Chapter 2
Physical And Mathematical Models

Static Physical Models

- Physical models are such models where the system attributes are represented by physical measurements such as voltage or position of shaft.
- Static physical model is a scaled down model of a system which does not change with time.
- Static physical model is the physical model which describes relationships that do not change with respect to time.
- Such models only **depict the object's characteristics at any instance of time, considering that the object's property will not change over time.**
- Example : An architectural model of a house, scale models and so on.
- The best known examples of physical models are scale models.
- In shipbuilding, making a scale model provides a simple way of determining the exact measurements of the plates covering the hull, rather than having to produce drawings of complicated, three-dimensional shapes.

- Scientists have used models in which spheres represent atoms, and rods or specially shaped sheets of metal connect the spheres to represent atomic bonds.
- Scale models are also used in wind tunnels and water tanks in the course of designing aircraft and ships.
- **Sometimes, a static physical model is used as a means of solving equations with particular boundary conditions.**
- There are many examples in the field of mathematical physics where the same equations apply to different physical phenomena. For example, the flow of heat and the distribution of electric charge through space can be related by common equations.

Dynamic Physical Models

- Dynamic physical models rely upon an analogy between the **system being studied** and **some other system of a different nature**, the analogy usually depending upon an underlying similarity in the forces governing the behavior of the systems.
- To illustrate this type of physical model, consider the two systems shown in following figures below.

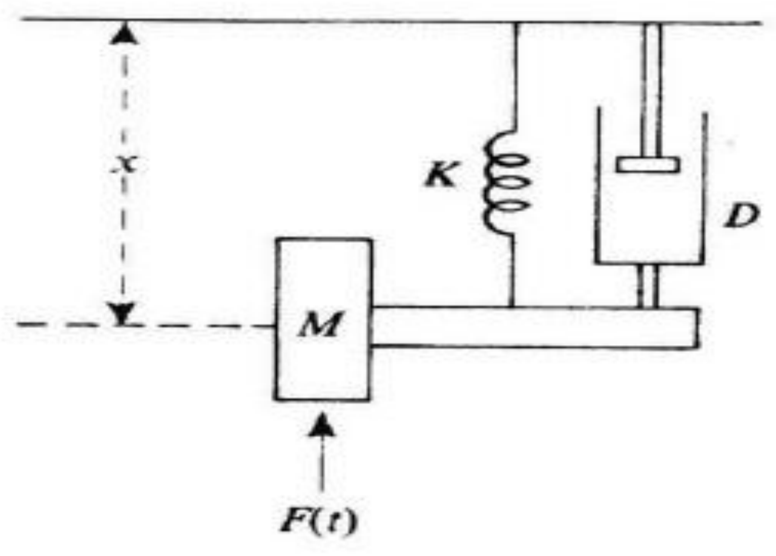

Figure 1: Mechanical System

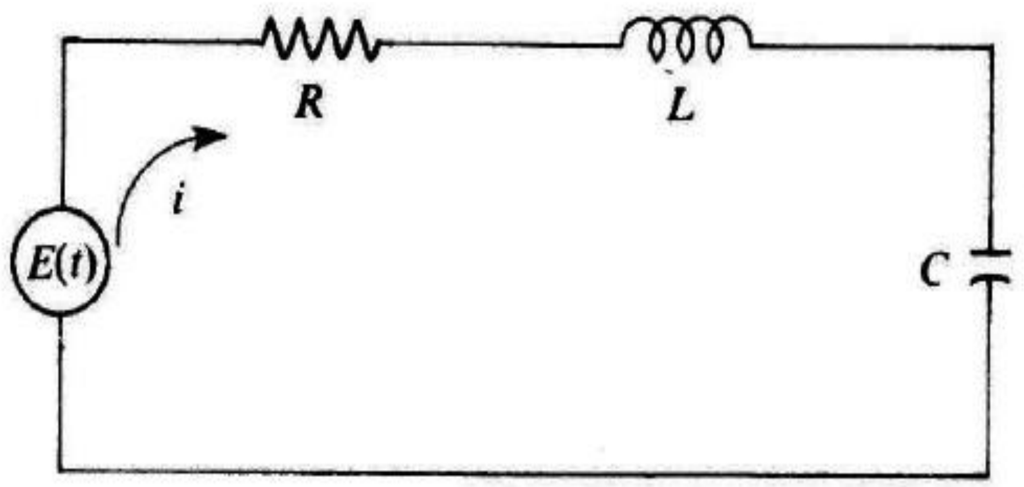

Figure 2: Electrical System

- Figure 1 shows a mass spring system where a mass M is subject to an applied force F(t) varying with time. The force on spring is directly proportional to expansion or contraction of the spring with spring constant k. There is also a shock absorber of damping constant D that exerts a damping force proportional to the velocity of mass.
- This system might represent an example of suspension of automobile wheel.
- The motion of the system can be described by the following differential equation

