillation

Simple harmonic motion (SHM):-

It is periodic motion in which acceleration is directly proportional to the displacement from mean position and displacement is directly proportional to restoring force. e.g. pendulum, spring mass system etc.



(a) Simple pendulum (b) Motion of a simple pendulum

Differential equation of simple harmonic motion:-

Consider a particle of mass 'm' is in SHM and displacement of the particle at that instant is 'x'. Then from definition of SHM restoring force is directly proportional to the displacement from mean position.

$$i.e.\ F \propto x$$

$$or, F = -kx \ldots\ldots\ldots (1)$$

Where, k is constant of proportionality called force constant and negative sign indicates that restoring force and displacement are in opposite direction.

Also, From Newton's second law of motion;

$$F = ma = m\frac{d^2x}{dt^2} \dots\dots\dots (2)$$

From equation (1) and (2);

$$m\frac{d^2x}{dt^2} = -kx$$

$$or,\ \ m\frac{d^2x}{dt^2} + kx = 0$$

$$or,\qquad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$or,\qquad \frac{d^2x}{dt^2} + \omega^2 x = 0 \dots\dots\dots (3)$$

Where, $\omega^2 = \frac{k}{m}$

$$\therefore \omega = \sqrt{\frac{k}{m}}\ \ is\ \ the\ angular\ velocity.$$

$$2\pi f = \sqrt{\frac{k}{m}}$$

$$\therefore f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Equation (3) represents the differential equation of SHM. The solution of equation (3) is; $x = Asin(\omega t + \emptyset)$

Velocity of the particle:-

We have displacement of the particle is;

$$x = Asin(\omega t + \emptyset)$$

$$v = \frac{dx}{dt} = Awcos(\omega t + \emptyset)$$

$$\therefore v = A\omega\sqrt{1^2 - sin^2(\omega t + \emptyset)}$$

$$or, \qquad v = A\omega\sqrt{1 - \frac{x^2}{A^2}}$$

$$\therefore v = \omega\sqrt{A^2 - x^2}$$

$$\therefore v_{max} = A\omega \quad (at\ mean\ position)$$

Acceleration of the particle:-

We have, $v = A\omega cos(\omega t + \emptyset)$

$$\therefore accelaration\ (a) = \frac{dv}{dt}$$

$$= -A\omega^2 \sin(\omega t + \emptyset)$$

$$\therefore a = -\omega^2 x$$

$$\therefore a_{max} = -\omega^2 A$$

Total Energy in SHM:-

The sum of kinetic and potential energies at any instant is the total energy of a particle executing SHM at that instant.

i.e. E = K. E. + P.E.

The displacement of a particle executing SHM at any instant is

$$x = Asin(\omega t + \emptyset)$$

The velocity at that instant is $v = \frac{dx}{dt}$ = Aω Cos (ωt ± ϕ)

and acceleration is

$$\mathrm{a} = -\omega^2 \mathrm{x}$$

Now, the kinetic energy at that instant is given by

$$K.E. = \frac{1}{2} m v^2 = \frac{1}{2} m A^2 \omega^2 \cos^2(\omega t + \emptyset)$$

According to Newton's second law of motion, the force produced on the accelerated particle is

F = mass × acceleration = $m\omega^2 x$.

If dx be the small displacement on the particle executing SHM then small amount of work done is given by

dU = force × displacement = $m\omega^2 x\, dx$

For 0 to x displacement, total amount of work done is given by

$U = \int_0^x m\omega^2 x\, dx = \frac{1}{2} m\omega^2 x^2 = \frac{1}{2} kx^2$

This amount of work done represent the potential energy associated at t.

$$i.e. P.E. = \frac{1}{2} kx^2$$

$$or, P.E. = \frac{1}{2} kA^2 \sin^2(\omega t + \emptyset)$$

$$\therefore P.E. = \frac{1}{2} mA^2\omega^2 \sin^2(\omega t + \emptyset)$$

Now, total energy $(E) = K.E. + P.E.$

$$= \frac{1}{2} mA^2\omega^2 \cos^2(\omega t + \emptyset) + \frac{1}{2} mA^2\omega^2 \sin^2(\omega t + \emptyset)$$

