First Order Differential Equations Reducible to Homogenous Form

If the differential equation is of the form $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$; then it can be solved by the following ways.

1. If the differential equation is of the form,
   $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$ and $\frac{a}{A} = \frac{b}{B} = \frac{1}{\ell}$
   Then the equation reduced to the form
   $\frac{dy}{dx} = \frac{(ax+by+c)}{\ell\,(ax+by)+C}$
   and put ax + by = v then $a + b\frac{dy}{dx} = \frac{dv}{dx}$
   Separating the variable and integrating we get, the required solution.
2. If the differential equation is of the form
   $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$ and $\frac{a}{A} \neq \frac{b}{B}$ then the equation reduced to the homogeneous form by putting x = X + h and y = Y + K. Where h, K are constant which will be interms of h and K, the equation are to be chose in such a way that the differential equation should be homogeneous.

Exercise - 22

Solve the following differential equations

1. $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$ ***[B. E. 2058]***

Sol^n. Given differential equation is,

$\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$ ……. (i)

Put x + y = v

Then, $1 + \frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{dy}{dx} = \frac{dv}{dx} - 1$

Equation (i) becomes,

$\frac{dv}{dx} - 1 = \frac{v+1}{v-1}$

or, $\frac{dv}{dx} = \frac{v+1}{v-1} + 1$

or, $\frac{dv}{dx} = \frac{v+1+v-1}{v-1} = \frac{2v}{v-1}$

or, $\int \frac{v-1}{2v}\,dv = \int dx$; Integrating

or, $\int\left(\frac{1}{2} - \frac{1}{2v}\right)dv = \int dx$

or, $\frac{1}{2}v - \frac{1}{2}\log v = x + C$

Restoring the value of v we get,

$\frac{1}{2}(x+y) - \frac{1}{2}\log(x+y) = x + C$

or, x + y – log (x + y) = 2x + 2C

or, y – x – log (x + y) = K where K = 2C is the required solution.

2. **(x + y + 1) dx – (2x + 2y + 1) dy = 0**

Sol^n. Given differential equation is,

(x + y + 1) dx – (2x + 2y + 1) dy = 0

or, $\frac{dy}{dx} = \frac{(x+y+1)}{2(x+y)+1}$ ……… (i)

Put x + y = v then $1 + \frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{dy}{dx} = \frac{dv}{dx} - 1$

Now (i) becomes,

$\frac{dv}{dx} - 1 = \frac{v+1}{2v+1}$

or, $\frac{dv}{dx} = \frac{v+1}{2v+1} + 1 = \frac{v+1+2v+1}{2v+1} = \frac{3v+2}{2v+1}$

or, $\int \frac{2v+1}{3v+2}\,dv = \int dx$; Integrating

$\int \frac{\frac{2}{3}(3v+2) - \frac{4}{3} + 1}{3v+2}\,dv = \int dx$

or, $\frac{2}{3}\int \frac{3v+2}{3v+2}dv - \frac{1}{3}\int \frac{1}{3v+2}\,dv = \int dx$

or, $\frac{2}{3}v - \frac{1}{3}\cdot\frac{1}{3}\log(3v+2) = x + C$

$$2v - \frac{1}{3}\log(3v+2) = 3x + 3C$$

Restoring the value of v we get,

$$2x + 2y - \frac{1}{3}\log(3x+3y+2) = 3x + 3C$$

$$2y - x - \frac{1}{3}\log(3x+3y+2) = 3C$$

or, $6y - 3x - \log(3x + 3y + 2) = 6C$

or, $6y - 3x = \log(3x + 3y + 2) + K$ where $K = 6C$ is the required solution.

