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ACE 077 BCT 020.

Q. 1) Leibnitz's Rule

If $f(x, \alpha)$ & $\partial f(x, \alpha)$ be continuous function of x & α, then

$$\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)\,dx\right] = \int_a^b \frac{\partial f(x,\alpha)}{\partial\alpha}\,dx.$$

proof:

let $(\alpha) = \int_a^b f(x, \alpha)\,dx$, then

$$F(\alpha + F(\alpha)\,\delta\alpha) = \int_a^b f(x, \alpha + \delta\alpha)\,dx$$

Hence,

$$F(\alpha + \delta\alpha) - F(\alpha) = \int_a^b f(x, \alpha + \delta\alpha)\,dx - \int_a^b f(x, \alpha)\,dx.$$

$$= \int_a^b [F(x, \alpha + \delta\alpha) - f(x, \alpha)]\,dx$$

$$\therefore \frac{F(\alpha + \delta\alpha) - F(\alpha)}{\delta\alpha} = \int_a^b \frac{[F(x, \alpha + \delta\alpha) - f(x, \alpha)]}{\delta\alpha}\,dx$$

Taking the limiting $\delta\alpha \to 0$ on both sides, we get

$$\lim_{\delta\alpha \to 0} \frac{F(\alpha + \delta\alpha) - F(\alpha)}{\delta\alpha} = \lim_{\delta\alpha \to 0} \int_a^b \frac{[F(x, \alpha + \delta\alpha) - f(x, \alpha)]}{\delta\alpha}\,dx$$

$$\therefore \frac{dF}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial\alpha}\,dx$$

proved

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Q. 3)

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2} \quad \text{let} \quad \lim_{x\to 0} A = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$$

$$= \lim_{x\to 0} \frac{1}{x^2}\log\left(\frac{\sin x}{x}\right)$$

$$= \lim_{x\to 0} \frac{\log\left(\frac{\sin x}{x}\right)}{x^2} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\frac{\cos x}{\sin x} - \frac{1}{x}}{2x}$$

$$= \lim_{x\to 0} \frac{x\cos x - \sin x}{2x^2\sin x}$$

$$= \lim_{x\to 0} \frac{x\cos x - \sin x}{2x^3\left(\frac{\sin x}{x}\right)}$$

$$= \lim_{x\to 0} \frac{x\cos x - \sin x}{2x^3}$$

$$= \lim_{x\to 0} \frac{-x\sin x + \cancel{\cos x} - \cancel{\cos x}}{6x^2}$$

$$= \lim_{x\to 0} \frac{-x\sin x}{6x^2}$$

$$= \lim_{x\to 0} -\frac{1}{6}\left(\frac{\sin x}{x}\right)$$

$$= -\frac{1}{6}$$

$$\Rightarrow \lim_{x\to 0} A = e^{-1/6}$$

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6} \quad \text{Ans}$$

Q.4) Asymptotes

=> Sol^n. the eq^n of curve is,

$$x^3 + y^2 - xy^2 - x^2y + x^2 - y^2 = 1 \quad \text{——(i)}$$

The second degree eq^n in x & y. so, it has at most two asymptotes. Since x^3 & y^3 both present then, are no horizontal & vertical asymptote.

For the oblique asymptote, put $x = 1$ & $y = m$ in 3rd, 2nd & 1st degree terms in (1)

$\phi_3(m) = m^3 - m^2 - m + 1$ $\quad$ $\phi_2 = 1 - m^2$ $\quad$ $\phi_1(m) = 0$.

