UNIT 2: LINEAR PROGRAMMING PROBLEM (LPP) [10 Hrs]

Introduction to Linear Programming, Linear Programming Problem Formulation, Formulation with Different Types of Constraints, Graphical Analysis of Linear Programming, Graphical Linear Programming Solution, Multiple Optimal Solutions, Unbounded Solution, Infeasible Solution, Basics of Simplex Method, Simplex Method Computation, Simplex Method with More Than Two Variables, Primal and Dual Problems, Economic Interpretation

#Past Questions

- Write an algorithm to maximize the solution of LPP using Simplex method. – **2021**
- Write an algorithm to minimize the solution of LPP using Simplex method. -**2023**
- Write the meaning of duality. What are the major steps of formulating dual LPPs? Justify with example.-**2024**
- Write short notes on:
    - Duality theorem -2021
    - Importance and advantage of duality problem -2023
- What do you mean by mathematical formulation of LPP? A firm manufactures products A, B, C. Time to manufacture A is twice that of B and thrice that of C, in ratio 3:4:5. If the whole raw material is engaged in manufacturing product A, 1600 units of this product can be produced. There is a demand at least 300, 250, 200 units of product A,B and C and profit earned per unit: Rs.(50, 40, 70) respectively. Formulate the following LPP: **-2021**

Raw Material	Req per unit of product (kg)			Total Availability (kg)
	A	B	C	
P	6	5	9	5000
Q	4	7	6	6000

- A chair manufacturing company produces two types of chairs ,A and B, by using three machines X,Y and Z. The time required for each chair on each machine and max time available are given below: **-2023**

Machine	Time per chair A	Time per chair B	Max time available (hrs/week)
X	6	8	120
Y	8	4	100
Z	12	4	144

Long Questions

- Use duality to solve the LPP: **-2021**
  Minimize z = 4x1 + 2x2 + 3x3
  Subject to:
  $2x1 + 4x3 \geq 5$
  $2x1 + 3x2 + x3 \geq 4$
  $x1, x2, x3 \geq 0$
- Solve the duality problem: **-2023**
  Minimize Z = 20x1 + 40x2
  Subject to:
  $36x1 + 6x2 \geq 600$
  $3x1 + 12x2 \geq 36$
  $20x1 + 10x2 \geq 100$
  $x1, x2 \geq 0$

- Solve by simplex: **-2024**

  Maximize $z = 5x1 - 3x2$

  Subject to:

  $2x1 + x2 \leq 5$

  $x1 + x2 \leq 4$

  $x1, x2 \geq 0$

1. Introduction to Linear Programming

→ **Linear Programming (LP)** is a mathematical technique to find the best possible outcome (such as maximum profit or minimum cost) in a given mathematical model, where the relationships are linear.

→ **Linear Programming (LP)** is a mathematical technique used for optimization (maximizing or minimizing) of a linear objective function, subject to linear constraints (equalities or inequalities).

→ **LP** is a mathematical technique to find the best optimal solution for problems involving the allocation of limited resources to meet specific objectives.

Key Features/Components:

1. Decision Variables: These are the unknowns that need to be determined to solve the problem. They represent the choices to be made. For example, in a production problem, decision variables might be the number of units of each product to manufacture.

2. Objective Function: This is a mathematical expression (usually linear) that represents the quantity to be optimized (either maximized or minimized). For example, the objective function might represent the total profit to be maximized or the total cost to be minimized.

3. Constraints: These are linear inequalities or equations that restrict the values of the decision variables. They represent limitations on resources, requirements, or other restrictions of the problem.

4. Non-negativity Restrictions: These are typically included to ensure that the decision variables have realistic values (e.g., you can't produce a negative number of items).

Summary:

- **Objective function**: Linear equation to maximize or minimize
- **Constraints**: Set of linear inequalities or equations
- **Decision variables**: Unknowns to be solved for
- **Non-negativity**: Variables cannot be negative

Applications:

- **Manufacturing**: Maximizing profit by deciding product mix
- **Transportation**: Minimizing shipping costs
- **Resource Allocation**: Efficient use of limited resources
- **Diet Problems**: Finding minimum-cost diet satisfying nutrition

- **Staff Scheduling**: Minimizing labor cost while covering all shifts
- **Blending Problems**: Optimizing mixture of raw materials (e.g., in oil refineries)
- **Network Flow**: Maximizing flow in transportation or communication networks
- **Finance**: Portfolio optimization to maximize return or minimize risk
- **Project Selection**: Choosing best projects under budget constraints

Standard form of LPP:

Maximize or Minimize:

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq / \geq / = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq / \geq / = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq / \geq / = b_m$$

and

$$x_1, x_2, \ldots, x_n \geq 0$$

2. Linear Programming Problem Formulation

Formulating a real-life situation into a mathematical LPP involves:

Steps:

1. Identify decision variables (unknowns to be determined)
2. Write the objective function to be maximized/minimized
3. Identify and formulate constraints
4. Include non-negativity restrictions

Example 1:

A furniture company produces tables (Rs. 40 profit each) and chairs (Rs. 25 profit each).

- A table requires 3 hours of carpentry, 4 units of wood
- A chair requires 2 hours of carpentry, 3 units of wood
- Available: 60 hours of carpentry, 72 units of wood

Formulation:

Let,

x = number of tables produced

y = number of chairs produced

Objective Function:

Maximize $Z = 40x + 25y$

Subject to Constraints:

$3x + 2y \leq 60$ (Carpentry hours)

$4x + 3y \leq 72$ (Wood)

$x, y \geq 0$

Example 2:

Suppose an industry is manufacturing two types of products P1 and P2. The profits per Kg of the two products are Rs.30 and Rs.40 respectively. These two products require processing in three types of machines. The following table shows the available machine hours per day and the time required on each machine to produce one Kg of P1 and P2. Formulate the problem in the form of linear programming model.

Profit/Kg	P1 Rs.30	P2 Rs.40	Total available Machine hours/day
Machine 1	3	2	600
Machine 2	3	5	800
Machine 3	5	6	1100

Solution:

Let,

$x1$ = amount of P1

$x2$ = amount of P2

In order to maximize profits, we establish the objective function as

$$Z = 30x1 + 40x2$$

Since one Kg of P1 requires 3 hours of processing time in machine 1 while the corresponding requirement of P2 is 2 hours. So, the first constraint can be expressed as

$$3x1 + 2x2 \leq 600$$

Similarly, corresponding to machine 2 and 3 the constraints are

$$3x1 + 5x2 \leq 800$$

$$5x1 + 6x2 \leq 1100$$

In addition to the above there is no negative production, which may be represented algebraically as

$$x1 \geq 0 \; ; x2 \geq 0$$

Thus, the product mix problem in the linear programming model is as follows:

Maximize

$Z = 30x1 + 40x2$

Subject to:

$3x1 + 2x2 \leq 600$

$3x1 + 5x2 \leq 800$

$5x1 + 6x2 \leq 1100$

And $x1 \geq 0$, $x2 \geq 0$

3. Formulation with Different Types of Constraints

Constraints in LPP can be of three types:

Type	Example	Meaning
≤ (Less than)	$2x + y \leq 100$	Resource limitation (Max available)
≥ (Greater than)	$x + 3y \geq 150$	Minimum requirement (Production)
= (Equality)	$x + y = 80$	Exact resource utilization

Note: Constraints reflect real-world limitations or requirements.

The constraints in the previous example 2 are of "less than or equal to" type. In this section we are going to discuss the linear programming problem with different constraints, which is illustrated in the following Example 3.

Example 3:

A company owns two flour mills viz. A and B, which have different production capacities for high, medium and low-quality flour. The company has entered a contract to supply flour to a firm every month with at least 8, 12 and 24 quintals of high, medium and low quality respectively. It costs the company Rs.2000 and Rs.1500 per day to run mill A and B respectively. On a day, Mill A produces 6, 2 and 4 quintals of high, medium and low-quality flour, Mill B produces 2, 4 and 12 quintals of high, medium and low-quality flour respectively. How many days per month should each mill be operated in order to meet the contract order most economically?

Solution:

Let us define x1 and x2 are the mills A and B. Here the objective is to minimize the cost of the machine runs and to satisfy the contract order. The linear programming problem is given by

Minimize

$Z = 2000x1 + 1500x2$

Subject to:

$6x1 + 2x2 \geq 8$

$2x1 + 4x2 \geq 12$

$4x1 + 12x2 \geq 24$

$x1 \geq 0,\ x2 \geq 0$

Example 4:

A firm manufactures three products A,B, C. Time to manufacture A is twice that for B and thrice that for C and to be produced in the ratio 3 : 4 :5. The relevant data is given the following table. If the whole raw material is engaged in manufacturing product A,1600 units of this product can be produced. There is demand for at least 300, 250, 200 units A, B and C and profit earned unit is RS 50, Rs. 40, R s. 70, respectively. Formulate the problem as LPP.

Raw Material	A (kg)	B (kg)	C (kg)	Total Available (kg)
P	6	5	9	5000
Q	4	6	8	6000

Solution:

Step 1: Define Decision Variables

Let, x_1 = amount of product **A** produced (units)
x_2 = amount of product **B** produced (units)
x_3 = amount of product **C** produced (units)

Step 2: Establish the Objective Function

In order to maximize profit, we set up the objective function as:

Maximize $Z = 50x_1 + 40x_2 + 70x_3$

Where: Profit per unit of A = Rs. 50, B = Rs. 40 and C = Rs. 70

Step 3: Formulate the Constraints

Based on the raw material requirement table:
$6x_1 + 5x_2 + 9x_3 \leq 5000$ *(Raw Material P constraint)*
$4x_1 + 6x_2 + 8x_3 \leq 6000$ *(Raw Material Q constraint)*

Step 4: Demand Constraints

The minimum required production of products is:
$x_1 \geq 300$ *(Product A demand)*
$x_2 \geq 250$ *(Product B demand)*
$x_3 \geq 200$ *(Product C demand)*

Step 5: Non-Negativity Constraints

Production cannot be negative, so:
$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$

Production Time Ratio Interpretation

Time for **A** = 2 × Time for B = 3 × Time for C

Production ratio: **A : B : C = 3 : 4 : 5**

But for LPP, time ratio isn't directly affecting constraints unless explicitly needed for machine hour constraints. In this case, it seems only raw material and demand constraints are to be considered.

Additional Information

If only A is produced, 1600 units can be produced with available resources.
But this info is already covered in raw material constraints, so it's **not an extra constraint**, it's just additional information.

<u>Final LPP Model:</u>

Maximize
$Z = 50x_1 + 40x_2 + 70x_3$

Subject to:
$6x_1 + 5x_2 + 9x_3 \leq 5000$
$4x_1 + 6x_2 + 8x_3 \leq 6000$
$x_1 \geq 300$
$x_2 \geq 250$
$x_3 \geq 200$
$x_1, x_2, x_3 \geq 0$

Example 5:

. A chair manufacturing company produces two types of chairs, A and B, by using three machines, X, Y and Z. The time required for each chair on each machine and the maximum time available on are given below?

Message. Machines.	Time required for each type.		Maximum time available per week in hrs.
	A	B	
X	6	8	120
Y	8	4	100
Z	12	4	144

The profit on the A and B are rupees 500 and rupees 300 respectively. What combination of pains should be produced to obtain maximum profit?

