6th Class

Ex-2.3 Q.N. 13 $\frac{dy}{dx} + y = \cos x$.

Soln: Given diff. eqn is $\frac{dy}{dx} + y = \cos x$ —(1)

Which is linear diff. eqn on y

So Comparing it with $\frac{dy}{dx} + Py = Q$

$P = 1,\ Q = \cos x$

Now, $I.F. = e^{\int P dx} = e^{\int 1 \cdot dx} = e^{x}$

Multiplying eqn (1) by $I.F. = e^x$ on both sides we get;

$y \times I.F. = \int (Q \times I.F.) dx + C$

or, $y \times e^x = \int e^x \cos x \, dx + C$

* using $\int e^{ax} \cos bx \, dx = \frac{e^{ax}(a\cos bx + b \sin bx)}{a^2+b^2}$

** $\int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b\cos bx)}{a^2+b^2}$

$y \cdot e^x = \frac{e^x(1 \cdot \cos x + 1 \cdot \sin x)}{1^2+1^2} + C$

or, $y e^x = \frac{e^x(\cos x + \sin x)}{2} + C$

$\Rightarrow 2y = \cos x + \sin x + 2Ce^{-x}$

or, $2y = \cos x + \sin x + Ke^{-x}$

where $K = 2C$ which is the required general soln

Exact differential eqn

Exact differential eq

Exact diff. eq^n:- A diff. eq^n

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be exact if there exist a function $f(x,y)$ such that

$$\underline{M(x,y)\,dx + N(x,y)\,dy} = \underline{d\,f(x,y)}$$

e.g. $\underline{x\,dy} + \underline{y\,dx} = d(xy)$

i.e. the given diff. eq^n is exact if $M(x,y)\,dx + N(x,y)\,dy$ is exact or perfect differential.

Note:- The diff. eq^n

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

will be exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

where $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x}$ denotes the partial derivative.

$$U = ax^2 + 2hxy + by^2$$

diff. partial w.r.t. 'x'

$$\frac{\partial U}{\partial x} = 2ax + 2hy + 0$$

diff. partial w.r.t. y

$$\frac{\partial U}{\partial y} = 0 + 2hx + 2by$$

Note:- Every diff. eq^n

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Every diff. $\to$
$M(x,y)\,dx + N(x,y)\,dy = 0$
is not exact.

for e.g. $x^2 dy + 2xy\,dy = 0$ is exact
because
$$x^2 dy + 2xy\,dy = d(x^2 y)$$

Some formula

(1) $x\,dy + y\,dx = d(xy)$

(2) $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$

(3) $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$

and so on. — page 329

Ex - 25

Solve the following diff. eq^n by reducing to exact form

e.g. $x\,dy + y\,dx = 0$

Sol^n: Given diff. eq^n is

$x\,dy + y\,dx = 0$

or, $\int d(xy) = \int 0$ Integrating

$xy = c$ which is the required general sol^n #

OR $x\,dy + y\,dx = 0$

OR $x\,dy + y\,dx = 0$

or, $x\,dy = -y\,dx$

or, $\int \frac{1}{y}\,dy = -\int \frac{1}{x}\,dx$ Int.

$\log y = -\log x + \log c$

or, $\log y = \log\left(\frac{c}{x}\right) \Rightarrow y = \frac{c}{x}$

$\therefore xy = c$

Which is the required general soln of the given diff. eqn

Solve: Ex-21, 23, 24

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

e.g. $\frac{dy}{dx} = \frac{2x-y+1}{x+2y-3}$ Ex-22 (4)

$(x+2y-3)\,dy = (2x-y+1)\,dx$

or, $x\,dy + 2y\,dy - 3\,dy = 2x\,dx - y\,dx + dx$

or, $x\,dy + y\,dx + 2y\,dy - 3\,dy = 2x\,dx + dx$

or, $\int d(xy) + 2\int y\,dy - 3\int dy = 2\int x\,dx + \int dx$

Integrating

$xy + 2\cdot\frac{y^2}{2} - 3y = 2\frac{x^2}{2} + x + c$

a, $xy + y^2 - 3y - x^2 - x - c = 0$

$$y + y^2 - 3y - x^2 - x - c = 0$$

Ex-25

First order but not first degree differential equation

An eq^n of the form $f(x, y, p) = 0$ where $p = \frac{dy}{dx}$ is called first order but first degree diff. eq^n. The solution of such type of diff. eq^n contains only one constant.

