POISSON'S AND LAPLACE'S EQUATIONS

7.1 INTRODUCTION

In the previous chapter, $\vec{E}$ was determined by summation or integration of point charges, line charges, and other charge configurations. In another chapter, Gauss' law was applied to obtain $\vec{D}$, which then gave $\vec{E}$. While these two approaches are of value to an understanding of electromagnetic field theory, they both tend to be impractical because charge distributions are not usually known. In chapter 4, use of $\vec{E} = -\nabla V$ requires that the potential function throughout the region be known. But it is generally not known. Instead, conducting materials in the form of planes, curved surfaces, or lines are usually specified, and the voltage on one is known with respect to some reference, often one of the other conductors. Laplace's equation then provides a method whereby potential function V can be obtained subject to the conditions on the bounding conductors.

7.2 DERIVATION OF POISSON'S AND LAPLACE'S EQUATIONS

From the point form of Gauss' law,

$$\nabla.\vec{D} = \rho_v$$

or, $\nabla.(\varepsilon\vec{E}) = \rho_v$

where ε = permittivity of the medium

Also, we have,

$$\vec{E} = -\nabla V$$

So, $\nabla.(-\varepsilon\nabla V) = \rho_v$

or, $\boxed{\nabla.\nabla V = -\frac{\rho_v}{\varepsilon}}$ .... (i)

Equation (i) is called **Poisson's equation** and is true for a homogeneous region in which ε is constant. The term $\nabla.\nabla V$ is read as the divergence of gradient of potential.

$$\nabla.\nabla V = \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\boxed{\therefore \nabla^2 V = -\frac{\rho_v}{\varepsilon}}$$

Let's assume a volume charge density $\rho_v = 0$, but allowing point charges, line charge and surface charge density to exist at singular locations as sources of the field, then

$\nabla^2 V = 0$ which is **Laplace's equation** and the ∇^2 operation is called the Laplacian of V.

IMPORTANT EXPRESSIONS

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \quad \textbf{(rectangular)}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \quad \textbf{(cylindrical)}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \quad \textbf{(spherical)}$$

7.3 UNIQUENESS THEOREM

Any solution to Laplace's equation or Poisson's equation which also satisfies the boundary conditions must be the only solution that exists; it is unique. This is the statement of uniqueness theorem. The proof of uniqueness theorem for both Laplace's and Poisson's equations are shown here.

1. **For Laplace's equation**

   Laplace's equation is given as

   $$\nabla^2 V = 0$$

   Let us assume the two solutions of Laplace's equation, V_1 and V_2, both general functions of the coordinates used

   Therefore, $\nabla^2 V_1 = 0$

   $$\nabla^2 V_2 = 0$$

   or, $\nabla^2 (V_1 - V_2) = 0$ .... (i)

Each solution should satisfy the boundary conditions. Let the given potential values on the boundaries be V_b, then the value of V_1 on the boundary V_{1b} and the value of the V_2 on the boundary V_{2b} must both be equal to V_b,

$$V_{1b} = V_{2b} = V_b$$

$$\therefore \quad V_{1b} - V_{2b} = 0 \qquad \text{.........(ii)}$$

Consider a vector identity,

$$\nabla.(V\vec{D}) = V(\nabla.\vec{D}) + \vec{D}.(\nabla V) \qquad \text{..........(iii)}$$

which holds true for any scalar V and any vector $\vec{D}$.

If we represent the scalar V as $V_1 - V_2$ and the vector $\vec{D}$ as $\nabla(V_1 - V_2)$, the equation (iii) can be rewritten as

$$\nabla.[(V_1 - V_2)\nabla(V_1 - V_2)] = (V_1 - V_2)[\nabla.\nabla(V_1 - V_2)] + \nabla(V_1 - V_2).\nabla(V_1 - V_2)$$

Integrating this equation throughout the volume that is enclosed by the boundary surfaces, we get

$$\int_{vol} \nabla.[(V_1 - V_2)\nabla(V_1 - V_2)]\,dv = \int_{vol}(V_1 - V_2)[\nabla.\nabla(V_1 - V_2)]\,dv + \int_{vol}\nabla(V_1 - V_2).\nabla(V_1 - V_2)\,dv$$

or, $$\int_{vol} \nabla.[(V_1-V_2)\nabla(V_1 - V_2)]\,dv = \int_{vol}(V_1-V_2)[\nabla.\nabla(V_1-V_2)]\,dv + \int_{vol}[\nabla(V_1-V_2)]^2\,dv \qquad \text{...........(iv)}$$

Using the divergence theorem,

$$\int_{vol}\nabla.\vec{D}\,dv = \oint_S \vec{D}.d\vec{S}$$

Thus,

$$\int_{vol}\nabla.[(V_1 - V_2)\nabla(V_1 - V_2)]\,dv = \oint_S[(V_1 - V_2)\nabla(V_1 - V_2)].\,d\vec{S}$$

$$= \oint_S[(V_{1b} - V_{2b})\nabla(V_{1b} - V_{2b})].\,d\vec{S}$$

From equation (ii), we have, $V_{1b} - V_{2b} = 0$

$$\therefore \quad \int_{vol} \nabla.[(V_1 - V_2) \nabla (V_1 - V_2)] \, dv = 0$$

Now equation (iv) is reduced to

$$0 = \int_{vol} (V_1 - V_2) [\nabla.\nabla (V_1 - V_2)] \, dv + \int_{vol} [\nabla (V_1 - V_2)]^2 \, dv$$

$$\text{or,} \quad 0 = \int_{vol} (V_1 - V_2) [\nabla^2 (V_1 - V_2)] \, dv + \int_{vol} [\nabla (V_1 - V_2)]^2 \, dv$$

From equation (i), $\nabla^2 (V_1 - V_2) = 0$

$$\text{So,} \quad 0 = 0 + \int_{vol} [\nabla (V_1 - V_2)]^2 \, dv$$

$$\text{or,} \quad \int_{vol} [\nabla (V_1 - V_2)]^2 \, dv = 0$$

This integral will be zero if the integrand is everywhere zero, or if the integrand is positive in some regions and negative in others resulting zero. The first reason must be true because $[\nabla (V_1 - V_2)]^2$ cannot be negative though it may be positive in some region. Therefore,

$$[\nabla (V_1 - V_2)]^2 = 0$$

$$\text{or,} \quad \nabla (V_1 - V_2) = 0 \quad \text{..............} \quad \text{(v)}$$

The equation (v) shows the gradient of $V_1 - V_2$ is zero, hence $V_1 - V_2$ cannot change with any coordinates and

$$V_1 - V_2 = \text{constant}$$

Considering a point on the boundary, we find that $V_1 - V_2 = V_{1b} - V_{2b} = 0$ which results the constant to be zero.

$$V_1 - V_2 = 0$$

$$\therefore V_1 = V_2$$

which proves the two solutions we have assumed are identical.