$$\boxed{M\ddot{x} + D\dot{x} + Kx = KF(t)}$$

Where x is the distance moved,

M is the mass,

K is the stiffness of the spring or spring constant,

D is the damping factor of the shock absorber

- Figure 2 shows an electrical system that contains Resistance R, Capacitor of capacitance C and Inductor of inductance L and are connected in series with a voltage source E(t) which varies with time.
- If q is the charge on the capacitance, it can be shown that the behavior of the circuit is governed by the following differential equation

$$\boxed{L\ddot{q} + R\dot{q} + \frac{q}{C} = \frac{E(t)}{C}}$$

- Both these mechanical and electrical system exactly have the same form of equation and hence following equivalences occur between these system.

Mechanical System	Electrical System
Displacement (x)	Charge (q)
Force (F)	Voltage (E)
Mass (M)	Inductance (L)
Damping Factor (D)	Resistance (R)
Spring Stiffness (K)	1/Capacitance(C)

Hence the mechanical and electrical systems are analogous to each other.

- Since these systems are analogous, the performance of one can be studied with the other.
- In practice, it is simpler to modify the electrical system than to change the mechanical system, so it is more likely that the electrical system will have been built to study the mechanical system.

- If, for example, a car wheel is considered to bounce too much with a particular suspension system, the electrical model will demonstrate this fact by showing that the charge (and, therefore, the voltage) on the condenser oscillates excessively.
- To predict what effect a change in the shock absorber or spring will have on the performance of the car, it is only necessary to change the values of the resistance or condenser in the electrical circuit and observe the effect on the way the voltage varies.
- If in fact, the mechanical system were as simple as illustrated, it could be studied by solving the mathematical equation derived in establishing the analogy.
- However, effects can easily be introduced that would make the mathematical equation difficult to solve.

Static Mathematical Models

- Mathematical models use symbolic notation and mathematical equation to represent a system.
- A static model gives the relationships between the system attributes **when system is in equilibrium.**
- Static Mathematical Models are such mathematical models that give the relationships between the system attributes when system is in equilibrium.
- These are the mathematical models that represent system at a particular point of time.
- If the point of equilibrium is changed by altering any of the attribute values, the model enables the new values for all the attributes to be derived but does not show the way in which they changed to their new values.
- For example, in marketing a commodity there is a balance between the supply and demand for the commodity

- Both factors demand and supply depend upon price.
- **Let S represent Supply, Q represent Demand and P represents Price.**
- Demand for the commodity will be low when the price is high and it will increase as the price drops.
- The relationship between demand and price might be represented by the straight line marked "Demand“ in following figure.
- Similarly the supply can be expected to increase as the price increases, because the suppliers see an opportunity for more revenue.
- The relationship between supply and price might also be represented by the straight line marked “Supply” in following figure.

