Chapter - 6

DIELECTRICS AND CAPACITANCE

6.1 INTRODUCTION

After having studied the properties of media in the last chapter, we now turn our attention to insulating materials, or dielectrics. Such materials differ from conductors in that ideally, there is no free charge that can be transported within them to produce conduction current. Instead, all charge is confined to molecular or lattice sites by Coulomb forces. Applying electric field will displace the charges slightly, thereby creating electric dipoles. The extent to which ensemble of electric dipoles are formed is measured by the **relative permittivity**, or **dielectric constant.** The polarization of the medium may modify the electric field, whose magnitude and direction may differ from the values that it would have in a different dielectric or in free space. Boundary conditions for the fields at interfaces between dielectrics are developed in order to evaluate these differences.

6.2 DIELECTRIC MATERIALS

A dielectric in an electric field can be viewed as a free-space arrangement of microscopic electric dipoles which are composed of positive and negative charges whose centers do not quite coincide.

These are not free charges, and they cannot contribute to the conduction process. Rather, they are bound in place by atomic and molecular forces and can only shift positions slightly in response to external fields. They are called bound charges, in contrast to the free charges that determine conductivity. The bound charges can be treated as any other sources of the electrostatic field.

The characteristic which all dielectric materials have in common, whether they are solid, liquid, or gas, and whether or not they are crystalline in nature, is their ability to store electric energy. This storage takes place by means of a shift in the relative positions of the internal, bound positive and negative charges against the normal molecular and atomic forces. This displacement against a restraining force is analogous to lifting a weight or stretching a spring and represents potential energy. The source of the energy is the external field, the motion of the shifting charges resulting perhaps in a transient current through a battery which is producing the field.

The actual mechanism of the charge displacement differs in the various dielectric materials. Polar molecules have a permanent displacement existing between the centers of gravity of the positive and negative charges, and each pair of charges acts as a dipole. Normally, the dipoles are oriented in a random way throughout the interior of the material, and the action of the external field is to align these molecules, to some extent, in the same direction. On the other hand, a nonpolar molecule does not have this dipole arrangement until after a field is applied. The negative and positive charges shift in opposite directions against their mutual attraction and produce a dipole which is aligned with the electric field.

6.3 POLARIZATION

An atom of the dielectric can be considered as two superimposed positive and negative charge regions as shown in Figure 6.1. Upon the application of an electric field, the positive charge region moves in the direction of the applied field, and the negative charge region moves in the opposite direction, and the atom is said to be polarized. This phenomenon is called polarization. The direction of the polarization is from '–' to '+' charges, i.e., in the direction of $\vec{E}$ field.

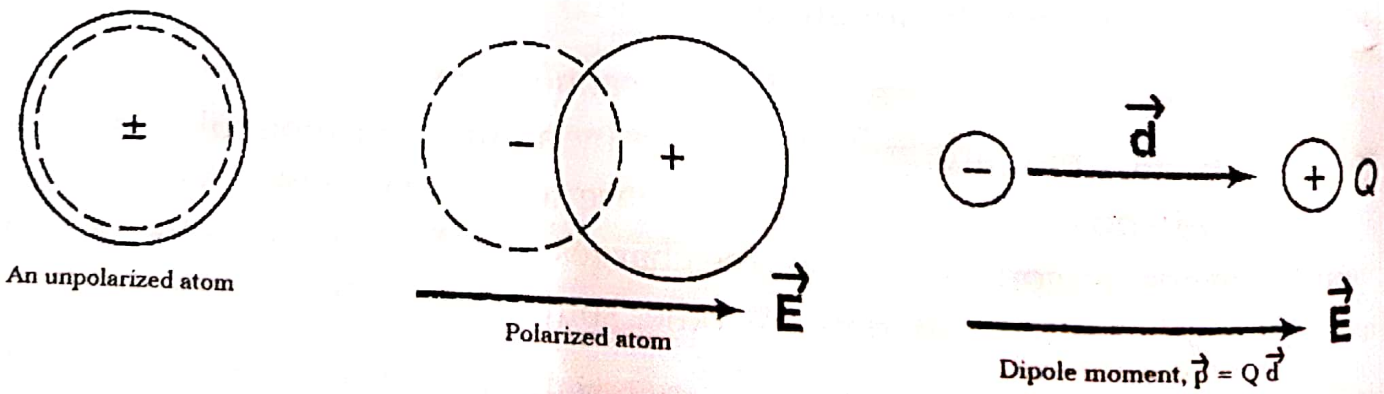


Figure 6.1 *Dipole moment.*

Dipole moment $\vec{p}$ is given as

$$\vec{p} = Q\vec{d} \text{ (Coulomb -meter)}$$

where Q = positive one of the two bound charges composing the dipole

$\vec{d}$ = vector from the negative to the positive charge.

Let n be the number of dipoles per unit volume, then there are $n\Delta v$ dipoles in a volume Δv. The total dipole moment is

$$\vec{p}_{total} = \sum_{i=1}^{n\Delta v} \vec{p}_i$$

The polarization $\vec{P}$ is defined as the dipole moment per unit volume.

$$\vec{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} \vec{p}_i \quad (\text{Coulomb/m}^2)$$

For isotropic dielectric materials, the linear relationship between $\vec{P}$ and $\vec{E}$ is

$$\boxed{\vec{P} = \chi_e \varepsilon_0 \vec{E}} \quad \text{................} \text{ (i)}$$

where χ_e (chi) is a dimensionless quantity called the electric susceptibility of the material.

If we take into account of polarization, the expression of electric flux density($\vec{D}$) is

$$\boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}} \quad \text{.............} \text{ (ii)}$$

Using equation (i) in (ii),

$$\vec{D} = \varepsilon_0 \vec{E} + \chi_e \varepsilon_0 \vec{E}$$

or, $\vec{D} = \varepsilon_0 \vec{E}\ (\chi_e + 1)$ which is in the form of $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$

$$\boxed{\therefore \varepsilon_r = \chi_e + 1}$$

where ε_r = dimensionless quantity known as the relative permittivity or dielectric constant of the material.

