Exact Differential equation

An equation Mdx + Ndy = 0 where M and N are function of x any y is said to be exact differential equation if there exist a function U(x, y) such that Mdx + Ndy = d{u (x, y)}
i.e. If Mdx + Ndy is perfect differential

Some exact differential equation

1. $xdy + y\,dx = d(xy)$
2. $\dfrac{xdy + ydx}{xy} = d\{\log(xy)\}$
3. $(xy)^n\,(xdy + ydx) = \dfrac{1}{(n+1)}d(xy)^{n+1}$
4. $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$
5. $\dfrac{xdy - ydx}{y^2} = -\,d\left(\dfrac{x}{y}\right)$
6. $\dfrac{xdy - ydx}{xy} = d\left\{\log\left(\dfrac{y}{x}\right)\right\}$
7. $\dfrac{xdy - ydx}{x^2 + y^2} = d\tan^{-1}\left(\dfrac{y}{x}\right)$
8. $\dfrac{ydx - xdy}{x^2 + y^2} = d\tan^{-1}\left(\dfrac{x}{y}\right)$
9. $\dfrac{xdx + ydy}{2} = \dfrac{1}{2}d(x^2 + y^2)$
10. $\dfrac{xdx + ydy}{x^2 + y^2} = \dfrac{1}{2}\left\{d\log(x^2 + y^2)\right\}$
11. $\dfrac{dx + dy}{x + y} = d\{\log(x + y)\}$
12. $\dfrac{2xydx - x^2dy}{y^2} = d\left(\dfrac{x^2}{y}\right)$
13. $\dfrac{2xydy - y^2dx}{x^2} = d\left(\dfrac{y^2}{x}\right)$

Exercise - 25

Solve the following equations

1. **(2ax + by) y dx + (ax + 2by) x dy = 0**

Sol^n. Given differential equation is

$(2ax + by)\, y\, dx + (ax + 2by)\, x\, dy = 0$

or, $2axy\, dx + by^2\, dx + ax^2 dy + 2bxydy = 0$

or, $2axydx + ax^2 dy + by^2 dx + 2bxy\, dy = 0$

or, $a\int d(x^2y) + b\int d(y^2x) = \int odx$ Integrating

or, $ax^2y + by^2x = c$ is the required solution.

2. **$(x^2 – ay)dx – (ax – y^2)\, dy = 0$**

Sol^n. Given differential equation is

$(x^2 – ay)dx – (ax – y^2)\, dy = 0$

or, $x^2dx – aydx – axdy + y^2dy = 0$

or, $x^2dx + y^2dy – a(ydx + xdy) = 0$

or, $\int x^2dx + \int y^2dy – a\int d(xy) = \int odx$ Integrating

or, $\frac{x^3}{3} + \frac{y^3}{3} - axy = c$

or, $x^3 + y^3 - 3axy = 3c$

pr, $x^3 + y^3 – 3axy = K$ where K is constant is the required solution.

3. $\mathbf{\frac{dy}{dx} = \frac{2x - y + 1}{x + 2y - 3}}$

Sol^n. Given differential equation is

$\frac{dy}{dx} = \frac{2x - y + 1}{x + 2y - 3}$

or, $xdy + 2ydy – 3dy = 2x\, dx – ydx + dx$

or, $xdy + ydx = 2x\, dx – 2ydy + dx + 3dy$

or, $\int d(xy) = \int 2xdx – \int 2ydy + \int dx + \int 3dy$; Integrating.

$xy = \frac{2x^2}{2} \frac{-2y^2}{2} + x + 3y + c$

or, $xy = x^2 – y^2 + x + 3y + c$

or, $y^2 – x^2 + xy – 3y – x = c$ is the required solution.

4. **(x + y) dy + (y – x) dx = 0**

Sol^n. Given differential equation is

$(x + y)\, dy + (y – x)\, dx = 0$

or, $xdy + ydy + ydx – xdx = 0$

or, $xdy + ydx = xdx – ydy$

or, $\int d\,(xy) = \int xdx – \int ydy$ Integrating

or, $xy = \frac{x^2}{2} - \frac{y^2}{2} + c$

or, $2xy = x^2 – y^2 + 2c$

or, $x^2 – y^2 – 2xy = – 2c$

or, $x^2 – y^2 – 2xy = k$ where $k = – 2c$ is the required solution.

