Change a Variable

If the differential equation is of the form $\frac{dy}{dx} = f(ax + by + c)$ …… (i)

Where a, b, c are constant

Then we put ax + by + c = v

$$a + b\frac{dy}{dv} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{1}{b}\frac{dv}{dx} - \frac{a}{b}$$

Now, equation (i) becomes,

$$\frac{1}{b}\frac{dv}{dx} - \frac{a}{b} = f(v)$$

Separating the variables and integrating we get the required solution.

Exercise - 20

Solve the following differential equations

1. $(x + y)^2 \frac{dy}{dx} = a^2$

Sol^n. Given differential equation is,

$$(x + y)^2 \frac{dy}{dx} = a^2$$

$$\therefore \frac{dy}{dx} = \frac{a^2}{(x+y)^2} \quad \text{...... (i)}$$

Put x + y = v

$$\therefore 1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Now (i) becomes,

$$\frac{dv}{dx} - 1 = \frac{a^2}{v^2}$$

or, $\frac{dv}{dx} = \frac{a^2}{v^2} + 1 = \frac{a^2 + v^2}{v^2}$

or, $\int \frac{v^2}{a^2 + v^2} dv = \int dx$ Integrating

or, $\int \left(1 - \frac{a^2}{a^2 + v^2}\right) dv = \int dx$

or, $v - a^2 . \frac{1}{a} \tan^{-1}\frac{v}{a} = x + C$

or, $v - a\tan^{-1}\left(\frac{v}{a}\right) = x + C$

Restoring the value of v we get,

$x + y - a\tan^{-1}\left(\frac{x+y}{a}\right) = x + C$

or, $y - a\tan^{-1}\frac{(x+y)}{a} = C$

$\Rightarrow y - C = a\tan^{-1}\left(\frac{x+y}{a}\right)$

or, $\frac{y-C}{a} = \tan^{-1}\left(\frac{x+y}{a}\right)$

$\Rightarrow x + y = a\tan^{-1}\left(\frac{y-C}{a}\right)$ is the required solution.

2. **cos (x + y) dy = dx**

Sol^n. Given differential equations is,

cos (x + y) dy = dx

or, $\frac{dy}{dx} = \frac{1}{\cos(x+y)}$ ……. (i)

Put x + y = v

$$\therefore 1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dv} = \frac{dv}{dx} - 1$$

Now (i) becomes,

$$\frac{dv}{dx} - 1 = \frac{1}{\cos v}$$

or, $\frac{dv}{dx} = \frac{1}{\cos v} + 1$

or, $\frac{dv}{dx} = \frac{1 + \cos v}{\cos v}$

or, $\int \frac{\cos v}{1 + \cos v} dv = \int dx$; Integrating

or, $\int \frac{\cos v + 1 - 1}{1 + \cos v} dv = \int dx$

or, $\int \left(1 - \frac{1}{1 + \cos v}\right) dv = \int dx$

or, $\int dv - \int \frac{1}{\sin^2\frac{v}{2} + \cos^2\frac{v}{2} + \cos^2\frac{v}{2} - \sin^2\frac{v}{2}} dv = \int dx$

or, $\int dv - \int \frac{1}{2\cos^2 \frac{v}{2}} dv = \int dx$

or, $\int dv - \frac{1}{2} \int \sec^2 \frac{v}{2} dv = \int dx$

or, $v - \frac{1}{2} \ 2\tan\frac{v}{2} = x + C$

Restoring the value of v

$x + y - \tan\frac{x+y}{2} = x + C$

$y - \tan\frac{(x+y)}{2} = C \Rightarrow y - C = \tan\left(\frac{x+y}{2}\right)$

or, $\tan\left(\frac{x+y}{2}\right) = y - C$ is the required solution.