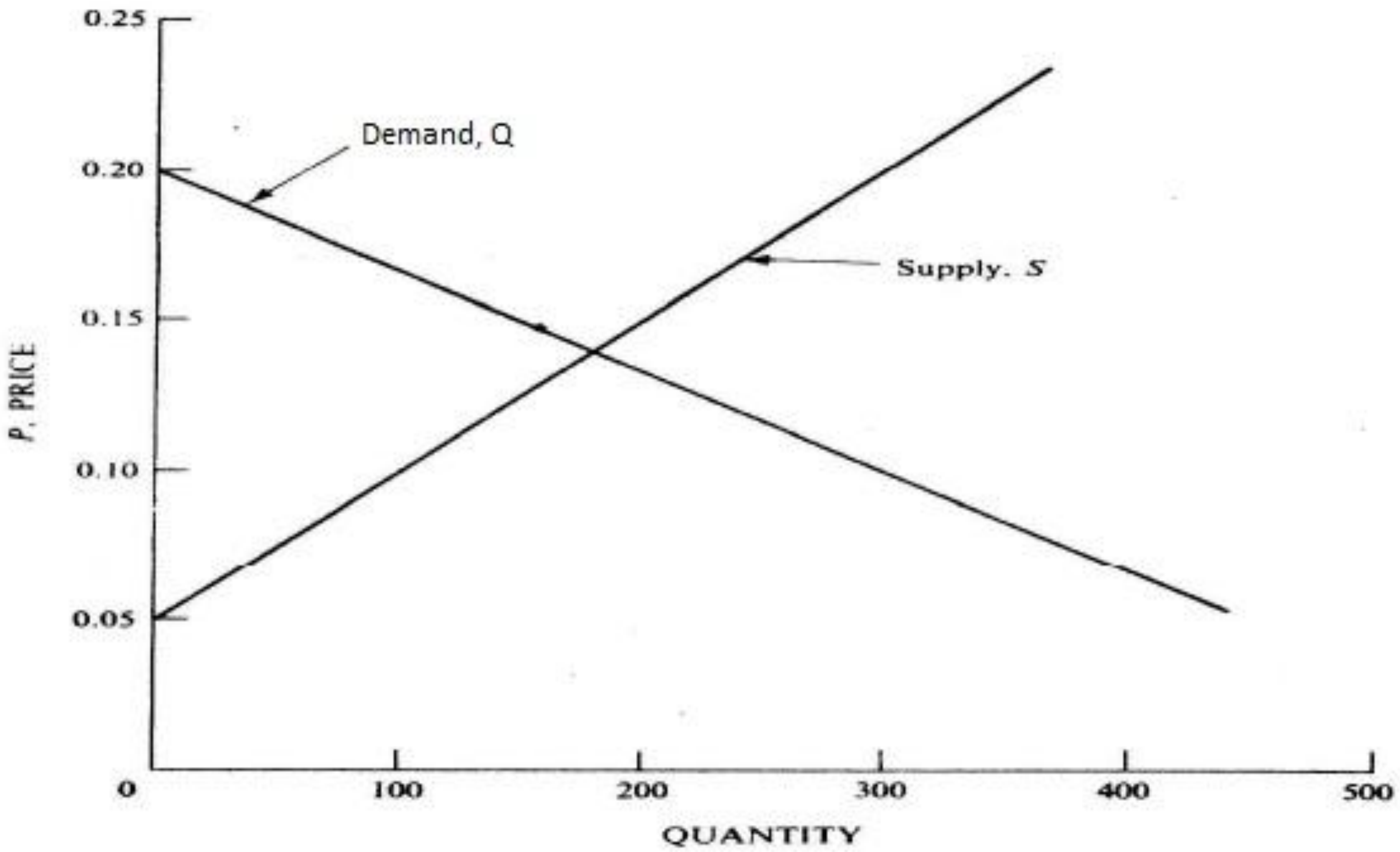

Figure: Linear Market Model

- If conditions remain stable, the price will settle to the point at which the two lines cross, because that is where the supply equals the demand.
- Since the relationships have been assumed linear, the complete market model can be written mathematically as follows:

$$Q = a - bP$$

$$S = c + dP$$

$$S = Q$$

- For the model to correspond to normal market conditions in which demand goes down and supply increases as price goes up the coefficients b and d need to be positive numbers.
- For realistic, positive results, the coefficient a must also be positive. Above figure has been plotted for the following values of the coefficients:

- a=600
- b=3000
- c=-100
- d=2000

- The fact that linear relationships have been assumed allows the model to be solved analytically. The equilibrium market price, in fact, is given by the following expression:

$$\boxed{P = \frac{a-c}{b+d}}$$

- With the chosen values, the equilibrium price is 0.14, which corresponds to a supply of 180.
- More usually, the demand will be represented by a curve that slopes downwards, and the supply by a curve that slopes upwards as shown below.