- Dielectrics
  - Isotropic (direction of $\vec{E}$ = direction of $\vec{P}$)
    - Linear ($\vec{E}$ & $\vec{P}$ are linearly related)
    - Nonlinear
  - Nonisotropic (direction of $\vec{E} \neq$ direction of $\vec{P}$)
    - Linear
    - Nonlinear

Figure 6.2 Classification of dielectrics

It is wisely noted that most engineering dielectrics are linear for moderate-to-large field strengths and are also isotropic.

6.4 BOUNDARY CONDITIONS FOR PERFECT DIELECTRIC MATERIALS

Consider the interface between two dielectrics having permittivities ε_1 and ε_2 and occupying the regions 1 and 2 as shown in the Figure 6.3.

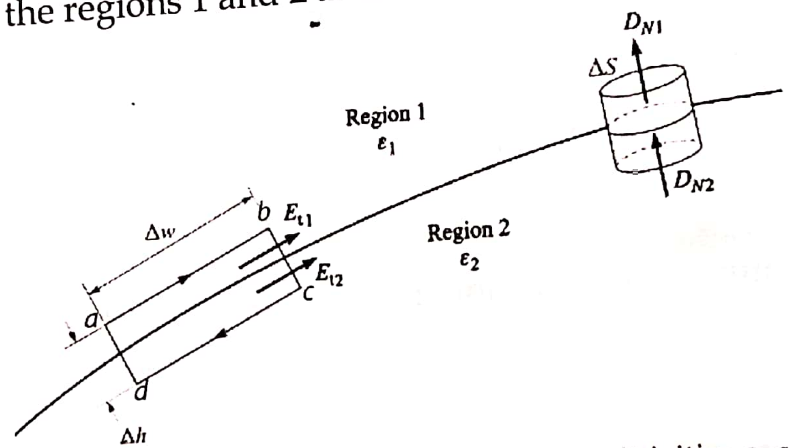


Figure 6.3 *The boundary between perfect dielectrics of permittivities ε_1 and ε_2. The continuity of D_N is shown by the gaussian surface on the right, and the continuity of E_t is shown by the line integral about the closed path at the left.*

Examining tangential components by using,

$$\oint_{abcda} \vec{E} \cdot d\vec{l} = 0$$

$$\text{or,} \int_a^b \vec{E} \cdot d\vec{l} + \int_b^c \vec{E} \cdot d\vec{l} + \int_c^d \vec{E} \cdot d\vec{l} + \int_d^a \vec{E} \cdot d\vec{l} = 0$$

$$\text{or, } E_{t1}\,\Delta w + 0 - E_{t2}\,\Delta w + 0 = 0$$

$$\boxed{\text{or, } E_{t1} = E_{t2}} \qquad \text{.........(i)}$$

which shows that the tangential component of the electric field intensity is continuous across the boundary.

Equation (i) may be rewritten as

$$\frac{D_{t1}}{\varepsilon_1} = \frac{D_{t2}}{\varepsilon_2}$$

$$\text{or,}\ D_{t1} = \frac{\varepsilon_1}{\varepsilon_2} D_{t2}$$

which means that the tangential component of electric flux density is discontinuous across the boundary.

Examining normal components by using,

$$\oint_S \vec{D}.\, d\vec{S} = \Delta Q$$

$$\text{or,}\ \int_{top} \vec{D}.\, d\vec{S} + \int_{bottom} \vec{D}.\, d\vec{S} + \int_{sides} \vec{D}.\, d\vec{S} = \Delta Q$$

$$\text{or,}\ D_{N1}\,\Delta S - D_{N2}\,\Delta S + 0 = \rho_s \Delta S$$

$$\text{or,}\ D_{N1} - D_{N2} = \rho_s$$

If no free charges exist at the interface (i.e., charges are not deliberately placed there), $\rho_s = 0$.

$$D_{N1} - D_{N2} = 0$$

$$\boxed{\text{or,}\ D_{N1} = D_{N2}} \qquad \text{............ (ii)}$$

This shows that the normal component of electric flux density is continuous.

Equation (ii) may be rewritten as

$$\varepsilon_1 E_{N1} = \varepsilon_2 E_{N2}$$

$$\text{or,}\ E_{N1} = \frac{\varepsilon_2}{\varepsilon_1} E_{N2}$$

which means that the normal component of electric field intensity is discontinuous.

As an illustration, let's consider the refraction of D at dielectric interface between two regions ε_1 and ε_2 ($\varepsilon_1 > \varepsilon_2$). Let $\vec{D}_1$ (and $\vec{E}_1$) makes an angle θ_1 with a normal to the surface.

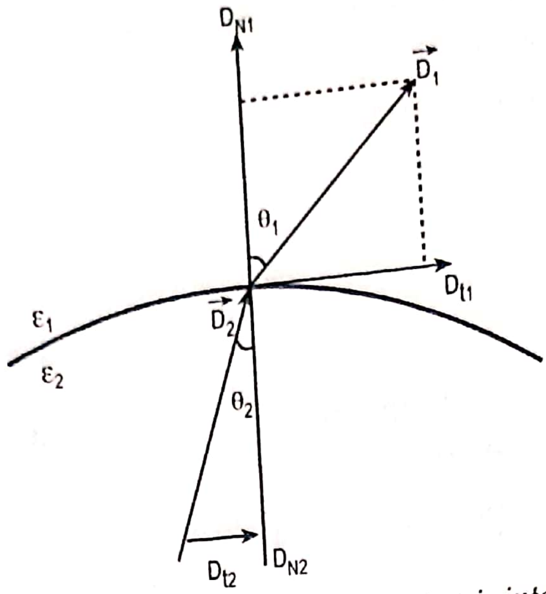


Figure 6.4 The refraction of D at a dielectric interface.

