Homogeneous Differential Equation of First Order

If the differential equation is of the form $\frac{dy}{dx} = \frac{f(x, y)}{\phi(x, y)}$ where f(x, y) and ϕ (x, y) are homogeneous function of the same degree n.
Then it can be solved by putting y = vx.
Differentiating w. r. t. 'x' we get,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So the differential equation reduce to

$$v + x\frac{dv}{dx} = f(v)$$

Then separating the variables and integrating we get the required solution.

Exercise - 21

Solve the following differential equations

1. $\mathbf{x + y\frac{dy}{dx} = 2y}$

Sol^n. Given differential equation is,

$$x + y\frac{dy}{dx} = 2y$$

$$\text{or, } \frac{dy}{dx} = \frac{2y - x}{y} \quad \text{……… (i)}$$

Equation (i) is homogeneous differential equation
So, Put y = vx

$$\text{Then } \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{2vx - x}{vx} = \frac{x(2v-1)}{vx}$$

$$\therefore v + x\frac{dv}{dx} = \frac{2v-1}{v}$$

$$x\frac{dv}{dx} = \frac{2v-1}{v} - v$$

$$= \frac{2v-1-v^2}{v} = -\frac{(v^2-2v+1)}{v}$$

$$\int \frac{v}{(v^2-2v+1)}\, dv = -\int \frac{1}{x}\, dx; \text{ Integrating}$$

$$\text{or, } \int \frac{\frac{1}{2}(2v-2)+1}{v^2-2v+1}\, dv = -\int \frac{1}{x}\, dx$$

$$\text{or, } \frac{1}{2}\int \frac{2v-2}{v^2-2v+1}\, dv + \int \frac{1}{(v-1)^2}\, dv = -\int \frac{1}{x}\, dx$$

$$\text{or, } \frac{1}{2}\log(v^2-2v+1) - \frac{1}{(v-1)} = -\log x + C$$

$$\text{or, } \frac{1}{2}\log(v-1)^2 - \frac{1}{(v-1)} = -\log x + C$$

$$\text{or, } \log(v-1) - \frac{1}{(v-1)} = -\log x + C$$

Restoring the value of $v = \frac{y}{x}$ we get,

$$\log\left(\frac{y}{x}-1\right) - \frac{1}{\left(\frac{y}{x}-1\right)} = -\log x + C$$

$$\text{or, } \log\left(\frac{y-x}{x}\right) - \frac{x}{y-x} = -\log x + C$$

$$\text{or, } \log(y-x) - \log x - \frac{x}{y-x} = -\log x + C$$

$\Rightarrow \log(y-x) = C + \frac{x}{y-x}$ is the required solution.

2. $(x^2 - y^2)\, dx + 2xy\, dy = 0$

Sol^n. Given differential equation is,

$(x^2 - y^2)\, dx + 2xy\, dy = 0$

$$\text{or, } \frac{dy}{dx} = \frac{-\left(x^2-y^2\right)}{2xy} \quad \text{……… (i)}$$

Equation (i) is homogeneous differential equation

So, put $y = vx$

Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{-\left(x^2-x^2v^2\right)}{2x\,.\,vx} = -\frac{\left(1-v^2\right)}{2v}$$

$$\text{or, } x\frac{dv}{dx} = \frac{-1+v^2}{2v} - v = \frac{-1+v^2-2v^2}{2v}$$

$$\text{or, } x\frac{dv}{dx} = \frac{-1-v^2}{2v} = \frac{-\left(v^2+1\right)}{2v}$$

$$\int \frac{2v}{v^2+1}\, dv = -\int \frac{1}{x}\, dx; \text{ Integrating}$$

or, $\log(v^2 + 1) = -\log x + \log C$

$\log(v^2 + 1) = \log\left(\frac{c}{x}\right)$

or, $v^2 + 1 = \frac{C}{x}$

or, $x(v^2 + 1) = C$

Restoring the value of $v = \frac{y}{x}$ we get,

$$x\left(\frac{y^2}{x^2}+1\right) = C$$

$$\text{or, } x\frac{\left(y^2+x^2\right)}{x^2} = C$$

or, $\frac{y^2+x^2}{x} = C \Rightarrow x^2 + y^2 = Cx$ is the required solution.

3. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

Sol^n. Given differential equation is,

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$$

$$\text{or, } \frac{dy}{dx} = \frac{y^2}{x^2} - \frac{y}{x} \quad \text{……. (i)}$$

Equation (i) is homogeneous differential equation

So, put $y = vx$

then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{v^2x^2}{x^2} - \frac{vx}{x}$$

$$\text{or, } v + x\frac{dv}{dx} = v^2 - v$$

or, $x\frac{dv}{dx} = v^2 - 2v$

or, $\int \frac{1}{v^2 - 2v}\, dv = \int \frac{1}{x}\, dx$; Integrating

or, $\int \frac{1}{v(v-2)}\, dv = \int \frac{1}{x}\, dx$

or, $\frac{1}{2}\int\left(\frac{-1}{v} + \frac{1}{(v-2)}\right)dv = \int \frac{1}{x}\, dx$

or, $\frac{1}{2}$ $\{(-\log v + \log (v-2)\} = \log x + \log C$

or, $\frac{1}{2}\log\frac{(v-2)}{v} = \log x\, C$

or, $\sqrt{\frac{v-2}{v}} = xC \Rightarrow \frac{v-2}{v} = x^2C^2$

Restoring the value of $v = \frac{y}{x}$ we get

or, $\frac{\frac{y}{x} - 2}{\frac{y}{x}} = x^2C^2 \Rightarrow \frac{y-2x}{y} = x^2C^2$

or, $y - 2x = x^2y\, C^2$

or, $2x - y = (-C^2)\, x^2y$

or, $2x - y = K\, x^2y$ where $K = -C^2$

is the required solution.

4. $\mathbf{(x^2 + y^2)\, dx = (x^2 + xy)\, dy}$

Sol^n. Given differential equation is,

$(x^2 + y^2)\, dx = (x^2 + xy)\, dy$

or, $\frac{dy}{dx} = \frac{(x^2 + y^2)}{(x^2 + xy)}$ …….. (i)

Equation (i) is homogeneous differential equation

So, put $y = vx$ then

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$v + x\frac{dv}{dx} = \frac{x^2 + x^2v^2}{x^2 + x^2v}$

or, $v + x\frac{dv}{dx} = \frac{1+v^2}{1+v}$

or, $x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v$

or, $x\frac{dv}{dx} = \frac{1+v^2-v-v^2}{1+v}$

or, $x\frac{dv}{dx} = \frac{1-v}{1+v}$

or, $\int \frac{1+v}{1-v}\, dv = \int \frac{1}{x}\, dx$; Integrating

or, $\int\left(-1 + \frac{2}{1-v}\right)dv = \int \frac{1}{x}\, dx$;

$-v - 2\log(1-v) = \log x + \log C$

$-v - \log(1-v)^2 = \log x\, C$

or, $-v = \log xC\,(1-v)^2$

$\Rightarrow xC\,(1-v)^2 = e^{-v}$

Restoring the value of $v = \frac{y}{x}$ we get,

or, $xC\left(1 - \frac{y}{x}\right)^2 = e^{-\frac{y}{x}}$

or, $xC\,\frac{(x-y)^2}{x^2} = e^{-\frac{y}{x}}$

or, $C\,(x-y)^2 = x\, e^{-\frac{y}{x}}$ is the required solution.

5. $\mathbf{x\,(x - y)\, dy = y\,(x + y)\, dx}$

Sol^n. Given differential equation is,

$x\,(x - y)\, dy = y\,(x + y)\, dx$

$\therefore \frac{dy}{dx} = \frac{y(x+y)}{x(x-y)}$ ……. (i)

Equation (i) is homogeneous differential equation

So, put $y = vx$

Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$v + x\frac{dv}{dx} = \frac{vx(x+vx)}{x(x-vx)}$

or, $v + x\frac{dv}{dx} = \frac{v(1+v)}{(1-v)}$

or, $x\frac{dv}{dx} = \frac{v+v^2}{1-v} - v$

or, $x\frac{dv}{dx} = \frac{v+v^2-v+v^2}{1-v} = \frac{2v^2}{1-v}$

or, $x\frac{dv}{dx} = \frac{2v^2}{1-v}$

$\int\frac{1-v}{2v^2}\,dv = \int\frac{1}{x}\,dx$; Integrating

or, $\int\left(\frac{1}{v^2}-\frac{1}{v}\right)dv = 2\int\frac{1}{x}\,dx$

or, $-\frac{1}{v} - \log v = 2\log x + \log C$

or, $-\frac{1}{v} = \log x^2Cv$

or, $x^2Cv = e^{-\frac{1}{v}}$

Restoring the value of $v = \frac{y}{x}$, we get,

$x^2C\,.\,\frac{y}{x} = e^{-\frac{x}{y}}$

or, $Cxy = e^{-\frac{x}{y}} \Rightarrow e^{-\frac{x}{y}} = C\,xy$ is the required solution.

