Orthonormal subsets of the Legendre-Haar quarklet system

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Abstract

The Legendre-Haar quarklet system is a set of piecewise polynomial and globally discontinous functions on the real line with cancellation properties. As we show in this paper, the Legendre-Haar quarklet system not only contains the Haar wavelet basis and the L_2-normalized truncated Legendre polynomials as orthonormal subsystems, but also a variety of other orthonormal bases which are piecewise polynomial with respect to a nonuniform partition of $\mathbb{R}$ into dyadic subintervals.

1 Orthogonality properties of Legendre-Haar quarklets

1.1 Legendre-Haar quarklets

Let us first define the Legendre-Haar quarklet system. To this end, for $p \in \mathbb{N}_0$, let L_p be the p-th orthogonal Legendre polynomial on $[-1, 1]$ with normalization $L_p(1) = 1$. It is well-known that

$$\int_0^1 L_p(x)L_q(x)\,\mathrm{d}x = \begin{cases} \frac{2}{2p+1}, & p = q, \\ 0, & p \neq q, \end{cases}$$

so that the truncated and renormalized Legendre polynomials

$$\varphi_p(x) := \sqrt{2p+1}L_p(2x-1)\chi_{[0,1)}(x), \quad p \in \mathbb{N}_0, \quad x \in \mathbb{R}, \tag{1}$$

induce the orthonormal basis

$$\big\{\varphi_p(\cdot - k) : p \in \mathbb{N}_0, k \in \mathbb{Z}\big\}$$

for $L_2(\mathbb{R})$. Let us call the functions φ_p *Legendre-Haar quarks*, which is motivated by the combination of Legendre polynomials L_p and the Haar scaling function $\chi_{[0,1)}$.

Furthermore, let us define the *Legendre-Haar quarklets*

$$\psi_p(x) := \varphi_p(2x) - \varphi_p(2x-1), \quad p \in \mathbb{N}_0, \quad x \in \mathbb{R}, \tag{2}$$

which is a generalization of the two-scale relation $\psi_0(x) = \varphi_0(2x) - \varphi_0(2x-1)$ of Haar wavelets. The *Legendre-Haar quarklet system* is then given as the set

$$\Psi := \big\{\varphi_p(\cdot - k) : p \in \mathbb{N}_0, k \in \mathbb{Z}\big\} \cup \big\{\psi_{p,j,k} := 2^{j/2}\psi_p(2^j \cdot -k) : p, j, \in \mathbb{N}_0, k \in \mathbb{Z}\big\} \tag{3}$$

of $L_2(\mathbb{R})$-normalized functions. Note that besides from the truncated and shifted Legendre polynomials, Ψ also contains the orthonormal Haar wavelet basis

$$\big\{\varphi_0(\cdot - k) : k \in \mathbb{Z}\big\} \cup \big\{\psi_{0,j,k} = 2^{j/2}\psi_0(2^j \cdot -k) : j \in \mathbb{N}_0, k \in \mathbb{Z}\big\}.$$

In Figure 1, we see some plots of the Legendre-Haar quarks and quarklets.

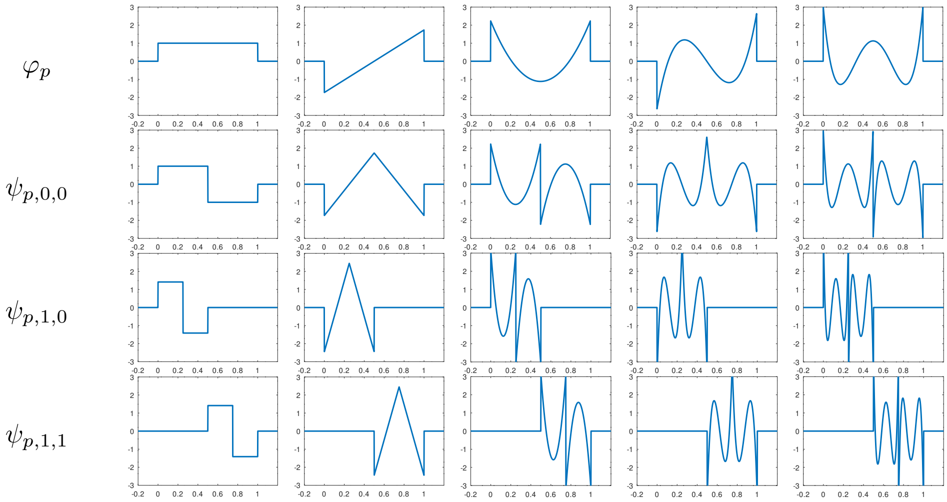

Figure 1: Some Legendre-Haar quarks φ_p and quarklets $\psi_{p,j,k}$ with $0 \le p \le 4$

1.2 Orthogonality properties between quarks and quarklets

We will now collect the orthogonality properties of various subsets of the Legendre-Haar quarklet system Ψ from (3). The following lemma covers the case $j = 0$.