$\phi_3'(m) = 3m^2 - 2m - 1$ $\quad$ $\phi_2' = -2m$

$\phi_3''(m) = 6m - 2$

for value of m put $\phi_3(m) = 0$.

$$m^3 - m^2 - m + 1 = 0$$

$$(m^2 - 1)(m - 1) = 0$$

$$m = -1, 1, 1.$$

Now we have $\quad c = -\dfrac{\phi_2(m)}{\phi_3'(m)} = \dfrac{1 - m^3}{-3m^2 - 2m - 1}$

$$c = -\frac{\phi_2(-1)}{\phi_3'(-1)} = \frac{1 - (-1)^3}{-3(-1)^2 - 2 \times (-1) + 1}$$

$$c = 0.$$

Here, $m = 1$ is repeated root & $\phi_3'(1) = 0$.

using

$$\frac{c^2}{2!}\phi_3''(1) + \frac{c}{1!}\phi_2'(1) + \phi_1(1) = 0$$

or $\frac{c^2}{2!} \times 4 + \frac{c}{1!} \times (+2) + 0 = 0$

$c^2 - c = 0$

$c = 0, 1.$

The oblique asymptote is $y = mx + c$

∴ Here, the asymptote $x - y = 0$, $x + y = 0$ & $x - y + 1 = 0$.

Q.5) Soln.

given curve is.

$y^2 = 4c(x+c)$

Differentiating.

$2y \frac{dy}{dx} =$

Q. 13)

$$(x^2-y^2)\,dx + 2xy\,dy = 0$$

Here
the give diff. eqn is

$$(x^2-y^2)\,dx + 2xy\,dy = 0 \quad -(i)$$

$$2xy\,dy = -(x^2-y^2)\,dx$$

or $$\frac{dy}{dx} = -\frac{(x^2-y^2)}{2xy}$$

$$\frac{dy}{dx} = \left(-\frac{\frac{x^2-y^2}{x^2}}{\frac{2xy}{x^2}}\right)$$

$$\frac{dy}{dx} = \frac{-\left(1-\frac{y^2}{x^2}\right)}{\frac{2y}{x}}$$

The given diff. eqn in y/x.
so, put

$$y = vx \Rightarrow v = y/x$$

diff. y. w.r.t. x we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Replacing $\frac{dy}{dx}$. dv we get in eqn (i).

$$v + x\frac{dv}{dx} = -\frac{(1-v^2)}{2v}$$

$$x\frac{dv}{dx} = -\frac{(1-v^2)}{2v} - v$$

$$= \frac{-1+v^2-2v^2}{2v}$$

$$= \frac{-1-v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{-(1+v^2)}{2v}$$

$$x\frac{dv}{dx} = \frac{-(1+v^2)}{2v}$$

$$x\frac{dv}{dx} = \frac{-(1+v^2)}{2v}$$

$$\therefore \frac{2v\,dv}{1+v^2} = \frac{-dx}{x}$$

Integrating both side

$$\int \frac{2v\,dv}{1+v^2} = -\int \frac{dx}{x}$$

$$\log(1+v^2) = -\log x + c$$

$$\log(1+v^2) + \log x = c$$

$$\log(1+v^2)\cdot x = c$$

taking Anti-log on both sid

$$(1+v^2)\,x = k$$

replacing v with y/x

$$\left(1+\frac{y^2}{x^2}\right)x = c$$

$$\left(\frac{x^2+y^2}{x^2}\right)x = c$$

$$\boxed{\therefore\ x^2+y^2 = xC}$$ is req^n sol^n.

Q.N. 8

$$\int_0^{\pi/4} \sin^4 x \cos^2 x \, dx = \frac{3\pi - 4}{192}$$

$$= \int_0^{\pi/4} (\sin^2 x)^2 . \cos^2 x \, dx$$

$$= \int_0^{\pi/4} \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{8} \int_0^{\pi/4} (1 - 2\cos 2x + \cos^2 2x)(1 + \cos 2x) dx.$$

$$= \frac{1}{8} \int_0^{\pi/4} (1 - \cos 2x) dx - \frac{1}{8} \int_0^{\pi/4} [\cos^2 x - \cos 2x] dx$$

$$= \frac{1}{8} \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/4} - \frac{1}{8} \int_0^{\pi/4} [\cos^2 2x - \cos^3 2x] dx$$