3. $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Sol^n. Given differential equation is,

$\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

or, $\frac{dy}{dx} = \sin(x + y)$ …… (i)

Put $x + y = v$

$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

Now equation (i) becomes,

$\frac{dv}{dx} - 1 = \sin v$

or, $\frac{dv}{dx} = \sin v + 1$

or, $\int \frac{1}{\sin v + 1} dv = \int dx$; Integrating

or, $\int \frac{(1 - \sin v)}{1 - \sin^2 v} dv = \int dx$

or, $\int \left(\frac{1 - \sin v}{\cos^2 v}\right) dv = \int dx$

or, $\int \sec^2 v \, dv - \int \sec v \tan v \, dv = \int dx$

or, $\tan v - \sec v = x + C$

Restoring the value of v we get,

$\tan(x + y) - \sec(x + y) = x + C$ is the required solution.

4. $\frac{dy}{dx} + 1 = e^{x+y}$

Sol^n. Given differential equations

$\frac{dy}{dx} + 1 = e^{x+y}$

or, $\frac{dy}{dx} = e^{x+y} - 1$ ……. (i)

Put $x + y = v$

$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

Now (i) becomes,

$\frac{dv}{dx} - 1 = e^v - 1$

or, $\frac{dv}{dx} = e^v$

or, $\frac{1}{e^v} dv = dx \Rightarrow \int e^{-v} dv = \int dx$; Integrating

or, $-e^{-v} = x + C$

Restoring the value of v we get,

$-e^{-(x+y)} = x + C$

or, $-1 = (x + C) e^{(x+y)}$

or, $(x + C) e^{x+y} - + 1 = 0$

or, $(x + C) e^x . e^y = -1$

or, $(x + C) e^x = -e^{-y}$

or, $(x + C) e^x + e^{-y} = 0$ is the required solution.

5. $\frac{dy}{dx} + 1 = e^{x-y}$

Sol^n. Given differential equations is,

$\frac{dy}{dx} + 1 = e^{x-y}$

or, $\frac{dy}{dx} = e^{x-y} - 1$ ……. (i)

Put $x - y = v$

$1 - \frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{dy}{dx} = 1 - \frac{dv}{dx}$

Now (i) becomes,

$1 - \frac{dv}{dx} = e^v - 1$

or, $\frac{dv}{dx} = -e^v + 2$

or, $\int \frac{1}{2-e^v}\, dv = \int dx$; Integrating

or, $\int \frac{e^{-v}}{2e^{-v}-1} dv = \int dx$

or, $\frac{1}{2} \int \frac{2e^{-v}}{2e^{-v}-1} dv = \int dx$

or, $-\frac{1}{2} \log (2e^{-v} - 1) = x + C$

or, $\log (2e^{-v} - 1) = -2 (x + C)$

or, $(2e^{-v} - 1) = e^{-2(x+C)}$

Restoring the values of v we get,

$(2e^{-x+y} - 1) = e^{-2(x+C)}$

$2e^{-x} . e^{y} - 1 = e^{-2x} . e^{-2C}$

$e^y = \left(\frac{e^{-2x} . e^{-2C}}{2e^{-x}}\right) + 1$

$= \left(\frac{e^{-2C}}{2}\right) e^{-x} + \frac{1}{2} e^x$

$e^y = \frac{1}{2} e^x + k e^{-x}$ where $K = \frac{e^{-2C}}{2}$ is the required solution.

6. $\frac{dy}{dx} - x \tan (y - x) = 1$

Sol^n. Given differential equation is,

$\frac{dy}{dx} - x \tan (y - x) = 1$ …….. (i)

Put $y - x = v$

$\frac{dy}{dx} - 1 = \frac{dv}{dx}$

or, $\frac{dy}{dx} = \frac{dv}{dx} + 1$

Now (i) becomes,

$\frac{dv}{dx} + 1 - x \tan v = 1$

or, $\frac{dv}{dx} = x \tan v$

or, $\frac{1}{\tan v} dv = x\, dx$

or, $\int \cot v\, dv = \int x\, dx$; Integrating

$\log \sin v = \frac{x^2}{2} + \log C$

Restoring the value of v we get,

$\log \sin (y - x) = \frac{x^2}{2} + \log C$

$\Rightarrow \sin (y - x) = C\, e^{\frac{x^2}{2}}$ is the required solution.

