Chapter 8
Analysis of Simulation Output

Simulation Output Introduction

- Whenever a random variable is introduced to the simulation model, all the system variables that describe its behavior become random or stochastic.
- The values of the variables involved in the system will fluctuate as the simulation proceeds.
- So, arbitrary measurement of the values of these variables can not represent the true value.
- In simulation study, it is assumed that the observations being made are mutually independent. But, in most of the real world problems, simulation results are mutually dependent.
- The various methods used to analyze simulation results are as follows:
  1. Estimation Methods
  2. Simulation Run Statistics
  3. Replication of Runs
  4. Elimination of Initial Bias

Estimation Methods

- Estimation Method estimates the range for the random variable so that the desired output can be achieved.
- It is assumed that the random variables are stationary and independent drawn from an infinite population with a finite mean μ and finite variance σ^2.
- Such random variables that meet all these conditions are called Independently and Identically Distributed (IID) random variable.
- The central limit theorem can be applied to IID data. It states that "the sum of n numbers of IID variables, drawn from a population that has a mean of μ and a variance of σ^2, is approximately distributed as a normal variable with a mean of $n\mu$ and a variance of $n\sigma^2$."
- Here Sample variable and time does not affect population distribution .

- Let x(i) be the n IID random variables i.e. random variables drawn from a sample of population with mean μ and variance σ^2, where (i=1,2,…,n) .
- Then

