Orthonormal subsets of the Legendre-Haar quarklet system

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Abstract

The Legendre-Haar quarklet system is a set of piecewise polynomial and globally discontinous functions on the real line with cancellation properties. As we show in this paper, the Legendre-Haar quarklet system not only contains the Haar wavelet basis and the L_2-normalized truncated Legendre polynomials as orthonormal subsystems, but also a variety of other orthonormal bases which are piecewise polynomial with respect to a nonuniform partition of $\mathbb{R}$ into dyadic subintervals.

1 Orthogonality properties of Legendre-Haar quarklets

1.1 Legendre-Haar quarklets

Let us first define the Legendre-Haar quarklet system. To this end, for $p \in \mathbb{N}_0$, let L_p be the p-th orthogonal Legendre polynomial on $[-1, 1]$ with normalization $L_p(1) = 1$. It is well-known that

$$\int_0^1 L_p(x)L_q(x)\,\mathrm{d}x = \begin{cases} \frac{2}{2p+1}, & p = q, \\ 0, & p \neq q, \end{cases}$$

so that the truncated and renormalized Legendre polynomials

$$\varphi_p(x) := \sqrt{2p+1}L_p(2x-1)\chi_{[0,1)}(x), \quad p \in \mathbb{N}_0, \quad x \in \mathbb{R}, \tag{1}$$

induce the orthonormal basis

$$\big\{\varphi_p(\cdot - k) : p \in \mathbb{N}_0, k \in \mathbb{Z}\big\}$$

for $L_2(\mathbb{R})$. Let us call the functions φ_p *Legendre-Haar quarks*, which is motivated by the combination of Legendre polynomials L_p and the Haar scaling function $\chi_{[0,1)}$.

Furthermore, let us define the *Legendre-Haar quarklets*

$$\psi_p(x) := \varphi_p(2x) - \varphi_p(2x-1), \quad p \in \mathbb{N}_0, \quad x \in \mathbb{R}, \tag{2}$$

which is a generalization of the two-scale relation $\psi_0(x) = \varphi_0(2x) - \varphi_0(2x-1)$ of Haar wavelets. The *Legendre-Haar quarklet system* is then given as the set

$$\Psi := \big\{\varphi_p(\cdot - k) : p \in \mathbb{N}_0, k \in \mathbb{Z}\big\} \cup \big\{\psi_{p,j,k} := 2^{j/2}\psi_p(2^j \cdot - k) : p, j, \in \mathbb{N}_0, k \in \mathbb{Z}\big\} \tag{3}$$

of $L_2(\mathbb{R})$-normalized functions. Note that Ψ contains the orthonormal Haar wavelet basis

$$\big\{\varphi_0(\cdot - k) : k \in \mathbb{Z}\big\} \cup \big\{\psi_{0,j,k} = 2^{j/2}\psi_0(2^j \cdot - k) : j \in \mathbb{N}_0, k \in \mathbb{Z}\big\}$$

as well as the orthonormal basis of truncated and shifted Legendre polynomials

$$\big\{\varphi_p(\cdot - k) : p \in \mathbb{N}_0, k \in \mathbb{Z}\big\}.$$

1.2 Orthogonality properties between quarks and quarklets

We will now collect the orthogonality properties of various subsets of the Legendre-Haar quarklet system Ψ from (3). The following lemma covers the case $j = 0$.

Lemma 1. *We have*

$$\big\langle \varphi_p, \varphi_q(\cdot - k)\big\rangle_{L_2(\mathbb{R})} = \begin{cases} 1, & k = 0 \text{ and } p = q \\ 0, & \text{otherwise} \end{cases}, \quad p, q \in \mathbb{N}_0, \quad k \in \mathbb{Z}. \tag{4}$$

Moreover, we have

$$\big\langle \varphi_p(2 \cdot -k), \varphi_p\big\rangle_{L_2(\mathbb{R})} = \begin{cases} 2^{-(p+1)}, & k \in \{0, 1\} \\ 0, & \text{otherwise} \end{cases}, \quad p \in \mathbb{N}_0, \quad k \in \mathbb{Z}. \tag{5}$$

By consequence, Legendre-Haar quarklets are orthogonal to Legendre-Haar quarks of lower degree,

$$\big\langle \psi_p, \varphi_q(\cdot - k)\big\rangle_{L_2(\mathbb{R})} = 0, \quad \text{for all } 0 \leq q \leq p, \quad k \in \mathbb{Z}. \tag{6}$$

In particular, each ψ_p has $p + 1$ vanishing moments, $\langle \psi_p, r\rangle_{L_2(\mathbb{R})} = 0$ for all $r \in \mathbb{P}_p$. Moreover, Legendre-Haar quarklets are orthogonal to Legendre-Haar quarklets of strictly lower degree,

$$\big\langle \psi_p, \psi_q(\cdot - k)\big\rangle_{L_2(\mathbb{R})} = \begin{cases} 1, & k = 0 \text{ and } p = q, \\ 0, & \text{otherwise} \end{cases}, \quad \text{for all } 0 \leq q \leq p, \quad k \in \mathbb{Z}. \tag{7}$$