2. For Poisson's equation

Poisson's equation is given as

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

Let us assume the two solutions of Poisson's equation, V_1 and V_2, both general functions of the coordinates used.

Therefore,

$$\nabla^2 V_1 = -\frac{\rho_v}{\varepsilon}, \quad \nabla^2 V_2 = -\frac{\rho_v}{\varepsilon}$$

or, $\nabla^2 (V_1 - V_2) = 0$

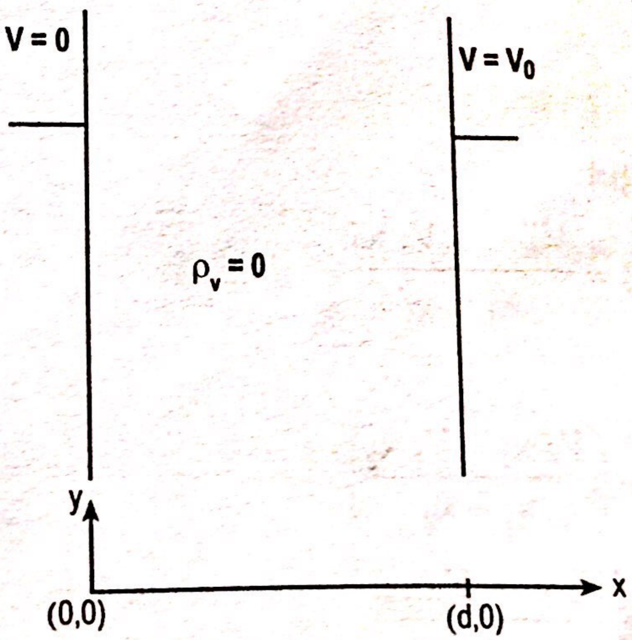
Then, proceed as done for Laplace's equation.

7.4 BOUNDARY VALUE PROBLEM IN ELECTROSTATICS

Boundary-value problems are the problems in which the potential and field are determined in a given region subject to stated values of potential or charge density on the boundary surfaces. We shall consider practical electrostatic problems where only electrostatic conditions (charge and potential) at some boundaries are known and it is desired to find $\vec{E}$ and V throughout the region. Such boundary-value problems are usually tackled using Poisson's or Laplace's equation or the method of images.

7.5 ONE DIMENSIONAL BOUNDARY VALUE PROBLEM

Let's consider a parallel plate capacitor with the potential V_o on one plate, and zero on the other. We calculate V in between the boundaries as a function of distance measured along x axis. Since V (x) is a function of one variable only, it is called one dimensional boundary value problem.



volume charge density

Figure 7.1 For illustrating one dimensional boundary value problem.

Because $\rho_v = 0$ between the boundaries, we use Laplace's equation

$$\nabla^2 V = 0$$

$$\text{or, } \frac{\partial^2 V}{\partial x^2} + \cancelto{0}{\frac{\partial^2 V}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 V}{\partial z^2}} = 0$$

$$\text{or, } \frac{\partial^2 V}{\partial x^2} = 0$$

Since V is not a function of y or z but only x, the partial derivative may be replaced by an ordinary derivative

$$\frac{d^2 V}{dx^2} = 0$$

Integrating twice,

$$\frac{dV}{dx} = A$$

$$\text{or, } V = Ax + B \qquad \text{............ (i)}$$

where A and B are constants of integration which are determined by using boundary conditions.

Let (in general),

$$V = V_1 \text{ at } x = x_1$$

$$V = V_2 \text{ at } x = x_2$$

Using these values in equation (i),

$$V_1 = Ax_1 + B$$

$$V_2 = Ax_2 + B$$

Solving for A & B, we get

$$A = \frac{V_1 - V_2}{x_1 - x_2}, \quad B = \frac{V_2 x_1 - V_1 x_2}{x_1 - x_2}$$

From equation (i),

$$V = \left(\frac{V_1 - V_2}{x_1 - x_2}\right) x + \left(\frac{V_2 x_1 - V_1 x_2}{x_1 - x_2}\right)$$

$$= \frac{(V_1 - V_2)\, x + (V_2 x_1 - V_1 x_2)}{x_1 - x_2}$$

$$\therefore \quad V = \frac{V_1 (x - x_2) - V_2 (x - x_1)}{x_1 - x_2}$$

But, as we had supposed initially,

$$V = 0 \text{ at } x = 0$$

$$V = V_o \text{ at } x = d$$

From equation (i), we get

$$0 = A \cdot 0 + B$$

$$V_o = A\,d + B$$

$$\therefore \quad B = 0,\ A = \frac{V_o}{d}$$

From equation (i) again,

$$V = \frac{V_o}{d}x + 0$$

$$\boxed{\therefore\ V = \frac{V_o}{d}x}$$

1. **Find the capacitance of a parallel-plate capacitor.**

Solution:

The steps are:

(i) Given V, use $\vec{E} = -\nabla V$ to find $\vec{E}$

(ii) Use $\vec{D} = \varepsilon\,\vec{E}$ to find $\vec{D}$

(iii) Evaluate $\vec{D}$ at either capacitor plate, $\vec{D} = \vec{D_S} = D_N\,\hat{a}_N$

(iv) Recognize that $\rho_s = D_N$

(v) Find Q by a surface integration over the capacitor plate, and then calculate capacitance.

We have,

$$V = V_o\frac{x}{d}$$

(i) $\vec{E} = -\nabla V$

Writing ∇V in rectangular coordinate system,

$$\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right)$$

$$= -\left[\frac{\partial\left(V_o \frac{x}{d}\right)}{\partial x}\hat{a}_x + \frac{\partial\left(V_o \frac{x}{d}\right)}{\partial y}\hat{a}_y + \frac{\partial\left(V_o \frac{x}{d}\right)}{\partial z}\hat{a}_z\right]$$

$$= -\frac{V_o}{d}\hat{a}_x$$

(ii) $\vec{D} = \varepsilon\,\vec{E} = \varepsilon\left(-\frac{V_o}{d}\hat{a}_x\right) = -\varepsilon\frac{V_o}{d}\hat{a}_x$

(iii) $\vec{D}_s = \vec{D}\,\Big|_{x=0} = -\varepsilon\frac{V_o}{d}\hat{a}_x\Big|_{x=0} = -\varepsilon\frac{V_o}{d}\hat{a}_x$

