First Order Linear Differential Equation

An equation is of the form $\frac{dy}{dx} + py = Q$ where P and Q are function of x or constant is called first order linear differential equation.

To solve the linear differential equation. We multiply $e^{\int pdx}$ on the both sides of the given differential equation.
So the equation becomes,

$$e^{\int pdx}\frac{dy}{dx} + py\, e^{\int pdx} = Qe^{\int pdx}$$

or, $\frac{d}{dx}\left(y \times e^{\int pdx}\right) = Q^{\int pdx}$ Integrating

or, $\int d\left(y \times e^{\int pdx}\right) = \int Qe^{\int pdx}$ Integrating

or , $y \times e^{\int pdx} = \int Qe^{\int pdx}\, dx + C$

or, $y\times e^{\int pdx} = \int Qe^{\int pdx} + C$ is the required solution.

So its formula is remembered as $y \times e^{\int pdx} = \int Qe^{\int pdx} dx + c$

The factor $e^{\int pdx}$ is called Integrating factor i.e. I.F. = $e^{\int pdx}$
So, Y × I.F = ∫(Q× I.F) dx + C
Similarly the first ordor differential equation is of the form $\frac{dx}{dy} + px = Q$ Where p and Q are function of y or constant is called linear differential equation in x and its I. F = $e^{\int pdy}$ Then its solution is x × I.F = ∫ (Q × I.F.) dy + c .

Exercise - 23

Solve the following differential equation.

1. $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2}$

Solution:

Given differential equation is

$$\frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{1}{(1+x^2)^2} \quad ....(1)$$

eq^n (1) is linear differential eq^n form

$$P=\frac{2x}{1+x^2}, Q=\frac{1}{(1+x^2)^2}$$

$$\therefore \text{ I.F} = e^{\int p dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)}$$

$$= (1+x^2)$$

Multiplying eq^n (1) by I.F. we get

$$y.(x^2+1) = \int \frac{1}{(1+x^2)^2}.(1+x^2)+c$$

$$= \int \frac{1}{(1+x^2)}dx + c$$

$y.(x^2+1) = \tan^{-1}x + c$ is the required solution.

2. $\mathbf{\frac{dy}{dx}+\frac{y}{x^2}=\frac{1}{x^2}}$

Sol^n. Given differential equation is

$$\frac{dy}{dx}+\frac{y}{x^2}y=\frac{1}{x^2} \quad ....(1)$$

eq^n(1) is linear differential eq^n form,

$$P=\frac{1}{x^2}, Q=\frac{1}{x^2}$$

$$\therefore \text{ I.F} = e^{\int \frac{1}{x^2}dx} = e^{-\frac{1}{x}}$$

multiplying eq^n (1) by I.F. we get

$$y.\ e^{-\frac{1}{x}} = \int \frac{1}{x^2}e^{-\frac{1}{x}}dx + C$$

$$= -\int e^{-\frac{1}{x}} d\left(\frac{1}{x}\right) + c$$

$$y.e^{-\frac{1}{x}} = e^{-\frac{1}{x}} + c$$

or, $y = 1 + ce^{\frac{1}{x}}$ is the required solution.

3. $\mathbf{(1-x^2)\frac{dy}{dx} - xy = 1}$

Sol^n. Given differential eq^n is

$$(1-x^2)\frac{dy}{dx}-xy=1$$

or, $\frac{dy}{dx}-\frac{x}{1-x^2}.y=\frac{1}{1-x^2}$ ....(1)

eq^n(1) is linear differential eq^n form

$$p=\frac{-x}{1-x^2}, Q=\frac{1}{1-x^2}$$

$$\therefore \text{ I.F.} = e^{-\int \frac{x}{1-x^2}dx} = e^{\frac{1}{2}\log(1-x^2)} = \sqrt{(1-x^2)}$$

Multiplying eq^n (1) by I.F. we get

$$y\sqrt{(1-x^2)} = \int \frac{1}{(1-x^2)}.\sqrt{(1-x^2)}dx + c$$

$$= \int \frac{1}{\sqrt{1-x^2}}dx + c$$

$y\sqrt{1-x^2} = \sin^{-1}x + c$ is the required solution.