Solution:

Let,
x = number of chairs of type A produced per week
y = number of chairs of type B produced per week

Objective Function:
Maximize Profit: **Z = 500x + 300y**
Subject to Constraints:
$6x + 8y \leq 120$
$8x + 4y \leq 100$
$12x + 4y \leq 144$
$x \geq 0, y \geq 0$ (solve using graphical or simplex method to obtain max profit)

Methods to Solve LPP:

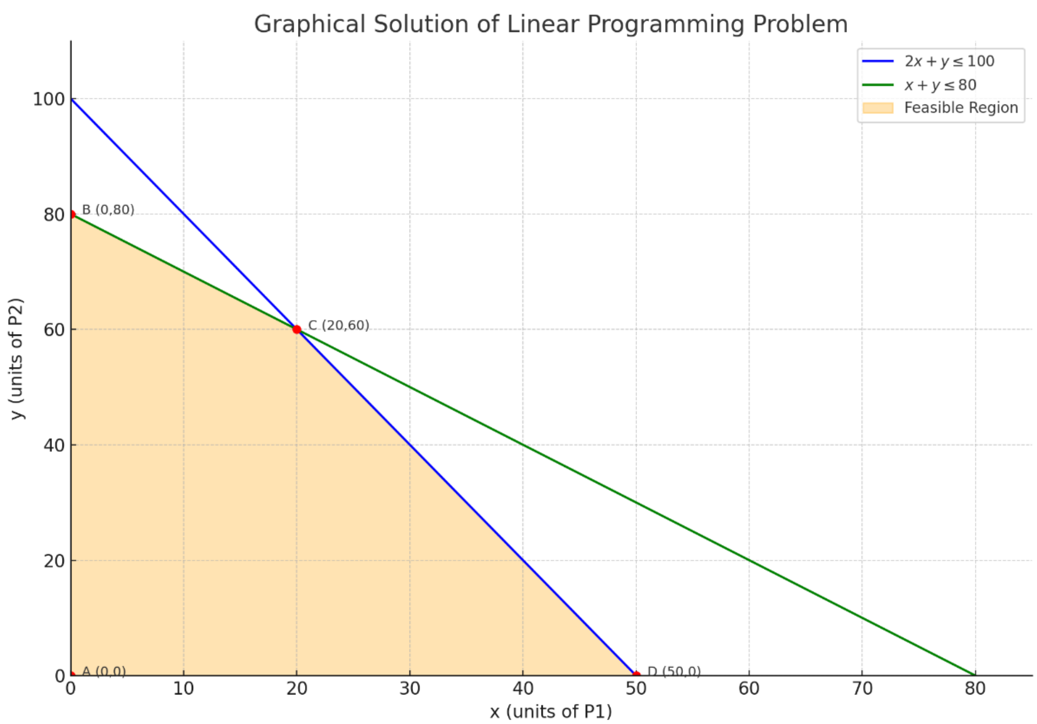
1. **Graphical Method** (for 2 variables only)
2. **Simplex Method** (for more than 2 variables)

4. Graphical Analysis of Linear Programming

Graphical Method is a visual way to solve LPP with **two decision variables**.

Steps:

- Plot each constraint as a straight line on a graph
- Shade feasible region satisfying all constraints
- Identify corner points (intersections and intercepts)
- Compute Z(objective function) at each corner points
- The point giving max/min objective value is the optimal solution



Here is the **graphical analysis** of the Linear Programming Problem:

- The two lines represent the constraints:
  - **Blue Line:** $2x + y \leq 100$
  - **Green Line:** $x + y \leq 80$
- The **shaded orange region** is the **feasible region** where all constraints are satisfied.
- The **red points** (A, B, C, D) are the **corner points** of the feasible region.

  These points are tested in the objective function **Z=40x+30y** to find the **maximum profit**.
- **Maximum profit occurs at point C (20, 60)** where **Z=2600.**

5. Graphical Linear Programming Solution

Example 1:Solve the given linear programming problems graphically:
Maximize: Z = 8x + y
Constraints are,

- **$x + y \leq 40$**
- **$2x + y \leq 60$**
- **$x \geq 0, y \geq 0$**

Solution:
Step 1: *The equation associated with above Constraints are,*

- $x + y = 40$ *...(i)*
- $2x + y = 60$ *...(ii)*

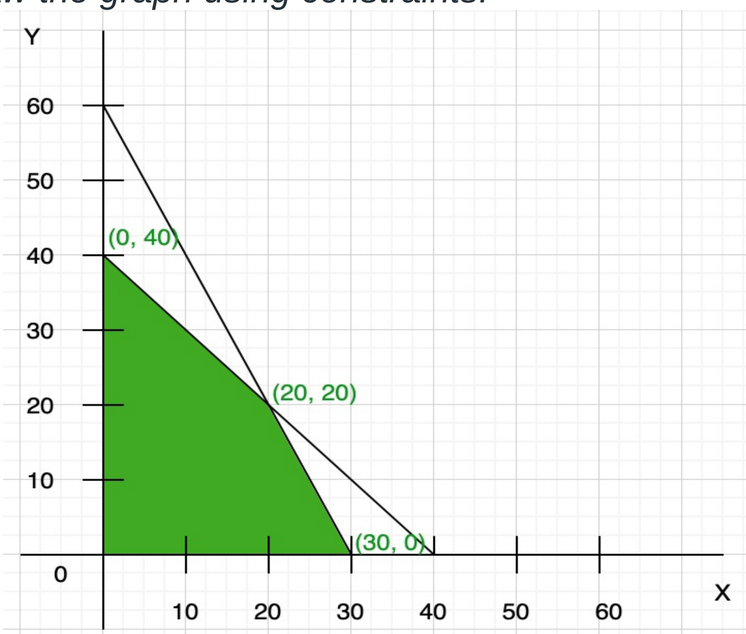
Form equation ---(i)

X	*0*	*40*
y	*40*	*0*

Form equation ---(ii)

X	*0*	*30*
y	*60*	*0*

Step 2: *Draw the graph using constraints.*



Here both the constraints are less than or equal to, so they satisfy the below region (towards origin). You can find the vertex of the feasible region by graph, or you can calculate using the given constraints:

- $x + y = 40$ *...(i)*
- $2x + y = 60$ *...(ii)*

Now solving eq(i) and (ii), we get
$x = 20$
Now, put the value of x in any of the equations, and we get
$y = 20$
So, the third point of the feasible region is (20, 20)
Step 3: *To find the maximum value of $Z = 8x + y$. Compare each intersection point of the graph to find the maximum value*

Points	Z = 8x + y
(0, 0)	0
(0, 40)	40
(20, 20)	180
(30, 0)	240

So, the maximum value of Z = 240 at point x = 30, y = 0.

Example 2: One kind of cake requires 200 g of flour and 25g of fat, and another kind of cake requires 100 g of flour and 50 g of fat Find the maximum number of cakes that can be made from 5 kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients, used in making the cakes.

Solution:

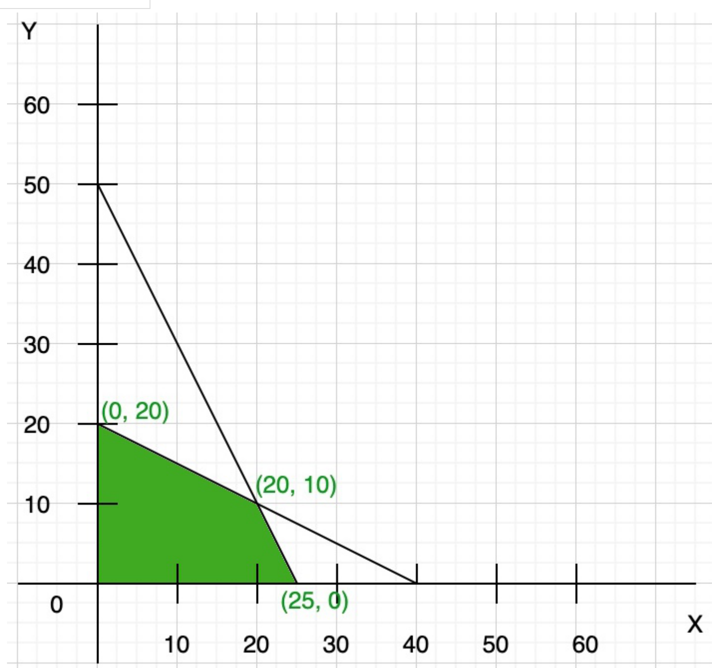
Step 1: *Create a table like this for easy understanding (not necessary).*

	Flour(g)	Fat(g)
Cake of first kind (x)	200	25
Cake of second kind (y)	100	50
Availability	5000	1000

Step 2: *Create a linear equation using inequality*

- *$200x + 100y \le 5000$ or $2x + y \le 50$*
- *$25x + 50y \le 1000$ or $x + 2y \le 40$*
- *Also, $x > 0$ and $y > 0$*

Step 3: *Create a graph using the inequality*



Step 4: *To find the maximum number of cakes (Z) = x + y. Compare each intersection point of the graph to find the maximum number of cakes that can be baked.*

x	y	Z = (x+y)
0	20	20
20	10	30
25	0	25

Z is maximum at coordinate (20, 10). So, the maximum number of cakes that can be baked is Z = 20 + 10 = 30.

Example 3: Solve the given linear programming problems graphically:
Minimize: Z = 20x + 10y
Constraints are,

- **$x + 2y \leq 40$**
- **$3x + y \geq 30$**
- **$4x + 3y \geq 60$**
- **$x \geq 0, y \geq 0$**

Solution:

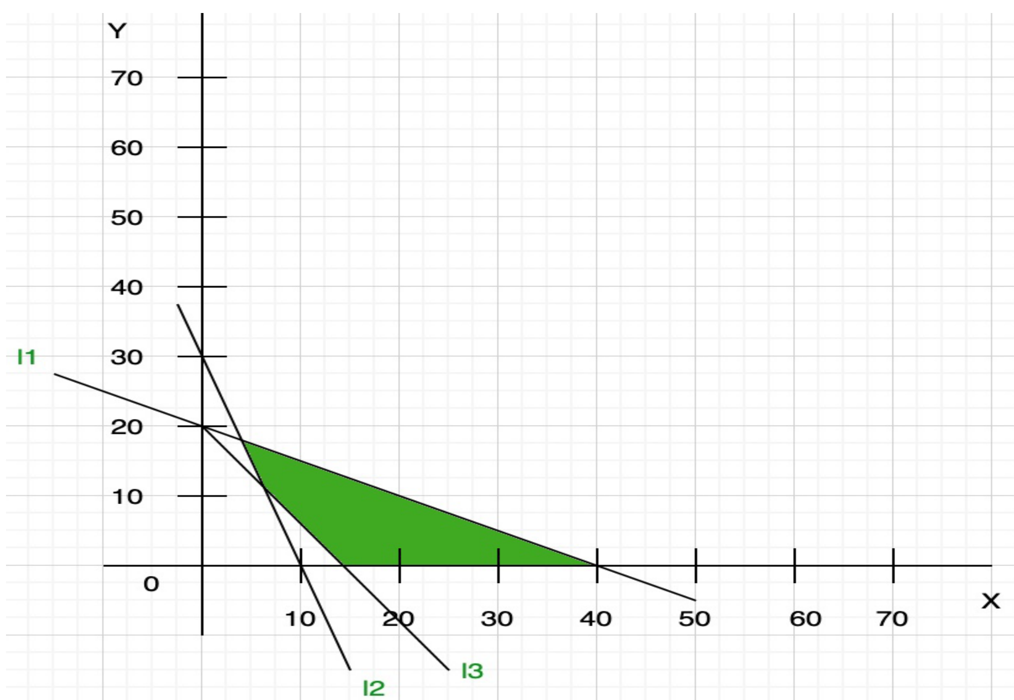
Step 1: *The corresponding equations of given constraints are:*
$l_1 = x + 2y = 40$ ....(i)
$l_2 = 3x + y = 30$ ....(ii)
$l_3 = 4x + 3y = 60$ ....(iii)

From eq(i),
When x = 0, y = 20
When y = 0, x = 40
So, the points are (0, 20) and (40, 0)

Similarly, in eq(ii)
When x = 0, y = 30
When y = 0, x = 10
So, the points are (0, 30) and (10, 0)

Similarly, in eq(iii)
When x = 0, y = 20
When y = 0, x = 15
So, the points are (0, 20) and (15, 0)

Step 2: *Now, plot these points in the graph and find the feasible region.*



Step 3: *Find the coordinates of point new intersection point (A):*
As you can see from the graph, at point A, l2 and l3 lines intersect, so we find the coordinate of point A by solving these equations:
$l_2 = 3x + y = 30$ ....(ii)
$l_3 = 4x + 3y = 60$ ....(iii)
Now multiply eq(ii) with 3 and solving with (iii), we get coordinates (6, 12)

Similarly, for point (B)
$l_1 = x + 2y = 40$ ....(i)
$l_2 = 3x + y = 30$ ....(ii)
Now multiply eq(i) by 3 and solving with (ii), we get coordinates (4,18)
***Step 5:** To find the minimum value of Z = 20x +10 y. Compare each intersection point of the graph to find the minimum value*

Points	Z = 20x + 10y
O(0, 0)	0
A(6, 12)	240
B(4,18)	260
C(40, 0)	800
D(15, 0)	300

Thus, the minimum value of Z with the given constraint is 240 at x = 6 and y = 12.