$\hat{a}_N = \hat{a}_x$

$\therefore\ \vec{D}_s = -\varepsilon\frac{V_o}{d}\hat{a}_N = D_N\,\hat{a}_N$

and $D_N = -\varepsilon\frac{V_o}{d}$

(iv) $\rho_s = D_N = -\varepsilon\frac{V_o}{d}$

(v) $Q = \int_S \rho_s\, dS = \int_S\left(-\varepsilon\frac{V_o}{d}\right)dS = -\frac{\varepsilon V_o}{d}\int_S dS = -\frac{\varepsilon V_o}{d}S$

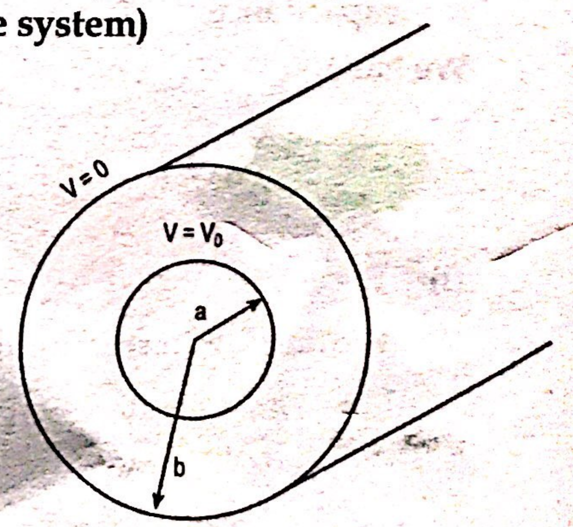
The capacitance is given as

$$C = \frac{|Q|}{V_o} = \frac{\left|-\frac{\varepsilon V_o}{d}S\right|}{V_o} = \frac{\frac{\varepsilon V_o}{d}S}{V_o}$$

$$\boxed{\therefore\ C = \frac{\varepsilon S}{d}}$$

2. Find the capacitance of a co-axial capacitor using Laplace's equation. (One dimensional problem in cylindrical coordinate system)

Solution:



Assume that V is a function of ρ only.

Laplace's equation is

$\nabla^2 V = 0$

Here, $\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \overset{0}{\cancel{\frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial\phi^2}\right)}} + \overset{0}{\cancel{\frac{\partial^2 V}{\partial z^2}}}$

or, $\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = 0$

or, $\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = 0$

Integrating

$$\rho\frac{\partial V}{\partial\rho} = A \Rightarrow \frac{\partial V}{\partial\rho} = \frac{A}{\rho}$$

Integrating again,

$$V = A\,ln(\rho) + B \qquad \text{..............} \ (i)$$

where A and B are constants of integration which are determined by using boundary conditions.

Let $V = 0$ at $\rho = b$

$V = V_o$ at $\rho = a,\ a < b$

From equation (i),

$$0 = A\,ln(b) + B$$

$$V_o = A\,ln(a) + B$$

Solving,

$$A = \frac{V_o}{ln\left(\frac{a}{b}\right)} = \frac{-V_o}{ln\left(\frac{b}{a}\right)}, \quad B = \frac{V_o\,ln(b)}{ln\left(\frac{b}{a}\right)}$$

Now, equation (i) becomes

$$V = \left[\frac{-V_o}{ln\left(\frac{b}{a}\right)}\right] ln(\rho) + \left[\frac{V_o\,ln(b)}{ln\left(\frac{b}{a}\right)}\right]$$

or, $V = \frac{V_o}{ln\left(\frac{b}{a}\right)}[ln(b) - ln(\rho)]$

$$\therefore \quad V = V_o\frac{ln\left(\frac{b}{\rho}\right)}{ln\left(\frac{b}{a}\right)}$$

(i) $\vec{E} = -\nabla V$

Writing ∇V in cylindrical coordinate system,

$$\vec{E} = -\left(\frac{\partial V}{\partial\rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial\phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z\right)$$

$$= -\frac{\partial V}{\partial \rho}\,\hat{a}_\rho$$

$$= -\frac{\partial}{\partial \rho}\left[\frac{V_o\, ln\left(\frac{b}{\rho}\right)}{ln\left(\frac{b}{a}\right)}\right]\hat{a}_\rho = \frac{-V_o}{ln\left(\frac{b}{a}\right)}\,\frac{\partial\left[ln\left(\frac{b}{\rho}\right)\right]}{\partial \rho}\,\hat{a}_\rho$$

$$\vec{E} = -\frac{V_o}{ln\left(\frac{b}{a}\right)}\left[\frac{\partial\, ln\,(b)}{\partial \rho} - \frac{\partial\, ln\,(\rho)}{\partial \rho}\right]\hat{a}_\rho$$

$$= \frac{V_o}{ln\left(\frac{b}{a}\right)}\,\frac{\partial\, ln\,(\rho)}{\partial \rho}\,\hat{a}_\rho = \frac{V_o}{ln\left(\frac{b}{a}\right)}\,\frac{1}{\rho}\,\hat{a}_\rho$$

(ii) $\vec{D} = \varepsilon\vec{E} = \varepsilon\left[\frac{V_o}{ln\left(\frac{b}{a}\right)}\,\frac{1}{\rho}\,\hat{a}_\rho\right] = \frac{\varepsilon V_o}{\rho}\,\frac{1}{ln\left(\frac{b}{a}\right)}\,\hat{a}_\rho$

(iii) $\vec{D}_s = \vec{D}\,|_{\rho = a} = \left.\frac{\varepsilon V_o}{\rho}\,\frac{1}{ln\left(\frac{b}{a}\right)}\,\hat{a}_\rho\right|_{\rho = a} = \frac{\varepsilon V_o}{a}\,\frac{1}{ln\left(\frac{b}{a}\right)}\,\hat{a}_\rho$

$\hat{a}_N = \hat{a}_\rho$

$\therefore\ \vec{D}_s = \frac{\varepsilon V_o}{a}\,\frac{1}{ln\left(\frac{b}{a}\right)}\,\hat{a}_N = D_N\,\hat{a}_N$

and $D_N = \frac{\varepsilon V_o}{a}\,\frac{1}{ln\left(\frac{b}{a}\right)}$

(iv) $\rho_s = D_N = \frac{\varepsilon V_o}{a}\,\frac{1}{ln\left(\frac{b}{a}\right)}$

(v) $Q = \int_S \rho_s\, dS = \int_S\left[\frac{\varepsilon V_o}{a}\,\frac{1}{ln\left(\frac{b}{a}\right)}\right]dS = \frac{\varepsilon V_o}{a}\,\frac{1}{ln\left(\frac{b}{a}\right)}\int_S dS$