We will discuss the following first order but not first degree diff. eq^n

→ Solvable for p
→ Solvable for y
→ Solvable for x
→ Clairaut's Equation

Solvable for p

An eq^n of the form $f(x, y, p) = 0$ where $p = \frac{dy}{dx}$ can be factorized into linear factors such as

$$\{p - f_1(x, y)\}\{p - f_2(x, y)\} \cdots \cdots \{p - f_n(x, y)\} = 0$$

Such type of first order but not first degree diff. eq^n is called solvable for p.

"Solvable for p."

It's general soln is

1. $p^2+p-6=0$ or, $\left(\frac{dy}{dx}\right)^2+\frac{dy}{dx}-6=0$ where $\frac{dy}{dx}=p$

Soln. Given diff. eqn is

$p^2+p-6=0$

or, $p^2+3p-2p-6=0$

or, $p(p+3)-2(p+3)=0$

or, $(p+3)(p-2)=0$

Either $p+3=0$

or, $\frac{dy}{dx}+3=0$

or, $\int dy=-\int 3dx$ Int.

$\Rightarrow y=-3x+c$

or, $y+3x-c=0$

or $p-2=0$

$\frac{dy}{dx}-2=0$

or, $\int dy=\int 2dx$ Int.

$\Rightarrow y=2x+c$

or, $y-2x-c=0$

$\therefore$ The required general sol^n is

$$(y+3x-c)(y-2x-c) = 0$$

(2) $p^2+2px-3x^2=0$

Sol^n: Given diff. eq^{ns}

(3) $p^2-p(e^x+e^{-x})+1=0$

$f(x,y,p)=0$

$\{p-f_1(x,y)\}$

Sol^n: Given diff. eq^n

$$p^2-p(e^x+e^{-x})+1=0$$

or, $p^2-pe^x-pe^{-x}+1=0$

or, $p(p-e^x)-e^{-x}\left(p-\frac{1}{e^{-x}}\right)=0$

or, $p(p-e^x)-e^{-x}(p-e^x)=0$

or, $(p-e^x)(p-e^{-x})=0$

Either $p-e^x=0$ or $p-e^{-x}=0$

$\frac{dy}{dx}=e^x$

$\int dy=\int e^x dx$ Int.

$y=e^x+c$

$y-e^x-c=0$

$\int dy=\int e^{-x}dx$ Int.