7. $\frac{dy}{dx} = (4x + y + 1)^2$

Sol^n. Given differential equation is,

$\frac{dy}{dx} = (4x + y + 1)^2$ ………. (i)

Put $4x + y + 1 = v$ differentiating w. r. t. 'x' we get,

$4 + \frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{dy}{dx} = \frac{dv}{dx} - 4$

Now (i) becomes

$\frac{dv}{dx} - 4 = v^2$

or, $\frac{dv}{dx} = v^2 + 4$

or, $\int \frac{1}{v^2+4} dv = \int dx$; Integrating

$\frac{1}{2} \tan^{-1} \frac{v}{2} = x + C$

Restoring the value of v we get,

$\frac{1}{2} \tan^{-1} \left(\frac{4x+y+1}{2}\right) = (x + C)$

or, $\tan^{-1} \left(\frac{4x+y+1}{2}\right) = 2 (x + C)$

$\Rightarrow \left(\frac{4x+y+1}{2}\right) = \tan^{-1} \{2 (x + C)\}$

or, $(4x + y + 1) = 2 \tan^{-1} (2x + 2C)$

$(4x + y + 1) = 2 \tan^{-1} (2x + k)$; where $k = 2C$ is the required solution.

8. $(x^2 + y^2 + 2xy + 1)\, dy = (x + y)\, dx$

Sol^n. Given differential equation is,

$(x^2 + y^2 + 2xy + 1)\, dy = (x + y)\, dx$

or, $\frac{dy}{dx} = \frac{(x+y)}{(x+y)^2+1}$ ……. (i)

Put x + y = v; differentiating w. r. t. x we get,

$1 + \frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{dy}{dx} = \frac{dv}{dx} - 1$

Now (i) becomes,

$\frac{dv}{dx} - 1 = \frac{v}{v^2+1}$

or, $\frac{dv}{dx} = \frac{v}{v^2+1} + 1$

or, $\frac{dv}{dx} = \frac{v+v^2+1}{v^2+1}$

or, $\int \frac{v^2+1}{v^2+v+1} dv = \int dx$; Integrating

or, $\int \frac{(v^2+v+1)-v}{(v^2+v+1)} dv = \int dx$

or, $\int dv - \int \frac{v}{v^2+v+1} dv = \int dx$

or, $\int dv - \int \frac{\frac{1}{2}(2v+1)-\frac{1}{2}}{(v^2+v+1)} dv = \int dx$

or, $\int dv - \frac{1}{2} \int \frac{(2v+1)}{(v^2+v+1)} dx + \frac{1}{2} \int \frac{1}{v^2+2.v.\frac{1}{2}+\left(\frac{1}{2}\right)^2 - \frac{1}{4}+1} dv = \int dx$

or, $\int dv - \frac{1}{2} \int \frac{(2v+1)}{(v^2+v+1)} dx + \frac{1}{2} \int \frac{1}{\left(v+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dv = \int dx$

or, $v - \frac{1}{2} \log (v^2 + v + 1) + \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1} \left\{\frac{v+1}{2} \cdot \frac{2}{\sqrt{3}}\right\} = x + C$

or, $v - \frac{1}{2} \log (v^2 + v + 1) + \frac{1}{\sqrt{3}} \tan^{-1} \left\{\frac{2v+1}{\sqrt{3}}\right\} = x + C$

Restoring the value of v we get,

$x + y - \frac{1}{2} \log \{(x + y)^2 + x + y + 1\}$

$+ \frac{1}{\sqrt{3}} \tan^{-1} \left\{\frac{2(x+y)+1}{\sqrt{3}}\right\} = x + C$

or, $y - \frac{1}{2} \log (x + y)^2 + x + y + 1\} + \frac{1}{\sqrt{3}} \tan^{-1} \left\{\frac{2x+2y+1}{\sqrt{3}}\right\} = C$

is the required solution.