Since the normal components of $\vec{D}$ are continuous,

$$D_{N_1} = D_{N_2}$$

or, $\quad D_1 \cos\theta_1 = D_2 \cos\theta_2$

$$\Rightarrow \frac{D_1}{D_2} = \frac{\cos\theta_2}{\cos\theta_1} \qquad \ldots\ldots\ldots\ldots\ldots \text{(x)}$$

Since the tangential components of $\vec{D}$ are discontinuous,

$$D_{t1} = \frac{\varepsilon_1}{\varepsilon_2} D_{t2}$$

or, $\quad \dfrac{D_{t1}}{D_{t2}} = \dfrac{\varepsilon_1}{\varepsilon_2}$

or, $\quad \dfrac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \dfrac{\varepsilon_1}{\varepsilon_2} \qquad \ldots\ldots\ldots\ldots\ldots \text{(y)}$

Putting equation (x) in (y),

$$\frac{\cos\theta_2 \sin\theta_1}{\cos\theta_1 \sin\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

or, $\quad \dfrac{\tan\theta_1}{\tan\theta_2} = \dfrac{\varepsilon_1}{\varepsilon_2}$

Since $\varepsilon_1 > \varepsilon_2$, $\tan\theta_1 > \tan\theta_2$ and therefore $\theta_1 > \theta_2$.

The magnitude of $\vec{D}$ in region 2 can be obtained using equation (x) and (y) as

$$D_2 = D_1 \sqrt{\cos^2\theta_1 + \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2 \sin^2\theta_1}$$

and the magnitude of $\vec{E}$ in region 2 is

$$E_2 = E_1 \sqrt{\sin^2\theta_1 + \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 \cos^2\theta_1}$$

6.5 CAPACITANCE

A capacitor is a device that stores electric potential energy and electric charge. To make a capacitor, two conductors have to be insulated from each other. To store energy in this device, charge has to be transferred from one conductor to the other so that one has a negative charge and the other has an equal amount of positive charge. Work must be done to move the charges through the resulting potential difference between the conductors, and the work done is stored as electric potential energy.

Capacitors have a tremendous number of practical applications in devices such as electronic flash units for photography, pulsed lasers, air bag sensors for cars, and radio and television receivers.

Let's consider two conductors embedded in a homogeneous dielectric as shown in the Figure 6.5.

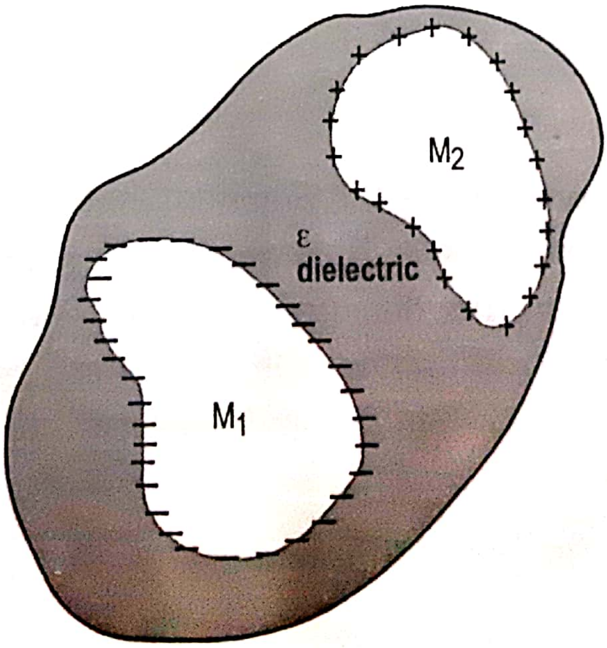


Figure 6.5 *Two oppositely charged conductors M_1 and M_2 surrounded by a uniform dielectric. The ratio of the magnitude of the charge on either conductor to the magnitude of the potential difference between them is the capacitance C.*

Conductor M_2 carries a total positive charge Q, and M_1 carries an equal negative charge. There are no other charges present, and the total charge of the system is zero.

We now know that the charge is carried on the surface as a surface charge density and also that the electric field is normal to the conductor surface. Each conductor is, moreover, an equipotential surface. Since M_2 carries the positive charge, the electric flux is directed from M_2 to M_1, and M_2 is at the more positive potential. In other words, work must be done to carry a positive charge from M_1 to M_2.

Let's designate the potential difference between M_2 and M_1 as V_o. We may now define the capacitance of this two-conductor system as the ratio of the magnitude of the total charge on either conductor to the magnitude of the potential difference between conductors.

$$C = \frac{Q}{V_o}$$

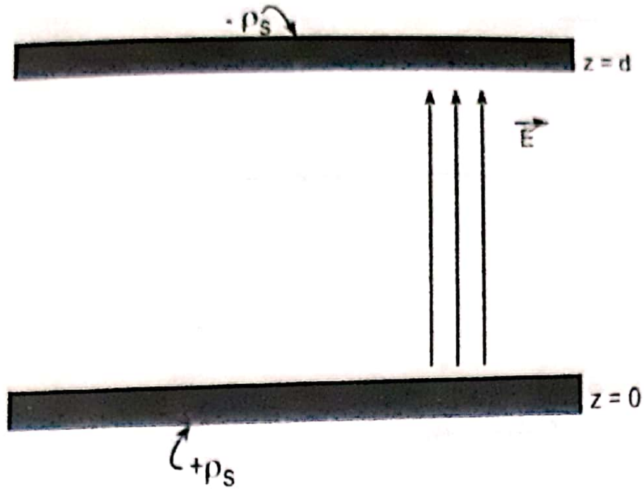
In general terms, we determine Q by a surface integral over the positive conductors, and we find V_o by carrying a unit positive charge from the negative to the positive surface.

$$C = \frac{\oint_S \varepsilon \vec{E} \cdot d\vec{S}}{-\int_{-}^{+} \vec{E} \cdot d\vec{l}}$$

The capacitance is independent of the potential and total charge, for their ratio is constant. If the charge density is increased by a factor of N, Gauss' law indicates that the electricity flux density or electric field intensity also increases by N, as does the potential difference. The capacitance is a function only of the physical dimensions of the system of conductors and of the permittivity of the homogeneous dielectric.

