Ex-18

Prove the following Legendre's function

(1) $\frac{2}{5} P_3(x) + \frac{3}{5} P_1(x) = x^3$

Solⁿ: we know the Legendre's polynomials are:

$$P_1(x) = x^1$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Now

$$L.H.S. = \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x)$$

$$= \frac{2}{5} \frac{1}{2}(5x^3 - 3x) + \frac{3}{5} x$$

$$= x^3 - \frac{3}{5}x + \frac{3}{5}x$$

$$= x^3 \quad R.H.S.$$

② $x^5 = \frac{8}{63}\left[P_5(x) + \frac{7}{2} P_3(x) + \frac{27}{8} P_1(x)\right]$

sol^n: we know,

$$P_n(x) = \frac{1}{2^n . n!} \frac{d^n}{dx^n} (x^2-1)^n$$

Now,

$n = 1,$

$$P_1(x) = \frac{1}{2} \frac{d}{dx} (x^2-1)$$

or, $P_1(x) = \frac{1}{2} \times 2x$

$P_1(x) = x$

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Put $n = 3$,

$$P_3(x) = \frac{1}{2^3 \cdot 3!} \frac{d^3 (x^2-1)^3}{dx^3}$$

or, $$P_3(x) = \cancel{\frac{1}{48} \times 3(x^2-1)^2}\ \frac{1}{2}(5x^3 - 3x)$$

and,

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

Now, Taking R.H.S.

$$= \frac{8}{63}\left[\frac{1}{8}(63x^5 - 70x^3 + 15x) + \frac{7}{2} \times \frac{1}{2}(5x^3 - 3x) + \frac{27}{8}x\right]$$

$$= \frac{8}{63}\left[\frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x + \frac{35}{4}x^3 - \frac{21}{4}x + \frac{27}{8}x\right]$$

$$= \frac{8}{63}\left[\frac{63}{8}x^5 + \frac{42}{8}x - \frac{21}{4}x\right]$$

$$= \frac{8}{63}\left[\frac{63}{8}x^5 + \frac{21}{4}x - \frac{21}{4}x\right]$$

$$= \frac{8}{63} \times \frac{63}{8}x^5$$

$$= x^5$$

= L.H.S proved.

Express the following in the terms of legendre's polynomial

3. $f(x) = 1 + x - x^2$

sol: we know the legendre's polynomials are;

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$ or, $2P_2(x) = 3x^2 - 1$

$2P_2(x) + 1 = 3x^2$

or, $\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x) = x^2$ //

Now,

$f(x) = 1 + x - x^2$ can be expressed as

$$f(x) = P_0(x) + P_1(x) - \frac{2}{3} P_2(x) - \frac{1}{3} P_0(x)$$

$$f(x) = \frac{2}{3} P_0(x) + P_1(x) - \frac{2}{3} P_2(x)$$

4. $f(x) = 5x^3 + x$

sol: we know,

$P_1(x) = x$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

or, $2P_3(x) = 5x^3 - 3x$

or, $2P_3(x) = 5x^3 - 3P_1(x)$

or, $5x^3 = 2P_3(x) + 3P_1(x)$

Now,

$f(x) = 2P_3(x) + 3P_1(x) + P_1(x)$

$\therefore f(x) = 2P_3(x) + 4P_1(x)$

5. $f(x) = x^3 - 5x^2 + 6x + 1$

sol: we know,

$P_0(x) = 1$

$P_1(x) = x$

or, $6P_1(x) = 6x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$2P_2(x) = 3x^2 - P_0(x)$

or,

or, $2P_2(x) + P_0(x) = 3x^2$

or, $\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x) = x^2$

$5x^2 = \frac{10}{3}P_2(x) + \frac{5}{3}P_0(x)$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

or, $2P_3(x) = 5x^3 - 3x$

or, $2P_3(x) = 5x^3 - 3P_1(x)$

or $5x^3 = 2P_3(x) + 3P_1(x)$

$x^3 = 2/5\, P_3(x) + 3/5\, P_1(x)$

Now,

$$f(x) = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) - \frac{10}{3}P_2(x) - \frac{5}{3}P_0(x) + 6P_1(x) + P_0(x)$$

$$f(x) = \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{33}{5}P_1(x) - \frac{2}{3}P_0(x)$$