$$E = \frac{1}{2} mA^2\omega^2$$

$$\therefore E = \frac{1}{2} kA^2 = constant.$$

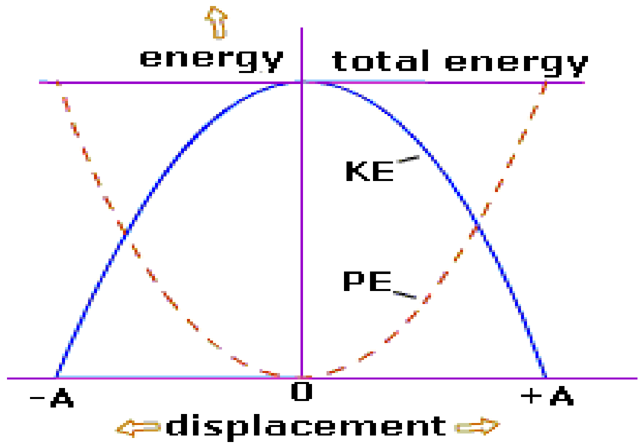
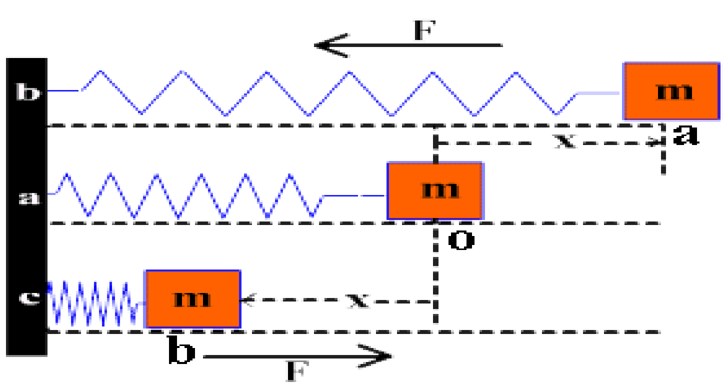


Figure. P.E. , K.E. and Total energy as a function of displacement.

Figure shows that P.E. become maximum, twice during a time period. For x=0, P.E. becomes zero. Where, $x = \mp A,\ \frac{1}{2}kx^2\ becomes\ \frac{1}{2}kA^2$ which is maximum (total) energy and vice versa, for K.E.

Spring mass system:-



Consider a mass less spring having spring constant ‘k’. One end of spring is connected to mass ‘m’ and other is connected to the rigid support. If 'x' is the displacement of the body, then the restoring force exerted by the spring at any instant is given by;

$$F = -kx \ldots \ldots \ldots (1)$$

And also from Newton's second law of motion;

$$F = ma = m\frac{d^2x}{dt^2} \ldots \ldots \ldots (2)$$

From equation (1) and (2)

$$m\frac{d^2x}{dt^2} = -kx$$

$$or,\ m\frac{d^2x}{dt^2} + kx = 0$$

$$or,\qquad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$or,\qquad \frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$where, \omega^2 = \frac{k}{m}\ and\ \omega = \sqrt{\frac{k}{m}}$$

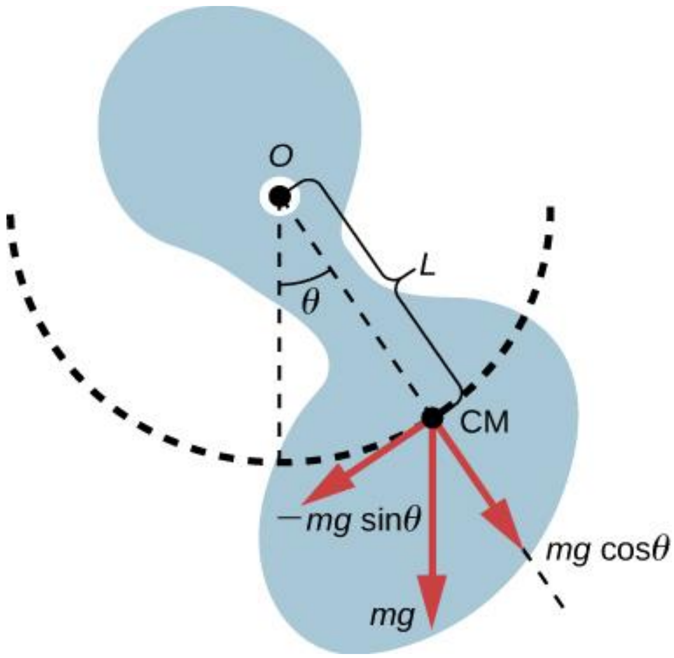
$$\therefore 2\pi f = \sqrt{\frac{k}{m}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$\therefore T = \frac{2\pi}{w} = 2\pi\sqrt{\frac{m}{k}}$$

$\therefore T = 2\pi\sqrt{\frac{m}{k}}$ is the time period for the spring mass system.

Physical pendulum:-



A physical pendulum or compound pendulum is just a rigid body of whatever shape. It is capable of oscillating about horizontal axis passing through it. All real pendulums are physical pendulum. Let the pendulum be given small angular displacement 'θ'. So that it's CG takes new position. Now restoring torque from above figure is;

$$\tau = -mglsin\theta \ldots\ldots\ldots (1)$$

(-)ve sign indicates that torque is in opposite direction to the displacement ‘θ’.