3. $(4x + 6y + 5)\,dy = (3y + 2x + 4)\,dx$

Sol^n. Given differential equation is,

$(4x + 6y + 5)\,dy = (3y + 2x + 4)\,dx$

$$\text{or, } \frac{dy}{dx} = \frac{(3y+2x+4)}{(4x+6y+5)}$$

$$\text{or, } \frac{dy}{dx} = \frac{(3y+2x+4)}{2(2x+3y)+5} \;\;\ldots\ldots\text{(i)}$$

Put $2x + 3y = v$ then,

$$2 + 3\frac{dy}{dx} = \frac{dv}{dx}$$

$$\text{or, } \frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$$

Now (i) becomes,

$$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{v+4}{2v+5}$$

$$\text{or, } \frac{dv}{dx} - 2 = \frac{3v+12}{2v+5}$$

$$\text{or, } \frac{dv}{dx} = \frac{2v+12}{2v+5} + 2$$

$$\frac{dv}{dx} = \frac{3v+12+4v+10}{2v+5} = \frac{7v+22}{2v+5}$$

$$\text{or, } \int\left(\frac{2v+5}{7v+22}\right)dv = \int dx\text{; Integrating}$$

$$\text{or, } \int \frac{\frac{2}{7}(7v+22) - \frac{44}{7} + 5}{7v+22}\,dv = \int dx$$

$$\text{or, } \frac{2}{7}\int\frac{(7v+22)}{(7v+22)}dv - \frac{9}{7}\int\frac{1}{7v+22}dv = \int dx$$

$$\text{or, } \frac{2}{7}v - \frac{9}{7}\cdot\frac{1}{7}\log(7v+22) = x + C$$

$$\text{or, } 2v - \frac{9}{7}\log(7v+22) = 7x + 7C$$

Restoring the value of v we get,

$$\text{or, } 4x + 6y - \frac{9}{7}\log(14x+21y+22) = 7x + 7C$$

$$\text{or, } 6y - 3x - \frac{9}{7}\log(14x+21y+22) = 7C$$

or, $42y - 21x - 9\log(14x + 21y + 22) = 49C$

or, $42y - 21x - 9\log(14x + 21y + 22) = K$

Where $K = 49C$

or, $7(6y - 3x) - 9\log(14x + 21y + 22) = K$ is the required solution.

4. $(2x + 2y + 3)\,dy - (x + y + 1)\,dx = 0$ ***[B.E. 2057/2060]***

Sol^n. Given differential equation is,

$(2x + 2y + 3)\,dy - (x + y + 1)\,dx = 0$

$$\text{or, } \frac{dy}{dx} = \frac{x+y+1}{2(x+y)+3} \;\;\ldots\ldots\text{(i)}$$

$$\text{Put } x + y = v \text{ then } 1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\text{or, } \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Now (i) becomes,

$$\frac{dv}{dx} - 1 = \frac{v+1}{2v+3}$$

$$\text{or, } \frac{dv}{dx} = \frac{v+1}{2v+3} + 1$$

$$\frac{dv}{dx} = \frac{v+1+2v+3}{2v+3} = \frac{3v+4}{2v+3}$$

$$\text{or, } \int\frac{2v+3}{3v+4}dv = \int dx\text{; Integrating}$$

$$\int\frac{\frac{2}{3}(3v+4) - \frac{8}{3} + 3}{3v+4}dv = \int dx$$

$$\text{or, } \frac{2}{3}\int\frac{3v+4}{3v+4}dv + \frac{1}{3}\int\frac{1}{3v+4}dv = \int dx$$

$$\text{or, } \frac{2}{3}v + \frac{1}{3}\cdot\frac{1}{3}\log(3v+4) = x + C$$

$$\text{or, } 2v + \frac{1}{3}\log(3v+4) = 3x + 3C$$

Restoring the value of v

$$2x + 2y + \frac{1}{3}\log(3x + 3y + 4) = 3x + 3C$$

or, $2y - x + \frac{1}{3}\log(3x + 3y + 4) = 3C$

or, $6y - 3x + \log(3x + 3y + 4) = 9C$

or, $6y - 3x + \log(3x + 3y + 4) = K$ where $K = 9C$ is the required solution.

