Unit 3: Transportation and Assignment Problem (8 Hrs.)

Transportation Problems definition, linear form, Solution methods: North West comer method, least cost method, Vogel's approximation method. Degeneracy in transportation, Modified Distribution method, unbalanced problems and profit maximization problems. Transshipment Problems. Assignment Problem Structure and Solution: Short-Cut Method (Hungarian Method), Unbalanced Assignment Problem, Infeasible Assignment Problem, Maximization in an Assignment Problem, Crew Assignment Problem.

#Past Questions

1. Find the optimal solution for the following transportation problem using any methods. -2021

1	2	1	4	30
3	3	2	1	50
4	2	5	9	20
20	40	30	40	100

2. Write Hungarian algorithm to solve the assignment problem. -2021/2023
3. Given the transportation framework, find the optimal transportation cost. – 2023

From↓ | To→	A	B	C	Plant Capacity
W	40	80	80	55
X	160	-	160	25
Y	80	160	240	35
Requirement	35	45	35	115

4. A city corporation has decided to carry out Maintenance work in the city. There are 4 tasks to be completed using 5 contractors . Which contractors should be assigned those tasks? Solve the assignment problem to get best result. -2023 long

Contractor	Task 1	Task 2	Task 3	Task 4
A	19	24	29	25
B	17	27	30	29
C	19	28	31	28
D	20	12	28	29
E	20	25	31	26

5. Carew's machine shop has 4 machines on which 3 jobs have to be done. Each job can be assigned to one and only one machine. The cost(in Rs) of each job on each machine is given below:

	Machines			
Job	**P**	**Q**	**R**	**S**
A	45	60	70	80
B	20	32	42	47
C	25	37	47	55

Req: What are the job assignments which will minimize total cost?

6. Write short notes on (any two)
   a. Meaning of Degeneracy in Transportation Problem.

b. Assignment Problem Algorithm

7. 11. Jack Evan owns several trucks used to haul crushed stone to road project iun the country. The road contractor for whom jack hauls, N Teer, has given jack this schedule for next week:

Project	Requirement per week	Plant	Available per week
A	50	W	45
B	75	X	60
C	50	Y	60

Jack figures his cost from the crushing plant to each of the road projects to be these:

Cost information (in RS)

To /From	A	B	C
W	4	8	3
X	6	7	9
Y	8	2	5

Req: Compute Jack's optimal hauling schedule for next week and his transportation cost.

1. Transportation Problems

A **Transportation Problem** involves transporting goods from multiple **sources** (e.g., factories) to multiple **destinations** (e.g., warehouses) in such a way that:

- The **total transportation cost is minimized** (or profit is maximized),
- Supply and demand constraints are satisfied.

Mathematical (Linear) Formulation:

Let:

- m = number of sources
- n = number of destinations
- c_{ij} = cost of transporting one unit from source i to destination j
- x_{ij} = quantity transported from source i to destination j
- a_i = supply at source i
- b_j = demand at destination j

Objective function (minimize cost):

$$\text{Minimize } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Subject to:

- $\sum_{j=1}^{n} x_{ij} = a_i$ (supply constraints)
- $\sum_{i=1}^{m} x_{ij} = b_j$ (demand constraints)
- $x_{ij} \geq 0$

The transportation problem can be stated in the following tabular form:

		Destinations				
	1	2	3	…	n	supply
1	(x_{11}) C_{11}	(x_{12}) C_{12}	(x_{13}) C_{13}	…	(x_{1n}) C_{1n}	a_1
2	(x_{21}) C_{21}	(x_{22}) C_{22}	(x_{23}) C_{23}	…	(x_{2n}) C_{2n}	a_2
Origins ⋮		⋮	⋮	⋮	⋮	⋮
m	(x_{m1}) C_{m1}	(x_{m2}) C_{m2}	(x_{m3}) C_{m3}	…	(x_{mn}) C_{mn}	a_m
demand	b_1	b_2	b_3	…	b_n	

Example 1: A company has three production factories *S*1, *S*2 and *S*3 with production capacity of 7, 9 and 18 units

(in 100s) per week of a product, respectively. These units are to be shipped to four warehouses *D*1, *D*2, *D*3 and *D*4 with requirement of 5, 8, 7 and 14 units (in 100s) per week, respectively. The transportation costs (in rupees) per unit between factories to warehouses are given in the table below:

	D1	D2	D3	D4	Supply (availability)
*S*1	19	30	50	10	7
*S*2	70	30	40	60	9
*S*3	40	8	70	20	18
Demand (requirement)	5	8	7	14	34

Formulate this transportation problem as an LP model to minimize the total transportation cost.

