Homogeneous Linear Differential Equation

An equation is of the form $x^2\frac{d^2y}{dx^2} + P_1x\frac{dy}{dx} + P_2y = Q$

or, $(x^2D^2 + P_1xD + P_2)\,y = Q$ where P_1 and P_2 are constant and Q is function of x or constant is called second order homogeneous linear differential equation.

Than can be reduced to the linear differential equation with constant coefficient by the substitution.

$x = e^z$ or $z = \log x$ and $\frac{dz}{dx} = \frac{1}{x}$

Exercise - 31

Solve the following differential equations.

1. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = \frac{1}{x}$

Sol^n. Given differential equation is,

$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = \frac{1}{x}$

or, $(x^2D^2 - 2xD + 2)\,y = \frac{1}{x}$ …….. (i)

Equation (i) is homogeneous differential equation.

So, put $x = e^z \Rightarrow z = \log x$

$x\,Dy = \delta y$, $x^2D^2y = (\delta^2 - \delta)\,y$ where $\delta = \frac{d}{dz}$ in equation (i)

Now equation (i) becomes,

$(\delta^2 - \delta - 2\delta + 2)\,y = \frac{1}{e^z}$

or, $(\delta^2 - 3\delta + 2)\,y = e^{-z}$

So, it's A. E. is;

$m^2 - 3m + 2 = 0$

or, $m^2 - 2m - m + 2 = 0$

or, $m(m - 2) - 1(m - 2) = 0$

or, $(m - 1)(m - 2) = 0$

$\therefore$ m = 1, m = 2

Hence, C. F. = $C_1e^z + C_2e^{2z}$

and P. I. = $\frac{1}{(\delta^2 - 3\delta + 2)}\,e^{-z}$

$= \frac{1}{(-1)^2 - 3(-1) + 2}\,e^{-z} = \frac{1}{6}e^z$

Thus, y = C. F. + P. I.

or, $y = C_1e^z + C_2e^{2z} + \frac{1}{6} - e^z$

or, $y = C_1\,e^{\log x} + C_2e^{2\log x} + \frac{1}{6x}$

or, $y = C_1x + C_2x^2 + \frac{1}{6x}$

is the required general solution.

2. $\mathbf{(x^2D^2 + xD - 1)\,y = x^2}$

Sol^n. Given differential equation is,

$(x^2D^2 + xD - 1)\,y = x^2$ ……. (i)

Equation (i) is homogeneous differential equation

So, put $x = e^z \Rightarrow \log x = z$

$xDy = \delta y$, $x^2D^2y = (\delta^2 - \delta)\,y$ where $\delta = \frac{d}{dz}$ in (i)

Now (i) becomes,

$(\delta^2 - \delta + \delta - 1)\,y = e^{2z}$

or, $(\delta^2 - 1)\,y = e^{2z}$

So, it's A. E. is, $m^2 - 1 = 0 \Rightarrow m = \pm 1$

Hence, C. F. = $C_1e^z + C_1e^{-z}$

and P. I. = $\frac{1}{(\delta^2 - 1)}e^{2z} = \frac{1}{(2^2 - 1)}e^{2z} = \frac{e^{2z}}{(4 - 1)} = \frac{e^{2z}}{3}$

Thus, y = C. F. + P. I.

or, $y = C_1e^z + e_2\,e^{-z} + \frac{e^{2z}}{3}$

or, $y = C_1e^{\log x} + C_2e^{-1\log x} + \frac{e^{2\log x}}{3}$

or, $y = C_1x + \frac{C_2}{x} + \frac{x^2}{3}$ is the required solution.

3. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 4x^3$

Sol^n. Given differential equation is,

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 4x^3$$

or, $(x^2D^2 - 2xD + 2)\, y = 4x^3$ …… (i)

Equation (i) is homogeneous linear differential equation.

So, put $x = e^z \Rightarrow \log x = z$

$x\,Dy = \delta y$, $x^2D^2y = (\delta^2 - \delta)\,y$ whre, $\delta = \frac{d}{dz}$ in equation (i)

Now, (i) becomes,

$(\delta^2 - \delta - 2\delta + 2)\, y = 4e^{3z}$

or, $(\delta^2 - 3\delta + 2)\, y = 4e^{3z}$

So, it's A. E. is,

$m^2 - 3m + 2 = 0$

or, $m^2 - m - 2m + 2 = 0$

or, $m(m - 1) - 2(m - 1) = 0$

or, $(m - 1)(m - 2) = 0$

$\Rightarrow m = 1, 2$

Hence, C. F. $= C_1e^z + C_2e^{2z}$

and P. I. $= \frac{1}{(\delta^2 - 3\delta + 2)}\, 4e^{3z}$

$= \frac{1}{(3^2 - 3.3 + 2)}\, 4e^{3z} = \frac{4}{2}\, e^{3z} = 2e^{3z}$

Thus, y = C. F + P. I.

or, $y = C_1e^z + C_2e^{2z} + 2e^{3z}$

or, $y = C_1e^{\log x} + C_2e^{2\log x} + 2e^{3\log x}$

or, $y = C_1x + C_2x^2 + 2x^3$ is the required solution.