5. $\mathbf{\dfrac{dy}{dx} = \dfrac{x+y}{x+y-2}}$

Sol^n. Given differential equation is,

$$\frac{dy}{dx} = \frac{x+y}{x+y-2} \quad \text{......... (i)}$$

Put $x + y = v$ then,

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

or, $\dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$`

Now (i) becomes,

$$\frac{dv}{dx} - 1 = \frac{v}{v-2}$$

or, $\dfrac{dv}{dx} = \dfrac{v}{v-2} + 1$

$$\frac{dv}{dx} = \frac{v+v-2}{v-2} = \frac{2v-2}{v-2} = \frac{2(v-1)}{v-2}$$

or, $\int \dfrac{v-2}{v-1}\,dv = 2\int dx$; Integrating

or, $\int \dfrac{v-1-1}{v-1}\,dv = 2\int dx$

or, $\int 1\,dv - \int \dfrac{1}{v-1}$`$dv = 2\int dx$

or, $v - \log(v - 1) = 2x + C$

Restoring the value of v we get,

$x + y - \log(x + y - 1) = 2x + C$

or, $y - x - \log(x + y - 1) = C$ is the required solution.

6. $\mathbf{(6x - 5y + 4)\,dy + (y - 2x - 1)\,dx = 0}$

Sol^n. Given differential equation is,

$(6x - 5y + 4)\,dy + (y - 2x - 1)\,dx = 0$

or, $\dfrac{dy}{dx} = -\dfrac{(y-2x-1)}{(6x-5y+4)}$ .... (i)

Put $x = X + h$, and $y = Y + K$, where h and K are constant

So that $dx = dX$ and $dy = dY$

Then $\dfrac{dy}{dx} = \dfrac{dY}{dX}$

Now equation (i) becomes,

$$\frac{dY}{dX} = -\frac{(Y+K)-2(X+h)-1}{6(X+h)-5(Y+k)+4} \quad \text{...... (ii)}$$

$$= -\frac{(Y-2X)+(K-2h-1)}{(6X-5Y)+(6h-5K+4)}$$

Let us choose h and K such that

$K - 2h - 1 = 0$ and $6h - 5K + 4 = 0$

Solving these, we get, $h = -\dfrac{1}{4}$, $k = \dfrac{1}{2}$

Now equation (ii) reduces to

$$\frac{dY}{dX} = -\frac{(Y-2X)}{(6X-5Y)} \quad \text{........ (iii)}$$

Equation (iii) is homogeneous differential equation

So, put $Y = vX$ then

$$\frac{dY}{dX} = V + X\frac{dV}{dX}$$

Now (iii) becomes,

$$V + X\frac{dv}{dX} = \frac{-(Xv-2X)}{(6X-5vX)} = -\frac{(V-2)}{(6-5v)}$$

or, $x\dfrac{dv}{dx} = -\dfrac{(v-2)}{(6-5v)} - V$

$$= \frac{-v+2-6v+5v^2}{6-5V} \times \frac{dv}{dx} = \frac{5v^2-7v+2}{6-5v}$$

or, $\displaystyle\int \frac{6-5v}{5v^2-7v+2}\,dv = \int \frac{1}{X}\,dX$; Integrating

or, $\displaystyle\int \frac{-\frac{1}{2}(10v-7)-\frac{7}{2}+6}{5v^2-7v+2}\,dv = \int \frac{1}{X}\,dX$

or, $\displaystyle\frac{-1}{2}\int \frac{10v-7}{5v^2-7v+2}\,dv + \frac{5}{2.5}\int \frac{1}{v^2-2.v.\frac{7}{10}+\left(\frac{7}{10}\right)^2+\frac{2}{5}}\,dv$

$$= \int \frac{1}{X}\,dX$$

or, $\displaystyle\frac{-1}{2}\int \frac{10v-7}{5v^2-7v+2}\,dv + \frac{1}{2}\int \frac{1}{\left(v-\frac{7}{10}\right)^2-\left(\frac{3}{10}\right)^2}\,dv = \int \frac{1}{X}\,dX$

or, $-\frac{1}{2}\log(5v^2-7v+2)+\frac{1}{2}\cdot\frac{1}{2\cdot\frac{3}{10}}\log\frac{\left(v-\frac{7}{10}-\frac{3}{10}\right)}{\left(v-\frac{7}{10}+\frac{3}{10}\right)}=\log X$

