Orthonormal subsets of the Legendre-Haar quarklet system

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Abstract

The Legendre-Haar quarklet system is a set of piecewise polynomial and globally discontinous functions on the real line with cancellation properties. As we show in this paper, the Legendre-Haar quarklet system not only contains the Haar wavelet basis and the L_2-normalized truncated Legendre polynomials as orthonormal subsystems, but also a variety of other orthonormal bases which are piecewise polynomial with respect to a nonuniform partition of $\mathbb{R}$ into dyadic subintervals.

1 Orthogonality properties of Legendre-Haar quarklets

1.1 Legendre-Haar quarklets

Let us first define the Legendre-Haar quarklet system. To this end, for $p \in \mathbb{N}_0$, let L_p be the p-th orthogonal Legendre polynomial on $[-1, 1]$ with normalization $L_p(1) = 1$. It is well-known that

$$\int_0^1 L_p(x)L_q(x)\,\mathrm{d}x = \begin{cases} \frac{2}{2p+1}, & p = q, \\ 0, & p \neq q, \end{cases}$$

so that the truncated and renormalized Legendre polynomials

$$\varphi_p(x) := \sqrt{2p+1}L_p(2x-1)\chi_{[0,1)}(x), \quad p \in \mathbb{N}_0, \quad x \in \mathbb{R}, \tag{1}$$

induce the orthonormal basis

$$\{\varphi_p(\cdot - k) : p \in \mathbb{N}_0, k \in \mathbb{Z}\}$$

for $L_2(\mathbb{R})$. Let us call the functions φ_p *Legendre-Haar quarks*, which is motivated by the combination of Legendre polynomials L_p and the Haar scaling function $\chi_{[0,1)}$.

Furthermore, let us define the *Legendre-Haar quarklets*

$$\psi_p(x) := \varphi_p(2x) - \varphi_p(2x-1), \quad p \in \mathbb{N}_0, \quad x \in \mathbb{R}, \tag{2}$$

which is a generalization of the two-scale relation $\psi_0(x) = \varphi_0(2x) - \varphi_0(2x-1)$ of Haar wavelets. The *Legendre-Haar quarklet system* is then given as the set

$$\Psi := \{\varphi_p(\cdot - k) : p \in \mathbb{N}_0, k \in \mathbb{Z}\} \cup \{\psi_{p,j,k} := 2^{j/2}\psi_p(2^j \cdot -k) : p, j, \in \mathbb{N}_0, k \in \mathbb{Z}\} \tag{3}$$

of $L_2(\mathbb{R})$-normalized functions. Note that Ψ contains the orthonormal Haar wavelet basis

$$\{\varphi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi_{0,j,k} = 2^{j/2}\psi_0(2^j \cdot -k) : j \in \mathbb{N}_0, k \in \mathbb{Z}\}$$

as well as the orthonormal basis of truncated and shifted Legendre polynomials

$$\{\varphi_p(\cdot - k) : p \in \mathbb{N}_0, k \in \mathbb{Z}\}.$$

1.2 Orthogonality properties between quarks and quarklets

We will now collect the orthogonality properties of various subsets of the Legendre-Haar quarklet system Ψ from (3). The following lemma covers the case $j = 0$.

Lemma 1. *We have*

$$\big\langle \varphi_p, \varphi_q(\cdot - k)\big\rangle_{L_2(\mathbb{R})} = \begin{cases} 1, & k = 0 \textit{ and } p = q \\ 0, & \textit{otherwise} \end{cases}, \quad p, q \in \mathbb{N}_0, \quad k \in \mathbb{Z}. \tag{4}$$

Moreover, we have

$$\big\langle \varphi_p(2\cdot -k), \varphi_p\big\rangle_{L_2(\mathbb{R})} = \begin{cases} 2^{-(p+1)}, & k \in \{0,1\} \\ 0, & \textit{otherwise} \end{cases}, \quad p \in \mathbb{N}_0, \quad k \in \mathbb{Z}. \tag{5}$$