Also, from Newton’s second law of motion;

$$\tau = I\alpha \ldots\ldots\ldots (2)$$

Where, ‘I’ is moment of inertia and α is the angular acceleration.

$$i.e. \alpha = \frac{d^2\theta}{dt^2}$$

From equation (1) and (2)

$$\frac{Id^2\theta}{dt^2} = -mglsin\theta$$

Now, for small displacement $sin\theta \cong \theta$

$$\therefore \frac{Id^2\theta}{dt^2} = -mgl\theta$$

$$or, \qquad \frac{Id^2\theta}{dt^2} + mgl\theta = 0$$

$$or, \qquad \frac{d^2\theta}{dt^2} + \frac{mgl}{I}\theta = 0$$

If k is the radius of gyration of pendulum, then from parallel axis theorem, the total momentum of inertia of pendulum about the axis through point of suspension is;

$$I = mk^2 + ml^2 = m(k^2 + l^2)$$

$$\therefore \quad \frac{d^2\theta}{dt^2} + \frac{mgl}{m(k^2 + l^2)}\theta = 0$$

which is required differential equation for physical or compound pendulum.

Now, comparing this equation with standard differential equation of SHM we get; $\omega^2 = \frac{gl}{(k^2+l^2)}$

$$\therefore \omega = \sqrt{\frac{gl}{k^2 + l^2}}$$

$$\therefore T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{k^2 + l^2}{gl}}$$

$$\therefore T = 2\pi\sqrt{\frac{\frac{k^2}{l} + l}{g}}$$

which is required time period for physical pendulum or compound pendulum.

When,

$$\frac{k^2}{l} + l = L, (length\ of\ the\ pendulum)$$

Then physical pendulum is equivalent to simple pendulum.

Bar pendulum:-

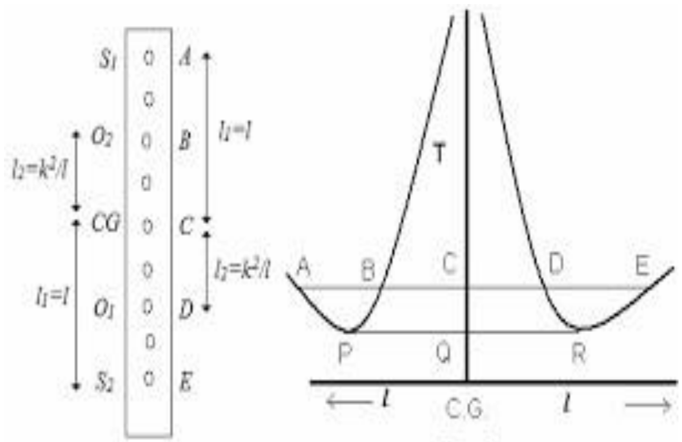


Fig.1   Fig. 2

A bar pendulum is the simplest form of a compound pendulum and consists of a uniform metal bar having equally spaced holes along its length on either side of its CG. We have time period of the bar pendulum is;

$$T = 2\pi\sqrt{\frac{k^2 + l^2}{gl}}$$

It can be seen from the graph that there are two point’s on each sides of the bar, where the time period are equal.

We have,

$$T = 2\pi\sqrt{\frac{k^2 + l^2}{gl}}$$

Squaring both sides;

$$T^2 = 4\pi^2(\frac{k^2 + l^2}{lg})$$

$$4\pi^2 l^2 + 4\pi^2 k^2 - T^2 lg = 0$$

$$4\pi^2 l^2 - T^2 lg + 4\pi^2 k^2 = 0$$

Solving this equation we get;

$$l = \frac{T^2 g \mp \sqrt{T^4 g^2 - 64\pi^4 k^2}}{8\pi^2}$$

From this equation it is clear that there are two different points having equal volume of time period on each sides of the bar pendulum. Therefore, we can say that there are four collinear points on bar.

Determination of Radius of gyration:-

Radius of gyration is the square root of mean square distance from axis of rotation. Two points one is point of suspension and other is point of oscillation on bar pendulum. Let their distance be l_1 and l_2 from the CG as shown in figure. Thus the equivalent length of the pendulum is;

$$L = l_1 + l_2$$

Now, using the relation of time period;

$$T = 2\pi\sqrt{\frac{k^2 + l_1^2}{g l_1}} = 2\pi\sqrt{\frac{\frac{k^2}{l_1} + l_1}{g}}$$

In other way, $T = 2\pi\sqrt{\frac{L}{g}}$

$$so, \qquad \frac{k^2}{l_1} + l_1 = L = l_1 + l_2$$

$$or, \qquad \frac{k^2}{l_1} = l_2$$

$$or, \qquad k^2 = l_1 l_2$$

$$\therefore K = \sqrt{l_1 l_2}$$

Point of oscillation and point of suspension are interchangeable on bar pendulum:-

Let the time period be T_1 for a point at distance l_1 from CG. Similarly, the time period T_2 for a point at distance l_2 from CG, then time periods are;

$$T_1 = 2\pi\sqrt{\frac{k^2 + l_1^2}{g l_1}} = 2\pi\sqrt{\frac{\frac{k^2}{l_1} + l_1}{g}}$$

$$and, \qquad T_2 = 2\pi\sqrt{\frac{\frac{k^2}{l_2} + l_2}{g}}$$

But, we know that;

$$k^2 = l_1 l_2$$

$$so, \quad \frac{k^2}{l_1} + l_1 = l_1 + l_2$$

$$and, \quad \frac{k^2}{l_2} + l_2 = l_1 + l_2$$

$$\therefore T_1 = T_2$$

This proves that point of suspension and point of oscillation are interchangeable.

