P-series or Harmonic Series

feb - 04

1) $\sum(\sqrt{n^2+1}-n)$

=> sol^n.

general term of the given series

$$u_n = \sqrt{n^2+1} - n$$

$$= \frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{(\sqrt{n^2+1}+n)}$$

~~taking~~ $u_n = \frac{1}{\sqrt{n^2+1}+n}$

taking ~~the~~ series $\sum v_n = \sum \frac{1}{n}$ & its general term

$$v_n = \frac{1}{n}$$

$$\lim \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}+n}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}+1}$$

$= \frac{1}{2}$ is finite & non-zero

~~Here~~ ~~we~~ we know that $\sum v_n$ is divergent.
There fore by the comparison test $\sum u_n$ is also divergent.

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6) $\Sigma\left(\sqrt[3]{x^3+1} - n\right)$

→ given,

$$\Sigma\left(\sqrt[3]{x^3+1} - n\right)$$

The general term of the given series.

$$U_n = (n^3+1)^{1/3} - n.$$

$$= \frac{\{(n^3+1)^{1/3} - n\}\{\{(n^3+1)^{1/3}\}^2 + (n^3+1)^{1/3}.n + n^2\}}{[\{(n^3+1)^{1/3}\}^2 + (n^3+1)^{1/3}.n + n^2]}$$

$$= \frac{\{(n^3+1)^{1/3}\}^3 - n^3}{[\{(n^3+1)^{1/3}\}^2 + (n^3+1)^{1/3}.n + n^2]}$$

$$= \frac{\cancel{n^3}+1-\cancel{n^3}}{[\{(n^3+1)^{1/3}\}^2 + n.(n^3+1)^{1/3} + n^2]}$$

$$U_n = \frac{1}{\{(n^3+1)^{1/3}\}^2 + n(n^3+1)^{1/3} + n^2]}$$

taking the series $\Sigma V_n = \Sigma \frac{1}{n^2}$
whose general term in $V_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \left\{ \frac{1}{[\{(n^3+1)^{1/3}\}^2 + n(n^3+1)^{1/3} + n^2]} \cdot \frac{n^2}{1} \right]$$

$$= \lim_{n \to \infty} \left\{ \frac{n^2}{n^2 \left[\left\{ \left(1+\frac{1}{n^3}\right)^{1/3} \right\}^2 + \left(1+\frac{1}{n^3}\right)^{1/3} + 1 \right]} \right.$$

$$= \frac{1}{\left\{ \left(1+\frac{1}{\infty^3}\right)^{1/3} \right\}^2 + \left(1+\frac{1}{\infty^3}\right)^{1/3} + 1}$$

$\lim \frac{U_n}{V_n} = \frac{1}{3}$ which is finite & none-zero

since

$\Sigma V_n = \Sigma \frac{1}{n^2}$ convergent by p-test

Then by limit comparison test.

ΣU_n in also comparsion test.

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Q) $\Sigma \sqrt{n^4-1} - n^2)$

$\Rightarrow$ given

$$\Sigma\left(\sqrt{n^4-1} - n^2\right)$$

The general term of given series

$$u_n = \sqrt{n^4-1} - n^2$$

$$= \frac{\left(\sqrt{n^4-1} - n^2\right)\left(\sqrt{n^4-1} + n^2\right)}{\sqrt{n^4-1} + n^2}$$

$$= \frac{n^4 - 1 - n^4}{\sqrt{n^4-1} + n^2}$$

$$= \frac{-1}{\sqrt{n^4-1} + n^2}$$

taking the series $\Sigma v_n = \Sigma \frac{1}{n^2}$
whose second last is $\frac{1}{n^2}$

now

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \frac{-1}{\sqrt{n^4-1} + n^2} \cdot n^2$$

$$= \lim_{n \to \infty} \frac{-n^2}{n^2\left\{\left(1 - \frac{1}{n^4}\right)^{1/2} + 1\right\}}$$

$$= \frac{-1}{\left(1 - \frac{1}{\infty^4}\right)^{1/2} + 1}$$

$$= \frac{-1}{1+1}$$

$$= \frac{-1}{2}$$

which is finite & non-zero

$\Sigma U_n = \Sigma \sqrt{n^4 - 1} - n^2$ is also convergent by limit comparision test.

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De' Alembert Ratio test

If ΣU_n be the series of positive term

If $\lim_{n \to \infty} \frac{U_{n+1}}{V_n} = l$ then the series.

- convergent for $l < 1$
- divergent for $l > 1$
- ~~convergent~~ test for $l = 1$

Root test :- let ΣU_n be the series of positive term. If $\lim_{n \to \infty} \sqrt[n]{u_r} = l$ then the series is.

convergent for $l < 1$

divergent for $l > 1$

f test for $l = 1$

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Ex-28

5) $x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + \dots + \frac{n^2-1}{n^2+1}x^n + \dots, x>0$