By consequence, Legendre-Haar quarklets are orthogonal to Legendre-Haar quarks of lower degree,

$$\big\langle \psi_p, \varphi_q(\cdot - k)\big\rangle_{L_2(\mathbb{R})} = 0, \quad \textit{for all } 0 \le q \le p, \quad k \in \mathbb{Z}. \tag{6}$$

In particular, each ψ_p has $p+1$ vanishing moments, $\langle \psi_p, r\rangle_{L_2(\mathbb{R})} = 0$ for all $r \in \mathbb{P}_p$. Moreover, Legendre-Haar quarklets are orthogonal to Legendre-Haar quarklets of strictly lower degree,

$$\big\langle \psi_p, \psi_q(\cdot - k)\big\rangle_{L_2(\mathbb{R})} = \begin{cases} 1, & k = 0 \textit{ and } p = q, \\ 0, & \textit{otherwise} \end{cases}, \quad \textit{for all } 0 \le q \le p, \quad k \in \mathbb{Z}. \tag{7}$$

Proof. (4) follows from $\operatorname{supp} \varphi_p = [0,1]$ and from the orthonormality of the truncated, shifted and renormalized Legendre polynomials. As concerns (5), let $k \in \{0,1\}$. On $\operatorname{supp} \varphi_p(2\cdot -k) = [\frac{k}{2}, \frac{k+1}{2}]$, φ_p is a polynomial of degree p the leading coefficient of which coincides with 2^{-p} times the leading coefficient of $\varphi_p(2\cdot -k)$. Since $\varphi_p(2\cdot -k)$ is orthogonal to all polynomials of degree $p-1$ on its support, it follows

$$\big\langle \varphi_p(2\cdot -k), \varphi_p\big\rangle_{L_2(\mathbb{R})} = 2^{-p}\big\langle \varphi_p(2\cdot -k), 2^p\varphi_p\big\rangle_{L_2(\mathbb{R})} = 2^{-p}\big\|\varphi_p(2\cdot -k)\big\|^2_{L_2(\mathbb{R})} = 2^{-(p+1)}.$$

If $k \notin \{0,1\}$, the supports of $\varphi_p(2\cdot -k)$ and φ_p are disjoint, hence (5) follows. Combining (2) with (5), we compute for $0 \le q \le p$ and any $k \in \mathbb{Z}$ that

$$\big\langle \psi_p, \varphi_q(\cdot - k)\big\rangle_{L_2(\mathbb{R})} = \big\langle \varphi_p(2\cdot +2k) - \varphi_p(2\cdot +2k-1), \varphi_q\big\rangle_{L_2(\mathbb{R})} = 0,$$

which yields (6) and, in turn, the vanishing moment property $\psi_p \perp \mathbb{P}_p$. Finally, (7) follows from (4) after inserting the definition of ψ_p,

$$\begin{aligned} \big\langle \psi_p, \psi_q(\cdot - k)\big\rangle_{L_2(\mathbb{R})} &= \big\langle \varphi_p(2\cdot) - \varphi_p(2\cdot -1), \varphi_q(2\cdot -2k) - \varphi_q(2\cdot -2k-1)\big\rangle_{L_2(\mathbb{R})} \\ &= \big\langle \varphi_p(2\cdot), \varphi_q(2\cdot -2k) - \varphi_q(2\cdot -2k-1)\big\rangle_{L_2(\mathbb{R})} \\ &\quad - \big\langle \varphi_p(2\cdot -1), \varphi_q(2\cdot -2k) - \varphi_q(2\cdot -2k-1)\big\rangle_{L_2(\mathbb{R})} \\ &= \begin{cases} 1, & k = 0 \text{ and } p = q, \\ 0, & \text{otherwise} \end{cases}, \quad 0 \le q \le p, \quad k \in \mathbb{Z}. \end{aligned}$$

$\square$

As concerns the orthogonality properties between dilated quarks and quarklets, the situation is more complicated because low-degree, fine-scale quark(let)s are in general *not* orthogonal to high-degree, low-scale quark(lets). In the following lemma, we collect some positive results on orthogonality.