Capacitance is measured in farads (F), where a farad is defined as one coulomb per volt.

1. **Calculate the capacitance of the parallel-plate capacitor and the total energy stored in it. Let $+\rho_s$ and $-\rho_s$ be the surface charge density of two conductors.**



Solution:

$Q = \rho_s S$; S = surface area of each conductor

$$V_0 = -\int_{upper}^{lower} \vec{E} \cdot d\vec{l}$$

The electric field between the parallel plates is

$$\vec{E} = \frac{\rho_s}{\varepsilon}\,\hat{a}_z$$

$$\therefore V_o = -\int_{upper}^{lower} \frac{\rho_s}{\varepsilon}\,\hat{a}_z \cdot dz\,\hat{a}_z$$

$$= -\int_{d}^{0} \frac{\rho_s}{\varepsilon}\,dz = \frac{\rho_s}{\varepsilon}\,d$$

Now, $C = \dfrac{Q}{V_o} = \dfrac{\rho_s S}{\frac{\rho_s}{\varepsilon} d} = \dfrac{\varepsilon S}{d}$

Finally, the total energy stored in the capacitor is

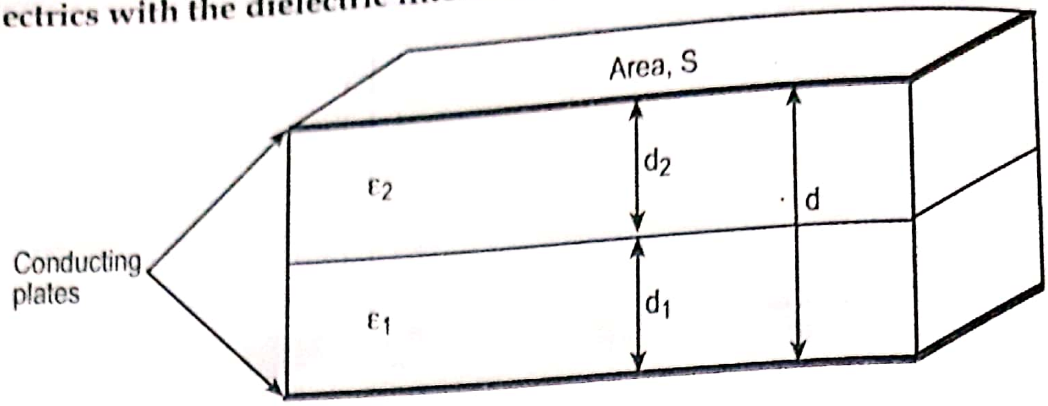
$$W_E = \frac{1}{2}\int_{vol} \varepsilon E^2\, dv$$

$$= \frac{1}{2}\int_0^S \int_0^d \varepsilon E^2\, dz\, dS$$

$$= \frac{1}{2}\int_0^S \int_0^d \varepsilon \frac{\rho_s^2}{\varepsilon^2}\, dz\, dS$$

$$= \frac{\rho_s^2}{2\varepsilon}\int_0^S \int_0^d dz\, dS = \frac{\rho_s^2}{2\varepsilon}\int_0^S [z]_o^d\, dS = \frac{\rho_s^2 d}{2\varepsilon} S = \frac{1}{2}\frac{\rho_s^2}{\varepsilon} Sd$$

2. **Calculate the capacitance of a parallel-plate capacitor containing two dielectrics with the dielectric interface parallel to the conducting plates.**



Solution:

$$V_o = E_1 d_1 + E_2 d_2$$

At the dielectric interface, E is normal and $D_{N1} = D_{N2}$, or $\varepsilon_1 E_1 = \varepsilon_2 E_2$

$$\text{So, } V_o = E_1 d_1 + \left(\frac{\varepsilon_1}{\varepsilon_2}\right) E_1 d_2$$

$$\text{or, } E_1 = \frac{V_o}{d_1 + \left(\frac{\varepsilon_1}{\varepsilon_2}\right) d_2}$$

$$\therefore \rho_{s1} = D_1 = \varepsilon_1 E_1 = \frac{V_o}{\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}}$$

Since $D_1 = D_2$, the magnitude of the surface charge is the same on each plate. The capacitance is then

$$C = \frac{Q}{V_o} = \frac{\rho_s S}{V_o} = \frac{1}{\frac{d_1}{\varepsilon_1 S} + \frac{d_2}{\varepsilon_2 S}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

6.6 GRAPHICAL FIELD PLOTTING: USING FIELD SKETCHES TO ESTIMATE CAPACITANCE IN TWO-DIMENSIONAL PROBLEMS

In capacitance problems in which the conductor configurations cannot be easily described using a single coordinate system, we apply a technique that involves making sketches of field lines and equipotential surface to allow fair, quick estimates of capacitance while providing a useful picture of the field configuration. This method is applicable only to fields in which no variation exists in the direction normal to the plane of the sketch. The procedure is based on the following facts.