6. $\mathbf{(x^2 + y^2)\,dy = xy\,dx}$

Sol^n. Given differential equation is,

$(x^2 + y^2)\,dy = xy\,dx$

or, $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$ ……… (i)

Equation (i) is homogeneous differential equation

So, put $y = vx$

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$v + x\frac{dv}{dx} = \frac{x\,.\,vx}{x^2+v^2x^2}$

or, $v + x\frac{dv}{dx} = \frac{v}{1+v^2}$

or, $\frac{xdv}{dx} = \frac{v}{1+v^2} - v$

or, $x\frac{dv}{dx} = \frac{v-v-v^3}{1+v^2}$

or, $\frac{xdv}{dx} = \frac{-v^3}{1+v^2}$

or, $\int\frac{1+v^2}{v^3}\,dv = \int\frac{1}{x}\,dx$; Integrating

or, $\int\left(\frac{1}{v^3}+\frac{1}{v}\right)dv = -\int\frac{1}{x}\,dx$

or, $-\frac{1}{2v^2} + \log v = -\log x + \log C$

or, $\log v + \log x - \log C = \frac{1}{2v^2}$

or, $\log\frac{vx}{C} = \frac{1}{2v^2}$

or, $\frac{vx}{C} = e^{\frac{1}{2}v^2}$

or, $vx = C\,e^{\frac{1}{2v^2}}$

Restoring the value of $v = \frac{y}{x}$ we get,

$\frac{y}{x}\,.\,v = C\,e^{\frac{x^2}{2y^2}}$

or, $y = C\,e^{\frac{x^2}{2y^2}}$ is the required solution.

7. $\mathbf{\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}}$ ***(B.E. 2059, 072)***

Sol^n. Given differential equation is,

$\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ ……. (i)

Equation (i) is homogeneous differential equation

So, put $y = vx$

So, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$v + x\frac{dv}{dx} = \frac{vx}{x} + \tan\frac{vx}{x}$

or, $v + x\,\frac{dv}{dx} = v + \tan v$

or, $x\frac{dv}{dx} = \tan v$

or, $\int \frac{1}{\tan v} dv = \int \frac{1}{x} dx$; Integrating

or, $\int \cot v \, dv = \int \frac{1}{x} dx$

or, log sinv = log x + log C

or, log sinv = log x C

or, sinv = xC

Restoring the value of $v = \frac{y}{x}$ we get,

$\sin\frac{y}{x} = xC$ is the required solution.

8. $x^2dy + y(x + y)dx = 0$

Sol^n. Given differential equation is,

$x^2dy + y(x + y)dx = 0$

or, $\frac{dy}{dx} = -\frac{y(x+y)}{x^2}$ ……… (i)

Equation (i) is homogeneous differential equation

So, put y = vx

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$v + x\frac{dv}{dx} = \frac{-vx(x+vx)}{x^2} = -v(1 + v)$

or, $x\frac{dv}{dx} = -v - v^2 - v$

or, $x\frac{dv}{dx} = -(v^2 + 2v)$

or, $\int \frac{1}{v^2+2v} dv = -\int \frac{1}{x} dx$; Integrating

or, $\int \frac{1}{v(v+2)} dv = -\int \frac{1}{x} dx$

or, $\frac{1}{2}\int\left[\frac{1}{v} - \frac{1}{(v+2)}\right] dv = -\int \frac{1}{x} dx$

or, $\int\left(\frac{1}{v} - \frac{1}{v+2}\right) dv = -2\int \frac{1}{x} dx$

or, log v – log (v + 2) = – 2 log x + log C

or, $\log\left(\frac{v}{v+2}\right) = \log\frac{C}{x^2}$

or, $v = (v + 2)\frac{C}{x^2}$

Restoring the value of $v = \frac{y}{x}$ we get,

$\frac{y}{x} = \left(\frac{y}{x} + 2\right)\frac{C}{x^2}$

or, $y = \frac{(y+2x)}{x^2}C$

$\Rightarrow x^2y = C(y + 2x)$ is the required solution.

