Unit - III
Ordinary Differential Equation and its Applications

Chapter - 14 Ordinary Differential Equations

Chapter - 15 First Order and First Degree Differential Equations

Chapter - 16 First Order but not First Degree Differential Equations

Chapter - 17 Second order Linear differential Equations with Constant Coefficients

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Ordinary Differential Equations

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Definitions : An equation which involves differentials or differential coefficients is called the differential equation.

Example

(i) $\frac{dy}{dx} = \cos x$

(ii) $x^3 \frac{d^3y}{dx^3} + x\frac{dy}{dx} - y = x$

(iii) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3 = 0$

Differential equations are two types

(i) Ordinary differential equations : The differential equations which involves only one independent variable is called ordinary differential equations.

(ii) Partial differential equations : The differential equation which involves two or more than two independent variables is called partial differential equations.

For example : $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 5z$ is a partial differential equations.

Order : The order of the differential equation is the order of the highest derivative in the equation.

Example :

$$\left.\begin{array}{l} \frac{dy}{dx} = \sin x \\ \left(\frac{dy}{dx}\right)^2 = px^2 + qx + c \end{array}\right\} \text{First order}$$

$$\frac{d^2y}{dx^2} = x^2 \quad \text{Second order}$$

Degree : The degree of differential equation is the degree of derivative of highest order. It has been made free from radicals of the derivatives.

Exercise - 18

1. Determine the order and degree of each of the following differential equations.

(i) (x + 3y – 2) dx + (2x – 3y + 5) dy = 0

Sol^n. Given differential equation is,

$(x + 3y - 2)\, dx + (2x - 3y + 5)\, dy = 0$

or, $\frac{dy}{dx} + \frac{x + 3y - 2}{2x - 3y + 5} = 0$

Which is first order and first degree differential equation.

(ii) $\mathbf{y = x\frac{d^2y}{dx^2} + \frac{k}{\frac{d^2y}{dx^2}}}$

Sol^n. Given differential equation is,

$$y = x\frac{d^2y}{dx^2} + \frac{k}{\frac{d^2y}{dx^2}}$$

or, $y\frac{d^2y}{dx^2} = x\left(\frac{x^2y}{dx^2}\right)^2 + k$

or, $x\left(\frac{d^2y}{dx^2}\right)^2 - y\frac{d^2y}{dx^2} + k = 0$

Which is second degree and second order differential equation.

(iii) $\mathbf{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = k\frac{d^2y}{dx^2}}$

Sol^n. Given differential equation is,

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = k\frac{d^2y}{dx^2}$$

Squaring both sides we get,

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = k^2\left(\frac{d^2y}{dx^2}\right)^2$$

Which is second order and second degree differential equation.

(iv) $\mathbf{x^2\frac{d^2y}{dx^2}+2xy\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^3=0}$

Sol^n. Give, differential equation is,

$$x^2\frac{d^2y}{dx^2}+2xy\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^3=0$$

Which is second order and fist degree differential equation.

2. Form the differential equations from the following equations.

(i) y = a logx + b

Sol^n. Given equation is,

$y = a\log x + b$

Differentiating both sides w.r. to x we get,

$$\frac{dy}{dx}=a.\frac{1}{x}$$

Again differentiating both sides w. r. to x we get,

$$\frac{d^2y}{dx^2}=-\frac{a}{x^2}$$

$$\text{or, } x\frac{d^2y}{dx^2}=-\frac{a}{x}=-\frac{dy}{dx}$$

$$\therefore x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$$

Which is the required differential equation.

(ii) xy = a + bx

Sol^n. Given equation is,

$$xy=a+bx \Rightarrow y=\frac{a}{x}+b$$

Differentiating both sides w. r. to x we get,

$$\frac{dy}{dx}=-\frac{a}{x^2}$$

Again differentiating both sides w. r. to x we get,

$$\frac{d^2y}{dx^2}=-\frac{2a}{x^3}$$

$$\text{or, } x\frac{d^2y}{dx^2}=-\frac{2a}{x^2}=-2\frac{dy}{dx}$$

$$\therefore x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$$

Which is the required differential equation.

(iii) $\mathbf{xy = Ae^x + Be^{-x}}$

Sol^n. Given equation is,

$xy = Ae^x + Be^{-x}$

Diffrentiating both sides w. t. to x we get,

$$x\frac{dy}{dx}+y\,.\,1=A\,e^x-B\,e^{-x}$$

Again, differentiating both sides w. r. to x we get,

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}.1+\frac{dy}{dx}=A\,e^x+B\,e^{-x}$$

$$\text{or, } x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=xy$$

$$\text{or, } \frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}=y$$

$$\therefore \frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}-y=0$$

Which is the required differential equation.