so put $2x = \theta$, $2dx = d\theta$

when $x = 0$, $\theta = 0$. when $x = \frac{\pi}{4}$, $\theta = \frac{\pi}{2}$

$$= \frac{1}{8}\left[\frac{\pi}{4} - \sin \frac{1}{2}\right] - \frac{1}{8} \int_0^{\pi/4} (\cos^2\theta - \cos^3\theta) . \frac{d\theta}{2}$$

$$= \frac{\pi}{32} - \frac{1}{16} - \frac{1}{16} \int_0^{\pi/4} (\cos^2\theta - \cos^3\theta) d\theta$$

$$= \frac{\pi}{32} - \frac{1}{16} - \frac{1}{16} - \frac{\sqrt{\pi}}{2} \frac{\sqrt{(2+1)/2}}{\sqrt{(2+2)/2}} + \frac{1}{16} \frac{\sqrt{\pi}}{2} \frac{\sqrt{(3+2)/2}}{\sqrt{(3+2)/2}}$$

$$= \frac{\pi}{32} - \frac{1}{16} - \frac{1}{32} \frac{\sqrt{\pi} . \frac{1}{2} . \sqrt{\pi}}{1} + \frac{1}{32} \frac{\sqrt{\pi} . 1}{\frac{3}{2} . \frac{1}{2} \sqrt{\pi}}$$

$$= \frac{\pi}{32} - \frac{1}{16} - \frac{\sqrt{\pi}}{64} + \frac{1}{24}$$

$$= \frac{\pi}{64} - \frac{1}{48} = \frac{3\pi - 4}{192}$$ #

Q.N.(7)

⇒ Sol^n.

we have $I = \int_0^{\pi/2} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$

$$= \int_0^{\pi/2} \frac{\sec^2 x\, dx}{a^2\tan^2 x + b^2}$$

$$= \frac{1}{b} \cdot \frac{1}{a}\left[\tan^{-1} \frac{a\tan x}{b}\right]_0^{\pi/2}$$

$$= \frac{1}{ab}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2ab}$$

$$I = \frac{\pi}{2ab}$$

$$\int_0^{\pi/2} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{\pi}{2ab} \quad \text{---} (i)$$

Now,

diff. both side with respect to "a", we get

$$\int_0^{\pi/2} \frac{-2a\sin^2 x\, dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{-\pi}{2a^2b}$$

$$\int_0^{\pi/2} \frac{\sin^2 x\, dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{\pi}{4a^3b} \quad \text{---} (2)$$

again, diff. both side of (1) with respect to b we get

$$\int_0^{\pi/2} \frac{\cos^2 x\, dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{\pi}{4ab^3} \quad \text{---} (3)$$

adding (2) & (3) & putting $\sin^2 x + \cos^2 x = 1$, we get,

$$\int_0^{\pi/2} \frac{dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{\pi}{4a^3b} + \frac{\pi}{4ab^3}$$

$$= \frac{\pi(a^2+b^2)}{4a^3b^3} \quad \#$$

Q.N. (10)

The eq^n is

$$x^2 + 2\sqrt{3}xy - y^2 = 2a^2 \quad \text{---(i)}$$

Sine the axes is turned through angle $\theta = 30°$.

$$x = x\cos\theta - y\sin\theta$$
$$= x\cos 30 - y\sin 30$$
$$= x \cdot \frac{\sqrt{3}}{2} - y \cdot \frac{1}{2}$$

$$y = y\cos\theta + x\sin\theta$$
$$= y\cos 30 + x\sin 30$$
$$= y\frac{\sqrt{3}}{2} + x\frac{1}{2}$$

eq^n (i) becomes

$$\Rightarrow \left(\frac{\sqrt{3}x}{2} - \frac{y}{2}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}}{2}x - \frac{y}{2}\right)\left(\frac{\sqrt{3}}{2}y + \frac{x}{2}\right) - \left(\frac{\sqrt{3}}{2}y + \frac{x}{2}\right)^2 = 2a^2$$