5. **$2xydx – (x^2 – y^2)\, dy = 0$**

Sol^n. Given differential equation is

$2xydx – (x^2 – y^2)\, dy = 0$

or, $2xydx = (x^2 – y^2)\, dy$

or, $2xydx = x^2dy – y^2dy$

or, $2xydx – x^2dy = – y^2dy$

Dividing both sides by y^2

$\frac{2xydx - x^2dy}{y^2} = -dy$

or, $\int d\left(\frac{x^2}{y}\right) = -\int dy$ Integrating

or, $\frac{x^2}{y} = -y + c$

$x^2 + y^2 = yc$ is the required solution.

6. **$(x^2 + y^2 + 2x\,)dx\ + xy\, dy = 0$**

Sol^n. Given differential equation is

$(x^2 + y^2 + 2x\,)dx\ + xy\, dy = 0$

or, $x^2dx + y^2dx + 2xdx + xydy = 0$

or, $x^2dx + 2xdx + y^2dx + xydy = 0$

Multiplying both sides by x

$x^3dx + 2x^2dx + xy^2dx + x^2ydy = 0$

$\int x^3dx + \int 2x^2dx + \frac{1}{2}\int d(x^2y^2) = \int odx$ Integrating

or, $\frac{x^4}{4} + \frac{2x^3}{2} + \frac{1}{2}x^2y^2 = c$

or, $3x^4 + 8x^3 + 6x^2y^2 = 12c$

or, $3x^4 + 8x^3 + 6x^2y^2 = K$ where $K = 12c$ is the required solution.

7. $\mathbf{x\frac{dy}{dx} + y = y^2 \log x}$

Sol^n. Given differential equation is

$x\frac{dy}{dx} + y = y^2 \log x$

or, $xdy + ydx = y^2\log x\, dx$

Dividing both sides by x^2y^2 we get,

or, $\left(\dfrac{1}{xy^2}\right)dy + \left(\dfrac{1}{x^2y}\right)dx = \left(\dfrac{1}{x^2}\log x\right)dx$

or, $-\left\{\dfrac{1}{x}\left(-\dfrac{1}{y^2}\right)dy + \dfrac{1}{y}\left(\dfrac{-1}{x^2}\right)dx\right\} = \left(\dfrac{1}{x^2}\log x\right)dx$

or, $-\int d\left(\dfrac{1}{x}.\dfrac{1}{y}\right) = \int \dfrac{\log x}{x^2}dx$. Integrating

$\dfrac{1}{xy} = \int \dfrac{\log x}{x^2} + c$ ....(1)

or, $-\dfrac{1}{xy} = \log x.\left(-\dfrac{1}{x}\right) - \int \dfrac{1}{x}.\left(-\dfrac{1}{x}\right)dx + c$

or, $-\dfrac{1}{xy} = -\dfrac{\log x}{x} + \int \dfrac{1}{x^2}\ dx + c$

or, $\dfrac{-1}{xy} = -\dfrac{\log x}{x} - \dfrac{1}{x} + c$

or, – 1 = – y (logx + 1) + cxy

or, y (logx + 1) = cxy + 1 is the required solution.

8. $xdy - ydx + a(x^2 + y^2)\,dx = 0$ ***(B. E. 2058)***

Sol^n. Given differential equation is

$xdy - ydx + a(x^2 + y^2)\,dx = 0$

or, $\dfrac{xdy - ydx}{x^2 + y^2} + adx = 0$

or, $\dfrac{\dfrac{xdy - ydx}{x^2}}{1 + \left(\dfrac{y}{x}\right)^2} + adx = 0$

or, $\int \dfrac{d\left(\dfrac{y}{x}\right)}{1 + \left(\dfrac{y}{x}\right)^2} + a\int dx = \int odx$ Integrating

or $\tan^{-1}\left(\dfrac{y}{x}\right) + ax = c$ is the required solution.