Minimum time period:-

We know that,

$$T = 2\pi\sqrt{\frac{k^2 + l_1^2}{g l_1}} = 2\pi\sqrt{\frac{\frac{k^2}{l_1} + l_1}{g}}$$

In the equation of time period T be minimum if $\frac{k^2}{l_1} + l_1$ is minimum.

Differentiating $\frac{k^2}{l_1} + l_1$ with respect to l_1 we get;

$$\frac{d}{dl_1}\left[\frac{k^2}{l_1} + l_1\right] = -\frac{k^2}{l_1^2} + 1$$

For the time period to be minimum $-\frac{k^2}{l_1^2} + 1$ is should be zero.

$$so, \qquad -\frac{k^2}{l_1^2} + 1 = 0$$

$$\therefore \; k^2 = l_1^2$$

But we have,

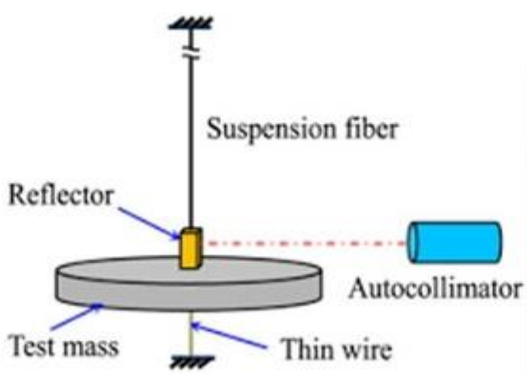
$$k^2 = \; l_1 l_2$$

$$\therefore l_1 l_2 = \; l_1^2$$

$$\therefore \; l_1 = l_2$$

That is point of suspension and point of oscillation should be equal distance from CG. And time period will be maximum at $l = 0$.

Torsional pendulum:-



A heavy body such as a disc is suspended at the one end of wire and the other end of the wire is fixed at a rigid support is known as torsional pendulum. When a disc is rotated the wire is twisted by an angle θ. In this case the restoring torque is created the restoring torque is directly proportional to the angular displacement of the wire.

$$i.e. \tau \propto \theta$$

$$or, \qquad \tau = -C\theta \ldots\ldots\ldots (1)$$

Where 'C' is tortional constant $i.e. C = \frac{\pi\eta r^4}{2l}$

The rotational form of Newton's second law of motion is;

$$\tau = I\alpha$$

$$\tau = I\frac{d^2\theta}{dt^2} \ldots\ldots\ldots (2)$$

From equation (1) and (2)

$$I\frac{d^2\theta}{dt^2} + C\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$$

which is required differential equation of tortional pendulum.

Now, comparing this equation with standard differential equation of SHM, we get,

$$\omega^2 = \frac{C}{I}$$

$$\therefore \omega = \sqrt{\frac{C}{I}}$$

$$or, \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{C}}$$

$$\therefore T = 2\pi\sqrt{\frac{I}{C}} \ldots\ldots\ldots (3)$$

which is required time period for torsional pendulum.

For regular body having moment of inertia I_1 is placed on the disc. Then,

$$T_1 = 2\pi\sqrt{\frac{I + I_1}{C}} \ldots\ldots\ldots. (4)$$

And, for irregular body having moment of inertia I_2 is placed on the disc. Then,

$$T_2 = 2\pi\sqrt{\frac{I + I_2}{C}} \ldots\ldots\ldots. (5)$$

From equation (3) and (4)

$$\frac{T_1^2}{T^2} = \frac{I + I_1}{I}$$

$$\frac{T_1^2}{T^2} = 1 + \frac{I_1}{I}$$

$$\frac{T_1^2 - T^2}{T^2} = \frac{I_1}{I} \dots \dots \dots (6)$$

From equation (3) and (5)

$$\frac{T_2^2}{T^2} = \frac{I + I_2}{I}$$

$$\frac{T_2^2}{T^2} = 1 + \frac{I_2}{I}$$

$$\frac{T_2^2 - T^2}{T^2} = \frac{I_2}{I} \dots \dots \dots (7)$$

From equation (6) and (7)

$$\frac{T_2^2 - T^2}{T_1^2 - T^2} = \frac{I_2}{I_1}$$

$$\therefore\ I_2 = \frac{T_2^2 - T^2}{T_1^2 - T^2} \times I_1$$

Free Oscillation:-

Consider a simple pendulum in vacuum, there is no loss of energy by friction. The amplitude of such vibration remains unchanged. Such vibration is called free vibration or un-damped vibration. From the definition of simple harmonic motion, for displacement y we have;