9. $x\frac{dy}{dx} = y - \sqrt{x^2 + y^2}$

Sol^n. Given differential equation is,

$x\frac{dy}{dx} = y - \sqrt{x^2 + y^2}$

or, $\frac{dy}{dx} = \frac{y - \sqrt{x^2 - y^2}}{x}$ …….. (i)

Equation (i) is homogeneous differential equation

So put y = vx

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$v + x\frac{dv}{dx} = \frac{vx - \sqrt{x^2 + x^2v^2}}{x}$

or, $v + x\frac{dv}{dx} = v - \sqrt{1+v^2}$

or, $x\frac{dv}{dx} = -\sqrt{1+v^2}$

or, $\int \frac{1}{\sqrt{1+v^2}} dv = -\int \frac{1}{x} dx$; Integrating

or, $\log\left(v + \sqrt{1+v^2}\right) = -\log x + \log C$

or, $v + \sqrt{1+v^2} = \frac{C}{x}$

Restoring the value of $v = \frac{y}{x}$ we get,

$\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = \frac{c}{x}$

or, $y + \sqrt{x^2 + y^2} = C$ is the required solution.

10. $\mathbf{x\sin\frac{y}{x}\ dy = \left(y\sin\frac{y}{x} - x\right)dx}$

Sol^n. Given differential equations is,

$$x\sin\left(\frac{y}{x}\right)dy = \left(y\sin\left(\frac{y}{x}\right) - x\right)dx$$

$$\text{or, } \frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right) - x}{x\sin\left(\frac{y}{x}\right)} \quad \ldots\ldots\ldots \text{(i)}$$

Equation (i) is homogeneous differential equation
So, put y = vx

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{vx\sin\left(\frac{vx}{x}\right) - x}{x\sin\left(\frac{vx}{x}\right)}$$

$$\text{or, } v + x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v}$$

$$\text{or, } x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v} - v$$

$$\text{or, } x\frac{dv}{dx} = \frac{v\sin v - 1 - v\sin v}{\sin v}$$

$$\text{or, } x\frac{dv}{dx} = -\frac{1}{\sin v}$$

or, $\int \sin v\, dv = -\int \frac{1}{x}dx$; Integrating

or, $-\cos v = -\log x + C$
or, $\log x = \cos v + C$

Restoring the value of $v = \frac{y}{x}$ we get,

$\log x = \cos\left(\frac{y}{x}\right) + C$ is the required solution.

11. $\mathbf{x\,dy - y\,dx = \sqrt{x^2 + y^2}\ dx}$ ***(B. E. 2061, 062)***

Sol^n. Given differential equation is,

$$x dy - y\,dx = \sqrt{x^2 + y^2}\ dx$$

$$\text{or, } x\,dy = y\,dx + \sqrt{x^2 + y^2}\ dx$$

$$\text{or, } x\,dy = \left(y + \sqrt{x^2 + y^2}\right)dx$$

$$\text{or, } \frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} \quad \ldots\ldots\ldots \text{(i)}$$

Equation (i) is homogeneous differential equation
So, put y = vx

then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x}$$

$$v + x\frac{dv}{dx} = v + \sqrt{1 + v^2}$$

$$\text{or, } x\frac{dv}{dx} = \sqrt{1 + v^2}$$

or, $\int \frac{1}{\sqrt{1 + v^2}}dv = \int \frac{1}{x}dx$; Integrating

$$\log\left(v + \sqrt{1 + v^2}\right) = \log x + \log C$$

$$\text{or, } v + \sqrt{1 + v^2} = xC$$

Restoring the value of $v = \frac{y}{x}$ we get,

$$\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = xC$$

$$\text{or, } y + \sqrt{x^2 + y^2} = x^2C$$

$$\text{or, } \sqrt{x^2 + y^2} = x^2C - y$$

Squaring on both sides,
$x^2 + y^2 = (Cx^2 - y)^2$ is the required solution.