$+\log C$

or, $\log(5v^2-7v+2)-\frac{5}{3}\log\left(\frac{v-1}{5v-2}\right)+\log X=-2\log C$

or, $\log\left\{X^2\left(5v^2-7v+2\right)\left(\frac{5v-2}{v-1}\right)^{\frac{5}{3}}\right\}=\log\frac{1}{C^2}$

or, $X^2(5v^2-7v+2)\frac{(5v-2)^{\frac{5}{3}}}{(v-1)^{\frac{5}{3}}}=\frac{1}{C^2}$

or, $X^2\frac{(5v-2)(v-1)(5v-2)^{\frac{5}{3}}}{(v-1)^{\frac{5}{3}}}=\frac{1}{C^2}$

or, $X^2\frac{(5v-2)^{\frac{8}{3}}}{(v-1)^{\frac{2}{3}}}=\frac{1}{C^2}$

or, $X^2\left(5v-2\right)^{\frac{8}{3}}=\left(v-1\right)^{\frac{2}{3}}\cdot\frac{1}{C^2}$

Restoring the value of $V=\frac{Y}{X}$ we get,

$X^2\left(\frac{5Y}{X}-2\right)^{\frac{8}{3}}=\left(\frac{Y}{X}-1\right)^{\frac{2}{3}}\cdot\frac{1}{C^2}$

or, $\frac{X^2\left(5Y-2X\right)^{\frac{8}{3}}}{X^2\cdot X^{\frac{2}{3}}}=\frac{(Y-X)^{\frac{2}{3}}}{X^{\frac{2}{3}}}\cdot\frac{1}{C^2}$

or, $\left(5Y-2X\right)^{\frac{8}{3}}=\frac{(Y-X)^{\frac{2}{3}}}{C^2}$

or, $(5Y-2X)^8=\frac{\left(Y-X\right)^2}{C^6}$

or, $(5Y-2X)^8=(Y-X)^2\cdot\frac{1}{C^2}$

or, $\left[5\left(y-\frac{1}{2}\right)-2\left(x+\frac{1}{4}\right)\right]^4=\frac{1}{C^2}\left[\left(y-\frac{1}{2}\right)-\left(x+\frac{1}{4}\right)\right]$

or, $(5y-2x-3)^4=K(4y-4x-3)$ is the required solution.

7. $\frac{\mathbf{dy}}{\mathbf{dx}}=\frac{\mathbf{x+2y-3}}{\mathbf{2x+y-3}}$

Sol^n. Given differential equation is,

$\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$ …… (i)

Put $x = X + h$, $y = Y + K$ so that $dx = dX$, $dy = dY$ where h and K are constant.

Then $\frac{dy}{dx}=\frac{dY}{dX}$

Hence equation (i) will reduces to

$\frac{dY}{dX}=\frac{(x+h)+2(Y+K)-2}{2(X+h)+(Y+K)-3}=\frac{(X+2Y)+(h+2K-3)}{(2X+Y)+(2h+K-3)}$ ….. (ii)

Let us choose h and K such that,

$h + 2K - 3 = 0$ and $2h + K - 3 = 0$

Solving these,

$h = 1, K = 1$

Now equation (ii) becomes,

$\frac{dY}{dX}=\frac{X-2Y}{2X+Y}$ ……. (iii)