$$F = -ky \ldots\ldots\ldots (1)$$

And also from Newton's second law of motion;

$$F = ma = m\frac{d^2y}{dt^2} \ldots\ldots\ldots (2)$$

From equation (1) and (2)

$$m\frac{d^2y}{dt^2} = -ky$$

$$or,\ m\frac{d^2y}{dt^2} + ky = 0$$

$$or, \qquad \frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

$$or, \qquad \frac{d^2y}{dt^2} + w^2y = 0$$

$$where, w^2 = \frac{k}{m}\ and\ w = \sqrt{\frac{k}{m}}$$

$$\therefore 2\pi f = \sqrt{\frac{k}{m}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

And solution is; $y = A \sin wt$

Damped Oscillation:-

It is such type of oscillation in which energy is loss by external resistance or from internal friction within a system. Particularly all oscillations are damped. The energy of such oscillation decreases in time, this causes the amplitude also decreases in time.

For a pendulum there is frictional force of air. Hence is dissipated in each vibration. The dissipative force is proportional to velocity of the particle at that instant and velocity decreases exponentially in the time.

$$\text{i.e. dissipative force (F)} \propto \text{v}$$

$$or, F = -bv \quad where, b\ is\ damping\ constant$$

$$or, \qquad F = -b\frac{dx}{dt}$$

The differential equation of motion in this case will be;

$$F = -kx - b\frac{dx}{dt}$$

$$or, \qquad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$or, \qquad \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \ldots\ldots\ldots (1)$$

Which is required differential equation for damped harmonic oscillation. The solution of above equation is;

$$x = ae^{-\left(\frac{b}{2m}\right)t}\sin(\omega^I t)$$

$$or, \qquad A = A_0 e^{-\left(\frac{b}{2m}\right)t}$$

$$or, \qquad A = A_0 e^{-\left(\frac{t}{2\tau}\right)} \dots\dots\dots(2) \ \ where, \frac{1}{\tau} = \frac{b}{m}$$

The damped angular velocity is;

$$\omega^I = \sqrt{\frac{K}{m} - \frac{b^2}{4m^2}}$$

And frequency is;

$$f^I = \frac{1}{2\pi}\sqrt{\frac{K}{m} - \frac{b^2}{4m^2}}$$

Which is the less than the natural frequency f, and damped energy is;

$$E_{avg} = \frac{1}{2}m\omega^2 A^2 e^{-\left(\frac{bt}{m}\right)}$$

$$i.e. E_{avg} = E_0 e^{-t/\tau}$$

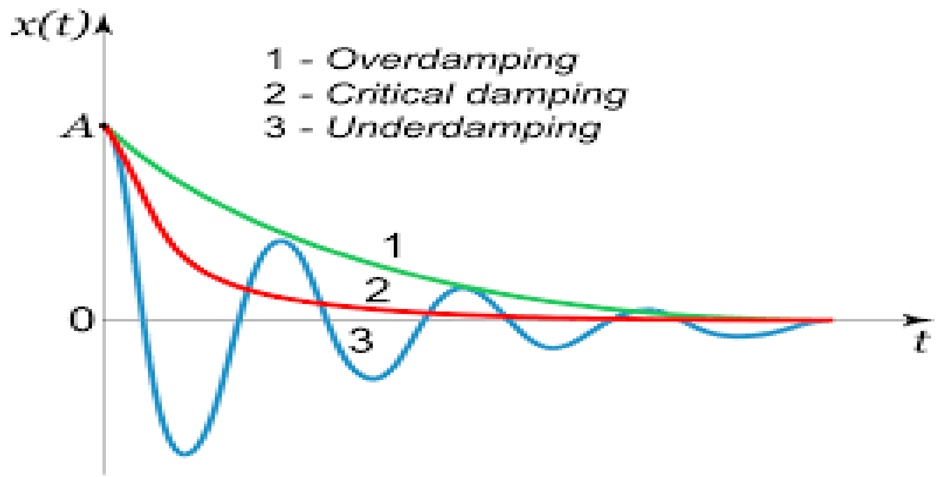


Figure:- Damped oscillation

We have,

$$\omega^I = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$$

1) When $\omega_0 > \frac{b}{2m}$, ω^I is positive, the oscillation is under damped.

2) When $\omega_0 = \frac{b}{2m}$, ω^I is 0, the oscillation is critically damped.

3) When $\omega_0 < \frac{b}{2m}$, ω^I is negative, the oscillation is over damped.

Forced oscillation:-

The phenomenon of setting a body into continuous vibration with the help of strong period force having a frequency different from the natural frequency of the bodies is called forced oscillation.