Lemma 1. *We have*

$$\big\langle \varphi_p, \varphi_q(\cdot - k)\big\rangle_{L_2(\mathbb{R})} = \begin{cases} 1, & k = 0 \text{ and } p = q \\ 0, & \text{otherwise} \end{cases}, \quad p, q \in \mathbb{N}_0, \quad k \in \mathbb{Z}. \tag{4}$$

Moreover, we have

$$\big\langle \varphi_p(2 \cdot -k), \varphi_p\big\rangle_{L_2(\mathbb{R})} = \begin{cases} 2^{-(p+1)}, & k \in \{0,1\} \\ 0, & \text{otherwise} \end{cases}, \quad p \in \mathbb{N}_0, \quad k \in \mathbb{Z}. \tag{5}$$

By consequence, Legendre-Haar quarklets are orthogonal to Legendre-Haar quarks of the same or of lower degree,

$$\big\langle \psi_p, \varphi_q(\cdot - k)\big\rangle_{L_2(\mathbb{R})} = 0, \quad \text{for all } 0 \le q \le p, \quad k \in \mathbb{Z}. \tag{6}$$

In particular, each ψ_p has $p+1$ vanishing moments, $\langle \psi_p, r\rangle_{L_2(\mathbb{R})} = 0$ for all $r \in \mathbb{P}_p$. Moreover, Legendre-Haar quarklets are orthogonal to Legendre-Haar quarklets of lower degree,

$$\big\langle \psi_p, \psi_q(\cdot - k)\big\rangle_{L_2(\mathbb{R})} = \begin{cases} 1, & k = 0 \text{ and } p = q, \\ 0, & \text{otherwise} \end{cases}, \quad \text{for all } 0 \le q \le p, \quad k \in \mathbb{Z}. \tag{7}$$

Proof. (4) follows from $\operatorname{supp} \varphi_p = [0,1]$ and from the orthonormality of the truncated, shifted and renormalized Legendre polynomials. As concerns (5), let $k \in \{0,1\}$. On $\operatorname{supp} \varphi_p(2 \cdot -k) = [\frac{k}{2}, \frac{k+1}{2}]$, φ_p is a polynomial of degree p the leading coefficient of which coincides with 2^{-p} times the leading coefficient of $\varphi_p(2 \cdot -k)$. Since $\varphi_p(2 \cdot -k)$ is orthogonal to all polynomials of degree $p-1$ on its support, it follows

$$\big\langle \varphi_p(2 \cdot -k), \varphi_p\big\rangle_{L_2(\mathbb{R})} = 2^{-p}\big\langle \varphi_p(2 \cdot -k), 2^p \varphi_p\big\rangle_{L_2(\mathbb{R})} = 2^{-p}\big\|\varphi_p(2 \cdot -k)\big\|^2_{L_2(\mathbb{R})} = 2^{-(p+1)}.$$

If $k \notin \{0,1\}$, the supports of $\varphi_p(2\cdot -k)$ and φ_p are disjoint, hence (5) follows. Combining (2) with (5), we compute for $0 \le q \le p$ and any $k \in \mathbb{Z}$ that

$$\big\langle \psi_p, \varphi_q(\cdot - k)\big\rangle_{L_2(\mathbb{R})} = \big\langle \varphi_p(2\cdot +2k) - \varphi_p(2\cdot +2k-1), \varphi_q\big\rangle_{L_2(\mathbb{R})} = 0,$$

which yields (6) and, in turn, the vanishing moment property $\psi_p \perp \mathbb{P}_p$. Finally, (7) follows from (4) after inserting the definition of ψ_p,