$y=-e^{-x}+c$

$\Rightarrow y+e^{-x}-c=0$

$\therefore$ The required general sol^n is,

$$(y-e^x-c)(y+e^{-x}-c)=0 \quad \#$$

(8) $p^3-p(x^2+xy+y^2)+x^2y+xy^2=0$

Sol^n: Given diff. eq^n is

$$p^3-px^2-pxy-py^2+x^2y+xy^2=0$$

Sol: Given diff eq

$$p^3 - px^2 - pxy - py^2 + x^2y + xy^2 = 0$$

or, $p^3 - px^2 - pxy + x^2y - py^2 + xy^2 = 0$

or, $p(\underbrace{p^2 - x^2}_{=(p-x)(p+x)}) - xy(p-x) - y^2(p-x) = 0$

or, $(p-x)\{p(p+x) - xy - y^2\} = 0$

or, $(p-x)(p^2 + px - xy - y^2) = 0$

or, $(p-x)(p^2 - y^2 + px - xy) = 0$

or, $(p-x)\{(p-y)(p+y) + x(p-y)\} = 0$

or, $(p-x)(p-y)(p+y+x) = 0$

Either

$$\frac{dy}{dx} + y + x = 0$$

or, $\frac{dy}{dx} + y = -x \quad \text{(1)}$

Which is linear diff. eq^n on y

So it's I.F. $= e^{\int 1 \cdot dx} = e^x$

Multiplying eq^n (1) by I.F. $= e^x$ on both sides we get

$$y \times I.F. = \int (Q \times I.F.)\, dx + C$$

$$y \cdot e^x = -\int x e^x\, dx + C$$

$$= -\left(x e^x - \int 1 \cdot e^x\, dx\right) + C$$

$$= -\left(xe^{x} - \int 1.e^{x}\,dx\right) + C$$

$$y.e^{x} = -xe^{x} + e^{x} + C$$

$$\text{or, } y = -x + 1 + Ce^{-x}$$

$$\text{or, } y + x - 1 - Ce^{-x} = 0$$

$\therefore$ The required general solⁿ is,

$$(\quad)(\quad)(y + x - 1 - Ce^{-x}) = 0$$

#

$$Ax^2 + Bx + C = 0$$

$$\Rightarrow x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2.A}$$

e.g. $p^3 + 3xp^2 - y^3p^2 - 3xy^3p = 0$

$$\int \frac{6 - 5v^2}{(5v^2 - 7v + 2)}\,dv$$

$$\int \frac{-\frac{1}{2}(10v - 7) - \frac{7}{2} + 6}{(5v^2 - 7v + 2)}\,dv$$

$$= -\frac{1}{2}\int \frac{10v - 7}{5v^2 - 7v + 2}\,dv + \frac{5}{2}\int \frac{1}{(5v^2 - 7v + 2)}\,dv$$

$$= -\frac{1}{2}\log(5v^2 - 7v + 2) + \frac{5}{2}.\frac{1}{5}\int \frac{1}{v^2 - 7}\,dv$$

$$= \text{"} + \frac{1}{2}\int \frac{1}{\qquad}\,dv$$

$$\frac{2}{5} + \frac{2}{5}$$

$$= \text{"} + \frac{1}{2}\int \frac{1}{(v - 7)^2 - (B)^2}\,dv$$

$$= '' + \frac{1}{9}\int \frac{1}{\left(v-\frac{7}{10}\right)^2-\left(\frac{3}{10}\right)^2}\,dv$$

$$* \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + c$$

BCT – A First year 7th Class 2078-02-26

First order but not first degree diff. eqn

→ Solvable for y: An eqn of the form $y = f(x, p)$ where $p = \frac{dy}{dx}$. Such type of first order but not first degree diff. eqn is called solvable for y.

The method of solvable for y

$p = \frac{dy}{dx}$

Let $y = f(x, p)$ —— (1)

$= x^2 + p^2 = f(x, p)$

diff. (1) w.r.t. 'x'

$= 2x + 2p\frac{dp}{dx} + c$

$= \phi\left(x, p, \frac{dp}{dx}\right)$

$$\frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right)$$

or

$$p = \phi\left(x, p, \frac{dp}{dx}\right) \quad \text{—— (2)}$$

or which is the diff. eqn in the two variables x and p

$\frac{dy}{dx} = \sin x$

$\int dy = \int \sin x\,dx$

$y = -\cos x + c$

or $y + \cos x + c = 0$

or $F(x, y, c) = 0$

Suppose its soln is

$$F(x, p, c) = 0 \quad \text{—— (3)}$$

$F(x, p, c) = 0 \quad \hookrightarrow (3)$

$y + \tan x + c = 0$
or $F(x, y, c) = 0$

Eliminating p from (1) and (3) we get the required general solⁿ.

If p can not be eliminated from two given eqⁿ (1) and (3), Then the eqⁿ (1) and (3) together give the required general solⁿ.

Ex-27.

Find the general solⁿ of the following diff. eqⁿ.

(5) $y + px = x^4p^2$ where $p = \frac{dy}{dx}$

Solⁿ: Given diff. eqⁿ $y + px = x^4p^2$

or, $y = x^4p^2 - px \quad \text{---} (1)$

which is of the form $y = f(x, p)$

So it can be solve by the method for solvable for y.

$\therefore$ diff. (1) w.r.t. x

$$\frac{dy}{dx} = x^4 2p\frac{dp}{dx} + p^2 4x^3 - p.1 - x\frac{dp}{dx}$$

$$\text{or, } p = 2px^4\frac{dp}{dx} + 4p^2x^3 - p - x\frac{dp}{dx}$$

$$p - 4p^2x^3 + p = 2px^4\frac{dp}{dx} - x\frac{dp}{dx}$$

$$\text{or, } (2p - 4p^2x^3) = (2px^4 - x)\frac{dp}{dx}$$

dx

$$\text{or, } \frac{(2p - 4p^2x^3)}{(2px^4 - x)} = \frac{dp}{dx}$$

$$\text{or, } \frac{2p(1 - \cancel{2}px^3)}{-x(1 - \cancel{2}px^3)} = \frac{dp}{dx}$$

$$\text{or, } \frac{2p}{-x} = \frac{dp}{dx}$$

$$\text{or, } \int \frac{2}{x}\,dx = -\int \frac{1}{p}\,dp \quad \text{Integrating}$$