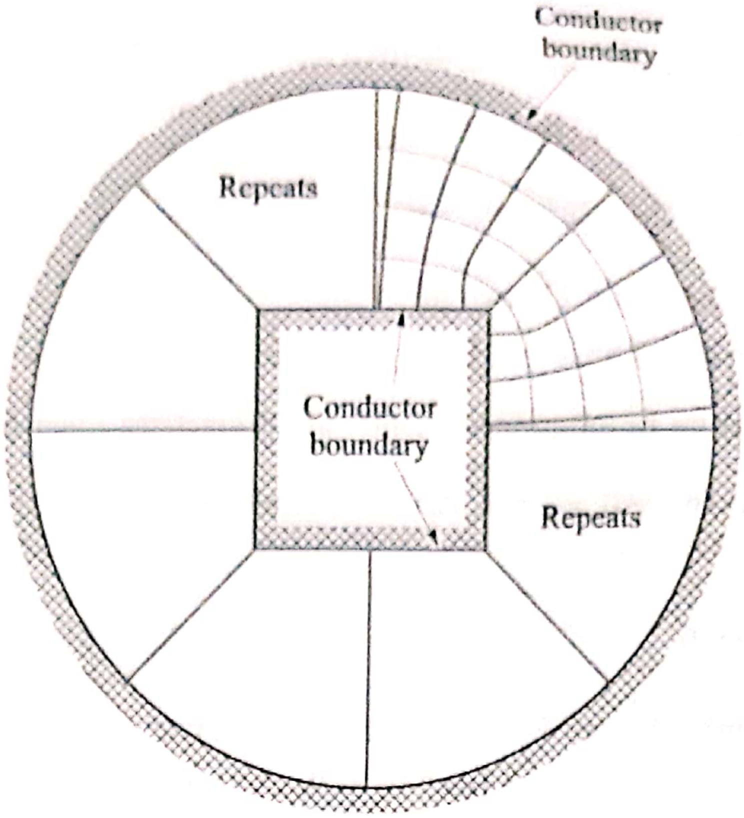


Figure 6.7 An example of a curvilinear-square field map. The side of the square is two-thirds the radius of the circle. $N_v = 4$ and $N_Q = 8\times 3.25 = 26$, and therefore $C = \varepsilon_o N_Q/N_v = 57.6$ pF/m.

$$C = \frac{N_Q \Delta Q}{N_V \Delta V} = \frac{N_Q}{N_V}\varepsilon\frac{\Delta L_t}{\Delta L_N}$$

$$\therefore\ C = \varepsilon\frac{N_Q}{N_V} \text{ for } \frac{\Delta L_t}{\Delta L_N} = 1$$

From the figure, we get

$$C = \varepsilon_o \frac{8\times 3.25}{4} = 57.6 \text{ pF/m}$$

PROBLEMS SOLVED AND SCRAMBLED

1. **The electric field intensity in polystyrene ($\varepsilon_r = 2.55$) filling the space between the plates of a parallel-plate capacitor is 10 kV/m. The distance between the plates is 1.5 mm. Calculate:**

   (a) D

   (b) P

   (c) The surface charge density of free charge on the plates

   (d) The surface density of polarization charge

   (e) The potential difference between the plates.

Solution:

a. $D = \varepsilon_o \varepsilon_r E = 8.854\times10^{-12} \times 2.55\times10^{4} = 225.4\times10^{-9} = 225.4\ nC/m^2$

b. $P = \chi_e \varepsilon_o E = (\varepsilon_r - 1)\,\varepsilon_o E = (2.55 - 1) \times 8.854 \times 10^{-12} \times 10^{4} = 137\ nC/m^2$

c. $\rho_s = \vec{D}\,.\hat{a}_N = D_N = 225.4\ nC/m^2$

d. $\rho_{sp} = \vec{P}\,.\hat{a}_N = P_N = 137\ nC/m^2$

e. $V = Ed = 10^{4}\times(1.5\times10^{-3}) = 15\ V$

2. **Two extensive homogeneous isotropic dielectrics meet on plane z=0. For z>0, $\varepsilon_{r1} = 4$ and for z<0, $\varepsilon_{r2} = 3$. A uniform electric field $\vec{E_1} = 5\,\hat{a}_x - 2\hat{a}_y + 3\,\hat{a}_z$ kV/m exists for $z \geq 0$. Find:**

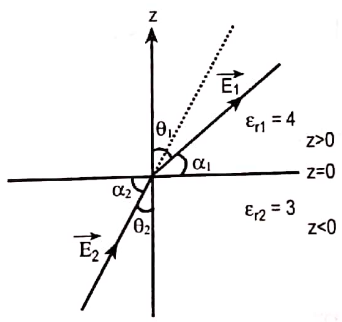
(a) $\vec{E_2}$ for $z \leq 0$.

(b) The angles $\vec{E_1}$ and $\vec{E_2}$ make with the interface.

(c) The energy densities (J/m³) in both dielectrics.

(d) The energy within a cube of side 2m centered at (3,4, -5).

Solution:



a. $\vec{E_2} = \vec{E}_{t2} + \vec{E}_{N2}$ .... (i)

Given, $\vec{E_1} = (5\hat{a}_x - 2\hat{a}_y + 3\hat{a}_z)\times10^3$ V/m

Since z = 0 separates two medium,

$E_{N1} = \vec{E_1}\,.\hat{a}_z = (5\hat{a}_x - 2\hat{a}_y + 3\hat{a}_z) \times 10^3 .\ \hat{a}_z = 3 \times 10^3$ V/m

$\vec{E}_{N1} = E_{N1}\,\hat{a}_z = 3\times10^3\,\hat{a}_z$ V/m

$\therefore\ \vec{E}_{t1} = \vec{E_1} - \vec{E}_{N1} = (5\hat{a}_x - 2\hat{a}_y)\times10^3$ V/m