Lemma 2. *If $0 \le p \le p'$ and $0 \le j \le j'$, we have*

$$\langle \varphi_{p,j,k}, \varphi_{p',j',k'} \rangle_{L_2(\mathbb{R})} = \begin{cases} 2^{(j-j')(p+1/2)}, & 2^{-j'}[k', k'+1] \subseteq 2^{-j}[k, k+1], \quad p' = p, \\ 0, & \textit{otherwise}, \end{cases} \tag{8}$$

and

$$\langle \varphi_{p,j,k}, \psi_{p',j',k'} \rangle_{L_2(\mathbb{R})} = 0. \tag{9}$$

Proof. $\operatorname{supp} \varphi_{p,j,k} = 2^{-j}[k, k+1]$ is an interval of length 2^{-j}. Therefore, the support sets of two different quarks $\varphi_{p,j,k}$ and $\varphi_{p',j',k'}$ with $j \le j'$ either intersect on a set of Lebesgue measure zero, or $\operatorname{supp} \varphi_{p',j',k'} \subseteq \operatorname{supp} \varphi_{p,j,k}$.

Assume now that $2^{-j'}[k', k'+1] \subseteq 2^{-j}[k, k+1]$ and let $0 \le p \le p'$. Then $\varphi_{p,j,k}$ is a polynomial of degree p on the support of $\varphi_{p',j',k'}$, so that the inner product between $\varphi_{p,j,k}$ and $\varphi_{p',j',k'}$ vanishes if $p < p'$. If $p = p'$, we can proceed just as in the proof of Lemma 1: on its support, $\varphi_{p,j,k} = 2^{j/2}\varphi_p(2^j \cdot -k)$ is a polynomial of degree p the leading coefficient of which coincides with $2^{(j-j')(p+1/2)}$ times the leading coefficient of $\varphi_{p,j',k'} = 2^{j'/2}\varphi_p(2^{j'} \cdot -k')$, so that by the orthogonality properties of the Legendre polynomials, we get

$$\begin{aligned} \langle \varphi_{p,j,k}, \varphi_{p,j',k'} \rangle_{L_2(\mathbb{R})} &= 2^{(j-j')(p+1/2)} \langle 2^{(j'-j)(p+1/2)} \varphi_{p,j,k}, \varphi_{p',j',k'} \rangle_{L_2(\mathbb{R})} \\ &= 2^{(j-j')(p+1/2)} \|\varphi_{p,j',k'}\|^2_{L_2(\mathbb{R})} \\ &= 2^{(j-j')(p+1/2)}. \end{aligned}$$

By using the two-scale relation

$$\psi_{p',j',k'} = \frac{1}{\sqrt{2}}(\varphi_{p',j'+1,2k'} - \varphi_{p',j'+1,2k'+1})$$

and (8), we can compute inner products between Legendre-Haar quarks and quarklets of higher degree and higher scale because regardless of whether $2^{-j'}[k', k'+1] \subseteq 2^{-j}[k, k+1]$ holds true (which entails $2^{-(j'+1)}[2k', 2k'+1], 2^{-(j'+1)}[2k'+1, 2k'+2] \subseteq 2^{-j}[k, k+1]$) or not, we get

$$\langle \varphi_{p,j,k}, \psi_{p',j',k'} \rangle_{L_2(\mathbb{R})} = \frac{1}{\sqrt{2}} \langle \varphi_{p,j,k}, \varphi_{p',j'+1,2k'} - \varphi_{p',j'+1,2k'+1} \rangle_{L_2(\mathbb{R})} = 0.$$

□