$$\text{or, } 2\log x = -\log p + \log c$$

$$\text{or, } \log x^2 = \log \frac{c}{p}$$

$$\Rightarrow x^2 = \frac{c}{p}$$

$$\text{or, } p = \frac{c}{x^2} \quad \text{——} \quad (2)$$

$$y = x^4p^2 - px \quad \text{——} \quad (1)$$

Eliminating p from (1) and (2)

$$\text{or, } y = x^4 \cdot \left(\frac{c}{x^2}\right)^2 - \frac{c}{x^2} \cdot x$$

$$\text{or, } y = c^2 - \frac{c}{x}$$

$\Rightarrow xy = c^2x - c$ which is the required general soln of the given diff. eqn.

Solvable for x

An eq^n of the form $x = f(y, p)$ where $p = \frac{dy}{dx}$ such type of first order but not first degree diff. eq^n is called solvable for x.

The method of solvable for x

Let $x = f(y, p)$ —— (1)

diff. (1) w.r.t. 'y'

$$\frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right) \qquad \left[\frac{dy}{dx} = p, \quad \frac{dx}{dy} = \frac{1}{p}\right]$$

$$\therefore \frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right) \quad \text{—— (2)}$$

which is the diff. eq^n in two variable y and p.

Suppose its solution is

$$F(y, p, c) = 0 \quad \text{—— (3)}$$

Eliminating p from (1) and (3) gives the required general sol^n.

If p can not be eliminated from two given eq^n (1) and (3) together give required general sol^n of the given diff. eq^n

Ex-27.

Find the general sol^n of the following diff. eq^n.

(a)

(2) $y = 2px + p^3 y^2$

Soln: Given diff. eqn $y = 2px + p^3 y^2$

or, $2px = y - p^3 y^2$

or, $2x = \dfrac{y - p^3 y^2}{p} = \dfrac{y}{p} - p^2 y^2$

or, $2x = \dfrac{y}{p} - p^2 y^2$ —— (1)

which is of the form $x = f(y, p)$

So it can be solve by the method for solvable for x

$\therefore$ diff. (1) w.r.t. 'y'

$$2\frac{dx}{dy} = \frac{p \cdot \overbrace{\frac{dy}{dy}}^{=1} - y\frac{dp}{dy}}{p^2} - p^2 . 2y - y^2 . 2p\frac{dp}{dy}$$

or, $2 \cdot \dfrac{1}{p} = \dfrac{1}{p} - \dfrac{y}{p^2}\dfrac{dp}{dy} - 2p^2 y - 2py^2\dfrac{dp}{dy}$

or, $\dfrac{2}{p} - \dfrac{1}{p} + 2p^2 y = -\left(\dfrac{y}{p^2} + 2py^2\right)\dfrac{dp}{dy}$

or, $\dfrac{(2 - 1 + 2p^3 y)}{p} = -\dfrac{(y + 2p^3 y^2)}{p^2}\dfrac{dp}{dy}$

or, $\dfrac{\cancel{(1 + 2p^3 y)}}{\cancel{p}} = -y\dfrac{\cancel{(1 + 2p^3 y)}}{p^{\cancel{2}}}\dfrac{dp}{dy}$

or, $1 = -\dfrac{y}{p}\dfrac{dp}{dy}$

or, $\int \dfrac{1}{y}\,dy = -\int \dfrac{1}{p}\,dp$ Ind.

$\log y = -\log p + \log C$

$$\log y = -\log p + \log c$$

or $\log y = \log\left(\frac{c}{p}\right)$

or $y = \frac{c}{p}$

or, $p = \frac{c}{y}$ —— (2)

Eliminating p from (1) and (3) we get

$$2x = \frac{y}{\left(\frac{c}{y}\right)} - \left(\frac{c}{y}\right)^2 \cdot y^2$$

$$\Rightarrow 2x = \frac{y^2}{c} - c^2$$

$$\Rightarrow 2cx = y^2 - c^3$$

which is the required general soln of the given diff. eqn.