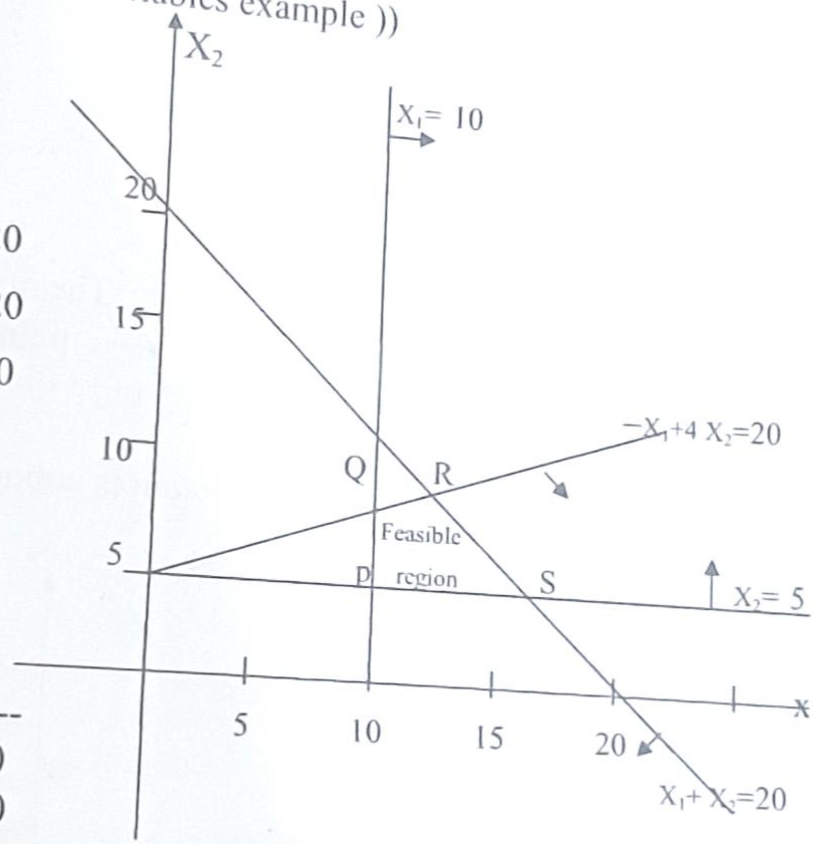
1- linear programming
((two variables example))

a ♣ Graphical method

Example:- (1)

Min. $Z = -3X_1 - 4X_2$

Subject to :-
$$X_1 + X_2 \leq 20$$
$$-X_1 + 4X_2 \leq 20$$
$$X_1 \geq 10$$
$$X_2 \geq 5$$
$$X \geq 0$$



Sol: -

Point	X_1, X_2	Z
Q	10 , 7.5	- 60
P	10 , 5	- 50
R	12 , 8	- 68
S	15, 5	- 65

The min. value of z = – 68 at point R (12 , 8)

6. Multiple Optimal Solutions

The optimal value of the objective function occurs at more than 1 extreme points, then the problem has multiple optimal solution.

Example: **Find solution using graphical method**

MAX z = $10 x1 + 6x2$

subject to:

$5x1 + 3x2 \leq 30$

$x1 + 2x2 \leq 18$

and $x1, x2 \geq 0;$

Solution:

The corresponding equations of given constraints are:

- $5x1+3x2=30$ -----(i)
- $x1+2x2=18$ ------(ii)

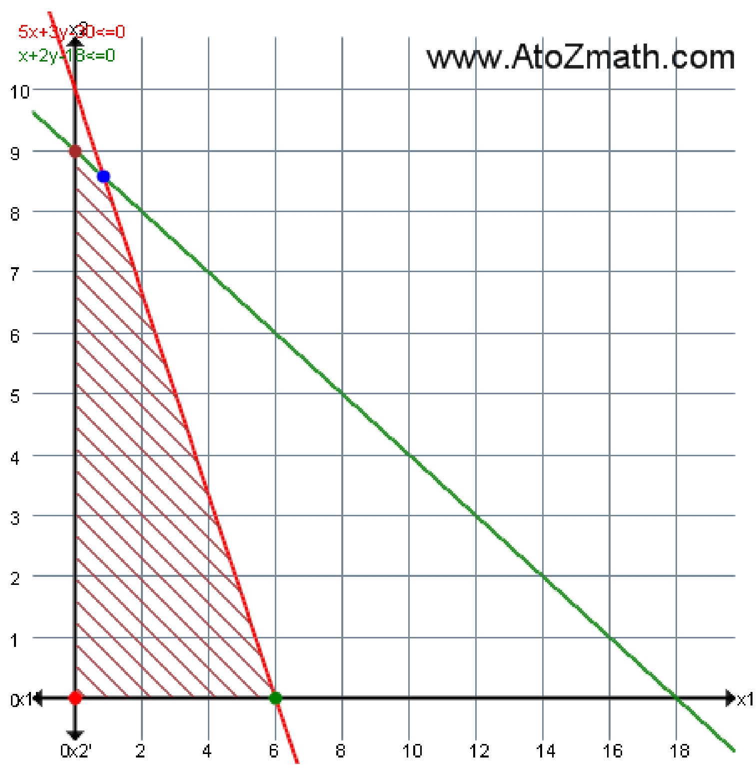
From eq(i)

$x1$	0	6
$x2$	10	0

From eq(ii)

$x1$	0	18
$x2$	9	0

Here both the constraints are less than or equal to, so they satisfy the below region (towards origin).



Coordinates (x_1,x_2)	Objective function value **$Z=10x_1+6x_2$**
$O(0,0)$	$10(0)+6(0)=0$
$A(6,0)$	$10(6)+6(0)=60$
$B(0.86,8.57)$	$10(0.86)+6(8.57)=60$
$C(0,9)$	$10(0)+6(9)=54$

The maximum value of the objective function **$Z=60$** occurs at 2 extreme points.

Hence, problem has multiple optimal solutions and max $Z=60$.

7. Unbounded Solution

If feasible region is **open and extends infinitely** in the direction of optimization (maximization or minimization), the solution is unbounded.

Meaning, there is **no finite maximum or minimum**, the objective function can increase or decrease indefinitely.

Example: Feasible region is not enclosed, and objective function improves as you move further along a direction.

Find solution using graphical method
MAX Z = 5X1 + 4X2
subject to
X1 - 2X2 ≤ 1
X1 + 2X2 ≥ 3
and X1,X2 ≥ 0

Solution:
The corresponding equations of given constraints are:
$X_1 - 2X_2 = 1$ -----(i)
$X_1 + 2X_2 = 3$ ------(ii)

From eq (i)

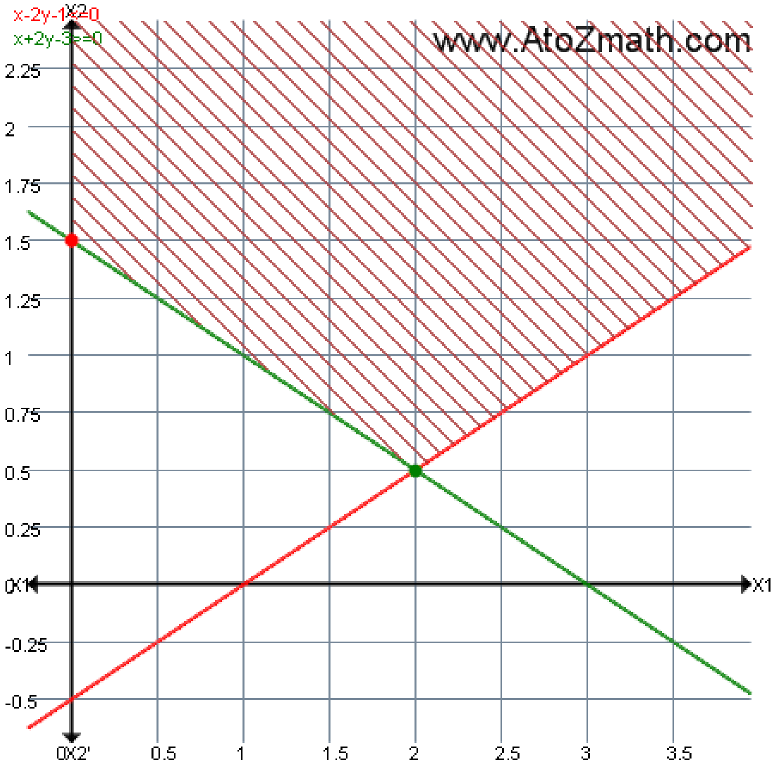
X_1	0	1
X_2	-0.5	0

Taking (0,0) as testing point on **X1 - 2X2 ≤ 1 => 0 ≤ 1** (which is true).so, it contains origin.

From eq (ii)

X_1	0	3
X_2	1.5	0

Taking (0,0) as testing point on **X1 + 2X2 ≥ 3 => 0 ≤ 1** (which is false).so, it doesn't contain origin.



Coordinates (X_1,X_2)	Objective function value $Z=5X_1+4X_2$
$A(0,1.5)$	$5(0)+4(1.5)=6$
$B(2,0.5)$	$5(2)+4(0.5)=12$

In above maximization problem, the shaded area is open-ended that the maximization is not possible and the LPP has no finite solution. Hence the solution of the given problem is unbounded.

8. Infeasible Solution

If there is **no region that satisfies all constraints simultaneously**, the problem is infeasible.

This occurs when constraints contradict each other, leaving no valid solution.

Example:

Find solution using graphical method
MAX Z = 6X1 - 4X2
subject to
2X1 + 4X2 ≤ 4
4X1 + 8X2 ≥ 16
and X1,X2 ≥ 0

Solution:
The corresponding equations of given constraints are:
2X1 + 4X2 = 4 -----(i)
4X1 + 8X2 = 16 -----(ii)

From eq(i)

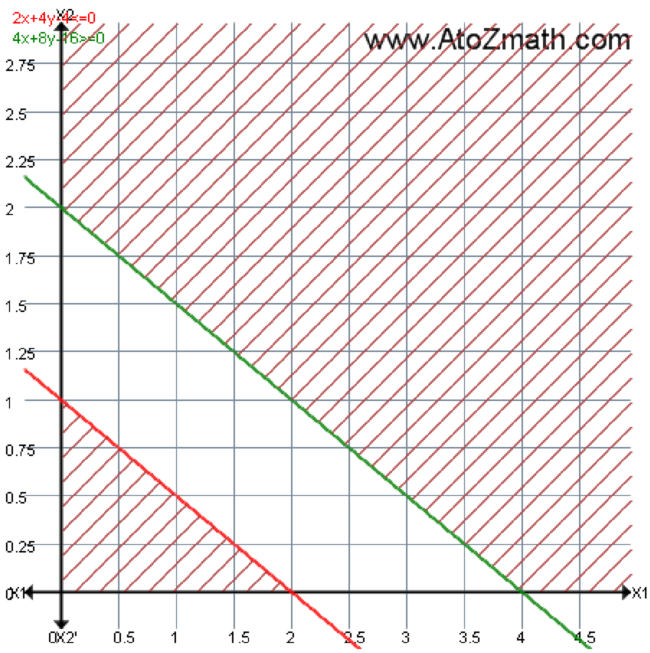
X_1	0	2
X_2	1	0

Taking (0,0) as testing point on $2X1 + 4X2 \leq 4 \Rightarrow 0 \leq 4$ (which is true).so, it contains origin.

From eq(ii)

X_1	0	4
X_2	2	0

Taking (0,0) as testing point on $4X1 + 8X2 \geq 16 \Rightarrow 0 \geq 16$ (which is false).so, It doesn't contain origin.



This Problem has an infeasible solution.