Equation (iii) is homogenous differential equation

So, put $Y = vX$

Then, $\frac{dY}{dX}=v+X\frac{dv}{dX}$

Hence, equation (iii) becomes,

$v+X\frac{dv}{dX}=\frac{X+2vX}{2X+vX}$

or, $v+X\frac{dv}{dX}=\frac{1+2v}{2+v}$

or, $X\frac{dv}{dX}=\frac{1+2v}{2+v}-v=\frac{1+2v-2v-v^2}{2+v}$

or, $X\frac{dv}{dX}=\frac{1-v^2}{2+v}$

or, $\int\frac{dX}{X}=\int\frac{2+v}{1-v^2}dv$ Integrating

$=\int\frac{2+v}{(1-v)(1+v)}dv=\int\frac{1}{2}\left[\frac{3}{1-v}+\frac{1}{1+v}\right]dv$

(By using the partial function)

$\therefore$ log X = $\frac{1}{2}$ [– 3 log (1 – v) + log (1 + v)] + $\frac{1}{2}$ log K
or, 2 logX = – 3 log (1 – v) + log (1 + v) + log K
or, log X^2 = log $(1 - v)^{-3}$ + log (1 + v) + log K
or, log X^2 = log $(1 - v)^{-3}$ (1 + v) K
$\Rightarrow X^2 = (1 - v)^{-3} (1 + v) K$

$\Rightarrow X^2 = \frac{K(1+v)}{(1-v)^3}$

or, $X^2 (1 - v)^3 = K (1 + v)$

Restoring the value of v = $\frac{Y}{X}$ we get,

$\Rightarrow X^2 \left(1 - \frac{Y}{X}\right)^3 = k\left(1 + \frac{Y}{X}\right)$

$\frac{(X-Y)^3}{X} = K\frac{(X+Y)}{X}$

$\Rightarrow (X - Y)^3 = K (X + h)$ ……. (iv)
But x = X + h = X + 1 and y = Y + K = y + 1
$\therefore$ X = x – 1, Y = y – 1
Hence, equation (iv) becomes,
$\{(x - 1) - (y - 1)^3\} = k (x - 1 + y - 1)$
$\Rightarrow (x - y)^3 = k (x + y - 2)$
$\Rightarrow x + y - 2 = C (x - y)^3$

Where C = $\frac{1}{K}$ which is the required solution.

8. $\mathbf{\frac{dy}{dx} = \frac{2x+3y+1}{3x+5y-1}}$

Sol^n. Given differential equation is,

$\frac{dy}{dx} = \frac{2x+3y+1}{3x+5y-1}$ …….. (i)

Put x = X + h and y = Y + K where h and K are constant.
So, that dx = dX and dy = dY

$\therefore \frac{dy}{dx} = \frac{dY}{dX}$

Now equation (i) becomes,

$\frac{dY}{dX} = \frac{+2(X+h)+3(Y+x)+1}{3(X+h)+5(Y+x)-1} = 0$

or, $\frac{dY}{dX} = -\frac{(2X+3Y)+(2h+3k+1)}{(3X+5Y)+(3h+5K-1)}$ …….. (ii)

Let us choose h and k such that
2h + 3K + 1 = 0 and 3h + 5K – 1 = 0 solving these

We get, h = – 8, K = 5
Now (ii) becomes,

$\frac{dY}{dX} = \frac{-(2X+3Y)}{(3X+5Y)}$ …….. (iii)

Equation (iii) is homogenous differential equation,
So, put Y = vX

then $\frac{dY}{dX} = v + X\frac{dv}{dX}$

From (iii)

$\therefore v + X\frac{dv}{dX} = -\frac{(2X+3vX)}{(3X+5vX)} = -\frac{(2+3v)}{(3+5v)}$

or, $X\frac{dv}{dX} = \frac{-2-3v}{3+5v} - v$

or, $X\frac{dv}{dX} = \frac{-2-3v-3v-5v^2}{3+5v} = \frac{-5v^2-6v-2}{3+5v}$

or, $\int \frac{(3+5v)}{(5v^2+5v+2)} dv = -\int \frac{1}{X} dX$; Integrating

$\frac{1}{2}$ log $(5v^2 + 6v + 2)$ = – log X + log C
log $(5v^2 + 6v + 2)$ = – 2 log X + 2 log C
or, log $(5v^2 + 6v + 2)$ = – log X^2 + log C^2
or, log $(5v^2 + 6v + 2) X^2$ = log C^2
or, $(5v^2 + 6v + 2) x^2 = C^2$