For radius = a, length = L

$\int_S dS = 2\pi rh = 2\pi aL$

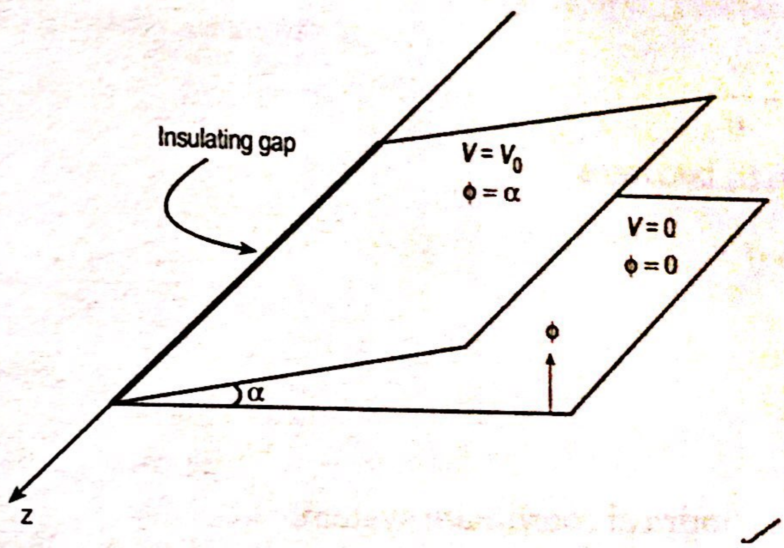
$\therefore\ Q = \frac{\varepsilon V_o}{a}\,\frac{1}{ln\left(\frac{b}{a}\right)}\,2\pi aL = \frac{2\pi\,\varepsilon V_o L}{ln\left(\frac{b}{a}\right)}$

The capacitance is given as

$$C = \frac{|Q|}{V_o} = \frac{\left|\dfrac{2\pi\varepsilon V_o L}{\ln\left(\dfrac{b}{a}\right)}\right|}{V_0} = \frac{\dfrac{2\pi\varepsilon V_o L}{\ln\left(\dfrac{b}{a}\right)}}{V_0}$$

$$\boxed{\therefore C = \frac{2\pi\,\varepsilon L}{\ln\left(\dfrac{b}{a}\right)}}$$

3. **Find the electric field intensity due to two infinite radial planes as shown in figure with an interior angle α. An infinitesimal insulating gap exists at ρ = 0.**



Solution:

Given that V is a function of ϕ only.

Laplace's equation is

$\nabla^2 V = 0$

or, $\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial V}{\partial\rho}\right) + \dfrac{1}{\rho^2}\left(\dfrac{\partial^2 V}{\partial\phi^2}\right) + \dfrac{\partial^2 V}{\partial z^2} = 0$

or, $\dfrac{1}{\rho^2}\dfrac{\partial^2 V}{\partial\phi^2} = 0$

Excluding $\rho = 0$, we have

$$\frac{\partial^2 V}{\partial\phi^2} = 0$$

Since V is not a function of ρ or z but only ϕ, the partial derivative may be replaced by an ordinary derivative.

$$\frac{d^2 V}{d\phi^2} = 0$$

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Integrating twice,

$$\frac{dV}{d\phi} = A$$

or, $V = A\,\phi + B$ .... (i)

where A and B are constants of integration which are determined by using boundary conditions

Let, $V = 0$ at $\phi = 0$

$V = V_o$ at $\phi = \alpha$

From equation (i),

$$0 = 0 + B$$

$$V_0 = A\alpha + B$$

Solving,

$$B = 0, \quad A = \frac{V_o}{\alpha}$$

Now, equation (i) becomes

$$V = \frac{V_o}{\alpha}\phi + 0$$

$$\therefore \quad V = \frac{V_o}{\alpha}\phi$$

$$\vec{E} = -\nabla V$$

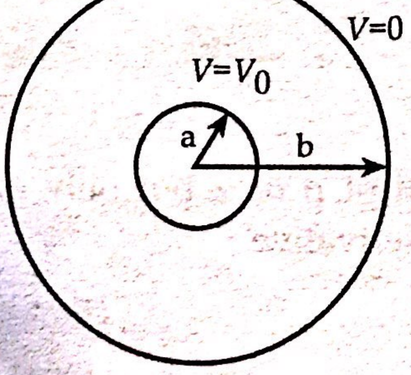
Writing ∇V in cylindrical coordinate system,

$$\vec{E} = -\left(\frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z\right)$$

$$= -\frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi \quad = -\frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{V_o}{\alpha}\phi\right)\hat{a}_\phi = -\frac{V_o}{\rho\alpha}\frac{\partial \phi}{\partial \phi}\hat{a}_\phi$$

$\boxed{\therefore\ \vec{E} = -\frac{V_o}{\rho\,\alpha}\hat{a}_\phi}$ which is the required expression.

4. Find the capacitance of a spherical capacitor using Laplace's equation. (One dimensional problem in spherical coordinate system)



Solution:

Assume that V is a function of r only,

Laplace's equation is

$$\nabla^2 V = 0$$

$$\text{or,} \quad \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}\right] = 0$$

$$\text{or,} \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0$$

$$\text{or,} \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0$$

Since V is not a function of θ or ϕ but only r, the partial derivative may be replaced by an ordinary derivative.

$$\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0$$

Integrating,

$$r^2\frac{dV}{dr} = A \Rightarrow \frac{dV}{dr} = \frac{A}{r^2}$$

Integrating again,

$$V = -\frac{A}{r} + B \quad \text{.............. (i)}$$

Let, $V = 0$ at $r = b$

$V = V_o$ at $r = a$, $\quad a < b$

From equation (i),

$$0 = -\frac{A}{b} + B$$

$$V_o = -\frac{A}{a} + B$$

Solving,

$$A = \frac{V_o}{\left(\frac{1}{b} - \frac{1}{a}\right)}, \qquad B = \frac{V_o}{b\left(\frac{1}{b} - \frac{1}{a}\right)}$$

Now equation (i) becomes

$$V = -\frac{V_o}{r\left(\frac{1}{b} - \frac{1}{a}\right)} + \frac{V_o}{b\left(\frac{1}{b} - \frac{1}{a}\right)} = \frac{V_o}{\left(\frac{1}{b} - \frac{1}{a}\right)}\left[\frac{1}{b} - \frac{1}{r}\right]$$

$$\therefore\ V = V_o \frac{\left(\frac{1}{b} - \frac{1}{r}\right)}{\left(\frac{1}{b} - \frac{1}{a}\right)}$$