Solution: Let xij = number of units of the product to be transported from a production factory i (i = 1, 2, 3) to a warehouse j (j = 1, 2, 3, 4)

The transportation problem is stated as an LP model as follows:

Minimize (total transportation cost) $Z = 19\,x11 + 30\,x12 + 50\,x13 + 10\,x14 + 70\,x21 + 30\,x22 + 40\,x23 + 60\,x24 + 40\,x31 + 8\,x32 + 70\,x33 + 20\,x34$

subject to the constraints

$$\left.\begin{aligned} x11 + x12 + x13 + x14 &= 7 \\ x21 + x22 + x23 + x24 &= 9 \\ x31 + x32 + x33 + x34 &= 18 \end{aligned}\right] \qquad \text{(Supply)}$$

$$\left.\begin{aligned} x11 + x21 + x31 &= 5 \\ x12 + x22 + x32 &= 8 \\ x13 + x23 + x33 &= 7 \\ x14 + x24 + x34 &= 14 \end{aligned}\right] \qquad \text{(Demand)}$$

and $xij \geq 0$ for $i = 1, 2, 3$ and $j = 1, 2, 3$, and 4.

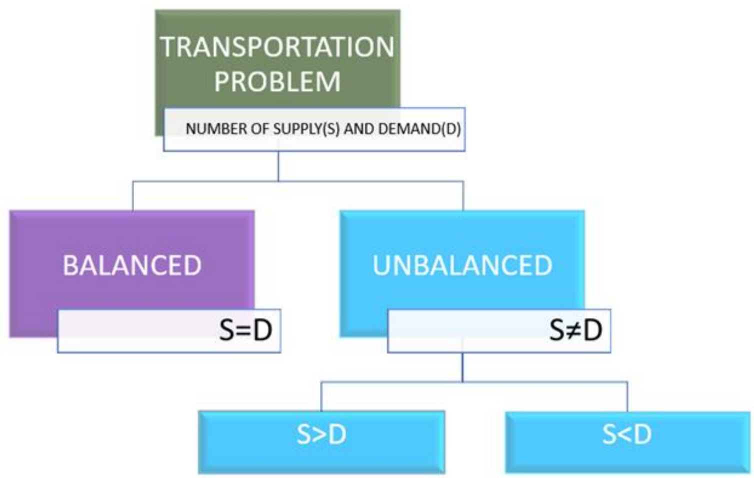
In the above LP model, there are $m \times n = 3 \times 4 = 12$ decision variables, xij and $m + n = 7$ constraints, where m are the number of rows and n are the number of columns in a general transportation table.

There are only ($m + n -1$) basic variable out of $m \times n$

Therefore, BFS (Basic Feasible Solution) of Transportation Problem consist of at most ($m + n -1$) positive variables other being zero.

Types of Transportation Problems:

- **Balanced:** Total supply equals total demand.
- **Unbalanced:** Total supply does not equal total demand. In this case, a dummy source or destination is added to balance the problem.



2. Solution Methods for Transportation Problem

A transportation problem can be solved by using two most familiar methods:

a. Simplex Method
b. Transportation Method

But our interest is in only transportation method i.e. we will discuss only transportation method.

Transportation Algorithm

The steps of the transportation algorithm are exact parallels of the simplex algorithm, they are:

Step 1: Determine a starting basic feasible solution, using any one of the following three methods

1. North West Corner Method
2. Least Cost Method
3. Vogel Approximation Method

Step 2: Determine the optimal solution using the following method

1. MODI (Modified Distribution Method) or UV Method.
2. Stepping Stone Method

Basic Feasible Solution of a Transportation Problem

a. North West Corner Method (NWCM):

<u>Algorithm:</u>

1. Start in cell $(1,1)$ (north-west).
2. Allocate $x_{11} = \min(s_1, d_1)$.
3. Subtract that from the corresponding supply or demand (whichever hits zero "exhausts" first).
4. If s_1 becomes zero, move down to cell $(2,1)$; if d_1 becomes zero, move right to $(1,2)$.
5. Repeat until all supplies/demands are allocated.

Question: Use North-West Corner Method (NWCM) to find an initial basic feasible solution to the transportation problem using data of **Example 1.**

Solution: The cell (1,1) is the north-west corner cell in the given transportation table. The min values for row S1 and column D1 are compared i.e. min(s1,d1). The smaller of the two (5,7), i.e. 5, is assigned as the first allocation

	D1	D2	D3	D4	Supply
S1	19 5	30	50	10	~~7~~ 2
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	~~5~~	8	7	14	34

	D2	D3	D4	Supply
S1	30 2	50	10	~~2~~
S2	30	40	60	9
S3	8	70	20	18
Demand	~~8~~ 6	7	14	

	D2	D3	D4	Supply
S2	30 6	40	60	~~9~~ 3
S3	8	70	20	18
Demand	~~6~~	7	14	

	D3	D4	Supply
S2	40 3	60	~~3~~
S3	70	20	18
Demand	~~7~~ 4	14	

	D3	D4	Supply
S3	70 4	20 14	~~18~~ ~~14~~
Demand	~~4~~	~~14~~	

Once the procedure is over, count the number of positive allocations. These allocations (occupied cells) should be equal to $m + n - 1 = 3 + 4 - 1 = 6$. If yes, then solution is non-degenerate feasible solution, Otherwise degenerate solution.

