ELECTRIC FLUX DENSITY, GAUSS' LAW, AND DIVERGENCE

3.1 INTRODUCTION

After having investigated Coulomb's law, we will now try to explore some concepts on Gauss' law and its applications. But before that we should have clear understanding of electric flux and electric flux density. That is why, we will start with Faraday's experment.

3.2 FARADAY'S EXPERIMENT

For his experiment, he constructed a pair of concentric metallic spheres, the outer sphere was a combination of two hemispheres.

The steps consisted

(i) The inner sphere was given a known positive charge

(ii) The two hemispheres were joined to form a single and were kept around the charged sphere. He kept about 2 cm of dielectric material between them.

(iii) The outer sphere was discharged by connecting it momentarily to ground.

(iv) The outer sphere was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.

Faraday found that the total charge on the outer sphere was equal in magnitude to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres.

This led to a conclusion that there was some sort of "displacement" from the inner sphere to the outer sphere which was independent of the medium & this is referred now as displacement, displacement flux, or simply electric flux.

The electric flux generated was directly proportional to the charge on the inner sphere. In SI units, the constant of proportionality is unity which results

$$\psi = Q$$

where ψ (psi) = electric flux (in coulombs), Q = total charge in the inner sphere

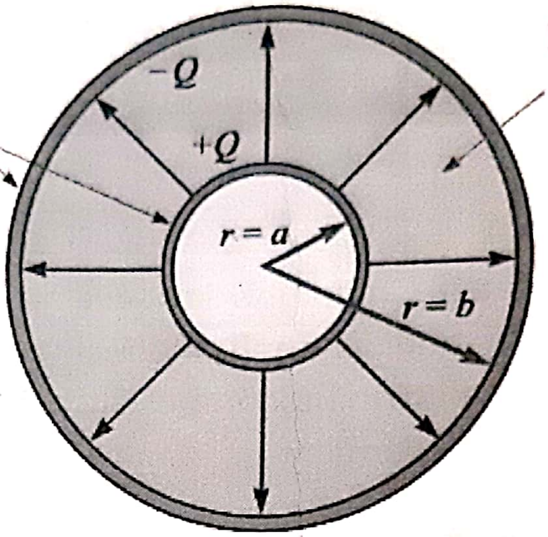


Figure 3.1 The electric flux in the region between a pair of charged concentric spheres. The direction and magnitude of D are not functions of the dielectric between the spheres.

Referring to Figure 3.1, the electric flux density is in the radial direction and has a value of

$$\vec{D}\,|_{r=a} = \frac{Q}{4\pi a^2}\,\hat{a}_r \text{ (inner sphere)}, \quad \vec{D}\,|_{r=b} = \frac{Q}{4\pi b^2}\,\hat{a}_r \text{ (outer sphere)}$$

At a radial distance r,

$$\vec{D} = \frac{Q}{4\pi r^2}\,\hat{a}_r, \qquad a \le r \le b$$

3.3 ELECTRIC FLUX AND FLUX DENSITY

By definition, electric flux Ψ (a scalar field) originates on positive charge and terminates on negative charge. In the absence of negative charge, the flux Ψ terminates at infinity.

In Figure 3.2 (a), the lines leave + Q and terminate on – Q. This assumes that the two charges are of equal magnitude. The case of positive charge with no negative charge in the region is illustrated in Figure 3.2 (b).

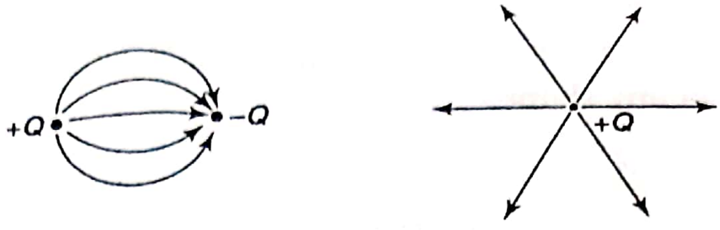


Figure 3.2 Illustration of electric flux.

If in the neighbourhood of point P, the lines of flux have the direction of the unit vector $\hat{a}_R$ (see Figure 3.3) and if an amount of flux $d\Psi$ crosses the differential area dS, which is normal to $\hat{a}_R$, then the electric flux density at P is

$$\vec{D} = \frac{d\Psi}{dS}\,\hat{a}_R \quad (C/m^2)$$

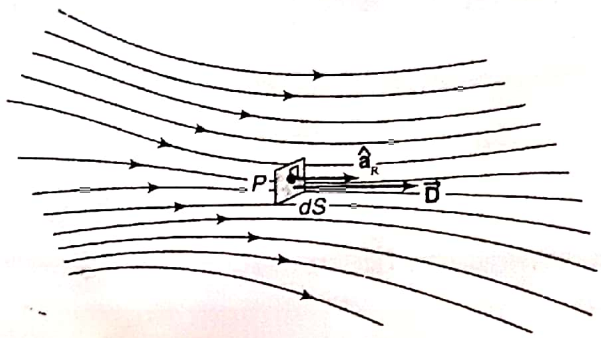


Figure 3.3 Illustration of electric flux density.

Thus, the magnitude of electric flux density denoted by $\vec{D}$ is given by the number of flux lines crossing a surface normal to the lines divided by the surface area.

The relation between electric flux density and electric field intensity is

$$\vec{D} = \varepsilon_0 \vec{E} \text{ (for free space)}$$

$$\vec{D} = \varepsilon \vec{E} \text{ (for any other medium)}$$

3.4 GAUSS' LAW

The generalizations of Faraday's experiment is Gauss' law. It states that the electric flux passing through any closed surface is equal to the total charge enclosed by the surface.

Mathematical Explanation

Consider a distribution of charge, let's say a cloud of point charges surrounded by a closed surface of any shape.

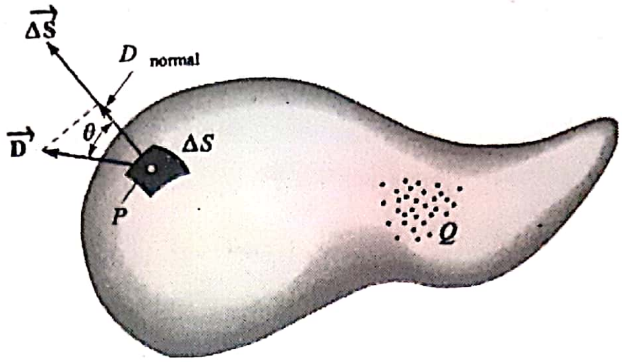


Figure 3.4 The electric flux density $\vec{D}$ at P due to charge Q. The total flux passing through $\Delta\vec{S}$ is $\vec{D}.\Delta\vec{S}$.

Let the total charge enclosed is Q. At every point on the surface the electric flux density $\vec{D}$ will have some value. Generally, $\vec{D}$ varies in magnitude & direction from one point on the surface to another.

Now, consider an incremental element of surface ΔS whose direction is specified by the normal to the surface. Small amount of flux crossing incremental area ΔS is

$$\Delta\psi = \text{flux crossing } \Delta S = D_{normal}\,\Delta S$$

From Figure 3.4, $D_{normal} = D\cos\theta$ $\quad\therefore\ \Delta\psi = D\cos\theta\,\Delta S = \vec{D}.\vec{\Delta S}$

The total flux passing through the closed surface is obtained by adding the differential contributions crossing each surface element $\vec{\Delta S}$.

$$\psi = \int d\psi = \oint_{\text{closed surface}} \vec{D}.d\vec{S}$$

$$\therefore\quad \psi = \oint_S \vec{D}.d\vec{S} = \text{charged enclosed} = Q$$

which is the mathematical statement of Gauss' law.

As we use $Q = \int_{vol} \rho_v\,dv$ expression in most of our problems, Gauss' law can now be written in terms of charge distribution as

$$\psi = Q = \oint_S \vec{D}.d\vec{S} = \int_{vol} \rho_v \, dv$$

$$\therefore \oint_S \vec{D}.\,d\vec{S} = \int_{vol} \rho_v \, dv$$

Proof:

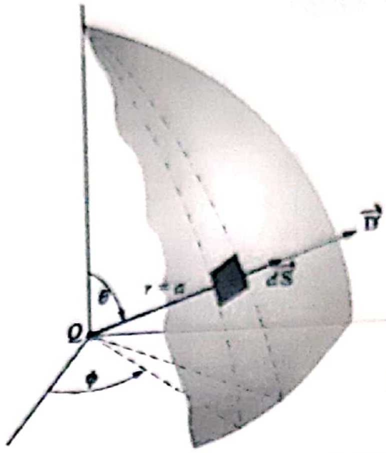


Figure 3.5 Application of Gauss' law to the field of a point charge Q on a spherical closed surface of radius a.

Consider a point charge Q at the origin of a spherical co-ordinate system, radius of sphere being a. The electric flux density $\vec{D}$ is everywhere normal to the spherical surface and has a constant magnitude at every point on it.

The electric field intensity of the point charge is

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{a}_r$$

$$\therefore \vec{D} = \varepsilon_0 \vec{E} = \frac{Q}{4\pi r^2}\,\hat{a}_r$$

At the surface of the sphere, r = a

$$\therefore \vec{D} = \frac{Q}{4\pi a^2}\,\hat{a}_r$$

In spherical coordinate system,

$$d\vec{S} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r = a^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r$$

The total electric flux passing through the closed surface can be obtained as

$$\psi = \oint_S \vec{D}.\,d\vec{S}$$

$$= \oint_S \left(\frac{Q}{4\pi a^2}\hat{a}_r\right) . (a^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r)$$

$$= \oint_S \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

$$= \frac{Q}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \, d\theta \, d\phi$$

$$= \frac{Q}{4\pi} \int_{\phi=0}^{2\pi} d\phi \, [-\cos\theta]_0^{\pi}$$

$$= \frac{Q}{4\pi} \int_{\phi=0}^{2\pi} 2 \, d\phi = \frac{Q}{2\pi}[\phi]_0^{2\pi} = \frac{Q}{2\pi} \times (2\pi - 0)$$

$\therefore \psi = Q$ *proved.*

3.5 APPLICATIONS OF GAUSS' LAW: SOME SYMMETRICAL CHARGE DISTRIBUTIONS

Before applying Gauss' law, it is noteworthy that closed surface should be chosen in such a way that

(i) $\vec{D}$ is everywhere normal or tangent to the surface, so that

$\vec{D} . d\vec{S} = |\vec{D}| \, |d\vec{S}| \cos 0° = D \, dS$, when $\vec{D}$ is normal to the surface

$\vec{D} . d\vec{S} = |\vec{D}| \, |d\vec{S}| \cos 90° = 0$, when $\vec{D}$ is tangent to the surface

(ii) $\vec{D}$ will be a constant if not equal to zero.