9. Basics of Simplex Method

The Simplex Method is a systematic, algebraic approach for solving LPPs, especially those with:

- More than two decision variables
- Complex or numerous constraints

Core Concepts

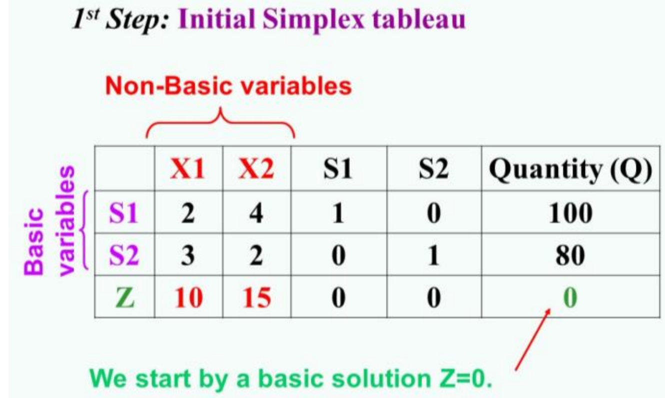
1. **Standard Form Requirement:**

- Objective Function: Must be **Maximization** (Convert Min Z to Max(-Z)).
- Constraints: All must be **equations** (=), not inequalities.
- Right-Hand Side (RHS): All constants must be **non-negative** (≥ 0).
- Variables: All decision variables are **non-negative** (≥ 0).

2. **Basic and Non-Basic Variables:**

- **Basic Variables (BV):** Variables that are "in solution" (positive). In the initial tableau, slack variables are usually basic. The number of BVs equals the number of constraints. Their columns form an **identity matrix** in the simplex tableau .Their values appear in the **RHS (Right Hand Side)** of the tableau.

- **Non-Basic Variables (NBV):** Variables set to zero for a given solution. Initially, the original decision variables.
  - Basic variables = Currently active in the solution
  - Non-basic variables = Inactive (value = 0) but may enter the basis in future steps
- **Basic Feasible Solution (BFS):** A solution where, it satisfies all constraints (feasible).



	X1	X2	S1	S2	Quantity (Q)
S1	2	4	1	0	100
S2	3	2	0	1	80
Z	10	15	0	0	0

3. **Slack, Surplus, and Artificial Variables**

- **Slack Variables:** Added to convert ≤ constraints to equalities

  Example: $x + y \leq 10 \rightarrow x + y + s = 10$

- **Surplus Variables:** Subtracted to convert ≥ constraints to equalities

  Example: $x + y \geq 5 \rightarrow x + y - e = 5$

- **Artificial Variables:** Temporarily introduced to maintain feasibility in ≥ or = constraints (especially in special cases).

  Example: $x + y \geq 5 \rightarrow x + y - e + A = 5$

  Used in **Big-M** or **Two-Phase Simplex** methods.

- **Big-M Method**: Coefficient = -M (maximization) or +M (minimization), where M is a large positive number.

- **Two-Phase Method**:
  - **Phase I**: Minimize sum of artificial variables.
  - **Phase II**: Use the feasible solution from Phase I to solve the original problem.

Key: Slack/surplus stay; artificial must vanish for feasibility.

Example:

```
Maximize:
Z = 3x + 2y
Subject to:
x + y ≤ 4
x + 3y ≤ 6
x, y ≥ 0
```

Introduce slack variables s_1 and s_2:

$x + y + s_1 = 4$

$x + 3y + s_2 = 6$

Objective function becomes:

$Z - 3x - 2y = 0$

10. Simplex Method Computation

Algorithm/Steps Involved to maximize the solution of LPP:

1. **Convert the LPP to Standard Form:**
   - Introduce **slack variables** (s1, s2, ….,Sn) to convert inequalities to equalities:

- Rewrite the objective function:

2. **Initialize the Simplex Tableau:**
   - Construct the initial tableau with the coefficients of the constraints and the objective function:
     - Rows: Constraints + Objective function.
     - Columns: Decision variables, slack variables, and RHS (Right-Hand Side).
3. **Check for Optimality:**
   - Compute Zj and $z_j - c_j$ values for this solution
   - If all coefficients in the $z_j - c_j$ are **non-negative**, the current solution is optimal. Stop.
   - Else, proceed to the next step.
4. **Select the Pivot Column (Entering Variable):**
   - Choose the **most negative coefficient** in the objective row ($z_j - c_j$). The corresponding variable is the **entering variable**.
5. **Select the Pivot Row (Leaving Variable):**
   - Compute the **ratio** $\frac{\text{RHS constant}}{\text{Pivot Column Coefficient}}$ for each row (ignore negative or zero denominators).
   - The row with the **smallest non-negative ratio** is the pivot row. The corresponding slack variable is the **leaving variable**.
6. **Perform Pivot Operation:**
   - The intersection of the pivot column and pivot row is the **pivot element**.
   - Divide the pivot row by the pivot element to make the pivot element equal to 1.
   - Use row operations to make all other entries in the pivot column equal to 0.
7. **Update the Tableau:**
   - Replace the leaving variable in the basis with the entering variable.
   - Update the tableau with the new coefficients.
8. **Repeat:**
   - Go back to Step 3 and repeat the process until the optimality condition is met.

Example : Find solution using Simplex method

MAX Z = 30x1 + 40x2
subject to
3x1 + 2x2 <= 600
3x1 + 5x2 <= 800
5x1 + 6x2 <= 1100
and x1,x2 >= 0

Solution:

Introducing slack variables

Max $Z = 30x_1 + 40x_2 + 0S_1 + 0S_2 + 0S_3$

subject to

$$3x_1 + 2x_2 + S_1 = 600$$
$$3x_1 + 5x_2 + S_2 = 800$$
$$5x_1 + 6x_2 + S_3 = 1100$$

and $x_1, x_2, S_1, S_2, S_3 \geq 0$

Iteration-1		C_j	30	40	0	0	0	
B.v	C_B	**b**	x_1	x_2	S_1	S_2	S_3	**Min Ratio** $\frac{b}{x_2}$
S_1	0	600	3	2	1	0	0	600/2=300
S_2	0	800	3	(5)	0	1	0	800/5=160→
S_3	0	1100	5	6	0	0	1	1100/6=183.33
		Z_j	**0**	**0**	**0**	**0**	**0**	
		Z_j-C_j	-30	-40↑	0	0	0	

Negative minimum Z_j-C_j is -40 and its column index is 2. So, the entering variable is x_2.

Minimum ratio is 160 and its row index is 2. So, the leaving basis variable is S_2.

∴ The pivot element is 5.

Entering $=x_2$, Departing $=S_2$, Key Element $=5$

Applying row operations:

$R_2(\text{new}) = R_2(\text{old}) \div 5$

$R_1(\text{new}) = R_1(\text{old}) - 2R_2(\text{new})$

$R_3(\text{new}) = R_3(\text{old}) - 6R_2(\text{new})$

Iteration-2		C_j	30	40	0	0	0	
B.v	C_B	b	x_1	x_2	S_1	S_2	S_3	**Min Ratio** $\frac{b}{x_1}$
S_1	0	280	1.8	0	1	-0.4	0	280/1.8=155.55
x_2	40	160	0.6	1	0	0.2	0	160/0.6=266.66
S_3	0	140	(1.4)	0	0	-1.2	1	140/1.4=100→
Z=6400		Z_j	**24**	**40**	**0**	**8**	**0**	
		Z_j-C_j	-6↑	0	0	8	0	

Negative minimum Z_j-C_j is -6 and its column index is 1. So, the entering variable is x_1.

Minimum ratio is 100 and its row index is 3. So, the leaving basis variable is S_3.
∴ The pivot element is 1.4.

Entering $=x_1$, Departing $=S_3$, Key Element $=1.4$

$R_3(\text{new}) = R_3(\text{old}) \div 1.4$

$R_1(\text{new}) = R_1(\text{old}) - 1.8R_3(\text{new})$

$R_2(\text{new}) = R_2(\text{old}) - 0.6R_3(\text{new})$

Iteration-3		C_j	30	40	0	0	0	
B	C_B	***B***	x_1	x_2	S_1	S_2	S_3	**Min Ratio**
S_1	0	100	0	0	1	1.1429	-1.2857	
x_2	40	100	0	1	0	0.7143	-0.4286	
x_1	30	100	1	0	0	-0.8571	0.7143	
$Z=7000$		Z_j	**30**	**40**	**0**	**2.8571**	**4.2857**	
		Z_j-C_j	0	0	0	2.8571	4.2857	

Since all $Z_j-C_j \geq 0$ .Hence, optimal solution is arrived with value of variables as :
$x_1=100, x_2=100$
Max $Z=7000$

11. Simplex Method with More Than Two Variables

Graphical method cannot handle more than two variables. **Simplex Method** easily solves problems with:

- 3 or more decision variables
- Multiple constraints
- Complex real-world problems

Example : Find solution using Simplex method

```
MAX Z = 22x1 + 6x2 + 2x3
subject to
10x1 + 2x2 + x3 <= 100
7x1 + 3x2 + 2x3 <= 72
2x1 + 4x2 + x3 <= 80
and x1,x2,x3 >= 0
```

Solution:
Introducing slack variables

Max $Z = 22x_1 + 6x_2 + 2x_3 + 0S_1 + 0S_2 + 0S_3$

subject to

$$10x_1 + 2x_2 + x_3 + S_1 = 100$$
$$7x_1 + 3x_2 + 2x_3 + S_2 = 72$$
$$2x_1 + 4x_2 + x_3 + S_3 = 80$$

and $x_1,x_2,x_3,S_1,S_2,S_3 \geq 0$

Iteration-1		C_j	22	6	2	0	0	0	
B	**C_B**	b	**x_1**	**x_2**	**x_3**	**S_1**	**S_2**	**S_3**	**Min Ratio** **b/x_1**
S_1	0	100	**(10)**	2	1	1	0	0	100/10=10→
S_2	0	72	7	3	2	0	1	0	72/7=10.28
S_3	0	80	2	4	1	0	0	1	80/2=40
		Z_j	**0**	**0**	**0**	**0**	**0**	**0**	
		Z_j-C_j	-22↑	-6	-2	0	0	0	

Negative minimum Z_j-C_j is -22 and its column index is 1. So, the entering variable is x_1.

Minimum ratio is 10 and its row index is 1. So, the leaving basis variable is S_1.

∴ The pivot element is 10.

Entering $=x_1$, Departing $=S_1$, Key Element $=10$

$R_1(\text{new}) = R_1(\text{old}) \div 10$

$R_2(\text{new}) = R_2(\text{old}) - 7R_1(\text{new})$

$R_3(\text{new}) = R_3(\text{old}) - 2R_1(\text{new})$

Iteration-2		C_j	22	6	2	0	0	0	
B	**C_B**	**X_B**	**x_1**	**x_2**	**x_3**	**S_1**	**S_2**	**S_3**	**MinRatio** $\frac{X_B}{x_2}$
x_1	22	10	1	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	0	0	$\frac{10}{\frac{1}{5}} = 50$
S_2	0	2	0	$\left(\frac{8}{5}\right)$	$\frac{13}{10}$	$-\frac{7}{10}$	1	0	$\frac{2}{\frac{8}{5}} = \frac{5}{4} = 1.25$ →
S_3	0	60	0	$\frac{18}{5}$	$\frac{4}{5}$	$-\frac{1}{5}$	0	1	$\frac{60}{\frac{18}{5}} = \frac{50}{3} = 16.6667$
$Z = 220$		**Z_j**	**22**	$\frac{22}{5}$	$\frac{11}{5}$	$\frac{11}{5}$	**0**	**0**	
		$Z_j - C_j$	0	$-\frac{8}{5}$ ↑	$\frac{1}{5}$	$\frac{11}{5}$	0	0	

Negative minimum Zj-Cj is -8/5 and its column index is 2. So, the entering variable is $x2$.

Minimum ratio is 1.25 and its row index is 2. So, the leaving basis variable is $S2$.

∴ The pivot element is 85.

