

Orthonormal subsets of the Legendre-Haar quarklet system

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Abstract

The Legendre-Haar quarklet system is a set of piecewise polynomial and globally discontinuous functions on the real line with cancellation properties. As we show in this paper, the Legendre-Haar quarklet system not only contains the Haar wavelet basis and the L_2 -normalized truncated Legendre polynomials as orthonormal subsystems, but also a variety of other orthonormal bases which are piecewise polynomial with respect to a nonuniform partition of \mathbb{R} into dyadic subintervals.

1 Orthogonality properties of Legendre-Haar quarklets

1.1 Legendre-Haar quarklets

Let us first define the Legendre-Haar quarklet system. To this end, for $p \in \mathbb{N}_0$, let L_p be the p -th orthogonal Legendre polynomial on $[-1, 1]$ with normalization $L_p(1) = 1$. It is well-known that

$$\int_0^1 L_p(x)L_q(x) dx = \begin{cases} \frac{2}{2p+1}, & p = q, \\ 0, & p \neq q, \end{cases}$$

so that the truncated and renormalized Legendre polynomials

$$\varphi_p(x) := \sqrt{2p+1}L_p(2x-1)\chi_{[0,1)}(x), \quad p \in \mathbb{N}_0, \quad x \in \mathbb{R}, \quad (1)$$

induce the orthonormal basis

$$\{\varphi_p(\cdot - k) : p \in \mathbb{N}_0, k \in \mathbb{Z}\}$$

for $L_2(\mathbb{R})$. Let us call the functions φ_p *Legendre-Haar quarks*, which is motivated by the combination of Legendre polynomials L_p and the Haar scaling function $\chi_{[0,1)}$.

Furthermore, let us define the *Legendre-Haar quarklets*

$$\psi_p(x) := \varphi_p(2x) - \varphi_p(2x-1), \quad p \in \mathbb{N}_0, \quad x \in \mathbb{R}, \quad (2)$$

which is a generalization of the two-scale relation $\psi_0(x) = \varphi_0(2x) - \varphi_0(2x-1)$ of Haar wavelets. The *Legendre-Haar quarklet system* is then given as the set

$$\Psi := \{\varphi_p(\cdot - k) : p \in \mathbb{N}_0, k \in \mathbb{Z}\} \cup \{\psi_{p,j,k} := 2^{j/2}\psi_p(2^j \cdot -k) : p, j, k \in \mathbb{N}_0, k \in \mathbb{Z}\} \quad (3)$$

of $L_2(\mathbb{R})$ -normalized functions. Note that Ψ contains the orthonormal Haar wavelet basis

$$\{\varphi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi_{0,j,k} = 2^{j/2}\psi_0(2^j \cdot -k) : j \in \mathbb{N}_0, k \in \mathbb{Z}\}$$

as well as the orthonormal basis of truncated and shifted Legendre polynomials

$$\{\varphi_p(\cdot - k) : p \in \mathbb{N}_0, k \in \mathbb{Z}\}.$$

1.2 Orthogonality properties between quarks and quarklets

We will now collect the orthogonality properties of various subsets of the Legendre-Haar quarklet system Ψ from (3). The following lemma covers the case $j = 0$.

Lemma 1. *We have*

$$\langle \varphi_p, \varphi_q(\cdot - k) \rangle_{L_2(\mathbb{R})} = \begin{cases} 1, & k = 0 \text{ and } p = q \\ 0, & \text{otherwise} \end{cases}, \quad p, q \in \mathbb{N}_0, \quad k \in \mathbb{Z}. \quad (4)$$

Moreover, we have

$$\langle \varphi_p(2 \cdot -k), \varphi_p \rangle_{L_2(\mathbb{R})} = \begin{cases} 2^{-(p+1)}, & k \in \{0, 1\} \\ 0, & \text{otherwise} \end{cases}, \quad p \in \mathbb{N}_0, \quad k \in \mathbb{Z}. \quad (5)$$

By consequence, Legendre-Haar quarklets are orthogonal to Legendre-Haar quarks of lower degree,

$$\langle \psi_p, \varphi_q(\cdot - k) \rangle_{L_2(\mathbb{R})} = 0, \quad \text{for all } 0 \leq q \leq p, \quad k \in \mathbb{Z}. \quad (6)$$

In particular, each ψ_p has $p + 1$ vanishing moments, $\langle \psi_p, r \rangle_{L_2(\mathbb{R})} = 0$ for all $r \in \mathbb{P}_p$. Moreover, Legendre-Haar quarklets are orthogonal to Legendre-Haar quarklets of strictly lower degree,

$$\langle \psi_p, \psi_q(\cdot - k) \rangle_{L_2(\mathbb{R})} = 0, \quad \text{for all } 0 \leq q < p, \quad k \in \mathbb{Z}. \quad (7)$$

Proof. (4) follows from $\text{supp } \varphi_p = [0, 1]$ and from the orthogonality of the Legendre polynomials. As concerns (5), let $k \in \{0, 1\}$. On $\text{supp } \varphi_p(2 \cdot -k) = [\frac{k}{2}, \frac{k+1}{2}]$, φ_p is a polynomial of degree i the leading coefficient of which coincides with 2^{-p} times the leading coefficient of $\varphi_p(2 \cdot -k)$. Since $\varphi_p(2 \cdot -k)$ is orthogonal to all polynomials of degree $p - 1$ on its support, it follows

$$\langle \varphi_p(2 \cdot -k), \varphi_p \rangle_{L_2(\mathbb{R})} = 2^{-p} \langle \varphi_p(2 \cdot -k), 2^p \varphi_p \rangle_{L_2(\mathbb{R})} = 2^{-p} \|\varphi_p(2 \cdot -k)\|_{L_2(\mathbb{R})}^2 = 2^{-(p+1)}.$$

If $k \notin \{0, 1\}$, the supports of $\varphi_p(2 \cdot -k)$ and φ_p are disjoint, hence (5) follows. Combining (2) with (5), we compute for $0 \leq q \leq p$ and any $k \in \mathbb{Z}$ that

$$\langle \psi_p, \varphi_q(\cdot - k) \rangle_{L_2(\mathbb{R})} = \langle \varphi_p(2 \cdot + 2k) - \varphi_p(2 \cdot + 2k - 1), \varphi_q \rangle_{L_2(\mathbb{R})} = 0,$$

which yields (6) and, in turn, the vanishing moment property $\psi_p \perp \mathbb{P}_p$. Finally, (7) follows from (4) after inserting the definition of ψ_p ,

$$\begin{aligned} \langle \psi_p, \psi_q(\cdot - k) \rangle_{L_2(\mathbb{R})} &= \langle \varphi_p(2 \cdot) - \varphi_p(2 \cdot - 1), \varphi_q(2 \cdot - 2k) - \varphi_q(2 \cdot - 2k - 1) \rangle_{L_2(\mathbb{R})} \\ &= 0, \quad 0 \leq q < p, \quad k \in \mathbb{Z}. \end{aligned}$$

□