4. $\mathbf{x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x^6}$

Sol^n. Given differential equation is,

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x^6$$

or, $(x^2D^2 - 3xD + 4)\, y = x^6$ ……. (i)

Equation (i) in homogeneous differential equation,

So, put $x = e^z \Rightarrow \log x = z$

$xDy = \delta y$, $x^2D^2y = (\delta^2 - \delta)\, y$

Where, $\delta = \frac{d}{dz}$ in (i)

Now, (i) becomes,

$(\delta^2 - \delta - 3\delta + 4)\, y = e^{6z}$

or, $(\delta^2 - 4\delta + 4)\, y = e^{6z}$

or, So its, A. E. is

$m^2 - 4m + 2 = 0$

or, $(m - 2)^2 = 0 \Rightarrow m = 2, 2$

Hence, C. F. $= (C_1 + C_2z)\, e^{2z}$

and P. I. $= \frac{1}{(\delta^2 - 4\delta + 4)}\, e^{6z}$

$= \frac{1}{(6^2 - 4.6 + 4)}\, e^{6z} = \frac{1}{(36 - 24 + 4)}\, e^{6z} = \frac{1}{16}\, e^{6z}$

Thus, y = C. F. + P. I.

$= (C_1 + C_2z)\, e^{2z} + \frac{1}{16}\, e^{6z}$

$= (C_1 + C_2\log x)\, e^{2\log x} + \frac{1}{16}\, e^{6\log x}$

or, $y = (C_1 + C_2 \log x)\, x^2 + \frac{1}{16}\, x^6$ is the required solution.

5. $\mathbf{x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 6x}$ ***(B. E. 2071)***

Sol^n. Given differential equation is,

$$x\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 6x$$

Multiplying both sides by x we get,

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 6x^2$$

or, $(x^2D^2 + 2xD)\, y = 6x^2$ …….. (i)

Equation (i) is homogeneous differential equation,

So, put $x = e^z \Rightarrow \log x = z$

$xDy = \delta y$; $x^2D^2y = (\delta^2 - \delta)\, y$

When, $\delta = \frac{d}{dz}$ in (i)

Now, (i) becomes,

$(\delta^2 - \delta + 2\delta)\, y = 6e^{2z}$

or, $(\delta^2 + \delta)\, y = 6e^{2z}$

So, it A. E. is,

$m^2 + m = 0 \Rightarrow m(m + 1) = 0$

or, $m = 0, -1$

Hence, C. F. $= C_1 + C_2e^{-z}$

and P. I. $= \frac{1}{\delta^2 + \delta}\, 6e^{2z} = \frac{1}{2^2 + 2}\, 6e^{2z} = \frac{6}{6}\, e^{2z} = e^{2z}$

Thus, y = C. F. + P. I.

$= C_1 + C_2e^{-z} + e^{2z}$

$= C_1 + C_2\, e^{-\log x} + e^{2\log x}$

or, $y = C_1 + \frac{C_2}{x} + x^2$ is the required solution.

6. $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$ ***(B. E. 2072)***

Sol^n. Given differential equation is,

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$$

or, $(x^2D^2 - 2xD - 4)\, y = x^4$ ……… (i)

Equation (i) is homogeneous differential equation.

So, put, $x = e^z \Rightarrow \log x = z$

$xDy = \delta y$; $x^2D^2y = (\delta^2 - \delta)\, y$

Where, $\delta = \frac{d}{dz}$ in equation (i)

Now, equation (i) becomes,

$(\delta^2 - \delta - 2\delta - 4)\, y = e^{4z}$

or, $(\delta^2 - 3\delta - 4)\, y = e^{4z}$

So, it's A. E. is,

$m^2 - 3m - 4 = 0$

or, $m^2 - 4m + m - 4 = 0$

or, $m(m - 4) + 1(m - 4) = 0$

or, $(m + 1)(m - 4) = 0$

$\Rightarrow m = -1, 4$

Hence, C. F. $= C_1e^{-z} + C_2e^{4z}$

and P.I. $= \frac{1}{(\delta^2 - 3\delta - 4)} . e^4 = \frac{z}{(2\delta - 3)} e^{4z} = \frac{3}{(2.4 - 3)} e^{4z} = \frac{ze^{4z}}{5}$

Thus, y = C. F. + P. I.

$= C_1e^{-z} + C_2e^{4z} + \frac{z}{5} e^{4z}$

or, $C_1e^{-\log x} + C_2e^{4\log x} + \frac{\log x}{5} e^{4\log x}$

or, $y = \frac{C_1}{x} + C_2x^4 + \frac{\log x}{5} x^4$ is the required solution.

7. $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^2$

Sol^n. Given differential equation is,

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^2$$

or, $(x^2D^2 + xD - 4)\, y = x^2$ …….. (i)

Equation (i) in homogeneous differential equations

So, put $x = e^z \Rightarrow \log x = z$

$xDy = \delta y$; $x^2D^2y = (\delta^2 - \delta)\, y$

Where, $\delta = \frac{d}{dz}$ in equation (i)

Now, (i) becomes,

$(\delta^2 - \delta + \delta - 4)\, y = e^{2z}$

or, $(\delta^2 - 4)\, y = e^{2z}$

So, it A. E. is $m^2 - 4 = 0 \Rightarrow m = \underline{+}\, 2$

Hence, C. F. $= C_1e^{2z} + C_2e^{-2z}$

and P.I. $= \frac{I}{(\delta^2 - 4)} e^{2z} = z . \frac{1}{2\delta} . e^{2z} = \frac{z}{2.2} e^{2z} = \frac{z}{4} e^{2z}$

Thus, y = C. F. + P. I.