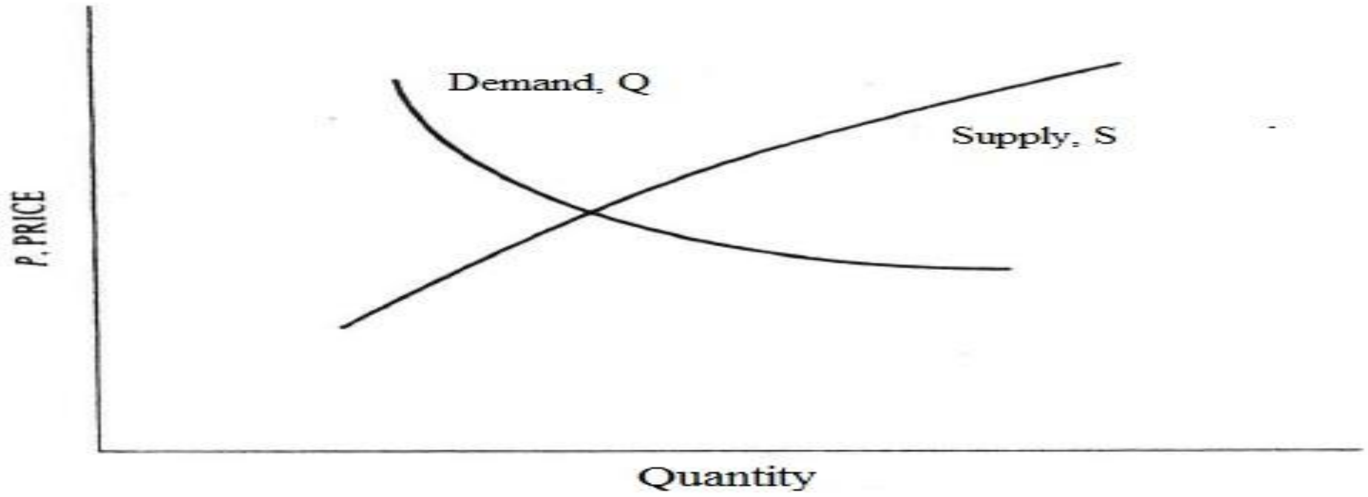

Figure: Non-Linear Market Model

- For such scenario, it may not then be possible to express the relationships by equations that can be solved.
- **Drawing the curves to scale and determining graphically where they intersect** is one such method. In practice, **it is difficult to get precise values** for the coefficients of the model.
- **Observations over an extended period of time can be performed.** The values depend upon economic factors, so the observations will usually attempt to correlate the values with the economy allowing the model to be used as a means of forecasting changes in market conditions for anticipated economic changes.

Dynamic Mathematical Models

- A dynamic mathematical model allows the changes of system attributes to be derived as a function of time.
- The derivation may be made with an analytical solution or with a numerical computation, depending upon the complexity of the model.
- The equation that was derived to describe the behavior of a **car wheel is an example of a dynamic mathematical model**. In this case, an equation that can be solved analytically.
- The equation is written as,

$$\boxed{\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = \omega^2 F(t)}$$

where $\zeta \rightarrow$ Zeta, $2\zeta\omega = \frac{D}{M}$ and $\omega^2 = \frac{K}{M}$

- Expressed in this form, solutions can be given in terms of the variable ωt.
- The plot between x and ωt for various values of ζ is shown in figure below.