4. $\mathbf{\frac{dy}{dx}+2y=4x}$

Sol^n. Given differential equation is

$$\frac{dy}{dx}+2y=4x \quad ....(1)$$

eq^n (1) is linear differential equation form

$p = 2, Q = 4x$

$$\therefore \text{I.F.} = e^{\int 2dx} = e^{2x}$$

Multiplying eq^n (1) my I.F. we get

$$y.\ e^{2x} = \int 4x.e^{2x}dx. + c$$

$$= 4x.\frac{e^{2x}}{2} - 4\int 1.\frac{e^{2x}}{2}dx + c$$

$$= 2xe^{2x} - 2\int fe^{2x}dx + c$$

$$= 2x\ e^{2x} - 2\frac{e^{2x}}{2} + c$$

or, $y.e^{2x} = 2xe^{2x} - e^{2x} + c$

or, $y = 2x - 1 + ce^{-2x}$ is the required sol^n

5. $\mathbf{(1+x)\frac{dy}{dx} - xy = 1 - x}$

Sol^n. Given differential equation is

$$(1+x)\frac{dy}{dx} - xy = 1 - x$$

or, $\frac{dy}{dx}-\frac{x}{1+x}.y=\frac{1-x}{1+x}$ ....(1)

eq^n (1) is linear differential equation form

$P=\frac{-x}{1+x}, Q=\frac{1-x}{1+x}$

$\therefore$ I. F = $e^{\int -\frac{x}{1+x}dx}=e^{\int\left(-1+\frac{1}{1+x}\right)dx}$

$= e^{-x+\log x} = e^{-x}.e^{\log(1+x)} = e^{-x}(1+x)$

Multiplying eq^n (1) by I.F. we get

or, $y.e^{-x}(1+x)=\int\frac{(1-x)}{1+x)}(1+x)e^{-x}dx+c$

$=\int(1-x)e^{-x}dx+c$

$=\int e^{-x}dx-\int xe^{-x}dx+c$

$=-e^{-x}-\frac{xe^{-x}}{-1}+\int 1.\frac{e^{-x}}{-1}dx+c$

$y.e^{-x}(1+x) = -e^{-x} + xe^{-x} + e^{-x} + c$

or, $ye^{-x}(1+x) = xe^{-x} + c$

or, $y(1+x) = x + ce^{x}$ is the required solution.

6. $\mathbf{(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}}$

Sol^n. Given differential equation is

$(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$

or, $\frac{dy}{dx}+\frac{1}{1+x^2}.y=\frac{e^{\tan^{-1}x}}{1+x^2}$ ....(1)

eq^n (1) is linear differential equation form

$P=\frac{1}{1+x^2}, Q=\frac{e^{\tan^{-1}x}}{1+x^2}$

I.F. = $e^{\int pdx}=e^{\int\frac{1}{1+x^2}dx}=e^{\tan^{-1}x}$

Multiplying eq^n (1) by I.F. we get

$y.\ e^{\tan^{-1}x}=\int\frac{e^{\tan^{-1}x}}{1+x^2}.e^{\tan^{-1}x}dx+c$

$=\int\frac{e^{2\tan^{-1}x}}{(1+x^2)}dx+c$

Put $\tan^{-1}x = t$

Then $\left(\frac{1}{1+x^2}\right)dx=dt$

$\therefore\ ye^{\tan^{-1}x}=\int e^{2t}dt+c=\frac{1}{2}e^{2t}+c=\frac{1}{2}e^{2\tan^{-1}x}+c$

or, $ye^{\tan^{-1}x}=\frac{1}{2}e^{2\tan^{-1}x}+c$

or, $2ye^{\tan^{-1}x}=e^{2\tan^{-1}x}+2C$

$2ye^{\tan^{-1}x}=e^{2\tan^{-1}x}+K$ Where K = 2c is the required solution.

7. $\mathbf{x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1}$

Sol^n. Given differential equation is

$x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1$

or, $\frac{dy}{dx}+\frac{(x\sin x+\cos x)}{x\cos x}.y=\frac{1}{x\cos x}$ ....(1)

eq^n (1) is linear differential equation form

$P=\frac{(x\sin x+\cos x)}{x\cos x}, Q=\frac{1}{x\cos x}$

$\therefore$ I.F. = $e^{\int pdx}=e^{\int\frac{(x\sin x+\cos x)}{x\cos x}dx}$

$=e^{\int\left(\tan x+\frac{1}{x}\right)dx}=e^{(\log\sec x+\log x)}=e^{\log x\sec x}$

= xsecx

Multiplying eq^n (1) by I. F we get

$y.\ x\sec x=\int\frac{1}{x\cos x}.x\sec x dx+c$

$=\int\sec^2 x dx + C$

or, xy see x = tanx + c is the required solution

8. $\mathbf{\cos^2 x\frac{dy}{dx}+y=\tan x}$

sol^n. Given differential equation is $\cos^2 x\frac{dy}{dx}+y=\tan x$

or, $\frac{dy}{dx}+\frac{1}{\cos^2 x}.y=\frac{\tan x}{\cos^2 x}$

or, $\frac{dy}{dx}+\sec^2 x.y=\tan x\sec^2 x$ ....(1)

eq^n (1) is linear differential eq^n form, $p=\sec^2 x$. $Q=\tan x\sec^2 x$

$\therefore$ I. F = $e^{\int pdx} = e^{\int \sec^2 xdx} = e^{\tan x}$
Multiplying eq^n (1) by I.F. we get
y. $e^{tanx} = \int$tanx. sec^{2}x . e^{tanx}dx + c
Put tanx = t then sec^{2}dx = dt
$\therefore$ y. $e^{tanx} = \int te^t\, dt + c$
$= te^t - e^t + c$
or, $ye^{tanx} = \tan x\, e^{tanx} - e^{tanx} + c$
or, $(y - \tan x + 1)\, e^{tanx} = c$ is the required solution.