Otherwise, the calculations may turn even more complex.

Thus, we must choose a surface that has some of the symmetry exhibited by the charge distribution. The choice of an appropriate gaussian surface, where there is symmetry in the charge distribution comes from intuitive reasoning and a slight degree of maturity in the application of Coulomb's law.

(a) Field due to a Point Charge

Consider a point charge Q at the origin of a spherical co-ordinate system. To determine $\vec{D}$ at a point P, we choose a gaussian surface to be a spherical surface containing that point. $\vec{D}$ is everywhere normal to the surface and has the same value at all points on the surface.

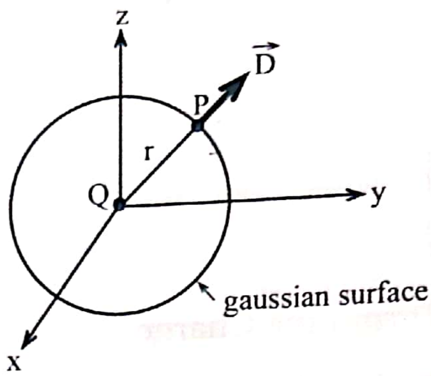


Figure 3.6 Gaussian surface about a point charge.

From Gauss' law,

$$\psi = \oint_S \vec{D}.\, d\vec{S} = Q \;\ldots\ldots\ldots\; \text{(i)}$$

We have $\vec{D} = D_r \hat{a}_r$, $d\vec{S} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r$ (for spherical coordinate system)

$$\therefore \quad \oint_S \vec{D}.\, d\vec{S} = \oint_S (D_r \hat{a}_r) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r)$$

$$= D_r r^2 \oint_S \sin\theta \, d\theta \, d\phi$$

$$= D_r r^2 \int_{\phi=0}^{2\pi} [-\cos\theta]_0^{\pi} \, d\phi$$

$$= D_r r^2 \int_{\phi=0}^{2\pi} 2 \, d\phi = 2D_r r^2 [\phi]_0^{2\pi} = 2D_r r^2 [2\pi - 0] = D_r 4\pi r^2$$

From equation (i),

$$\oint_S \vec{D}.\, d\vec{S} = Q$$

or, $D_r 4\pi r^2 = Q$

$$\therefore \quad D_r = \frac{Q}{4\pi r^2}$$

Thus, $\vec{D} = \frac{Q}{4\pi r^2}\,\hat{a}_r$

Now, electric field intensity is given as

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0}$$

$$\boxed{\therefore\ \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{a}_r}$$

(b) Field due to the Uniform Line Charge

Consider the uniform line charge distribution having line charge density ρ_L, lying along the z-axis extending from $-\infty$ to $+\infty$. To determine $\vec{D}$ at a point, let's take the gaussian surface (closed surface) as a right circular cylinder of length L and radius ρ containing that point.

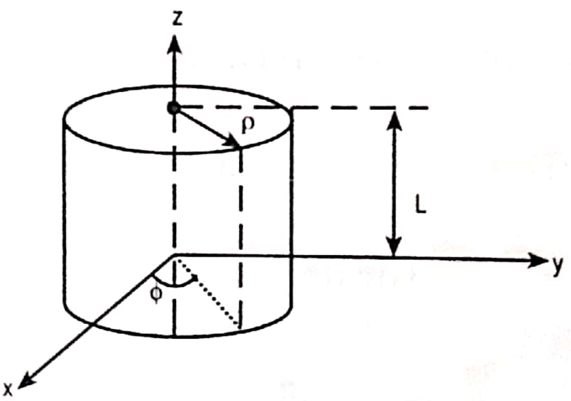


Figure 3.7 An infinite uniform line charge is enclosed by a gaussian surface (right circular cylinder).

The electric flux density $\vec{D}$ is constant in magnitude and everywhere perpendicular to the cylindrical surface. Since the line charge is infinitely long, it is noticeable that only the radial components of $\vec{D}$ exist i.e.,

$$\vec{D} = D_\rho\,\hat{a}_\rho \ \ \& \ \ \vec{D} = f(\rho)$$

From Gauss' law,

$$\psi = \oint_{cyl} \vec{D}.\vec{dS} = Q \text{ .... (i)}$$

To find $d\vec{S}$, we first observe that the directions of the top and the bottom surfaces are respectively toward + z and – z axis while the direction of the side surface is radially outwards.

From equation (i),

$$\int_{top} \vec{D}.d\vec{S} + \int_{bottom} \vec{D}.d\vec{S} + \int_{side} \vec{D}.d\vec{S} = Q$$

For top & bottom surfaces, $\theta = 90^o$

$$\text{So, } \vec{D}.d\vec{S} = D\,dS\cos\theta = D\,dS\cos 90^o = 0$$

For side surfaces, $\theta = 0°$

$$\text{So, } \vec{D}.d\vec{S} = D\,dS\cos 0° = D\,dS$$

Now, we have

$$0 + 0 + \int_{sides} \vec{D}.d\vec{S} = Q$$

$$\text{or, } \int_{sides} D\,dS = Q$$

For a cylindrical surface, $dS = \rho\, d\phi\, dz$

$$\text{or, } \int_{z=0}^{L} \int_{\phi=0}^{2\pi} D(\rho\, d\phi\, dz) = Q$$

As $D = D_\rho$ = constant, we write

$$D_\rho \rho \int_{z=0}^{L} \int_{\phi=0}^{2\pi} d\phi\ dz = Q$$

$$\text{or, } D_\rho \rho \int_{z=0}^{L} 2\pi\, dz = Q$$

$$\text{or, } D_\rho\, \rho\, 2\pi\, L = Q$$

$$\text{or, } D_\rho = \frac{Q}{2\pi\rho L}$$; Q = total charge enclosed by gaussian surface = $\rho_L\, L$

$$\text{or, } D_\rho = \frac{\rho_L\, L}{2\pi\, \rho\, L}$$

$$\text{and } E_\rho = \frac{}{\varepsilon_0} = \frac{}{2\pi\varepsilon_0\rho}$$

In vector form,

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\hat{a}_\rho}$$

Field due to the Infinite Coaxial Cable with Air Dielectric

Consider a coaxial cable of an infinite length with two coaxial cylindrical conductors, the inner conductor of radius a and the outer of radius b. Let ρ_s be the value of uniform surface charge density on the outer surface of the inner conductor. Consider the gaussian surface as a cylindrical surface of radius ρ and length L.

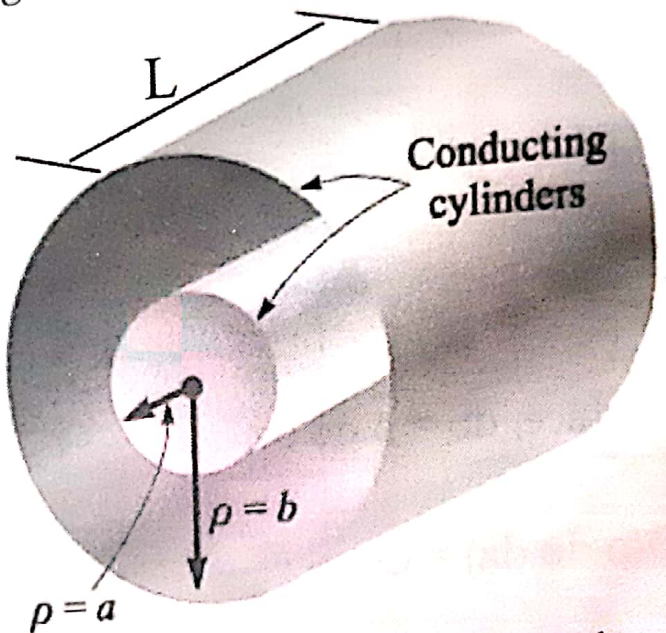


Figure 3.8 The two coaxial cylindrical conductors forming a coaxial cable.

We notice that by symmetry, only radial components of $\vec{D}$ exist i.e.,

$$\vec{D} = D_\rho \hat{a}_\rho \;\; \& \;\; \vec{D} = f(\rho)$$

se-I: For a < ρ < b

From Gauss' law,

$$\psi = \oint_{cyl} \vec{D}.d\vec{S} = Q \qquad \text{............ (i)}$$

We proceed like we have done before and obtain

$$\oint_{cyl} \vec{D}.d\vec{S} = D_\rho 2\pi\rho L \qquad \text{............... (ii)}$$

and Q = total charge enclosed by gaussian surface = $\int \rho_s \, dS$

where dS is the elemental surface area of the inner cylinder

$$\text{or, } Q = \rho_s \int_{\phi=0}^{2\pi} \int_{z=0}^{L} a \, d\phi \, dz$$

$$= a \, \rho_s \int_{\phi=0}^{2\pi} d\phi \, [z]_0^L \quad = a \, \rho_s \, L \, [\phi]_0^{2\pi}$$

$$\therefore \quad Q = 2\pi a L \, \rho_s \quad \text{..............} \text{(iii)}$$

Form equation (i), (ii), (iii),

$$D_\rho \, 2\pi\rho L = 2\pi \, a \, L \, \rho_s$$

$$\text{or, } D_\rho = \frac{a \, \rho_s}{\rho}$$

$$\text{or, } E_\rho = \frac{D_\rho}{\varepsilon_o} = \frac{a \, \rho_s}{\varepsilon_o \rho}$$

The problem of a coaxial cable is almost identical with that of the line charge and is an example which is extremely difficult to solve from the standpoint of Coulomb's law.

In vector form, we have

$$\boxed{\vec{E} = \frac{a \, \rho_s}{\varepsilon_o \rho} \, \hat{a}_\rho}$$

Case-II: For $\rho > b$, $\rho < a$

For $\rho > b$, the total charge enclosed would be zero, for there are equal and opposite charges on each conducting cylinder. This is because every line of eletric flux starting from the charge on the inner cylinder must terminate on a negative charge on the inner surface of the outer cylinder.

$$\psi = \oint_{cyl} \vec{D} \cdot d\vec{S} = Q$$

$$\text{or, } D_\rho \, 2\pi\rho L = 0$$

$$\therefore \quad D_\rho = 0 \Rightarrow \boxed{E_\rho = 0}$$

Also for $\rho < a$; $Q = 0$

$$\therefore \quad D_\rho = 0 \Rightarrow \boxed{E_\rho = 0}$$

This proves that the coaxial cable or capacitor has no external field and there is no field within the center conductor.