Entering $=x2$, Departing $=S2$, Key Element $=8/5$
Applying row operations:

- $R_2(\text{new}) = R_2(\text{old}) \times \frac{5}{8}$

- $R_1(\text{new}) = R_1(\text{old}) - \frac{1}{5} R_2(\text{new})$

- $R_3(\text{new}) = R_3(\text{old}) - \frac{18}{5} R_2(\text{new})$

Iteration-3		C_j	22	6	2	0	0	0
B	**C_B**	**X_B**	**x_1**	**x_2**	**x_3**	**S_1**	**S_2**	**S_3**
x_1	22	$\frac{39}{4}$	1	0	$-\frac{1}{16}$	$\frac{3}{16}$	$-\frac{1}{8}$	0
x_2	6	$\frac{5}{4}$	0	1	$\frac{13}{16}$	$-\frac{7}{16}$	$\frac{5}{8}$	0
S_3	0	$\frac{111}{2}$	0	0	$-\frac{17}{8}$	$\frac{11}{8}$	$-\frac{9}{4}$	1
$Z = 222$		**Z_j**	**22**	**6**	**$\frac{7}{2}$**	**$\frac{3}{2}$**	**1**	**0**
		$Z_j - C_j$	0	0	$\frac{3}{2}$	$\frac{3}{2}$	1	0

Since all $Z_j - C_j \geq 0$

Hence, optimal solution is arrived with value of variables as :

$x_1 = \frac{39}{4}, x_2 = \frac{5}{4}, x_3 = 0$

Max $Z = 222$

Special Cases

- **Minimization**: Convert to maximization: Min Z=Max $(-Z)$.
- **Big-M Method**: For ≥ constraints, add artificial variables with penalty M in the objective.
- **Two-Phase Method**:
    - **Phase I**: Minimize sum of artificial variables.
    - **Phase II**: Use the feasible solution from Phase I to solve the original problem.
- **Unbounded Solution**: If pivot column has no positive coefficients, $Z \to \infty$.
- **Infeasibility**: If artificial variables remain positive in Phase I or Big-M.

Big M Method

The **Big-M Method** is a way to handle "≥" (greater-than) constraints (and equalities) in the Simplex Method by introducing artificial variables that are heavily penalized in the objective.
Artificial variables are introduced to start the Simplex Method when slack or surplus variables alone are insufficient .

When to Use Big M Method

- Constraints like:
    - **≥ Type Constraints:** Require surplus and artificial variables
    - **= Type Constraints:** Require artificial variables

Artificial variables are not real decision variables; they help to start the simplex process but should be removed from the final solution.

Steps of Big M Method

Step 1: Formulate the LPP

Convert all constraints into equalities using slack, surplus, and artificial variables.

Step 2: Modify the Objective Function

- For **Maximization Problems**:

  ➔ Assign **-M** times artificial variables in the objective function

- For **Minimization Problems**:

  ➔ Assign **+M** times artificial variables

  OR: Convert to maximization: Min Z=Max $(-Z)$ and do same for max.

Where **M = a very large positive number**

Step 3: Construct the Simplex Tableau

Include slack, surplus, artificial variables, and objective function with 'M' terms.

Step 4: Perform Simplex Iterations

Proceed with usual Simplex Method operations:

- Identify entering and leaving variables
- Perform pivot operations
- Repeat until optimality condition is met

Step 5: Check for Artificial Variables

- In the final solution, if any artificial variable has a non-zero value → **Infeasible Solution**
- If all artificial variables are **zero**, valid optimal solution is obtained

Simple Example

Maximize:

$Z = 3x_1 + 2x_2$

Subject to:

$x_1 + x_2 \geq 4$

$x_1 + 2x_2 \leq 6$

$x_1, x_2 \geq 0$

Step 1: Convert to Standard Form

First constraint (≥) → Subtract surplus, add artificial variable:

$x_1 + x_2 - s_1 + A_1 = 4$

Second constraint (≤) → Add slack variable:

$x_1 + 2x_2 + s_2 = 6$

Step 2: Modified Objective Function

Maximize:

$Z = 3x_1 + 2x_2 - MA_1$

Where **M = very large positive number**

Rest of the Process:

Set up the tableau, perform iterations, and solve as per the usual Simplex method.

- Build the tableau, note the large $-M$ entry in the artificial column of the net-evaluation row.
- Pivot to remove A_1 from the basis first (its negative cost will drive it out).
- Continue until all $z_j - c_j \geq 0$.

Example 1 : Find solution using Simplex(Big M) method
MIN Z = x1 + x2
subject to
2x1 + 4x2 >= 4
x1 + 7x2 >= 7
and x1,x2 >= 0

Solution:
Introducing surplus, artificial variables

Min $Z = x_1 + x_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

subject to

$2x_1 + 4x_2 - S_1 + A_1 = 4$

$x_1 + 7x_2 - S_2 + A_2 = 7$

and $x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

OR: you can do by converting Min into Max as:

Max Z` = Min(-Z) = -x1 – x2

Note: For Min ➔ + M . A

For Max ➔ - M . A

Iteration-1		C_j	1	1	0	0	M	M	
B	C_B	X_B	x_1	x_2	S_1	S_2	A_1	A_2	MinRatio $\frac{X_B}{x_2}$
A_1	M	4	2	4	-1	0	1	0	$\frac{4}{4} = 1$
A_2	M	7	1	(7)	0	-1	0	1	$\frac{7}{7} = 1 \rightarrow$
$Z = 0$		Z_j	$3M$	$11M$	$-M$	$-M$	M	M	
		$C_j - Z_j$	$-3M + 1$	$-11M + 1 \uparrow$	M	M	0	0	

Negative minimum $C_j - Z_j$ is $-11M + 1$ and its column index is 2. So, the entering variable is x_2.

Minimum ratio is 1 and its row index is 2. So, the leaving basis variable is A_2.

∴ The pivot element is 7.

Entering $= x_2$, Departing $= A_2$, Key Element $= 7$

R_2(new) $= R_2$(old)$\div 7$

R_1(new) $= R_1$(old)$- 4R_2$(new)

Note: *After removing the Artificial variable from basic variable you can also remove its corresponding column.*
Form above table we will remove A2 column and continue the process in others table also.

Iteration-2		C_j	1	1	0	0	M	
B	C_B	X_B	x_1	x_2	S_1	S_2	A_1	MinRatio $\frac{X_B}{x_1}$
A_1	M	0	$\left(\frac{10}{7}\right)$	0	-1	$\frac{4}{7}$	1	$\frac{0}{\frac{10}{7}} = 0 \rightarrow$
x_2	1	1	$\frac{1}{7}$	1	0	$-\frac{1}{7}$	0	$\frac{1}{\frac{1}{7}} = 7$
$Z = 1$		Z_j	$\frac{10M}{7} + \frac{1}{7}$	1	$-M$	$\frac{4M}{7} - \frac{1}{7}$	M	
		$C_j - Z_j$	$-\frac{10M}{7} + \frac{6}{7} \uparrow$	0	M	$-\frac{4M}{7} + \frac{1}{7}$	0	

Negative minimum $C_j - Z_j$ is $-\frac{10M}{7} + \frac{6}{7}$ and its column index is 1. So, the entering variable is x_1.

Minimum ratio is 0 and its row index is 1. So, the leaving basis variable is A_1.

∴ The pivot element is 10/7. Entering $=x_1$, Departing $=A_1$, Key Element $=10/7$

$R_1(\text{new})=R_1(\text{old})\times 7/10$

$R_2(\text{new}) = R_2(\text{old}) - \frac{1}{7}R_1(\text{new})$

Iteration-3		C_j	1	1	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	MinRatio $\frac{X_B}{S_2}$
x_1	1	0	1	0	$-\frac{7}{10}$	$\left(\frac{2}{5}\right)$	$\frac{0}{\frac{2}{5}} = 0 \rightarrow$
x_2	1	1	0	1	$\frac{1}{10}$	$-\frac{1}{5}$	---
$Z = 1$		Z_j	1	1	$-\frac{3}{5}$	$\frac{1}{5}$	
		$C_j - Z_j$	0	0	$\frac{3}{5}$	$-\frac{1}{5} \uparrow$	

Negative minimum $C_j - Z_j$ is $-\frac{1}{5}$ and its column index is 4. So, the entering variable is S_2.

Minimum ratio is 0 and its row index is 1. So, the leaving basis variable is x_1.

∴ The pivot element is $2/5$.

Entering $= S_2$, Departing $= x_1$, Key Element $= \frac{2}{5}$

R_1(new) $= R_1$(old) $\times \frac{5}{2}$

R_2(new) $= R_2$(old)$+ \frac{1}{5} R_1$(new)

Iteration-4		C_j	1	1	0	0	
B	**C_B**	**X_B**	**x_1**	**x_2**	**S_1**	**S_2**	**MinRatio**
S_2	0	0	$\frac{5}{2}$	0	$-\frac{7}{4}$	1	
x_2	1	1	$\frac{1}{2}$	1	$-\frac{1}{4}$	0	
$Z = 1$		Z_j	$\mathbf{\frac{1}{2}}$	**1**	$\mathbf{-\frac{1}{4}}$	**0**	
		$C_j - Z_j$	$\frac{1}{2}$	0	$\frac{1}{4}$	0	

Since all $C_j - Z_j \geq 0$. Hence, optimal solution is arrived with value of variables as:
$x_1=0$, $x_2=1$
Min $Z=1$

Example 2: Find solution using Simplex method (Big M method)
MIN Z = 5x1 + 3x2
subject to
2x1 + 4x2 <= 12
2x1 + 2x2 = 10
5x1 + 2x2 >= 10
and x1,x2 >= 0

Solution:

The problem is converted to canonical form by adding slack, surplus and artificial variables as appropriate

1. As the constraint 1 is of type '≤' we should add slack variable S_1

2. As the constraint 2 is of type '=' we should add artificial variable A_1

3. As the constraint 3 is of type '≥' we should subtract surplus variable S_2 and add artificial

variable A_2

After introducing slack, surplus, artificial variables

Min $Z = 5x_1 + 3x_2 + 0.S_1 + 0.S_2 + MA_1 + MA_2$

subject to

$2x_1 + 4x_2 + S_1 = 12$

$2x_1 + 2x_2 + A_1 = 10$

$5x_1 + 2x_2 - S_2 + A_2 = 10$

and $x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

Iteration-1		C_j	5	3	0	0	M	M	
B	**C_B**	**X_B**	**x_1**	**x_2**	**S_1**	**S_2**	**A_1**	**A_2**	**Min Ratio** **$\frac{X_B}{x_1}$**
S_1	0	12	2	4	1	0	0	0	$\frac{12}{2}=6$
A_1	M	10	2	2	0	0	1	0	$\frac{10}{2}=5$
A_2	M	10	**(5)**	2	0	-1	0	1	$\frac{10}{5}=2 \rightarrow$
		Z_j	**$7M$**	**$4M$**	**0**	**$-M$**	**M**	**M**	
		Z_j-C_j	$7M-5 \uparrow$	$4M-3$	0	$-M$	0	0	

Positive maximum Z_j-C_j is $7M-5$ and its column index is 1. So, the entering variable is x_1.

Minimum ratio is 2 and its row index is 3. So, the leaving basis variable is A_2.

∴ The pivot element is 5.

Entering $=x_1$, Departing $=A_2$, Key Element $=5$

$R_3(\text{new}) = R_3(\text{old}) \div 5$

$R_1(\text{new}) = R_1(\text{old}) - 2R_3(\text{new})$

$R_2(\text{new}) = R_2(\text{old}) - 2R_3(\text{new})$

Iteration-2		C_j	5	3	0	0	M	
B	C_B	X_B	x_1	x_2	S_1	S_2	A_1	**MinRatio** $\frac{X_B}{x_2}$
S_1	0	8	0	$\left(\frac{16}{5}\right)$	1	$\frac{2}{5}$	0	$\frac{8}{\frac{16}{5}} = \frac{5}{2} = 2.5 \rightarrow$
A_1	M	6	0	$\frac{6}{5}$	0	$\frac{2}{5}$	1	$\frac{6}{\frac{6}{5}} = 5$
x_1	5	2	1	$\frac{2}{5}$	0	$-\frac{1}{5}$	0	$\frac{2}{\frac{2}{5}} = 5$
$Z = 6M + 10$		Z_j	5	$\frac{6M}{5} + 2$	0	$\frac{2M}{5} - 1$	M	
		$Z_j - C_j$	0	$\frac{6M}{5} - 1 \uparrow$	0	$\frac{2M}{5} - 1$	0	

Positive maximum $Z_j - C_j$ is $6M/5-1$ and its column index is 2. So, the entering variable is x_2.