Applying boundary condition,

$$\vec{D}_{N1} = \vec{D}_{N2}$$

$$\text{or, } \varepsilon_0\varepsilon_{r1}\vec{E}_{N1} = \varepsilon_0\varepsilon_{r2}\vec{E}_{N2}$$

$$\therefore \vec{E}_{N2} = 4\times10^3\hat{a}_z \text{ V/m}$$

Applying boundary condition,

$$\vec{E}_{t1} = \vec{E}_{t2}$$

$$\text{or, } (5\hat{a}_x - 2\hat{a}_y)\times10^3 = \vec{E}_{t2}$$

$$\therefore \vec{E}_{t2} = (5\hat{a}_x - 2\hat{a}_y)\times10^3 \text{ V/m}$$

From equation (i),

$$\vec{E}_2 = \vec{E}_{t2} + \vec{E}_{N2} = (5\hat{a}_x - 2\hat{a}_y + 4\hat{a}_z)\times10^3 \text{ V/m}$$

b. $\theta_2 = \cos^{-1}\dfrac{E_{N2}}{E_2} = \cos^{-1}\dfrac{4\times10^3}{3\sqrt{5}\times10^3} = 53.39°$

$$\alpha_2 = 90° - 53.39° = 36.60°$$

$$\theta_1 = \cos^{-1}\frac{E_{N1}}{E_1} = \cos^{-1}\frac{3\times10^3}{\sqrt{38}\times10^3} = 60.87°$$

$$\alpha_1 = 90° - 60.87° = 29.121°$$

c. $\dfrac{dW_{E1}}{dv} = w_{E1} = \dfrac{1}{2}\varepsilon E_1^2 = \dfrac{1}{2}\varepsilon_0\varepsilon_{r1}E_1^2 = 672\ \mu\text{J/m}^3$

$$\frac{dW_{E2}}{dv} = w_{E2} = \frac{1}{2}\varepsilon E_2^2 = \frac{1}{2}\varepsilon_0\varepsilon_{r2}E_2^2 = 597\ \mu\text{J/m}^3$$

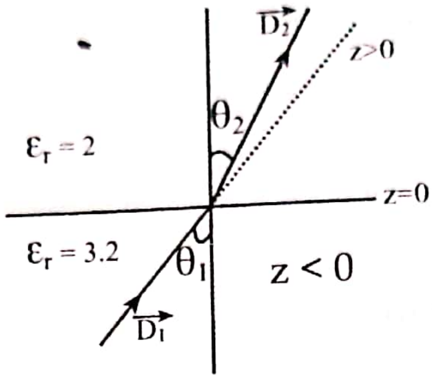
d. At the center (3, 4, -5) of the cube of side 2m, the cube is in region $2\le x\le 4$, $3\le y\le 5$, $-6\le z\le -4$. The z coordinate of the center of cube i.e., $z = -5<0$ implies the cube is in region 2.

$$\therefore W_{E2} = \int w_{E2}\,dv = w_{E2}\iiint dx\,dy\,dz = w_{E2}\int_{x=2}^{4}\int_{y=3}^{5}\int_{z=-6}^{4} dx\,dy\,dz$$

$$= w_{E2}\,(2)\,(2)\,(2) = 4.776 \text{ mJ}$$

3. Consider the region $z < 0$ be composed of a uniform dielectric material for which $\varepsilon_r = 3.2$ while the region $z > 0$ is characterized by $\varepsilon_r = 2$. Let $\vec{D}_1 = -30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z$ nC/m². Find: (a) $\vec{D}_{t1}$ (b) Polarization $\vec{P}_1$ (c) E_{n2} (d) E_{t2}, (e) $\vec{D}_2$ (f) θ_1 (g) θ_2. [2065 Chaitra]

Solution:



$\vec{D}_1 = (-30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z) \times 10^{-9}$ C/m²

a. $D_{n1} = \vec{D}_1 \cdot \hat{a}_z = 70 \times 10^{-9}$ C/m²

$\vec{D}_{n1} = D_{n1}\,\hat{a}_z = 70 \times 10^{-9}\,\hat{a}_z$ C/m²

$\vec{D}_{t1} = \vec{D}_1 - \vec{D}_{n1} = (-30\hat{a}_x + 50\hat{a}_y) \times 10^{-9}$ C/m²

b. Polarization $(\vec{P}_1) = \chi_e\, \varepsilon_o\, \vec{E}_1 = \dfrac{\varepsilon_{r_1} - 1}{\varepsilon_{r_1}}\vec{D}_1$

$$= \frac{(3.2-1)}{3.2}(-30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z)$$

$$= (-20.625\,\hat{a}_x + 34.4\,\hat{a}_y + 48.1\,\hat{a}_z) \times 10^{-9}\ \text{C/m}^2$$

c. Using boundary condition,

$\vec{D}_{n1} = \vec{D}_{n2}$

$\vec{D}_{n1} = 70\,\hat{a}_z$ nC/m²

$\vec{D}_{n2} = \varepsilon_o\, \varepsilon_{r2}\, \vec{E}_{n2}$

$70\hat{a}_z = \varepsilon_o\, \varepsilon_{r2}\, \vec{E}_{n2}$

$\therefore \vec{E}_{n2} = \dfrac{70 \times 10^{-9}}{2 \times 8.854 \times 10^{-12}}\hat{a}_z = 3.95 \times 10^3\,\hat{a}_z$ V/m

$E_{n2} = \sqrt{(0)^2 + (0)^2 + (3.95 \times 10^3)^2} = 3.95 \times 10^3$ V/m

d. Using boundary condition,

$\vec{E}_{t2} = \vec{E}_{t1}$

or, $\vec{E}_{t2} = \dfrac{\vec{D}_{t1}}{\varepsilon_0\varepsilon_{r1}} = (-1.058\hat{a}_x + 1.764\hat{a}_y) \times 10^3$ V/m

$\therefore E_{t2} = \sqrt{(-1.058 \times 10^3)^2 + (1.764 \times 10^3)^2} = 2.056 \times 10^3$ V/m

e. $\vec{D}_{t2} = \varepsilon_0\varepsilon_{r2}\ \vec{E}_{t2}$

$= 8.854 \times 10^{-12} \times 2\,[-1.058 \times 10^3\,\hat{a}_x + 1.764 \times 10^3\,\hat{a}_y)$