Expression of ρ_s for outer cylinder

Using, $Q_{outer\ cyl} = -Q_{inner\ cyl}$

$$Q_{outer\ cyl} = -2\pi aL\ \rho_{s,\ inner\ cyl}$$

But, $Q_{outer\ cyl}$ is also expressed as

$$Q_{outer\ cyl} = 2\pi bL\ \rho_{s,\ outer\ cyl}$$

or, $2\pi b L \rho_{s,\ outer\ cyl} = -2\pi a L \rho_{s,\ inner\ cyl}$

$$\boxed{\therefore \rho_{s,\ outer\ cyl} = \frac{-a}{b} \rho_{s,\ inner\ cyl}}$$

(d) Field near an Infinite Plate with Surface Charge Density ρ_s

Let ρ_s be the surface charge density on each side of the infinite plate. The fluxes are emanating from an infinite conductor plate on each side as shown. Consider the gaussian surface as a cylindrical surface enclosing a small area ΔS on the plate.

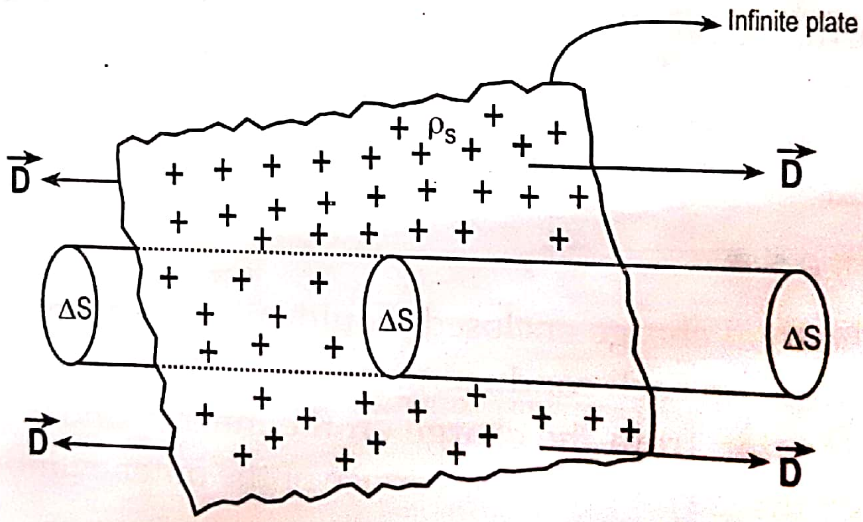


Figure 3.9 Illustration of an infinite plate with surface charge density on each side.

From Gauss' law,

$$\psi = \oint_{cyl} \vec{D}.d\vec{S} = Q \qquad \text{.............. (i)}$$

To find $d\vec{S}$, we first observe that the directions of top and the bottom surfaces of cylinder is parallel to the direction of the surface area of the plate while the direction of side surface of cylinder makes normal to the direction of the surface area of the plate.

From equation (i),

$$\int_{top} \vec{D}.d\vec{S} + \int_{bottom} \vec{D}.d\vec{S} + \int_{sides} \vec{D}.d\vec{S} = Q$$

For top & bottom surfaces, $\theta = 0^\circ$

So, $\vec{D}.d\vec{S} = D\,dS\cos\theta = D\,dS\cos 0^\circ = D\,dS$

For side surfaces, $\theta = 90^\circ$

$\therefore \quad \vec{D}.d\vec{S} = D\,dS\cos\theta = D\,dS\cos 90^\circ = 0$

$$\therefore \quad \int_{top} \vec{D}.d\vec{S} + \int_{bottom} \vec{D}.d\vec{S} + 0 = Q$$

or, $\int_{top} D\,dS + \int_{bottom} D\,dS = Q$

or, $D\,\Delta S + D\,\Delta S = Q$

or, $2D\,\Delta S = Q$

But, $Q = \rho_s\,\Delta S + \rho_s\,\Delta S$ (for both sides of the plate)

$= 2\rho_s\Delta S$

or, $2D\,\Delta S = 2\rho_s\,\Delta S \Rightarrow D = \rho_s$

and $E = \frac{D}{\varepsilon_o} = \frac{\rho_s}{\varepsilon_o}$

In vector form, $\boxed{\vec{E} = \frac{\rho_s}{\varepsilon_o}\hat{a}_N}$

When only one side of the plate is considered, surface charge reduces to half.

$$\boxed{\therefore\ \vec{E} = \frac{\rho_s}{2\varepsilon_o}\hat{a}_N}$$

3.6 APPLICATION OF GAUSS' LAW: DIFFERENTIAL VOLUME ELEMENT

In this section, we apply Gauss' law choosing a gaussian surface that is not symmetrical.

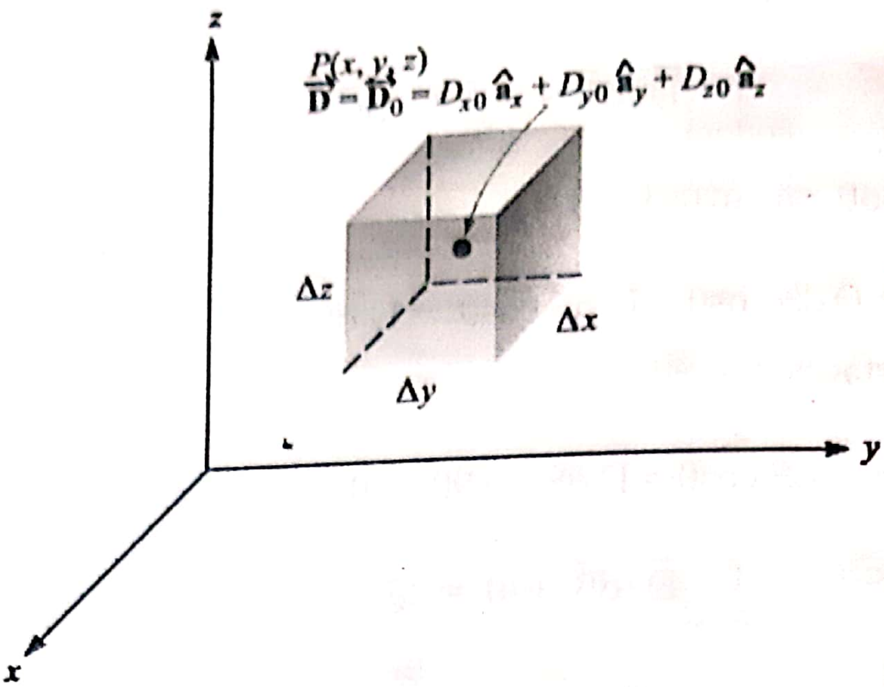


Figure 3.10 A differential-sized gaussian surface about the point P is used to investigate the space rate of change of $\vec{D}$ in the neighbourhood of P.

Consider any point P located by a rectangular coordinate system and let the value of $\vec{D}$ at the point P is

$$\vec{D}_o = D_{xo}\,\hat{a}_x + D_{yo}\,\hat{a}_y + D_{zo}\,\hat{a}_z$$

Let us choose a small rectangular box as a gaussian surface having sides of lengths Δx, Δy, and Δz.

From Gauss' law,

$$\psi = \oint_S \vec{D}.\,d\vec{S} = Q \qquad \text{..........} \ (x)$$

Here,

$$\oint_S \vec{D}.\,d\vec{S} = \int_{front} + \int_{back} + \int_{left} + \int_{right} + \int_{top} + \int_{bottom} \qquad \text{................} \ (y)$$

Now, $\int_{front} = \vec{D}_{front} \,.\, \Delta\vec{S}_{front} = (D_{x,\,front}\,\hat{a}_x)\,.\,(\Delta y\,\Delta z\,\hat{a}_x) = D_{x,\,front}\,\Delta y\,\Delta z$

where $D_{x,\,front} = D_{xo} + \dfrac{\Delta x}{2} \times$ rate of change of D_x with $x = D_{xo} + \dfrac{\Delta x}{2}\dfrac{\partial D_x}{\partial x}$

D_x varies along x, y, and z but we are concerned only the variance of D_x in x direction and this is the reason we take partial derivative while calculating $D_{x,front}$.

$$\therefore \int_{front} = \left(D_{xo} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y\,\Delta z \quad \text{.............} (i)$$

Again, $\int_{back} = \vec{D}_{back} . \Delta\vec{S}_{back} = (D_{x,\,back}\,\hat{a}_x) . (-\Delta y\,\Delta z\,\hat{a}_x) = -\ D_{x,\,back}\,\Delta y\,\Delta z$

where $D_{x,\,back} = D_{xo} - \dfrac{\Delta x}{2}\dfrac{\partial D_x}{\partial x}$

$$\therefore \int_{back} = -\left(D_{xo} - \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y\,\Delta z = \left(-D_{xo} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y\,\Delta z \quad \text{.............} (ii)$$

Adding equation (i), (ii)

$$\int_{front} + \int_{back} = \left(D_{xo} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y\,\Delta z + \left(-D_{xo} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y\,\Delta z$$

$$= \frac{\partial D_x}{\partial x}\Delta x\,\Delta y\,\Delta z \quad \text{..................} (iii)$$

Similarly, $\displaystyle\int_{right} + \int_{left} = \frac{\partial D_y}{\partial y}\Delta x\,\Delta y\,\Delta z$ .... (iv)

$$\int_{top} + \int_{bottom} = \frac{\partial D_z}{\partial z}\Delta x\,\Delta y\,\Delta z \quad \text{..................} (v)$$

Putting equation (iii), (iv), (v) in equation (y), we get

$$\oint_S \vec{D}.\,d\vec{S} = \frac{\partial D_x}{\partial x}\Delta x\,\Delta y\,\Delta z + \frac{\partial D_y}{\partial y}\Delta x\,\Delta y\,\Delta z + \frac{\partial D_z}{\partial z}\Delta x\,\Delta y\,\Delta z$$

$$= \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x\,\Delta y\,\Delta z$$

$$= \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta v$$

Hence, equation (x) becomes

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta v = Q$$

where Q = charged enclosed in volume Δv.

3.7 DIVERGENCE

The divergence of the vector flux density $\vec{A}$ is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero. Mathematically, divergence is illustrated as

$$\boxed{\operatorname{div}\vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A}.d\vec{S}}{\Delta v}}$$

where $\vec{A}$ may represent velocity, temperature gradient, force, or any other vector field.

The divergence is an operation which is performed on a vector, but that the result is a scalar. It should be noted that divergence only tells us how much flux is leaving a small volume on a per unit–volume basis; no direction is associated with it.

Derivation

We have,

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta v = \oint_S \vec{D}.d\vec{S} = Q$$

$$\text{or,} \quad \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \frac{\oint_S \vec{D}.d\vec{S}}{\Delta v} = \frac{Q}{\Delta v}$$

Taking limit as $\Delta v \to 0$,

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S \vec{D}.d\vec{S}}{\Delta v} = \lim_{\Delta V \to 0} \frac{Q}{\Delta v}$$

$$\text{or,} \quad \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S \vec{D}.d\vec{S}}{\Delta v} = \rho_v \quad (\rho_v = \text{charge density})$$

$$\therefore \quad \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S \vec{D}.d\vec{S}}{\Delta v} \quad \text{..........} (i)$$

$$\text{and}\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\rho_v \quad \text{.........} \quad \text{(ii)}$$

Considering equation (i), we write more general term

$$\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \vec{A}.\,d\vec{S}}{\Delta v}$$

where $\vec{A}$ may represent velocity, temperature gradient, force, or any other vector field.