Minimum ratio is 2.5 and its row index is 1. So, the leaving basis variable is S_1.

∴ The pivot element is 16/5.

Entering $=x_2$, Departing $=S_1$, Key Element $=16/5$

$$R_1(\text{new}) = R_1(\text{old}) \times \frac{5}{16}$$

$$R_2(\text{new}) = R_2(\text{old}) - \frac{6}{5} R_1(\text{new})$$

$$R_3(\text{new}) = R_3(\text{old}) - \frac{2}{5} R_1(\text{new})$$

Iteration-3		C_j	5	3	0	0	M	
B	C_B	X_B	x_1	x_2	S_1	S_2	A_1	MinRatio $\frac{X_B}{S_2}$
x_2	3	$\frac{5}{2}$	0	1	$\frac{5}{16}$	$\frac{1}{8}$	0	$\frac{\frac{5}{2}}{\frac{1}{8}} = 20$
A_1	M	3	0	0	$-\frac{3}{8}$	$\left(\frac{1}{4}\right)$	1	$\frac{3}{\frac{1}{4}} = 12 \rightarrow$
x_1	5	1	1	0	$-\frac{1}{8}$	$-\frac{1}{4}$	0	---
$Z = 3M + \frac{25}{2}$		Z_j	5	3	$-\frac{3M}{8} + \frac{5}{16}$	$\frac{M}{4} - \frac{7}{8}$	M	
		$Z_j - C_j$	0	0	$-\frac{3M}{8} + \frac{5}{16}$	$\frac{M}{4} - \frac{7}{8} \uparrow$	0	

Positive maximum Z_j-C_j is $\frac{M}{4} - \frac{7}{8}$ and its column index is 4. So, the entering variable is S_2.

Minimum ratio is 12 and its row index is 2. So, the leaving basis variable is A_1.

∴ The pivot element is 1/4.
Entering $=S_2$, Departing $=A_1$, Key Element $=1/4$

$$R_2(\text{new}) = R_2(\text{old}) \times 4$$

$$R_1(\text{new}) = R_1(\text{old}) - \frac{1}{8} R_2(\text{new})$$

$$R_3(\text{new}) = R_3(\text{old}) + \frac{1}{4} R_2(\text{new})$$

Iteration-4		C_j	5	3	0	0	
B	**C_B**	**X_B**	**x_1**	**x_2**	**S_1**	**S_2**	**Min Ratio**
x_2	3	1	0	1	12	0	
S_2	0	12	0	0	-32	1	
x_1	5	4	1	0	-12	0	
$Z=23$		**Z_j**	**5**	**3**	**-1**	**0**	
		Z_j-C_j	0	0	-1	0	

Since all Z_j-$C_j \leq 0$ .Hence, optimal solution is arrived with value of variables as:
$x_1=4$, $x_2=1$
Min $Z = 23$

Two-Phase Method

The **Two-Phase Method** is a structured approach to solving Linear Programming Problems (LPPs) that contain:

- "≥" (greater-than) constraints,
- "=" (equality) constraints,
  where **artificial variables** are needed to start the simplex method.

Unlike the **Big-M method**, which penalizes artificial variables using a large constant M, the **Two-Phase Method** splits the process into:

- **Phase 1:** Find a **feasible solution** by minimizing the **sum of artificial variables** (ignoring the original objective).
- **Phase 2:** Use that feasible solution to **maximize (or minimize) the original objective.**

It avoids directly using a large penalty **M** like in the **Big M Method.**

Algorithm

Two-Phase Method Steps (Rule)	
Step-1:	**Phase-1** a. Form a new objective function by assigning zero to every original variable (including slack and surplus variables) and -1 to each of the artificial variables. e.g. Max Z = - A1 - A2 b. Using simplex method, try to eliminate the artificial variables from the basis. c. The solution at the end of Phase-1 is the initial basic feasible solution for Phase-2.
Step-2:	**Phase-2** a. The original objective function is used and coefficient of artificial variable is 0 (so artificial variable is removed from the calculation process). b. Then simplex algorithm is used to find optimal solution.

Example
Find solution using Two-Phase method
MIN Z = x1 + x2
subject to
2x1 + x2 >= 4
x1 + 7x2 >= 7
and x1,x2 >= 0

Solution:

-->Phase-1<--

Introducing surplus, artificial variables

Min $Z = A_1 + A_2$

subject to

$2x_1 + x_2 - S_1 + A_1 = 4$

$x_1 + 7x_2 - S_2 + A_2 = 7$

and $x_1, x_2, S_1, S_2, A_1, A_2 \ge 0$

Iteration-1		C_j	0	0	0	0	1	1	
B	C_B	X_B	x_1	x_2	S_1	S_2	A_1	A_2	MinRatio $\frac{X_B}{x_2}$
A_1	1	4	2	1	-1	0	1	0	$\frac{4}{1} = 4$
A_2	1	7	1	(7)	0	-1	0	1	$\frac{7}{7} = 1 \rightarrow$
$Z = 0$		Z_j	3	8	-1	-1	1	1	
		$C_j - Z_j$	-3	-8 $\uparrow$	1	1	0	0	

Negative minimum $C_j - Z_j$ is -8 and its column index is 2. So, the entering variable is x_2.

Minimum ratio is 1 and its row index is 2. So, the leaving basis variable is A_2.

∴ The pivot element is 7.

Entering $= x_2$, Departing $= A_2$, Key Element $= 7$

R_2(new)$= R_2$(old)$\div 7$

R_1(new)$= R_1$(old)$- R_2$(new)

Iteration-2		C_j	0	0	0	0	1	
B	C_B	X_B	x_1	x_2	S_1	S_2	A_1	MinRatio $\frac{X_B}{x_1}$
A_1	1	3	$\left(\frac{13}{7}\right)$	0	-1	$\frac{1}{7}$	1	$\frac{3}{\frac{13}{7}} = 1.62 \rightarrow$
x_2	0	1	$\frac{1}{7}$	1	0	$-\frac{1}{7}$	0	$\frac{1}{\frac{1}{7}} = 7$
$Z = 0$		Z_j	$\frac{13}{7}$	0	-1	$\frac{1}{7}$	1	
		$C_j - Z_j$	$-\frac{13}{7} \uparrow$	0	1	$-\frac{1}{7}$	0	

Negative minimum $C_j - Z_j$ is $-\frac{13}{7}$ and its column index is 1. So, the entering variable is x_1.

Minimum ratio is 1.62 and its row index is 1. So, the leaving basis variable is A_1.

∴ The pivot element is 13/7.

Entering $= x_1$, Departing $= A_1$, Key Element $= 13/7$

R_1(new)$=R_1$(old)$\times 7/13$

R_2(new)$=R_2$(old)$-1/7 * R_1$(new)

Iteration-3		C_j	0	0	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	MinRatio
x_1	0	$\frac{21}{13}$	1	0	$-\frac{7}{13}$	$\frac{1}{13}$	
x_2	0	$\frac{10}{13}$	0	1	$\frac{1}{13}$	$-\frac{2}{13}$	
$Z = 0$		Z_j	0	0	0	0	
		$C_j - Z_j$	0	0	0	0	

Since all $C_j - Z_j \geq 0$. Hence, optimal solution is arrived with value of variables as :
$x_1 = 21/13, x_2 = 10/13$. Min $Z = 0$

-->Phase-2<--

we eliminate the artificial variables and change the objective function for the original,

Iteration-1		C_j	1	1	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	**MinRatio**
x_1	1	$\frac{21}{13}$	1	0	$-\frac{7}{13}$	$\frac{1}{13}$	
x_2	1	$\frac{10}{13}$	0	1	$\frac{1}{13}$	$-\frac{2}{13}$	
$Z=\frac{31}{13}$		Z_j	1	1	$-\frac{6}{13}$	$-\frac{1}{13}$	
		C_j-Z_j	0	0	$\frac{6}{13}$	$\frac{1}{13}$	

Since all $C_j-Z_j \geq 0$. Hence, optimal solution is arrived with value of variables as :
$x_1=21/13$, $x_2=10/13$
Min $Z=31/13$

Degeneracy (Tie for leaving basic variable)

During solving LP problem, a situation may arise in which there is a tie between, 2 or more basic variables for leaving the basis. (means minimum ratios are same). It is called degeneracy and to resolve this we can select any of them arbitrarily. But if artificial variable is present then it must be removed first.

Example:
MAX Z = 3x1 + 9x2
subject to
x1 + 4x2 ≤ 8
x1 + 2x2 ≤ 4
and x1,x2 >= 0

Solution:
After introducing slack variables

$$\text{Max } Z = 3x_1 + 9x_2 + 0S_1 + 0S_2$$

subject to

$$x_1 + 4x_2 + S_1 = 8$$

$$x_1 + 2x_2 + S_2 = 4$$

and $x_1,x_2,S_1,S_2 \geq 0$

Iteration-1		C_j	3	9	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	MinRatio $\frac{X_B}{x_2}$
S_1	0	8	1	4	1	0	$\frac{8}{4}=2$
S_2	0	4	1	(2)	0	1	$\frac{4}{2}=2 \rightarrow$
$Z=0$		Z_j	0	0	0	0	
		$C_j - Z_j$	3	$9 \uparrow$	0	0	

Positive maximum C_j-Z_j is 9 and its column index is 2. So, the entering variable is x_2.
Here, what we see that the element of the ratio column is same (i.e. 2) means there is tie in Min. ratio. Now we need to break the tie either leaving s1 or s2 as both the variables slack variable. Let, variable s2 leave the Basic .So that, the leaving basis variable is S_2.

∴ The pivot element is 2.

Entering $=x_2$, Departing $=S_2$, Key Element $=2$

$R_2(\text{new})=R_2(\text{old})\div 2$

$R_1(\text{new})=R_1(\text{old})-4R_2(\text{new})$

Iteration-2		C_j	3	9	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	MinRatio
S_1	0	0	-1	0	1	-2	
x_2	9	2	$\frac{1}{2}$	1	0	$\frac{1}{2}$	
$Z=18$		Z_j	$\frac{9}{2}$	9	0	$\frac{9}{2}$	
		$C_j - Z_j$	$-\frac{3}{2}$	0	0	$-\frac{9}{2}$	

Since all C_j-$Z_j \leq 0$

Hence, optimal solution is arrived with value of variables as :

$x_1=0, x_2=2$

Max $Z=18$

Multiple optimal solution

In the final simplex table when all $z_j - c_j$ imply optimal solution (for maximization all $z_j - c_j \leq 0$ and for minimization all $z_j - c_j \geq 0$). Opposite for $c_j - z_j$

but if $z_j - c_j 0$ for some non-basic variable column, then this indicates that there are more than 1 optimal solution of the problem. Thus, by entering this variable into the basis, we may obtain another alternative optimal solution.

Example:

Maximize $Z = 4x_1 + 3x_2$
Subject to:
$8x_1 + 6x_2 \leq 25$
$3x_1 + 4x_2 \leq 15$
$x_1, x_2 \geq 0$

Introduce slack variables s_1 and s_2:

Z = 4x1+3x2+0.s1+0.s2
$8x_1 + 6x_2 + s_1 = 25$
$3x_1 + 4x_2 + s_2 = 15$
$x_1, x_2, s_1, s_2 \geq 0$

Simplex Table 1

C_B	Basis	b	x_1	x_2	s_1	s_2	Ratio
0	s_1	25	8	6	1	0	25/8
0	s_2	15	3	4	0	1	5→
	z_j		0	0	0	0	
c_j			4	3	0	0	
$z_j - c_j$			-4↑	-3	0	0	

Entering variable: x_1

Leaving variable: s_1

- R1 = R1/8
- R2 = R2−3× R1(new)

Simplex Table 2

C_B	Basis	b	x_1	x_2	s_1	s_2
4	x_1	25/8	1	3/4	1/8	0
0	s_2	45/8	0	7/4	-3/8	1
	z_j		4	3	1/2	0
	c_j		4	3	0	0
	z_j-c_j		0	0	1/2	0

Since, all $z_j - c_j \geq 0 \rightarrow$ Optimal solution reached. Where,
$x_1 = 25/8$
$x_2 = 0$
$s_1 = 0$, $s_2 = 45/8$
Max Z = 25/2

Check for Multiple Optimal Solutions (Algebraically)

In the **final simplex tableau**, we look at the **non-basic variables** (those not in the basis) and see if any of them have:

- $z_j - c_j = 0$

In our case, the final row of $z_j - c_j$ was:

Variable	**Zj -Cj**
x1	0 (basic)
x2	**0** (non-basic)
s1	-0.5
s2	0

Since **x2** is a **non-basic variable** (value = 0), and $z_j - c_j = 0$,

This indicates that there are more than 1 optimal solution of the problem.
Thus by entering *x*2 into the basis, we may obtain another alternative optimal solution.

