Chapter - 10

THE UNIFORM PLANE WAVE AND EQUATIONS

10.1 INTRODUCTION

A plane wave is a wave whose phase is constant over a set of planes. A uniform plane wave is a wave whose magnitude and phase are both constant. A nonuniform plane wave is a wave whose amplitude (not phase) may vary within a plane normal to the direction of propagation. Consequently, the electric and magnetic fields are no longer in time phase.

Electromagnetic waves in free space are typical uniform plane waves. The electric and magnetic fields are mutually perpendicular to each other and to the direction of propagation of the waves. The phases of the two fields are always in time phase and their magnitudes are always constant. The stored energies are equally divided between the two fields, and the energy flow is transmitted by the two fields in the direction of propagation. Thus, a uniform plane wave is a transverse electromagnetic wave or a TEM wave. A uniform plane wave cannot exist physically because it stretches to infinity and would represent an infinite energy. However, such waves are characteristically simple but fundamentally important as they serve as approximations to practical waves.

A wave of energy having a frequency within the electromagnetic spectrum and propagated as a periodic disturbance of the electromagnetic field when an electric charge oscillates or accelerates is called an electromagnetic wave. Typical examples of EM waves include radio waves, TV signals, radar beams, and light rays.

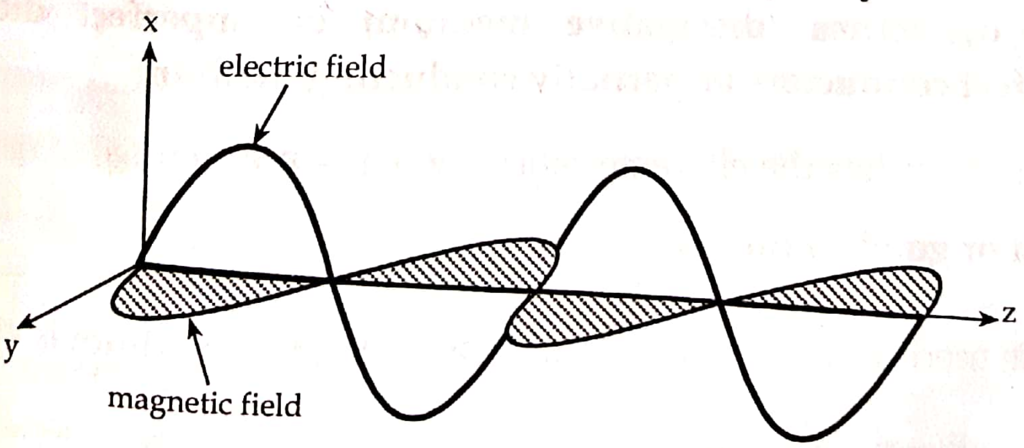


***Figure 10.1** An electromagnetic wave*

The following table illustrates uses of different EM waves:

Table 10.1 EM phenomena and examples of uses

(Source: Elements of electromagnetics by M.N.O. Sadiku, 4th ed.)

EM Phenomena	Examples of Uses
Cosmic rays	Physics, astronomy
Gamma rays	Cancer therapy
X-rays	X-ray examination
Ultraviolet radiation	Sterilization
Visible light	Human vision
Infrared radiation	Photography
Microwave waves	Rader, satellite communication
Radio waves	UHF television, VHF television, FM radio, Short-wave radio, AM radio

10.2 CHARACTERISTICS OF DIFFERENT MEDIA

1. **Free space**

   Free space is characterized by $\sigma = 0$, $\varepsilon = \varepsilon_o$, $\mu = \mu_o$.

2. **Lossless or perfect dielectrics or good dielectrics**

   In this case, $\sigma << \omega\varepsilon$ so that $\frac{\sigma}{\omega\varepsilon} << 1$. Therefore, this medium characterized by $\sigma = 0$, $\varepsilon = \varepsilon_o\varepsilon_r$, $\mu = \mu_o\mu_r$.

3. **Lossy dielectrics (dissipative medium) or imperfect dielectrics imperfect conductors or partially conducting medium**

   This medium has the characteristics $\sigma \neq 0$, $\varepsilon = \varepsilon_o\varepsilon_r$, $\mu = \mu_o\mu_r$.

4. **Perfect or good conductors**

   For this medium, $\sigma >> \omega\varepsilon$ so that $\frac{\sigma}{\omega\varepsilon} >> 1$. So, this medium is characteri[zed] by $\sigma \simeq \infty$, $\varepsilon = \varepsilon_o$, $\mu = \mu_o\mu_r$.

10.3 PROPAGATION CONSTANT

Propagation constant is a complex constant and is expressed as

$$\gamma = \alpha + j\beta$$

The real part of the propagation constant (α) is defined as the attenuation constant while the imaginary part (β) is defined as the phase constant.

(i) Attenuation constant (α)

Attenuation constant or attenuation factor of the medium is a measure of the spatial rate of the decay of the wave as the wave propagates in the medium, measured in nepers per meter (Np/m) or in decibels per meter (dB/m).

1 Np = 8.686 dB

(ii) Phase constant (β)

Phase constant or wave number is a measure of the phase shift per length of the wave as the wave propagates in the medium, measured in radian per meter (rad/m).

For a medium with permittivity ε, permeability μ, and conductivity σ, the expression for propagation constant is

$$\boxed{\gamma = \sqrt{j\omega\mu\ (\sigma + j\omega\varepsilon)}}$$ This is a general expression

or, $\gamma^2 = j\omega\mu\ (\sigma + j\omega\varepsilon)$

or, $(\alpha + j\beta)^2 = j\omega\mu\sigma - \omega^2\mu\varepsilon$

or, $(\alpha^2 + 2j\alpha\beta - \beta^2) = -\omega^2\mu\varepsilon + j\omega\mu\sigma$

or, $(\alpha^2 - \beta^2) + 2j\alpha\beta = -\omega^2\mu\varepsilon + j\omega\mu\sigma$

Equating real and imaginary parts, we have

$\alpha^2 - \beta^2 = -\omega^2\ \mu\varepsilon$ ----(i)

$2\alpha\beta = \omega\mu\sigma$ -----(ii)

Using $\gamma^2 = j\omega\mu\ (\sigma + j\omega\varepsilon)$

or, $|\gamma^2| = \omega\mu\sqrt{\sigma^2 + \omega^2\varepsilon^2}$

Also, $\gamma = \alpha + j\beta$

or, $|\gamma| = \sqrt{\alpha^2 + \beta^2}$

or, $|\gamma|^2 = |\gamma^2| = \alpha^2 + \beta^2$

Therefore, $\alpha^2+\beta^2 = \omega\mu\sqrt{\sigma^2+\omega^2\varepsilon^2}$ ....(iii)

From equation (i) & (iii),

$$(\alpha^2-\beta^2) + (\alpha^2+\beta^2) = -\omega^2\mu\varepsilon + \omega\mu\sqrt{\sigma^2+\omega^2\varepsilon^2}$$

$$\text{or, } 2\alpha^2 = -\omega^2\mu\varepsilon + \omega\mu\sqrt{\sigma^2+\omega^2\varepsilon^2}$$

$$\text{or, } 2\alpha^2 = -\omega^2\mu\varepsilon + \omega\mu\,\omega\varepsilon\sqrt{\frac{\sigma^2}{\omega^2\varepsilon^2}+1}$$

$$\text{or, } \alpha^2 = -\omega^2\frac{\mu\varepsilon}{2} + \omega^2\frac{\mu\varepsilon}{2}\sqrt{\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2+1}$$

$$\text{or, } \alpha^2 = \omega^2\left[\left\{\frac{\mu\varepsilon}{2}\sqrt{\left(\frac{\sigma}{\omega\varepsilon}\right)^2+1}\right\}-1\right]$$

$$\therefore \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} \text{ .... (iv)}$$

Similar simplification will yield

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]} \text{ .... (v)}$$

Equations (iv) and (v) characterizes any medium and are general expressions.

A **dissipative medium** ($\sigma \neq 0$, $\varepsilon = \varepsilon_r\varepsilon_o$, $\mu = \mu_r\mu_o$) can be considered as a gener one whose α and β are defined by expressions (iv) and (v) respectively.

For perfect conductors,

$$\varepsilon = \varepsilon_o,\ \mu = \mu_o\,\mu_r,\ \sigma >> \omega\varepsilon \left(\frac{\sigma}{\omega\varepsilon}\to\infty\right)$$

$$\text{So, } \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} \simeq \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]}$$

$$= \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\frac{\sigma}{\omega\varepsilon}-1\right]}$$

$$\simeq \omega\sqrt{\frac{\mu\varepsilon}{2}\,\frac{\sigma}{\omega\varepsilon}} = \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\text{Similarly, } \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$$

For **perfect dielectrics,** we put $\sigma = 0$ in (iv) & (v) to obtain

$$\alpha = 0,\ \beta = \omega\sqrt{\mu\varepsilon}$$

For free space, $\sigma = 0, \mu = \mu_0, \varepsilon = \varepsilon_0$

$$\therefore \alpha = 0, \beta = \omega\sqrt{\mu_0 \varepsilon_0}$$

10.4 WAVE IMPEDANCE AND INTRINSIC IMPEDANCE

The wave impedance of an electromagnetic wave is the ratio of the transverse components of the electric and magnetic fields, and is designated 'Z'. Intrinsic impedance is the impedance of the medium that the wave propagates in. It is a characteristic of the medium, and is designated 'η'. For a transverse electromagnetic plane wave traveling through a homogeneous medium, the wave impedance is everywhere equal to the intrinsic impedance of the medium.

For a medium with permittivity ε, permeability μ, and conductivity σ, the expression for intrinsic impedance is

$$\boxed{\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}}$$ This is a general expression

Hence, for a **dissipative medium**

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

As $\eta = |\eta| \angle\theta$, $|\eta|$ and θ can be calculated as follows:

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon\left(\frac{\sigma}{j\omega\varepsilon} + 1\right)}} = \sqrt{\frac{\mu/\varepsilon}{1 - j\frac{\sigma}{\omega\varepsilon}}} = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}}$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{\sqrt{(1)^2 + \left(\frac{-\sigma}{\omega\varepsilon}\right)^2}}} = \frac{\sqrt{\mu/\varepsilon}}{\sqrt[4]{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2}}$$

$$\tan 2\theta = \frac{\sigma}{\omega\varepsilon}, \ 0^\circ \angle \theta \angle 45^\circ$$

For **perfect conductors**, $\sigma >> \omega\varepsilon$ $(\omega\varepsilon << \sigma)$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon\left(\frac{\sigma}{j\omega\varepsilon} + 1\right)}} = \sqrt{\frac{\mu/\varepsilon}{1 - j\frac{\sigma}{\omega\varepsilon}}} = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}}$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{\sqrt{(1)^2 + \left(\frac{-\sigma}{\omega\varepsilon}\right)^2}}} = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}}} \simeq \frac{\sqrt{\mu/\varepsilon}}{\sqrt{\sqrt{\left(\frac{\sigma}{\omega\varepsilon}\right)^2}}} \quad (\because \sigma >> \omega\varepsilon)$$

$$= \sqrt{\frac{\omega\mu}{\sigma}}$$

$$\tan 2\theta = \frac{\sigma}{\omega\varepsilon}$$

For $\omega\varepsilon << \sigma$, $\tan 2\theta \rightarrow \infty$ and thus, $\theta \rightarrow 45^o$

For **perfect dielectrics**, we put $\sigma = 0$ in the general expression to get

$$\eta = \sqrt{\mu/\varepsilon}$$

$$|\eta| = \sqrt{\mu/\varepsilon}, \theta = 0^o$$

For **free space**, $\sigma = 0$, $\mu = \mu_o$, $\varepsilon = \varepsilon_o$

$$\eta = \sqrt{\mu_o/\varepsilon_o}$$

$$|\eta| = \sqrt{\mu_o/\varepsilon_o} = \sqrt{\frac{4\pi\times10^{-7}}{8.854\times10^{-12}}} \approx 120\,\pi\,\Omega$$

10.5 PHASOR

The phasor in the EM field is analogous to the logarithm (log) in the number field. As the log simplifies the calculation for numbers, the phasor simplifies the calculation involved in the analysis of EM waves. In general, a phasor could be a scalar or vector.

If the expression for a time-varying x-component of the electric field is

$$E_x = E_{xo} \cos(\omega t + \theta)$$

then, the phasor of E_x is written as E_{xs} and it's expression is

$$E_{xs} = E_{xo} e^{j\theta}$$

To recover the E_x from E_{xs} (like taking antilog in the number field), multiply E_{xs} by $e^{j\omega t}$ and take the real part. That is,

$$E_x = \text{Re}\,[(E_{xs})\, e^{j\omega t}]$$

$$= \text{Re}\,[(E_{xo}\, e^{j\theta})\, e^{j\omega t}]$$

$$= \text{Re}\,[E_{xo}\, e^{j(\omega t + \theta)}]$$

$$= \text{Re}\,[E_{xo}\,\{\cos(\omega t+\theta) + j\sin(\omega t + \theta)\}]$$

$\therefore \quad E_x = E_{xo} \cos(\omega t + \theta)$

The phasor also replaces the cumbersome differentiation with a simple multiplication. It is due to the fact that the multiplication of a phasor quantity by 'jω' is equivalent to the differentiation of that quantity in the time domain.

For $E_x = E_{xo} \cos(\omega t + \theta)$,

$$\frac{\partial E_x}{\partial t} = -\omega E_{xo} \sin(\omega t + \theta)$$

Also,

$$\begin{aligned} \mathrm{Re}\left[(j\omega E_{xs}) e^{j\omega t}\right] &= \mathrm{Re}\left[j\omega E_{xo} e^{j\theta} e^{j\omega t}\right] \\ &= \mathrm{Re}\left[j\omega E_{xo} e^{j(\omega t + \theta)}\right] \\ &= \mathrm{Re}\left[j\omega E_{xo} \{\cos(\omega t + \theta) + j \sin(\omega t + \theta)\}\right] \\ &= \mathrm{Re}\left[j\omega E_{xo} \cos(\omega t + \theta) - \omega E_{xo} \sin(\omega t + \theta)\right] \\ &= -\omega E_{xo} \sin(\omega t + \theta) \end{aligned}$$

$$\therefore \frac{\partial \vec{E}_x}{\partial t} \rightarrow j\omega \vec{E}_{xs}$$

Similarly, $\int \vec{A}\, \partial t \rightarrow \dfrac{\vec{A}_s}{j\omega}$

If $\vec{E} = 200 \sin 10^9 t \sin 20z\, \hat{a}_x$ V/m, find $\vec{E}_S$

Solution:

$$\begin{aligned} \vec{E} &= 200 \sin 10^9 t \sin 20z\, \hat{a}_x \\ &= 200 \cos(90° - 10^9 t) \sin 20z\, \hat{a}_x \\ &= 200 \cos[-(90° - 10^9 t)] \sin 20z\, \hat{a}_x \\ &= 200 \cos(10^9 t - 90°) \sin 20z\, \hat{a}_x \end{aligned}$$

In phasor form,

$$\begin{aligned} \vec{E}_S &= 200\, e^{-j90°} \sin 20z\, \hat{a}_x \\ &= 200 (\cos 90° - j \sin 90°) \sin 20z\, \hat{a}_x \\ &= -j\, 200 \sin 20z\, \hat{a}_x \text{ V/m} \end{aligned}$$

10.6 WAVE EQUATIONS

The wave equation is the fundamental mathematical expression which governs the phenomena involved in the electromagnetic waves, the backbone of all the communications. This includes expression for both electric and magnetic fields.

1. Wave Equation in Perfect Dielectric or Lossless Dielectric

$(\sigma = 0, \mu=\mu_0\mu_r, \varepsilon = \varepsilon_0 \varepsilon_r)$

Expression for electric field:

Consider an identity,

$$\nabla \times (\nabla \times \vec{E}_S) = \nabla (\nabla.\vec{E}_S) - \nabla^2 \vec{E}_S \text{ .... (i)}$$

In most of the cases such as in free space, the medium is free of any charge (ρ_v = 0), thus

$$\nabla.\vec{E}_S = 0$$

Therefore, equation (i) is reduced to

$$\nabla \times (\nabla \times \vec{E}_S) = - \nabla^2 \vec{E}_S \text{ .... (ii)}$$

Maxwell's equation in phasor is

$$\nabla \times \vec{E}_S = - j\omega\mu \vec{H}_S$$

Substituting this value in equation (ii), we get

$$\nabla \times (- j\omega\mu \vec{H}_S) = - \nabla^2\vec{E}_S$$

$$\text{or,} \quad - j\omega\mu (\nabla \times \vec{H}_S) = - \nabla^2\vec{E}_S \text{ .... (iii)}$$

Maxwell's equation in phasor gives

$$\nabla \times \vec{H}_S = \vec{J}_s + \vec{J}_{ds}$$

where $\vec{J}_s$ =conduction current density, $\vec{J}_{ds}$= displacement current density

For perfectly dielectric medium, conduction current density is zero and only the displacement current density has to be considered.

$$\therefore \quad \nabla \times \vec{H}_S = 0 + j\omega \varepsilon \vec{E}_S = j\omega \varepsilon \vec{E}_S$$

Now, equation (iii) becomes

$$- j\omega\mu (j\omega \varepsilon \vec{E}_S) = - \nabla^2 \vec{E}_S$$

$$\text{or,} \quad \omega^2\mu\varepsilon \vec{E}_S = - \nabla^2\vec{E}_S$$

or, $\nabla^2\vec{E_s} = -\omega^2\mu\varepsilon\,\vec{E_s}$ which is known as the vector Helmholtz equation.

This is a second order differential equation with three components of the electric field intensity. Thus, an attempt to solve this equation for all the components at once might lead to confusion rather than making the basics clear. Hence, we take only the x-component of the electric field intensity.

$$\nabla^2\,\vec{E}_{xs} = -\omega^2\mu\varepsilon\,\vec{E}_{xs}$$

or, $\nabla^2 E_{xs} = -\omega^2\mu\varepsilon\,E_{xs}$

or, $\dfrac{\partial^2 E_{xs}}{\partial x^2} + \dfrac{\partial^2 E_{xs}}{\partial y^2} + \dfrac{\partial^2 E_{xs}}{\partial z^2} = -\omega^2\mu\varepsilon\,E_{xs}$

Let E_{xs} varies along the z-direction only.

$$0 + 0 + \frac{\partial^2 E_{xs}}{\partial z^2} = -\omega^2\mu\varepsilon\,E_{xs}$$

or, $\dfrac{\partial^2 E_{xs}}{\partial z^2} = -\omega^2\mu\varepsilon\,E_{xs}$

The solution of the above equation is

$$E_{xs} = A\,e^{-j\beta z},\ \beta = \omega\sqrt{\mu\varepsilon}$$

In time domain,

$$E_x = \text{Re}\,[(E_{xs})\,e^{j\omega t}]$$

$$= \text{Re}\,[A\,e^{-j\beta z}\,e^{j\omega t}]$$

$$= \text{Re}\,[A\,e^{j(\omega t - \beta z)}]$$

$$= \text{Re}\,[A\,\{\cos(\omega t - \beta z) + j\sin(\omega t - \beta z)\}]$$

$$= A\cos(\omega t - \beta z)$$

The constant 'A' may be replaced with the maximum value of the electric field intensity, E_{xo} that is obtained with $t = 0$ and $z = 0$.

Hence, $E_x = E_{xo}\cos(\omega t - \beta z)$

In vector form,

$$\boxed{\vec{E}_x = E_{xo}\cos(\omega t - \beta z)\,\hat{a}_x}$$

Expression for magnetic field:

Since we are dealing with the transverse electromagnetic (TEM) wave, the possible magnetic component associated with $\vec{E}_x$ is $\vec{H}_y$ for the wave propagating in the z-direction.