$$\Rightarrow \frac{3}{4}x^2 - \frac{2\sqrt{3}}{4}xy + \frac{y^2}{4} + 2\sqrt{3}\left(\frac{3}{4}xy + \frac{\sqrt{3}}{4}x^2 - \frac{\sqrt{3}}{4}y^2 - \frac{xy}{4}\right)$$
$$- \frac{3y^2}{4} - \frac{2\sqrt{3}}{4}xy - \frac{x^2}{4} = 2a^2$$

$$\Rightarrow \frac{3}{4}x^2 - \frac{2\sqrt{3}}{4}xy + \frac{y^2}{4} + \frac{6\sqrt{3}}{4}xy + \frac{6x^2}{4} - \frac{6y^2}{4} - \frac{2\sqrt{3}}{4}xy -$$
$$\frac{3}{4}y^2 - \frac{2\sqrt{3}}{4}xy - \frac{x^2}{4} = 2a^2$$

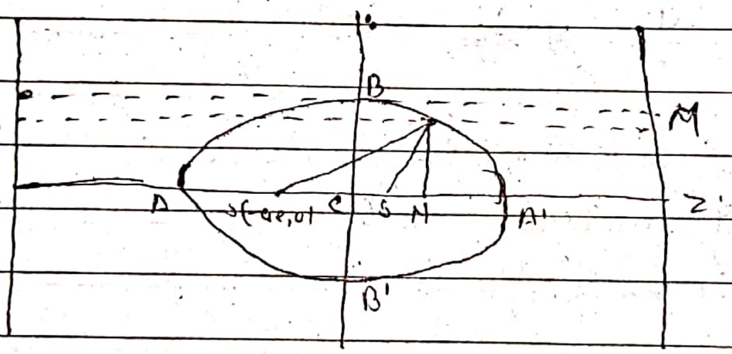
$\Rightarrow 3x^2 - 2\sqrt{3}xy + y^2 + 6\sqrt{3}xy + 6x^2 - 6y^2 - 2\sqrt{3}xy = 3y^2 - 2\sqrt{3}xy - x^2 = 3a^2$

$\Rightarrow 8x^2 - 8y^2 = 8a^2$

$\therefore x^2 - y^2 = a^2$

Q.N. 11

let S be fixed point & ZM be the fixed st. line
Draw SZ perpendicular to ZM.



By definition of ellipse.

Let A be a point on the locus such that
$SA = eAZ$ — (i)

Also the point 'A'' be on the locus such that
$SA' = eA'Z$ — (2)

let C be the middle point of AA' & $AA' = 2a$ then.

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$$CA = CA' = a$$

On addition (1) & (2)

$$SA + SA' = e(AZ + A'Z)$$
$$AA' = e[(CZ - CA) + CA' + CZ]$$
$$2a = e[CZ - a + a + CZ]$$
$$2a = 2e \cdot CZ$$
$$CZ = \frac{a}{e}$$

subtraction (1) & (2)

$$SA - SA' = e(AZ - A'Z)$$
$$(CS - CA) - (CA' - CS) = e \cdot AA'$$
$$2CS = 2 \cdot ae$$
$$CS = ae$$

let C be the origin, CA be along the axis of x, a line through C perpendicular to ACA' be the y-axis so that the coordinate of foci S & S' $S(-ae, 0)$ & $S'(ae, 0)$.

let $P(x, y)$ be any point on the ellipse

$$CN = x, \quad PN = y$$

by definition of the ellipse

$$SP = e\,PM$$
$$SP^2 = e^2 PM^2 = e^2 (ZN)^2 = e^2 (CZ + CN)^2$$
$$(x + ae)^2 + (y - 0)^2 = e^2 \left(\frac{a}{e} + x\right)^2 = (a + ex)^2$$
$$x^2 + 2aex + a^2e^2 + y^2 = a^2 + 2aex + e^2x^2$$
$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

when $b^2 = a^2(1-e^2)$

This is the req'd. eq^n of ellipse in standard form.