9. $xdx + ydy + (x^2 + y^2)\,dy = 0$

Sol^n. Given differential equation is

$xdx + ydy + (x^2 + y^2)\,dy = 0$

or, $\dfrac{xdx + ydy}{x^2 + y^2} + dy = 0$

or, $\dfrac{1}{2}\int \dfrac{d(x^2 + y^2)}{(x^2 + y^2)} + \int dy = \int 0dx$ Integrating

or, $\dfrac{1}{2}\log(x^2 + y^2) + y = 0$

or, $\log(x^2 + y^2) + 2y = 2c$

or, $\log(x^2 + y^2) + 2y = k$ where k = 2c is the required solution.

10. $(1 + xy)\,y\,dx + (1 - xy)\,xdy = 0$

Sol^n. Given differential equation is

$(1 + xy)\,y\,dx + (1 - xy)\,xdy = 0$

or, $ydx + xy^2dx + xdy - x^2y\,dy = 0$

or, $ydx + xdy + xy^2dx - x^2ydy = 0$

dividing both sides by x^2y^2 we get.

$\dfrac{1}{x^2y}dx + \dfrac{1}{xy^2}dy + \dfrac{1}{x}dx - \dfrac{1}{y}dy = 0$

or, $-\int d\left(\dfrac{1}{x}.\dfrac{1}{y}\right) + \int \dfrac{1}{x}dx = \int \dfrac{1}{y}dy$; Integrating

or, $-\dfrac{1}{xy} + \log x = \log y + c$

or, $-\dfrac{1}{xy} + \log\left(\dfrac{x}{y}\right) = c$

or, $\log\left(\dfrac{x}{y}\right) = c + \dfrac{1}{xy}$ is the required solution.

11. $\text{Sinx}\,dy - y\cos x dx + y^2dx = 0.$

Sol^n. Given differential equation is

$\text{Sinx}\,dy - y\cos xdx + y^2dx = 0$

or, $y^2dx = y\cos x\,dx - \sin x\,dy$

dividing both sides by y^2 we get,

$dx = \dfrac{y\cos xdx - \sin xdy}{y^2}$

or, $\int dx = \int d\left(\dfrac{\sin x}{y}\right)$; Integrating

or, $x = \dfrac{\sin x}{y} + c$

or, $xy = \sin x + cy$ is the required solution.

12. $\mathbf{x\frac{dy}{dx} = y + x^2 logx}$

Sol^n. Given differential equation is

or, $x\frac{dy}{dx} = y + x^2 \log x$

or, $xdy = (y + x^2logx)\ dx$

or, $xdy = ydx + x^2\ logx\ dx$

or, $xdy - ydx = x^2logx\ dx$

Dividing both sides by x^2 we get

or, $\frac{xdy - ydx}{x^2} = \log xdx$

or, $\int d\left(\frac{y}{x}\right) = \int \log xdx$ Integrating

or, $\frac{y}{x} = x \log x - x + c$

or, $y = x^2\ logx - x^2 + cx$ is the required solution.

13. $\mathbf{x\cos\left(\frac{y}{x}\right)(ydx + xdy) = y\sin\left(\frac{y}{x}\right)(xdy - ydx)}$

Sol^n. Given differential equation is

$x\cos\left(\frac{y}{x}\right)(ydx + xdy) = y\sin\left(\frac{y}{x}\right)(xdy - ydx)$

Dividing both sides by 'x' we get

$\cos\left(\frac{y}{x}\right)(ydx + xdy) = y\sin\left(\frac{y}{x}\right)\frac{(xdy - ydx)}{x}$

or, $\cos\left(\frac{y}{x}\right)(ydx + xdy) = xy\sin\left(\frac{y}{x}\right)\frac{(xdy - ydx)}{x^2}$

or, $\cos\left(\frac{y}{x}\right)(ydx + xdy) = xy\sin\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)$

or, $\cos\left(\frac{y}{x}\right)d(xy) = (xy)\sin\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)$

or, $\int \frac{1}{xy}d(xy) = \int \frac{\sin\left(\frac{y}{x}\right)}{\cos\left(\frac{y}{x}\right)}d\left(\frac{y}{x}\right)$; Integrating

$\log(xy) = -\log\cos\left(\frac{y}{x}\right) + \log c$

$\log(xy) + \log\cos\left(\frac{y}{x}\right) = \log c$

or, $\log(xy)\cos\left(\frac{y}{x}\right) = \log c$

or, $(xy)\cos\left(\frac{y}{x}\right) = c$ is the required sol^n.