Note:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

Ex-18

⑥ $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

we know,

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m}$$

Put $n = 1/2$

$$J_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\frac{1}{2}+1)} \left(\frac{x}{2}\right)^{\frac{1}{2}+2m}$$

or, $$J_{\frac{1}{2}}(x) = \left(\frac{x}{2}\right)^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+\frac{3}{2})} \left(\frac{x}{2}\right)^{2m}$$

or, $$J_{\frac{1}{2}}(x) = \sqrt{\frac{x}{2}} \left[\frac{(-1)^0}{0!\,\Gamma(0+3/2)} \cdot \left(\frac{x}{2}\right)^{2\cdot 0} + \frac{(-1)^1}{1!\,\Gamma(1+\frac{3}{2})} \left(\frac{x}{2}\right)^{2\cdot 1} + \dots \right]$$

or, $$J_{\frac{1}{2}}(x) = \sqrt{\frac{x}{2}} \left[\frac{1}{\frac{1}{2}\sqrt{\pi}} - \frac{1}{\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}} \cdot \frac{x^2}{4} + \frac{1}{2\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} \frac{x^4}{16} - \dots \right]$$

or, $$J_{\frac{1}{2}}(x) = \sqrt{\frac{x}{2}} \times \frac{1}{\frac{\sqrt{\pi}}{2}} \left[1 - \frac{x^2}{3\cdot2\cdot1} + \frac{x^4}{5\cdot4\cdot3\cdot2\cdot1} - \dots \right]$$

or, $$J_{\frac{1}{2}}(x) = \sqrt{\frac{x}{2}} \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{1}{x} \left[x - \frac{x^3}{3\cdot2\cdot1} + \frac{x^5}{5\cdot4\cdot3\cdot2\cdot1} - \dots \right]$$

or, $$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right]$$

$$\therefore J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

⑦ $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

soln: we have,

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m}$$

Put $n = -\frac{1}{2}$

$$J_{-\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m-\frac{1}{2}+1)} \cdot \left(\frac{x}{2}\right)^{-\frac{1}{2}+2m}$$

or $$J_{-\frac{1}{2}}(x) = \left(\frac{x}{2}\right)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\frac{1}{2})} \cdot \left(\frac{x}{2}\right)^{2m}$$

or $$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\frac{1}{2})} \left(\frac{x}{2}\right)^{2m}$$

or $$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{x}} \left[\frac{1}{\sqrt{\pi}} - \frac{x^2}{\frac{1}{2}\sqrt{\pi} \times 4} + \frac{x^4}{4 \times 16 \times 2 \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi}} - \cdots \right]$$

or $$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{x}} \times \frac{1}{\sqrt{\pi}} \left[1 - \frac{x^2}{2.1} + \frac{x^4}{4.3.2.1} - \cdots \right]$$

or $$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right]$$

$$\therefore \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos x$$

(2) $$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$$

Sol^n: we know recurrence relation of the Bessel's function, is,

$$2n\, J_n(x) = x \left[J_{n-1}(x) + J_{n+1}(x) \right] \quad -(1)$$

Put $n = \frac{1}{2}$ in (1)

$$2\cdot\frac{1}{2}\,J_{\frac{1}{2}}(x) = x\left[J_{-\frac{1}{2}}(x) + J_{\frac{3}{2}}(x)\right]$$

$$\therefore\ J_{\frac{1}{2}}(x) = x\,J_{-\frac{1}{2}}(x) + x\,J_{\frac{3}{2}}(x)$$

$$\therefore\ x\,J_{\frac{3}{2}}(x) = -x\,J_{-\frac{1}{2}}(x) + J_{\frac{1}{2}}(x)$$