Consider Maxwell's equation,

$$\nabla \times \vec{E}_S = -j\omega\mu\vec{H}_S$$

Taking only the y-component of the magnetic field and hence, the x-component of the electric field, we have

$$\nabla \times \vec{E}_{xs} = -j\omega\mu\vec{H}_{ys}$$

$$\text{or,} \quad \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) \times \vec{E}_{xs} = -j\omega\mu\vec{H}_{ys}$$

Let $\vec{E}_{xs}$ varies along the z-direction only.

$$\text{or,} \quad \frac{\partial E_{xs}}{\partial z} = -j\omega\mu\, H_{ys}$$

$$\text{or,} \quad \frac{\partial(E_{xo}\, e^{-j\beta z})}{\partial z} = -j\omega\mu\, H_{ys}$$

$$\text{or,} \quad -j\beta\, E_{xo}\, e^{-j\beta z} = -j\omega\mu\, H_{ys}$$

$$\text{or,} \quad H_{ys} = \frac{\beta}{\omega\mu} E_{xo}\, e^{-j\beta z}$$

$$\text{or,} \quad H_{ys} = \frac{\omega\sqrt{\mu\varepsilon}}{\omega\mu} E_{xo}\, e^{-j\beta z}$$

$$\text{or,} \quad H_{ys} = \sqrt{\frac{\varepsilon}{\mu}}\, E_{xo}\, e^{-j\beta z}$$

In time domain,

$$H_y = \text{Re}\left[(H_{ys})\, e^{j\omega t}\right]$$

$$= \text{Re}\left[\sqrt{\frac{\varepsilon}{\mu}}\, E_{xo}\, e^{-j\beta z}\, e^{j\omega t}\right]$$

$$= \text{Re}\left[\sqrt{\frac{\varepsilon}{\mu}}\, E_{xo}\, e^{j(\omega t - \beta z)}\right]$$

$$\text{or, } H_y = \sqrt{\frac{\varepsilon}{\mu}} E_{xo} \cos(\omega t - \beta z)$$

In vector form,

$$\boxed{\vec{H}_y = \sqrt{\frac{\varepsilon}{\mu}} E_{xo} \cos(\omega t - \beta z)\, \hat{a}_y}$$

Perfect dielectric : $\vec{J}_s = 0,\ \vec{J}_{ds} \neq 0$

Perfect conductor : $\vec{J}_s \to \infty,\ \vec{J}_{ds} = 0$

Lossy medium : $\vec{J}_s \neq 0,\ \vec{J}_{ds} \neq 0$

2. Wave Equation in Free Space

$(\sigma = 0, \mu = \mu_o, \varepsilon = \varepsilon_o)$

For free space, $\sigma = 0$, $\mu = \mu_o$, $\varepsilon = \varepsilon_o$. To find the expression for electric field and magnetic field, we repeat the process as done for perfect dielectric but replacing μ and ε by μ_o and ε_o respectively.

3. Wave Equation in Dissipative Medium (Lossy Dielectric)

$$(\sigma \neq 0, \varepsilon = \varepsilon_r \varepsilon_o, \mu = \mu_r \mu_o)$$

Expression for electric field:

Consider an identity

$$\nabla \times (\nabla \times \vec{E}_s) = \nabla (\nabla . \vec{E}_s) - \nabla^2 \vec{E}_s \ ....(i)$$

If we consider a linear, isotropic, homogeneous, lossy dielectric medium that is charge free (macroscopic $\rho_v = 0$), then $\nabla . \vec{E}_s = 0$.

For $\nabla . \vec{E}_s = 0$, we can write

$$\nabla \times (\nabla \times \vec{E}_s) = -\nabla^2 \vec{E}_s \ .... (ii)$$

Maxwell's equation in phasor is

$$\nabla \times \vec{E}_s = -j\omega\mu \vec{H}_s$$

Substituting this value in equation (ii), we get

$$\nabla \times (-j\omega\mu \vec{H}_s) = -\nabla^2 \vec{E}_s$$

$$\text{or, } -j\omega\mu (\nabla \times \vec{H}_s) = -\nabla^2 \vec{E}_s \ .... (iii)$$

From Maxwell's equation, we have

$$\nabla\times\vec{H_s} = \vec{J_s} + \vec{J_{ds}}$$

In case of dissipative (lossy) medium, both conduction current density and displacement current density have to be considered.

$$\therefore\ \nabla\times\vec{H_s} = \vec{J_s} + \vec{J_{ds}} = \sigma\,\vec{E_s} + j\omega\varepsilon\,\vec{E_s} = (\sigma+j\omega\varepsilon)\vec{E_s}$$

Now, equation (iii) becomes

$$-j\omega\mu\left[(\sigma+j\omega\varepsilon)\vec{E_s}\right] = -\nabla^2\,\vec{E_s}$$

$$\text{or, } \nabla^2\,\vec{E_s} = j\omega\mu\,(\sigma+j\omega\varepsilon)\vec{E_s}$$

$$\text{or, } \nabla^2\,\vec{E_s} - \gamma^2\,\vec{E_s} = 0$$

where $\gamma = \sqrt{j\omega\mu\,(\sigma+j\omega\varepsilon)}$ is propagation constant of the medium.

We take only the x-component of the electric field intensity which will result

$$\nabla^2\,\vec{E_{xs}} - \gamma^2\vec{E_{xs}} = 0$$

$$\text{or, } \nabla^2\,E_{xs} - \gamma^2\,E_{xs} = 0$$

$$\text{or, } \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)E_{xs} - \gamma^2\,E_{xs} = 0$$

Let's assume that E_{xs} varies along the z-direction only.

$$0 + 0 + \frac{\partial^2 E_{xs}}{\partial z^2} - \gamma^2\,E_{xs} = 0$$

$$\text{or, } \frac{\partial^2 E_{xs}}{\partial z^2} - \gamma^2\,E_{xs} = 0$$

The solution of above differential equation is

$$E_{xs} = E_{xo}\,e^{-\gamma z}$$

$$\text{or, } E_{xs} = E_{xo}\;e^{-(\alpha+j\beta)z}$$

or, $E_{xs} = E_{xo}\;e^{-\alpha z}\,e^{-j\beta z}$. This is phasor notation.

In time domain, we have

$$E_x = \mathrm{Re}\left[(E_{xs})\,e^{j\omega t}\right] = \mathrm{Re}\left[(E_{xo}\,e^{-\alpha z}\,e^{-j\beta z})\,e^{j\omega t}\right]$$

$$= E_{xo}\;e^{-\alpha z}\,\mathrm{Re}\left[e^{j(\omega t-\beta z)}\right] = E_{xo}\;e^{-\alpha z}\cos(\omega t-\beta z)$$

The equation is in agreement to the fact that the wave attenuates exponentially as it propagates in the dissipative medium.

In vector form, $\boxed{\vec{E_x} = E_{xo}\ e^{-\alpha z} \cos(\omega t-\beta z)\ \hat{a}_x}$

Expression for magnetic field:

Consider Maxwell's equation in phasor

$$\nabla\times\vec{E_s} = -j\omega\mu\vec{H_s}$$

Taking only the y-component of the magnetic field, and x-component of the electric field, we have

$$\nabla \times \vec{E_{xs}} = -j\omega\mu\vec{H_{ys}}$$

$$\text{or, } \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial y}\hat{a}_z\right) \times \vec{E_{xs}} = -j\omega\mu\vec{H_{ys}}$$

Let's assume that $\vec{E_{xs}}$ varies along the z-direction only.

$$\frac{\partial}{\partial z}\hat{a}_z \times \vec{E_{xs}} = -j\omega\mu\vec{H_{ys}}$$

$$\text{or, } \frac{\partial}{\partial z}\hat{a}_z \times E_{xs}\,\hat{a}_x = -j\omega\mu\, H_{ys}\,\hat{a}_y$$

$$\text{or, } \frac{\partial E_{xs}}{\partial z} = -j\omega\mu\, H_{ys}$$

Here, $E_{xs} = E_{xo}\ e^{-\gamma z}$

$$\frac{\partial(E_{xo}\ e^{-\gamma z})}{\partial z} = -j\omega\mu\, H_{ys}$$

$$\text{or, } -\gamma E_{xo}\ e^{-\gamma z} = -j\omega\mu\, H_{ys}$$

$$\text{or, } \sqrt{j\omega\mu(\sigma+j\omega\varepsilon)}\ E_{xo}\ e^{-\gamma z} = -j\omega\mu\, H_{ys}$$

$$\text{or, } \sqrt{\frac{\sigma+j\omega\varepsilon}{j\omega\mu}}\ E_{xo}\ e^{-\gamma z} = H_{ys}$$

$$\text{or, } H_{ys} = \frac{E_{xo}}{\eta}\ e^{-\gamma z}$$

where $\eta = \sqrt{\dfrac{j\omega\mu}{\sigma+j\omega\varepsilon}}$, $|\eta| = \dfrac{\sqrt{\mu/\varepsilon}}{\sqrt[4]{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2}}$, $\tan 2\theta = \dfrac{\sigma}{\omega\varepsilon}$

Hence, $H_{ys} = \frac{E_{xo}}{|\eta| e^{j\theta}} e^{-\gamma z} = \frac{E_{xo}}{|\eta|} e^{-j\theta} e^{-\alpha z} e^{-j\beta z} = \frac{E_{xo}}{|\eta|} e^{-\alpha z} \; e^{-j(\theta+\beta z)}$

In time-domain, we have

$$H_y = \text{Re}\left[(H_{ys})e^{j\omega t}\right] = \text{Re}\left[\left\{\frac{E_{xo}}{|\eta|} e^{-\alpha z} e^{-j(\theta+\beta z)}\right\} e^{j\omega t}\right]$$

$$= \frac{E_{xo}}{|\eta|} e^{-\alpha z}\, \text{Re}\left[e^{j(\omega t-\beta z-\theta)}\right] = \frac{E_{xo}}{|\eta|} e^{-\alpha z} \cos(\omega t-\beta z-\theta)$$

In vector form, $\boxed{\vec{H}_y = \frac{E_{xo}}{|\eta|} e^{-\alpha z} \cos(\omega t-\beta z-\theta)\, \hat{a}_y}$

4. Wave Equation for Perfect Conductors (Good Conductors)

$(\sigma \simeq \infty, \varepsilon = \varepsilon_o, \mu = \mu_r\mu_o)$

In order to determine wave equation for perfect conductions, we first write the wave equation for a dissipative medium, and then substitute θ, α, β which defines perfect conductors.

For a **dissipative medium,**

$$\vec{E}_x = E_{xo}\; e^{-\alpha z} \cos(\omega t-\beta z)\, \hat{a}_x, \quad \vec{H}_y = \frac{E_{xo}}{|\eta|}\; e^{-\alpha z} \cos(\omega t-\beta z-\theta)\, \hat{a}_y$$

A perfect or good conductor is a special case of dissipative medium for which

$$|\eta| = \sqrt{\frac{\omega\mu}{\sigma}}, \theta = 45^\circ \left(\frac{\pi^c}{4}\right), \text{and } \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$$

Thus, for perfect conductors

$$\boxed{\begin{gathered}\vec{E}_x = E_{xo}\; e^{-\alpha z} \cos(\omega t-\beta z)\, \hat{a}_x \\ \vec{H}_y = \frac{E_{xo}}{|\eta|}\; e^{-\alpha z} \cos\left(\omega t-\beta z - \frac{\pi^c}{4}\right) \hat{a}_y; |\eta| = \sqrt{\frac{\omega\mu}{\sigma}}, \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}\end{gathered}}$$

10.7 POYNTING'S THEOREM AND WAVE POWER

Poynting Vector

The cross product of electric field intensity $(\vec{E})$ and magnetic field intensity $(\vec{H})$ is called Poynting vector and is denoted by $\vec{S}$ or $\wp$ or $\vec{P}$. It represents the instantaneous power density, measured in watts per square meter (W/m²).

$$\vec{S} = \vec{E} \times \vec{H} \quad W/m^2$$

The direction of the vector $\vec{S}$ indicates the direction of the instantaneous power flow at a point. Since $\vec{S}$ is given by the cross product of $\vec{E}$ and $\vec{H}$, the direction of power flow at any point is normal to both the $\vec{E}$ and $\vec{H}$ vectors.

If $\vec{E} = E_x \hat{a}_x$ and $\vec{H} = H_y \hat{a}_y$, then

$$\vec{S} = \vec{E} \times \vec{H}$$

$$= E_x \hat{a}_x \times H_y \hat{a}_y = E_x H_y \hat{a}_z$$

or, $\vec{S} = S_z \hat{a}_z$

The unit vector $\hat{a}_z$ shows the direction of power flow.

Poynting's Theorem

From Maxwell's equation for a conductive medium

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Taking scalar product with $\vec{E}$ on both sides,

$$\vec{E}.\nabla \times \vec{H} = \vec{E}.\vec{J} + \vec{E}.\frac{\partial \vec{D}}{\partial t} \quad \text{.............} \quad \text{(i)}$$

But, we know that

$$\nabla.(\vec{E} \times \vec{H}) = -\vec{E}.\nabla \times \vec{H} + \vec{H}.\nabla \times \vec{E}$$

or, $\vec{E}.\nabla \times \vec{H} = \vec{H}.\nabla \times \vec{E} - \nabla.(\vec{E} \times \vec{H})$

So, equation (i) will now be

$$\vec{H}.\nabla \times \vec{E} - \nabla.(\vec{E} \times \vec{H}) = \vec{J}.\vec{E} + \vec{E}.\frac{\partial \vec{D}}{\partial t}$$

As $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, we have

$$-\vec{H}.\frac{\partial \vec{B}}{\partial t} - \nabla.(\vec{E}\times\vec{H}) = \vec{J}.\vec{E} + \vec{E}.\frac{\partial \vec{D}}{\partial t}$$

$$\text{or, } -\nabla.(\vec{E}\times\vec{H}) = \vec{J}.\vec{E} + \vec{E}.\frac{\partial \vec{D}}{\partial t} + \vec{H}.\frac{\partial \vec{B}}{\partial t}$$

$$\text{or, } -\nabla.(\vec{E}\times\vec{H}) = \vec{J}.\vec{E} + \varepsilon\vec{E}.\frac{\partial \vec{E}}{\partial t} + \mu\vec{H}.\frac{\partial \vec{H}}{\partial t} \quad \text{................ (ii)}$$

$$\frac{\partial\left(\frac{1}{2}\vec{D}.\vec{E}\right)}{\partial t} = \frac{\partial\left(\frac{1}{2}\varepsilon\vec{E}.\vec{E}\right)}{\partial t} = \frac{1}{2}\varepsilon\frac{\partial E^2}{\partial t} = \frac{1}{2}\varepsilon\frac{\partial E^2}{\partial E}\frac{\partial E}{\partial t} = \frac{1}{2}2\varepsilon E\frac{\partial E}{\partial t} = \varepsilon\vec{E}.\frac{\partial \vec{E}}{\partial t}$$

On rearranging, we get

$$\varepsilon\vec{E}.\frac{\partial \vec{E}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}.\vec{E}\right)$$

$$\mu\vec{H}.\frac{\partial \vec{H}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}.\vec{H}\right)$$

Substituting these values in equation (ii),

$$-\nabla.(\vec{E}\times\vec{H}) = \vec{J}.\vec{E} + \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}.\vec{E}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}.\vec{H}\right)$$

Integrating above equation throughout a volume,

$$-\int_{vol}\nabla.(\vec{E}\times\vec{H})\,dv = \int_{vol}\vec{J}.\vec{E}\,dv + \int_{vol}\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}.\vec{E}\right)dv + \int_{vol}\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}.\vec{H}\right)dv$$

Applying divergence theorem which converts the volume integral into an integral over the surface that encloses the volume, the L.H.S. is written as

$$-\int_{vol}\nabla.(\vec{E}\times\vec{H})\,dv = -\oint_{area}(\vec{E}\times\vec{H}).d\vec{S}$$

Therefore,

$$-\oint_{area}(\vec{E}\times\vec{H}).d\vec{S} = \int_{vol}\vec{J}.\vec{E}\,dv + \int_{vol}\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}.\vec{E}\right)dv + \int_{vol}\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}.\vec{H}\right)dv$$

Finally,

$$\boxed{-\oint_{area}(\vec{E}\times\vec{H}).d\vec{S} = \int_{vol}\vec{J}.\vec{E}\,dv + \frac{d}{dt}\int_{vol}\frac{1}{2}\vec{D}.\vec{E}\,dv + \frac{d}{dt}\int_{vol}\frac{1}{2}\vec{B}.\vec{H}\,dv}$$

which is the mathematical expression for Poynting's theorem.

On the R.H.S., the first integral gives the total instantaneous ohmic power dissipated within the volume, the second integral is the measure of the total energy in the electric field, and the third integral gives the total energy stored in the magnetic field. The sum of the expressions on the right must therefore be the total power flowing into this volume and so the total power flowing out of the volume is

$$\oint_{area} (\vec{E} \times \vec{H}) \cdot d\vec{S} \ \ W$$

The cross product $(\vec{E} \times \vec{H})$ in the expression is known as the Poynting vector or power density vector denoted by $\vec{S}$ or $\wp$ or $\vec{P}$.

Calculation of Time Average Power Density

In a **perfect dielectric**, the $\vec{E}$ and $\vec{H}$ field amplitudes are given by

$$E_x = E_{xo} \cos(\omega t - \beta z), \ \ H_y = \frac{E_{xo}}{\eta} \cos(\omega t - \beta z)$$

The power density amplitude is expressed as

$$S_z = E_x H_y = \frac{E_{xo}^2 \cos^2(\omega t - \beta z)}{\eta}$$

The time average power density is denoted as $\langle S_z \rangle$ or $\wp_{ave.}$

$$\langle S_z \rangle = \frac{1}{T} \int_0^T S_z \, dt$$

$$= \frac{1}{T} \int_0^T \frac{E_{xo}^2 \cos^2(\omega t - \beta z)}{\eta} dt$$

$$= \frac{E_{xo}^2}{T\eta} \int_0^T \frac{1 + \cos 2(\omega t - \beta z)}{2} dt = \frac{E_{xo}^2}{2T\eta} \left[\int_0^T 1 \, dt + \int_0^T \cos 2(\omega t - \beta z) \, dt \right]$$

$$\boxed{\therefore \langle S_z \rangle = \frac{E_{xo}^2}{2\eta}}$$

In a **lossy dielectric**, we have

$$E_x = E_{xo} e^{-\alpha z} \cos(\omega t - \beta z), \ H_y = \frac{E_{xo}}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta)$$

$$\therefore \quad S_z = E_x H_y$$

$$= \frac{1}{|\eta|} E_{xo}^2 e^{-2\alpha z} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta)$$

$$= \frac{1}{2|\eta|} E_{xo}^2 e^{-2\alpha z} [\cos(2\omega t - 2\beta z - \theta) + \cos\theta]$$

The time-average power density $\langle S_z \rangle$ is calculated as

$$\langle S_z \rangle = \frac{1}{T}\int_0^T S_z \, dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2}\frac{E_{xo}^2}{|\eta|} e^{-2\alpha z} [\cos(2\omega t - 2\beta z - \theta) + \cos\theta]\, dt$$

$$= \frac{1}{T}\frac{1}{2}\frac{E_{xo}^2}{|\eta|} e^{-2\alpha z}\left[\frac{1}{2\omega}\sin(2\omega t - 2\beta z - \theta) + t\cos\theta\right]_0^T$$

$$= \frac{1}{T}\frac{1}{2}\frac{E_{xo}^2}{|\eta|} e^{-2\alpha z}\left[T\cos\theta + \frac{1}{2\omega}\sin(2\omega T - 2\beta z - \theta) - \frac{1}{2\omega}\sin(-2\beta z - \theta)\right]$$

Substituting $\omega = \frac{2\pi}{T}$ and simplifying, we will obtain

$$\boxed{\langle S_z \rangle = \frac{1}{2}\frac{E_{xo}^2}{|\eta|} e^{-2\alpha z} \cos\theta}$$

Alternatively, we can obtain the above expression using

$$\boxed{\langle \vec{S} \rangle = \frac{1}{2}\mathrm{Re}\,[\vec{E}_s \times \vec{H}_s^*] \quad \mathrm{W/m^2}}$$

which is a general formula for computing the average power density in a propagating wave.

10.8 LOSS TANGENT

The ratio of the magnitude of the conduction current density $\vec{J}$ to that of the displacement current density $\vec{J}_d$ in a lossy medium is

$$\frac{|\vec{J}_s|}{|\vec{J}_{ds}|} = \frac{|\sigma \vec{E}_s|}{|j\omega\varepsilon \vec{E}_s|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta$$

where tanθ is known as the loss tangent and θ is the loss angle of the medium as illustrated in Figure 10.2. Although a line of demarcation between good conductors and lossy dielectrics is not easy to make, tan θ or θ may be used to determine how lossy a medium is. A medium is said to be good (lossless or perfect) dielectric if tan θ is very small ($\sigma << \omega\varepsilon$) or a good conductor if tan θ s very large ($\sigma >> \omega\varepsilon$).

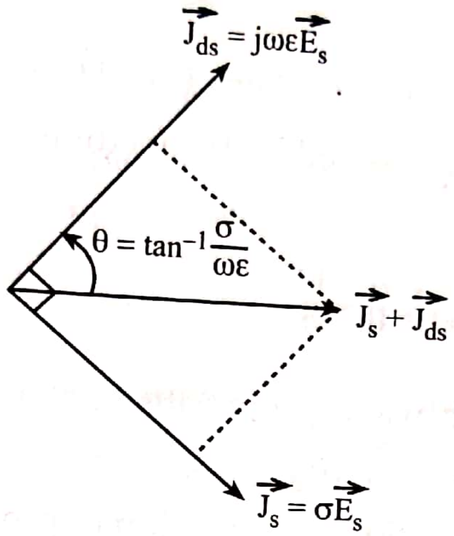


Figure 10.2 Loss angle of a lossy medium.