(4) Find the general soln of following diff. eqn

$$\sin y \cos px - \cos y \sin px = p$$

Soln: Given diff. eqn is

$$\sin y \cos px - \cos y \sin px = p$$

or, $\sin(y - px) = p$

or, $y - px = \sin^{-1} p$

$$y = px + \sin^{-1}(p) \quad \text{—— (1)}$$

diff. (1) w.r.t. (x)

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$$

or, $\cancel{p} = \cancel{p} + x\frac{dp}{dx} + \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$

or, $0 = \left[x + \frac{1}{\sqrt{1-p^2}}\right]\frac{dp}{dx}$

or, $\left[x + \frac{1}{\sqrt{1-p^2}}\right]\frac{dp}{dx} = 0$

[-x-28]

Either $\frac{dp}{dx} = 0$

or, $\int dp = \int 0 \cdot dx$ Int.

$\Rightarrow p = c$

putting the value of $p = c$ in (1)

$$\boxed{y = cx + \sin^{-1} c}$$

which is the required general soln of the given diff. eqn #

(6) $p^2 - py + x = 0$

Soln. Given diff. eqn $p^2 - py + x = 0$

or, $py = p^2 + x$

or, $y = p + \frac{x}{p}$ —(1)

diff. (1) w.r.t. x,

$$\frac{dy}{dx} = \frac{dp}{dx} + \frac{p \cdot 1 - x \cdot \frac{dp}{dx}}{p^2}$$

or, $p = \frac{dp}{dx} + \frac{1}{p} - \frac{x}{p^2}\frac{dp}{dx}$

or, $p - \frac{1}{p} = \left(1 - \frac{x}{p^2}\right)\frac{dp}{dx}$

or, $(p^2 - 1)$ ...

or, $\left(\frac{p^2-1}{\cancel{p}}\right) = \left(\frac{p^2-x}{p\cancel{x}}\right)\frac{dp}{dx}$

or, $\frac{dp}{dx} = \frac{p(p^2-1)}{(p^2-x)}$

or, $\frac{dx}{dp} = \frac{p^2-x}{p(p^2-1)}$

or, $\frac{dx}{dp} = \frac{p\cancel{2}}{\cancel{p}(p^2-1)} - \frac{x}{p(p^2-1)}$

or, $\frac{dx}{dp} + \frac{1}{p(p^2-1)}\cdot x = \frac{p}{(p^2-1)}$ — (2)

Which is linear diff. eqn in x

So it's I.F. $= e^{\int \frac{1}{p(p^2-1)}dp}$

$= e^{\int\left[\frac{-1}{p} + \frac{p}{p^2-1}\right]dp}$

$= e^{-\log p + \frac{1}{2}\log(p^2-1)}$

$= e^{-\log p + \log(p^2-1)^{1/2}}$

$= e^{\log\frac{\sqrt{p^2-1}}{p}}$

$= \frac{\sqrt{p^2-1}}{p}$

multiplying eqn (2) by I.F. $= \frac{\sqrt{p^2-1}}{p}$ on both sides we get,

$x \times \frac{\sqrt{p^2-1}}{p} = \int \frac{\cancel{p}}{(p^2-1)}\cdot\frac{\sqrt{p^2-1}}{\cancel{p}}dp + C$

$$\frac{\cdots}{p} = \int \frac{\cdots}{(p^2-1)} \cdot \frac{\cdots}{\cancel{p}} \cdots + c$$

$$= \int \frac{1}{\sqrt{p^2-1}}\,dp + c$$

using $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\{x + \sqrt{x^2-a^2}\}$

$$\therefore \frac{x\sqrt{p^2-1}}{p} = \log\left(p+\sqrt{p^2-1}\right) + c$$

$$\Rightarrow x = \frac{p\log\left(p+\sqrt{p^2-1}\right)}{\sqrt{p^2-1}} + \frac{cp}{\sqrt{p^2-1}} \quad \text{--- (3)}$$

Since p cannot be easily eliminated from (1) and (3) so (1) and (3) together gives the required general soln.

i.e.

$$\left.\begin{aligned} x &= \frac{p\log\left(p^2+\sqrt{p^2-1}\right)}{\sqrt{p^2-1}} + \frac{cp}{\sqrt{p^2-1}} \\ y &= p + \frac{x}{p} \end{aligned}\right\} \text{ Ans.}$$

Solve:- (1) $y = yp^2 + 2px$

(2) $y - 2px + ayp^2 = 0$

(3) $p^3 - 4xyp + 8y^2 = 0$

(5) $xp^2 - 2yp + ax = 0$

Ex- 27 (1) (2) (4) (5) old question

Ex-27 1 2 4 5 old question