The left hand side got name, divergence & hence we write

$$\text{Divergence of } \vec{A} = \text{div } \vec{A} = \lim_{\Delta v\to 0}\frac{\oint_S \vec{A}.\,d\vec{S}}{\Delta v}$$

Physical Interpretation of Divergence

Divergence is applied to see whether a point in the field associated with vector acts as a source, sink, or neither source nor sink. A positive divergence for any vector quantity indicates a source of that vector quantity at that point. A negative divergence indicates a sink (absorber) and a zero divergence means neither source nor sink.

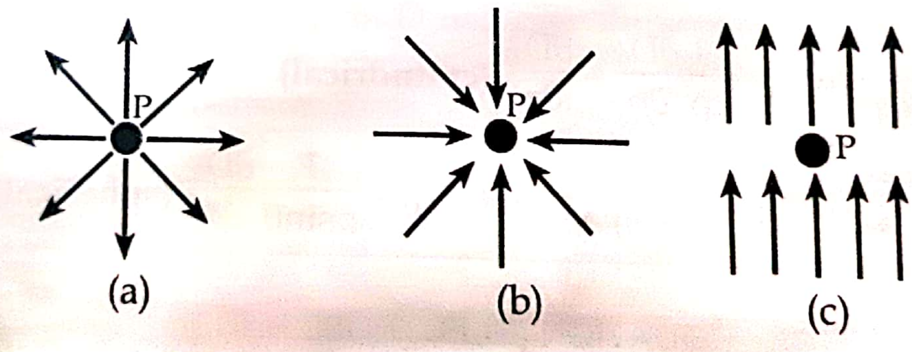


Figure 3.11 Illustration of the divergence of a vector field at P (a) positive divergence (b) negative divergence (c) zero divergence.

Physically, we may regard the divergence of the vector field $\vec{A}$ at a given point as a measure of how much the field diverges or emanates from that point. Figure 3.11 (a) shows that the divergence of a vector field at point P is positive because the vector diverges (or spreads out) at P. In Figure 3.11 (b) a vector field has negative divergence (or convergence) at P, and in Figure 3.11 (c) a vector field has zero divergence at P. The divergence of a vector field can also be viewed as simply the limit of the field's source strength per unit volume (or source density); it is positive at a source point in the field, and negative at a sink point, or zero where there is neither sink nor source.

Specific examples:

- If the divergence of the velocity of air is considered in a tire which has just been punctured by a nail, then we observe that the air is expanding as the pressure drops and that consequently there is a net outflow from any closed surface lying within the tire. Hence, the divergence of this velocity is greater than zero and each interior point may be considered as a source.
- When an evacuated light bulb is broken, there is momentarily a negative value for divergence in the space that was formerly the interior of the bulb. This makes every point as sink where divergence is negative.
- If the divergence of the velocity of water is considered in a hollow pipe with water entering one end, and leaving from the other end, then the divergence of this velocity is zero. Hence, any point within the pipe is therefore, neither a source nor a sink.

IMPORTANT EXPRESSIONS

$$\text{div } \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \quad \textbf{(rectangular)}$$

$$\text{div } \vec{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \quad \textbf{(cylindrical)}$$

$$\text{div } \vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi} \quad \textbf{(spherical)}$$

3.8 MAXWELL'S FIRST EQUATION (ELECTROSTATICS)

From Gauss' law,

$$\oint_S \vec{D}.\, d\vec{S} = Q$$

$$\text{or,} \quad \frac{\oint_S \vec{D}.\, d\vec{S}}{\Delta v} = \frac{Q}{\Delta v}$$

Let the volume shrinks to zero,

$$\lim_{\Delta v \to 0} \frac{\oint_S \vec{D}.d\vec{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$

$$\boxed{\text{or, div } \vec{D} = \rho_v}$$

which is **Maxwell's first equation** and is also called the **point form of Gauss' law**. Because that the divergence may be expressed as the sum of three partial derivatives i.e.,

div $\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$, Maxwell's first equation is also known as the **differential form of Gauss' law** and conversely, Gauss' law recognized as the integral form of Maxwell's first equation.

3.9 THE VECTOR OPERATOR ∇ AND THE DIVERGENCE THEOREM

The Vector Operator ∇

We know that divergence is an operation which is performed on a vector giving a scalar result just like two vectors giving scalar result. So, let's define something which when dotted with $\vec{D}$ will give the scalar $\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$.

We now define the del operator, ∇ as

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

$$\nabla.\vec{D} = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right).(D_x\hat{a}_x + D_y\hat{a}_y + D_z\hat{a}_z) = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

We have already obtained

$$\text{div } \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Hence, $\text{div } \vec{D} = \nabla.\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$

Divergence Theorem

Divergence theorem may be stated as follows:

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

This theorem is true for any vector field for which the appropriate partial derivatives exist. If we specifically let the electric flux density as a vector field, then divergence theorem has the form

$$\oint_S \vec{D} . d\vec{S} = \int_{vol} \nabla . \vec{D} \, dv$$

Proof:

From Gauss' law,

$$\oint_S \vec{D} . d\vec{S} = Q$$

$$\text{or, } \oint_S \vec{D} . d\vec{S} = \int_{vol} \rho_v \, dv$$

$$\text{or, } \oint_S \vec{D} . d\vec{S} = \int_{vol} (\nabla . \vec{D}) \, dv \quad (\because \nabla . \vec{D} = \text{div } \vec{D} = \rho_v)$$

Hence, proved.

PROBLEMS SOLVED AND SCRAMBLED

1. **A point charge of 20 nC is located at (4, –1, –3) & a uniform line charge ρ_L of -25 nC/m lies along the intersection of planes x = –4 and z = 6. Calculate the electric flux density $\vec{D}$ in cylindrical coordinate system at point (3, 1, 0).**

[2062 Bhadra]

Solution:

Electric field due to point charge is

$$\vec{E}_P = \frac{Q}{4\pi\varepsilon_o} \frac{\vec{R}}{R^3}$$

where $\vec{R} = (3, 1, 0) - (4, -1, -3) = -1\hat{a}_x + \hat{a}_y + 3\hat{a}_z$; $R = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

$$\therefore \quad \vec{E}_P = \frac{2\times10^{-9}\times9\times 10^{9} (-\hat{a}_x + 2\hat{a}_y + 3\hat{a}_z)}{(\sqrt{14})^3} = -3.436\hat{a}_x + 6.872\hat{a}_y + 10.309\hat{a}_z \text{ V/m}$$

Electric field due to line charge is

$$\vec{E}_L = \frac{\rho_L}{2\pi\varepsilon_o R}\hat{a}_R = \frac{\rho_L}{2\pi\varepsilon_0}\frac{\vec{R}}{R^2}$$

$$\vec{R} = \text{F.P.} - \text{S.P.} = (3, 1, 0) - (-4, 1, 6) = 7\hat{a}_x - 6\hat{a}_z\,;\ R = \sqrt{49 + 36} = 9.219$$

$$\therefore\quad \vec{E}_L = \frac{2 \times 9 \times 10^9 \times (-25) \times 10^{-9}\,(7\,\hat{a}_x - 6\hat{a}_z)}{(9.219)^2} = -37.059\,\hat{a}_x + 31.765\,\hat{a}_z\ \text{V/m}$$

Total electric field is given by

$$\vec{E} = \vec{E}_P + \vec{E}_L = (-3.436\,\hat{a}_x + 6.8728\,\hat{a}_y + 10.309\,\hat{a}_z) + (-37.059\,\hat{a}_x + 31.765\,\hat{a}_z)$$
$$= -40.495\,\hat{a}_x + 6.872\,\hat{a}_y + 42.075\hat{a}_z\ \text{V/m}$$

Electric flux density is given by

$$\vec{D} = \varepsilon_o\vec{E} = 8.854 \times 10^{-12}\,(-40.495\,\hat{a}_x + 6.872\,\hat{a}_y + 42.075\,\hat{a}_z)$$
$$= -358.380\,\hat{a}_x + 60.817\,\hat{a}_y + 372.36\hat{a}_z)\ \text{pC/m}^2$$

Expressing $\vec{D}$ in cylindrical coordinate system at (3, 1, 0), we have

$$\rho = \sqrt{3^2 + 1^2 + 0^2} = \sqrt{10},\ \phi = \tan^{-1}\left(\frac{1}{3}\right) = 18.435^o,\ z = 0$$

$$\vec{D} = D_\rho\,\hat{a}_\rho + D_\phi\,\hat{a}_\phi + D_z\,\hat{a}_z$$

where $D_\rho = \vec{D}.\,\hat{a}_\rho = D_x\,\hat{a}_x.\,\hat{a}_\rho + D_y\,\hat{a}_y.\,\hat{a}_\rho + D_z\,\hat{a}_z.\,\hat{a}_\rho$

$$= -358.380\cos(18.435^o) + 60.817\sin(18.438^o) + 0$$
$$= -339.98 + 19.232 \quad = -320.757$$

$$D_\phi = \vec{D}.\,\hat{a}_\phi = D_x\,\hat{a}_x.\,\hat{a}_\phi + D_y\,\hat{a}_y.\,\hat{a}_\phi + D_z\,\hat{a}_z.\,\hat{a}_\phi$$
$$= (-358.38)(-\sin 18.435^o) + 60\cos(18.435^o) = 171.026$$

$$D_z = 372.364$$

$$\therefore\quad \vec{D} = -320.787\,\hat{a}_\rho + 171.026\,\hat{a}_\phi + 372.364\,\hat{a}_z\ \text{pC/m}^2$$

2. A point charge of 12 nC is located at the origin. Four uniform line charges are located in the x = 0 plane as follows: 80 nC/m at y = –1 and –5m, –50 nC/m at y = –2 and –4 m. Find the electric flux density $\vec{D}$ in spherical coordinate system at P(0, –3, 2).