[Note: A medium that is regarded as a good conductor at low frequencies may be a good dielectric at high frequencies.]

We have,

$$\nabla\times\vec{H}_s = \vec{J}_s + \vec{J}_{ds} = \sigma\vec{E}_s + j\omega\varepsilon\vec{E}_s = (\sigma+j\omega\varepsilon)\vec{E}_s = j\omega\varepsilon\left(\frac{-j\sigma}{\omega\varepsilon}+1\right)\vec{E}_s = j\omega\varepsilon\left[1-\frac{j\sigma}{\omega\varepsilon}\right]\vec{E}_s = j\omega\varepsilon_c\vec{E}_s$$

where $\varepsilon_c = \varepsilon\left[1-j\frac{\sigma}{\omega\varepsilon}\right]$ which is in the form of $\varepsilon_c = \varepsilon' - j\varepsilon''$

$\therefore\ \varepsilon' = \varepsilon,\ \varepsilon'' = \frac{\sigma}{\omega\varepsilon}$; ε_c is called the complex permittivity of the medium. We observe that the ratio of ε'' to ε' is the loss tangent of the medium; that is,

$$\boxed{\tan\theta = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon}}$$

10.9 PROPAGATION IN GOOD CONDUCTORS: SKIN EFFECT

In this section, we will study the behavior of a good conductor when a uniform plane wave is established in it. The electromagnetic fields in a good conductor suffer dissipative loss arising from the large conduction currents it generates.

Skin Depth

The electric field intensity and magnetic field intensity in a good conductor are expressed as $\vec{E}_x = E_{xo}\, e^{-\alpha z} \cos(\omega t - \beta z)\, \hat{a}_x$

$$\vec{H}_y = \frac{E_{xo}}{|\eta|} e^{-\alpha z} \cos\left(\omega t - \beta z - \frac{\pi}{4}\right) \hat{a}_y$$

Both of these equations reveal that when $\vec{E}$ (or $\vec{H}$) wave travels in a conducting medium, its amplitude is attenuated by the factor $e^{-\alpha z}$.

$$\text{For } z = 0,\ e^{-\alpha z} = e^0 = 1$$

$$\text{For } z = \frac{1}{\alpha},\ e^{-\alpha z} = e^{-1} = 0.368$$

This means that when the wave propagates distance $z = \frac{1}{\alpha}$, the amplitude of wave decreases to 0.368 of the initial value. This distance is denoted by δ and is termed the "depth of penetration", or the "skin depth".

As stated earlier, for a good conductor

$$\alpha = \beta = \sqrt{\pi f \mu \sigma}$$

$$\therefore \delta = z = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

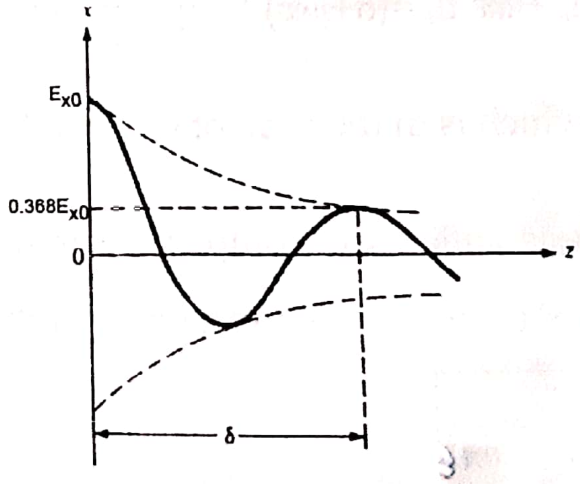


Figure 10.3 Illustration of skin depth.

The skin depth is a measure of the depth to which an EM wave can penetrate the medium. It is an important parameter in describing conductor behaviour in electromagnetic fields. The skin depth decreases with increase in frequency.

If we consider the equations,

$$\vec{E}_x = E_{xo}\, e^{-\alpha z} \cos(\omega t - \beta z)\, \hat{a}_x = E_{xo}\, e^{-z/\delta} \cos(\omega t - \frac{z}{\delta})\, \hat{a}_x$$

$$\vec{H}_y = H_{yo}\, e^{-\alpha z} \cos(\omega t - \beta z - \frac{\pi}{4})\, \hat{a}_y = H_{yo}\, e^{-z/\delta} \cos(\omega t - \frac{z}{\delta} - \frac{\pi}{4})\, \hat{a}_y$$

we notice that electric field and magnetic field can hardly propagate through good conductors.

Skin Effect

Any current density or electric field intensity established at the surface of a good conductor decays rapidly as we progress into the conductor. Electromagnetic energy is not transmitted in the interior of a conductor; it travels in the region surrounding the conductor, while the conductor merely guides the waves.

The phenomenon whereby field intensity in a conductor rapidly decreases is known as "skin effect". The fields and associated currents are confined to a very thin layers (the skin) of the conductor surface. For a wire of radius a, for example, it is a good approximation at high frequencies to assume that all of the current flows in the circular ring of thickness δ as shown in Figure 10.4. Skin effect appears in different guises in such problems as attenuation in waveguides, effective or ac resistance of transmission lines, and electromagnetic shielding.

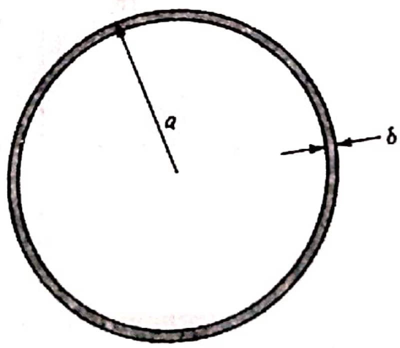


Figure 10.4 Skin depth at high frequencies, δ<<a.

Skin effect is the reason why hollow tubular conductors are used instead of solid conductors in outdoor television antennas. For the same reason, a silver-plated brass component (instead of a pure silver component) is used to reduce the material cost of waveguide components.

Please note that the dc resistance and the ac resistance (due to skin effect) are given respectively by the equations

$$R_{dc} = \frac{l}{\sigma S}$$

$$R_{ac} = \frac{l}{\sigma \delta w}$$

where l = length of the conductor, w = width of the conductor, and $S \simeq \delta w$.

For a conductor of radius a (see Figure 10.4), $w = 2\pi a$.

So, $R_{dc} = \frac{l}{\sigma \pi a^2}$, $R_{ac} = \frac{l}{\sigma \delta 2\pi a}$.

10.10 REFLECTION OF UNIFORM PLANE WAVES AT NORMAL INCIDENCE

Consider a uniform plane wave is incident on the boundary between regions composed of two different materials at normal incidence. Normal incidence simply means that the wave propagation direction is perpendicular to the boundary. This is illustrated in Figure 10.5.

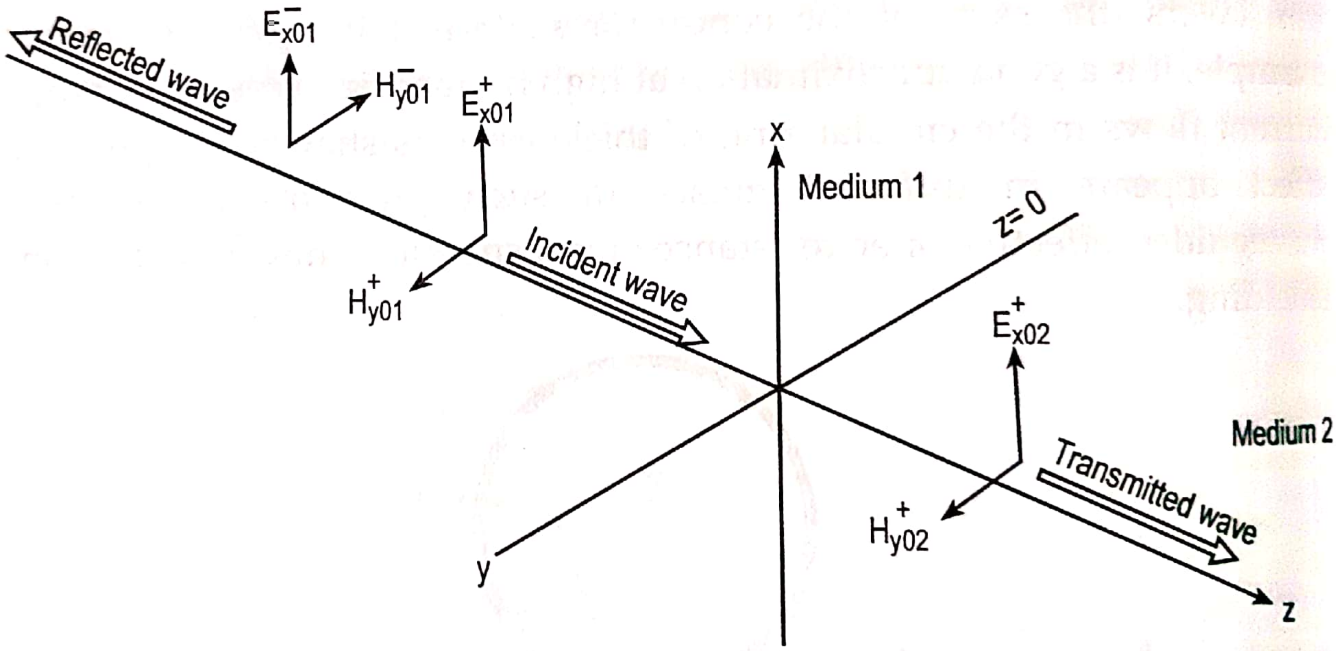


Figure 10.5 The incident, reflected and the transmitted waves. Let the '+' and the '–' subscripts denote the incident or transmitted and the reflected waves respectively. The medium 1 and medium 2 are respectively denoted by the subscripts '1' and '2' respectively.

Reflection Coefficient (Γ)

The wave equation in dissipative medium is given by

$$E_x = E_{xo}\, e^{-\alpha z} \cos(\omega t - \beta z)$$

In phasor,

$$E_{xs} = E_{xo}\, e^{-\gamma z}$$

where γ = propagation constant

The incident wave in medium 1 thus may be written as

$$E^{+}_{xs1} = E^{+}_{xo_1} e^{-\gamma_1 z}$$

The corresponding wave equation for magnetic field is

$$H^{+}_{ys_1} = \frac{E^{+}_{xo_1}}{\eta_1} e^{-\gamma_1 z}$$

In a similar way, the wave equation for reflected wave in medium 1 may be written as

$$E^{-}_{xs_1} = E^{-}_{xo_1} e^{+\gamma_1 z}$$

$$H^{-}_{ys_1} = -\frac{E^{-}_{xo_1}}{\eta_1} e^{+\gamma_1 z}$$

We now, note here that '+' sign on the superscript of e in the above two equations are meant to denote the wave travelling in the –z direction. The '–' sign included in the expression of H_{ys1} comes from the fact that the direction of the magnetic field component is reversed according to the right hand rule because the direction of the propagation is changed. And, as we have assumed, the '+' sign superscripted to H_{ys1} & E_{xs1} is for incident or transmitted wave and '–' sign for reflected waves.

From boundary conditions, we know that the tangential components of the electric field at the boundary z = 0 are equal.

$$E_{xs1} = E_{xs2}$$

$$\text{or, } E^{+}_{xs_1} + E^{-}_{xs1} = E^{+}_{xs_2}$$

Considering the maximum amplitude only,

$$E^{+}_{xo_1} + E^{-}_{xo_1} = E^{+}_{xo_2} \qquad \text{.............. (i)}$$

Similarly, for the magnetic field component,

$$H_{ys1} = H_{ys2}$$

$$\text{or, } H^{+}_{ys_1} + H^{-}_{ys_1} = H^{+}_{ys_2}$$

$$\text{or, } H^{+}_{yo_1} + H^{-}_{yo_1} = H^{+}_{y02}$$

$$\text{or, } \frac{E^{+}_{xo_1}}{\eta_1} - \frac{E^{-}_{xo_1}}{\eta_1} = \frac{E^{+}_{xo_2}}{\eta_2}$$

$$\text{or, } E^{+}_{xo_2} = \frac{\eta_2}{\eta_1} E^{+}_{xo_1} - \frac{\eta_2}{\eta_1} E^{-}_{xo_1} \quad \text{.................. (ii)}$$

From equation (i) and (ii),

$$E^{+}_{xo_1} + E^{-}_{xo_1} = \frac{\eta_2}{\eta_1} E^{+}_{xo_1} - \frac{\eta_2}{\eta_1} E^{-}_{xo_1}$$

$$\text{or, } E^{+}_{xo_1} - \frac{\eta_2}{\eta_1} E^{+}_{xo_1} = -\frac{\eta_2}{\eta_1} E^{-}_{xo_1} - E^{-}_{xo_1}$$

$$\text{or, } \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E^{+}_{xo_1} = E^{-}_{xo_1}$$

$$\text{or, } \frac{E^{-}_{xo_1}}{E^{+}_{xo_1}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad \text{............. (iii)}$$

This ratio of the amplitude of the reflected wave to that of the incident wave is known as the reflection coefficient, Γ.

$$\boxed{\therefore \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}} \; ; \; \Gamma = |\Gamma| e^{j\phi} \text{ and } \phi \text{ is the reflective phase shift}$$

Transmission coefficient (τ)

We have,

$$E^{+}_{xo_1} + E^{-}_{xo_1} = E^{+}_{xo_2}$$

$$\text{or, } E^{-}_{xo_1} = E^{+}_{xo_2} - E^{+}_{xo_1}$$

Substituting it in equation (iii),

$$\frac{E^{+}_{xo_2} - E^{+}_{xo_1}}{E^{+}_{xo_1}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$\text{or, } E^{+}_{xo_2}(\eta_2 + \eta_1) - E^{+}_{xo_1}(\eta_2 + \eta_1) = E^{+}_{xo_1}(\eta_2 - \eta_1)$$

$$\text{or, } E^{+}_{xo_2}(\eta_2 + \eta_1) = E^{+}_{xo_1}(\eta_2 - \eta_1) + E^{+}_{xo_1}(\eta_2 + \eta_1)$$

or, $\overset{+}{E}_{xo_2}(\eta_2 + \eta_1) = \overset{+}{E}_{xo_1}(\eta_2 - \eta_1 + \eta_2 + \eta_1)$

or, $\dfrac{\overset{+}{E}_{xo_2}}{\overset{+}{E}_{xo_1}} = \dfrac{2\eta_2}{\eta_2 + \eta_1}$

This ratio of the amplitude of the transmitted wave to that of the incident wave is referred to as the transmission coefficient, τ.

$$\boxed{\therefore \tau = \frac{2\eta_2}{\eta_2 + \eta_1}}$$

10.11 STANDING WAVE

Consider the case where the wave propagating in a perfect dielectric (medium 1) meets the surface of a perfect conductor (medium 2).

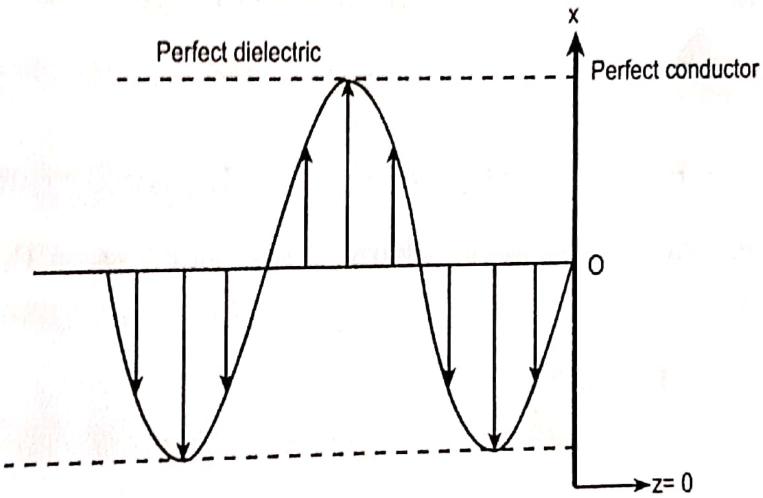


Figure 10.6 A standing wave.

We have,

$$\eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\varepsilon_2}}$$

For perfect conductor, $\sigma_2 \to \infty$

$\therefore \quad \eta_2 = 0$

As $\dfrac{\overset{+}{E}_{xo_2}}{\overset{+}{E}_{xo_1}} = \tau = \dfrac{2\eta_2}{\eta_2 + \eta_1}$, we now can write $\overset{+}{E}_{xo_2} = 0$

This concludes that no time-varying field exists in the perfect conductor.

Also,

$$\frac{E^-_{xo_1}}{E^+_{xo_1}} = \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

And, for $\eta_2 = 0$

$$\frac{E^-_{xo_1}}{E^+_{xo_1}} = \frac{0 - \eta_1}{0 + \eta_1}$$

$$\text{or,}\quad E^-_{xo_1} = -E^+_{xo_1}$$

That is, the reflected wave has the same amplitude as that of the incident wave, but is opposite in sign (direction). Since, the entire incident wave is reflected, the total field in the perfect dielectric (medium 1) is given by

$$E_{xs1} = E^+_{xs_1} + E^-_{xs_1}$$

$$= E^+_{xo_1}\, e^{-\gamma_1 z} + E^-_{xo_1}\, e^{+\gamma_1 z} = E^+_{xo_1}\, e^{-(\alpha_1 + j\beta_1)z} + E^-_{xo_1}\, e^{+(\alpha_1 + j\beta_1)z}$$

In the perfect dielectric, there is no ohmic losses i.e., $\alpha_1 = 0$. Therefore, above equation reduces to

$$E_{xs_1} = E^+_{xo_1}\, e^{-j\beta_1 z} + E^-_{xo_1}\, e^{+j\beta_1 z}$$

Using $E^-_{xo_1} = -E^+_{xo_1}$,

$$E_{xs_1} = E^+_{xo_1}\, e^{-j\beta_1 z} - E^+_{xo_1}\, e^{+j\beta_1 z}$$

$$= E^+_{xo_1}\left(\frac{e^{-j\beta_1 z} - e^{+j\beta_1 z}}{2j}\right) \times 2j \;=\; -E^+_{xo_1}\left(\frac{e^{+j\beta_1 z} - e^{-j\beta_1 z}}{2j}\right) 2j$$

$$\text{or,}\quad E_{xs_1} = -2j\, E^+_{xo_1} \sin\beta_1 z$$

In time domain,

$$E_{x_1} = \mathrm{Re}\,[E_{xs_1}\, e^{j\omega t}]$$

$$= \mathrm{Re}\,[\{-2j\, E^+_{xo_1} \sin\beta_1 z\}\, e^{j\omega t}]$$

$$= -2\, E^+_{xo_1} \sin\beta_1 z\;\; \mathrm{Re}\,[\, j e^{j\omega t}\,]$$

$$= -2\,\overset{+}{E}_{xo_1} \sin\beta_1 z \ \mathrm{Re}\,[\,j\,(\cos\omega t + j\sin\omega t)]$$

$$= -2\,\overset{+}{E}_{xo_1} \sin\beta_1 z \ \mathrm{Re}\,[j\cos\omega t - \sin\omega t]$$

$$= -2\,\overset{+}{E}_{xo_1} \sin\beta_1 z\,(-\sin\omega t)$$

or, $$E_{x1} = 2\,\overset{+}{E}_{xo1} \sin\beta_1 z \ \sin\omega t \qquad \text{...............} \ (iv)$$

The wave represented by equation (iv) is known as the **standing wave.**

The maximum amplitudes that the wave may attain at different z are calculated below.