(i) $\vec{E} = -\nabla V$

$$= -\left(\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi\right)$$

$$= -\frac{\partial V}{\partial r}\hat{a}_r$$

$$= -\frac{\partial}{\partial r}\left[V_o \frac{\left(\frac{1}{b} - \frac{1}{r}\right)}{\left(\frac{1}{b} - \frac{1}{a}\right)}\right]\hat{a}_r$$

$$= \frac{-V_o}{\frac{1}{b} - \frac{1}{a}}\frac{\partial}{\partial r}\left(\frac{1}{b} - \frac{1}{r}\right)\hat{a}_r = \frac{-V_o}{\frac{1}{b} - \frac{1}{a}}\left(0 + \frac{1}{r^2}\right)\hat{a}_r = \frac{-V_o}{r^2\left(\frac{1}{b} - \frac{1}{a}\right)}\hat{a}_r \qquad = \frac{V_o}{r^2\left(\frac{1}{a} - \frac{1}{b}\right)}\hat{a}_r$$

(ii) $\vec{D} = \varepsilon\,\vec{E} = \varepsilon \dfrac{V_o}{r^2\left(\frac{1}{a} - \frac{1}{b}\right)}\hat{a}_r = \dfrac{\varepsilon V_o}{r^2}\dfrac{1}{\left(\frac{1}{a} - \frac{1}{b}\right)}\hat{a}_r$

(iii) $\vec{D}s = \vec{D}\ \Big|_{r = a} = \dfrac{\varepsilon V_o}{r^2}\dfrac{1}{\left(\frac{1}{a} - \frac{1}{b}\right)}\hat{a}_r\Bigg|_{r = a} = \dfrac{\varepsilon V_o}{a^2}\dfrac{1}{\left(\frac{1}{a} - \frac{1}{b}\right)}\hat{a}_r$

$\hat{a}_N = \hat{a}_r$

$\therefore\ \vec{D}_s = \dfrac{\varepsilon V_o}{a^2}\dfrac{1}{\left(\frac{1}{a} - \frac{1}{b}\right)}\hat{a}_N = D_N\,\hat{a}_N$

and $D_N = \dfrac{\varepsilon V_o}{a^2}\dfrac{1}{\left(\frac{1}{a} - \frac{1}{b}\right)}$

(iv) $\rho_s = D_N = \dfrac{\varepsilon V_o}{a^2}\dfrac{1}{\left(\frac{1}{a} - \frac{1}{b}\right)}$

(v) $Q = \int_S \rho_s \, dS = \int_S \frac{\varepsilon V_o}{a^2} \frac{1}{\left(\frac{1}{a} - \frac{1}{b}\right)} dS = \frac{\varepsilon V_o}{a^2} \frac{1}{\left(\frac{1}{a} - \frac{1}{b}\right)} \int_S dS$

For radius = a, $\int_S dS = 4\pi r^2 = 4\pi a^2$

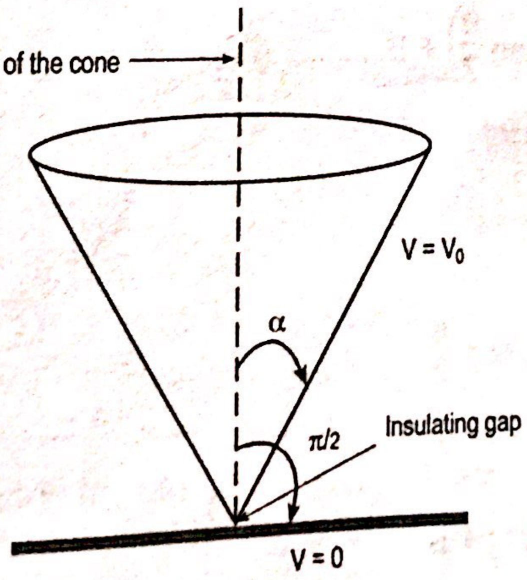
$$\therefore Q = \frac{\varepsilon V_o}{a^2} \frac{1}{\left(\frac{1}{a} - \frac{1}{b}\right)} 4\pi a^2 = \frac{4\pi \varepsilon V_o}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

The capacitance is given as

$$C = \frac{|Q|}{V_o} = \frac{\left|\frac{4\pi\varepsilon V_o}{\left(\frac{1}{a} - \frac{1}{b}\right)}\right|}{V_0} = \frac{\frac{4\pi\varepsilon V_o}{\left(\frac{1}{a} - \frac{1}{b}\right)}}{V_0}$$

$\boxed{\therefore C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}}$ which is the required expression.

5. **Find the capacitance between a conducting cone with its vertex separated from a conducting plane by an infinitesimal insulating gap and its axis normal to the plane. The cone is described by $\theta = \alpha$ with $V = V_0$ and the plane by $\theta = \frac{\pi}{2}$ with $V = 0$. Assume that V varies with θ only.**



Solution:

Given that V is a function of θ only

Laplace's equation is

$$\nabla^2 V = 0$$

In spherical coordinate system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

or, $$\frac{1}{r^2 \sin\theta}\ \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$$

Excluding r = 0 and θ = 0 or π,

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$$

Since V is not a function of ϕ or z, but only θ, the partial derivative may be replaced by an ordinary derivative.

$$\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) = 0$$

Integrating,

$$\sin\theta\frac{dV}{d\theta} = A \Rightarrow \frac{dV}{d\theta} = \frac{A}{\sin\theta}$$

Integrating again,

$$V = \int\frac{A}{\sin\theta}\,d\theta + B$$

or, $$V = A\,ln\left(\tan\frac{\theta}{2}\right) + B \text{ .... (i)}$$

Given,

$$V = 0 \text{ at } \theta = \frac{\pi}{2}$$

$$V = V_o \text{ at } \theta = \alpha,\ \alpha < \frac{\pi}{2}$$

From equation (i),

$$0 = A\,ln\left(\tan\frac{\pi/2}{2}\right) + B$$

$$V_o = A\,ln\left(\tan\frac{\alpha}{2}\right) + B$$

Solving, we get

$$A = \frac{V_o}{ln\left(\tan\frac{\alpha}{2}\right)}, \; B = 0$$

From equation (i),

$$V = \left[\frac{V_o}{ln\left(\tan\frac{\alpha}{2}\right)}\right] ln\left(\tan\frac{\theta}{2}\right) + 0$$

$$\therefore \quad V = V_o \frac{ln\left(\tan\frac{\theta}{2}\right)}{ln\left(\tan\frac{\alpha}{2}\right)}$$

