Derivation of the Mixed-Radix Decimation-in-Time Fast Fourier Transform

Start with the definition of the Discrete Fourier Transform *F* for the time sequence *f*. The length of *f* is a composite number N=mp.

$$F_N(k,f)=\sum_{n=0}^{N-1} f(n)e^{-j\frac{2\pi nk}{N}}$$

Reorder the summation as a double summation.

$$F_{mp}(k,f)=\sum_{q=0}^{m-1}\sum_{r=0}^{p-1} f(qp+r)e^{-j\frac{2\pi k(qp+r)}{N}}$$

Split the complex exponential into two parts.

$$F_{mp}(k,f)=\sum_{q=0}^{m-1}\sum_{r=0}^{p-1} f(qp+r)e^{-j\frac{2\pi kqp}{N}}e^{-j\frac{2\pi kr}{N}}$$

Pull one of the exponents out of the inner summation.

$$F_{mp}(k,f)=\sum_{r=0}^{p-1} e^{-j\frac{2\pi kr}{N}}\sum_{q=0}^{m-1} f(qp+r)e^{-j\frac{2\pi kqp}{N}}$$

Factor out p from the inner exponent. Notice how similar the inner summation looks to the original definition above.

$$F_{mp}(k,f)=\sum_{r=0}^{p-1} e^{-j\frac{2\pi kr}{N}}\sum_{q=0}^{m-1} f(qp+r)e^{-j\frac{2\pi qk}{m}}$$

Observe the time sequence *f(qp+r)* is a decimation of the original sequence. This is where the term *Decimation in Time (DIT)* originates.

Let $\hat{f}_r$ be the time-decimated sequence, such that $\hat{f}_r(q)=f(qp+r)$.

The symmetry of complex exponentials tells us that

$$F_N(k,f)=F_N(k\,mod\,N,f)$$

In this way, the DFT of size *mp* can be constructed from *p* DFTs of size *m*.

$$F_{mp}(k,f)=\sum_{r=0}^{p-1} e^{-j\frac{2\pi kr}{N}} F_m(k\,mod\,m,\hat{f}_r)$$