At $z = -\dfrac{\lambda_1}{8}$; Maximum amplitude $= 2\,\overset{+}{E}_{xo_1} \sin\left[\dfrac{2\pi}{\lambda_1}\left(-\dfrac{\lambda_1}{8}\right)\right]$

$$= -\frac{2}{\sqrt{2}}\,\overset{+}{E}_{xo_1} = -1.44\,\overset{+}{E}_{xo_1}$$

At $z = -\dfrac{2\lambda_1}{8}$; Maximum amplitude $= -2\,\overset{+}{E}_{xo_1}$

At $z = -\dfrac{3\lambda_1}{8}$; Maximum amplitude $= -1.44\,\overset{+}{E}_{xo_1}$

At $z = -\dfrac{4\lambda_1}{8}$; Maximum amplitude $= 0$

At $z = -\dfrac{5\lambda_1}{8}$; Maximum amplitude $= +1.44\,\overset{+}{E}_{xo_1}$

At $z = -\dfrac{6\lambda_1}{8}$; Maximum amplitude $= +2\,\overset{+}{E}_{xo_1}$

At $z = -\dfrac{7\lambda_1}{8}$; Maximum amplitude $= +1.44\,\overset{+}{E}_{xo_1}$

At $z = -\dfrac{8\lambda_1}{8}$; Maximum amplitude $= 0$

From equation (iv), we notice that, for $\beta_1 z = n\pi$ ($n = 0, \pm 1, \pm 2, \ldots\ldots$), $E_{x_1} = 0$ whatever the value of t be

$$\beta_1 z = n\pi \Rightarrow \frac{2\pi}{\lambda_1} z = n\pi \Rightarrow z = \frac{n\lambda_1}{2}$$

Hence, the wave possesses the zero amplitude at the positive and the negative integer multiple of half wavelength all the time. The maximum amplitude the wave may attain is twice the incident amplitude, and occurs midway between the zero amplitude points.

This wave consists of two traveling waves in the opposite directions but the wave as a whole is not traveling. Any point on the wave moves up and down without shifting its position laterally. Therefore, this wave is termed as the standing wave.

The expression for the magnetic field in the standing wave may be derived as follows:

$$H_{ys_1} = \overset{+}{H}_{ys_1} + \overset{-}{H}_{ys_1}$$

$$= \frac{\overset{+}{E}_{xo_1} e^{-\gamma_1 z}}{\eta_1} - \frac{\overset{-}{E}_{xo_1} e^{+\gamma_1 z}}{\eta_1} = \frac{\overset{+}{E}_{xo_1} e^{-(\alpha + j\beta_1)z}}{\eta_1} - \frac{\overset{-}{E}_{xo_1} e^{+(\alpha + j\beta_1)z}}{\eta_1}$$

In case of perfect dielectric, $\alpha = 0$. So,

$$H_{ys_1} = \frac{\overset{+}{E}_{xo_1} e^{-j\beta_1 z}}{\eta_1} - \frac{\overset{-}{E}_{xo_1} e^{+j\beta_1 z}}{\eta_1}$$

$$= \frac{\overset{+}{E}_{xo_1} e^{-j\beta_1 z}}{\eta_1} + \frac{\overset{+}{E}_{xo_1} e^{+j\beta_1 z}}{\eta_1} \qquad (\because \overset{+}{E}_{xo_1} = -\overset{-}{E}_{xo_1})$$

$$= \frac{\overset{+}{E}_{xo_1}}{\eta_1} (e^{-j\beta_1 z} + e^{+j\beta_1 z}) \frac{2}{2} = \frac{2\overset{+}{E}_{xo_1}}{\eta_1} \cos(\beta_1 z)$$

In time domain,

$$H_{y_1} = \text{Re}\,[H_{ys_1} e^{j\omega t}]$$

$$= \text{Re}\left[\left\{\frac{2\overset{+}{E}_{xo_1}}{\eta_1} \cos\beta_1 z\right\} e^{j\omega t}\right] = \frac{2\overset{+}{E}_{xo_1}}{\eta_1} \cos\beta_1 z \;\; \text{Re}\,[e^{j\omega t}]$$

$$\text{or,}\;\; H_{y_1} = \frac{2\overset{+}{E}_{xo_1}}{\eta_1} \cos\beta_1 z \;\; \cos\omega t \qquad \text{............... (v)}$$

From equations (iv) and (v), it is seen that when the electric field attains the maximum amplitude, the magnetic field attains the minimum (zero) and vice versa.

Standing Wave Ratio (SWR)

We have,

$$E_{xs1}^{+} = E_{xo1}^{+}\, e^{-j\beta_1 z}$$

$$E_{xs1}^{-} = E_{xo1}^{-}\, e^{+j\beta_1 z}$$

$$= \Gamma E_{xo_1}^{+}\, e^{+j\beta_1 z} \quad \left(\because \Gamma = \frac{E_{xo_1}^{-}}{E_{xo_1}^{+}}\right) \quad = |\Gamma| e^{j\phi} E_{xo1}^{+}\, e^{+j\beta_1 z}$$

Now, $E_{xs1} = E_{xs1}^{+} + E_{xs1}^{-}$

$$= E_{xo1}^{+}\, e^{-j\beta_1 z} + |\Gamma| e^{j\phi} E_{xo1}^{+}\, e^{+j\beta_1 z} = E_{xo1}^{+}\, (e^{-j\beta_1 z} + |\Gamma|\, e^{j\phi}\, e^{+j\beta_1 z})$$

The E_{xs1} will be maximum when the phase angles of the terms in the larger parenthesis are equal (the phase difference is zero) because they will be directed in the same direction, and their magnitudes are therefore added constructively. Hence,

$E_{xs1,max} = (1 + |\Gamma|)\, E_{xo1}^{+}$ which is true for $-\beta_1 z - (\beta_1 z + \phi) = 0 + 2n\pi$ ($n = 0, \pm1, \pm2, \ldots\ldots$)

or, $-2\beta_1 z = 2n\pi + \phi$

or, $-2 \times \dfrac{2\pi}{\lambda_1} z = 2n\pi + \phi$

or, $z_{max} = \dfrac{-\lambda_1}{2}\left(n + \dfrac{\phi}{2\pi}\right)$

$$\boxed{\text{or, } z_{max} = \frac{-1}{2\beta_1}(\phi + 2n\pi)}$$

Similarly, the E_{xs1} will be minimum when the phase angles of the terms in the larger parenthesis differ by 180°, because they will be directed in the opposite direction, and their magnitudes are therefore added destructively. Hence,

$$E_{xs1,min} = (1 - |\Gamma|)\, E_{xo1}^{+}$$

and this occurs when

$$-\beta_1 z-(\beta_1 z+\phi)=\pi+2n\pi \quad (n=0, \pm 1, \pm 2, \ldots\ldots\ldots)$$

$$\text{or, } -2\beta_1 z=2n\pi+\phi+\pi$$

$$\text{or, } z_{min}=-\frac{\lambda_1}{2}\left(n+\frac{\phi}{2\pi}+\frac{1}{2}\right)$$

$$\boxed{\text{or, } z_{min}=\frac{-1}{2\beta_1}[\phi+(2n+1)\pi]}$$

The ratio of the maximum to the minimum electric field is known as the standing wave ratio, SWR.

$$\boxed{SWR=\frac{E_{xs_1,max}}{E_{xs_1,min}}=\frac{(1+|\Gamma|)\,E^{+}_{x01}}{(1-|\Gamma|)\,E^{+}_{x01}}=\frac{1+|\Gamma|}{1-|\Gamma|}}$$

Since $|\Gamma| \le 1$, we have $1 \le SWR < \infty$. The higher the SWR, the greater the portions of the standing wave in the wave comprising of both the travelling and the standing waves.

10.12 INPUT INTRINSIC IMPEDANCE

We have already studied that, when a plane uniform EM wave propagates as a travelling wave (without reflections) in a medium 1, the ratio of its electric to the magnetic field is the intrinsic impedance, η_1. When the wave propagates in the +z direction, both the electric and the magnetic fields are taken as positive, and hence $+\eta_1$. However, it would be $-\eta_1$ when the wave propagates in the -z direction for the magnetic field reverses its direction of variation while that of the electric field does not change. When the wave propagating in a medium 1 strikes another medium 2, and reflects then, both the incident and the reflected waves exist in the medium 1. If still the ratio of the electric to the magnetic field is sought, then the total electric and the magnetic fields must be considered. This is because the total electric or the magnetic field, for example vary from zero (the minimum) to twice the incident amplitude (the maximum) when the wave propagating in a medium 1 strikes the perfect conductor. Thus, the ratio of the electric to the magnetic field may vary from zero to infinity depending upon the location z it is measured for. This ratio is referred to as the input intrinsic impedance, and is denoted as η_{in}.

Referring Figure 10.5,

$$E_{xs_1} = E^{+}_{xs1} + E^{-}_{xs1}$$

$$= E^{+}_{xo1}\, e^{-j\beta_1 z} + E^{-}_{xo1}\, e^{+j\beta_1 z}$$

$$= E^{+}_{xo_1}\, e^{-j\beta_1 z} + \Gamma E^{+}_{xo_1}\, e^{+j\beta_1 z}$$

$$\text{or, } E_{xs_1} = (e^{-j\beta_1 z} + \Gamma e^{+j\beta_1 z})\, E^{+}_{xo_1} \quad \text{........... (i)}$$

$$\text{Again, } H_{ys_1} = H^{+}_{ys1} + H^{-}_{ys1}$$

$$= \frac{E^{+}_{xs1}}{\eta_1} - \frac{E^{-}_{xs1}}{\eta_1}$$

$$= \frac{E^{+}_{xo1}\, e^{-j\beta_1 z}}{\eta_1} - \frac{E^{-}_{xo1}\, e^{+j\beta_1 z}}{\eta_1}$$

$$= \frac{E^{+}_{xo1}\, e^{-j\beta_1 z}}{\eta_1} - \frac{\Gamma E^{+}_{xo1}\, e^{+j\beta_1 z}}{\eta_1} \quad \left(\because \Gamma = \frac{E^{-}_{xo1}}{E^{+}_{xo1}}\right)$$

$$\text{or, } H_{ys1} = (e^{-j\beta_1 z} - \Gamma e^{+j\beta_1 z})\, \frac{E^{+}_{xo1}}{\eta_1} \quad \text{............. (ii)}$$

The interface between the two media is taken at z equals zero. Thus, the values of z is equation (i) & (ii) are always less than zero because these equations are applied in the medium 1 ($z < 0$) where both the incident and the reflected waves exist. Thus, equations (i) & (ii) are modified using $z = -l$ so that the positive distance l may be used onwards.

$$\text{Thus, } E_{xs_1} = (e^{j\beta_1 l} + \Gamma e^{-j\beta_1 l})\, E^{+}_{xo1}$$

$$H_{ys_1} = (e^{j\beta_1 l} - \Gamma e^{-j\beta_1 l})\, \frac{E^{+}_{xo_1}}{\eta_1}$$

$$\eta_{in} = \frac{E_{xs1}}{H_{ys1}}\Bigg|_{z=-l} = \eta_1 \frac{e^{j\beta_1 l} + \Gamma e^{-j\beta_1 l}}{e^{j\beta_1 l} - \Gamma e^{-j\beta_1 l}} \quad \text{.......... (iii)}$$

Putting $\Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ in equation (iii) , we get

$$\eta_{in} = \eta_1 \frac{e^{j\beta_1 l} + \left(\dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right) e^{-j\beta_1 l}}{e^{j\beta_1 l} - \left(\dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right) e^{-j\beta_1 l}}$$

$$= \eta_1 \frac{(\eta_2 + \eta_1)\, e^{j\beta_1 l} + (\eta_2 - \eta_1) e^{-j\beta_1 l}}{(\eta_2 + \eta_1)\, e^{j\beta_1 l} - (\eta_2 - \eta_1) e^{-j\beta_1 l}}$$

$$= \eta_1 \frac{(\eta_2 + \eta_1)\,(\cos\beta_1 l + j\sin\beta_1 l) + (\eta_2 - \eta_1)\,(\cos\beta_1 l - j\sin\beta_1 l)}{(\eta_2 + \eta_1)\,(\cos\beta_1 l + j\sin\beta_1 l) - (\eta_2 - \eta_1)\,(\cos\beta_1 l - j\sin\beta_1 l)}$$

$$= \eta_1 \left[\frac{(\eta_2\cos\beta_1 l + \eta_1\cos\beta_1 l + j\eta_2\sin\beta_1 l + j\eta_1\sin\beta_1 l) + (\eta_2\cos\beta_1 l - \eta_1\cos\beta_1 l - j\eta_2\sin\beta_1 l + j\eta_1\sin\beta_1 l)}{(\eta_2\cos\beta_1 l + \eta_1\cos\beta_1 l + j\eta_2\sin\beta_1 l + j\eta_1\sin\beta_1 l) - (\eta_2\cos\beta_1 l - \eta_1\cos\beta_1 l - j\eta_2\sin\beta_1 l + j\eta_1\sin\beta_1 l)}\right]$$

$$= \eta_1 \left[\frac{2\eta_2\cos\beta_1 l + 2j\eta_1\sin\beta_1 l}{2\eta_1\cos\beta_1 l + 2j\eta_2\sin\beta_1 l}\right] = \eta_1 \left[\frac{2\cos\beta_1 l\,(\eta_2 + j\eta_1\tan\beta_1 l)}{2\cos\beta_1 l\,(\eta_1 + j\eta_2\tan\beta_1 l)}\right]$$

$$\boxed{\therefore\ \eta_{in} = \eta_1 \frac{(\eta_2 + j\,\eta_1\tan\beta_1 l)}{(\eta_1 + j\eta_2\tan\beta_1 l)}}$$

which is the expression for input intrinsic impedance. It reveals that

(a) When the transmission system is matched ($\eta_2 = \eta_1$), then $\eta_{in} = \eta_1$

(b) When the second medium is a perfect conductor ($\eta_2 = 0$), then $\eta_{in} = j\eta_1 \tan\beta_1 l$. This is to say, $\eta_{in} = 0$ for $\beta_1 l = n\pi$. Hence, $E_{xs1} = 0$ at $l = \dfrac{n\pi}{\beta_1}$, or $z = -l = -\dfrac{n\pi}{\beta_1}$. Likewise, $\eta_{in} = \infty$ for $\beta_1 l = \dfrac{n\pi}{2}$. Hence, $H_{ys1} = 0$ at $l = \dfrac{n\pi}{2\beta_1}$, or $z = -l = \dfrac{-n\pi}{2\beta_1}$.

Importance of Input Intrinsic Impedance

Consider a case in which a radar dish antenna is needed to be covered with a transparent plastic to protect it from weather and this arrangement has to be done so that there is no reflection of the signal back to the antenna. In order to prevent reflection, we should match the transmission system to the outside world, and for this, we must have $\eta_{in} = \eta_2 = \eta_o$. Take $\eta_2 = 377\ \Omega$.

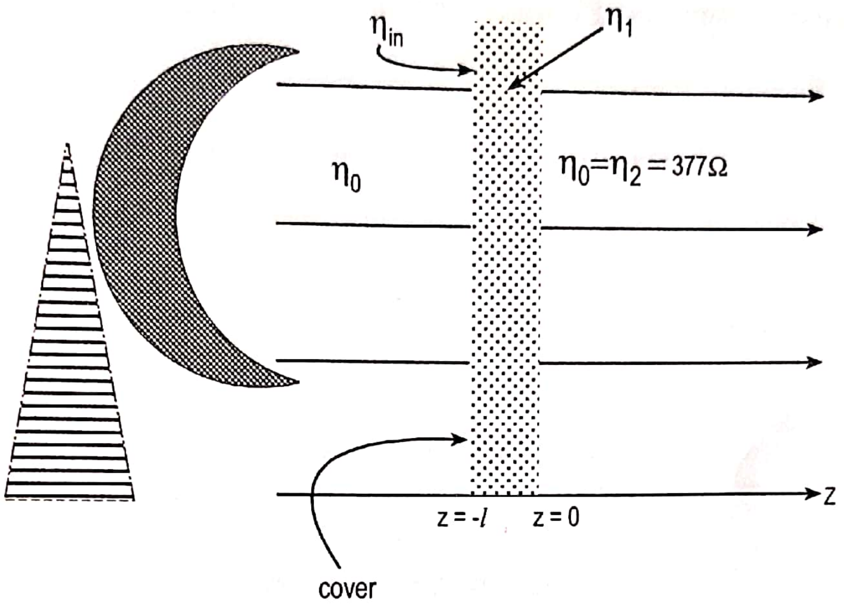
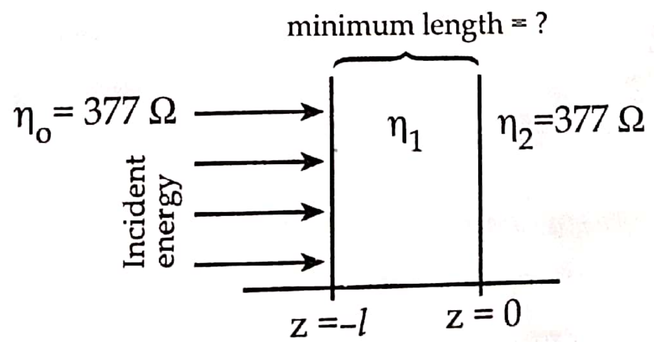


Figure 10.7 Matching the transmission system to the outside world.

Now, let's have a practical insight with a problem solved below.

1. **A radar dish antenna is to be covered with a RADOME (transparent plastic) to protect it from weather without any reflection of the signal to the antenna. Calculate the minimum length of RADOME for this condition. Assume ε_r for RADOME is 2.5.**



Solution:

For no reflection,

$\eta_{in} = \eta_2 = \eta_o = 377\ \Omega$

We have,

$$\eta_{in} = \eta_1 \frac{(\eta_2 + j\eta_1 \tan\beta_1 l)}{(\eta_1 + j\eta_2 \tan\beta_1 l)}$$

$$\text{or, } 377 = \eta_1 \frac{(377 + j\eta_1 \tan\beta_1 l)}{(\eta_1 + j377 \tan\beta_1 l)}$$

$$\text{or, } j(377)^2 \tan\beta_1 l = j\, \eta_1^2 \tan\beta_1 l$$

Since $\eta_1 < 377$, we can satisfy this equation only by selecting $\beta_1 l = n\pi$.

The thinnest radome is obtained with $n = 1$

$$\beta_1 l = n\pi = \pi$$

$$\text{or, } l = \frac{\pi}{\beta_1}$$

$$\text{or, } l = \frac{\pi\lambda_1}{2\pi}$$

$$\text{or, } l = \frac{\lambda_1}{2}$$

$$\text{or, } l = \frac{v_1}{2f_1}$$

$$\text{or, } l = \frac{\frac{1}{\sqrt{\mu\varepsilon}}}{2f_1}$$

$$\text{or, } l = \frac{\frac{1}{\sqrt{\mu_o\mu_r\,\varepsilon_o\varepsilon_r}}}{2f_1}$$

$$\text{or, } l = \frac{\frac{c}{\sqrt{\mu_r\varepsilon_r}}}{2f_1}$$

$$\text{or, } l = \frac{\frac{c}{\sqrt{\varepsilon_r}}}{2f_1} \text{ taking } \mu_r = 1$$

$$\therefore l = \frac{c}{2\sqrt{\varepsilon_r}\, f_1} = 1\text{cm}$$

Hence, the minimum length of radome required is 1 cm.

10.13 REFLECTION OF UNIFORM PLANE WAVES AT OBLIQUE INCIDENCE

Consider a uniform plane wave that is incident obliquely on a plane boundary between two different dielectric media as shown.

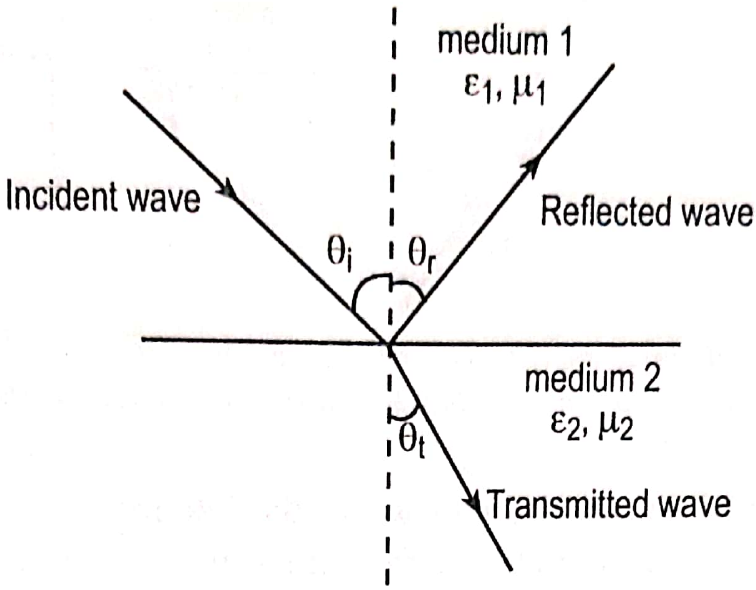


Figure 10.8 A uniform plane wave incident obliquely at the interface between two media.

We have two cases to consider

Case-I: When the electric field vector of the wave is linearly polarized parallel to the interface.

Case-II: When the magnetic field vector of the wave is linearly polarized parallel to the interface.