(i) $\vec{E} = -\nabla V$

Writing ∇V in spherical coordinate system,

$$\vec{E} = -\left(\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi\right)$$

$$= -\frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta$$

$$= -\frac{1}{r}\frac{\partial}{\partial \theta}\left[V_o \frac{ln\left(\tan\frac{\theta}{2}\right)}{ln\left(\tan\frac{\alpha}{2}\right)}\right]\hat{a}_\theta$$

$$= -\frac{V_o}{r}\frac{1}{ln\left(\tan\frac{\alpha}{2}\right)}\frac{\partial}{\partial \theta}\left[ln\left(\tan\frac{\theta}{2}\right)\right]\hat{a}_\theta \qquad = \frac{-V_o}{r\,ln\left(\tan\frac{\alpha}{2}\right)}\frac{1}{\sin\theta}\hat{a}_\theta$$

(ii) $\vec{D} = \varepsilon\,\vec{E} = \dfrac{-\varepsilon V_o}{r\,ln\left(\tan\frac{\alpha}{2}\right)}\dfrac{1}{\sin\theta}\hat{a}_\theta$

(iii) $\vec{D}_s = \vec{D}\,|_{\theta=\alpha} = \left.\dfrac{-\varepsilon V_o}{r\sin\theta\, ln\left(\tan\frac{\alpha}{2}\right)}\hat{a}_\theta\right|_{\theta=\alpha} = \dfrac{-\varepsilon V_o}{r\sin\alpha\; ln\left(\tan\frac{\alpha}{2}\right)}\hat{a}_\theta$

$\hat{a}_N = \hat{a}_\theta$

$$\therefore \vec{D}_s = \frac{-\varepsilon V_o}{r\sin\alpha\; ln\left(\tan\frac{\alpha}{2}\right)}\hat{a}_N = D_N\,\hat{a}_N$$

and $D_N = \dfrac{-\varepsilon V_o}{r \sin\alpha \; ln\left(\tan\dfrac{\alpha}{2}\right)}$

(iv) $\rho_s = D_N = \dfrac{-\varepsilon V_o}{r \sin\alpha \; ln\left(\tan\dfrac{\alpha}{2}\right)}$

(v) $Q = \int\limits_S \rho_s \, dS \;\; = \int\limits_S \left[\dfrac{-\varepsilon V_o}{r \sin\alpha \; ln\left(\tan\dfrac{\alpha}{2}\right)}\right] dS = \dfrac{-\varepsilon V_o}{\sin\alpha \; ln\left(\tan\dfrac{\alpha}{2}\right)} \int\limits_S \dfrac{dS}{r}$

For spherical coordinate system in the direction of $\hat{a}_\theta$, $dS = r \sin\theta \, dr \, d\phi = r \sin\alpha$ $dr \, d\phi$.

$$Q = \frac{-\varepsilon V_o}{\sin\alpha \; ln\left(\tan\frac{\alpha}{2}\right)} \int\limits_{r=0}^{\infty} \int\limits_{\phi=0}^{2\pi} \frac{r \sin\alpha \, dr \, d\phi}{r}$$

$$= \frac{-\varepsilon V_o \sin\alpha}{\sin\alpha \; ln\left(\tan\frac{\alpha}{2}\right)} \int\limits_{r=0}^{\infty} \int\limits_{\phi=0}^{2\pi} dr \, d\phi$$

$$= \frac{-\varepsilon V_o}{ln\left(\tan\frac{\alpha}{2}\right)} \int\limits_{r=0}^{\infty} dr \;\; \phi \Big|_0^{2\pi}$$

$$= \frac{-\varepsilon V_o}{ln\left(\tan\frac{\alpha}{2}\right)} 2\pi \int\limits_{r=0}^{\infty} dr$$

$$= \frac{-2\pi \varepsilon V_o}{ln\left(\tan\frac{\alpha}{2}\right)} \int\limits_{r=0}^{\infty} dr$$

Let's figure out the physical conical surface extending from $r = 0$ to $r = r_1$ rather than that from $r = 0$ to $r = \infty$.

$$\therefore \quad Q = \frac{-2\pi\varepsilon V_o}{ln\left(\tan\frac{\alpha}{2}\right)} \int\limits_{r=0}^{r_1} dr$$

$$= \frac{-2\pi\varepsilon V_o}{ln\left(\tan\frac{\alpha}{2}\right)} \, r \Big|_0^{r_1}$$

$$= \frac{-2\pi\varepsilon V_o r_1}{\ln\left(\tan\frac{\alpha}{2}\right)} = \frac{2\pi\varepsilon V_o r_1}{-\ln\left(\tan\frac{\alpha}{2}\right)} = \frac{2\pi\varepsilon V_o r_1}{\ln\left(\tan\frac{\alpha}{2}\right)^{-1}} = \frac{2\pi\varepsilon V_o r_1}{\ln\left(\cot\frac{\alpha}{2}\right)}$$

The capacitance is given as

$$C = \frac{|Q|}{V_o} = \frac{\left|\frac{2\pi\varepsilon V_o r_1}{\ln\left(\cot\frac{\alpha}{2}\right)}\right|}{V_0} = \frac{\frac{2\pi\varepsilon V_o r_1}{\ln\left(\cot\frac{\alpha}{2}\right)}}{V_0}$$

$$\boxed{\therefore C = \frac{2\pi\,\varepsilon r_1}{\ln\left(\cot\frac{\alpha}{2}\right)}}$$ which is the required expression

7.6 TWO DIMENSIONAL BOUNDARY VALUE PROBLEM

In two dimensional boundary value problem, the potential function is a function of two variables i.e., $V(x, y)$; which gives the potential at any point within the boundaries and it is also satisfied when the potential of the boundaries are substituted.

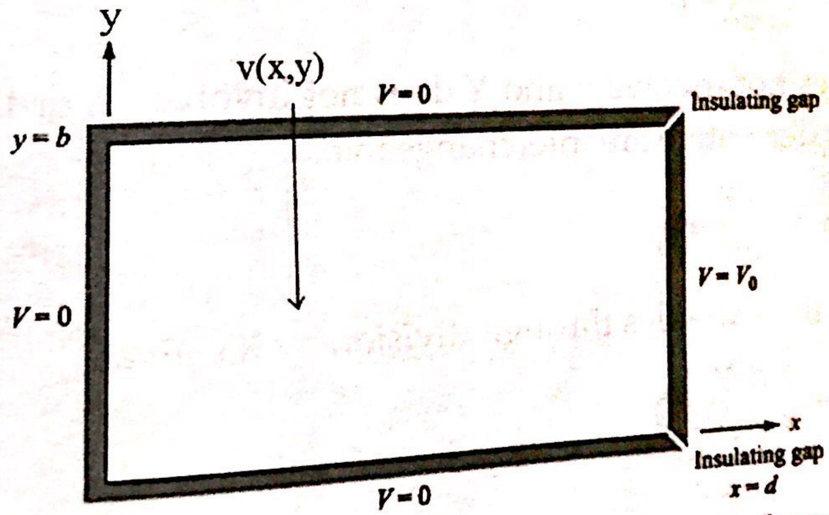


Figure 7.2 Illustration of a two dimensional boundary value problem.