Let the applied force is sinusoidal and represented as;

$$F_{ext}^{I} = F_0 \sin \omega t$$

Where w is angular frequency applied externally.

Now, resulting force becomes;

$$F = -kx - b\frac{dx}{dt} + F_0 \sin \omega t$$

$$or, \qquad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \sin \omega t$$

$$or, \qquad \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\sin \omega t \ldots\ldots\ldots (1)$$

And solution of above equation is;

$$x = A_0 \sin(\omega t + \emptyset_0)$$

$$\text{Where, } A_0 = \frac{F_0}{m\sqrt{(\omega^2 - \omega_0^2)^2 + \frac{\omega^2 b^2}{m^2}}}$$

$$\text{And, } \emptyset_0 = tan^{-1}[\frac{(\omega^2 - \omega_0^2)}{\omega(\frac{b}{m})}]$$

When natural frequency $\omega_0$is equal to applied frequency ωthen the condition is called resonance condition.

Quality factor (Q-factor):-

Quality factor of a damped oscillation is defined as the quality of oscillator. Less is the damping, higher is the quality factor.

$$\mathrm{Q-factor} = \frac{\text{energy stored}}{\text{energy loss per cycle}}$$

$$= 2\pi \frac{E_{avg}}{P \times T} \ldots\ldots\ldots (1)$$

Rate of loss of energy is power dissipation.

$$i.e. P = \frac{-dE_{avg}}{dt}$$

We know that energy in damped harmonic oscillator is;

$$E_{avg} = \frac{1}{2} m\omega^2 x_m^2 e^{-\left(\frac{bt}{m}\right)}$$

$$as, \frac{1}{\tau} = \frac{b}{m} = 2\delta$$

2δ is the logarithmic decrement and τ is relaxation time.

$$E_{avg} = \frac{1}{2} m\omega^2 x_m^2 e^{-2\delta t}$$

$$E_{avg} = E_0 e^{-2\delta t}$$

$$\therefore \; P = \frac{-dE_{avg}}{dt} = \frac{-dE_0 e^{-2\delta t}}{dt}$$

$$P = 2\delta E_0 e^{-2\delta t}$$

$$\therefore P = 2\delta E_{avg} \dots \dots \dots (2)$$

From equation (1) and (2);

$$Q = \frac{2\pi}{T} . (\frac{1}{2\delta})$$

$$Q = \frac{\omega}{2\delta}$$

$$\therefore Q = \omega\tau$$

Numerical Examples:-

1. **An oscillatory motion of a body is represented by $\boldsymbol{y = ae^{i\omega t}}$ where $\boldsymbol{y}$ is displacement in time $\boldsymbol{t}$, $\boldsymbol{a}$ is amplitude and $\boldsymbol{\omega}$ is angular frequency. Show that the motion is simple harmonic.**

Solution:-

We have $y = ae^{i\omega t}$

Now differentiating with respect to time,

$$\frac{dy}{dt} = \frac{d\left(ae^{i\omega t}\right)}{dt} = i\omega ae^{i\omega t}$$

Again differentiating with respect to time,

$$\frac{d^2y}{dt^2} = i^2\omega^2 ae^{i\omega t}$$

$$or, \qquad \frac{d^2y}{dt^2} = -\omega^2 y$$

$\therefore \frac{d^2y}{dt^2} + \omega^2 y = 0$ represent the motion is simple harmonics.

2. A linear spring whose force constant is $0.2\ N/m$ hangs vertically supporting a $1\ kg$ mass at rest. The mass is pulled down a distance $0.2\ m$ and then released. What will be its maximum velocity and also find the frequency of vibration?

Solution:-

$$\text{Force constant } (k) = 0.2\ N/m$$

$$\text{Mass } (m) = 1\ kg$$

$$\text{Amplitude } (A) = 0.2\ m$$

$$\text{Maximum velocity } (v_{max}) = ?$$

$$\text{Frequency } (f) = ?$$

We know that;

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.2}{1}} = 0.447\ rev/sec$$

$$Now, \qquad (v_{max}) = \omega A = 0.089\ m/sec$$

$$and, \qquad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 0.071\ Hz$$

3. A small body of mass $0.1\ kg$ under going SHM of amplitude $0.1\ m$ and time period 2 sec. (i) What is the maximum force on the body? (ii) If the oscillations are produced on the spring, what should be the force constant?