$= C_1e^{2z} + C_2e^{-2z} + \frac{z}{4} e^{2z}$

or, $y = C_1e^{2\log x} + C_2e^{-2\log x} + \frac{\log x}{4} e^{2\log x}$

or, $y = C_1x^2 + \frac{C_2}{x^2} + \frac{x^2}{4} \log x$ is the required solution.

8. $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Sol^n. Given differential equation is,

$$x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$$

or, $(x^2D^2 - xD + 1)y = \log x$ …….. (i)

Equation (i) is homogeneous differential equation is,

So, put, $x = e^z \Rightarrow \log x = 3$

$xDy = \delta y$; and $x^2D^2y = (\delta^2 - \delta)\, y$

Where, $\delta = \frac{d}{dz}$ in equation (i)

Now (i) becomes,

$(\delta^2 - \delta - \delta + 1)\, y = z$

or, $(\delta^2 - 2\delta + 1)\, y = z$

so, it's A. E. is, $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$

Hence, C. F. $= (C_1 + C_2z)\, e^z$

and P. I. $= \frac{1}{(\delta^2 - 2\delta + 1)} z = \frac{1}{(1 - \delta)^2} z = [1 - \delta]^{-2} z$

$= [1 + 2\delta + 3\delta^2 + ……]\, z$

P. I. $= (z + 2)$

Thus, y = C. F. + P. I.

or, $y = (C_1 + C_2z)\, e^z + z + 2$

or, $y = (C_1 + C_2 \log x)\, e^{\log x} + \log x + 2$

or, $y = (C_1 + C_2\log x) + \log x + 2$ is the required solution.

9. $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$ ***(B. E. 2068, 2069)***

Sol^n. Given differential equation is,

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$$

or, $(x^2D^2 + 4xD + 2)\,y = e^x$ ……… (i)

Equation (i) is homogeneous differential equation,

So, put, $x = e^z \Rightarrow \log x = z$

$xDy = \delta y;\ x^2D^2y = (\delta^2 - \delta)\,y$

Where, $\delta = \frac{d}{dz}$ in equation (i)

Now (i) becomes,

$(\delta^2 - \delta + 4\delta + 2)\,y = e^{e^z}$

or, $(\delta^2 + 3\delta + 2)\,y = e^{e^z}$

So, It's A. E. is

$m^2 + 3m + 2 = 0$

or, $m^2 + 2m + m + 2 = 0$

or. $(m + 2) + 1\,(m + 2) = 0$

or, $(m + 1)\,(m + 2) = 0$

$\Rightarrow m = -1, -2$

Hence, C. F. $= (C_1e^{-z} + C_2e^{-2z})$

and P. I. $= \frac{1}{(\delta^2+3\delta+2)}\,e^{e^z} = \frac{1}{(\delta+1)(\delta+2)}\,e^{2z}$

$= \left[\frac{1}{(\delta+1)} - \frac{1}{(\delta+2)}\right] e^{e^z}$ (by using partial fraction)

P.I. $= \frac{1}{(\delta+1)}e^{e^z} - \frac{1}{(\delta+2)}\,e^{e^z}$

We know, $\frac{X}{D-a} = e^{ax}\int e^{-ax}\,; X\,dx$

$\therefore$ P.I. $= \left(\frac{e^{e^z}}{\delta+1} - \frac{e^{e^z}}{\delta+2}\right) = e^{-z}\int e^z\,e^{e^z}\,dz - e^{-2z}\int e^{2z}\,e^{e^z}\,dz$ ……. (ii)

Now, $\int e^z\,e^{e^z}\,dz$

Put $e^z = t \Rightarrow e^z\,dz = dt$

$= \int e^t dt = e^t = e^{e^z}$

Also, $\int e^{2z}\,e^{e^z}\,dz$

Put, $e^z = t \Rightarrow e^z\,dz = dt$

$= \int e^z \,.\, e^z\, e^{e^z}\,dz$

$= \int t \,.\, e^t\,dt$

$= t\,e^t - e^t = e^z\,e^{e^z} - e^{e^z}$

$\therefore$ P. I. $= e^{-z}\,e^z - e^{-2z}\,[e^z\,e^{e^z} - e^{e^z}]$

$= e^{-z}\,e^z - e^{-z}\,e^z + e^{-2z}\,e^z$

P.I. $= e^{-2z}\,e^{e^z}$

Thus, y = C. F. + P. I.

$= C_1e^{-z} + C_2e^{-2z} + e^{-2z}\,e^{e^z}$

or, $y = C_1\,e^{-\log x} + C_2e^{-2\log x} + e^{-2\log x}\,.\,e^{\log x}$

or, $y = \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{1}{x^2}\,e^x$ is the required solution.

10. $(x^2D^2 + xD + 1)\,y = \sin(\log x^2)$

Sol^n. Given differential equation is,

$(x^2D^2 + xD + 1)\,y = \sin(\log x^2)$

Equation (i) is homogeneous differential equation,

So, put, $x = e^z \Rightarrow \log x = z$

$xDy = \delta y;\ x^2D^2y = (\delta^2 - \delta)\,y$

Where, $\delta = \frac{d}{dz}$ in equation (i)

Now (i) becomes,

$(\delta^2 - \delta + \delta + 1)\,y = \sin 2z$

or, $(\delta^2 + 1)\,y = \sin 2z$

So, It's A. E. is $m^2 + 1 = 0 \Rightarrow m = \pm\, i$

Hence, C. F. $= A\cos z + B\sin z$

and P. I. $= \frac{1}{(\delta^2+1)}\sin 2z = \frac{1}{-2^2+1}\sin 2z = -\frac{1}{3}\sin 2z$

Thus, y = C. F. + P. I.