Case-I:

The electric field vector is parallel to the interface (x = 0 plane). The plane of incidence (i.e., the plane containing the normal to the interface) and the propagation vectors, is assumed to be in the xz-plane, so that the electric field vectors are entirely in the y-direction. The corresponding magnetic field vectors are then as shown in the figure (i) so as to be consistent with the condition that

$\vec{E}, \vec{H}$ & $\vec{\beta}$ form a right-handed mutually orthogonal set of vectors. Since the electric field vectors are perpendicular to the plane of incidence, this case corresponds to perpendicular polarization.

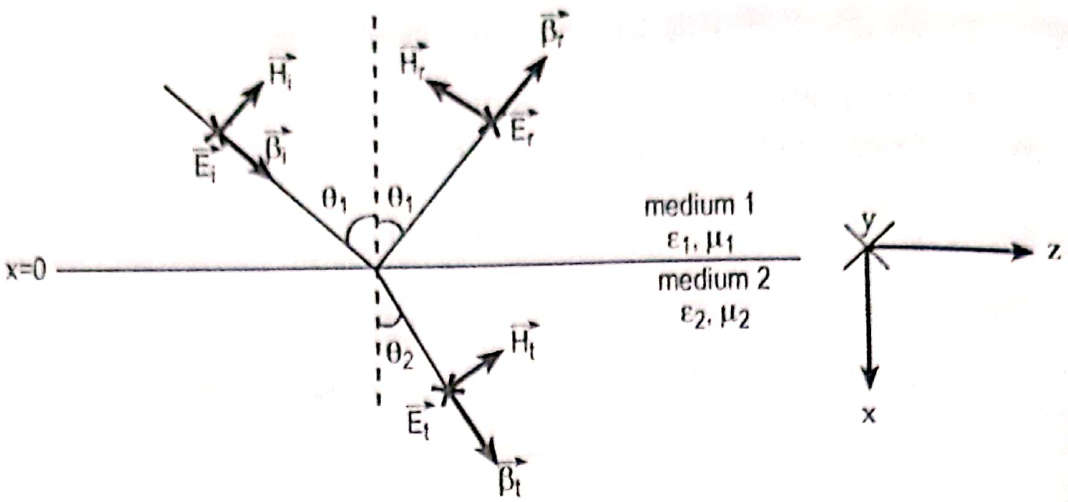


Figure 10.9 For obtaining the reflection and transmission coefficients for an obliquely incident uniform plane wave on a dielectric interface with its electric field perpendicular to the plane of incidence.

The boundary conditions to be satisfied at the interface x = 0 are that (i) the tangential component of the electric field intensity be continuous and (ii) the tangential component of the magnetic field intensity be continuous. Thus, at x = 0, we haveuous and (ii) the tangential component of the magnetic field intensity be continuous. Thus, at x = 0, we have

$E_{yi} + E_{yr} = E_{yt}$;   i → for incident wave

$H_{zi} + H_{zr} = H_{zt}$ r → for reflected wave

t → for transmitted wave

Rewriting these equations in terms of the total fields, we have

$$E_i + E_r = E_t \;.................. \text{(i)}$$

$$H_i \cos\theta_1 - H_r \cos\theta_1 = H_t \cos\theta_2 \;........... \text{(ii)}$$

But, $$\frac{E_i}{H_i} = \frac{E_r}{H_r} = \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} \;........................ \text{(iii)}$$

$$\frac{E_t}{H_t} = \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} \;................................ \text{(iv)}$$

Substituting (iii) & (iv) in (ii), we have

$$E_i - E_r = E_t \frac{\eta_1}{\eta_2} \frac{\cos\theta_2}{\cos\theta_1} \;........................... \text{(v)}$$

Solving (i) & (v), we get

$$E_i = \frac{E_t}{2}\left(1 + \frac{\eta_1 \cos\theta_2}{\eta_2 \cos\theta_1}\right) \quad \text{..................... (vi)}$$

$$E_r = \frac{E_t}{2}\left(1 - \frac{\eta_1 \cos\theta_2}{\eta_2 \cos\theta_1}\right) \quad \text{..................... (vii)}$$

Reflection coefficient, $\Gamma_\perp$ and the transmission coefficient, $\tau_\perp$ are defined as

$$\Gamma_\perp = \frac{E_r}{E_i} = \frac{E_{yr}}{E_{yi}}, \quad \tau_\perp = \frac{E_t}{E_i} = \frac{E_{yt}}{E_{yi}}$$

where the subscript $\perp$ refers to perpendicular polarization.

Finally, rearranging equations (vi) & (vii) will result

$$\Gamma_\perp = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}, \quad \tau_\perp = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

Case-II:

The magnetic field vector is parallel to the interface (x = 0 plane). The geometry pertinent to this case is shown in fig (ii). Here again the plane of incidence is chosen to be the xz-plane, so that the magnetic field vectors are entirely in the y-direction. The corresponding electric field vectors are then as shown in the figure so as to be consistent with the condition that $\vec{E}$, $\vec{H}$ & $\vec{\beta}$ form a right-handed mutually orthogonal set of vectors. Since, the electric field vectors are parallel to the plane of incidence, this case is parallel polarization.

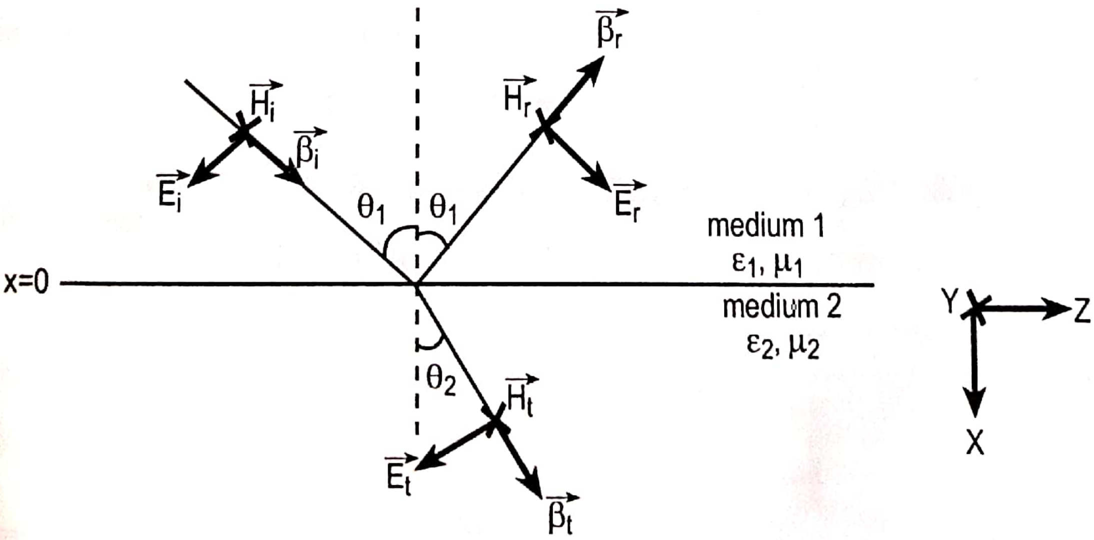


***Figure* 10.10** *For obtaining the reflection and transmission coefficients for an obliquely incident uniform plane wave on a dielectric interface with its electric field parallel to the plane of incidence.*

Applying the boundary conditions at the interface

$$E_{zi} + E_{zr} = E_{zt} \text{ ....} \quad \text{(i)}$$

$$H_{yi} + H_{yr} = H_{yt} \text{ ....} \quad \text{(ii)}$$

Expressing these equations in terms of the total fields and using

$$\frac{E_i}{H_i} = \frac{E_r}{H_r} = \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$\frac{E_t}{H_t} = \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

We will obtain the following two equations

$$E_i - E_r = E_t \frac{\cos\theta_2}{\cos\theta_1} \text{ ....} \quad \text{(iii)}$$

$$E_i + E_r = E_t \frac{\eta_1}{\eta_2} \text{ ....} \quad \text{(iv)}$$

Solving equations (iii) & (iv), we get

$$E_i = \frac{E_t}{2}\left(\frac{\eta_1}{\eta_2} + \frac{\cos\theta_2}{\cos\theta_1}\right) \text{ ....} \quad \text{(v)}$$

$$E_r = \frac{E_t}{2}\left(\frac{\eta_1}{\eta_2} - \frac{\cos\theta_2}{\cos\theta_1}\right) \text{ ....} \quad \text{(vi)}$$

We now define the reflection coefficient, $\Gamma_{||}$ and the transmission coefficient, as

$$\Gamma_{||} = -\frac{E_r}{E_i}, \quad \tau_{||} = \frac{E_t}{E_i}$$

where the subscript | | refers to parallel polarization.

A simple simplification of equations (v) & (vi) will result

$$\boxed{\Gamma_{||} = \frac{\eta_2\cos\theta_2 - \eta_1\cos\theta_1}{\eta_2\cos\theta_2 + \eta_1\cos\theta_1}},$$

$$\boxed{\tau_{||} = \frac{2\eta_2\cos\theta_1}{\eta_2\cos\theta_2 + \eta_1\cos\theta_1}}$$

1. **Find the skin depth δ at a frequency of 1.6 MHz in aluminum, where $\sigma = 38.2$ MS/m and $\mu_r = 1$. Also find γ and the wave velocity.**

Solution:

For perfect or good conductors,

$$\alpha = \beta = \sqrt{\frac{\mu\omega\sigma}{2}} = \sqrt{\pi f\mu\sigma} = \frac{1}{\delta}$$

$$\therefore \delta = \frac{1}{\sqrt{\pi f\mu\sigma}} = \frac{1}{\sqrt{\pi f\mu_o\mu_r\sigma}} = \frac{1}{\sqrt{\pi\times(1.6\times10^{6}\times(1)\times(4\pi\times10^{-7})\times(38.2\times10^{6}))}}$$

$$= 6.4376\times10^{-5}\text{m}$$

$$\alpha = \beta = \frac{1}{\delta} = 15533.57 = 1.553\times10^{4}$$

$$\therefore \gamma = \alpha + j\beta = 1.553\times10^{4} + j1.553\times10^{4}\ \text{m}^{-1}$$

$$\text{Wave velocity (v)} = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = \frac{2\pi\times(1.6\times10^{6})}{1.553\times10^{4}} = 64.7.334\ \text{m/s}$$

2. **Determine the propagation constant γ for a material having $\mu_r = 1$, $\varepsilon_r = 8$, and $\sigma = 0.25$ pS/m, if the wave frequency is 1.6 MHz.**

Solution:

$$\text{Here, } \frac{\sigma}{\omega\varepsilon} = \frac{0.25\times10^{-12}}{2\pi(1.6\times10^{6})\,(8)\,(8.854\times10^{-12})} \approx 10^{-9} \approx 0$$

so that

$$\alpha \approx 0,\ \beta = \omega\sqrt{\mu\varepsilon} = (2\pi\times1.6\times10^{6})\sqrt{(1\times4\pi\times10^{-7})\,(8\times8.854\times10^{-12})}$$

$$= 9.48\times10^{-2}\ \text{rad/m}$$

$\therefore\ \gamma = \alpha + j\beta \approx j9.48\times10^{-2}\ \text{m}^{-1}$. The material behaves like a perfect dielectric at the given frequency.

3. **A uniform plane wave propagating in a medium has $\vec{E} = 2e^{-\alpha z}\sin(10^{8}t - \beta z)\,\hat{a}_y$ V/m. If the medium is characterized by $\varepsilon_r = 1$, $\mu_r = 20$, and $\sigma = 3$ S/m, find α, β.**

Solution:

$$\text{Loss tangent is } \frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon_o\varepsilon_r} = \frac{1}{(10^{8})\times(8.854\times10^{-12})\times(1)} = 3393 >> 1$$

This shows that the medium may be regarded as a good conductor at the frequency of operation.

$$\therefore \alpha = \beta = \sqrt{\frac{\mu\omega\sigma}{2}} = \sqrt{\frac{\mu_0 \mu_r \omega\sigma}{2}} = \sqrt{\frac{(4\pi \times 10^{-7})(20)(10^8)(3)}{2}} = 61.4$$

Hence, $\alpha = 61.4$ Np/m, $\beta = 61.4$ rad/m

4. **Assume that dry soil has conductivity equal to 10^{-4} S/m, $\varepsilon = 3\varepsilon_0$ and $\mu = \mu_0$. Determine the frequency at which the ratio of the magnitudes of the conduction current density and displacement current density is unity.** [2069 Chaitra]

Solution:

$$\frac{\vec{J}_s}{\vec{J}_{ds}} = \frac{\sigma \vec{E}_s}{j\omega\varepsilon \vec{E}_s} = \frac{\sigma}{j\omega\varepsilon} = \frac{\sigma + j0}{0 + j\omega\varepsilon}$$

$$\frac{|\vec{J}_s|}{|\vec{J}_{ds}|} = \frac{\sqrt{(\sigma)^2 + 0^2}}{\sqrt{0^2 + (\omega\varepsilon)^2}} = \frac{\sigma}{\omega\varepsilon}$$

Given that, $\dfrac{|\vec{J}_s|}{|\vec{J}_{ds}|} = 1$

$$\text{or, } 1 = \frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{2\pi f (3\varepsilon_o)} \Rightarrow f = \frac{\sigma}{6\pi\varepsilon_o} = \frac{10^{-4}}{6\pi \times 8.854 \times 10^{-12}} = 0.599 \text{ MHz.}$$

5. **A 9.375 GHz uniform plane wave is propagating in polyethylene ($\varepsilon_r = 2.26, \mu_r = 1$). If the amplitude of the electric field intensity is 500 V/m and the material is assumed to be lossless, find: (a) the phase constant (b) the wavelength in the polyethylene (c) the velocity of propagation (d) the intrinsic impedance (e) the amplitude of the magnetic field intensity.**

Solution:

(a) $\beta = \omega\sqrt{\mu\varepsilon} = 2\pi f\sqrt{\mu_o\mu_r\varepsilon_o\varepsilon_r}$

$$= 2\pi \times 9.375 \times 10^9 \sqrt{4\pi \times 10^{-7} \times 1 \times 8.854 \times 10^{-12} \times 2.26} = 295.312 \text{ rad/m}$$

(b) $\beta = \dfrac{2\pi}{\lambda} \Rightarrow \lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{295.312} = 0.02127\text{m} = 2.127 \text{ cm}$

(c) $v = \lambda \times f = 0.02127 \times 9.375 \times 10^9 = 1.994 \times 10^8$ m/s

(d) $\eta = \sqrt{\dfrac{\mu}{\varepsilon}} = \sqrt{\dfrac{\mu_o\mu_r}{\varepsilon_o\varepsilon_r}} = \sqrt{\dfrac{4\pi \times 10^{-7} \times 1}{8.854 \times 10^{-12} \times 2.26}} = 250.656\ \Omega$

(e) $\eta = \dfrac{E_o}{H_o} \Rightarrow H_o = \dfrac{E_o}{\eta} = \dfrac{500}{250.656} = 1.994$ A/m

6. A uniform plane wave in free space is given by $\vec{E}_s = (200\angle 30^o)\ e^{-j250z}\ \hat{a}_x$ V/m. Find: (a) β (b) ω (c) f (d) λ (e) η (f) $\vec{H}_s$ (g) $|\vec{E}|$ at z = 8 mm, t = 6 ps.

Solution:

(a) The coefficient of z is the value of β

$\therefore \beta = 250$ rad/m

(b) $\beta = \omega\sqrt{\mu_o\varepsilon_o} \Rightarrow \omega = \dfrac{\beta}{\sqrt{\mu_o\varepsilon_o}} = 74.9489\times10^9$ rad/s = 74.9489 G rad/s.

(c) $\omega = 2\pi f \Rightarrow f = \dfrac{\omega}{2\pi} = 11.9285$ GHz.

(d) $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{250} = 0.025\ 132$ m = 25.132 mm

(e) $\eta = \sqrt{\dfrac{\mu_o}{\varepsilon_o}} = 376.7343\ \Omega$ (a constant for free space)

(f) Since -z is involved in the equation for $\vec{E}_s$, the direction of $\vec{P}$ is $\hat{a}_z$.

$$\vec{P} = \vec{E} \times \vec{H}$$

$$\hat{a}_z = \hat{a}_x \times ?$$

As $\hat{a}_x \times \hat{a}_y = \hat{a}_z$, the direction of $\vec{H}$ is $\hat{a}_y$.

$$\eta = \frac{E_o}{H_o} \Rightarrow H_o = \frac{E_o}{\eta} = \frac{200}{376.7343} = 0.531 \text{ A/m.}$$

$$\therefore \vec{H}_s = 0.531\angle 30^o\ e^{-j250z}\ \hat{a}_y \text{ A/m.}$$

(g) $\vec{E}_s = 200\angle 30^o\ e^{-j250z}\ \hat{a}_x = 200\ e^{j30^o}\ e^{-j250z}\ \hat{a}_x = 200\ e^{j(30^o-250z)}\ \hat{a}_x$ (This is phasor form)

In time domain,

$$\vec{E} = \text{Re}\left[(\vec{E}_s)e^{j\omega t}\right] = \text{Re}\left[200\ e^{j(30^o-250z)}e^{j\omega t}\right]\hat{a}_x = 200\ \text{Re}\left[e^{j(\omega t-250+30^o)}\right]\hat{a}_x$$

$$= 200\text{Re}\left[\cos(\omega t-250z+30^o) + j\sin(\omega t-250z+30^o)\right]\hat{a}_x$$

$$= 200\cos(\omega t - 250z+30^o)\ \hat{a}_x \text{ A/m}$$

At t = 6 ps = 6×10^{-12} s, z = 8 mm = 8×10^{-3} m,

$\vec{E} = 200 \cos [74.9483 \times 10^9 \times 6 \times 10^{-12} - 250 \times 8 \times 10^{-3} + 30^o]\ \hat{a}_x$

$= 200 \cos (-1.550^c + 30^o)\ \hat{a}_x$

$= 200 \cos (-88.826^o + 30^o)\ \hat{a}_x$

$= 200 \cos (-58.826^o)\ \hat{a}_x = 103.52768 \hat{a}_x$ V/m

$\therefore |\vec{E}| = 103.52768$ V/m

7. A uniform plane wave in free space is propagating in the $-\hat{a}_y$ direction at a frequency of 10 MHz. If $\vec{E} = 400 \cos \omega t\ \hat{a}_z$ V/m at y = 0, write expressions for: (a) $\vec{E}$ (x, y, z, t) (b) $\vec{E}_s$ (x, y, z) (c) $\vec{H}_s$ (x, y, z) (d) $\vec{H}$ (x, y, z, t)

Solution:

(a) Since the wave is propagating in the $-\hat{a}_y$ direction, +y must be involved in the equation for $\vec{E}$.

$\therefore\ \vec{E} = 400 \cos (\omega t + \beta y)\ \hat{a}_z$

Here, $\omega = 2\pi f = 2\pi \times 10 \times 10^6 =$ rad/s.

$\beta = \omega\sqrt{\mu_o \varepsilon_o} = 62.832 \times 10^6 \sqrt{4\pi \times 10^{-7} \times 8.854 \times 10^{-12}} = 0.21$ rad/m

$\vec{E} = 400 \cos (62.832 \times 10^6\ t + 0.21y)\ \hat{a}_z$ V/m.

(b) The phasor form of $E_x = E_{x0} \cos (\omega t + \theta)$ is written as $E_{xs} = E_{xo}\, e^{j\theta}$

Similarly, the phasor equivalent of $\vec{E} = 400 \cos (62.832 \times 10^6 t + 0.21y)\ \hat{a}_z$ is

$\vec{E}_s = 400\, e^{j0.21y}\ \hat{a}_z$ V/m $= 400\angle 0^o\ e^{j0.21y}\ \hat{a}_z$ V/m

(c) $\eta = 376.7343\ \Omega$ (for free space)

$$\eta = \frac{E_o}{H_o} \Rightarrow H_o = \frac{E_o}{\eta} = \frac{400}{376.7343} = 1.06176 \text{ A/m}$$

The direction of $\vec{P}$ is $-\hat{a}_y$ (given) and that of $\vec{E}$ is $\hat{a}_z$.

$\vec{P} = \vec{E} \times \vec{H}$

$-\hat{a}_y = \hat{a}_z \times ?$

Since $\hat{a}_z \times (-\hat{a}_x) = -\hat{a}_y$, the direction of $\vec{H}$ must be $-\hat{a}_x$.

$\therefore\ \vec{H}_s = 1.06176\, e^{j0.21y} (-\hat{a}_x) = -1.06176\, e^{j0.21y}\ \hat{a}_x$ A/m.

(d) $\vec{H}_s = -1.06176\, e^{j0.21y}\, \hat{a}_x$ A/m.