9. $\mathbf{Sinx\frac{dy}{dx} + ycosx = xsinx}$

Sol^n. Given differential equation is

$$\sin x\frac{dy}{dx} + y\cos x = x\sin x$$

or, $\frac{dy}{dx} + \frac{\cos x}{\sin x}.y = \frac{x\sin x}{\sin x}$

or, $\frac{dy}{dx} + \cot xy = x$ ....(1)

eq^n (1) is linear differential equation form
P = cotx , Q = x
$\therefore$I. F = $e^{\int pdx} = e^{\int \cot x\, dx}$ = log sinx
= sinx
Multiplying eq^n (1) by I. F. we get
y. sinx = $\int$x sinx dx + c
= – x cosx + $\int$cosx dx + c
ysinx = – x cosx + sinx + c
or, y sinx + x cosx – sinx = c is the required solution.

10. $\mathbf{\frac{dy}{dx} + ycotx = 2cosx}$ ***(B. E. 2068)***

Sol^n. Given differential equation is

$\frac{dy}{dx} + y\cot x = 2\cos x$ ....(1)

eq^n (1) is linear differential equation
P = cotx, Q = 2cosx
$\therefore$ I.F = $e^{\int pdx} = e^{\int cotxdx} = e^{\log \sin x}$ = sinx
Multiplying eq^n (1) by I.F. we get

y × sinx = $\int 2\cos x \sin x\, dx + c$

or, y sinx = $\int$sin2x dx + c

$= -\frac{\cos 2x}{2} + c$

or, 2y sinx + cos2x + 2c

2ysinx = k – cos2x where K = 2c is the required solution

11. $\mathbf{\frac{dy}{dx} + ytanx = secx}$

Sol^n. Given differential equation is

$\frac{dy}{dx} + y\tan x = \sec x$ ....(1)

eq^n (1) is linear differential equation form
P = tanx, Q = secx
$\therefore$ I.F = e $e^{\int pdx}$
$= e^{\int \tan x\, dx} = e^{\log \sec x}$
= secx
Multiplying eq^n (1) by I.F. we get
y. secx = $\int$secx. secx dx + c
= $\int$sec^{2}x dx + c
or, y secx = tanx + C
y = sinx + c cosx is the required solution.

12. $\mathbf{x\log x\frac{dy}{dx} + y = 2\log x}$ ***[B.E. 2061]***

Sol^n. Given differential equation is

$$x\log x\frac{dy}{dx} + y = 2\log x$$

or, $\frac{dy}{dx} + \frac{1}{x\log x}.y = \frac{2\log x}{x\log x}$

or, $\frac{dy}{dx} + \frac{1}{x\log x}.y = \frac{2}{x}$ ....(1)

eq^n (1) is linear differential equation form

P = $\frac{1}{x\log x}$. Q = $\frac{2}{x}$

$\therefore$ I.F = $e^{\int pdx} = e^{\int \frac{1}{x\log x}dx}$

Put logx = t Then $\frac{1}{x}dx = dt$

$\therefore$I.F. = $e^{\int \frac{1}{t}dt} = e^{\log t} = t = \log x$

Multiplying eq^n (1) by I.F. we get,

y . logx = $\int \frac{2}{x}\log x dx + c$

$= 2\int \frac{\log x}{x}dx + c$

Put $\log x = t$ Then $\frac{1}{x}dx = dt$

$y \log x = 2\int t\, dt + c$

$= \frac{2t^2}{2} + c$

or, $y \log x = (\log x)^2 + c$ is the required solution.

13. $\frac{dy}{dx} + y = \cos x$

Sol^n. Given differential equation is

$\frac{dy}{dx} + y = \cos x$ ....(1)

eq^n (1) is linear differential equation form,

$P = 1,\ Q = \cos x$

$\therefore$ I. F $= e^{\int pdx} = e^x$

Multiplying eq^n (1) by I.F. we get

$ye^x = \int \cos x\, e^x dx + c$

$= \int e^x \cos x\, dx$

$ye^x = e^x \frac{(\cos x + \sin x)}{2} + c$

$2y = \cos x + \sin x + 2ce^{-x}$

or, $2y = \cos x + \sin x + ke^{-x}$ where $K = 2c$ is the required solution.