$= A\cos z + B\sin z - \frac{1}{3}\sin 2z$

$= A\cos(\log x) + B\sin(\log x) - \frac{1}{3}\sin(\log x^2)$ is the required solution.

11. $(x^2D^2 - 2)\,y = x^2 + \frac{1}{x}$

Sol^n. Given differential equation is,

$(x^2D^2 - 2)\,y = x^2 + \frac{1}{x}$ ……… (i)

Equation (i) is homogeneous differential equaton.

So, Put, $x = e^z \Rightarrow \log x = z$

$xDy = \delta y;\ x^2D^2y = (\delta^2 - \delta)\,y$

Where, $\delta = \frac{d}{dz}$ in equation (i)

Now (i) becomes,

$(\delta^2 - \delta - 2)y = e^{2z} + e^{-z}$

So, It's A. E. = $m^2 - m - 2 = 0$

or, $m^2 - 2m + m - 2 = 0$

or, $m(m-2) + 1(m-2) = 0$

or,. $(m+1)(m-2) = 0$

or, $m = -1, 2$

Hence, C. F. = $C_1e^{-z} + C_2e^{2z}$

and P. I. = $\frac{1}{\left(\delta^2 - \delta - 2\right)} e^{2z} + e^{-z}$

$$= \frac{1}{\left(\delta^2 - \delta - 2\right)} e^{2z} + \frac{1}{\left(\delta^2 - \delta - 2\right)} e^{-z}$$

$$= \frac{z}{2\delta - 1} e^{2z} + \frac{z}{2\delta - 1} e^{-z}$$

$$= \frac{z}{3} e^{2z} + \frac{z}{-3} e^{-z}$$

$$= \frac{z}{3}(e^{2z} - e^{-z})$$

Thus, y = C. F. + P. I.

or, $y = C_1e^{-z} + C_2 e^{2z} + \frac{z}{3}(e^{2z} - e^{-z})$

or, $y = C_1e^{-\log x} + C_2 e^{2\log x} + \frac{\log x}{3}(e^{2\log x} - e^{-\log x)}$

or, $y = \frac{C_1}{x} + C_2x^2 + \frac{\log x}{3}\left(x^2 - \frac{1}{x}\right)$ is the required solution.

12. $\mathbf{(x+p)^2 \frac{d^2y}{dx^2} - 4(x+p)\frac{dy}{dx} + 6y = x}$

Sol^n. Given differential equation is,

$$(x+p)^2 \frac{d^2y}{dx^2} - 4(x+p)\frac{dy}{dx} + 6y = x$$

or, $(x+p)^2 \frac{d^2y}{dx^2} - 4(x+p)\frac{dy}{dx} + 6y = x$

or, $\{(x+p)^2 D^2 - 4(x+p)D + 6\}\, y = x$ …….. (i)

equation (i) is homogeneous differential equation is,

So, put, $(x+p) = e^z \Rightarrow z = \log(x+p)$

$(x+p)Dy = \delta y$, $(x+p)^2D^2y = (\delta^2 - \delta)y$

Which $\delta = \frac{d}{dz}$ in equation (i)

Now, equation (i) becomes,

$(\delta^2 - \delta - 4\delta + 6)\, y = (e^z - p)$

or, $(\delta^2 - 5\delta + 6)\, y = e^z - p$

So, It's, A. E. is

$m^2 - 5m + 6 = 0$

or, $m^2 - 2m - 3m + 6 = 0$

or, $m(m-2) - 3(m-2) = 0$

or, $(m-2)(m-3) = 0$

or, m (2, 3

Hence, C. F. = $C_1e^{2z} + C_2e^{3z}$

and P.I. = $\frac{1}{\left(\delta^2 - 5\delta + 6\right)}(e^z - p)$

$$= \frac{1}{\left(\delta^2 - 5\delta + 6\right)} e^z - \frac{z}{\left(\delta^2 - 5\delta + 6\right)} e^{0.z}$$

P.I. = $\frac{1}{2} e^z - \frac{p}{6}$

Thus, y = C. F. + P. I.

$= C_1e^{2z} + C_2 e^{3z} + \frac{1}{2} e^z - \frac{p}{6}$

or, $y = C_1e^{2\log(x+p)} + C_2 e^{3\log(x+p)} + \frac{1}{2} e^{\log(x+p)} - \frac{p}{6}$

or, $y = C_1(x+p)^2 + C_2(x+p)^3 + \frac{1}{2}(x+p) - \frac{p}{6}$

or, $y = C_1(x+p)^2 + C_2(x+p)^3 + \frac{\left(3x + 2p\right)}{6}$ is the required solution.