(iv) $\mathbf{y = ax^3 + bx^2}$

Sol^n. Given equation is,

$y = ax^3 + bx^2$

Differentiating both sides w. r. to x we get,

$$\frac{dy}{dx}=3ax^2+abx \;\ldots\ldots\ldots\text{(i)}$$

Again differentiating both sides w. r. to x we get,

$$\frac{d^2y}{dx^2}=6ax+2b \;\ldots\ldots\text{(ii)}$$

$$\text{From (i), } x\frac{dy}{dx}=3ax^3+2bx^2 \;\ldots\text{(iii)}$$

$$\text{From (ii), } x^2\frac{d^2y}{dx^2}=6ax^3+2bx^2 \;\ldots\ldots\text{(iv)}$$

From (iii) and (iv)

$$4x\frac{dy}{dx}-x^2\frac{d^2y}{dx^2}=12ax^3+8bx^2-6ax^3-2bx^2$$

$$\text{or, } 4x\frac{dy}{dx}-x^2\frac{d^2y}{dx^2}=6ax^3+6bx^2$$

$$\text{or, } 4x\frac{dy}{dx}-x^2\frac{d^2y}{dx^2}=6(ax^3+bx^2)=6y$$

$\therefore\ x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = 0$

Which is the required differential equation.

(v) a cos(logx) + bsin (logx)

Sol^n. Given equation is,

$y = a\cos(\log x) + b\sin(\log x)$

Differentiating both sides w. r. to x we get,

$\dfrac{dy}{dx} = -a\sin(\log x).\dfrac{1}{x} + b\cos(\log x).\dfrac{1}{x}$

$\therefore\ \dfrac{dy}{dx} = \dfrac{-a}{x}\sin(\log x) + \dfrac{b}{x}\cos(\log x)$ …….. (i)

Again differentiating both sides w. r. to x we get,

$$\frac{d^2y}{dx^2} = \frac{-a}{x}\cos(\log x).\frac{1}{x} + \frac{a}{x^2}\sin(\log x) - \frac{b}{x}\sin(\log x).\frac{1}{x} - \frac{b}{x^2}\cos(\log x)$$

or, $\dfrac{d^2y}{dx^2} = \dfrac{-a}{x^2}\cos(\log x) + \dfrac{a}{x^2}\sin(\log x) - \dfrac{b}{x^2}\sin(\log x) - \dfrac{b}{x^2}\cos(\log x)$

$$x\frac{d^2y}{dx^2} = -\frac{a}{x}\cos(\log x) + \frac{a}{x}\sin(\log x) - \frac{b}{x}\sin(\log x) - \frac{b}{x}\cos(\log x) \ ……… (ii)$$

From (i) and (ii)

$x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = -\dfrac{a}{x}\cos(\log x) - \dfrac{b}{x}\sin(\log x)$

or, $\dfrac{xd^2y}{dx^2} + \dfrac{dy}{dx} = -\dfrac{1}{x}[a\cos(\log x) + b\sin(\log x)] = -\dfrac{1}{x}y$

$\therefore\ x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$

Which is the required differential equation.

3. Obtain the differential equation of all circles of radius a and centre (h, k).

Sol^n. We have the equation of circle of raidous a and centre (h, k) is

$(x-h)^2 + (y-k)^2 = a^2$ …….. (i)

Differential equation (i) w. r. t. 'x' we get, $(x-h) + (y-k)y_1 = 0$ …. (ii)

Again differential equation $1 + y_1^2 + (y-k)y_2 = 0$

or, $y - k = \dfrac{-(1+y_1^2)}{y_2}$ …….. (iii)

Put the value of $y-k$ in (ii)

$x - h = -(y-k)y_1 = \dfrac{(1+y_1^2)}{y_2}, y_1$ …….. (iv)

Using (iii) and (iv) in (i) we get,

$$\left[\frac{(1+y_1^2)}{y_2}y_1\right]^2 + \left[\frac{(1+y_1^2)}{y_2}\right]^2 = a^2$$

$\Rightarrow \dfrac{(1+y_1^2)^2}{y_2^2}(1+y_1^2) = a^2$

or, $(1+y_1^2)^3 = a^2y_2^2$

$$\therefore \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx}\right)^2$$

which is the required differential equation.

4. From a differential equation of simple harmonic motion given by x = Acos(nt + α)

Sol^n. Given equation is,

$x = A\cos(nt+\alpha)$

Differentiating both sides w. r. to 't' we get,

$\dfrac{dx}{dt} = -A\sin(nt+\alpha).n$

Again differentiating both sides w.r. to t,

$\dfrac{d^2x}{dt^2} = -A\cos(nt+\alpha).n^2 = -n^2x$

$\therefore\ \dfrac{d^2x}{dt^2} + n^2x = 0$

Which is the required differential equation.