12. $\mathbf{(1 + e^{x./y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0}$

Sol^n. Given differential equation is,

$$(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$$

or, $\dfrac{dx}{dy} = \dfrac{-e^{x/y}\left(1-\dfrac{x}{y}\right)}{\left(1+e^{\frac{x}{y}}\right)}$ …….. (i)

Equation (i) is homogeneous differential equation
So, put x = vy

Then $\dfrac{dx}{dy} = v + \dfrac{dv}{dy}$

Now (i) becomes,

$$v + \frac{dv}{dy} = \frac{-e^{\frac{vy}{y}}\left(1-\frac{vy}{y}\right)}{\left(1+e^{\frac{vy}{y}}\right)}$$

or, $v + \dfrac{dv}{dy} = \dfrac{-e^{v}(1-v)}{\left(1+e^{v}\right)}$ or, $v + \dfrac{dv}{dy} = \dfrac{-e^{v}+ve^{v}}{1+e^{v}} - v$

or, $\dfrac{dv}{dy} = \dfrac{-e^{v}+ve^{v}-v-ve^{v}}{1+e^{v}}$ or, $\dfrac{dv}{dy} = \dfrac{-e^{v}-v}{1+e^{v}}$

or, $\int \dfrac{1+e^{v}}{v+e^{v}}\, dv = -\int \dfrac{1}{y}\, dy$; Integrating

or, $\log (v + e^v) = -\log y + \log C$

or, $\log (v + e^v) = \log \dfrac{C}{y}$

or, $v + e^v = \dfrac{C}{y}$ or, $y (v + e^v) = C$

Restoring the value of $v = \dfrac{x}{y}$ we get,

$$y\left(\frac{x}{y} + e^{\frac{x}{y}}\right) = C$$

or, $\left(x + y\, e^{\frac{x}{y}}\right) = C$ is the required solution.

13. $\boldsymbol{\dfrac{dy}{dx} = \dfrac{3xy+y^2}{3x^2}}$

Sol^n. Given differential equation is,

$\dfrac{dy}{dx} = \dfrac{3xy+y^2}{3x^2}$ …….. (i)

Equation (i) is homogeneous differential equation
So, put y = vx

Then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now equation (i) becomes,

$v + x\dfrac{dv}{dx} = \dfrac{3x\,.\,vx + v^2x^2}{3x^2} = \dfrac{3v+v^2}{3}$

or, $x\dfrac{dv}{dx} = \dfrac{3v+v^2}{3} - v$ or, $x\dfrac{dv}{dx} = \dfrac{3v+v^2-3v}{3}$

or, $x\dfrac{dv}{dx} = \dfrac{v^2}{3}$ or, $\int \dfrac{1}{v^2}\, dv = \int \dfrac{1}{3x}\, dx$; Intgrating

or, $-\dfrac{1}{v} = \dfrac{1}{3}\log x + C$

Restoring the value of $v = \dfrac{y}{x}$ we get,

$-\dfrac{x}{y} = \dfrac{1}{3}\log x + C$

or, $-3x = y\log x + 3C\, y$ or, $3x + y\log x = (-3C)\, y$
or, $3x + y\log x = Ky$ where $K = -3C$ is the required solution.

14. $\boldsymbol{(x^2 + 2xy^2)\, dx + (2x^2y + y^2)\, dy = 0}$

Sol^n. Given differential equation is,
$(x^2 + 2xy^2)\, dx + (2x^2y + y^2)\, dy = 0$

or, $\dfrac{dy}{dx} = -\dfrac{\left(x^2+2xy^2\right)}{\left(2x^2y+y^2\right)}$ or, $2x^2y\, dy + y^2dy = -x^2dx - 2xy^2\, dx$

or, $x^2\, dx + y^2dy + 2xy^2\, dx + 2yx^2\, dy = 0$
or, $x^2dx + y^2dy = -(2xy^2dx + 2yx^2dy)$

or, $\int d\left(\dfrac{x^3}{3}\right) + \int d\left(\dfrac{y^3}{3}\right) = -\int d\,(x^2y^2)$; Integrating

or, $\dfrac{x^3}{3} + \dfrac{y^3}{3} = -x^2y^2 + C$

or, $x^3 + y^3 + 3x^2y^2 = 3C$
or, $x^3 + y^3 + 3x^2y^2 = K$ where $K = 3C$ is the required solution.