9. $(x + y + 1) \frac{dy}{dx} = 1$

Sol^n. Given differential equation is,

$(x + y + 1) \frac{dy}{dx} = 1$

or, $\frac{dy}{dx} = \frac{1}{x+y+1}$ ……. (i)

Put x + y + 1 = v differentiating w. r. t. 'x' we get,

or, $1 + \frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{dy}{dx} = \frac{dv}{dx} - 1$

Now (i) becomes,

$\frac{dv}{dx} - 1 = \frac{1}{v}$ or, $\frac{dv}{dx} = \frac{1}{v} + 1 = \frac{1+v}{v}$

or, $\int \frac{v}{1+v} dv = \int dv$; Integrating

or, $\int \left(1 - \frac{1}{1+v}\right) dv = \int dx$

or, v – log (1 + v) = x + C

Restoring the value of v we get,

x + y + 1 – log (1 + x + y + 1) = x + C

or, y + 1 – log (x + y + 2) = C

or, y – log (x + y + 2) = C – 1 = K where K = c – 1

is the required solution.

10. $\frac{dy}{dx} = \sqrt{y - x}$

Sol^n. Given differential equation is,

$\frac{dy}{dx} = \sqrt{y-x}$ …….. (i)

Put $y - x = v^2$ differentiating w. r. t. 'x'

$\frac{dy}{dx} - 1 = 2v \frac{dv}{dx}$

or, $\frac{dy}{dx} = 2v \frac{dv}{dx} + 1$

Now (i) becomes,

$$2v\frac{dv}{dx} + 1 = v$$

or, $\frac{dv}{dx} = \frac{v-1}{2v}$

or, $\int \frac{2v}{v-1}\, dv = \int dx$; Integrating

or, $2\int\left(1+\frac{1}{v-1}\right) dv = \int dx$

or, $2v + 2\log(v-1) = x + C$

Restoring the value of v we get,

$2\sqrt{y-x} + 2\log\left(\sqrt{y-x}-1\right) = x + C$

or, $\sqrt{y-x} + \log\left(\sqrt{y-x}-1\right) = \frac{x}{2}+\frac{C}{2}$

or, $\sqrt{y-x} + \log\left(\sqrt{y-x}-1\right) = \frac{x}{2} + K$ where $K = \frac{C}{2}$

is the required solution.

11. $\mathbf{x^2(xdx + y\,dy) + 2y(x\,dy - y\,dx) = 0}$

Sol^n. Given differential equation is,

$x^2(xdx + y\,dy) + 2y(x\,dy - y\,dx) = 0$ …….. (i)

Put $x = r\cos\theta$ and $y = r\sin\theta$

Then, $x^2 + y^2 = r^2$ and $\frac{y}{x} = \tan\theta$

$\therefore \frac{xdx - ydx}{x^2} = \sec^2\theta$

and $x\,dx + y\,dy = r\,dr$

Now equation (i) becomes

$x\,dx + y\,dy + 2y\frac{xdy - ydx}{x^2} = 0$

$rdr + 2r\sin\theta \,.\, \sec^2\theta = 0$

$\int dr + 2\int \tan\theta\sec\theta\, d\theta = \int 0$; Integrating

$r + 2\sec\theta = C$

Restoring the value of r and θ we get,

$\sqrt{x^2+y^2} + 2\sqrt{1+\frac{y^2}{x^2}} = C$

or, $\sqrt{x^2+y^2} + 2\frac{\sqrt{x^2+y^2}}{x} = C$

or, $\sqrt{x^2+y^2}\left(1+\frac{2}{x}\right) = C$

or, $(x + 2)\sqrt{x^2+y^2} = Cx$ is the required solution.