Since $\rho_v = 0$ within the boundaries, the Laplace's equation will be used.

Laplace's equation is

$$\nabla^2 V = 0$$

In rectangular coordinate system,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\text{So, } \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad \text{.................. (i)}$$

We have two methods to solve this second order differential equation. These are:

1. *Separation of variables method*
2. *Numerical iteration method*

1. **Separation of Variables Method**

Let's suppose the solution of (i) is

$$V = XY$$

where X is a function of x only and Y is a function of y only.

It is then substituted in equation (i) to give

$$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial y^2} = 0$$

$$\text{or, } Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

Since X does not involve y and Y does not involves x, ordinary derivative and partial derivative are interchangeable,

$$Y\frac{d^2 X}{dx^2} + X\frac{d^2 Y}{dy^2} = 0$$

Separating the variables through division by XY gives

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} = 0$$

$$\text{or, } \frac{1}{X}\frac{d^2 X}{dx^2} = -\frac{1}{Y}\frac{d^2 Y}{dy^2}$$

L.H.S. does not involve y and R.H.S. does not involves x. And since L.H.S. = R.H.S, we conclude that L.H.S. & R.H.S must each be a constant. Let the constant be α^2.

$$\frac{1}{X}\frac{d^2 X}{dx^2} = \alpha^2, \qquad -\frac{1}{Y}\frac{d^2 Y}{dy^2} = \alpha^2$$

Solving these equations through a complicated mathematical process, we get

$X = A \cosh \alpha x + B \sinh \alpha x$

$Y = C \cos \alpha y + D \sin \alpha y$

$V = XY = (A \cosh \alpha x + B \sinh \alpha x)(C \cos \alpha y + D \sin \alpha y)$

The constants A, B, C and D are determined by using the boundary conditions. Passing through several levels of complicated mathematical procedures, the final result is obtained as.

$$\boxed{V(x,y) = \frac{4V_o}{\pi} \sum_{m=1,\,odd}^{\infty} \frac{1}{m} \frac{\sinh(m\pi x/b)}{\sinh(m\pi d/b)} \sin \frac{m\pi y}{b}}$$

The electric field distribution within the enclosure is obtained by the relation

$$\vec{E}(x,y) = -\nabla V(x,y)$$

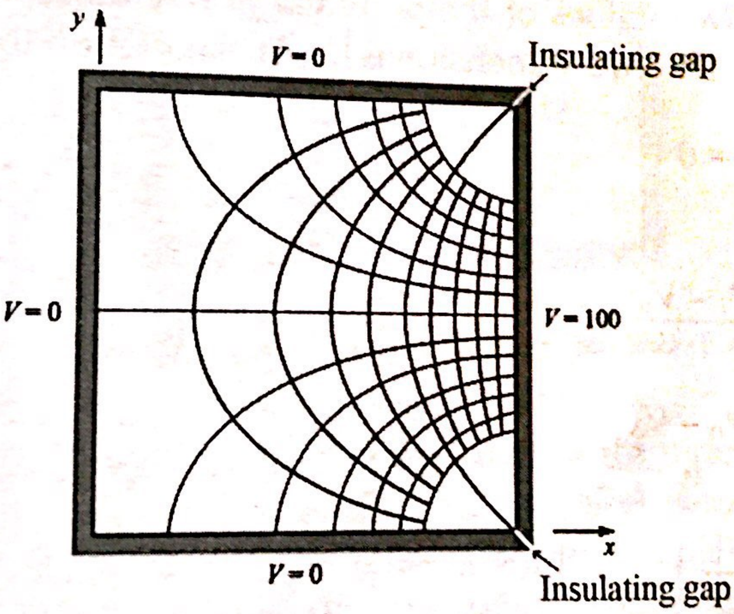


Figure 7.3 The potential field map corresponding to $V = \frac{40_o}{\pi} \sum_{m=1,\,odd}^{\infty} \frac{1}{m} \frac{\sinh(m\pi x/b)}{\sinh(m\pi l/b)} \sin \frac{m\pi y}{b}$ *with* $b = d$ *and* $V_0 = 100$ V.

2. **Numerical Iteration**

Consider a two-dimensional problem in which the potential is a function of x and y coordinate. Let's divide the interior of a cross section of the region where we want to determine the potential into squares of length h on a side. A portion of this region is illustrated in the Figure 7.4.

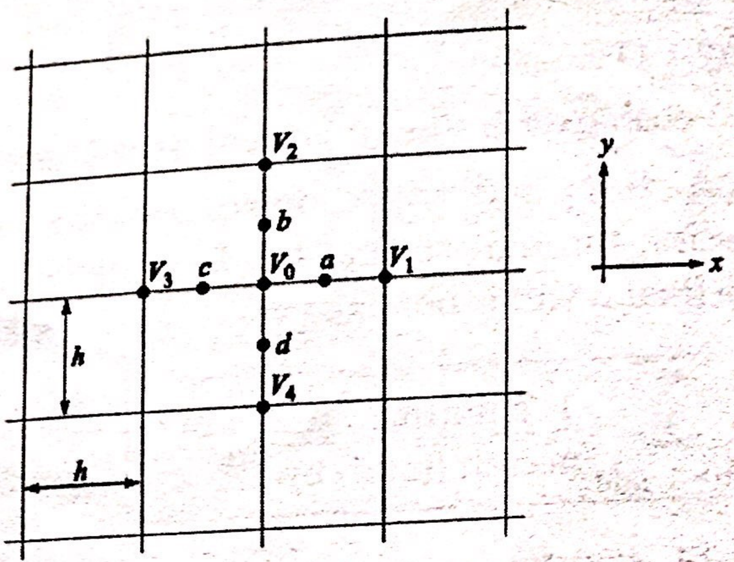


Figure 7.4 A portion of a region containing a two-dimensional potential field, divided into squares of side h. The potential V_0 is approximately equal to the average of the potentials at the four neighboring points.