In time domain,

$$\vec{H} = \mathrm{Re}\left[(\vec{H}_s)e^{j\omega t}\right] = \mathrm{Re}\left[(-1.06176\, e^{j0.21y})e^{j\omega t}\right]\hat{a}_x = -1.06176\, \mathrm{Re}\left[e^{j(\omega t+0.21y)}\right]\hat{a}_x$$

$$\therefore\ \vec{H} = -1.06176 \cos(\omega t + 0.21y)\, \hat{a}_x \text{ A/m}$$

8. **The electric field intensity of a 300- MHz uniform plane wave in free space is given as $\vec{E}_s = (20+j50)\,(\hat{a}_x + 2\hat{a}_y)\, e^{-j\beta z}$ V/m.**

(a) Find $\omega, \lambda, v, \beta$ (b) Find $\vec{E}$ at $t = 1$ ns, $z = 10$ cm (c) What is $|\vec{H}|_{max}$?

Solution:

(a) $\omega = 2\pi f = 2\pi\times300\times10^6 = 1.885\times10^9$ rad/s $= 1.885$ G rad/s.

For free space, $\beta = \omega\sqrt{\mu_o\varepsilon_o} = 6.287468$ rad/m

$$\lambda = \frac{2\pi}{\beta} = 0.999318 \text{ m}$$

For free space, $v = \dfrac{1}{\sqrt{\mu_o\,\varepsilon_o}} = 3\times10^8$ m/s

(b) $\vec{E}_s = (20+j50)\,(\hat{a}_x+2\hat{a}_y)\, e^{-j\beta z}$

$= 53.852\angle 68.2^\circ\,(\hat{a}_x+2\hat{a}_y)\, e^{-j\beta z}$

$= 53.852\, e^{j68.2^\circ}\,(\hat{a}_x+2\hat{a}_y)\, e^{-j\beta z} = 53.852\,(\hat{a}_x+2\hat{a}_y)\, e^{(j68.2^\circ-\beta z)}$ V/m

In time domain,

$$\vec{E} = \mathrm{Re}\left[(\vec{E}_s)e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[\{53.852(\hat{a}_x+2\hat{a}_y)\, e^{(j68.2^\circ-\beta z)}\}e^{j\omega t}\right]$$

$$= 53.852\,(\hat{a}_x+2\hat{a}_y)\,\mathrm{Re}\left[e^{j(\omega t-\beta z+68.2^\circ)}\right]$$

$$= 53.852\,(\hat{a}_x+2\hat{a}_y)\cos(\omega t - \beta z + 68.2^\circ)$$

At $t = 1$ ns $= 1\times10^{-9}$ s, $\quad z = 10$ cm $= 0.1$ m

$$\vec{E} = 53.852\,(\hat{a}_x+2\hat{a}_y)\cos(1.885\times10^9\times10^{-9} - 6.287468\times0.1 + 68.2^\circ)$$

$= 53.852 \ (\hat{a}_x+2\hat{a}_y) \cos (1.25625 + 68.2^o)$

$= 53.852 \ (\hat{a}_x+2\hat{a}_y) \cos (71.978^o + 68.2^o) \ (\therefore 1.25625^C = 71.978^o)$

$= -41.36 \ (\hat{a}_x+2\hat{a}_y)$

$\therefore \ \vec{E} = -41.36 \ \hat{a}_x - 82.721 \ \hat{a}_y \ V/m$

(c) We have, $\eta = \dfrac{|\vec{E}|_{max}}{|\vec{H}|_{max}}$

or, $|\vec{H}|_{max} = \dfrac{|\vec{E}|_{max}}{\eta}$ ......(i)

Here, $\eta = 376.7343 \ \Omega$

$|\vec{E}|_{max}$ can be found from the expression,

$\vec{E} = 53.852 \ (\hat{a}_x+2\hat{a}_y) \cos (\omega t - \beta z + 68.2^o)$

where $|\vec{E}|_{max} = 53.852\sqrt{(1)^2 + (2)^2} = 120.4167 \ V/m$

Now, from equation (i),

$|\vec{H}|_{max} = \dfrac{120.4167}{376.7343} = 0.3196 \ A/m.$

9. **A wave propagating in a lossless dielectric has the components, $\vec{E} = 500 \cos (10^7 t - \beta z) \ \hat{a}_x$ V/m and $\vec{H} = 1.1 \cos (10^7 t - \beta z) \ \hat{a}_y$ A/m. If the wave is travelling at $v = 0.5c$, find: (a) μ_r (b) ε_r (c) β (d) λ (e) η**

Solution:

Amplitude of electric field (E_o) = 500 V/m

Amplitude of magnetic field (H_o) = 1.1 A/m

$v = 0.5 \ c = 0.5 \times 3 \times 10^8 = 1.5 \times 10^8 \ m/s$

(a) $\eta = \dfrac{E_o}{H_o}$

or, $\sqrt{\dfrac{\mu_o \mu_r}{\varepsilon_o \varepsilon_r}} = \dfrac{E_o}{H_o}$

or, $\sqrt{\dfrac{4\pi \times 10^{-7} \ \mu_r}{8.854 \times 10^{-12} \ \varepsilon_r}} = \dfrac{500}{1.1}$

$$\text{or, } \sqrt{\frac{\mu_r}{\varepsilon_r}} = 1.20654$$

$$\text{or, } \frac{\mu_r}{\varepsilon_r} = 1.45574 \text{ .... (i)}$$

$$\text{Also, } v = \sqrt{\frac{1}{\mu_o \mu_r \varepsilon_o \varepsilon_r}}$$

$$\text{or, } 1.5\times10^8 = \sqrt{\frac{1}{4\pi\times10^{-7}\times8.854\times10^{-12}\times\mu_r\varepsilon_r}}$$

$$\text{or, } 1.5\times10^8 = \sqrt{\frac{1}{1.112626\times10^{-17}\mu_r\varepsilon_r}}$$

$$\text{or, } 0.5 = \sqrt{\frac{1}{\mu_r\varepsilon_r}} = \frac{1}{\sqrt{\mu_r\varepsilon_r}}$$

$$\text{or, } \sqrt{\mu_r\varepsilon_r} = \frac{1}{0.5} = 2$$

$$\text{or, } \mu_r\varepsilon_r = 4 \text{ .... (ii)}$$

From equation (i), (ii), we get

$$\frac{\mu_r}{\varepsilon_r} \times \mu_r\varepsilon_r = 1.45574 \times 4$$

$$\text{or, } \mu_r = 2.413$$

(b) $\varepsilon_r = \frac{4}{\mu_r}$ [from equation (ii)]

$$\therefore \varepsilon_r = \frac{4}{2.413} = 1.6576$$

(c) $\beta = \omega\sqrt{\mu\varepsilon}$

$$= \sqrt{\mu_o\mu_r\varepsilon_o\varepsilon_r}$$

$$= 10^7\sqrt{4\pi\times10^{-7}\times2.413\times8.854\times10^{-12}\times1.6576}$$

$$\therefore \beta = 0.0667 \text{ rad/m.}$$

(d) $\lambda = \frac{2\pi}{\beta} = 94.201$ m

(e) $\eta = \frac{E_o}{H_o} = \frac{500}{1.1} = 454.545\ \Omega$.

10. **Let $\vec{E}_s$ = (1000 $\hat{a}_x$ + 400 $\hat{a}_z$) e^{-j10y} V/m for a 250 - MHz uniform plane wave propagating in a perfect dielectric. If the maximum amplitude of $\vec{H}$ is 3 A/m, find β, η, λ, v, ε_r, μ_r, and $\vec{E}$ (x, y, z, t).**

Solution:

β = coefficient of y in the equation = 10 rad/m.

$$\eta = \frac{|\vec{E}|_{max}}{|\vec{H}|_{max}}$$

Here,

$$|\vec{E}|_{max} = \sqrt{(1000)^2 + (400)^2} = 1077.03296 \text{ V/m}$$

$$|\vec{H}|_{max} = 3 \text{ A/m}$$

$$\therefore \eta = \frac{1077.03296}{3} = 359.011\Omega$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{10} = 0.6283 \text{ m}$$

$$v = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = \frac{2\pi \times 250 \times 10^6}{10} = 1.571 \times 10^8 \text{ m/s.}$$

To find ε_r, μ_r, we calculate first

$$\eta = \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_r}}$$

$$\text{or, } 359.011 = \sqrt{\frac{4\pi \times 10^{-7} \mu_r}{8.854 \times 10^{-12} \varepsilon_r}}$$

$$\text{or, } \frac{\mu_r}{\varepsilon_r} = 0.9081 \text{ ...... (i)}$$

Also,

$$v = \sqrt{\frac{1}{\mu_0 \mu_r \varepsilon_0 \varepsilon_r}}$$

$$\text{or, } 1.571 \times 10^8 = \sqrt{\frac{1}{4\pi \times 10^{-7} \times 8.854 \times 10^{-12} \times \mu_r \varepsilon_r}}$$

$$\text{or, } \mu_r \varepsilon_r = 3.64165 \text{ .... (ii)}$$

From equation (i) & (ii),

$$\frac{\mu_r}{\varepsilon_r} \times \mu_r \varepsilon_r = 0.9081 \times 3.64165$$

or, $\mu_r = 1.8185$

then, $\varepsilon_r = \frac{3.64165}{\mu_r}$ [from equation (ii)]

$$= \frac{3.64165}{1.8185} = 2.0025$$

$\vec{E}$ (x, y, z, t) requires the given expression of electric field (which is in phasor form) to be converted into me domain.

Given, $\vec{E}_s = (1000\,\hat{a}_x + 400\,\hat{a}_z)\, e^{-j10y}$ V/m

$$\vec{E}(x, y, z, t) = \text{Re}\left[(\vec{E}_s)e^{-j10y}\right]$$

$$= \text{Re}\left[(1000\,\hat{a}_x + 400\,\hat{a}_z)\, e^{-j10y}\, e^{j\omega t}\right]$$

$$= \text{Re}\left[(1000\,\hat{a}_x + 400\,\hat{a}_z)\, e^{j(\omega t - 10y)}\right]$$

$$= (1000\,\hat{a}_x + 400\,\hat{a}_z) \quad \text{Re}\left[e^{j(\omega t - 10y)}\right]$$

$$\therefore \vec{E}(x, y, z, t) = (1000\,\hat{a}_x + 400\,\hat{a}_z)\cos(\omega t - 10y) \text{ V/m}$$

11. **Uniform plane wave, $\vec{E}_s = (5\hat{a}_x + j10\,\hat{a}_y)\, e^{-j2z}$ V/m, has a frequency of 50 MHz in a lossless dielectric material for which $\mu_r = 1$. Find (a) $\beta, \omega, v, \lambda, \varepsilon_r, \eta$ (b) $\vec{E}$ at the origin for $\omega t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, and π.**

Solution:

(a) β = coefficient of z = 2 rad/m

$\omega = 2\pi f = 2\pi \times 50 \times 10^6 = 3.14159 \times 10^8$ rad/s.

$$v = \frac{\omega}{\beta} = \frac{3.14159 \times 10^8}{2} = 1.57079 \times 10^8 \text{ m/s.}$$

$$\lambda = \frac{2\pi}{\beta} = 3.14159 \text{ m.}$$

Using,

$$\beta = \omega\sqrt{\mu\,\varepsilon}$$

or, $\beta = \omega\sqrt{\mu_0 \mu_r \varepsilon_0 \varepsilon_r}$

or, $2 = 3.14159 \times 10^8 \sqrt{4\pi\times10^{-7}\times2.413\times8.854\times10^{-12}\times\varepsilon_r} \Rightarrow \varepsilon_r = 3.6426$

$$\eta = \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_r}} = \sqrt{\frac{4\pi\times10^{-7}\times1}{8.854\times10^{-12}\times3.6426}} = 197.392\ \Omega$$

(b) $\vec{E}_s = (5\hat{a}_x + j10\,\hat{a}_y)\, e^{-j2z}$

$= 5e^{-j2z}\,\hat{a}_x + j10\, e^{-j2z}\,\hat{a}_y$

$= 5e^{-j2z}\,\hat{a}_x + 10\angle 90^\circ\, e^{-j2z}\,\hat{a}_y$

$= 5e^{-j2z}\,\hat{a}_x + 10\, e^{j90^\circ}\, e^{-j2z}\,\hat{a}_y$

$= 5e^{-j2z}\,\hat{a}_x + 10\, e^{j(30^\circ-2z)}\,\hat{a}_y$

$\vec{E} = \text{Re}\left[(\vec{E}_s)\, e^{j\omega t}\right]$

$= \text{Re}\left[(5e^{-j2z}\,\hat{a}_x)\, e^{j\omega t}\right] + \text{Re}\left[\{10e^{j(90^\circ-2z)}\,\hat{a}_y\}e^{j\omega t}\right]$

$= 5\,\text{Re}\left[e^{j(\omega t-2z)}\right]\,\hat{a}_x + 10\,\text{Re}\left[e^{j(\omega t-2z+90^\circ)}\right]\,\hat{a}_y$

$\vec{E}(x, y, z, t) = 5\cos(\omega t-2z)\,\hat{a}_x + 10\cos(\omega t - 2z + 90^\circ)\,\hat{a}_y$ V/m

Now,

$\vec{E}(0, 0, 0, t) = 5\cos\omega t\,\hat{a}_x + 10\cos(\omega t+90^\circ)\,\hat{a}_y$

$= 5\cos\omega t\,\hat{a}_x - 10\sin\omega t\,\hat{a}_y$ V/m which is the expression for $\vec{E}$ at the origin.

$\vec{E}$ at origin (for $\omega t = 0$) $= 5\cos\omega t\,\hat{a}_x - 10\sin 0\,\hat{a}_y = 5\,\hat{a}_x$ V/m

$\vec{E}$ at origin (for $\omega t = \frac{\pi}{4} = 5\cos\frac{\pi}{4}\,\hat{a}_x - 10\sin\frac{\pi}{4}\,\hat{a}_y = 3.5355\,\hat{a}_x - 7.071\,\hat{a}_y$ V/m.

$\vec{E}$ at origin (for $\omega t = \frac{\pi}{2} = 5\cos\frac{\pi}{2}\,\hat{a}_x - 10\sin\frac{\pi}{2}\,\hat{a}_y = -10\hat{a}_y$ V/m.

$\vec{E}$ at origin (for $\omega t\,\frac{3\pi}{4}) = 5\cos\frac{3\pi}{4}\,\hat{a}_x - 10\sin\frac{3\pi}{4}\,\hat{a}_y$

$= -3.5355\,\hat{a}_x - 7.071\,\hat{a}_y$ V/m.

$\vec{E}$ at origin (for $\omega t = \pi$) $= 5\cos\pi\,\hat{a}_x - 10\sin\pi\,\hat{a}_y = -5\,\hat{a}_x$ V/m.

12. **An EM wave travels in free space with the electric field component $\vec{E} = (10\,\hat{a}_y + 5\,\hat{a}_z)\cos(\omega t + 2y - 4z)$ V/m. Find: (a) ω and λ (b) The magnetic field component (c)The time average power in the wave.**

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Solution:

a. $\vec{E} = (10\,\hat{a}_y + 5\,\hat{a}_z)\cos(\omega t + 2y - 4z)$ .... (i)

Comparing equation (i) with

$\vec{E} = \vec{E_o}\cos(\omega t - \beta_x x - \beta_y y - \beta_z z + \phi)$, we get

$\phi = 0,\ \beta_x = 0,\ \beta_y = -2,\ \beta_z = 4$

$\therefore \vec{\beta} = \beta_x\hat{a}_x + \beta_y\,\hat{a}_y + \beta_z\,\hat{a}_z = -2\hat{a}_y + 4\hat{a}_z$ rad/m

$$\beta = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2} = \sqrt{(0)^2 + (-2)^2 + (4)^2} = 4.472 \text{ rad/m}$$

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\,\varepsilon_o} = \frac{\omega}{c} \Rightarrow \omega = c\beta = 3\times10^8 \times 4.472 = 1.3416 \text{ Grad/s}$$

$$\beta = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{4.472} = 1.405 \text{ m}$$

b. $\vec{H_o} = \frac{1}{\omega\mu}\,\vec{\beta} \times \vec{E_o} = \frac{1}{\omega\mu_0}\,\vec{\beta} \times \vec{E_o}$

$$\vec{\beta} \times \vec{E_o} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & -2 & 4 \\ 0 & 10 & 5 \end{vmatrix} = -50\,\hat{a}_x$$

$$\therefore \vec{H_o} = \frac{1}{1.3416\times10^9\times4\pi\times10^{-7}}(-50\,\hat{a}_x) = -0.02965\,\hat{a}_x \text{ A/m}$$

The magnetic field component is therefore given as

$\vec{H} = \vec{H_o}\cos(\omega t + 2y - 4z) = -0.02965\cos(\omega t + 2y - 4z)\hat{a}_x$ A/m

Time average power in the wave is given as

$$\wp_{ave} = \frac{1}{2}\text{Re}\left[\vec{E_s} \times \vec{H_s}^*\right] = \frac{E_o^2\,\hat{a}_\beta}{2\eta}$$

$$\hat{a}_\beta = \frac{\vec{\beta}}{\beta} = \frac{-2\hat{a}_y + 4\hat{a}_z}{4.472} = -0.447\hat{a}_y + 0.894\hat{a}_z$$

$E_o = \sqrt{(10)^2 + (5)^2} = 11.18$ V/m

$n = 377$

$$\therefore \wp_{ave} = \frac{(11.18)^2(-0.447\hat{a}_y + 0.894\hat{a}_z)}{2 \times 377} = -0.073\hat{a}_y + 0.147\hat{a}_z \text{ W/m}^2$$

13. Consider a 30 MHz uniform plane wave propagating in free space and given by the electric field vector

$\vec{E} = 5\left(\hat{a}_x + \sqrt{3}\,\hat{a}_y\right)\cos\left[6\pi\times10^7 t - 0.05\pi(3x - \sqrt{3}y + 2z)\right]$ **V/m. Find β, $\vec{H}$.**

Solution:

Here, $\vec{E} = 5\left(\hat{a}_x + \sqrt{3}\,\hat{a}_y\right)\cos\left[6\pi\times10^7 t - 0.05\pi(3x - \sqrt{3}y + 2z)\right]$ ..... (i)

We have the expression for the electric field as

$$\vec{E} = \vec{E}_o \cos\left(\omega t - \vec{\beta}\cdot\vec{r} + \phi\right)$$

where $\vec{\beta} = \beta_x\hat{a}_x + \beta_y\hat{a}_y + \beta_z\hat{a}_z$ is the propagation vector, $\vec{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ is the position vector drawn from the origin to the point (x, y, z), and ϕ is the phase at the origin at t= 0.

$\therefore \vec{E} = \vec{E}_o \cos[\omega t - \beta_x x - \beta_y y - \beta_z z + \phi]$ .... (ii)

Comparing equation (i) with (ii), we get

$\beta_x = 0.05\pi\times3 = 0.4712$, $\beta_y = -0.05\pi\times\sqrt{3} = -0.272$, $\beta_z = 0.05\pi\times2 = 0.3141$

$\vec{\beta} = 0.4712\hat{a}_x - 0.272\hat{a}_y + 0.3141\hat{a}_z$; $\beta = \sqrt{(0.4712)^2 + (-0.272)^2 + (0.3141)^2} = 0.6282$ rad/m

The direction of propagation is given by the unit vector

$$\hat{a}_\beta = \frac{\vec{\beta}}{\beta} = \frac{0.4712\hat{a}_x - 0.272\hat{a}_y + 0.3141\hat{a}_z}{0.6282} = 0.75\,\hat{a}_x - 0.4329\,\hat{a}_y + 0.5\,\hat{a}_z$$

Using the relation

$$\vec{H}_o = \frac{1}{\omega\mu}\vec{\beta}\times\vec{E}_o = \frac{1}{\omega\mu_o}\vec{\beta}\times\vec{E}_o$$

$$\vec{\beta}\times\vec{E}_o = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0.4712 & -0.272 & 0.3141 \\ 5 & 5\sqrt{3} & 0 \end{vmatrix}$$

$$= -2.72\,\hat{a}_x + 1.5705\,\hat{a}_y + 5.4407\,\hat{a}_z$$

$$\therefore \vec{H}_o = \frac{1}{6\pi\times10^7\times4\pi\times10^{-7}}(-2.72\,\hat{a}_x + 1.5705\,\hat{a}_y + 5.4407\,\hat{a}_z)$$

$$= -0.01148\,\hat{a}_x + 6.6302\times10^{-3}\,\hat{a}_y + 0.02296\,\hat{a}_z\ \text{A/m}$$

The expression for magnetic field is therefore given as

$$\vec{H} = \vec{H}_o \cos\left[6\pi\times10^7 t - 0.05\pi(3x - \sqrt{3}y + 2z)\right]$$

$$= (-0.01148\hat{a}_x + 6.6302\times10^{-3}\hat{a}_y + 0.02296\hat{a}_z)\cos\left[6\pi\times10^7 t - 0.05\pi(3x - \sqrt{3}y + 2z)\right]\ \text{A/m}$$

14. An EM wave travels in free space with the electric field component $\vec{E}_s = 100\, e^{j(0.866y+0.5z)}\,\hat{a}_x$ V/m. Determine: (a) ω and λ (b) The magnetic field component (c) The time average power in the wave.