Restoring the value of v = $\frac{Y}{X}$ we get,

$\left(5\frac{Y^2}{X^2} + \frac{6Y}{X} + 2\right) X^2 = C^2$

or, $(5Y^2 + 6XY + 2X^2) = C^2$
or, $5 (y - 5)^2 + 6 (x + 8) (y - 5) + 2 (x + 8)^2 = C^2$
or, $5 (y^2 - 10y + 25) + 6 (xy - 5x + 8y - 40) + 2 (x^2 + 16x + 64) = C^2$
or, $5y^2 - 50y + 125 + 6xy - 30x + 48y - 240 + 2x^2 + 32x + 128 = C^2$
or, $5y^2 + 2x^2 - 2y + 2x + 6xy + 13 = C^2$
or, $2x^2 + 5y^2 + 6xy - 2y + 2x = K$ where $K = C^2 - 13$ is the required solution.

9. (x – 3y + 4) dy + (7y – 5x) dx = 0

Sol^n. Given differential equation is,
(x – 3y + 4) dy + (7y – 5x) dx = 0

or, $\frac{dy}{dx} = -\frac{(7y-5x)}{(x-3y+4)}$ …….. (i)

Put x = X + h and y = Y + K where h and K are constant.

So that dy = dX and dy = dY

$$\therefore \frac{dy}{dx} = \frac{dY}{dX}$$

Now equation (i) becomes,

$$\frac{dY}{dX} = -\frac{7(Y+K)-5(X+h)}{(X+h)-3(Y+K)+4}$$

$$\frac{dY}{dX} = \frac{(5X-7Y)+(5h-7K)}{(X-3Y)+(h-3K+4)} \quad \text{........ (ii)}$$

Let us choose h and K such that

5h – 7K = 0 and h – 3K + 4 = 0

Solving these we get,

$$h = \frac{7}{2} \text{ and } K = \frac{5}{2}$$

Now, equation (ii) reduce to

$$\frac{dY}{dX} = \frac{(5X-7Y)}{X-3Y} \quad \text{....... (iii)}$$

Equation (iii) is homogenous differential equation

So, put Y = vX

$$\text{Then } dY = v + X\frac{dv}{dX}$$

Now equation (iii) becomes,

$$v + X\frac{dv}{dX} = \frac{(5X-7vX)}{(X-3vX)} = \frac{(5-7v)}{(1-3v)}$$

$$X\frac{dv}{dX} = \frac{5-7v}{1-3v} - v$$

$$= \frac{5-7v-v+3v^2}{1-3v} = \frac{3v^2-8v+5}{1-3v}$$

$$\text{or, } \int \frac{1-3v}{3v^2-8v+5}\,dv = \int \frac{1}{X}\,dX; \text{ Integrating}$$

$$\text{or, } \int \frac{(1-3v)}{(v-1)\,(3v-5)}\,dv = \int \frac{1}{X}\,dX$$

$$\text{or, } \int \left(\frac{1}{v-1} - \frac{6}{3v-5}\right) dv = \int \frac{1}{X}\,dX$$

or, log (v – 1) – 2log (3v – 5) = logX + log C

$$\text{or, } \frac{(v-1)}{(3v-5)^2} = X\,C$$

or, $(v - 1) = (3v - 5)^2\,XC$

Restoring the value of $v = \dfrac{Y}{X}$ then, we get,

$$\left(\frac{Y}{X} - 1\right) = \left(\frac{3Y}{X} - 5\right)^2 XC$$

or, $(Y - X) = (3Y - 5X)^2\,C$

$$\text{or, } \left[\left(y-\frac{5}{2}\right)-\left(x-\frac{7}{2}\right)\right] = \left[3\left(y-\frac{5}{2}\right)-5\left(x-\frac{7}{x}\right)\right]^2 C$$

$$\text{or, } (y-x+1) = \frac{\left(6y-10x-15+35\right)^2}{2} C$$

$$= \frac{(6y-10x+20)^2}{2} C$$

$$= \left\{\frac{2(3y-5x+10}{2}\right\}^2 C$$

$\Rightarrow (3y - 5x + 10)^2\,C = (y - x + 1)$ is the required solution.