Solution:-

$$\text{Mass } (m) = 0.1\ kg$$

$$\text{Amplitude } (A) = 0.1\ m$$

$$\text{Time period } (T) = 2\ sec$$

$$\text{We have time period } (T) = 2\pi\sqrt{\frac{m}{k}}$$

$$or, \qquad 2 = 2\pi\sqrt{\frac{0.1}{k}}$$

$$\therefore\ force\ constant\ (k) = 0.987\ N/m$$

$$\text{Now, Maximum force } (F_{max}) = kA = 0.987 \times 0.1$$

$$\therefore\ F_{max} = 0.0987\ N$$

4. **An oscillating block spring system has mechanical energy of $1.18\,J$ and amplitude of $9.84\,cm$ and maximum speed of $1.22\,m/s$, find (i) The force constant of spring (ii) The mass of the block (iii) The frequency of oscillation.**

Solution:-

$$\text{Total energy } (E) = 1.18\,J$$

$$\text{Amplitude } (A) = 9.84\,cm = 0.0984\,m$$

$$\text{Maximum speed } (v_{max}) = 1.22\,m/s$$

$$(i)\ we\ have, E = \frac{1}{2}kA^2$$

$$or, \qquad 1.18 = \frac{1}{2} \times k \times (0.0984)^2$$

$$\therefore force\ constant(k) = 243.72\,N/m$$

$$(ii) Now,\ v_{max} = \omega A$$

$$or, \qquad 1.22 = \omega \times 0.0984$$

$$\therefore \omega = 12.39\,rad/sec$$

$$\text{Also we have, } \omega^2 = \frac{k}{m}$$

$$or, \qquad m = \frac{k}{w^2} = \frac{243.72}{153.71}$$

$$\therefore mass\ of\ the\ block\ (m) = 1.58\,kg$$

$$(iii) Also,\ \omega = 2\pi f$$

$$or, \qquad f = \frac{w}{2\pi} = \frac{12.39}{2\pi}$$

$$\therefore frequency\ of\ oscillation\ (f) = 1.97\ Hz$$

5. **The balance wheel of watch oscillates with angular amplitude of $\boldsymbol{\pi\ rad}$ and the period of $\mathbf{0.5}\ \boldsymbol{sec}$. Find (i) maximum angular speed of wheel, (ii) The angular speed of wheel, when the displacement is $\boldsymbol{\frac{\pi}{2}\ rad}$. And (iii) Magnitude of angular acceleration of wheel, when its displacement is $\boldsymbol{\frac{\pi}{4}\ rad}$.**

Solution:-

$$\text{Amplitude } (A) = \theta_{max} = \pi\ rad$$

$$\text{Time period } (T) = 0.5\ sec$$

$$we\ have, \qquad w = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{0.5}$$

$$\therefore w = 12.56\ rad/sec$$

$$(i) maximum\ angular\ speed\ (v_{max}) = w\theta_{max}$$
$$= 12.56 \times \pi$$

$$\therefore\ v_{max} = 39.438\ m/sec$$

$$(ii)\ angular\ speed\ (v) = w\sqrt{{\theta_{max}}^2 - \theta^2}$$

$$= 12.56\ \sqrt{\pi^2 - (\frac{\pi}{2})^2}$$

$$= 34.154\ rad/sec$$

$$(iii)\ angular\ acceleration\ (\alpha) = \ \omega^2\theta = (12.56)^2 \times \frac{\pi}{4}$$

$$\therefore\ \alpha = 123.84\ rad/sec^2$$

6. A mass of 2 kg is suspended from a spring of spring constant 18 N/m. If the un-damped frequency is $2/\sqrt{3}$ times the damped frequency, what will be its damping factor?

Solution:-

Mass (m) = 2 kg

Spring constant (k) = 18 N/m

Damping factor (b) = ?

Given that; undamped frequency (f_0)

$$= \frac{2}{\sqrt{3}} \text{damped frequency} (f^{I})$$

$$or, \qquad \frac{f_0}{f^{I}} = \frac{2}{\sqrt{3}}$$

or, $$\frac{\frac{1}{2\pi}\sqrt{\frac{k}{m}}}{\frac{1}{2\pi}\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}}=\frac{2}{\sqrt{3}}$$

$or,$ $$\frac{\frac{k}{m}}{\frac{k}{m}-\frac{b^2}{4m^2}}=\frac{4}{3}$$

$or,$ $$3\frac{k}{m}=4\frac{k}{m}-\frac{b^2}{m^2}$$

$or,$ $$\frac{b^2}{m^2}=\frac{k}{m}$$

$or,$ $$b^2=km=18\times 2=36$$

$\therefore b=6\ N.s/m$

7. **If the relaxation time of a damped harmonic oscillator is 50 sec. Find the time in which the amplitude falls to $\frac{1}{e^3}$ times the initial value and energy of the system falls to $\frac{1}{e^4}$ of its initial value.**

Solution:-

Given that; $\mathrm{A}=\frac{1}{e^3}\mathrm{A}_0$ and $\mathrm{E}=\frac{1}{e^4}\mathrm{E}_0$

Relaxation time $(\tau)=50$ sec

Now we know that;