Then the ratio column will take value 25/6 and 45/14. Here 45/15 is min ratio so, s2 will leave the basic and x2 enters . The key element is 7/4.

- R2 = R2× 4/7
- R1 = R1−3/4× R1(new)

C_B	**Basis**	**b**	**x_1**	**x_2**	**s_1**	**s_2**
4	x_1	5/7	1	0	2/7	-3/7
3	x_2	45/14	0	1	-3/14	4/7
	z_j		4	3	1/2	0
	c_j		4	3	0	0
	$z_j - c_j$		0	0	1/2	0

Since, all $z_j - c_j \geq 0 \rightarrow$ Optimal solution reached. Where,
$x_1 = 5/7$
$x_2 = 45/14$
$s_1 = 0$, $s_2 = 0$
Max Z = 25/2

Unbounded solution

In simplex table, if a variable should enter into the basis, but all the coefficients in that column are negative or zero. So, this variable cannot be entered into the basis, because for minimum ratio, negative value in denominator cannot be considered and zero value in denominator would result ∞.

Hence, the solution to the given problem is unbounded.

Example

Max $Z = 3x_1 + 5x_2$

subject to

$x_1 - 2x_2 \leq 6$

$x_1 \leq 10$

$x_2 \geq 1$

and $x_1, x_2 \geq 0$;

Solution:

After introducing slack, surplus, artificial variables

Max $Z = 3x_1 + 5x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1$

subject to

$x_1 - 2x_2 + S_1 = 6$

$x_1 + S_2 = 10$

$x_2 - S_3 + A_1 = 1$

and $x_1, x_2, S_1, S_2, S_3, A_1 \geq 0$

Iteration-1		C_j	3	5	0	0	0	$-M$	
B	C_B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	MinRatio $\frac{X_B}{x_2}$
S_1	0	6	1	-2	1	0	0	0	---
S_2	0	10	1	0	0	1	0	0	---
A_1	$-M$	1	0	(1)	0	0	-1	1	$\frac{1}{1} = 1 \rightarrow$
$Z = 0$		Z_j	0	$-M$	0	0	M	$-M$	
		$C_j - Z_j$	3	$M + 5 \uparrow$	0	0	$-M$	0	

Positive maximum $C_j - Z_j$ is $M+5$ and its column index is 2. So, the entering variable is x_2.

Minimum ratio is 1 and its row index is 3. So, the leaving basis variable is A_1.

∴ The pivot element is 1.

Entering $= x_2$, Departing $= A_1$, Key Element $= 1$

$R_3(\text{new}) = R_3(\text{old})$

$R_1(\text{new}) = R_1(\text{old}) + 2R_3(\text{new})$

$R_2(\text{new}) = R_2(\text{old})$

Iteration-2		C_j	3	5	0	0	0	
B	**C_B**	**X_B**	**x_1**	**x_2**	**S_1**	**S_2**	**S_3**	**MinRatio** $\frac{X_B}{S_3}$
S_1	0	8	1	0	1	0	**(-2)**	---
S_2	0	10	1	0	0	1	0	---
x_2	5	1	0	1	0	0	-1	---
$Z = 5$		Z_j	**0**	**5**	**0**	**0**	**-5**	
		$C_j - Z_j$	3	0	0	0	5 ↑	

Variable S_3 should enter into the basis, but all the coefficients in the S_3 column are negative or zero. So S_3 cannot be entered into the basis.

Hence, the solution to the given problem is unbounded

Infeasible solution

If there is no any solution that satisfies all the constraints, then it is called Infeasible solution.

In the final simplex table when all ***cj-zj*** imply optimal solution but at least one artificial variable present in the basis with positive value. Then the problem has no feasible solution.

Example

Max $Z = 6x_1 + 4x_2$

subject to

$x_1 + x_2 \le 5$

$x_2 \ge 8$

and $x_1, x_2 \ge 0$;

Solution:
After introducing slack, surplus, artificial variables

Max $Z = 6x_1 + 4x_2 + 0.S_1 + 0.S_2 - MA_1$

$x_1 + x_2 + S_1 = 5$

$x_2 - S_2 + A_1 = 8$

and $x_1, x_2, S_1, S_2, A_1 \ge 0$

Iteration-1		C_j	6	4	0	0	$-M$	
B	C_B	X_B	x_1	x_2	S_1	S_2	A_1	MinRatio $\frac{X_B}{x_2}$
S_1	0	5	1	(1)	1	0	0	$\frac{5}{1} = 5 \rightarrow$
A_1	$-M$	8	0	1	0	-1	1	$\frac{8}{1} = 8$
$Z = 0$		Z_j	0	$-M$	0	M	$-M$	
		$C_j - Z_j$	6	$M+4 \uparrow$	0	$-M$	0	

Positive maximum C_j-Z_j is M+4 and its column index is 2. So, the entering variable is x_2.

Minimum ratio is 5 and its row index is 1. So, the leaving basis variable is S_1.

∴ The pivot element is 1.

Entering $=x_2$, Departing $=S_1$, Key Element =1

R_1(new)=R_1(old)

R_2(new)=R_2(old)-R_1(new)

Iteration-2		C_j	6	4	0	0	$-M$
B	C_B	X_B	x_1	x_2	S_1	S_2	A_1
x_2	4	5	1	1	1	0	0
A_1	$-M$	3	-1	0	-1	-1	1
$Z = 20$		Z_j	$M+4$	4	$M+4$	M	$-M$
		$C_j - Z_j$	$-M+2$	0	$-M-4$	$-M$	0

Since all $C_j-Z_j \leq 0$. Hence, optimal solution is arrived with value of variables as :
$x_1=0, x_2=5$
Max Z=20
But this solution is not feasible
because the final solution violates the 2nd constraint $x_2 \geq 8$.

and the artificial variable A_1 appears in the basis with positive value 3

12. Primal and Dual Problems

There is always a corresponding Linear Programming Problem (LPP) associated to every LPP, which is called as dual problem of the original LPP(Primal Problem).

- **Duality** is a fundamental concept in linear programming that states **every linear programming problem (called the primal)** has an associated **dual problem**.
- **Statement of the Duality Theorem “**If the primal problem has an optimal solution, then its dual also has an optimal solution, and the two optimal objective values are equal”.

Why Duality is Important

- It provides **bounds** on the optimal value of the primal.
- If solving the primal is hard, the dual might be **easier to solve**.
- Used in **sensitivity analysis** to understand how changes in constraints affect the solution.
- Helps in **economic interpretation** of constraints and decision variables.

Advantages of Duality

- The dual may be simpler and quicker to solve than the primal.
- The optimal solution of Max and Min both can be achieved from same attempt.
- Provides insight into resource values, constraint tightness, and marginal utility.
- Strong duality theorem ensures that solving either problem leads to the same optimal value.
- Changes in the primal constraints can be studied using the dual variables.
- Sometimes, the dual better represents the true objective, such as cost minimization instead of profit maximization.

Characteristics of Duality

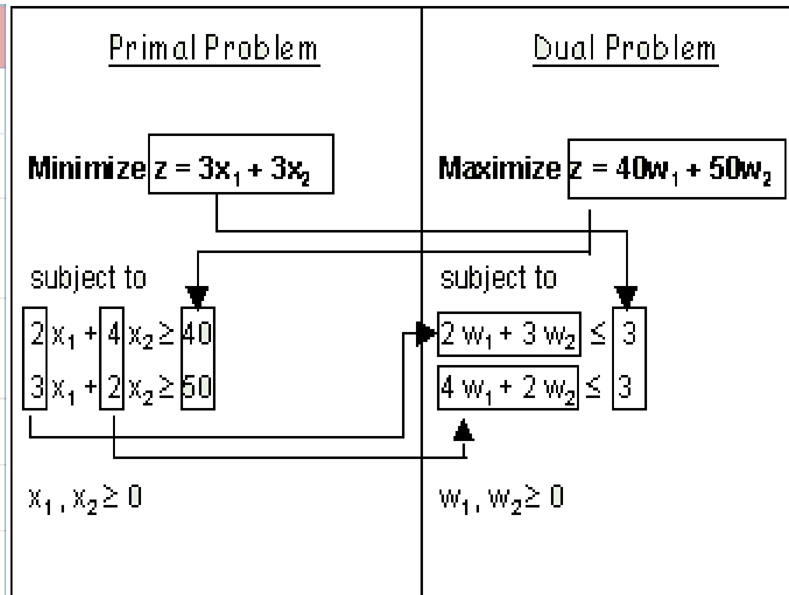
- The dual of the dual is primal.
- If any of the two problems (primal or dual problems) have a solution, the other must have a solution as well, and their optimum values must be equal.
- If either of the two problems has only an infeasible solution, the value of the other’s objective function will be unbounded.
- If either the primary or dual problem has an unbounded solution, the solution to the other problem is infeasible.
- If the primal problem has a feasible solution but the dual does not have, then the primal would not have a finite optimum solution, and vice-versa.

Primal Dual Relationship

- The number of constraints in the primal problem is equal to the number of dual variables, and *vice versa*.
- If the primal problem is a maximization problem, then the dual problem is a minimization problem and *vice versa*.

- If the primal problem has greater than or equal to type constraints, then the dual problem has less than or equal to type constraints and *vice versa*.
- The profit coefficients of the primal problem appear on the right-hand side of the dual problem.
- The rows in the primal become columns in the dual and *vice versa*.

PRIMAL	CONVERSION	DUAL
Maximization Problem	↔	Minimization Problem
Minimization Problem	↔	Maximization Problem
Objective Coefficients	↔	Right Hand Side (RHS) values
Right Hand Side (RHS) values	↔	Objective Coefficients
Number of Variables	↔	Number of Constraints
Number of Constraints	↔	Number of Variables
Variables are in terms of X n	↔	Variables are in terms of Y n



Primal LP :

$$\text{Max } z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \le b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \le b_m$$

$$x_1 \ge 0,\ x_2 \ge 0, \dots\dots x_j \ge 0, \dots\dots, x_n \ge 0.$$

Associated Dual LP :

$$\text{Min. } z = b_1y_1 + b_2y_2 + \dots + b_my_m$$

subject to:

$$a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m \ge c_1$$
$$a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m \ge c_2$$
$$\vdots$$
$$a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m \ge c_n$$

$$y_1 \ge 0,\ y_2 \ge 0, \dots\dots y_j \ge 0, \dots\dots, y_m \ge 0.$$

Steps to Formulate the Dual LPP

1. **Write the primal problem in standard form**
   - All constraints should be in **'≤' (for max)** or **'≥' (for min)** form.
   - All variables **must be non-negative**.
2. **Identify dual variables**
   - Number of dual variables = number of primal constraints.
   - Denote them as y1,y2 ,...ym **or** w1,w2 ,…..wm.
3. **Form the dual objective function**

- Use the **right-hand side values** of the primal constraints as coefficients.
- If primal is Max → **Dual** is Min, and vice versa.

4. **Form the dual constraints**
   - Number of dual constraints = number of primal variables.
   - Coefficients come from the **transpose** of the primal constraint matrix.
   - Inequality direction reverses (Max → '≥', Min → '≤').
5. **Non-negativity and signs**
   - If primal variables ≥ 0 → dual constraints are inequalities.
   - If primal variables are unrestricted → dual constraints are equalities.
   - Dual variables are **≥ 0**, unless specified.