13. $\mathbf{(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = (2x+3)(2x+4)}$

Sol^n. Given differential equation is,

$$(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = (2x+3)(2x+4)$$

or, $\{(x+1)^2 D^2 + (x+1)D\}\, y = (2x+3)(2x+4)$ ……. (i)

Equation (i) is homogeneous differential equation,

So, put $(x+1) = e^z \Rightarrow z = \log(x+1)$

or, $(x+1)Dy = \delta y$; $(x+1)^2 D^2y = (\delta^2 - \delta)\, y$

Where, $\delta = \frac{d}{dz}$ in equation (i)

Now equation (i) becomes,

$(\delta^2 - \delta + \delta)\, y = \{2(e^z - 1) + 3\}\, \{2(e^z - 1) + 4\}$

or, $\delta^2 y = (2e^z + 1)(2e^z + 2)$

$= 2(2e^z + 1)(e^z + 1)$
$= 2(2e^{2z} + 3e^z + 1)$
$= 4(e^{2z} + 6e^z + 2)$

It,s A. E. is, $m^2 = 0 \Rightarrow m = 0, 0$

Hence, C. F. $= C_1 + C_2z$

and P. I. $= \frac{1}{\delta^2}(4e^{2z} + 6e^z + 2) = \frac{4e^{2z}}{4} + 6e^z + z^2$

Thus, y = C. F. + P. I.

or, $y = C_1 + C_2z + e^{2z} + 6e^z + z^2$

or, $y = C_1 + C_2\log(x + 1) + e^{2\log(x+1)} + 6e^{\log(x+1)} + \{\log(x + 1)\}^2$

or, $y = C_1 + C_2\log(x + 1) + (x + 1)^2 + 6(x + 1) + \{\log(x + 1)\}^2$

or, $y = C_1 + C_2\log(x + 1) + x^2 + 2x + 1 + 6x + 6 + \{\log(x + 1)\}^2$

or, $y = C_1 + C_2\log(x + 1) + \{\log(x + 1)\}^2 + 8x + 7$ is the required solution.

14. $(2x+3)^2\frac{d^2y}{dx^2} + 2(2x+3)\frac{dy}{dx} - 4y = 8x$

Sol^n. Given differential equation is,

$(2x+3)^2\frac{d^2y}{dx^2} + 2(2x+3)\frac{dy}{dx} - 4y = 8x$

or, $(2x + 3)^2D^2 + 2(2x + 3)D - 4\}y = 8x$ …….. (i)

Equation (i) is homogeneous differential equation.

So, put $(2x + 3) = e^z \Rightarrow z = \log(2x + 3)$

and $(2x + 3)Dy = 2\delta y$; $(2x + 3)D^2y = 4(\delta^2 - \delta)$

Where $\delta = \frac{d}{dz}$ in equation (i)

Now equation (i) becomes,

$\{4(\delta^2 - \delta) + 4\delta - 4\}y = \frac{8}{2}(e^z - 3)$

or, $(4\delta^2 - 4\delta + 4\delta - 4)y = 4(e^z - 3)$

or, $(4\delta^2 - 4)y = 4(e^z - 3)$

So, Its, A. E. is,

$4m^2 - 4 = 0 \Rightarrow m = \underline{+}1$

Hence, C. F. $= C_1e^z + C_2e^{-z}$

and P. I. $= \frac{1}{(4\delta^2 - 4)} \cdot 4(e^z - 3)$

$= \left[\frac{1}{(\delta^2 - 1)}e^z - \frac{z}{(\delta^2 - 1)}\right]$

$= \left[\frac{z}{2\delta}e^z + 3\right]$

$= \frac{z}{2}e^z + 3$

Thus, y = C.F. + P. I.

$= (C_1e^z + C_2e^{-z}) + \frac{z}{2}e^z + 3$

or, $y = C_1e^{\log(2x+3)} + C_2e^{-\log(2x+3)} + \frac{\log(2x+3)}{2}e^{\log(2x+3)} + 3$

or, $y = C_1(2x + 3) + \frac{C_2}{(2x+3)} + \frac{(2x+3)}{2}\log(2x + 3) + 3$ is the required solution.

15. $(2x+1)^2\frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$

Sol^n. Given differential equation is,

$(2x+1)^2\frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$

or, $(2x + 1)^2D^2y - 6(2x + 1)Dy + 16y = 8(2x + 1)^2$

or, $\{(2x + 1)^2D^2 - 6(2x + 1)D + 16\}y = 8(2x + 1)^2$ ….. (i)

Equation (i) is homogeneous differential equation

So, put, $(2x + 1) = e^z \Rightarrow z = \log(2x + 1)$

$(2x + 1)Dy = 2\delta$; $(2x + 1)^2D^2y = 4(\delta^2 - \delta)$

Where, $\delta = \frac{d}{dz}$ in (i)

Now, equation (i) becomes,

$\{4(\delta^2 - \delta) - 12\delta + 16\}y = 8e^{2z}$

or, $(4\delta^2 - 4\delta - 12\delta + 16)y = 8e^{2z}$

or, $(4\delta^2 - 16\delta + 16)y = 8e^{2z}$

or, $(\delta^2 - 4\delta + 4) = 2e^{2z}$

So, It's A. E. is $m^2 - 4m + 4 = 0$

or, $(m - 2)^2 = 0 \Rightarrow m = 2, 2$

Hence, C. F. $= (C_1 + C_2z)e^{2z}$

and P. I. $= \frac{1}{(\delta^2 - 4\delta + 4)}2e^{2z} = \frac{z}{(2z - 4)}2e^{2z} = \frac{z^2}{2} \cdot 2e^{2z}$

P. I. $= z^2e^{2z}$

Thus, y = C. F. + P. I.

$= (C_1 + C_2z)e^{2z} + z^2e^{2z}$

or, $y = \{C_1 + C_1\log(2x + 1)\}e^{2\log(2x+1)} + \{\log(2x + 1)\}^2e^{2\log(2x+1)}$

or, $y = C_1 + C_1\log(2x + 1)\}(2x + 1)^2 + \{\log(2x + 1)\}^2(2x + 1)^2$

$= (2x + 1)^2[\{\log(2x + 1)\}^2 + C_1 + C_2\log(2x + 1)]$ is the required general solution.