Since, $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\ \sin x$, $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\ \cos x$

$$\therefore\ x\,J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\ \sin x - x\cdot\sqrt{\frac{2}{\pi x}}\ \cos x$$

or, $$x\,J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\sin x - x\cos x\right]$$

$$\therefore\ J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$$

(9) $$J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(-\frac{\cos x}{x} - \sin x\right)$$

soln: we have,

$$2n\,J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

Put $n = -1/2$

$$2\times -\frac{1}{2}\times J_{-\frac{1}{2}}(x) = x\left[J_{-1/2-1}(x) + J_{-1/2+1}(x)\right]$$

$$\therefore\ -J_{-\frac{1}{2}}(x) = x\,J_{-3/2}(x) + x\,J_{\frac{1}{2}}(x)$$

or, $$x\,J_{-3/2}(x) = -J_{-\frac{1}{2}}(x) - x\,J_{\frac{1}{2}}(x)$$

we have,

$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\ \sin x$, $J_{-1/2}(x)\ \sqrt{\frac{2}{\pi x}}\ \cos x$

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or, $$x J_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\cos x - x\sqrt{\frac{2}{\pi x}}\sin x$$

$\therefore$ $$x J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(-\cos x - x\sin x\right)$$

$\therefore$ $$J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(-\frac{\cos x}{x} - \sin x\right)$$

10) $J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{3-x^2}{x^2} \sin x - \frac{3}{x} \cos x \right]$

sol^n. we know,

$$2n J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right] \quad \text{--- (1)}$$

Put $n = \frac{3}{2}$

$$3 J_{\frac{3}{2}}(x) = x\left[J_{\frac{1}{2}}(x) + J_{\frac{5}{2}}(x)\right]$$

or, $J_{\frac{5}{2}}(x) = \frac{3}{x} J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x) \quad \text{--- (*)}$

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Putting $n = \frac{1}{2}$ in (1)

$$J_{\frac{1}{2}}(x) = x\left[J_{-\frac{1}{2}}(x) + J_{\frac{3}{2}}(x)\right]$$

or, $$J_{\frac{3}{2}}(x) = \frac{1}{x} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x) \quad -(2)$$

using (2) in (*)

$$J_{\frac{5}{2}}(x) = \frac{3}{x}\left[\frac{1}{x} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)\right] - J_{\frac{1}{2}}(x)$$

or, $$J_{\frac{5}{2}}(x) = \left(\frac{3}{x^2} - 1\right) J_{\frac{1}{2}}(x) - \frac{3}{x} J_{-\frac{1}{2}}(x)$$

or, $$J_{\frac{5}{2}}(x) = \left(\frac{3}{x^2} - 1\right)\sqrt{\frac{2}{\pi x}} \sin x - \frac{3}{x}\sqrt{\frac{2}{\pi x}} \cos x$$

or, $$J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3 - x^2}{x^2} \sin x - \frac{3}{x} \cos x\right]$$ proved

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(ii) $J_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x + \frac{3-x^2}{x^2}\cos x\right]$

solⁿ. we know,

$2n J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$ —①

put $n = -\frac{3}{2}$ —①

$\cancel{2}\times -\frac{3}{\cancel{2}} J_{\frac{3}{2}}(x) = x\left[J_{-5/2}(x) + J_{-\frac{1}{2}}(x)\right]$

$\therefore -3J_{-\frac{3}{2}}(x) = xJ_{-5/2}(x) + xJ_{-\frac{1}{2}}(x)$

$\Rightarrow xJ_{-5/2}(x) = -3J_{-3/2}(x) - xJ_{-\frac{1}{2}}(x)$ —②

$\cancel{xJ_{-5/2}(x)} \cancel{-\frac{3}{2}}$

Put $n = -1/2$ in ①

$2\left(-\frac{1}{2}\right) J_{-\frac{1}{2}}(x) = x\left[J_{-\frac{1}{2}-1}(x) + J_{\frac{1}{2}}(x)\right]$

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$-J_{-\frac{1}{2}}(x) = xJ_{-\frac{3}{2}}(x) + xJ_{\frac{1}{2}}(x)$

$\cdot \quad xJ_{-\frac{3}{2}}(x) = -J_{-\frac{1}{2}}(x) - xJ_{\frac{1}{2}}(x)$

Since, $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$ and $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

$xJ_{-\frac{3}{2}}(x) = -x\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$

$\therefore \quad J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[-\sin x - \frac{\cos x}{x}\right] \text{ — } ③$

from ② & ③

$xJ_{-\frac{5}{2}}(x) = -3\sqrt{\frac{2}{\pi x}}\left[-\sin x - \frac{\cos x}{x}\right] - x\cdot\sqrt{\frac{2}{\pi x}}\cos x$