$\Rightarrow$ sol^n: given

$$x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + \dots + \frac{n^2-1}{n^2+1}x^n + \dots$$

general term of the given series is

$$u_n = \frac{(n^2-1)}{(n^2+1)}x^n$$

$$u_{n+1} = \frac{\{(n+1)^2-1\}}{\{(n+1)^2+1\}}x^{n+1}$$

by ratio test

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left[\frac{\{(n+1)^2-1\}}{\{(n+1)^2+1\}}\cdot x^{n+1}\frac{(n^2+1)}{(n^2-1)\cdot x^n}\right]$$

$$= \lim_{n\to\infty}\left\{\frac{\cancel{n^2}}{\cancel{n^2}}\frac{[\{(1+\frac{1}{n})^2-\frac{1}{n^2}\}\cancel{n^2}(1+\frac{1}{n^2})]}{[\{(1+\frac{1}{n})^2+\frac{1}{n^2}\}\cancel{n^2}(1-\frac{1}{n^2})]}\cdot x\right\}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = 1.x. = x.$$

by ratio test Σu_n is convergent for $x<1$

divergent for $x>1$

& test fail for $x=1$

When $x=1$ given series become,

$$1+\frac{3}{5}+\frac{8}{10}+\frac{15}{17}+\cdots+\frac{n^2-1}{n^2+1}+\cdots$$

whose general term $U_n=\frac{n^2-1}{n^2+1}$

$$\lim_{n\to\infty} U_n=\lim_{n\to\infty}\frac{n^2-1}{n^2+1}=\lim_{n\to\infty}\frac{\cancel{n^2}\left(1-\frac{1}{n^2}\right)}{\cancel{n^2}\left(1+\frac{1}{n^2}\right)}$$

$\lim_{n\to\infty} U_n=1\neq 0$. which is divergent for $x=1$

$\therefore$ Thus the given seire, is convergent for $x<1$ & divergent for $x\geq 1$

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(v) $2x + \frac{9x^2}{2!} + \frac{64}{3!}x^3 + \cdots + \frac{(n+1)}{n^{n+1}}x^n$

=> Sol^n

$2x + \frac{9}{2!}x^2 + \frac{64}{3!}x^3 + \cdots \frac{(n+1)^2}{n^{n+1}}x^n$

the general term of the given series is

$$U_n = \frac{(n+1)^n x^n}{n^{n+1}}$$

$$U_{n+1} = \frac{(n+2)^{n+1} x^{n+1}}{(n+1)^{n+2}}$$

now by ratio test

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty}\left[\frac{(n+2)^{n+1}x^{n+1}}{(n+1)^{n+2}} \cdot \frac{n^{n+1}}{(n+1)^n x^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(n+2)^n (n+2)\cdot n^n \cdot n}{(n+1)^n (n+1)^2 (n+1)^n}\cdot x\right]$$

$$= \lim_{n\to\infty}\left[\frac{(n+2)^n (n+2)\cdot n}{(n+1)^{2n}\cdot n^2 (1+\frac{1}{n})^2}\cdot x\right]$$

$$= \lim_{n\to 0}\left[\frac{n^n\left[(1+\frac{2}{n})^n n (1+\frac{2}{n})\cdot n^n \cdot n x\right]}{n^n \cdot n^n (1+\frac{1}{n})^{2n} n^2 (1+\frac{1}{n})^2}\right]$$

$$= \lim_{n\to\infty} \frac{(1+\frac{2}{n})^n}{(1+\frac{1}{n})^{2n}} \cdot \lim_{n\to\infty} \frac{(1+\frac{2}{n})}{(1+\frac{1}{n})^2}\cdot x.$$

$$= \lim_{n\to\infty} \frac{(1+\frac{2}{n})^n}{(1+\frac{1}{n})^{2n}} 1\cdot x$$

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$$= \lim_{n\to\infty} \frac{\left\{\left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}}\right\}^2}{\left[\left(1+\frac{1}{n}\right)^n\right]^2} x$$

$$= \frac{e^2}{e^2} x$$

$$= x$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$$

by ratio test

given series is convergent for $x<1$

divergent for $x>1$

test fail for $x=1$

When $x=1$ given series become

$$2 + \frac{9}{8} + \frac{64}{81} + \cdots + \frac{(n+1)^n}{n^{n+1}} + \cdots$$

whose general term is.

$$u_n = \frac{(n+1)^n}{n^{n+1}}$$

taking the series $\Sigma v_n = \Sigma \frac{1}{n}$

whose general term $v_n = \frac{1}{n}$

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now

$$\lim_{n\to\infty} \left[\frac{(n+1)^{n}}{n^{n+1}} \cdot \frac{n}{1} \right]$$

$$= \lim_{n\to\infty} \frac{n^n \left(1+\frac{1}{n}\right)^n \cdot n}{n^n \cdot n} = e$$

$\therefore \lim_{n\to\infty} \frac{U_n}{V_n} = e$ which is finite & non-zero.

since $\Sigma V_n = \Sigma \frac{1}{n}$ is divergent by p-test

ie $p = 2$.

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13) $\Sigma \frac{1}{n^n}$

→ sol

given series $\cdot \Sigma \frac{1}{n^n}$

general term of the given series

$$U_n = \frac{1}{n^n}$$

by root test

$$\lim_{n\to\infty} \sqrt[n]{U_n} = \lim_{n\to\infty} (U_n)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n^n}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{n^{n \cdot \frac{1}{n}}}$$

$$= \lim_{n\to\infty} \cdot \frac{1}{n} = \frac{1}{\infty} = 0$$

$$\lim_{n\to\infty} \sqrt[n]{U_n} = 0 < 1$$

by the root test given series is convergent for all value of r.

10) $1 + \frac{2^2}{2!} + \frac{3^3}{3!} + \frac{4^4}{4!} + \cdots + \frac{n^n}{n!} + \cdots$

$\rightarrow$ sol^n,

$$U_n = \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

$$= 7e \quad = 2.7180\ldots \quad 71$$

the series is divergent

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