Important IOE Questions and Solution

1. Solve $(D^2 - 3D + 2)\, y = x^2 + x$ ***(B. E. 2067)***

Sol^n. Given differential equation is,

$(D^2 - 3D + 2)\, y = x^2 + x$

So it's a. F. is

$m^2 - 3m + 2 = 0$

or, $m^2 - 2m - m + 2 = 0$

or, $m(m-2) - 1(m-2) = 0$

or, $(m-1)(m-2) = 0$

$\Rightarrow m = 1, 2$

So, C. F. $= C_1 e^x + C_2 e^{2x}$

and P. I. $= \dfrac{1}{(D^2 - 3D + 2)} (x^2 + x)$

$$= \frac{1}{2}\left[1 + \frac{D^2 - 3D}{2}\right]^{-1} (x^2 + x)$$

$$= \frac{1}{2}\left[1 - \left(\frac{D^2 - 3D}{2}\right) + \left(\frac{D^2 - 3D}{2}\right)^2 - \ldots\right] (x^2 + x)$$

$$= \frac{1}{2}\left[x^2 + x - \frac{1}{2}(2 - 3.2x - 3) + \frac{9}{4}.2\right]$$

$$= \frac{1}{2}\left[x^2 + x + \frac{1}{2}(1 + 6x) + \frac{9}{2}\right]$$

$$= \frac{1}{2}\left[x^2 + x + \frac{1}{2} + 3x + \frac{9}{2}\right] = \frac{1}{2}[x^2 + 4x + 5]$$

Then, y = C. P. + P. I.

or, $y = C_1 e^x + C_2 e^{2x} + \frac{1}{2}(x^2 + 4x + 5)$

2. Solve the differential equation $x\dfrac{dy}{dx} + y \log y = xye^x$

(B. E. 2069)

Sol^n. Given differential equation is $x\dfrac{dy}{dx} + y \log y = xy\, e^x$

Dividing both sides by xy,

$$y^{-1}\frac{dy}{dx} + \frac{\log y}{x} = e^x$$

Put $\log y = v$

$$y^{-1}\frac{dy}{dx} = \frac{dv}{dx}$$

So the equation reduces to $\dfrac{dv}{dx} + \dfrac{v}{x} = e^x$

This is linear form, $p = \dfrac{1}{x}$, $Q = e^x$

$\therefore$ I. F. $= e^{\int p dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Its general solution is,

$v \times \text{I.F.} = \int Q \times (\text{I. F.})\, dx$

or, $vx = \int xe^x\, dx + C$

$\therefore$ $x \log y = xe^x - e^x + C$ is the required solution.

3. Solve the differential equation $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = e^x + e^{-x}$

(B. E. 2069)

Sol^n. Given differential equation is $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = e^x + e^{-x}$

or, $(D^2 + 2D)y = e^x + e^{-x}$

It's auxiliary equation is,

$m^2 + 2m = 0$

or, $m(m + 2) = 0$

Either $m = 0$ or $m + 2 = 0$, $m = -2$

$\therefore$ C. F. $= C_1 e^{0.x} + C_2 e^{-2x}$

$= C_1 + C_2\, e^{-2x}$

and P. I. $= \dfrac{1}{(D^2 + 2D)} (e^x + e^{-x})$

$$= \frac{1}{(D^2 + 2D)} e^x + \frac{1}{(D^2 + 2D)} e^{-x-}$$

$$= \frac{1}{(1^2 + 2.1)} e^x + \frac{1}{(-1)^2 + 2(-1)} e^{-x}$$

$$= \frac{e^x}{3} + \frac{e^{-x}}{1}$$

$\therefore$ The general solution is, y = C.F. + P. I.

$$y = C_1 + C_2 e^{-2x} + \frac{e^x}{3} + e^{-x}$$

4. Solve $y = px - \sqrt{m^2+p^2}$ where $p = \frac{dy}{dx}$ ***(B. E. 2069)***

Sol^n. Given differential equation is $y = px - \sqrt{m^2+p^2}$ .... (i)

This is clairaut's equation, differential equation (i) w. r. t. x

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{1}{2\sqrt{m^2+p^2}} . 2p\frac{dp}{dx}$$

$$\text{or, } p = p + \left(x + \frac{p}{\sqrt{m^2+p^2}}\right)\frac{dp}{dx}$$

$$\text{or, } 0 = \left(x + \frac{p}{\sqrt{m^2+p^2}}\right)\frac{dp}{dx}$$

Either $\frac{dp}{dx} = 0 \Rightarrow p = c$ .... (ii)

Eliminating p from (i) and (ii) we get,

$y = cx - \sqrt{m^2+c^2}$ is the required general solution.

or, $x - \frac{p}{\sqrt{m^2+p^2}} = 0$ .... (iii)

Eliminating p from (i) and (iii)

From (iii), $x^2 = \frac{p^2}{m^2+p^2}$

$\Rightarrow m^2x^2 + p^2x^2 = p^2$

$\Rightarrow m^2x^2 = p^2(1-x^2)$

$\Rightarrow p^2 = \frac{m^2x^2}{1-x^2} \quad \Rightarrow p = \frac{mx}{\sqrt{1-x^2}}$

From (i), $y = \frac{mx}{\sqrt{1-x^2}} . x - \sqrt{m^2 + \frac{m^2x^2}{1-x^2}}$

$$\text{or, } y = \frac{mx^2}{\sqrt{1-x^2}} - \sqrt{\frac{m^2 - m^2x^2 + m^2x^2}{1-x^2}}$$

$$\Rightarrow y = \frac{mx^2}{\sqrt{1-x^2}} - \frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow y = \frac{m\left(x^2-1\right)}{\sqrt{1-x^2}}$$

$$\Rightarrow y = \frac{-m\left(1-x^2\right)}{\sqrt{1-x^2}}$$

$\Rightarrow y = -m\sqrt{1-x^2}$

$\Rightarrow m^2(1-x^2) = y^2$

or, $y^2 + m^2x^2 = m^2$ is the required singular solution.