$$\boxed{Z = \frac{\sum_{i=1}^{n} x_i - n\mu}{\sigma\sqrt{n}}}$$

- In terms of sample mean

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

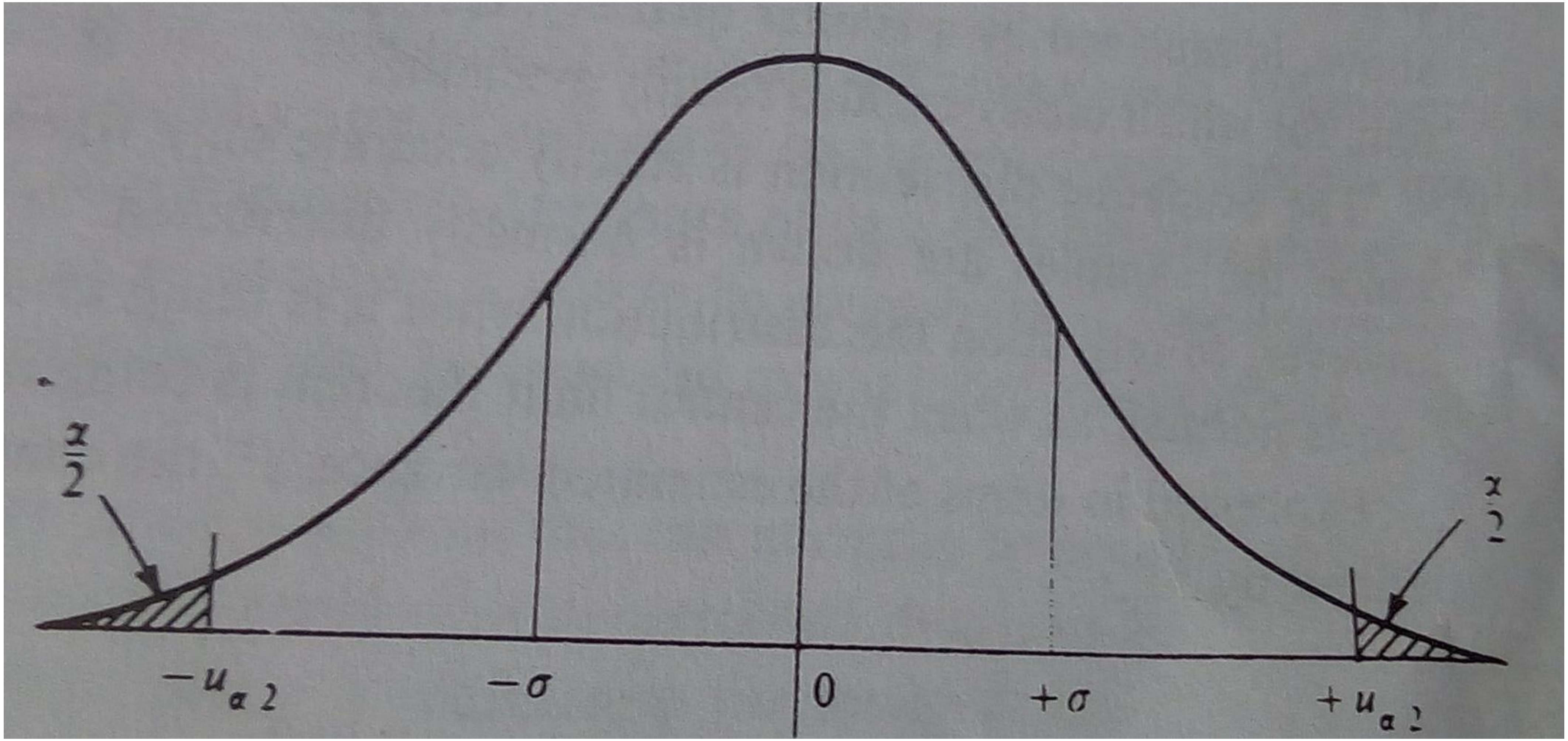

Figure: Probability Density Function of Standard Normal Variate

- For some constant α, known as level of significance, the probability that z lies between $-\mu_{\alpha/2}$ and $\mu_{\alpha/2}$ is given by:

$$\text{Prob}\{-\mu_{\alpha/2} \leq z \leq \mu_{\alpha/2}\} = 1 - \alpha$$

- In case of sample mean, this probability statement can be written as:

$$\text{Prob}\{\bar{x} + \frac{\sigma}{\sqrt{n}}\mu_{\alpha/2} \geq \mu \geq \bar{x} - \frac{\sigma}{\sqrt{n}}\mu_{\alpha/2}\} = 1 - \alpha$$

- The constant 1- α is call confidence level and the confidence interval is

$$\bar{x} \pm \frac{\sigma}{\sqrt{n}}\mu_{\alpha/2}$$

- Typically, the confidence level might be 90% in which case $\mu_{\alpha/2}$ is 1.65.
- Here the population variance σ^2 is usually not known. In this case it is can be replaced by an estimate calculated from the formula

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

- The normalized random variable based on σ^2 is replaced by a normalized random variable based on s^2.This has a **t distribution**, with n-1 degrees of freedom.
- The quantity $\mu_{\alpha/2}$ used in the distribution of a confidence interval given above is represented by a similar quantity $t_{n-1,\alpha/2}$ based on t-distribution.
- The t-distribution is strictly accurate only when the population from which the samples are drawn is normally distributed.

- Expressed in terms of the estimated variance s^2, the confidence interval for $\bar{x}$ is defined by

$$\bar{x} \pm \frac{s}{\sqrt{n}} t_{n-1,\alpha/2}$$

- Hence the estimation method gives the desired range of the sample variable taken from infinite population.

Example:

- Let us consider x(i) where i = 1, 2, 3, …………., n be the n number of random variables drawn from a sample of population with mean μ and variance σ^2.
- Using central limit theorem, and transforming to standard normal distribution, we get:

$$Z = \frac{\sum_{i=1}^{n} x_i - n\mu}{\sigma\sqrt{n}}$$

- Dividing top and bottom by n, we get:
- $Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ where $\bar{x}(n) = \frac{1}{n}\sum_{i=1}^{n} x_i$ = sample mean

Simulation Run Statistics

- One of the measure to analyze simulation result.
- This approach is used to obtain independent results by repeating the simulation.
- In most of the simulation study, the assumptions of stationary and mutually independent observations do not apply. An example of such case is queuing system.
- Correlation is necessary to analyze such scenario. In such cases, simulation run statistics method is used.
- Consider a single-server system in which the arrivals occur with a Exponential distribution and the service time has an exponential distribution.
- Suppose the study objective is to measure the mean waiting time, defined as the time entities spend waiting to receive service and excluding the service time itself.