$$\begin{aligned}
\big\langle \psi_p, \psi_q(\cdot - k)\big\rangle_{L_2(\mathbb{R})} &= \big\langle \varphi_p(2\cdot) - \varphi_p(2\cdot -1), \varphi_q(2\cdot -2k) - \varphi_q(2\cdot -2k-1)\big\rangle_{L_2(\mathbb{R})}\\
&= \big\langle \varphi_p(2\cdot), \varphi_q(2\cdot -2k) - \varphi_q(2\cdot -2k-1)\big\rangle_{L_2(\mathbb{R})}\\
&\quad - \big\langle \varphi_p(2\cdot -1), \varphi_q(2\cdot -2k) - \varphi_q(2\cdot -2k-1)\big\rangle_{L_2(\mathbb{R})}\\
&= \begin{cases} 1, & k=0 \text{ and } p=q,\\ 0, & \text{otherwise}\end{cases}, \quad 0 \le q \le p, \quad k \in \mathbb{Z}.
\end{aligned}$$

□

As concerns the orthogonality properties between dilated quarks and quarklets, the situation is more complicated because low-degree, fine-scale quark(let)s are in general *not* orthogonal to high-degree, low-scale quark(lets). In the following lemma, we collect some positive results on orthogonality between individual elements of the quarklet system Ψ.

Lemma 2. *Let $p, p', j, j' \ge 0$ and $k \in \mathbb{Z}$. Then the following implications hold true:*

$$p' \ge p \quad \Rightarrow \quad \langle \varphi_p, \psi_{p',j',k'}\rangle_{L_2(\mathbb{R})} = 0, \tag{8}$$

$$\left.\begin{aligned} p' \ge p \wedge j' > j\\ \textit{or } p' > p \wedge j' \ge j\\ \textit{or } \operatorname{meas}(2^{-j}[k,k+1] \cap 2^{-j}[k',k'+1]) = 0\end{aligned}\right\} \quad \Rightarrow \quad \langle \psi_{p,j,k}, \psi_{p',j',k'}\rangle_{L_2(\mathbb{R})} = 0. \tag{9}$$

Proof. If $p' \ge p$, then φ_p is a polynomial of degree at most $p \le p'$ on the support of $\psi_{p',j',k'}$, so that (6) yields $\langle \varphi_p, \psi_{p',j',k'}\rangle_{L_2(\mathbb{R})} = 0$, showing (8).

Similarly, if $j' > j$ and $p' \ge p$, then $\psi_{p,j,k}$ is a polynomial of degree at most $p \le p'$ on the support of $\psi_{p',j',k'}$, so that (6) yields $\langle \psi_{p,j,k}, \psi_{p',j',k'}\rangle_{L_2(\mathbb{R})} = 0$, showing the first implication in (9).

If $p' > p$ and $j' \ge j$, we can insert the definition (3) of the quarklets $\psi_{p,j,k}$ and obtain

$$\begin{aligned}
&\langle \psi_{p,j,k}, \psi_{p',j',k'}\rangle_{L_2(\mathbb{R})}\\
&= 2^{(j+j')/2}\big\langle \varphi_p(2^{j+1}\cdot -2k) - \varphi_p(2^{j+1}\cdot -2k-1), \varphi_{p'}(2^{j'+1}\cdot -2k') - \varphi_{p'}(2^{j'+1}\cdot -2k'-1)\big\rangle_{L_2(\mathbb{R})}\\
&= 2^{(j-j')/2-1}\big\langle \varphi_p(2^{j-j'}\cdot -2k) - \varphi_p(2^{j-j'}\cdot -2k-1), \varphi_{p'}(\cdot - 2k') - \varphi_{p'}(\cdot - 2k'-1)\big\rangle_{L_2(\mathbb{R})}.
\end{aligned}$$

In view of $p < p'$ and $j \le j'$, the function $\varphi_p(2^{j-j'}\cdot -2k) - \varphi_p(2^{j-j'}\cdot -2k-1)$ is a polynomial of degree at most p on the respective supports of $\varphi_{p'}(\cdot - 2k')$ and $\varphi_{p'}(\cdot - 2k'-1)$, so that $\langle \psi_{p,j,k}, \psi_{p',j',k'}\rangle_{L_2(\mathbb{R})}$ vanishes, showing the second implication in (8).

Finally, if $\operatorname{meas}(2^{-j}[k,k+1] \cap 2^{-j'}[k',k'+1]) = 0$, the support sets of $\psi_{p,j,k}$ and $\psi_{p',j',k'}$ do at most touch, so that $\langle \psi_{p,j,k}, \psi_{p',j',k'}\rangle_{L_2(\mathbb{R})} = 0$, and the proof is complete. □