5. A resistance of 100 ohms, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in the circuit as a function of time. ***(B. E. 2069, 070)***

Sol^n. Let i be the current flowing in a circuit at any time t, then by Krichhoff's first law.

Sum of voltage drops across R and L = E

or, $Ri + L\frac{di}{dt} = E$

or, $\frac{di}{dt} + \frac{R}{t}i = \frac{E}{L}$

This is first order line or differential equation and its general solution is,

$$i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

Here R = 100 ohms. L = 0.5 henry, E = 20 volts

$$i = \frac{20}{100}\left(1 - e^{-\frac{100t}{0.5}}\right)$$

$$= \frac{1}{5}\left(1 - e^{-200t}\right)$$

6. Solve the differential equation $y = yp^2 + 2px$ where $p = \frac{dy}{dx}$ ***(B. E. 2070)***

Sol^n. Given differential equation is,

$y = yp^2 + 2px$

or, $x = \frac{y}{2p} - \frac{yp^2}{2p}$ or, $x = \frac{1}{2}\frac{y}{p} - \frac{1}{2}yp$ .... (i)

Which is solvable for x,

Differentiate (i) w. r. t. y

$$\frac{dx}{dy} = \frac{1}{2p^2}\left[p - y\frac{dp}{dy}\right] - \frac{1}{2}\left(y\frac{dp}{dy} + p\right)$$

$$\text{or, } \frac{1}{p} = \frac{1}{2p^2} - \frac{1}{2}y\frac{dp}{dy}\left(\frac{1}{p^2} - 1\right) - \frac{p}{2}$$

or, $\frac{1}{p}-\frac{1}{2p}+\frac{p}{2}=\frac{-y}{2}\frac{dp}{dy}\left(\frac{1}{p^2+1}\right)$ or, $\frac{(p^2+1)}{2p}=-\frac{y}{2}\frac{dp}{dy}\frac{(p^2+1)}{p^2}$

or, $\int\frac{dy}{y}+\int\frac{dp}{p}=0$ Integrating.

$\log y+\log p=\log c$ or, $p=\frac{c}{y}$

Eliminating p from (i) and (ii)

We get, $y=y\frac{c^2}{y^2}+2\frac{c}{y}x$

or, $y^2=c^2+2cx$ is the required general solution.

7. Solve the differential equation $(D^2-2D+5)y=e^{2x}.\sin x$

(B. E. 2070)

Sol^n. Given, differential equation is $(D^2-2D+5)y=e^{2x}\sin x$

So its auxiliary equation is,

$m^2-2m+5=0$ or, $m=\frac{+2\pm\sqrt{4-20}}{2}$

$\therefore m=1\pm 2i$

So, C.F. $=e^x(A\cos 2x+B\sin 2x)$

Now, P.I. $=\frac{1}{(D^2-2D+5)}e^{2x}\sin x$

$=e^{2x}\frac{1}{\{(D+2)^2-2(D+2)+5\}}\sin x=e^{2x}\frac{1}{(D^2+4D+4-2D-4+5)}\sin x$

$=e^{2x}\frac{1}{(D^2+2D+5)}\sin x \quad =e^{2x}\frac{1}{(-1^2+2D+5)}\sin x$

$=e^{2x}\frac{1}{(2D+4)}\sin x=e^{2x}\frac{(\pi-2)}{2(D+2)(D-2)}\sin x$

$=\frac{e^{2x}}{2}\frac{(D-2)}{(D^2-4)}\sin x=\frac{e^{2x}}{2}\frac{(D-2)}{(-1^2-4)}\sin x$

$=-\frac{1}{10}e^{2x}(\cos x-2\sin x)$

Thus y = C. F. + P. I.

or, $y=e^x(A\cos 2x+B\sin 2x)-\frac{1}{10}e^{2x}(\cos x-2\sin x)$ is the general solution.

8. Solve the differential equation $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+2y=x\log x$

(B. E. 2070, 067)

Sol^n. Given differential equation is, $\frac{d^2y}{dx^2}-x\frac{dy}{dx}+2y=x\log x$

or, $(x^2D^2-xD+2)y=x\log x$ .... (i)

This is homogeneous differential equation

So put $x=e^z\Rightarrow z=\log x$

and $xDy=\delta y$, $x^2D^2y=(\delta^2-\delta)y$

Where $\delta=\frac{d}{dz}$ in (i)

So the equation (i) reduces to $(\delta^2-\delta-\delta+2)y=e^z.z$

or, $(\delta^2-2\delta+2)y=e^z.z$

Its auxiliary equation is $m^2-2m+2=0$

$m=\frac{2\pm\sqrt{4-8}}{2}=\frac{2\pm\sqrt{-4}}{2}=1\pm i$

$\therefore m=1\pm i$

So, C. F. $=e^z(A\cos z+B\sin z)$

$=x(A\cos\log x+B\sin\log x)$

and P. I. $=\frac{1}{(\delta^2-2\delta+2)}e^z.z=e^z\frac{1}{(\delta+1)^2-2(\delta+1)+2}z$

$=e^z\frac{1}{\delta^2+2\delta+1-2\delta-2+2}z=e^z\frac{1}{\delta^2+1}z$

$=e^z(1+\delta^2)-1z=e^z(1-\delta^2+\delta^4\text{ ....})z$

$=e^z z=x\log x$

Its general solution is, y = C. F. + P. I.