Solution:

$$\vec{D} = \varepsilon_o\vec{E} \quad \text{.....................} \quad \text{(i)}$$

$$\vec{E} = \vec{E}_P + \vec{E}_L \qquad \text{.................... (ii)}$$

$$\vec{E}_P = \frac{Q}{4\pi\varepsilon_o}\frac{\vec{R}}{R^3}$$

where $\vec{R} = (0, -3, 2) - (0, 0, 0) = -3\,\hat{a}_y + 2\,\hat{a}_z$; $R = \sqrt{(-3)^2 + (2)^2} = \sqrt{13}$

$$\therefore \quad \vec{E}_P = \frac{12 \times 10^{-9} \times 9 \times 10^{9}}{(\sqrt{13})^3}(-3\,\hat{a}_y + 2\,\hat{a}_z) = -6.9124\,\hat{a}_y + 4.6082\,\hat{a}_z \text{ V/m}$$

$$\vec{E}_L = \vec{E}_{L1} + \vec{E}_{L2} + \vec{E}_{L3} + \vec{E}_{L4}$$

$$= \frac{\rho_{L1}}{2\pi\varepsilon_o R_1^2}\vec{R}_1 + \frac{\rho_{L2}}{2\pi\varepsilon_o R_2^2}\vec{R}_2 + \frac{\rho_{L3}}{2\pi\varepsilon_o R_3^2}\vec{R}_3 + \frac{\rho_{L4}}{2\pi\varepsilon_o R_4^2}\vec{R}_4$$

$$\vec{R}_1 = (0, -3, 2) - (0, -1, 2) = -2\,\hat{a}_y\,;\; R_1 = \sqrt{(-2)^2} = 2$$

$$\vec{R}_2 = (0, -3, 2) - (0, -5, 2) = +2\,\hat{a}_y\,;\; R_2 = \sqrt{(2)^2} = 2$$

$$\vec{R}_3 = (0, -3, 2) - (0, -2, 2) = -\,\hat{a}_y\,;\; R_3 = \sqrt{(-1)^2} = 1$$

$$\vec{R}_4 = (0, -3, 2) - (0, -4, 2) = -\,\hat{a}_y\,;\; R_4 = \sqrt{(1)^2} = 1$$

$$\therefore \quad \vec{E}_L = \frac{80 \times 10^{-9}}{2\pi \times 8.85 \times 10^{-12} \times (2)^2}(-2\,\hat{a}_y)$$

$$+ \frac{80 \times 10^{-9}}{2\pi \times 8.85 \times 10^{-12} \times (2)^2}(+2\,\hat{a}_y)$$

$$+ \frac{-50 \times 10^{-9}}{2\pi \times 8.85 \times 10^{-12} \times (1)^2}(-\,\hat{a}_y)$$

$$+ \frac{-50 \times 10^{-9}}{2\pi \times 8.85 \times 10^{-12} \times (1)^2}(+\,\hat{a}_y)$$

$$= 0$$

From equation (ii),

$$\vec{E} = \vec{E}_P + \vec{E}_L = (-6.9124\hat{a}_y + 4.6082\hat{a}_z) + 0 = (-6.9124\,\hat{a}_y + 4.6082\,\hat{a}_z) \text{ V/m}$$

Using equation (i),

$$\vec{D} = \varepsilon_o\vec{E} = 8.85 \times 10^{-12}(-6.9124\,\hat{a}_y + 4.6082\,\hat{a}_z) \text{ C/m}^2$$

In spherical coordinate system,

$$\vec{D} = D_r\,\hat{a}_r + D_\theta\,\hat{a}_\theta + D_\phi\,\hat{a}_\phi$$

where $D_r = \vec{D} \cdot \hat{a}_r = \left[8.85 \times 10^{-12} (-6.9124\, \hat{a}_y + 4.6082\, \hat{a}_z)\right] \cdot \hat{a}_r$

$$= -6.1174 \times 10^{-11}\, \hat{a}_y \cdot \hat{a}_r + 4.0782 \times 10^{-11}\, \hat{a}_z \cdot \hat{a}_r$$

$$= -6.1174 \times 10^{-11} \sin\theta \sin\phi + 4.0782 \times 10^{-11} \cos\theta$$

$$D_\theta = \vec{D} \cdot \hat{a}_\theta = 6.1174 \times 10^{-11}\, \hat{a}_y \cdot \hat{a}_\theta + 4.0782 \times 10^{-11}\, \hat{a}_z \cdot \hat{a}_\theta$$

$$= -6.1174 \times 10^{-11} \cos\theta \sin\phi + 4.0782 \times 10^{-11} (-\sin\theta)$$

$$D_\phi = \vec{D} \cdot \hat{a}_\phi = -6.1174 \times 10^{-11}\, \hat{a}_y \cdot \hat{a}_\phi + 4.0782 \times 10^{-11}\, \hat{a}_z \cdot \hat{a}_\phi$$

$$= -6.1174 \times 10^{-11} \cos\phi + 0 = -6.1174 \times 10^{-11} \cos\phi$$

At (0, -3, 2),

$$\phi = 360^\circ - \tan^{-1}\frac{3}{0} = 270^\circ$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}} = \cos^{-1}\frac{2}{\sqrt{(0)^2 + (-3)^2 + (2)^2}} = 56.3099^\circ$$

Then,

$D_r = 7.3520 \times 10^{-11}$, $D_\theta \approx 0$, $D_\phi = 0$

$$\therefore \vec{D} = 7.352 \times 10^{-11}\, \hat{a}_r\ C/m^2$$

3. **A uniform line charge of 15 nC/m lies along z-axis and a uniform sheet of charge ρ_s of 4 nC/m² is located at z = 1. Find the electric flux density $\vec{D}$ in spherical coordinate system at point (2, 0, 0).** *[2068 Magh]*

Solution:

Electric flux density due to line charge is

$$\vec{D}_L = \varepsilon_o \vec{E}_L = \varepsilon_o \frac{\rho_L}{2\pi\varepsilon_o R}\, \hat{a}_R = \frac{\rho_L}{2\pi} \frac{\vec{R}}{R^2}$$

where $\vec{R} = (2,0,0) - (0,0,0) = 2\, \hat{a}_x$; $R = 2$

$$\therefore \vec{D}_L = \frac{15 \times 2\hat{a}_x}{2\pi \times 4} = \frac{30}{8\pi}\, \hat{a}_x\ nC/m^2 = 1.19366\, \hat{a}_x\ nC/m^2$$

Electric flux density due to surface charge is

$$\vec{D}_s = \frac{\rho_s}{2}\, \hat{a}_N = \frac{4}{2}(-\hat{a}_z) = -2\hat{a}_z\ nC/m^2$$

Total electric flux density is

$$\vec{D} = \vec{D}_L + \vec{D}_S = (1.19366\, \hat{a}_x - 2\hat{a}_z)\ nC/m^2$$

The point P (2, 0, 0) in spherical coordinate system is obtained as

$$r = \sqrt{x^2 + y^2 + z^2} = 2;\ \theta = \cos^{-1}\frac{z}{r} = 90^\circ;\ \phi = \tan^{-1}\frac{y}{x} = 0^\circ$$

$$P(x = 2, y = 0, z = 0) \rightarrow P(r = 2, \theta = 90^\circ, \phi = 0^\circ)$$

$$D_r = \vec{D} \cdot \hat{a}_r = 1.193666\ \hat{a}_x \cdot \hat{a}_r - 2\ \hat{a}_z \cdot \hat{a}_r$$
$$= 1.19366 \sin\theta \cos\phi - 2 \cos\theta$$
$$= 1.19366 \sin 90^\circ \cos 0^\circ - 2 \cos 90^\circ = 1.19366$$

$$D_\theta = \vec{D} \cdot \hat{a}_\theta = 1.19366\ \hat{a}_x \cdot \hat{a}_\theta - 2\ \hat{a}_z \cdot \hat{a}_\theta$$
$$= 1.19366 \cos\theta \cos\phi + 2 \sin\theta$$
$$= 1.19366 \cos 90^\circ \cos 0^\circ + 2 \sin 90^\circ = 2$$

$$D_\phi = \vec{D} \cdot \hat{a}_\phi = 1.19366\ \hat{a}_x \cdot \hat{a}_\phi - 2\ \hat{a}_z \cdot \hat{a}_\phi = -1.19366 \sin 0^\circ - 2\times 0 = 0$$

$$\therefore \vec{D}\ (2, 90^\circ, 0^\circ) = (1.19366\ \hat{a}_r + 2\ \hat{a}_\theta)\ nC/m^2$$

4. Surface charge densities of 200, –50, and ρ μC/ m² are located at r = 3, 5, and 7 cm respectively. Find $\vec{D}$ at (a) r = 1 cm (b) r = 4.8 cm (c) r = 6.9 cm. Find ρ if $\vec{D}$ = 0 at r = 9 cm. [2066 Magh]

Solution:

a. At r = 1 cm, no charge is enclosed so the flux leaving r = 1 cm is zero.

$$\therefore \vec{D} = \frac{\text{Flux}}{\text{Area}}\ \hat{a}_r = \frac{0}{4\pi(0.01)^2}\ \hat{a}_r = 0$$

b. At r = 4.8 cm, the sphere of radius 4.8 cm encloses the sphere of r = 3 cm.

So, charge enclosed by r = 4.8 cm is equal to charge present on r = 3 cm.

Hence,

$$Q = \Psi = \text{charge density} \times \text{area} = 200 \times 10^{-6}\ [4\pi\ (0.03)^2] = 0.72\pi \times 10^{-6}\ C$$

$$\therefore \vec{D}\ (\text{at } r = 4.8\text{ cm}) = \frac{\Psi}{A}\ \hat{a}_r = \frac{0.72\times\pi\times 10^{-6}}{4\pi\times(0.048)^2}\ \hat{a}_r = 78.125\ \hat{a}_r\ \mu C/m^2$$

c. At r = 6.9 cm, charge enclosed by r = 6.9 cm is equal to charge present on r = 3 cm & r = 5 cm.

$$\therefore Q = \Psi = 0.72\ \pi\times 10^{-6} - 50 \times 10^{-6}\ [4\pi \times (0.05)^2] = 0.22\ \pi \times 10^{-6}\ C$$

$$\therefore \vec{D}\ (r = 6.9\text{ cm}) = \frac{\Psi}{A}\ \hat{a}_r = \frac{0.22\pi\times 10^{-6}}{4\pi\times(0.069)^2}\ \hat{a}_r = 11.552\ \hat{a}_r\ \mu C/m^2$$

d. At $r = 9$ cm, $\vec{D} = \frac{Q}{A}\,\hat{a}_r = 0$ (given)

$\therefore Q = 0.$

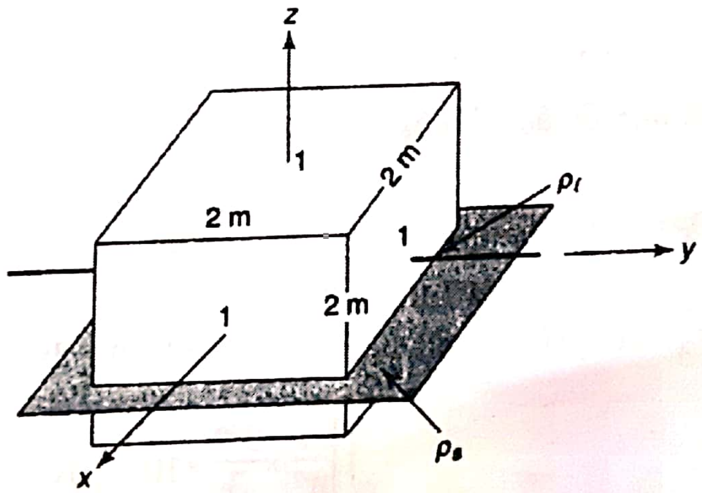
Charge enclosed by r = 9 cm is expressed as