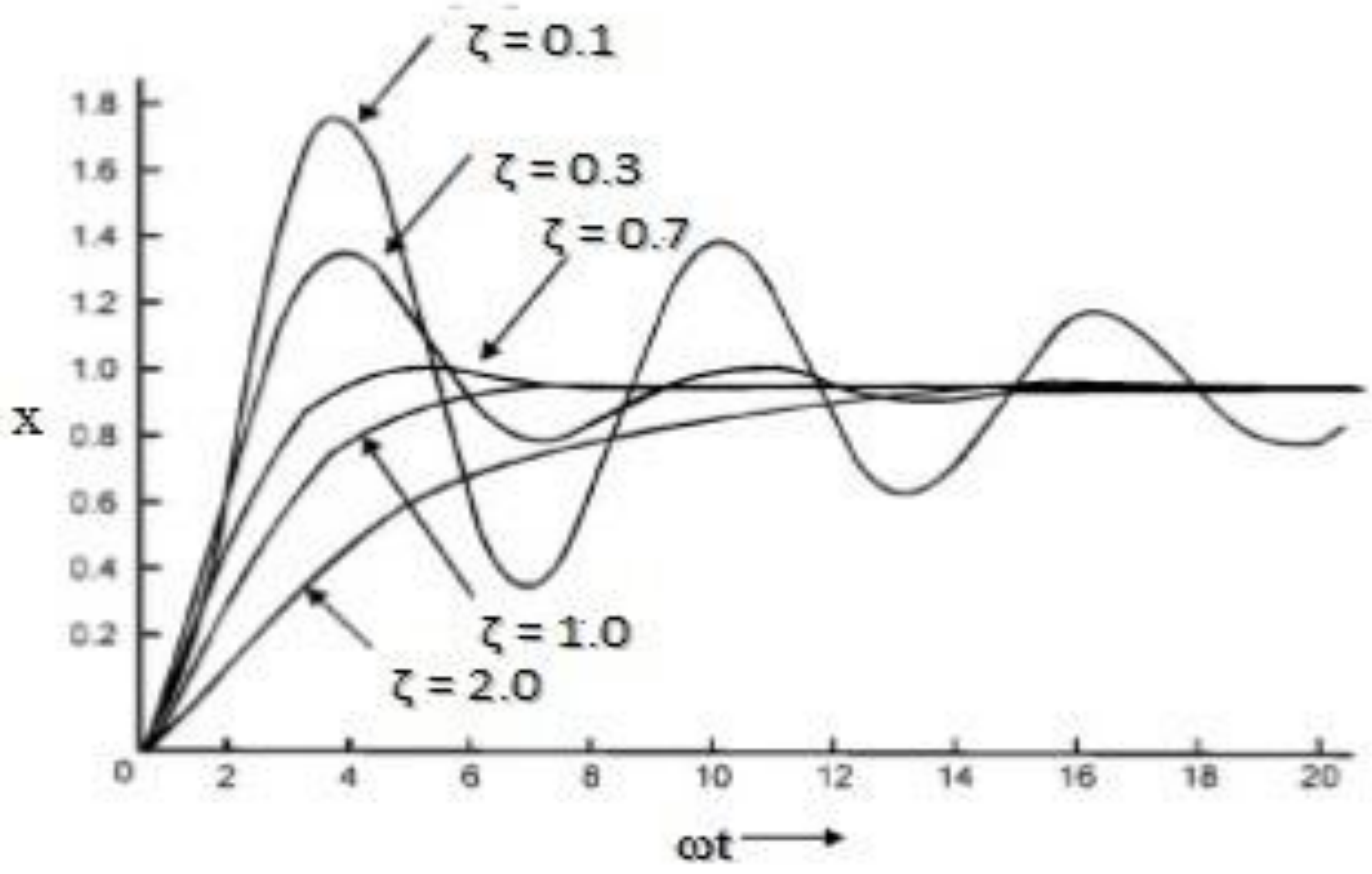

Figure: Solution of Second Order Differential Equations

- Solutions are shown for several values of ζ and it can be seen that when ζ is less than 1, the motion is oscillatory.
- The factor ζ is called the **damping ratio** and, when the motion is oscillatory, the frequency of oscillation is determined from the formula:

$$\boxed{\omega = 2\pi f}$$

where f is the number of cycles per second.

- The condition for the motion to occur without oscillation requires that $\zeta \geq 1$. It can be deduced from the definition of that the condition requires that $D^2 \geq 4MK$ since $2\zeta\omega = \frac{D}{M}$ and $\omega^2 = \frac{K}{M}$.

Principles Used In Modeling

1. **Block-Building**: The description of the system should be organized in **a series of blocks**. The main aim of these blocks is **to simplify the specifications of the interactions within the system.** The system as a whole can be described as the interconnections within the system.
2. **Relevance:** The model should only include those aspects that are relevant to the study objectives. Irrelevant information in the system might not do any harm, it should be excluded because it increases the complexity of the model.
3. **Accuracy:** The accuracy of the information gathered for the model should be considered. For example in the aircraft system the accuracy with which movement of the aircraft is described depends upon the representation of the airframe.
4. **Aggregation:** A further factor to be considered is the **extent to which the number of individual entities can be grouped together into larger entities.** In some cases, it may be necessary to construct artificial entities through the process of aggregation. Similar considerations of aggregation should be given to representation of activities.