$=Ax\text{ clos }(\log x)+Bx\sin(\log x)+x\log x$

9. Solve : $y-2px+ayp^2=0$ ***(B. E. 2070)***

Sol^n. Given differential equation,

$y-2px+ayp^2=0$

or, $y+ayp^2=2px$

or, $2x=\frac{y+ayp^2}{p}=\frac{y}{p}+apy$ .... (i)

Which is the form x = f(y, p)

So, it is solvable for x

$\therefore$ Diff. w. r. t. 'y'

$2\frac{dx}{xy}=\frac{p.\frac{dy}{dx}-y\frac{dp}{dy}}{p^2}+a\left(y\frac{dp}{dy}+p\frac{dy}{dy}\right)$

or, $\frac{2}{p}=\frac{1}{p}-\frac{y}{p^2}\frac{dp}{dy}+ay\frac{dp}{dy}+ap$

or, $\frac{2}{p}-\frac{1}{p}-ap=\left(ay-\frac{y}{p^2}\right)\frac{dp}{dy}$

or, $\frac{\left(1-ap^2\right)}{p}=-y\frac{(1-ap^2)}{p^2}\frac{dp}{dy}$

or, $1=-\frac{y}{p}\frac{dp}{dy}$

$-\int\frac{1}{y}dy=\int\frac{1}{p}dp$ integrating

$-\log y+\log c=\log p$

or, $\log p=\log\left(\frac{c}{y}\right)$ $\therefore p=\frac{c}{y}$ .... (ii)

Eliminating p from (i) and (ii)

$2x=\frac{y}{\frac{c}{y}}+ay.\frac{c}{y}$

$\Rightarrow 2x=\frac{y^2}{c}+ac$

or, $y^2+ac^2=2cx$ is the required general solution.

10. Solve : $(D^2-2D+5)y=e^{2x}.\sin x$ ***(B. E. 2070)***

Sol^n. Given differential equation,

$(D^2-2D+5)y=e^{2x}\sin x$

Its auxiliary equation is,

$m^2-2m+5=0$

$m=\frac{-(-2)\pm\sqrt{(-2)^2-4.1.5}}{2.1}$

$=\frac{2\pm\sqrt{4-20}}{2}=\frac{2\pm 4i}{2}$

$=1\pm 2i$

$\therefore$ C.F. $=e^x(A\cos 2x+B\sin 2x)$

and P. I. $=\frac{1}{(D^2-2D+5)}e^{2r}\sin x$

$=\frac{1}{\left\{(D+2)^2-2(D+2)+5^2\right\}}\sin x$

$=\frac{1}{(D^2+4D+4-2D-4+5)}\sin x=\frac{1}{(D^2+2D+5)}\sin x$

$=\frac{1}{(-1^2+2D+5)}\sin x=\frac{1}{(2D+4)}\sin x=\frac{1}{2}\frac{(D-2)}{(D+2)(D-2)}\sin x$

$=\frac{1}{2}\frac{(D-2)}{(D^2-4)}\sin x=\frac{1}{2}\frac{(D-2)}{(-1^2-4)}\sin x=\frac{1}{2}.\frac{1}{(-5)}(D-2)\sin x$

$=-\frac{1}{10}(\cos x-2\sin x)$

$\therefore$ The general solution is y = C. F. + P. I.

$=e^x(A\cos 2x+B\sin 2x)-\frac{1}{10}(\cos x-2\sin x)$

11. Solve : $xp^2-2yp+ax=0$ where $p=\frac{dy}{dx}$ ***(B. E. 2071)***

Sol^n. Given differential equation is

$xp^2-2yp+ax=0$ where $p=\frac{dy}{dx}$

or, $xp^2+ax=2yp$

$2y=\frac{xp^2+ax}{p}=xp+a\frac{x}{p}$ .... (i)

Which is of the form y = f(x. p)

So it is solvable for y

$\therefore$ Differential equation (i) w. r. t. 'x'

$2\frac{dy}{dx}=x\frac{dp}{dx}+p\frac{dx}{dx}+a\left(\frac{p\frac{dx}{dx}-x\frac{xp}{dx}}{p^2}\right)$

or, $2p=x\frac{dp}{dx}+p+a\frac{p}{p^2}-\frac{ax}{p^2}\frac{dp}{dx}\Rightarrow 2p-p-\frac{a}{p}=\left(x-\frac{ax}{p^2}\right)\frac{dp}{dx}$

$\Rightarrow\frac{(p^2-a)}{p}=\frac{x(p^2-a)}{p^2}\frac{dp}{dx}\Rightarrow 1=\frac{x}{p}\frac{dp}{dx}\Rightarrow pdx=xdp$

or, $\int\frac{1}{x}dx=\int\frac{1}{p}dy$, integrating

or, $\log x=\log c=\log p$

$\log y\ p=\log x\Rightarrow p=xc$ .... (ii)

Eliminating p from (i) and (ii)

$2y=x.(xc)+a\frac{x}{cx}\Rightarrow 2y=x^2c+\frac{a}{c}$ is the required general solution.