Q = charge enclosed on r = 3 cm + charge enclosed on r = 5 cm + charge enclosed on r = 7 cm

$0 = 0.7\,\pi\times10^{-6} - 0.5\pi\times10^{-6} + \rho\,[4\pi \times (0.07)^2]$

$\therefore \rho = -11.224\ \mu C/m^2$

5. **Charge in the form of a plane sheet with density $\rho_s = 40\ \mu C/m^2$ is located at z= -0.5 m, A uniform line charge of $\rho_L = -6\ \mu C/m$ lies along the y axis. What net flux crosses the surface of a cube 2m on an edge, centered at the origin, as shown in Figure ?**



Solution:

The charge enclosed from the plane is

$Q = (4\ m^2)\ (40\ \mu C/m^2) = 160\ \mu C$

The charge enclosed from the line is

$Q = (2\ m)\ (-6\ \mu C/\ m) = -12\ \mu C$

Thus, $Q_{enc} = \psi = 160 - 12 = 148\ \mu C.$

6. **Let $\vec{D} = \frac{r}{3}\,\hat{a}_r\,\frac{nC}{m^2}$ in free space.**

a. **Find $\vec{E}$ at r = 0.2 m.**

b. **Find the total charge within sphere r = 0.2 m.**

c. **Find the total electric flux leaving the sphere r = 0.3 m.**

Solution:

a. $\vec{E} = \frac{\vec{D}}{\varepsilon_o} = \frac{r/3 \times 10^{-9}\,\hat{a}_r}{8.854 \times 10^{-12}} = 37.6477\, r\, \hat{a}_r$

$\therefore \vec{E}|_{r=0.2m} = 37.6477 \times 0.2\, \hat{a}_r = 7.529\, \hat{a}_r$ V/m

b. The total charge within the sphere is

$Q = \int_{vol} \rho_v\, dv$; where dv (in spherical coordinate system) = $r^2 \sin\theta\, dr\, d\theta\, d\phi$

$= \int_{vol} (\nabla.\vec{D})\, dv$

$= \int_{vol} [\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}]\, dv$

Given,

$\vec{D} = \frac{r}{3} \times 10^{-9}\, \hat{a}_r = D_r\, \hat{a}_r + D_\theta\, \hat{a}_\theta + D_\phi\, \hat{a}_\phi$

$\therefore D_r = \frac{r}{3} \times 10^{-9}$, $D_\theta = 0$, $D_\phi = 0$

$$Q = \int\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \times \frac{r}{3} \times 10^{-9}) + 0 + 0\right] dv = \int\left[\frac{1}{r^2}\frac{\partial}{\partial r}(\frac{r^3}{3} \times 10^{-9})\right] dv$$

$$= \int\left[\frac{1}{r^2} \times \frac{3r^2}{3} \times 10^{-9}\right] dv$$

$$= 10^{-9}\int dv$$

$$= 10^{-9}\int\int\int r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 10^{-9} \times \frac{4}{3}\pi r^3$$

$$= 10^{-9} \times \frac{4}{3} \times \pi \times (0.2)^3 = 3.351 \times 10^{-11}\ \text{C}$$

c. The total flux leaving the sphere is $\Psi = \int \vec{D}.\, d\vec{S}$

$$= \int\int (\frac{r}{3} \times 10^{-9})\, \hat{a}_r . (r^2 \sin\theta\, d\theta\, d\phi\, \hat{a}_r)$$

$$= \frac{10^{-9}}{3}\int\int r^3 \sin\theta\, d\theta\, d\phi$$

$$= \frac{10^{-9}}{3} r^3 \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sin\theta \, d\theta \, d\phi$$

$$= \frac{10^{-9}}{3} r^3 \int_{\theta=0}^{\pi} \sin\theta \, d\theta \, [\phi]_0^{2\pi}$$

$$= \frac{10^{-9}}{3} \times 2\pi r^3 \int_{\theta=0}^{\pi} \sin\theta \, d\theta$$

$$= \frac{10^{-9}}{3} \times 2\pi r^3 [-\cos\theta]_0^{\pi}$$

$$= \frac{10^{-9}}{3} \times 2\pi r^3 [-\cos\pi - (-\text{Cos } 0)]$$

$$= \frac{10^{-9}}{3} \times 2\pi r^3 [1+1] = 10^{-9} \frac{4\pi}{3} r^3$$

For $r = 0.3$ m,

$$\Psi = 10^{-9} \times \frac{4}{3} \times \pi \times (0.3)^3 = 1.1309 \times 10^{-10} \text{ C}$$

7. For a vector field $\vec{A}$, show explicitly that $\nabla \cdot \nabla \times \vec{A} = 0$; that is, the divergence of the curl of any vector field is zero.

Solution:

Using Cartesian coordinates,

$$\nabla \cdot \nabla \times \vec{A} = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) \cdot \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

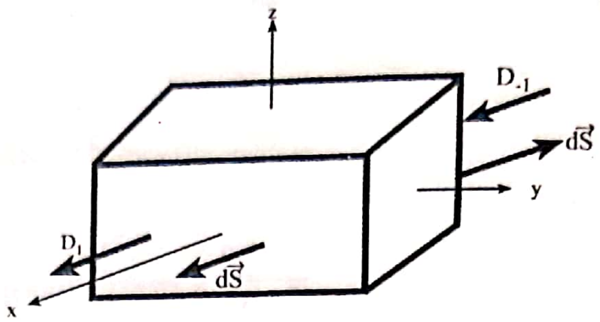
$$= \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) \cdot \left[\hat{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{a}_y\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_x}{\partial y \partial z} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}$$

Since $\frac{\partial^2 A_z}{\partial x \partial y} = \frac{\partial^2 A_z}{\partial y \partial x}$ and so on, $\nabla \cdot \nabla \times \vec{A} = 0$

8. Given that $\vec{D} = \frac{10x^3}{3}\hat{a}_x$ (C/m²), evaluate both sides of the divergence theorem for the volume of a cube, 2m on an edge, centered at the origin and with edges parallel to the axes.



Solution:

From divergence theorem,

$$\oint_S \vec{D} \cdot d\vec{S} = \int_{vol} (\nabla \cdot \vec{D})\, dv$$

Since $\vec{D}$ has only an x component, $\vec{D} \cdot d\vec{S}$ is zero on all but the faces at x = 1m and x = –1m (see Figure)

$$\text{L.H.S} = \oint_S \vec{D} \cdot d\vec{S}$$

$$= \int_{top} \vec{D} \cdot d\vec{S} + \int_{bottom} \vec{D} \cdot d\vec{S} + \int_{sides} \vec{D} \cdot d\vec{S}$$

$$= \iint \left(\frac{10x^3}{3}\hat{a}_x\right) \cdot (dx\, dy\hat{a}_z) + \iint \left(\frac{10x^3}{3}\hat{a}_x\right) \cdot \{dx\, dy\, (-\hat{a}_z)\}$$

$$+ \left[\underbrace{\iint \left(\frac{10x^3}{3}\hat{a}_x\right) \cdot (dy\, dz\, \hat{a}_x)}_{(\text{for } x = 1)} + \underbrace{\iint \left(\frac{10x^3}{3}\hat{a}_x\right) \cdot \{dy\, dz\, (-\hat{a}_x)\}}_{(\text{for } x = -1)}\right]$$

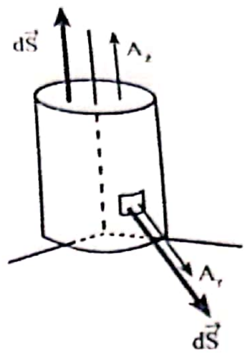
$$= 0 + 0 + \left[\int_{-1}^{1}\int_{-1}^{1} \frac{10(1)^3}{3} dydz + \int_{-1}^{1}\int_{-1}^{1} \frac{-10(-1)^3}{3} dydz\right] = \frac{80}{3}\, C$$

$$\nabla \cdot \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \frac{\partial\left(\frac{10x^3}{3}\right)}{\partial x} + 0 + 0 = 10x^2$$

$$\text{R.H.S} = \int_{vol} (\nabla \cdot \vec{D})\, dv$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (10x^2)\, (dx\, dy\, dz) = \frac{80}{3}\, C$$

9. Given that $\vec{A} = 30\, e^{-r}\, \hat{a}_r - 2z\, \hat{a}_z$ in cylindrical coordinates, evaluate both sides of the divergence theorem for the volume enclosed by r = 2, z = 0, and z = 5.



Solution:

Divergence theorem is

$$\oint_S \vec{A} \cdot d\vec{S} = \int_{vol} (\nabla \cdot \vec{A})\, dv$$

It is noted that $A_z = 0$ for $z = 0$ and hence, $\vec{A} \cdot d\vec{S}$ is zero over that part of the surface.

$$\text{L.H.S.} = \oint_S \vec{A} \cdot d\vec{S}$$

$$= \int_{top} \vec{A} \cdot d\vec{S} + \int_{bottom} \vec{A} \cdot d\vec{S} + \int_{sides} \vec{A} \cdot d\vec{S}$$

$$= \iint (30e^{-r}\, \hat{a}_r - 2z\hat{a}_z) \cdot (r\, dr\, d\phi\, \hat{a}_z) + \iint (30e^{-r}\, \hat{a}_r - 2z\hat{a}_z) \cdot [r dr\, d\phi\, (-\hat{a}_z)] + \iint (30e^{-r}\, \hat{a}_r - 2z\hat{a}_z) \cdot (r\, d\phi\, dz\, \hat{a}_r)$$

$$= \iint_{(z=5)} -2z\, r dr\, df + \iint_{(z=0)} 2r\, z\, dr\, df + \iint_{(r=2)} r\, 30\, e^{-r}\, df\, dz$$

$$= \iint -10\, r\, dr\, d\phi + 0 + \iint 60\, e^{-2}\, d\phi\, dz$$

$$= \int_{\phi=0}^{2\pi} \int_{r=0}^{2} -10r\, dr\, d\phi + \int_{\phi=0}^{2\pi} \int_{z=0}^{5} 60\, e^{-2}\, d\phi\, dz$$

$$= 129.4$$

$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r\, 30e^{-r}) + 0 + \frac{\partial}{\partial z}(-2z) = \frac{30e^{-r}}{r} - 30e^{-r} - 2$$