The total transportation cost of the initial solution is obtained by multiplying the quantity *xij* in the occupied cells with the corresponding unit cost *cij* and adding all the values together. Thus, the total transportation cost of this solution is

Thus, Total cost = 5 × 19 + 2 × 30 + 6 × 30 + 3 × 40 + 4 × 70 + 14 × 20 = Rs 1,015

b. Least Cost Method (LCM):

- Choose the **cell with the least cost**.
- Allocate as much as possible.
- Cross out row/column when supply or demand becomes zero.
- Repeat for remaining cells.
- **More efficient than NWCM.**

c. Vogel's Approximation Method (VAM):

- Compute penalty for each row and column (difference between smallest and second smallest costs).
- Choose the row/column with **highest penalty**.
- Allocate to the cell with **least cost** in that row/column.
- Adjust supply and demand and repeat.
- **Gives better initial feasible solution, often closer to optimum.**

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3. Degeneracy in Transportation:

- A **basic feasible solution** in a transportation problem must have exactly m+n−1m + n - 1 allocations.
- If it has **less than m+n−1m + n - 1** allocations, it is **degenerate**.
- To resolve, add a very small quantity ε\varepsilon (epsilon) to an unoccupied cell to maintain feasibility.

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4. Modified Distribution Method (MODI or u-v Method):

- Used to **test optimality** of a solution.

- Calculates opportunity cost for unoccupied cells.
- If all opportunity costs Δij≥0\Delta_{ij} \geq 0, current solution is optimal.
- If any Δij<0\Delta_{ij} < 0, then a better solution exists. Make adjustments by forming **loops**.
- **Iteratively improve the solution** until optimal.

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5. Unbalanced Transportation Problem:

- When **total supply ≠ total demand**.
- Make it balanced by adding a **dummy row or column**:
    - Add dummy **destination** if supply > demand.
    - Add dummy **source** if demand > supply.
    - Assign **zero cost** to dummy cells.

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6. Profit Maximization Problems:

- If objective is to **maximize profit**, convert it into a **minimization problem**:
    - Subtract each profit value from the **maximum profit value**.
    - Solve using standard transportation method.

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7. Transshipment Problems:

- Involves **intermediate points** between sources and destinations.
- Goods can be routed through these transshipment points.
- Formulated as a transportation problem with additional nodes and constraints.
- Need to ensure **flow conservation** at each transshipment node.

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8. Assignment Problems

Definition:

- A **special case of transportation problem** where:

    - Number of sources = number of destinations.
    - Each task is assigned to one agent and vice versa.
- Objective: **Minimize total cost or maximize total profit**.

Structure:

- Represented as a **square matrix**.
- Each cell shows **cost/profit** of assigning agent to task.
- Decision variable xij=1x_{ij} = 1 if agent ii assigned to task jj, 0 otherwise.

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9. Solution Method

Hungarian Method (Short-Cut Method):

1. **Row reduction:** Subtract minimum value from each row.
2. **Column reduction:** Subtract minimum value from each column.
3. **Cover all zeros** with minimum number of horizontal/vertical lines.
4. If number of lines = order of matrix, **optimal assignment** found.
5. If not, adjust matrix:
    - Subtract smallest uncovered value from uncovered elements.
    - Add it to the elements covered twice.
    - Repeat step 3.

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10. Special Cases in Assignment Problems

a. Unbalanced Assignment Problem:

- Matrix is not square (e.g., more tasks than agents).
- Add **dummy row/column** with zero costs to make it square.

b. Infeasible Assignment Problem:

- Some assignments are not allowed (infinite or very large cost).
- Represent these with a **very high cost (M)**.

c. Maximization in Assignment Problem:

- Convert to **minimization** by subtracting each element from the **maximum value** in the matrix.

d. Crew Assignment Problem:

- Assign crew members (drivers, pilots, etc.) to tasks (flights, trips) based on cost/profit/time.
- Can be solved using the **Hungarian Method**.

✅ Example Use Cases:

- **Transportation:** Delivering goods from factories to warehouses.
- **Assignment:** Assigning jobs to workers or teachers to subjects.
- **Transshipment:** Multi-step delivery logistics with central depots.