$= (-18.726\,\hat{a}_x + 31.223\,\hat{a}_y)$ nC/m²

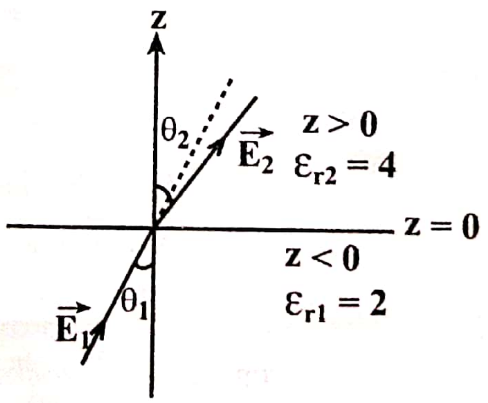
$\vec{D}_2 = \vec{D}_{n2} + \vec{D}_{t2} = (-18.726\,\hat{a}_x + 31.223\,\hat{a}_y + 70\,\hat{a}_z)$ nC/m²

f. $\theta_1 = \cos^{-1}\left(\dfrac{|\vec{D}_{n1}|}{|\vec{D}_1|}\right) = \cos^{-1}\left(\dfrac{70}{91.10}\right) = 39.79^o$

g. $\theta_2 = \cos^{-1}\left(\dfrac{|\vec{D}_{n2}|}{|\vec{D}_2|}\right) = \cos^{-1}\left(\dfrac{70}{78.902}\right) = 27.479^o$

4. The region z < 0 contains a dielectric material for which $\varepsilon_{R1} = 2$, while the region z > 0 is characterized by $\varepsilon_{R2} = 4$. Let $\vec{E}_1 = -30\,\hat{a}_x + 50\,\hat{a}_y + 70\,\hat{a}_z$ V/m & find (a) E_{n_1} (b) E_{t1} (c) θ_1 (d) D_{n2} (e) D_{t2} (f) $\vec{D}_2$ (g) $\vec{P}_1$ (h) θ_2 *[2064 Shrawan]*

Solution:



a. E_{n1} is component of $\vec{E}_1$ along z-axis.

So, $E_{n1} = \vec{E}_1 \cdot \hat{a}_z = 70$ V/m

and $\vec{E}_{n1} = E_{n1}\,\hat{a}_z = 70\,\hat{a}_z$ V/m

b. $\vec{E}_{t1}$ = component along x and y-axis

$= \vec{E}_1 - \vec{E}_{n1} = -30\hat{a}_x + 50\hat{a}_y$ V/m

$\therefore E_{t1} = \sqrt{(-30)^2 + 50^2} = 50.3095$ V/m

c. $\theta_1 = \cos^{-1}\left[\dfrac{|\vec{E}_{n1}|}{|\vec{E}_1|}\right]$

$|\vec{E}_{n1}| = 70\ V/m$

$|\vec{E}_1| = \sqrt{(-30)^2 + 50^2 + 70^2} = 91.104\ V/m$

$\therefore \theta_1 = \cos^{-1}\left[\dfrac{70}{91.104}\right] = 39.79^\circ$

d. $D_{n1} = \varepsilon_o \varepsilon_{R1} E_{n1}$

$= 8.854 \times 10^{-12} \times 2 \times 70 = 1239\ pC/m^2$

From boundary condition,

$D_{n2} = D_{n1}$

$= 1239\ pC/m^2$

e. From boundary condition,

$\vec{E}_{t2} = \vec{E}_{t1}$

$= -30\ \hat{a}_x + 50\ \hat{a}_y\ V/m$

$\vec{D}_{t2} = \varepsilon_o \varepsilon_{R2}\ \vec{E}_{t2} = 8.854 \times 10^{-12} \times 4 \times (-30\ \hat{a}_x + 50\ \hat{a}_y)$

$= -1062\ \hat{a}_x + 1770\ \hat{a}_y\ pC/m^2$

$\therefore |\vec{D}_{t2}| = \sqrt{(-1062)^2 + (1770)^2} = 2064.156\ pC/m^2$

f. $\vec{D}_2 = \vec{D}_{n1} + \vec{D}_{t2} = -1062\ \hat{a}_x + 1770\ \hat{a}_y + 1239\ \hat{a}_z\ pC/m^2$

g. Polarization $(\vec{P}_1) = \chi_e\ \varepsilon_o\ \vec{E}_1$

$= (\varepsilon_{R1} - 1)\varepsilon_o\ \vec{E}_1$

$= (2-1) \times 8.854 \times 10^{-12}(-30\ \hat{a}_x + 50\ \hat{a}_y + 70\ \hat{a}_z)$

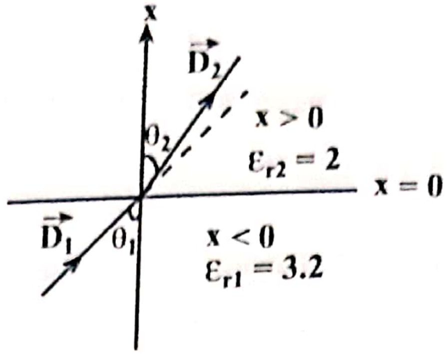
$= (-265.5\ \hat{a}_x + 442.5\ \hat{a}_y + 619.5\ \hat{a}_z)\ pC/m^2$

h. $\theta_2 = \cos^{-1}\left(\dfrac{|\vec{D}_{n2}|}{|\vec{D}_2|}\right) = \cos^{-1}\left(\dfrac{1239}{2407.46}\right) = 59.025^\circ$

5. The region $x < 0$ is composed of a uniform dielectric material for which $\varepsilon_{r1} = 3$ while region $x > 0$ is characterized by $\varepsilon_{r2} = 2$. The electric flux density at region $x < 0$ is $\vec{D} = 30\ \hat{a}_x + 50\ \hat{a}_y + 70\ \hat{a}_z\ nC/m^2$, then find polarization $(\vec{P})$ and electric field intensity $(\vec{E})$ in both regions.