$\cdot \quad xJ_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[3\sin x - \frac{3\cos x}{x}\right] - \sqrt{\frac{2}{\pi x}}\, x\cos x$

$\cdot \quad xJ_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[3\sin x - \frac{3}{x}\cos x - x\cos x\right]$

$\cdot \quad J_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x}{x} - \frac{3-x^2}{x^2}\cos x\right]$

22. $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$

soln: we have,

$$2n J_n(x) = x[J_{n-1}(x) + J_{n+1}(x)] \quad -①$$

Put $n = 3$,

$$6 J_3(x) = x[J_2(x) + J_4(x)]$$

or $6J_3(x) = xJ_2(x) + xJ_4(x)$

or $xJ_4(x) = 6J_3(x) - xJ_2(x)$

or $J_4(x) = \frac{6}{x} J_3(x) - J_2(x) \quad -②$

Put, $n = 2$ in ①

$$4J_2(x) = x[J_1(x) + J_3(x)]$$

or $4J_2(x) = xJ_1(x) + xJ_3(x)$

$\therefore J_3(x) = \frac{4}{x} J_2(x) - J_1(x) \quad -③$

From ② & ③

$$J_4(x) = \frac{6}{x}\left[\frac{4}{x} J_2(x) - J_1(x)\right] - J_2(x)$$

$$J_4(x) = \frac{24}{x^2} J_2(x) - J_2(x) - \frac{6}{x} J_1(x)$$

$$J_4(x) = \left(\frac{24}{x^2} - 1\right) J_2(x) - \frac{6}{x} J_1(x) - ④$$

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Put $n = 1$ in ①

$$2J_1(x) = x J_0(x) + x J_2(x)$$

$$\Rightarrow J_2(x) = \frac{2}{x} J_1(x) - J_0(x) \quad \text{—} ⑤$$

using ⑤ in ④

$$J_4(x) = \left(\frac{24}{x^2} - 1\right)\left(\frac{2}{x} J_1(x) - J_0(x)\right) - \frac{6}{x} J_1(x)$$

or, $$J_4(x) = \left(\frac{48}{x^3} - \frac{2}{x} - \frac{6}{x}\right) J_1(x) - \left(\frac{24}{x^2} - 1\right) J_0(x)$$

$$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x).$$

(14) $\cancel{4J_n''(x) =}$

$4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$

solu^n: we know,

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \quad -(1)$$

diff (1) w.r.t. x,

$$2J_n''(x) = J_{n-1}'(x) - J_{n+1}'(x)$$

Multiplying both sides by 2,

$$4J_n''(x) = 2J_{n-1}'(x) - 2J_{n+1}'(x) \quad -(2)$$

Put, n as $n-1$ in (1) and $n+1$ in 1

$$2J_{n-1}'(x) = J_{n-2}(x) - J_n(x) \quad -(3)$$

$$2J_{n+1}'(x) = J_n(x) - J_{n+2}(x) \quad -(4)$$

using (3) & (4) in (2)

$4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$ proved

18 and 15. $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$

Sol^n:- we know,

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \quad -①$$

Diff ① w.r.t. x,

$$2 J_n''(x) = J_{n-1}'(x) - J_{n+1}'(x) \quad -②$$

Multiplying both sides by 2

$$4 J_n''(x) = 2J_{n-1}'(x) - 2J_{n+1}'(x) \quad -③$$

Using relation ① for $2J_{n-1}'(x)$ and $2J_{n+1}'(x)$

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$$2J_{n-1}'(x) = J_{n-2}(x) - J_n(x)$$

$$2J_{n+1}'(x) = J_n(x) - J_{n+2}(x)$$

Using these in ③

$$4J_n''(x) = J_{n-2}(x) - J_n(x) - J_n(x) + J_{n+2}(x)$$

or $$4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$$

Diff. w.r.t x,

$$4J_n'''(x) = J_{n-2}'(x) - 2J_n'(x) + J_{n+2}'(x) \quad \text{—④}$$

Now using relation ① for $J_{n-2}'(x)$ and $J_{n+2}'(x)$

$$2J_{n-2}'(x) = J_{n-3}(x) - J_{n-1}(x)$$

$$2J_{n+2}'(x) = J_{n+1}(x) - J_{n+3}(x)$$

Using these in ④

$$4J_n'''(x) = \frac{1}{2}\left(J_{n-3}(x) - J_{n-1}(x)\right) - 2J_n'(x) + \frac{1}{2}\left(J_{n+1}(x) - J_{n+3}(x)\right)$$