- This system is commonly denoted by M/M/1 which indicates:
    - a. the inter-arrival time is distributed exponentially
    - b. the service time is distributed exponentially
    - c. there is one server
- In a simulation run, the simplest approach is to estimate the mean waiting time by accumulating the waiting time of n successive entities and dividing by n.
- This measure is denoted by $\bar{x}$(n) which emphasizes the fact that its value depends on the number of observations taken.
- If x_i for i=1,2,….,n are the individual waiting times then

$$\bar{x}(\mathrm{n}) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

- Whenever a waiting line forms, the waiting time of each entity on the line clearly depends upon the waiting time of its predecessors.
- Any series of data that has this property of having one value affect other values is said to be <u>autocorrelated</u>.
- The sample mean of autocorrelated data can be shown to approximate a normal distribution as the sample size increases.
- The value of $\bar{x}(\mathrm{n}) = \frac{1}{n}\sum_{i=1}^{n} x_i$ remains a satisfactory estimate for the mean of autocorrelated data.
- A simulation run is started with the system in some initial state, frequently the idle state, in which no service is being given and no entities are waiting.
- The early arrivals then have a more than normal probability of obtaining service quickly. So a sample mean that includes the early arrivals will be biased.

- For a given sample size starting from a given initial condition, the sample mean distribution is stationary.
- But if the distributions could be compared for different sample sizes, the distribution would be slightly different.
- The following figure is based on theoretical results, which shows how the expected value of sample mean depends upon the sample length, for the M/M/1 system, starting from an initial empty state, with a server utilization of 0.9.

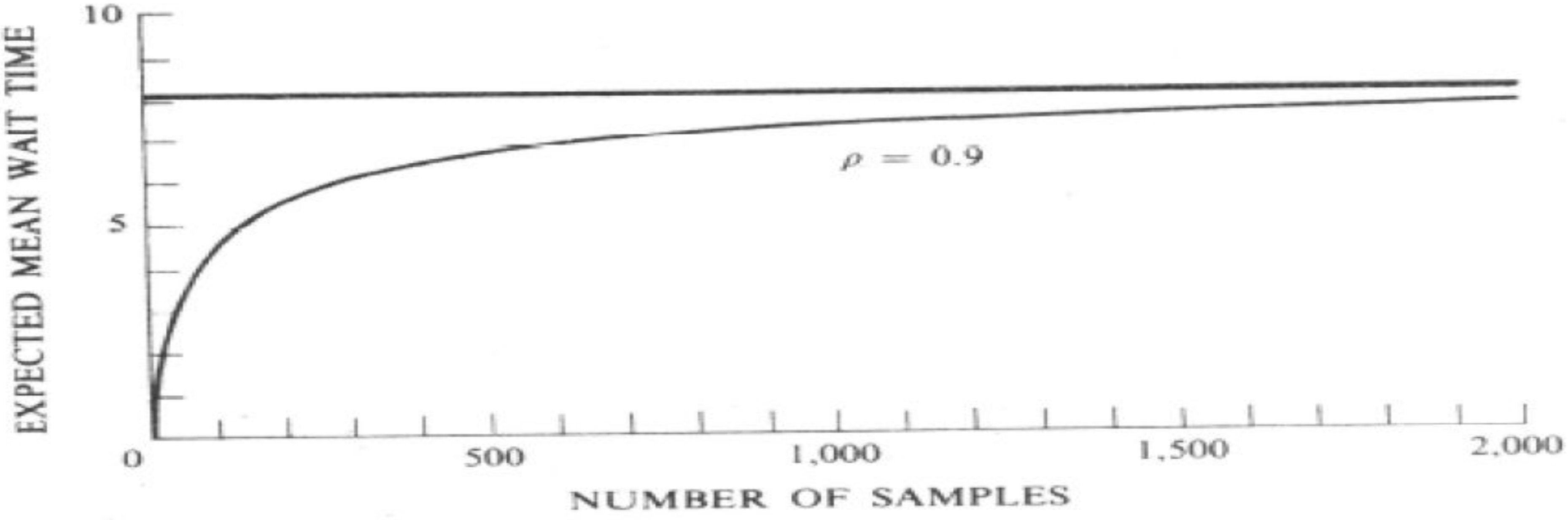

Figure: Mean waiting time in M/M/1 system for different sample size

Example

- Consider a system with Kendall's notation M/M/1/FIFO and the objective is to measure the mean waiting time.
- In simulation run approach, the mean waiting time is estimated by accumulating the waiting time of n successive entities and then it is divided by n. This measures the sample mean such that:
- $\bar{x}(n) = \frac{1}{n}\sum_{i=1}^{n} x_i$
- Such series of data in which one value affect other values is said to be auto-correlated.
- The sample mean of auto-correlated data can be shown to approximate a normal distribution as the sample size increases.