Proof. (4) follows from $\operatorname{supp}\varphi_p = [0, 1]$ and from the orthogonality of the Legendre polynomials. As concerns (5), let $k \in \{0, 1\}$. On $\operatorname{supp}\varphi_p(2 \cdot -k) = [\frac{k}{2}, \frac{k+1}{2}]$, φ_p is a polynomial of degree p the leading coefficient of which coincides with 2^{-p} times the leading coefficient of $\varphi_p(2 \cdot -k)$. Since $\varphi_p(2 \cdot -k)$ is orthogonal to all polynomials of degree $p - 1$ on its support, it follows

$$\big\langle \varphi_p(2 \cdot -k), \varphi_p\big\rangle_{L_2(\mathbb{R})} = 2^{-p}\big\langle \varphi_p(2 \cdot -k), 2^p\varphi_p\big\rangle_{L_2(\mathbb{R})} = 2^{-p}\big\|\varphi_p(2 \cdot -k)\big\|_{L_2(\mathbb{R})}^2 = 2^{-(p+1)}.$$

If $k \notin \{0, 1\}$, the supports of $\varphi_p(2 \cdot -k)$ and φ_p are disjoint, hence (5) follows. Combining (2) with (5), we compute for $0 \leq q \leq p$ and any $k \in \mathbb{Z}$ that

$$\big\langle \psi_p, \varphi_q(\cdot - k)\big\rangle_{L_2(\mathbb{R})} = \big\langle \varphi_p(2 \cdot +2k) - \varphi_p(2 \cdot +2k - 1), \varphi_q\big\rangle_{L_2(\mathbb{R})} = 0,$$

which yields (6) and, in turn, the vanishing moment property $\psi_p \perp \mathbb{P}_p$. Finally, (7) follows from (4) after inserting the definition of ψ_p,

$$\big\langle \psi_p, \psi_q(\cdot - k)\big\rangle_{L_2(\mathbb{R})} = \big\langle \varphi_p(2\cdot) - \varphi_p(2 \cdot -1), \varphi_q(2 \cdot -2k) - \varphi_q(2 \cdot -2k - 1)\big\rangle_{L_2(\mathbb{R})}$$

$$= \begin{cases} 1, & k = 0 \text{ and } p = q, \\ 0, & \text{otherwise} \end{cases}, \quad 0 \leq q \leq p, \quad k \in \mathbb{Z}.$$

$\square$

As concerns the orthogonality properties between dilated quarks and quarklets, the situation is more complicated.

Lemma 2. *We have*

$$\big\langle \varphi_{p,j,k}, \varphi_{p',j',k'}\big\rangle_{L_2(\mathbb{R})} = \begin{cases} 2^{(j-j')(p+1/2)}, & 2^{-j'}[k', k' + 1] \subseteq 2^{-j}[k, k + 1], \quad p' = p, \\ 2^{(j'-j)(p+1/2)}, & 2^{-j}[k, k + 1] \subseteq 2^{-j'}[k', k' + 1], \quad p' = p, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Proof. supp $\varphi_{p,j,k} = 2^{-j}[k, k+1]$ is an interval of length 2^{-j}. Therefore, the support sets of two different quarks $\varphi_{p,j,k}$ and $\varphi_{p',j',k'}$ either intersect on a set of Lebesgue measure zero, so that the inner product between the two quarks vanishes, or one of the two support sets is contained in the other.

Without loss of generality, assume that $2^{-j'}[k', k'+1] \subseteq 2^{-j}[k, k+1]$ and, hence, $j \leq j'$. Then $\varphi_{p,j,k}$ is a polynomial of degree p on the support of $\varphi_{p',j',k'}$, so that the inner product between $\varphi_{p,j,k}$ and $\varphi_{p',j',k'}$ vanishes if $p' \neq p$. If $p' = p$, we can proceed just as in the proof of Lemma 1: the leading coefficient of $\varphi_{p,j,k} = 2^{j/2}\varphi_p(2^j \cdot -j)$ coincides with $2^{(j-j')(p+1/2)}$ times the leading coefficient of $\varphi_{p,j',k'} = 2^{j'/2}\varphi_p(2^{j'} \cdot -k')$, so that

$$
\begin{aligned}
\langle \varphi_{p,j,k}, \varphi_{p,j',k'} \rangle_{L_2(\mathbb{R})} &= 2^{(j-j')(p+1/2)} \langle 2^{(j'-j)(p+1/2)} \varphi_{p,j,k}, \varphi_{p',j',k'} \rangle_{L_2(\mathbb{R})} \\
&= 2^{(j-j')(p+1/2)} \|\varphi_{p,j',k'}\|_{L_2(\mathbb{R})}^2 \\
&= 2^{(j-j')(p+1/2)}.
\end{aligned}
$$

$\square$