14. $(1 + y^2)\, dx = (\tan^{-1}y - x)\, dy$ ***(B. E. 2070)***

Sol^n. Given differential equation is

$(1 + y^2)\, dx = (\tan^{-1}y - x)\, dy$

or, $\frac{dx}{dy} = \frac{(\tan^{-1}y - x)}{1 + y^2}$

or, $\frac{dy}{dx} = \frac{\tan^{-1}y}{1 + y^2} - \frac{x}{1 + y^2}$

or, $\frac{dy}{dx} + \frac{1}{1 + y^2}.x = \frac{\tan^{-1}y}{1 + y^2}$ ....(1)

eq^n (1) is linear differential equation of the form $\frac{dx}{dy} + px = Q$

Here, $P = \frac{1}{1 + y^2}$. $Q = \frac{\tan^{-1}y}{1 + y^2}$

$\therefore$ I.F. $= e^{\int pdy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

Multiplying eq^n (1) by I.F. we get,

of, $e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1 + y^2}.e^{\tan^{-1}y}dy + c$ put $\tan^{-1}y = t$ Then

$\frac{1}{1 + y^2}dy = dt$

or, $xe^{\tan^{-1}y} = \int te^t\, dt + c$

$= te^t - e^t + c$

$xe^{\tan^{-1}y} = \tan^{-1}y\ e^{\tan^{-1}y} - e^{\tan^{-1}y} + c$

$x = (\tan^{-1}y - 1) + c\ e^{-\tan^{-1}y}$ is the required solution.

15. $(1 + y^2) + \left(x - e^{\tan^{-1}y}\right)\frac{dy}{dx} = 0$ ***(B. E. 2073- Shrawan, 070)***

Sol^n. Given differential equation is

$(1 + y^2) + \left(x - e^{\tan^{-1}y}\right)\frac{dy}{dx} = 0$

or, $\left(x - e^{\tan^{-1}y}\right)\frac{dy}{dx} = -(1 + y^2)$

$\frac{dy}{dx} = \frac{-(1 + y^2)}{(x - e^{\tan^{-1}}y)}$

$\therefore \frac{dx}{dy} = \frac{-(x - e^{\tan^{-1}}y)}{(1 + y^2)}$

or, $\frac{dx}{dy} = -\frac{x}{1 + y^2} + \frac{e^{\tan^{-1}}y}{1 + y^2}$

or, $\frac{dx}{dy} + \frac{1}{1 + y^2}.x = \frac{e^{\tan^{-1}}y}{1 + y^2}$ ....(1)

eq^n (1) is linear differential equation is of the form $\frac{dx}{dy} + px = Q$

$P = \frac{1}{1 + y^2}$. $Q = \frac{e^{\tan^{-1}}y}{1 + y^2}$

$\therefore$ I. F $= e^{\int pdy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}}y$

Multiplying eq^n (1) by I.F. we get,

$x.\, e^{\tan^{-1}}y = \int \frac{e^{\tan^{-1}}y}{1 + y^2}.e^{\tan^{-1}}y\, dy + c$

$$= \int \frac{e^{2\tan^{-1} y}}{1+y^2} dy + c$$

Put $\tan^{-1} y = t$ Then $\frac{1}{1+y^2} dy = dt$

$\therefore xe^{\tan^{-1} y} = \int e^{2t} dt + c$

$= \frac{1}{2} e^{2t} + c$

or, $x e^{\tan^{-1} y} = \frac{1}{2} e^{2\tan^{-1} y} + c$

or, $2 xe^{\tan^{-1} y} = e^{2\tan^{-1} y} + 2c$

or, $2 xe^{\tan^{-1} y} = e^{2\tan^{-1} y} + K$

Where K = 2c is the required solution.

16. Solve: $\frac{dy}{dx} + \frac{y}{x} = x^2$ If y = 1, When x = 1

Sol^n: Given differential equation is

$\frac{dy}{dx} + \frac{1}{x} y = x^2$ ....(1)

eq^n(1) is linear differential equation form

$p = \frac{1}{x}, Q = x^2$

$\therefore$ I. F. $= e^{\int p dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Multiplying eq^n (1) by I. F. we get,

$y. x = \int x^2 . x dx + c$

or, $xy = \int x^3 dx + c$

$xy = \frac{x^4}{4} + c$

$4xy = x^4 + 4c$ ....(2)

When x = 1, y = 1

4 = 1 + 4c

$\Rightarrow c = \frac{3}{4}$

Now (2) becomes

$4xy = x^4 + 4. \frac{3}{4}$

or, $4xy = x^4 + 3$ is the required solution.