10. (x – y) dy – (x + y + 1) dx = 0

Sol^n. Given differential equation is,

(x – y) dy – (x + y + 1) dx = 0

$$\text{or, } \frac{dy}{dx} = \frac{(x+y+1)}{(x-y)} \quad \text{........ (i)}$$

Put x = X + h and y = Y + K

Where h and K are constant.

So, that dx = dX and dy = dY

$$\therefore \frac{dy}{dx} = \frac{dY}{dX}$$

Now equation (i) becomes,

$$\frac{dY}{dX} = \frac{X+h+Y+K+1}{X+h-Y-K} = \frac{X+Y+(h+K+1)}{X-Y+(h-K)} \quad \text{...... (ii)}$$

Let us choose h and K such that,

h + k + 1 = 0 and h – k = 0

$$\text{Solving these } h = -\frac{1}{2} \text{ and } K = -\frac{1}{2}$$

Hence equation (ii) becomes,

$$\frac{dY}{dX} = \frac{X+Y}{X-Y} \quad \text{......... (iii)}$$

Equation (iii) is homoenous differential equation

So, put Y = vX

$$\text{Then, } \frac{dY}{dX} = v + X\frac{dv}{dX}$$

Now equation (iii) becomes,

$$v + X\frac{dv}{dX} = \frac{X+vX}{X-vx} = \frac{1+v}{1-v}$$

or, $X\dfrac{dv}{dX} = \dfrac{1+v}{1-v} - v = \dfrac{1+v-v+v^2}{1-v} = \dfrac{1+v^2}{1-v}$

or, $\displaystyle\int \frac{1-v}{1+v^2}\,dv = \int \frac{1}{X}\,dX$; Integrating

or, $\displaystyle\int \frac{1}{1+v^2}\,dv - \int \frac{v}{1+v^2}\,dv = \int \frac{1}{X}\,dX$

or, $\tan^{-1}v - \dfrac{1}{2}\log(1+v^2) = \log X + \log C$

or, $\tan^{-1}v = \log\left(\sqrt{1+v^2}\right) + \log X\,C$

$\tan^{-1}v = \log\left\{\sqrt{1+v^2}\,XC\right\}$

Restoring the value of $v = \dfrac{Y}{X}$ we get,

$\tan^{-1}\left(\dfrac{Y}{X}\right) = \log\left\{\sqrt{1+\dfrac{Y^2}{X^2}}\,.\,XC\right\}$

or, $\tan^{-1}\left(\dfrac{Y}{X}\right) = \log\left\{\sqrt{X^2+Y^2}\,.\,C\right\}$

Put $Y = y - K = y + \dfrac{1}{2}$ and $X = x - h + x + \dfrac{1}{2}$

Then $\tan^{-1}\left(\dfrac{y+\frac{1}{2}}{x+\frac{1}{2}}\right) = \dfrac{1}{2}\log\left\{C^{\frac{1}{2}}\left[\left(x+\dfrac{1}{2}\right)^2 + \left(y+\dfrac{1}{2}\right)^2\right]\right\}$

or, $\tan^{-1}\left(\dfrac{2y+1}{2x+1}\right) = \dfrac{1}{2}\log\left[K\left\{\left(x+\dfrac{1}{2}\right)^2 + \left(y+\dfrac{1}{2}\right)^2\right\}\right]$

Where $K = C^{\frac{1}{2}}$ is the required solution.