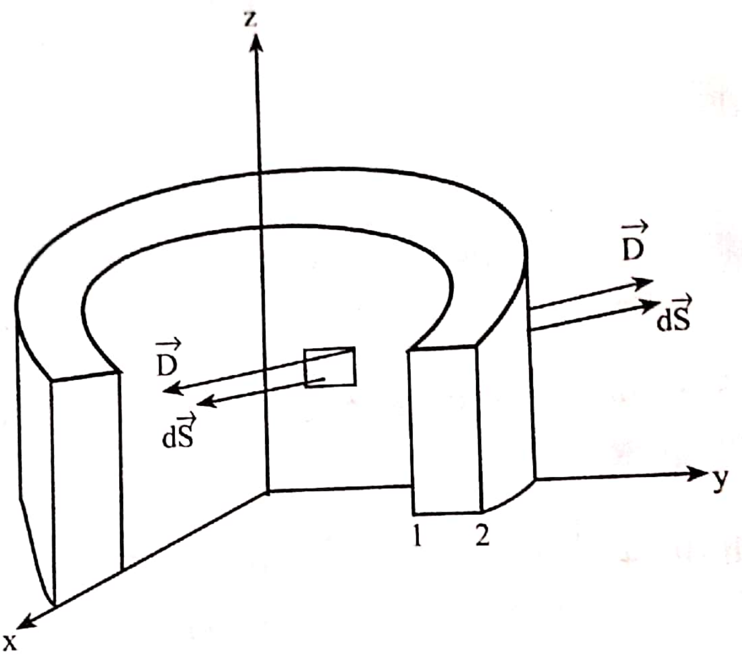
$$\text{R.H.S} = \int_{vol} (\nabla \cdot \vec{A})\, dv$$

$$= \int_{z=0}^{5}\int_{\phi=0}^{2\pi}\int_{r=0}^{2}\left(\frac{30e^{-r}}{r} - 30e^{-r} - 2\right)(r\,dr\,d\phi\,dz)$$

[$\because$ dv in cylindrical coordinate system = rdr dϕ dz]

$= 129.4$

10. **Given that $\vec{D} = (10r^3/4)\hat{a}_r$ (C/m²) in cylindrical coordinates, evaluate both sides of the divergence theorem for the volume enclosed by r = 1 m, r = 2 m, z = 0 and z = 10 m (see Figure).**



Solution:

From divergence theorem,

$$\oint_S \vec{D} \cdot d\vec{S} = \int_{vol} (\nabla \cdot \vec{D})\, dv$$

Since $\vec{D}$ has no z component, $\vec{D} \cdot d\vec{S}$ is zero for the top and bottom. On the inner cylindrical surface $d\vec{S}$ is in the direction $-\hat{a}_r$.

$$\text{L.H.S} = \oint_S \vec{D} \cdot d\vec{S} = \int_{top} \vec{D} \cdot d\vec{S} + \int_{bottom} \vec{D} \cdot d\vec{S} + \int_{sides} \vec{D} \cdot d\vec{S}$$

$$= 0 + 0 + \int_{sides} \vec{D} \cdot d\vec{S}$$

$$= \int_0^{10}\int_0^{2\pi} \frac{10}{4} r^3 \hat{a}_r \cdot r d\phi\, dz\, (-\hat{a}_r) + \int_0^{10}\int_0^{2\pi} \frac{10}{4} r^3 \hat{a}_r \cdot r d\phi\, dz\, \hat{a}_r$$

$$(r = 1) \qquad\qquad (r = 2)$$

($\because$ $dS = r\, d\phi\, dz$ in cylindrical coordinate system)

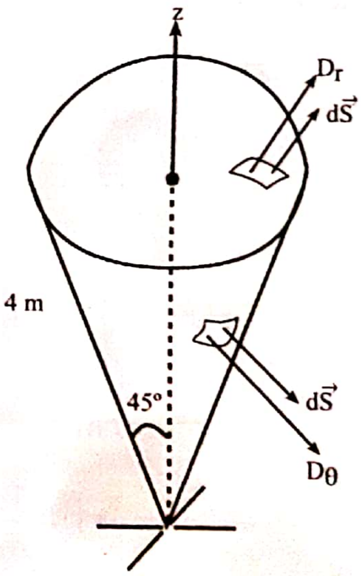
$$= \frac{-200\pi}{4} + 16\frac{200\pi}{4} = 750\pi \text{ C}$$

$$\nabla \cdot \vec{D} = \frac{1}{r}\frac{\partial}{\partial r}(rD_r) + \frac{1}{r}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{10r^3}{4}\right) + 0 + 0 = 10r^2$$

$$\text{R.H.S} = \int_{vol} (\nabla \cdot \vec{D})\, dv = \int_0^{10}\int_0^{2\pi}\int_1^{2} (10r^2)(r\, dr\, d\phi\, dz) = 750\pi \text{ C}$$

($\because$ $dv = r dr\, d\phi\, dz$ in cylindrical coordinate system)

11. Given that $\vec{D} = \left(\frac{5r^2}{4}\right)\hat{a}_r$ (C/ m²) in spherical coordinates, evaluate both sides of the divergence theorem for the volume enclosed by r = 4m and $\theta = \frac{\pi}{4}$ (see Fig.).



Solution:

From divergence theorem,

$$\oint_S \vec{D} \cdot d\vec{S} = \int_{vol} (\nabla \cdot \vec{D})\, dv$$

Since $\vec{D}$ has only a radial component, $\vec{D} \cdot d\vec{S}$ has a non zero value only on the surface r = 4 m.

$$\text{L.H.S} = \oint_S \vec{D} \cdot d\vec{S}$$

$$= \iint \left(\frac{5r^2}{4}\hat{a}_r\right) \cdot (r^2 \sin\theta\, d\theta\, d\phi\, \hat{a}_r)$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/4} \frac{5r^4}{4} \sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi/4} \frac{5(4)^4}{4} \sin\theta\, d\theta\, d\phi = 589.1 \text{ C}$$

$$(r = 4)$$

$$\nabla . \vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{5r^2}{4}\right) + 0 + 0 = 5r$$

$$\text{R.H.S} = \int_{vol} (\nabla . \vec{D})\, dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/4}\int_{r=0}^{4} (5r)\,(r^2 \sin\theta\, dr\, d\theta\, d\phi) = 589.1 \text{ C}$$

12. Given the flux density $\vec{D} = \left(\frac{2\cos\theta}{r^3}\right)\hat{a}_r + \left(\frac{\sin\theta}{r^3}\right)\hat{a}_\theta$ C/m², evaluate both sides of the divergence theorem for the region defined by $1<r<2$, $0<\theta<\frac{\pi}{2}$, $0<\phi<\frac{\pi}{2}$.

[2073 *Shrawan*]

Solution:

The divergence theorem is

$$\oint_S \vec{D} . d\vec{S} = \int_{vol} \nabla . \vec{D}\, dv$$

$$\text{R.H.S.} = \int_{vol} \nabla . \vec{D}\, dv$$

In spherical coordinate system,

$$\nabla . \vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$

$$D_r = \frac{2\cos\theta}{r^3},\; D_\theta = \frac{\sin\theta}{r^3},\; D_\phi = 0$$

$$\therefore \nabla . \vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{2\cos\theta}{r^3}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r^3}\sin\theta\right) + 0$$

$$= \frac{2\cos\theta}{r^2}\frac{\partial r^{-1}}{\partial r} + \frac{1}{r^4\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)$$

$$= \frac{2\cos\theta}{r^2}(-1)r^{-2} + \frac{1}{r^4\sin\theta}(2\sin\theta\cos\theta) = \frac{-2\cos\theta}{r^4} + \frac{2\cos\theta}{r^4} = 0$$

$$\therefore \text{R.H.S.} = \int_{vol} \nabla . \vec{D}\, dv = 0$$

$$\text{L.H.S.} = \oint_S \vec{D}\cdot d\vec{S}$$

$$= \int_{(r=1)} \vec{D}\cdot d\vec{S} + \int_{(r=2)} \vec{D}\cdot d\vec{S} + \int_{(\theta=0)} \vec{D}\cdot d\vec{S} + \int_{\left(\theta=\frac{\pi}{2}\right)} \vec{D}\cdot d\vec{S}$$

$$= \underset{(r=1)}{\int_0^{\pi/2}\int_0^{\pi/2} \vec{D}\cdot r^2\sin\theta\, d\theta\, d\phi\,(-\hat{a}_r)} + \underset{(r=2)}{\int_0^{\pi/2}\int_0^{\pi/2} \vec{D}\cdot r^2\sin\theta\, d\theta\, d\phi\,(+\hat{a}_r)}$$

$$+ \underset{(\theta=0)}{\int_1^{2}\int_0^{\pi/2} \vec{D}\cdot r\sin\theta\, dr\, d\phi\,(-\hat{a}_\theta)} + \underset{\left(\theta=\frac{\pi}{2}\right)}{\int_1^{2}\int_0^{\pi/2} \vec{D}\cdot r\sin\theta\, dr\, d\phi\,(+\hat{a}_\theta)}$$

Given, $\vec{D} = \left(\dfrac{2\cos\theta}{r^3}\right)\hat{a}_r + \left(\dfrac{\sin\theta}{r^3}\right)\hat{a}_\theta$

$$= -\underset{(r=1)}{\int_0^{\pi/2}\int_0^{\pi/2} \frac{2\cos\theta}{r^3} r^2\sin\theta\, d\theta\, d\phi} + \underset{(r=2)}{\int_0^{\pi/2}\int_0^{\pi/2} \frac{2\cos\theta}{r^3} r^2\sin\theta\, d\theta\, d\phi}$$

$$-\underset{(\theta=0)}{\int_1^{2}\int_0^{\pi/2} \frac{\sin\theta}{r^3} r\sin\theta\, dr\, d\phi} + \underset{\left(\theta=\frac{\pi}{2}\right)}{\int_1^{2}\int_0^{\pi/2} \frac{\sin\theta}{r^3} r\sin\theta\, dr\, d\phi}$$

$$= -\int_0^{\pi/2}\int_0^{\pi/2} \frac{\sin 2\theta}{(1)^3}(1)^2\, d\theta\, d\phi + \int_0^{\pi/2}\int_0^{\pi/2} \frac{\sin 2\theta}{(2)^3}(2)^2 d\theta\, d\phi$$

$$-0 + \int_1^{2}\int_0^{\pi/2} \frac{\sin^2(\pi/2)}{r^2} dr\, d\phi$$

$$= -\int_0^{\pi/2} \left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} d\phi + \frac{1}{2}\int_0^{\pi/2} \left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} d\phi + \int_1^2 \frac{1}{r^2} dr\, [\phi]_0^{\pi/2}$$

$$= \frac{-1}{2}\int_0^{\pi/2} [-\cos\pi + \cos 0]\, d\phi + \frac{1}{4}\int_0^{\pi/2} [-\cos\pi + \cos 0]\, d\phi + \frac{\pi}{2}\int_1^2 r^{-2} dr$$

$$= \frac{-1}{2}\int_0^{\pi/2} 2 d\phi + \frac{1}{4}\int_0^{\pi/2} 2\, d\phi + \frac{\pi}{2}\left[\frac{r^{-2+1}}{-2+1}\right]_1^2$$

$$= -1\left[\frac{\pi}{2} - 0\right] + \frac{1}{2}\left[\frac{\pi}{2} - 0\right] + \frac{\pi}{2}\left[\frac{r^{-1}}{-1}\right]_1^2$$

$$= -\frac{\pi}{2} + \frac{\pi}{4} - \frac{\pi}{2}[2^{-1} - 1^{-1}] = -\frac{\pi}{2} + \frac{\pi}{4} + \frac{\pi}{4} = 0$$