Problem that may arise in Simulation Run Statistics

- The distribution may not be stationary.
- A simulation run is started with the system in some initial idle state. In this case, the early arrivals will obtain service quickly deviating from normal distribution. Hence, the sample means of the early arrivals is known as **initial bias**.
- As the sample size increases and the length of run is long, the effect of bias dies and the normal distribution is again established.

Replication of Runs

- The precision of results of a dynamic stochastic can be increased by repeating the experiment with different random numbers strings.
- For each replication of a small sample size, the sample mean is determined.
- The sample means of the independent runs can be further used to estimate the variance of distribution.
- Let X_{ij} be the i^{th} observation in j^{th} run, then the sample mean and variance for the j^{th} run are:

$$\overline{x_j}(n) = \frac{1}{n}\sum_{i=1}^{n} x_{ij}$$

$$s_j^2 = \frac{1}{n-1}\sum_{i=1}^{n}[x_{ij} - \overline{x_j}(n)]^2$$

- When we have similar means and variances for m independent measurements, we can obtain the combined mean and variance for the population by combining them.
- The mean of the means and the mean of the variances are then used to estimate the confidence interval.
- Combining the results of p independent measurements gives the following estimates for the mean waiting time $\bar{x}$ and variance s^2 of the population:

$$\overline{x} = \frac{1}{p}\sum_{j=1}^{p}\overline{x_j(n)}$$

$$s^2 = \frac{1}{p}\sum_{j=1}^{p} s_j^2(n)$$

- In repetitions of run, the length of run of replications is so selected that when length of all runs are combined, it comes to be equal with the sample size N. i.e. p*n = N
- By increasing the number of replications and shortening their length of run, the confidence interval can be narrowed. Confidence interval is the range of possible values for the parameter based on a set of data (e.g. the simulation results.)
- In the replication of simulation runs, if the number of runs is increased at the cost of shortening the individual runs, the estimate of the mean will be more biased.
- The results obtained will not be accurate.
- Thus, a compromise has to be made. However, it is suggested that the number of replications should not be very large, and that the sample means should approximate a normal distribution.

Elimination of Initial Bias

- A simulation run is started with the system in some initial idle state. In this case, the early arrivals will obtain service quickly deviating from normal distribution. Hence, the sample means of the early arrivals is known as **initial bias**.
- The initial bias in simulation should be removed.
- Following approaches can be used to remove initial bias:

1. Ignore the initial bias occurred during the simulation run i.e. the first part of the simulation can be ignored.
2. The system should be started in a more representative state than in the empty state.
3. Start the simulation in the empty state, then stop after initial bias and then start again.
4. Run the simulation for such a long period of time so that the initial bias has no any significance in the output result.

- The ideal situation is to **know the steady state distribution for the system**, and **select the initial condition** from that distribution.
- In most of the existing systems, there may be information available on the expected conditions that makes it feasible to select better initial conditions.
- The most common approach to remove the initial bias is to illuminate the initial section of the run.
- The run is started from an idle state and stopped after a certain period of time.
- The entities existing in the system at that time are left as they are.
- The run is then restarted with the statistics being gathered from the point of restart.
- It is usual to program the simulation so that statistics are gathered from the beginning, and simply wipe out the statistics gathered up to the point of restart.

- No simple rules can be given to decide how long an interval should be eliminated.
- The disadvantage of eliminating the first part of a simulation run is that the estimate of the variance, needed to establish a confidence limit, must be based on less information.
- The reduction in bias, therefore, is obtained at the price of increasing the confidence interval size.
- We can also run the simulation for such a long period of time so that the initial bias has no any significance in the output result.