Damped amplitude (A) $= A_0e^{-\left(\frac{t}{2\tau}\right)}$

$$or, \quad \frac{1}{e^3}\mathrm{A}_0 = A_0 e^{-\left(\frac{t}{2\tau}\right)}$$

$$or, \quad e^{-3} = e^{-\left(\frac{t}{2\tau}\right)}$$

$$or, \quad \frac{t}{2\tau} = 3$$

$$\therefore t = 6\tau = 6 \times 50 = 300\, sec.$$

And damped energy (E) = $E_0 e^{-\left(\frac{t}{\tau}\right)}$

$$or, \quad \frac{1}{e^4}\mathrm{E}_0 = E_0 e^{-\left(\frac{t}{\tau}\right)}$$

$$or, \quad e^{-4} = e^{-\left(\frac{t}{\tau}\right)}$$

$$or, \quad \frac{t}{\tau} = 4$$

$$\therefore t = 4\tau = 4 \times 50 = 200\, sec.$$

Exercise:-

1. What is simple harmonic motion? Show that total energy of spring mass system which is oscillating in SHM is conserved.
2. What do you mean by centre of suspension and oscillation of a compound pendulum? Derive the time period of compound pendulum and also show that point of suspension and point of oscillation are interchangeable.
3. Deduce the time period of a simple harmonic vibration. Explain why a loaded bus is more comfortable than an empty bus.

4. Explain forced oscillation with its differential equation. Write the relation for the frequency dependent amplitude and hence give a rough sketch of the resonance curve.
5. Define compound pendulum. Show that the motion of torsional pendulum follows angular SHM. Use it to find modulus of rigidity of a given wire.
6. Write the difference between mechanical oscillation and e.m. oscillation. Set up the differential equation of damped harmonic mechanical oscillation. Obtain the relation for frequency of such oscillation. Hence explain the condition for different types of damped oscillation.
7. Differentiate between bar pendulum and torsional pendulum. Prove that there exists four collinear points in bar pendulum.
8. In simple harmonic motion, when the displacement is one – half the amplitude, what fraction of the total energy is KE and what fraction is PE? At what displacement is the energy half KE and half PE?
9. What do you mean by torsional pendulum? Using a torsional pendulum, derive a relation for modulus of rigidity of the metallic wire.
10. Show that in a bar pendulum, minimum time period is achieved if radius of gyration is equal to the distance of point of suspension or point of oscillation from centre of gravity.

11. Derive a relation to find the moment of inertia of a rigid body about an axis passing through its centre of gravity using the torsional pendulum.
12. Distinguish between free and force vibrations. Write the differential equation of forced oscillation. Determine the amplitude of oscillation for forced oscillation and hence explain sharpness of the resonance.
13. Obtain an expression for the time period of a compound pendulum and show that its time period is unaffected by the fixing of a small additional mass to it at its centre of suspension.
14. Differentiate between linear and angular harmonic motion. Show that the motion of torsional pendulum is angular harmonic motion. Also find its time period.
15. Explain the theory of a simple mass spring system. Developed the relation for the time period and frequency of a two springs having spring constants K_1 and K2 supporting a mass 'M' between them on a frictionless horizontal table.
16. Derive a relation to determine the radius of gyration of a compound pendulum. Why determination of acceleration due to gravity is more accurate from a compound pendulum then a simple pendulum?
17. Define the quality factor. Derive a relation of quality factor from the damped harmonic motion and show that the quality factor is inversely proportional to damping constant.

18. What are drawbacks of simple pendulum? Show that the period of torsional oscillations remains unaffected even if the amplitude be large, provided that the elastic limits of the suspension wire is not exceeded.

19. A meter stick suspended from one end swings as a physical pendulum (a) what is the period of oscillation. (b) What would be the length of the simple pendulum that would have a same time period.

20. A uniform circular disc of radius R oscillates in a vertical plane about a horizontal axis. Show that disc will oscillate with the minimum time period when the distance of the axis of rotation from the centre is $\frac{R}{\sqrt{2}}$.

21. A simple pendulum of length 40 cm and mass 50 gm is suspended in a car that is travelling with a constant speed 40 m/sec around a circle of radius 100 m. If the pendulum goes small oscillation in a radial direction about its equilibrium position, what will be its frequency of oscillation?

22. The amplitude of lightly damped oscillator decreases by 3 % during each cycle. What fraction of the energy of the oscillator is lost in each full oscillation?

23. In damped harmonic motion, calculate the time in which (i) its amplitude and (ii) its energy falls to 1/e of its un-damped value if the mass of the system is 0.25 gm and damping constant is 0.01g/s?

24. A particle is moving with simple harmonic motion in a straight line. If it has a speed v_1 when the displacement is x_1 and speed v_2 when the displacement is x_2 then show that the amplitude of motion is, $a = [\frac{v_2^2 x_1^2 - v_1^2 x_2^2}{v_2^2 - v_1^2}]^{1/2}$.

25. A 750 gm block oscillates on the end of a spring whose force constant is 56 N/m. The mass moves in a fluid which offers a resistive force $F = -bv$, where $b = 0.162\ \mathrm{Ns/m}$. What is the period of the oscillation?