Solution: Here,

$$\vec{E}_s = 100\, e^{j(0.866y+0.5z)}\,\hat{a}_x \ \ldots\ldots\text{(i)}$$

a. Comparing equation (i) with

$$\vec{E}_s = \vec{E}_0\, e^{j\vec{\beta}\cdot\vec{r}} = E_0\, e^{j\vec{\beta}\cdot\vec{r}}\,\hat{a}_x = E_0\, e^{j(\beta_x x + \beta_y y + \beta_z z)}\,\hat{a}_x\text{, we get}$$

$$E_0 = 100,\ \beta_x = 0,\ \beta_y = 0.866,\ \beta_z = 0.5$$

$$\therefore \beta = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2} = \sqrt{(0)^2 + (0.866)^2 + (0.5)^2} = 1$$

$$\beta = \omega\sqrt{\mu_o\,\varepsilon_o} \Rightarrow \omega = \frac{\beta}{\sqrt{\mu_o\,\varepsilon_o}} = \beta c = 1\times3\times10^8 = 3\times10^8\ \text{rad/s}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{1} = 6.283\ \text{m}$$

b. $$\vec{H}_s = \frac{1}{\omega\mu}\vec{\beta}\times\vec{E}_s = \frac{1}{\omega\mu_o}\vec{\beta}\times\vec{E}_s$$

$$\vec{\beta}\times\vec{E}_s = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0.866 & 0.5 \\ 100\, e^{j(0.866y+0.5z)} & 0 & 0 \end{vmatrix} = 50\, e^{j(0.866y+0.5z)}\,\hat{a}_y - 86.6\, e^{j(0.866y+0.5z)}\,\hat{a}_z$$

$$\vec{H}_s = \frac{1}{3\times10^8\times4\pi\times10^{-7}}\left[e^{j(0.866y+0.5z)}(50\,\hat{a}_y - 86.6\,\hat{a}_z)\right]$$

$$= (0.1326\,\hat{a}_y - 0.2297\,\hat{a}_z)\, e^{j(0.866y+0.5z)}\ \text{A/m}$$

c. The time average power is

$$\mathcal{P}_{ave} = \frac{1}{2}\text{Re}\left[\vec{E}_s \times \vec{H}_s^*\right] = \frac{E_o^2\, \hat{a}_\beta}{2\eta}$$

$$E_0 = 100,\ \eta = 377,\ \hat{a}_\beta = \frac{\vec{\beta}}{\beta} = \frac{0.866\,\hat{a}_y + 0.5\,\hat{a}_z}{1} = 0.866\,\hat{a}_y + 0.5\,\hat{a}_z$$

$$\therefore \mathcal{P}_{ave} = \frac{100^2}{2\times 377}(0.866\,\hat{a}_y + 0.5\,\hat{a}_z) = 11.4854\,\hat{a}_y + 6.6312\,\hat{a}_z\ \text{W/m}^2$$

15. In a lossless dielectric for which $\eta = 60\pi$, $\mu_r = 1$ and $\vec{H} = -0.1\cos(\omega t - z)\,\hat{a}_x + 0.5\sin(\omega t - z)\,\hat{a}_y$ A/m, calculate ε_r, ω, and $\vec{E}$. *[2072 Chaitra]*

Solution:

From the given expression of $\vec{H}$, we note $\beta = 1$ rad/m.

For a lossless dielectric,

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\,\mu_r}{\varepsilon_0\,\varepsilon_r}} = 377 \times \frac{1}{\sqrt{\varepsilon_r}} \Rightarrow \varepsilon_r = \left(\frac{377}{\eta}\right)^2 = \left(\frac{377}{60\pi}\right)^2 = 4.$$

$$\beta = \omega\sqrt{\mu\varepsilon} \Rightarrow \omega = \frac{\beta}{\sqrt{\mu\varepsilon}} = \frac{\beta}{\sqrt{\mu_0\,\mu_r\,\varepsilon_0\,\varepsilon_r}} = \frac{\beta c}{\sqrt{\mu_r\,\varepsilon_r}} = \frac{1\times 3\times 10^8}{\sqrt{1\times 4}} = 1.5\times 10^8\ \text{rad/s}$$

The expression for electric field is calculated as follows:

$$\text{Since } \sigma = 0,\ \nabla\times\vec{H} = \vec{J}_c + \frac{\partial\vec{D}}{\partial t} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t} = \varepsilon\frac{\partial\vec{E}}{\partial t} \Rightarrow \vec{E} = \frac{1}{\varepsilon}\int\left(\nabla\times\vec{H}\right)dt$$

$$\nabla\times\vec{H} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -0.1\cos(\omega t - z) & 0.5\sin(\omega t - z) & 0 \end{vmatrix} = 0.5\cos(\omega t - z)\hat{a}_x - 0.1\sin(\omega t - z)\hat{a}_y$$

$$\therefore \vec{E} = \frac{1}{\varepsilon}\int\left[0.5\cos(\omega t - z)\hat{a}_x - 0.1\sin(\omega t - z)\,\hat{a}_y\right]dt$$

$$= \frac{1}{\varepsilon_0\,\varepsilon_r}\int 0.5\cos(\omega t - z)\hat{a}_x\,dt - \frac{1}{\varepsilon_0\,\varepsilon_r}\int 0.1\sin(\omega t - z)\,\hat{a}_y\,dt$$

$$= \frac{0.5}{\varepsilon_0\,\varepsilon_r\,\omega}\sin(\omega t - z)\,\hat{a}_x + \frac{0.1}{\varepsilon_0\,\varepsilon_r\,\omega}\cos(\omega t - z)\,\hat{a}_y$$

$$= \frac{0.5}{8.854\times 10^{-12}\times 4\times 1.5\times 10^8}\sin(\omega t - z)\hat{a}_x + \frac{0.1}{8.854\times 10^{-12}\times 4\times 1.5\times 10^8}\cos(\omega t - z)\,\hat{a}_y$$

$$\vec{E} = 94.119\sin(\omega t - z)\hat{a}_x + 18.823\cos(\omega t - z)\,\hat{a}_y\ \text{V/m}.$$

Alternative Method

Given, $\vec{H}_1 = -0.1 \cos(\omega t - z)\,\hat{a}_x$ A/m

$\vec{H}_2 = 0.5 \sin(\omega t - z)\,\hat{a}_y$ A/m

The direction of $\vec{E}_1$ is $\hat{a}_y$. For amplitude of $\vec{E}_1$, we use

$$\eta = \frac{E_{o1}}{H_{o1}}$$

or, $E_{o1} = 60\pi \times 0.1$ [$\because \eta = 60\pi$ given]

or, $E_{o1} = 18.84$ V/m

$\therefore \vec{E}_1 = 18.84 \cos(\omega t - z)\,\hat{a}_y$

The direction of $\vec{E}_2$ is $\hat{a}_x$. For amplitude of $\vec{E}_2$, we use

$$\eta = \frac{E_{o2}}{H_{o2}}$$

or, $E_{o2} = 60\pi \times 0.5 = 94.24$ V/m

$\therefore \vec{E}_2 = 94.24 \sin(\omega t - z)\,\hat{a}_x$

Hence, $\vec{E} = \vec{E}_1 + \vec{E}_2 = 18.84 \cos(\omega t - z)\,\hat{a}_y + 94.24 \sin(\omega t - z)\,\hat{a}_x$ V/m

16. The electric field in free space is given by

$\vec{E} = 50 \cos(10^8 t + \beta x)\hat{a}_y$ V/m

a. Find the direction of wave propagation.

b. Calculate β and the time it takes to travel a distance of $\lambda/2$.

c. Sketch the wave at t = 0, T/4, and T/2.

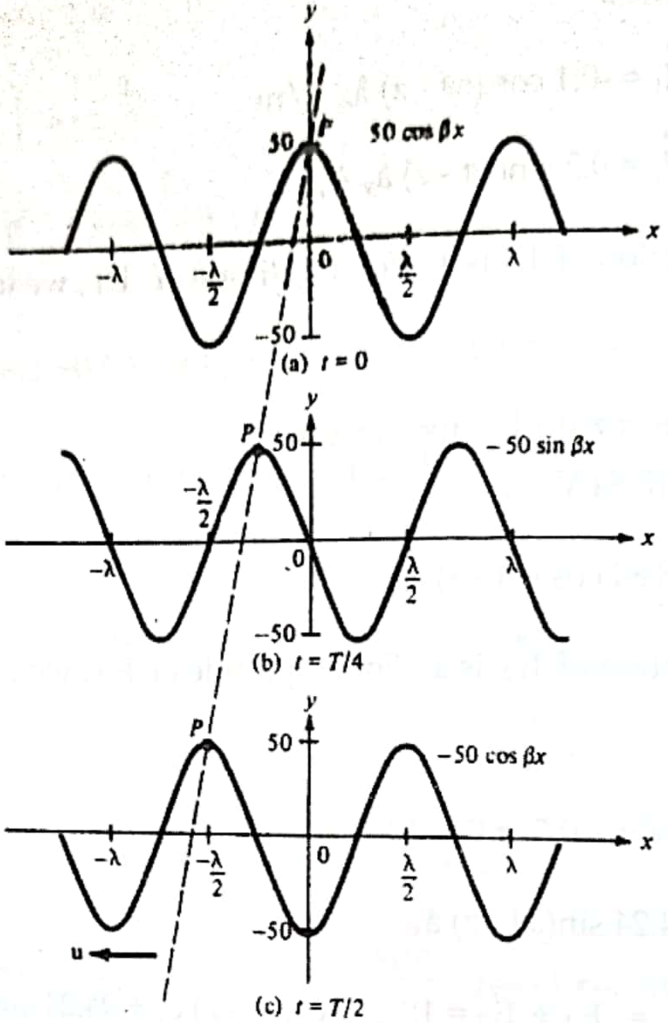
Solution:

a. From the positive sign in $(\omega t + \beta x)$, we infer that the wave is propagating along $-\hat{a}_x$.

b. For free space,

$$\beta = \omega\sqrt{\mu_o \varepsilon_o} = 10^8\sqrt{(4\pi\times10^{-7}) \times (8.854\times10^{-12})} = 0.3333 \text{ rad/m}$$

To travel a distance of $\lambda/2$, it will take T/2 seconds (taking T as the time period of wave).



$\therefore$ Time taken (t) = $T/2 = \frac{1}{2} \times \frac{2\pi}{\omega} = \frac{\pi}{\omega} = \frac{3.141516}{10^8} = 31.42$ ns

c. At t = 0, $E_y = 50 \cos \beta x$

At t = T/4, $E_y = 50 \cos (10^8 T/4 + \beta x) = 50 \cos \left(10^8 \frac{2\pi}{4\omega} + \beta x\right)$

$$= 50\cos\left(10^8 \frac{\pi}{2\times 10^8} + \beta x\right)$$

$$= 50\cos(\beta x + \pi/2) = -50 \sin \beta x$$

At t = T/2, $E_y = 50 \cos (10^8 T/2 + \beta x) = 50 \cos \left(10^8 \frac{2\pi}{2\omega} + \beta x\right)$

$$= 50 \cos \left(10^8 \frac{\pi}{10^8} + \beta x\right)$$

$$= 50 \cos (\pi + \beta x) = -50 \cos \beta x$$

E_y at t = 0, T/4, T/2 is plotted against x as shown in Figure.

17. In a nonmagnetic medium, $\vec{E} = 4\sin(2\pi\times10^7 t - 0.8x)\,\hat{a}_z$ V/m. Find: (a) ε_r, η (b) The time average power carried by the wave (c) The total power crossing 100 cm² of plane $2x+y = 5$.

Solution:

For a nonmagnetic medium, $\mu = \mu_o$, and $\varepsilon = \varepsilon_o\,\varepsilon_r$

a. $$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\,\varepsilon_o\,\varepsilon_r} = \frac{\omega}{c}\sqrt{\varepsilon_r} \Rightarrow \varepsilon_r = \frac{c^2\beta^2}{\omega^2} = \frac{(3\times10^8)^2\times(0.8)^2}{(2\pi\times10^7)^2} = 14.59$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_o}{\varepsilon_o\,\varepsilon_r}} = \frac{377}{\sqrt{\varepsilon_r}} = \frac{377}{\sqrt{14.59}} = 98.699\ \Omega$$

b. Nonmagnetic medium is a lossles medium and for a lossless medium, the time average power is

$$\wp_{ave} = \frac{1}{T}\int_0^T \wp\,dt$$

where $\wp = \vec{E}\times\vec{H}$ is the Poynting vector.

$\hat{a}_x = \hat{a}_z \times ?$

As $\hat{a}_z\times(-\hat{a}_y) = -\,\hat{a}_x$, the direction of $\vec{H}$ is $-\hat{a}_y$.

$$\therefore\ \wp = [4\sin(2\pi\times10^7t-0.8x)\,\hat{a}_z]\times\left[\frac{4}{\eta}\sin(2\pi\times10^7t-0.8x)(-\hat{a}_y)\right]$$

$$= \frac{16}{98.699}\sin^2(2\pi\times10^7t-0.8x)\,\hat{a}_x = 0.1621\sin^2(2\pi\times10^7t-0.8x)\,\hat{a}_x$$

$$\wp_{ave} = \frac{1}{T}\int_0^T 0.1621\sin^2(2\pi\times10^7t-0.8x)\,\hat{a}_x\,dt = 0.1621\,\hat{a}_x\frac{1}{T}\int_0^T \sin^2(2\pi\times10^7t-0.8x)\,\hat{a}_x\,dt$$

$$= 0.1621\,\hat{a}_x\times\frac{1}{2}\ \therefore\left(\frac{1}{T}\int_0^T \sin^2\theta\,d\theta = \frac{1}{2}\right)$$

$= 0.081\,\hat{a}_x$ W/m².

c. The total power crossing a given surface S is given by

$$P_{ave} = \int_S \wp_{ave}\,.\,d\vec{S} = \wp_{ave}\,.\,S\,\hat{a}_N$$

where

$\hat{a}_N$ = unit vector normal to the plane $2x+y=5 = \frac{2\,\hat{a}_x + \hat{a}_y}{\sqrt{(2)^2+(1)^2}} = \frac{2\,\hat{a}_x + \hat{a}_y}{\sqrt{5}}$

$$\therefore P_{ave} = (0.081\,\hat{a}_x) \cdot (100\times10^{-2})^2\left(\frac{2\,\hat{a}_x + \hat{a}_y}{\sqrt{5}}\right) = 0.07244\ \text{W}$$

18. In a medium characterized by $\sigma = 0$, $\mu = \mu_0$, $\varepsilon = 4\varepsilon_0$, and $\vec{E} = 20\sin(10^8t-\beta z)\,\hat{a}_y$ V/m. Calculate β and $\vec{H}$.

Solution:

This case resembles a lossless dielectric for which

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o(4\varepsilon_o)} = 10^8\sqrt{4\times4\pi\times10^{-7}\times8.854\times10^{-12}} = 0.6671\ \text{rad/m}$$

$$\eta = \frac{E_o}{H_o} \Rightarrow H_o = \frac{E_o}{\eta} = \frac{E_o}{\sqrt{\frac{\mu}{\varepsilon}}} = \frac{E_o}{\sqrt{\frac{\mu_o}{4\varepsilon_o}}} = \frac{20}{\sqrt{\frac{4\pi\times10^{-7}}{4\times8.854\times10^{-12}}}} = 0.1061\ \text{A/m}$$

As $\wp = \vec{E} \times \vec{H}$, the direction of $\vec{H}$ is $-\hat{a}_x$.

$\hat{a}_z = \hat{a}_y \times ?$

$$\therefore \vec{H} = 0.1061\sin(10^8t-\beta z)(-\hat{a}_x) = -0.1061\sin(10^8t-\beta z)\,\hat{a}_x\ \text{A/m}$$

19. The electric field intensity in free space is given as $\vec{E}(x, t) = 10\sin(10^8t + \beta z)\hat{a}_x$ V/m. Find:

a. Phase constant (β)

b. Wavelength (λ)

c. $|\vec{H}(x, y, z, t)|$ at P(−0.1, 0.2, −0.3) at t = 1 ns.

Solution:

a. $\beta = \omega\sqrt{\mu_o\varepsilon_o} = \frac{\omega}{c} = \frac{10^8}{3\times10^8} = 0.333$ rad/m

b. $\beta = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.333} = 18.868$ m

c. $\vec{P} = \vec{E} \times \vec{H}$

$-\hat{a}_z = \hat{a}_x \times ?$

Therefore, the direction of $\vec{H}$ is $-\hat{a}_y$.

The amplitude for magnetic field intensity is calculated as follows.

$$\eta = \frac{E_o}{H_o}$$

For free space, $\eta = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 377$

$$\therefore H_o = \frac{E_o}{\eta} = \frac{10}{377} = 0.0265\ A/m$$

Now, we write

$$\vec{H}\ (x, y, z, t) = 0.265 \sin(10^8 t + \ \beta z)(-\hat{a}_y)$$
$$= -0.265 \sin(10^8 t + 0.333z)\hat{a}_y\ A/m$$

At P(−0.1,0.2, −0.3) and t = 1ns,

$$\vec{H}\ (x, y, z, t) = -0.0265 \sin\ (10^8 \times 10^{-9} + 0.333 \times (-\ 0.3))\ \hat{a}_y$$
$$= -\ 0.0265 \sin\ (1 \times 10^{-4})^6 \hat{a}_y$$
$$= -2.65 \times 10^{-6} \hat{a}_y\ A/m$$

Hence, $|\vec{H}\ (x,y, z, t)| = 2.65 \times 10^{-6}\ A/m$

20. **A uniform plane wave with $\vec{E} = E_x \hat{a}_x$ propagates in a lossless simple medium (ε_r=4, μ_r=1, σ= 0) in the +z-direction. Assume that E_x is sinusoidal with a frequency 100 MHz and has a maximum value of $+10^{-4}$ V/m at t = 0 and z = $\frac{1}{8}$ m.**

   a. **Write the instantaneous expression for $\vec{E}$ for any t and z.**

   b. **Write the instantaneous expression for $\vec{H}$.**

   c. **Determine the locations where E_x is a positive maximum when t= 10^{-8}s.**

Solution:

a. Using cosωt as the reference, we find the instantaneous expression for $\vec{E}$ to be

$$\vec{E}\ (x, t) = E_x\ \hat{a}_x = 10^{-4} \cos\ (2\pi 10^8 t - \beta z + \phi)\hat{a}_x$$

where $\beta = \ \omega\sqrt{\mu\varepsilon} = \sqrt{\mu_o\mu_r\varepsilon_o\varepsilon_r} = \frac{4\pi}{3}$ rad/m

To find ϕ, we use the condition that at t = 0 and z = $\frac{1}{8}$ m, $E_x = 10^{-4}$ V/m.

For this, we must have

$$\cos\ (2\pi 10^8 t - \beta z + \phi) = 1$$

i.e., $2\pi\, 10^8\, t - \beta z + \phi = 0$

or, $2\pi\, 10^8 \times 0 - \frac{4\pi}{3} \times \frac{1}{8} + \phi = 0$

$\therefore \phi = \frac{\pi}{6}$ rad

Thus, $\vec{E}(z, t) = 10^{-4} \cos\left(2\pi 10^8 t - \frac{4\pi}{3} z + \frac{\pi^c}{6}\right) \hat{a}_x V/m$

b. $\vec{P} = \vec{E} \times \vec{H}$

$\hat{a}_z = \hat{a}_x \times ?$

Therefore, the direction of $\vec{H}$ is $+\hat{a}_y$. The amplitude of magnetic field intensity is found as

$$\eta = \frac{E_o}{H_o}$$

For the given medium,

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_o \mu_r}{\varepsilon_o \varepsilon_r}} = 188.49\ \Omega$$

$$\therefore H_o = \frac{E_o}{\eta} = \frac{10^{-4}}{188.49} = 5.305 \times 10^{-7} A/m$$

Thus,

$$\vec{H}(z, t) = 5.3053 \times 10^{-7} \cos\left(2\pi 10^8 t - \frac{4\pi}{3} z + \frac{\pi}{6}\right) \hat{a}_y A/m$$

c. E_x is a positive maximum when

$$2\pi 10^8 t - \frac{4\pi}{3} z + \frac{\pi^c}{6} = \pm 2n\pi^c$$

For $t = 10^{-8}$, $z = z_m = \frac{13}{8} \pm \frac{3n}{2}$ m; $n = 0, 1, 2....$

21. The electric field intensity in free space is given as $\vec{E}(x, t) = 10 \sin(10^8 t + \beta z)\hat{a}_x V/m$. Find:

a. Phase constant (β)

b. Wavelength (λ)

c. $|\vec{H}(x, y, z, t)|$ at P(–0.1, 0.2, –0.3) at t = 1 ns.