[2068 Chaitra]

Solution:



Here, $\vec{D_1} = 30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z \text{ nC/m}^2$

Since boundary is at x = 0, normal component of $\vec{D_1}$ is

$D_{n1} = \vec{D_1} \cdot \hat{a}_x = (30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z) \cdot \hat{a}_x = 30 \text{ nC/m}^2$

$\therefore\ \vec{D_{n1}} = D_{n1}\hat{a}_x = 30\,\hat{a}_x \text{ nC/m}^2$

Tangential component $(\vec{D_{t1}}) = \vec{D_1} - \vec{D_{n1}} = (50\hat{a}_y + 70\hat{a}_z) \text{ nC/m}^2$

$$\therefore\ \vec{E_{t1}} = \frac{\vec{D_{t1}}}{\varepsilon_o \varepsilon_{r1}} = \frac{(50\hat{a}_y + 70\hat{a}_z)}{\varepsilon_o \varepsilon_{r1}}$$

From boundary conditions, $\vec{D_{n2}} = \vec{D_{n1}}$

$$\text{So,}\quad \vec{E_{n2}} = \frac{\vec{D_{n2}}}{\varepsilon_o \varepsilon_{r2}} = \frac{30\hat{a}_x}{2\varepsilon_o}$$

$$\vec{E_{t2}} = \vec{E_{n1}} = \frac{(50\hat{a}_y + 70\hat{a}_z)}{\varepsilon_o \times 3.2}$$

Electric field in region 1,

$$\vec{E_1} = \frac{\vec{D_1}}{\varepsilon_o \varepsilon_{r1}} = \frac{(30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z) \times 10^{-9}}{3.2 \times 8.854 \times 10^{-12}} \text{ V/m}$$

$$= (1059.32\hat{a}_x + 1765.53\hat{a}_y + 2471.75\hat{a}_z) \text{ V/m}$$

Electric field in region 2,

$$\vec{E_2} = \vec{E_{n2}} + \vec{E_{t2}}$$

$$= \frac{30\hat{a}_x}{2\varepsilon_o} + \frac{50\hat{a}_y + 70\hat{a}_z}{\varepsilon_o \times 3.2}$$

$$= \frac{30\,\hat{a}_x \times 10^{-9}}{2 \times 8.854 \times 10^{-12}} + \frac{50\hat{a}_y \times 10^{-9}}{8.854 \times 10^{-12} \times 3.2} + \frac{70\hat{a}_z \times 10^{-9}}{8.854 \times 10^{-12} \times 3.2} \text{ V/m}$$

$$= (1694.91\hat{a}_x + 1765.53\hat{a}_y + 2471.75\hat{a}_z) \text{ V/m}$$

Polarization in region 1 ($\vec{P_1}$) $= \frac{(\varepsilon_{r1}-1)}{\varepsilon_{r1}}\vec{D_1} = \frac{(3.2-1)}{3.2}[30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z]$

$= (20.625\hat{a}_x + 34.375\hat{a}_y + 48.125\hat{a}_z)\ nC/m^2$

Polarization vector in medium 2 ($\vec{P_2}$) $= \frac{\varepsilon_{r2}-1}{\varepsilon_{r2}}\vec{D_2} = \frac{(2-1)}{2}\varepsilon_0\varepsilon_{r2}\vec{E_2}$

$= \frac{(2-1)}{2}\varepsilon_0 \times 2 \times [1694.91\hat{a}_x + 1765.53\hat{a}_y + 2471.75\hat{a}_z]$

$= 8.854\times10^{-12}[169.91\hat{a}_x + 1765.53\hat{a}_y + 2471.75\hat{a}_z]$

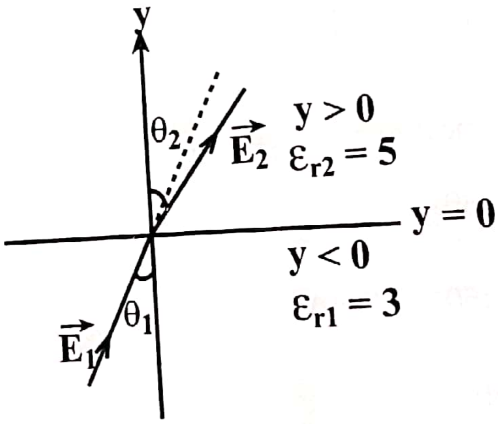
$= (14.99\hat{a}_x + 15.62\hat{a}_y + 21.87\hat{a}_z)\ nC/m^2$

Hammered Problems

1. Use the boundary condition to find $\vec{E_2}$ in the medium 2 with boundary located at plane y = 0. Medium one is perfect dielectric characterized by $\varepsilon_{r1} = 3$, medium 2 is perfect dielectric characterized by $\varepsilon_{r2} = 5$, electric field in medium

[2067 *Mangsir*]

1 is $\vec{E_1} = 3\hat{a}_x + 2\hat{a}_y + \hat{a}_z$ V/m.



Answer: $\vec{E_2} = 3\hat{a}_x + 1.2\hat{a}_y + \hat{a}_z$ V/m

2. A unit vector directed from region 1 to region 2 at the planar boundary between two perfect dielectrics is given as $\hat{a}_{N12} = -\frac{2}{7}\hat{a}_x + \frac{3}{7}\hat{a}_y + \frac{6}{7}\hat{a}_z$

Assume $\varepsilon_{r1} = 3$, $\varepsilon_{r2} = 2$, and $\vec{E_1} = 100\hat{a}_x + 80\hat{a}_y + 60\hat{a}_z$ V/m. Find $\vec{E_2}$.

Answer: $\vec{E_2} = 91.835\,\hat{a}_x + 92.245\,\hat{a}_y + 84.49\,\hat{a}_z$ V/m