Q.N. 9)

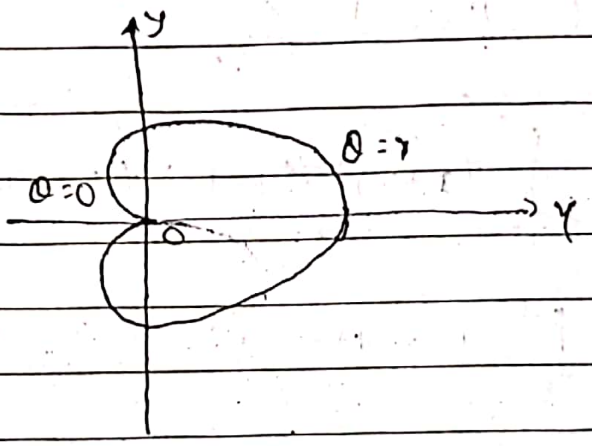
$$r = a(1+\cos\theta)$$

sol^n,

The eq^n of cardioid

$$r = a(1+\cos\theta)$$

The vol. generated revolving the cardioids about to initial line is defined as



$$V = \int_0^{\pi} y^2\,dx$$

put $x = r\cos\theta$, $y = r\sin\theta$;

$x = a(1+\cos\theta)\cdot\cos\theta$, $y = a(1+\cos\theta\cdot\sin\theta)$

Differentiating ~~yields~~

$$dx = a(-\sin\theta - 2\cos\theta\cdot\sin\theta)\,d\theta$$

$$dy = a(\cos^2\theta + \cos^2\theta - \sin^2\theta)\,d\theta$$

So the vol. generated by the cardioids revolving about initial line is defined

$$v = \int_0^{\pi} \pi y^2 \, dx$$

$$= \int_0^{\pi} \pi a^2 (1+\cos\theta)^2 \cdot \sin^2\theta \, (1+2\cos\theta)(-a\sin\theta)\, d\theta$$

put $\cos\theta = t$, $\sin\, d\theta = dt$

when, $\theta = \pi$, $t = -1$.

$$= \pi a^2 \int_t^{-1} (1+t)^2 (1-t^2)(1+2t)\, dt$$

$$= \pi a^2 \int_t^{-1} (1+4t+4t^2-2t^3-5t^4-2t^5)\, dt$$

$$= \pi a^2 \left[t + 2t^2 + \frac{4}{3}t^3 - \frac{1}{2}t^4 - t^5 - \frac{1}{3}t^6 \right]_1^{-1}$$

$$= \frac{8\pi a^3}{3} \text{ units.}$$

Q.N. (15)

$$(D^2-4D+4)\,y = x^2+e^{2x}$$

soln,

the $(D^2-4D+4)\,y = x^2+e^{2x}$

Its AE is

$m^2-4m+4=0$

either, $m=2$, or 2.

for CF

$= (C_1+C_2x)\,e^{2x}$.

for PI $= \dfrac{x^2+e^{2x}}{D^2-4D+4} = \dfrac{x^2}{D^2-4D+4} + \dfrac{e^{2x}}{D^2-4D+4}$

$= \dfrac{1}{4}\left(\dfrac{x^2}{1+\frac{D^2-4D}{4}}\right)x^2 + \dfrac{x^2e^{2x}}{2D-4D}$

$= \dfrac{1}{4}\left(1-\dfrac{D^2}{4}+\dfrac{4D}{4}+D^2\right)x^2 + \dfrac{x^2e^{2x}}{2}$

$= \dfrac{1}{4}\left(x^2-\dfrac{2}{4}+2x+2\right) + \dfrac{x^2\cdot e^{2x}}{2}$

$= \dfrac{1}{4}\left(x^2+2x+\dfrac{3}{2}\right) + \dfrac{x^2}{2}\,e^{2x}$

Thus,

the reqd. general soln is

$y = CF + PI$

$= (C_1+C_2x)\,e^{2x} + \dfrac{1}{4}\left(x^2+2x+\dfrac{3}{2}\right) + \dfrac{x^2e^{2x}}{2}$