13. **Verify the divergence theorem for the function $\vec{A} = r^2\, \hat{a}_r + r \sin\theta \cos\phi\, \hat{a}_\theta$, over the surface of quarter of a hemisphere defined by: $0<r<3$, $0<\phi<\frac{\pi}{2}$, $0 < \theta < \frac{\pi}{2}$.** [2069 Chaitra]

Solution:

The divergence theorem is $\oint_S \vec{A} \cdot d\vec{S} = \int_{vol} \nabla\cdot\vec{A}\, dv$

In spherical coordinate system,

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$A_r = r^2$, $A_\theta = r\sin\theta\cos\phi$, $A_\phi = 0$

$$\therefore \nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta\cos\phi\sin\theta) + 0$$

$$= \frac{1}{r^2} 4r^3 + \frac{r\cos\phi}{r\sin\theta}(2\sin\theta\cos\theta)$$

$$= 4r + 2\cos\theta\cos\phi$$

$$\int_{vol} \nabla.\vec{A}\, dv = \iiint (4r + 2\cos\theta\cos\phi)(r^2\sin\theta\, dr\, d\theta\, d\phi)$$

$$= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{3} 4r^3\sin\theta\, dr\, d\theta\, d\phi + \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{3} 2r^2\sin\theta\cos\theta\cos\phi\, dr\, d\theta\, d\phi$$

$$= 81\pi/2 + 9 = 136.234$$

$$\text{L.H.S} = \oint_S \vec{A}.d\vec{S} = \int_{(r=0)} \vec{A}.d\vec{S} + \int_{(r=3)} \vec{A}.d\vec{S} + \int_{(\theta=0)} \vec{A}.d\vec{S} + \int_{(\theta=\pi/2)} \vec{A}.d\vec{S}$$

Given, $\vec{A} = r^2\,\hat{a}_r + r\sin\theta\cos\phi\,\hat{a}_\theta$

$$= \iint_{(r=0)} \vec{A}.r^2\sin\theta\, d\theta\, d\phi\,(-\hat{a}_r) + \iint_{(r=3)} \vec{A}.r^2\sin\theta\, d\theta\, d\phi\,(+\hat{a}_r)$$

$$+ \iint_{(\theta=0)} \vec{A}.r\sin\theta\, dr\, d\phi\,(-\hat{a}_\theta) + \iint_{(\theta=\pi/2)} \vec{A}.r\sin\theta\, dr\, d\phi\,(+\hat{a}_\theta)$$

$$= \underbrace{\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} -r^4\sin\theta\, d\theta\, d\phi}_{(r=0)} + \underbrace{\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} r^4\sin\theta\, d\theta\, d\phi}_{(r=3)}$$

$$+ \underbrace{\int_{\phi=0}^{\pi/2}\int_{r=0}^{3} -r^2\sin^2\theta\cos\phi\, dr\, d\phi}_{(\theta=0)} + \underbrace{\int_{\phi=0}^{\pi/2}\int_{r=0}^{3} r^2\sin^2\theta\cos\phi\, dr\, d\phi}_{(\theta=\pi/2)}$$

$$= 0 + \frac{81\pi}{2} + 0 + 9 = 136.234$$

L.H.S. = R.H.S = 136.234. Hence, divergence theorem is verified.

14. Given the flux density $\vec{D} = 4\,xy\,\hat{a}_x + 2x^2\,\hat{a}_y$ C/m². Evaluate both sides of divergence theorem for the region of rectangular paralleopiped formed by the planes x = 0 & 1, y = 0 & 2, z = 0 to 3. *[2064 Shrawan]*

Solution:

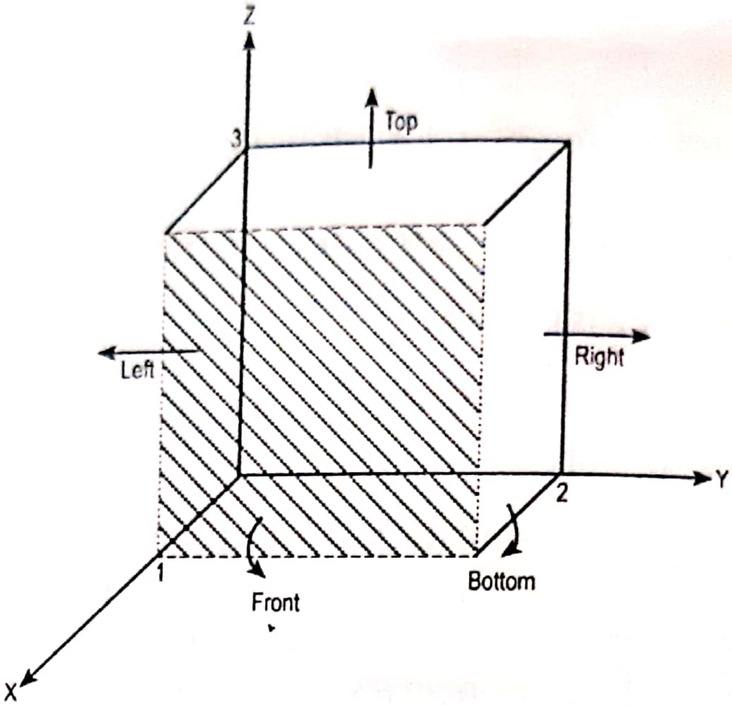
Divergence theorem is

$$\oint_S \vec{D}.d\vec{S} = \int_{vol} (\nabla.\vec{D})\, dv$$

$$\nabla.\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 4y$$

$$\text{R.H.S.} = \int_{vol} (\nabla.\vec{D})\, dv = \int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{3} 4y\, dx\, dy\, dz$$

$$= 1 \times 3 \times 4 \left[\frac{y^2}{2}\right]_0^2 = 12 \times 2 = 24\ C$$



$$\text{L.H.S.} = \oint_S \vec{D}.d\vec{S} = \left(\int_F + \int_B + \int_T + \int_{Bot} + \int_L + \int_R\right) \vec{D}.d\vec{S}$$

$$\int_T \vec{D}.d\vec{S} = \int_T (4xy\, \hat{a}_x + 2x^2\, \hat{a}_y).dx\, dy\, \hat{a}_z = 0$$

$$\int_{Bot} \vec{D}.d\vec{S} = 0$$

$$\int_F \vec{D}.d\vec{S} = \int_{\text{at } x=1} (4xy\, \hat{a}_x + 2x^2\, \hat{a}_y).\, dy\, dz\, (\hat{a}_x)$$

$$= \int_{\text{at } x=1} 4xy\, dy\, dz$$

$$= \int_{y=0}^{2}\int_{z=0}^{3} 4y\, dy\, dz = 4 \times 3 \times \left[\frac{y^2}{2}\right]_0^2 = 24$$

$$\int_B \vec{D}.d\vec{S} = \int_{\text{at } x=0} (4xy\, \hat{a}_x + 2x^2\, \hat{a}_y).dy\, dz\, (-\hat{a}_x)$$

$$= \int_{\text{at } x=0} -4xy\, dy\, dz = 0$$

$$\int_L \vec{D}.d\vec{S} = \int_{\text{at } y=0} (4xy\,\hat{a}_x + 2x^2\,\hat{a}_y).dx\,dz\,(-\hat{a}_y)$$

$$= -\int_{\text{at } y=0} 2x^2\,dx\,dz$$

$$= -\int_{x=0}^{1}\int_{z=0}^{3} 2x^2\,dx\,dz$$

$$= -2 \times 3 \times \left[\frac{x^3}{3}\right]_0^1 = -2$$

$$\int_R \vec{D}.d\vec{S} = \int_{\text{at } y=2} (2xy\,\hat{a}_x + 2x^2\,\hat{a}_y)\,dx\,dz\,(+\hat{a}_y) = \int_{\text{at } y=2} +2x^2\,dxdz$$

$$= +2\int_{x=0}^{1}\int_{z=0}^{3} x^2\,dx\,dz = +2$$

$$\therefore \oint_S \vec{D}.d\vec{S} = \left(\int_F + \int_B + \int_T + \int_L + \int_R + \int_{Bot}\right)\vec{D}.d\vec{S} = 24\text{ C}$$

Hammered Problems

1. Evaluate both sides of the divergence theorem for the region $r \le 2$, if $\vec{G} = 5r\sin^2\theta\cos^2\phi\ \hat{a}_r$.
   Answer: 167.47

2. Given $\vec{A} = x^2\,\hat{a}_x + xy\,\hat{a}_y + yz\,\hat{a}_z$, verify the divergence theorem over a cube one unit on each side. The cube is situated in the first octant of the Cartesian coordinate system with one corner at the origin. **Answer:** 2

3. Given that $\vec{D} = z\rho\cos^2\phi\,\hat{a}_z$ C/m², calculate the charge density at $(1, \frac{\pi}{4}, 3)$ and the total charge enclosed by the cylinder of radius 1 m with $-2 \le z \le 2$m.

   Answer: 0.5 C/m³, $\frac{4\pi}{3}$ C

4. If $\vec{D} = (2y^2 + z)\,\hat{a}_x + 4xy\,\hat{a}_y + x\hat{a}_z$ C/m², find:
   a. The volume charge density at (–1, 0, 3)
   b. The flux through the cube defined by $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.
   c. The total charge enclosed by the cube. **Answer:** (a) –4 C/m³ (b) 2 C (c) 2 C

5. A point charge of 6 μC is located at the origin, a uniform line charge density of 180 nC/m lies along the x-axis, and a uniform sheet of charge equal to 25 nC/m² lies in the z=0 plane. Find $\vec{D}$ at B(1, 2, 4). *[2070 Ashad]*

   Answer: $8.854 \times 10^{-12}\,(560.65\hat{a}_x + 1283.16\hat{a}_y + 3978.11\hat{a}_z$ C/m²)