~~$4J_n'''(x)$~~ Replacing n by 0,

$$4J_0'''(x) = \frac{1}{2}\left(J_{-3}(x) - J_{-1}(x)\right) - 2J_0'(x) + \frac{1}{2}\left(J_1(x) - J_3(x)\right)$$

or $$4J_0'''(x) = \frac{J_{-3}(x)}{2} - \frac{J_3(x)}{2} + \frac{J_1(x)}{2} - \frac{J_{-1}(x)}{2} - 2J_0'(x)$$

or, $$4J_0'''(x) = \frac{J_{-3}(x) - J_3(x)}{2} + \frac{J_1(x) - J_{-1}(x)}{2} - 2J_0'(x)$$

~~$4J_0'''(x)$~~ we know,

$$J_{-n}(x) = (-1)^n J_n(x)$$

Replacing n by 1 and 3

$$J_{-1}(x) = -J_1(x) \quad , \quad J_{-3}(x) = -J_3(x)$$

Then,

$$4J_0'''(x) = \frac{-2J_3(x)}{2} + \frac{2J_1(x)}{2} - 2J_0'(x)$$

or, $4J_0'''(x) = -J_3(x) + J_1(x) - 2J_0'(x)$

Also, $J_0'(x) = -J_1(x)$

or, $4J_0'''(x) = -J_3(x) - 3J_0'(x)$

$\therefore$ $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$

16. $J_0'(x) = -J_1(x)$

Solⁿ: we know the recurrence relation of the Bessel's function is,

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

Put $n=0$,

$$2J_0'(x) = J_{-1}(x) - J_1(x)$$

Since, $J_{-n}(x) = (-1)^n J_n(x)$

$$2J_0'(x) = (-1)^{-1} J_1(x) - J_1(x)$$

or $2J_0'(x) = -2J_1(x)$

$\therefore$ $J_0'(x) = -J_1(x)$ //

(17) $J_2(x) - J_0(x) = 2J_0''(x)$

sol^n From Q.N. 14,

$$4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$$

Put $n = 0$,

$$4J_0''(x) = J_{-2}(x) - 2J_0(x) + J_2(x)$$

since,

$$J_{-n}(x) = (-1)^n J_n(x)$$

$$4J_0''(x) = (-1)^2 J_2(x) - 2J_0(x) + J_2(x)$$

$$4J_0''(x) = 2J_2(x) - 2J_0(x)$$

$\therefore$ $J_2(x) - J_0(x) = 2J_0''(x)$

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19. $J_{\frac{-1}{2}}(x) = J_{1/2}(x) \cot x$

solⁿ: we know,

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{— (I)}$$

and,

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x \quad \text{— (II)}$$

Dividing (II) by (I)

$$\frac{J_{-\frac{1}{2}}(x)}{J_{\frac{1}{2}}(x)} = \frac{\sin x}{\cos x}$$

$\therefore$ $J_{-1/2}(x) = J_{1/2}(x) \cot x$ //

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20. $\left[J_{1/2}(x)\right]^2 + \left[J_{-1/2}(x)\right]^2 = \frac{2}{\pi x}$

soln: we know,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{— (i)}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x \quad \text{— (ii)}$$

Squaring and adding (i) and (ii)

$$\left[J_{1/2}(x)\right]^2 + \left[J_{-1/2}(x)\right]^2 = \frac{2}{\pi x}\sin^2 x + \frac{2}{\pi x}\cos^2 x$$

or, $\left[J_{1/2}(x)\right]^2 + \left[J_{-1/2}(x)\right]^2 = \frac{2}{\pi x}\left[\sin^2 x + \cos^2 x\right]$

$\therefore \left[J_{1/2}(x)\right]^2 + \left[J_{-1/2}(x)\right]^2 = \frac{2}{\pi x}$ proved.