14. **Cosx (cosx – siny) dx + cosy (cosy – sinx) dy = 0**

Sol^n. Given differential equation is

$\cos x(\cos x - \sin y)\ dx + \cos y(\cos y - \sin x)dy = 0$

or, $\cos^2 xdx - \sin y \cos x\ dx + \cos^2 y\ dy - \cos y \sin x\ dy = 0$

or, $\frac{(1+\cos 2x)}{2}dy + \left(\frac{1+\cos 2y}{2}\right)dy = \cos y \sin xdy + \sin y \cos dx$

or, $\int \frac{(1+\cos x)}{2}dx + \int\left(\frac{1+\cos 2y}{2}\right)dy = \int d(\sin x \sin y)$; integrating

or, $\frac{1}{2}x + \frac{\sin 2x}{4} + \frac{1}{2}y + \frac{\sin 2y}{4} = \sin x \sin y + c$

or, $2x + 2y + \sin 2x + \sin 2y - 4\sin x \sin y = 4c$

or, $2x + 2y + \sin 2x + \sin 2y - 4\sin x \sin y = K$ where $k = 4c$ is the required solution.

15. $\mathbf{(x + 2y^3)\ dy = ydx}$

Sol^n. Given differential equation is

$(x + 2y^3)\ d\ y = ydx$ or, $xdy + 2y^3dy = ydx$

or, $-2y^3dy = ydx - xdy$

Dividing both sides by y^2

$2ydy = \frac{ydx - xdy}{y^2}$

or, $\int 2ydy = \int d\left(\frac{x}{y}\right)$ Integrating

or, $\frac{2y^2}{2} = \frac{x}{y} + c$

or, $y^2 = \frac{x}{y} + c$

or, $y^3 = x + cy$ is the required soltion.

16. $\mathbf{x^2y^3\ dx + 3x^2y\ dy + 2ydx = 0}$

Sol^n. Given differential equation is

$x^2y^3\ dx + 3x^2y\ dy + 2ydx = 0$

or, $x^2y^3dx + 2ydx + 3x^2ydy = 0$

or, $y(x^2\ y^2 + 2)\ dx = -3x^2y\ dy$

or, $\frac{dy}{dx} = -\frac{y(x^2y^2+2)}{3x^2y}$ ....(1)

Put xy = v, Then

$y + x\frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{dy}{dx} = \left(\frac{dv}{dx} - y\right)\frac{1}{x}$

Now eq^n (1) becomes

$\left(\frac{dv}{dx} - y\right)\frac{1}{x} = \frac{-y(x^2y^2+2)}{2xy.x}$

Using xy = v and y = $\frac{v}{x}$

or, $\left(\frac{dv}{dx} - \frac{v}{x}\right)\frac{1}{x} = -\frac{v(v^2+2)}{x3vx}$

or, $\left(\frac{dv}{dx} - \frac{v}{x}\right)\frac{1}{x} = \frac{-(v^2+2)}{3x^2}$

or, $\frac{dv}{dx} = \frac{v}{x} - \frac{1}{3x}(v^2+2)$

or, $\frac{dv}{dx} = \frac{1}{x}\left[v - \frac{v^2+2}{3}\right]$

or, $\frac{dv}{dx} = \frac{1}{x}\frac{(3v-v^2+2)}{3}$

or, $\frac{3dv}{3v-v^2+2} = \frac{dx}{x}$

or, $\int\frac{3dv}{v^2-3v+2} = -\int\frac{dx}{x}$ Integrating

or, $\int\frac{3dv}{v^2-2.\frac{3}{2}v+\left(\frac{3}{2}\right)^2-\frac{1}{4}} = -\int\frac{dx}{x}$

or, $\int\frac{3dv}{\left(v-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2} = -\int\frac{dx}{x}$

or, $3.\frac{1}{2.\frac{1}{2}}\log\left\{\frac{\left(v-\frac{3}{2}\right)-\frac{1}{2}}{\left(v-\frac{3}{2}\right)+\frac{1}{2}}\right\} = -\log x + \log c$

or, $3\log\left(\frac{v-2}{v-1}\right) = -\log x + \log c$

or, $\log\left(\frac{v-2}{v-1}\right)^3 + \log x = \log c$

or, $\log\left(\frac{v-2}{v-1}\right)^3.x = \log c$ or, $\frac{(v-2)^3}{(v-1)^3}x = c$

or, $x(v-2)^3 = c(v-1)^3$

But xy = v

$\therefore$ $x(xy-2)^3 = c(xy-1)^3$ is the required solution.