Solution:

a. $\beta = \omega\sqrt{\mu_o \varepsilon_o} = \frac{\omega}{c} = \frac{10^8}{3 \times 10^8} = 0.333$ rad/m

b. $\beta = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.333} = 18.868$ m

c. $\vec{P} = \vec{E} \times \vec{H}$

$-\hat{a}_z = \hat{a}_x \times ?$

Therefore, the direction of $\vec{H}$ is $-\hat{a}_y$.

The amplitude for magnetic field intensity is calculated as follows.

$$\eta = \frac{E_o}{H_o}$$

For free space, $\eta = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 377$

$$\therefore H_o = \frac{E_o}{\eta} = \frac{10}{377} = 0.0265 \text{ A/m}$$

Now, we write

$\vec{H}(x, y, z, t) = 0.265 \sin(10^8 t + \beta z)(-\hat{a}_y) = -0.265 \sin(10^8 t + 0.333z)\hat{a}_y$ A/m

At P(−0.1, 0.2, −0.3) and t = 1ns,

$\vec{H}(x, y, z, t) = -0.0265 \sin(10^8 \times 10^{-9} + 0.333 \times (-0.3))\, \hat{a}_y = -2.65 \times 10^{-6} \hat{a}_y$ A/m

Hence, $|\vec{H}(x, y, z, t)| = 2.65 \times 10^{-6}$ A/m

22. A uniform plane wave with $\vec{E} = E_x \hat{a}_x$ propagates in a lossless simple medium (ε_r=4, μ_r=1, σ=0) in the +z-direction. Assume that E_x is sinusoidal with a frequency 100 MHz and has a maximum value of $+10^{-4}$ V/m at t = 0 and $z = \frac{1}{8}$ m.

a. Write the instantaneous expression for $\vec{E}$ for any t and z.

b. Write the instantaneous expression for $\vec{H}$.

c. Determine the locations where E_x is a positive maximum when t= 10^{-8}s.

Solution:

a. Using $\cos\omega t$ as the reference, we find the instantaneous expression for $\vec{E}$ to be

$$\vec{E}(x, t) = E_x \hat{a}_x = 10^{-4} \cos(2\pi 10^8 t - \beta z + \phi)\, \hat{a}_x$$

where $\beta = \omega\sqrt{\mu\varepsilon} = \sqrt{\mu_o\mu_r\varepsilon_o\varepsilon_r} = \frac{4\pi}{3}$ rad/m

To find ϕ, we use the condition that at $t = 0$ and $z = \frac{1}{8}$ m, $E_x = 10^{-4}$ V/m.

For this, we must have

$\cos(2\pi 10^8 t - \beta z + \phi) = 1$

i.e., $2\pi\, 10^8\, t - \beta z + \phi = 0$

or, $2\pi\, 10^8 \times 0 - \frac{4\pi}{3} \times \frac{1}{8} + \phi = 0$

$\therefore \phi = \frac{\pi}{6}$ rad

Thus, $\vec{E}\,(z,t) = 10^{-4} \cos\left(2\pi 10^8 t - \frac{4\pi}{3} z + \frac{\pi^c}{6}\right) \hat{a}_x$ V/m

b. $\vec{P} = \vec{E} \times \vec{H}$

$\hat{a}_z = \hat{a}_x \times ?$

Therefore, the direction of $\vec{H}$ is $+\hat{a}_y$. The amplitude of magnetic field intensity is found as

$$\eta = \frac{E_o}{H_o}$$

For the given medium,

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_o\mu_r}{\varepsilon_o\varepsilon_r}} = 188.49\ \Omega$$

$$\therefore H_o = \frac{E_o}{\eta} = \frac{10^{-4}}{188.49} = 5.305\times10^{-7}\ \text{A/m}$$

Thus, $\vec{H}\,(z, t) = 5.3053 \times 10^{-7} \cos\left(2\pi 10^8\, t - \frac{4\pi}{3} z + \frac{\pi^c}{6}\right) \hat{a}_y$ A/m

c. E_x is a positive maximum when

$$2\pi 10^8 t - \frac{4\pi}{3} z + \frac{\pi^c}{6} = \pm\, 2n\pi^c$$

For $t = 10^{-8}$, $z = z_m = \frac{13}{8} \pm \frac{3n}{2}$ m; $\quad n = 0,1,2....$

23. **An electric field $\vec{E}$ in free space is given as $\vec{E} = 800 \cos(10^8\, t - \beta y)\, \hat{a}_z$ V/m. Find (a) Propagation constant (β) (b) wavelength (λ) (c) magnetic field intensity $\vec{H}$ at P (0.1, 1.5, 0.4) at t = 8 ns** *[2062 Bhadra]*

Solution:

(a) $\omega = 10^8$

or, $2\pi f = 10^8$

or, $2\pi \frac{c}{\lambda} = 10^8 \Rightarrow \lambda = 6\pi$ m

$\therefore \beta = \frac{2\pi}{\lambda} = \frac{2\pi}{6\pi} = \frac{1}{3}$ rad/m

(b) Wavelength (λ) = 6π m

(c) $H_o = \frac{E_o}{\eta} = \frac{800}{377} = 2.122$ A/m ; $\eta = 377$ for free space

At (0.1, 1.5, 0.4) and t = 8 ns,

$\vec{H} = H_o\cos(10^8t - \beta y)\ \hat{a}_x = 2.122\cos\ (10^8 \times 8 \times 10^{-9} - \frac{1}{3} \times 1.5)\ \hat{a}_x = 2.121\ \hat{a}_x$ A/m

24. A uniform plane wave propagating in free space has $\vec{E} = 2\cos\ (10^7\pi t - \beta z)\ \hat{a}_x$, determine β & $\vec{H}$.

[2067 Mangsir]

Solution:

For free space, $v = 3 \times 10^8$ m/s

$f = \frac{\omega}{2\pi} = \frac{10^7\pi}{2\pi} = 5 \times 10^6$ Hz

$\lambda = \frac{v}{f} = \frac{3 \times 10^8}{5 \times 10^6} = 60$ m

$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{60} = 0.1047$ rad/m

Using , $\eta = \frac{E_o}{H_o}$

or, $H_o = \frac{E_o}{\eta} = \frac{2}{377}$ A/m

The direction of $\vec{H}$ is calculated as

$\vec{P} = \vec{E} \times \vec{H}$

$\hat{a}_z = \hat{a}_x \times ?$

Hence, the direction of $\vec{H}$ is $\hat{a}_y$.

$\therefore \quad \vec{H} = H_o \cos(10^7 \pi t - \beta z)\, \hat{a}_y$

$= 5.305 \times 10^{-3} \cos(10^7 \pi t - 0.1047\, z)\, \hat{a}_y$ A/m

25. **A z-polarized uniform plane wave with frequency 100 MHz propagates in air in the positive x-direction and impinges normally on a perfectly conducting plane at x = 0. Assuming the amplitude of electric field vector to be 3 mV/m. Determine phases & instantaneous expression for**

(a) Incident electric & magnetic field vectors

(b) Reflected electric & magnetic field vectors *[2067 Mangsir]*

Solution:

For the wave in air, $\varepsilon_r = 1.006 \simeq 1$. So, it can be considered as free space.

$$\text{So, } \beta_1 = \frac{\omega}{c} = \frac{2\pi \times f}{c} = \frac{2\pi \times 1 \times 10^8}{3 \times 10^8} = \frac{2\pi}{3}$$

$E_{io} = 3$ mV/m

$\therefore \quad \vec{E}_{is} = E_{io}\, e^{-j\beta x}\, \hat{a}_z$

$\therefore$ Incident electric field vectors

$\vec{E}_{is} = 3\, e^{-j\frac{2\pi}{3}x}\, \hat{a}_z$ mV/m is phasor form

$\vec{E}_i = 3 \cos(2\pi \times 1 \times 10^8 - \frac{2\pi}{3} x)\, \hat{a}_z$ mV/m in instantaneous form

For medium 1 (air), $\quad \eta_1 = 120\pi$

$$\eta_1 = \frac{E_{io}}{H_{io}}$$

$$\text{or, } H_{io} = \frac{E_{io}}{\eta_1} = \frac{3}{120\pi} = \frac{1}{40\pi} \text{ mA/m}$$

$\vec{H}_{is} = \frac{1}{40\pi} e^{-j\frac{2\pi}{3}x}\, \hat{a}_y$ mA/m in phasor form

$\vec{H}_i = \frac{1}{40\pi} \cos(2\pi \times 10^8 - \frac{2\pi}{3} x)\, \hat{a}_y$ mA/m in instantaneous form.

For reflected electric and magnetic field vectors

Since medium 2 is a perfect conductor

$$\frac{\sigma_2}{\omega\varepsilon_2} >> 1 \rightarrow \eta_2 << \eta_1$$

i.e. $\Gamma = -1$ & $\tau = 0$

$\therefore E_{ro} = \Gamma E_{io} = -E_{io} = -3$ mV/m

$\vec{E}_{rs} = E_{ro}\, e^{-j\frac{2\pi}{3}(-x)}\, \hat{a}_z = -3\, e^{j\frac{2\pi}{3}x}\, \hat{a}_z$ mV/m (in phasor form)

$\vec{E}_r = -3\cos\left(2\pi \times 10^8 + \frac{2\pi}{3}x\right)\, \hat{a}_z$ mV/m (in instantaneous from)

$\vec{H}_{rs} = H_{ro}\, e^{j\beta_x}\, (-\hat{a}_y)$

where $H_{ro} = \frac{E_{ro}}{\eta_1} = -\frac{3}{120\pi}$ mA/m $= -\frac{1}{40\pi}$ mA/m

$\therefore \vec{H}_{rs} = -\frac{1}{40\pi}\, e^{j\frac{2\pi}{3}x}\, (-\hat{a}_y)$ mA/m in phasor form

$\vec{H}_r = -\frac{1}{40\pi}\cos\left(2\pi \times 10^8 + \frac{2\pi}{3}x\right)(-\hat{a}_y)$ mA/m in instantaneous form

26. A time harmonic uniform plane wave $\vec{E}$ (x, y, z, t) with polarization in $\hat{a}_z$ direction & frequency 150 MHz is moving in free space in negative y direction & has maximum amplitude 2 V/m. Determine

(a) The angular frequency

(b) Phase constant β

(c) Expression for $\vec{E}$ (x, y, z, t)

(d) Expression for $\vec{H}$ (x, y, z, t)

[2068 Shrawan]

Solution:

(a) Angular frequency (ω) $= 2\pi f = 2\pi \times 10^6 \times 15 = 30\pi \times 10^6$ rad/s

(b) Phase constant (β) $= \frac{2\pi}{\lambda}$

$$\lambda = \frac{v}{f} = \frac{3 \times 10^8}{150 \times 10^6} = 2\text{m}$$

$$\therefore \beta = \frac{2\pi}{2} = \pi \text{ rad/m}$$

(c) $\vec{E}(x, y, z, t) = E_o \cos(2\pi ft + \beta y)\, \hat{a}_z$

$$= 2\cos(2\pi \times 150 \times 10^6 t + \pi y)\, \hat{a}_z \text{ V/m}$$

(d) $H(x, y, z, t) = H_o \cos(2\pi ft + \beta y)$

where $H_o = \frac{E_o}{\eta} = \frac{2}{377} = 5.305 \times 10^{-3}$ A/m

$\therefore H(x, y, z, t) = 5.30 \times 10^{-3} \cos(2\pi \times 150 \times 10^6 t + \pi y)$

$\vec{P} = \vec{E} \times \vec{H}$

$-\hat{a}_y = \hat{a}_z \times ?$

Hence, the direction of $\vec{H}$ is $-\hat{a}_x$.

$\therefore \vec{H}(x, y, z, t) = -5.305 \times 10^{-3} \cos(2\pi \times 150 \times 10^6 t + \pi y)\, \hat{a}_x$

$$= -5.305 \times 10^{-3} \cos(300\pi \times 10^6 t + \pi y)\, \hat{a}_x \text{ A/m}$$

27. For a non-magnetic material having $\varepsilon_r = 2.25$ and $\sigma = 10^{-4}$ mho/m. Find the numeric value at 5 MHz for

(a) the loss tangent **(b) the attenuation constant**
(c) phase constant **(d) the intrinsic impedance** *[2068 Baishakh]*

Solution:

(a) Loss tangent $(\tan\theta) = \frac{\sigma}{\varepsilon\omega} = \frac{\sigma}{\varepsilon_o \varepsilon_r \omega}$

where $\omega = 2\pi f = 2\pi \times 5 \times 10^6 = 3.1415 \times 10^7$ rad/s

$$\therefore \tan\theta = \frac{1 \times 10^{-4}}{8.854 \times 10^{-12} \times 2.25 \times 3.1415 \times 10^7} = 0.1597$$

(b) For non magnetic material, $\mu = \mu_o$

So, attenuation constant $(\alpha) = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}$

$$\therefore \alpha = 2\pi \times 5 \times 10^6 \sqrt{\frac{4\pi \times 10^{-7} \times 2.25 \times 8.854 \times 10^{-12}}{2}\left[\sqrt{1 + 0.14386^2 - 1}\right]}$$

$$= 0.011275 \text{ Np/m}$$

(c) Phase constant $(\beta) = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}$

$$\therefore \beta = 2\pi \times 5 \times 10^6 \sqrt{\frac{4\pi \times 10^{-7} \times 2.25 \times 8.854 \times 10^{-12}}{2}\left[\sqrt{1 + 0.14386^2} + 1\right]}$$

$$= 0.15755 \text{ rad/m}$$

This can be done by another way as

Propagation constant $(\gamma) = \sqrt{j\omega\mu\,(\sigma + j\omega\varepsilon)}$

$$\therefore \gamma = \sqrt{j2\pi\times5\times10^6\times4\pi\times10^{-7}(1\times10^{-4}+j3.1415\times10^7\times10^6\times2.25\times8.854\times10^{-12})}$$

$$= \sqrt{39.478\angle90°\times6.335119 \times 10^{-4}\angle80.91°}$$

$$= \sqrt{0.02500978 \angle 170.9178°}$$

$$= \sqrt{0.02500978} \angle \frac{170.9178°}{2}$$

$$= 0.158144 \angle 85.458° \;\; = 0.0125 + j\,0.15755 \text{ m}^{-1}$$

Hence, attenuation constant $(\alpha) = 0.0125$ Np/m

Phase constant $(\beta) = 0.15755$ rad/m

(d) Intrinsic impedance $(\eta) = \sqrt{\dfrac{j\omega\mu}{\sigma + j\omega\varepsilon}}$

$$= \sqrt{\frac{39.478 \angle 90°}{6.335119 \times 10^{-4} \angle 80.91°}}$$

$$= \sqrt{62316.11435 \angle 9.09°} \;\; = 249.63 \angle 4.545° \;\Omega$$

28. A uniform plane wave in free space is given by electric field $\vec{E}_S = (200 \angle 20°)\, e^{-j250z}\, \hat{a}_x$ V/m. Find:

a) angular frequency (ω) b) wave length (λ)

c) magnetic field intensity $\vec{H}$ in phasor form i.e., $\vec{H}_S$ *[2068 Magh]*

Solution:

$\vec{E}_S = (200\angle20°)\, e^{-j250z}\, \hat{a}_x$

Comparing it with the standard equation for an electric field in phasor form, we get

$E_o = 200,\ \beta = 250$

a. For free space, $v = c = 3 \times 10^8$ m/s

$\omega = \beta \times v = 3 \times 10^8 \times 250 = 7.5\times10^{10}$ rad/s

b. wavelength $(\lambda) = \frac{2\pi}{\beta} = \frac{2\pi}{250} = 0.02513$ m.

c. To find magnetic field, $\vec{H_s}$

$$H_o = \frac{E_o}{\eta} = \frac{200}{120\pi}$$

$$\therefore \vec{H_s} = H_o e^{-j\beta z}\,\hat{a}_y = \frac{200}{120\pi}\angle 20^\circ\, e^{-250z}\,\hat{a}_y \text{ A/m}$$

29. At 50 MHz, a lossy dielectric material is characterized by $\varepsilon = 3.6\varepsilon_o$, $\mu = 2.1\mu_o$ and $\sigma = 0.08$ S/m. It $\vec{E_s} = 6e^{-\gamma x}\,\hat{a}_z$ V/m, compute: (a) propagation constant (b) wavelength (c) $\vec{H_s}$

Solution:

$$\gamma = \sqrt{j\omega\mu\,(\sigma + j\omega\varepsilon)}$$

$$= \sqrt{[j(\pi\times10^8)\times(2.6389\times10^{-6})][0.08+j(\pi\times10^8)(3.1874\times10^{-11})]}$$

$$= 8.1744\angle 48.5672^\circ = 5.4093 + j6.1286 \text{ /m}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{6.1286} = 1.0252 \text{ m}$$

The amplitude of $\vec{H_s}$ is calculated as

$$H_o = \frac{E_o}{\eta} = \frac{6}{\eta}$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma+j\omega\varepsilon}} = \sqrt{\frac{829.0467\angle 90^\circ}{0.0806\angle 7.1345^\circ}} = 101.4196\angle 41.43^\circ\ \Omega$$

$$\therefore H_o = \frac{6}{101.4196\angle 41.43^\circ} = 0.0591\angle -41.43^\circ \text{ A/m}$$

The direction of $\vec{H_s}$ is calculated as

$$\vec{P} = \vec{E}\times\vec{H}$$

$$+\hat{a}_x = +\hat{a}_z \times ?$$

Therefore, the direction of $\vec{H_s}$ is $-\hat{a}_y$

Now, the expression for $\vec{H_s}$ is

$\vec{H}_s = 0.0591\angle{-41.43°}\ e^{-\gamma x}\,(-\hat{a}_y)$

$= -0.0591\ e^{-j41.43°}\ e^{-(5.4093 + j6.1286)x}\ \hat{a}_y$

$= -0.0591\ e^{-5.4093x}\ e^{-j(6.1286x + 41.43°)}\ \hat{a}_y$ A/m

Hammered Problems

1. **The electric field amplitude of a uniform plane wave propagating in the free space in $\hat{a}_z$ direction is 250 V/m. If $\vec{E} = E_x\,\hat{a}_x$ and ω = 1.00 M rad/s, find: (a) the frequency, (b) the wavelength, (c) the period, (d) the amplitude of H.**

   [2071 Shrawan]

   Answer: (a) 159.1549×10^3 Hz (b) 600πm (c) 6.2831×10^{-6} s (d) 0.6631 A/m

2. **A plane wave propagating through a medium with ε_r=8, μ_r=2 has $\vec{E} = 0.5\ e^{-z/3}$ sin (10^8t - βz) $\hat{a}_x$ V/m. Determine**
   a. **β**
   b. **The loss tangent**
   c. **Intrinsic impedence**
   d. **Wave velocity**
   e. **$\vec{H}$ field**

   Answer: (a) 1.374 rad/m (b) 0.5154 (c) 177.72 $\angle 13.63°$ Ω (d) 7.278×10^7 m/s (e) $2.817\ e^{-z/3}$ sin (10^8t-βz – 13.63°) $\hat{a}_y$ mA/m

3. **A plane wave in a non-magnetic medium has $\vec{E}$ = 50 sin (10^8t + 2z) $\hat{a}_y$ V/m. Find:**
   (a) **The direction of wave propagation**
   (b) **λ, f, and ε_r**
   (c) **$\vec{H}$**

   Answer: (a) in the -z direction (b) 3.142m, 15.92 MHz, 36 (c) 0.7958 sin (10^8t + 2z) $\hat{a}_x$ A/m

4. Let $\vec{H}_s$ = (2$\angle$–40°$\hat{a}_x$ – 3$\angle$20°$\hat{a}_y$) $e^{-j0.07z}$ A/m for a uniform plane wave traveling in free space. Find: (a) ω (b) H_x at P(1, 2, 3) at t = 31 ns (c) | $\vec{H}$ | at t = 0 at the origin.

   Answer: (a) 21.0 Mrad/s (b) 1.934 A/m (c) 3.22 A/m