Let the unknown values of the potential at five adjacent points be V_o, V_1, V_2, V_3 and V_4. The two-dimensional Laplace equation to be solved is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

We have,

$$\left.\frac{\partial V}{\partial x}\right|_a = \frac{V_1 - V_o}{h}$$

$$\left.\frac{\partial V}{\partial x}\right|_c = \frac{V_o - V_3}{h}$$

$$\therefore \left.\frac{\partial^2 V}{\partial x^2}\right|_0 = \frac{\left.\frac{\partial V}{\partial x}\right|_a - \left.\frac{\partial V}{\partial x}\right|_c}{h} = \frac{V_1 - V_o - V_o + V_3}{h^2} \quad \text{......... (i)}$$

Similarly,

$$\left.\frac{\partial^2 V}{\partial y^2}\right|_0 = \frac{V_2 - V_o - V_o + V_4}{h^2} \quad \text{.................. (ii)}$$

Combining equation (i) & (ii), we get

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{V_1 + V_2 + V_3 + V_4 - 4V_o}{h^2}$$

$$\text{or, } 0 = \frac{V_1 + V_2 + V_3 + V_4 - 4V_o}{h^2}$$

or, $V_0 = \frac{1}{4}(V_1 + V_2 + V_3 + V_4)$ .... (iii)

which is more exact as h approaches zero.

The iterative method only uses equation (iii) to determine the potential at the corner of every square subdivision in turn and then, the process is repeated over the entire region as many times as is necessary until the values no longer change.

As an example, consider a square region with conducting boundaries.

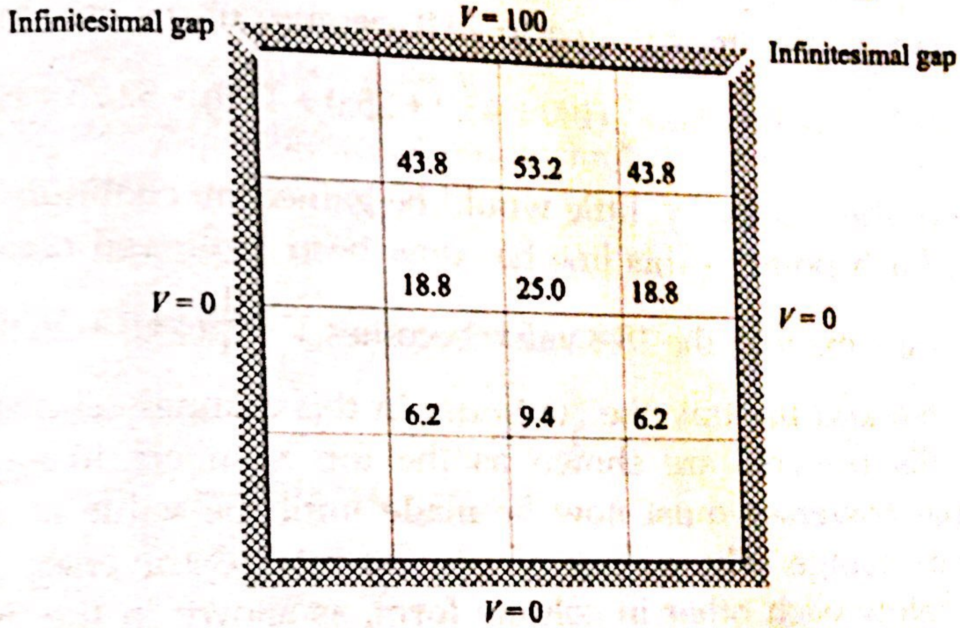


Figure 7.5 Cross section of a square trough with sides and bottom at zero potential and top at 100V.

The potential of the top is 100 V and that of the sides and bottom is zero. The region is divided first into 16 squares, and before we apply iterative method, we make initial estimate of the potential at every corner. The better the estimate, the shorter the solution, although the final result is independent of these initial estimates. Using equation (iii), the potential estimate at the center of the figure is $\frac{1}{4}(100 + 0 + 0 + 0) = 25.0$. Now, we apply equation (iii) along a diagonal set of axes to estimate the potential at the centers of the four double-sized squares. For the two upper double squares, we select a potential of 50V for the gap (the average of 0 and 100) and then,

$$V = \frac{1}{4}(50 + 100 + 25 + 0) = 43.75 = 43.8$$

For the lower ones,

$$V = \frac{1}{4}(0 + 25 + 0 + 0) = 6.25 = 6.2$$

The initial traverse is now made to obtain a corrected set of potentials, beginning in the upper left corner (with the 43.8 value, not with the boundary where the potentials are known and fixed), working across the row to the right and the dropping down to the second row and proceeding from left to right again. Thus, the 43.8 value changes to $\frac{1}{4}(100 + 53.2 + 18.8 + 0) = 43.0$. The best or newest potentials are always used when applying (iii), so both points marked 43.8 are changed to 43.0, because of the evident symmetry, and the 53.2 value becomes $\frac{1}{4}(100 + 43.0 + 25.0 + 43.0) = 52.75 = 52.8$.

Because of the symmetry, little would be gained by continuing across the top line. Each point of this line has now been improved once. Dropping down to the next line, the 18.8 value becomes $V = \frac{1}{4}(43.0 + 25.0 + 6.2 + 0) =$ $18.55 = 18.6$ and the traverse continues in this manner. The values at the end of this traverse are shown as the top numbers in each column. Additional traverses must now be made until the value at each corner shows no change. The values for the successive traverses are usually entered below each other in column form, as shown in the figure below and the final value is shown at the bottom of each column. Our example required only four traverses.

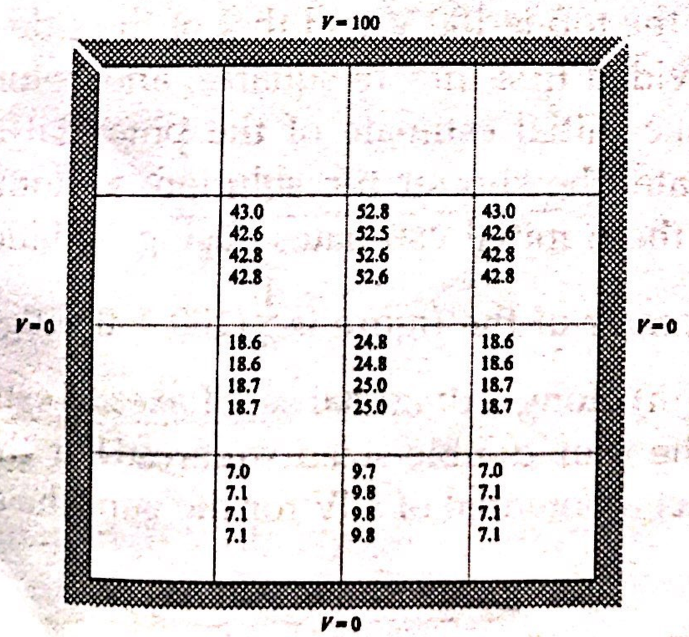


Figure 7.6 The results of each of the four necessary traverses of the problem of Fig. 7.8 are shown in order in the columns. The final values, unchanged in the last traverse, are at the bottom of each column.