17. $\mathbf{\frac{xdx+ydy}{xdy-ydx} = \frac{\sqrt{a^2-x^2-y^2}}{x^2+y^2}}$

Sol^n. Given differential equation is

$\frac{xdx+ydy}{xdy-ydx} = \frac{\sqrt{a^2-x^2-y^2}}{x^2+y^2}$

or, $\frac{xdx+ydy}{\sqrt{a^2-(x^2+y^2)}} = \frac{xdy-ydx}{x^2+y^2}$

or, $\frac{1}{2}\int\frac{d(x^2+y^2)}{\sqrt{a^2-\left(\sqrt{x^2+y^2}\right)^2}} = \int d\tan^{-1}\left(\frac{y}{x}\right)$; Integrating

or, $\sin^{-1}\frac{\sqrt{x^2+y^2}}{a} = 2\tan^{-1}\left(\frac{y}{x}\right)+c$

$\frac{\sqrt{x^2-y^2}}{a} = \sin\left\{2\tan^{-1}\left(\frac{y}{x}\right)+c\right\}$

or, $\sqrt{x^2-y^2} = a\sin\left\{2+\tan^{-1}\left(\frac{y}{x}\right)+c\right\}$

is the required solution.

18. $\mathbf{(x^2+y^2+2x)\,dx + 2ydy = 0}$

Sol^n. Given differential equation is

$(x^2+y^2+2x)\,dx + 2ydy = 0$

or, $(x^2+y^2)\,dx = -2(xdx+ydy)$

or, $\int dx = -2\int\frac{(xdx+ydy)}{(x^2+y^2)}$ Integrating

or, $x = -\log(x^2+y^2) + c$

or, $x + \log(x^2+y^2) = c$ is the required solution

19. $\mathbf{x^2dy + xydx + 2\sqrt{1-x^2y^2}dx = 0}$

Sol^n. Given differential equation is

$x^2dy + xydx + 2\sqrt{1-x^2y^2}dx = 0$

Dividing both sides by $\sqrt{1-x^2y^2}$ we get

$$\frac{x^2dy + xydx}{\sqrt{1-(xy)^2}} = -2dx$$

or, $\frac{x(xdy + ydx)}{\sqrt{1-(xy)^2}} = -2dx$ or, $\frac{xdy + ydx}{\sqrt{1-(xy)^2}} = -\frac{2}{x}dx$

$\int \frac{d(xy)}{\sqrt{1-(xy)^2}} = -\int \frac{2}{x}dx$ Integrating

or, $\sin^{-1}(xy) = -2\log x + c$ is the required solution.

20. $\mathbf{xdx + ydy = \frac{a^2(xdy - ydx)}{x^2 + y^2}}$

Sol^n. Given differential equation is

$xdx + ydy = \frac{a^2(xdy - ydx)}{x^2 + y^2}$

or, $\int xdx + \int ydy = a^2 \int d\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$ Integrating

or, $\frac{x^2}{2} + \frac{y^2}{2} = a^2\tan^{-1}\left(\frac{y}{x}\right) + c$

or, $x^2 + y^2 = 2a^2\tan^{-1}\left(\frac{y}{x}\right) + c$ is the required solution.

21. $\mathbf{(x^2 + y^2)\,dx - 2xy\,dy = 0}$

Sol^n. Given differential equation is

$(x^2 + y^2)\,dx - 2xy\,dy = 0$

$x^2dx + y^2dx - 2xydy = 0$

or, $y^2dx - 2xy\,dy = -x^2\,dx$

Dividing both sides by $-x^2$

$$\frac{2xydy - y^2dx}{x^2} = dx$$

or, $\int d\left(\frac{y^2}{x}\right) = \int dx$ Integrating or, $\frac{y^2}{x} = x + c$

or, $y^2 = x^2 + cx$ is the required solution.