Example:1. Find dual from primal conversion
MAX z = x1 - x2 + 3x3
subject to
x1 + x2 + x3 <= 10
2x1 - x2 - x3 <= 2
2x1 - 2x2 - 3x3 <= 6
and x1,x2,x3 >= 0
Solution:
Primal is

$$\text{MAX } z = x1 - x2 + 3x3$$

subject to

$$\begin{aligned} x1 + x2 + x3 &\leq 10 \\ 2x1 - x2 - x3 &\leq 2 \\ 2x1 - 2x2 - 3x3 &\leq 6 \end{aligned}$$

and $x1,x2,x3 \geq 0$;

→ In primal, there are 3 variables and 3 constraints, so in dual there must be 3 constraints and 3 variables.

→ In primal, the coefficient of objective function $c1=1,c2=-1,c3=3$ becomes right hand side constants in dual

→ In primal, the right-hand side constants $b1=10,b2=2,b3=6$ becomes coefficient of objective function in dual

→ In primal, objective function is maximizing, so in dual objective function must be minimizing

Let $y1,y2,y3$ be the dual variables

Dual is

$$\text{MIN } z = 10\,y1 + 2\,y2 + 6\,y3$$

subject to

$$\begin{aligned} y1 + 2y2 + 2y3 &\geq 1 \\ y1 - y2 - 2y3 &\geq -1 \\ y1 - y2 - 3y3 &\geq 3 \end{aligned}$$

and $y1,y2,y3 \geq 0$;

Example 2:

Min. $Z = 2x_2 + 5x_3$
Subject to:
$x_1 + x_2 \geq 2$ (1)
$2x_1 + x_2 + 6x_3 \leq 6$ (2)
$x_1 - x_2 + 3x_3 = 4$ (3)
$x_1, x_2, x_3 \geq 0$

Canonical Form: (all "≥" for minimization)

(2) ≤ 6 → multiply by −1: $-2x_1 - x_2 - 6x_3 \geq -6$
(3) = → split into two inequalities:
(3a) $x_1 - x_2 + 3x_3 \geq 4$
(3b) $-x_1 + x_2 - 3x_3 \geq -4$
So, the full canonical form is:

$$x_1 + x_2 \geq 2$$
$$-2x_1 - x_2 - 6x_3 \geq -6$$
$$x_1 - x_2 + 3x_3 \geq 4$$
$$-x_1 + x_2 - 3x_3 \geq -4$$
$$x_1, x_2, x_3 \geq 0$$

Dual Form:

Let dual variable per constraint: y_1, y_2, y_3, y_4 (all ≥ 0)
Max. $W = 2y_1 - 6y_2 + 4y_3 - 4y_4$ (Objective function from RHS)
Subject to:

$$y_1 - 2y_2 + y_3 - y_4 \leq 0$$
$$y_1 - y_2 - y_3 + y_4 \leq 2$$
$$-6y_2 + 3y_3 - 3y_4 \leq 5$$
$$y_1, y_2, y_3, y_4 \geq 0$$

Eliminate y_4

Let $y_3' = y_3 - y_4$ (which is unrestricted in sign)
Substituting, we eliminate y_4 and rewrite the dual as:
Maximize $W = 2y_1 - 6y_2 + 4y_3'$
Subject to:

$$y_1 - 2y_2 + y_3' \leq 0$$
$$y_1 - y_2 - y_3' \leq 2$$
$$-6y_2 + 3y_3' \leq 5$$
$y_1, y_2 \geq 0$; y_3' is unrestricted in sign

- **Primal problem**

$$\begin{aligned} \text{maximize } z &= 5x_1 + 12x_2 + 4x_3 \\ \text{subject to } \quad & x_1 + 2x_2 + x_3 \leq 10 \\ & 2x_1 - x_2 + 3x_3 = 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

- Primal in equation form

$$\begin{aligned} \text{maximize } z &= 5x_1 + 12x_2 + 4x_3 + 0x_4 \\ \text{subject to } \quad & x_1 + 2x_2 + x_3 + x_4 = 10 \\ & 2x_1 - x_2 + 3x_3 + 0x_4 = 8 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

- Dual

$$\begin{aligned} \text{minimize } w &= 10y_1 + 8y_2 \\ \text{subject to } \quad & y_1 + 2y_2 \geq 5 \\ & 2y_1 - y_2 \geq 12 \\ & y_1 + 3y_2 \geq 4 \\ & y_1 + 0y_2 \geq 0 \\ & y_1, y_2 \text{ unrestricted} \end{aligned}$$

- **Dual problem**

$$\begin{aligned} \text{minimize } w &= 10y_1 + 8y_2 \\ \text{subject to } \quad & y_1 + 2y_2 \geq 5 \\ & 2y_1 - y_2 \geq 12 \\ & y_1 + 3y_2 \geq 4 \\ & y_1 \geq 0 \\ & y_2 \text{ unrestricted} \end{aligned}$$

- **Primal problem**

$$\begin{aligned} \text{minimize } z &= 15x_1 + 12x_2 \\ \text{subject to } \quad x_1 + 2x_2 &\geq 3 \\ 2x_1 - 4x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

- Primal in equation form

$$\begin{aligned} \text{minimize } z &= 15x_1 + 12x_2 + 0x_3 + 0x_4 \\ \text{subject to } \quad x_1 + 2x_2 - x_3 + 0x_4 &= 3 \\ 2x_1 - 4x_2 + 0x_3 + x_4 &= 5 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

- Dual

$$\begin{aligned} \text{maximize } w &= 3y_1 + 5y_2 \\ \text{subject to } \quad y_1 + 2y_2 &\leq 15 \\ 2y_1 - 4y_2 &\leq 12 \\ -y_1 + 0y_2 &\leq 0 \\ 0y_1 + y_2 &\leq 0 \\ y_1, y_2 &\text{ unrestricted} \end{aligned}$$

- **Dual problem**

$$\begin{aligned} \text{maximize } w &= 3y_1 + 5y_2 \\ \text{subject to } \quad y_1 + 2y_2 &\leq 15 \\ 2y_1 - 4y_2 &\leq 12 \\ y_1 &\geq 0 \\ y_2 &\leq 0 \end{aligned}$$

- **Primal problem**

$$\begin{aligned} \text{maximize } z &= 5x_1 + 6x_2 \\ \text{subject to } \quad x_1 + 2x_2 &= 5 \\ -x_1 + 5x_2 &\geq 3 \\ 4x_1 + 7x_2 &\leq 8 \\ x_1 \text{ unrestricted, }, x_2 &\geq 0 \end{aligned}$$

- Primal equation form (here $x_1 = x_1^- - x_1^+$)

$$\begin{aligned} \text{maximize } z &= 5x_1^- - 5x_1^+ + 6x_2 \\ \text{subject to } \quad x_1^- - x_1^+ + 2x_2 &= 5 \\ -x_1^- + x_1^+ + 5x_2 - x_3 &= 3 \\ 4x_1^- - 4x_1^+ + 7x_2 + x_4 &= 8 \\ x_1^-, x_1^+, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

- Dual

$$\begin{aligned} \text{minimize } z &= 5y_1 + 3y_2 + 8y_3 \\ \text{subject to } \quad y_1 - y_2 + 4y_3 &\geq 5 \\ -y_1 + y_2 - 4y_3 &\geq -5 \\ 2y_1 + 5y_2 + 7y_3 &\geq 6 \\ -y_2 &\geq 0 \\ y_3 &\geq 0 \\ y_1, y_2, y_3 &\text{ unrestricted} \end{aligned}$$

- **Dual problem**

$$\begin{aligned} \text{minimize } z &= 5y_1 + 3y_2 + 8y_3 \\ \text{subject to } \quad y_1 - y_2 + 4y_3 &= 5 \\ 2y_1 + 5y_2 + 7y_3 &\geq 6 \\ y_2 &\leq 0 \\ y_3 &\geq 0 \\ y_1 &\text{ unrestricted} \end{aligned}$$

Dual Simplex Method

The Dual Simplex Method is a specific algorithm that works on the dual problem, often when the primal problem has certain characteristics (like an optimal but infeasible solution).

Algorithm

Step-1:	**Formulate the Problem** a. Formulate the mathematical model of the given linear programming problem. b. If the objective function is minimization type then change it into maximization type. c. all $\geq$ constraint to $\leq$ constraint multiplying by -1

	d. Transform every ≤ constraint into an = constraint by adding a slack variable to every constraint and assign a 0 cost coefficient in the objective function.
Step-2:	**Find out the Initial basic solution** Find the initial basic feasible solution by setting zero value to the decision variables
Step-3:	**Test for Optimality** a. If all the values of $X_B \geq 0$, then current solution is the optimal solution, terminate the process. b. If any $X_B < 0$, then select the Minimum negative X_B and this row is called key row. c. Find $\text{Ratio} = \dfrac{C_j - Z_j}{\text{KeyRow}_j}$ and $\text{KeyRow}_j < 0$. d. Find Minimum positive ratio, and this column is called key column. e. Find the new solution table. f. Repeat step-3.

Example 1: Find solution using dual-simplex method

$$\text{Min } Z = 2x1 + 3x2$$

subject to

$$2x1 - x2 - x3 \geq 3$$
$$x1 - x2 + x3 \geq 2$$

and $x1, x2, x3 \geq 0$;

Solution:
In order to apply the dual simplex method, convert Min Z to Max Z and all ≥ constraint to ≤ constraint by multiply -1.
The given LPP becomes:

$$\text{Max } Z = -2x1 - 3x2$$

subject to

$$-2x1 + x2 + x3 \leq -3$$
$$-x1 + x2 - x3 \leq -2$$

and $x1, x2, x3 \geq 0$;

After introducing slack variables

$$\text{Max } Z = -2x1 - 3x2 + 0x3 + 0S1 + 0S2$$

subject to

$$-2x1 + x2 + x3 + S1 = -3$$
$$-x1 + x2 - x3 + S2 = -2$$

and $x1, x2, x3, S1, S2 \geq 0$

Iteration-1		C_j	-2	-3	0	0	0
B	**C_B**	**X_B**	**x_1**	**x_2**	**x_3**	**S_1**	**S_2**
S_1	0	-3	**(-2)**	1	1	1	0
S_2	0	-2	-1	1	-1	0	1
$Z=0$		**Z_j**	**0**	**0**	**0**	**0**	**0**
		Z_j-C_j	2	3	0	0	0
		Ratio=Z_j-C_j/S_1 and $S_1<0$	-1↑	---	---	---	---

Minimum negative X_B is -3 and its row index is 1. So, the leaving basis variable is S_1.

Maximum negative ratio is -1 and its column index is 1. So, the entering variable is x_1.

∴ The pivot element is -2.

Entering $=x_1$, Departing $=S_1$, Key Element =-2

$$R_1(\text{new}) = R_1(\text{old}) \div (-2)$$

$$R_2(\text{new}) = R_2(\text{old}) + R_1(\text{new})$$

Iteration-2		C_j	-2	-3	0	0	0
B	**C_B**	**X_B**	**x_1**	**x_2**	**x_3**	**S_1**	**S_2**
x_1	-2	$\frac{3}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0
S_2	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\left(-\frac{3}{2}\right)$	$-\frac{1}{2}$	1
$Z=-3$		**Z_j**	**-2**	**1**	**1**	**1**	**0**
		Z_j-C_j	0	4	1	1	0
		Ratio $=\dfrac{Z_j-C_j}{S_2,j}$ and $S_2,j<0$	---	---	-0.6667 ↑	-2	---

Minimum negative X_B is -1/2 and its row index is 2. So, the leaving basis variable is S_2.

Maximum negative ratio is -0.6667 and its column index is 3. So, the entering variable is x_3.

∴ The pivot element is -3/2.

Entering $=x_3$, Departing $=S_2$, Key Element = -3/2

$$R_2(\text{new}) = R_2(\text{old}) \times \left(-\frac{2}{3}\right)$$

$$R_1(\text{new}) = R_1(\text{old}) + \frac{1}{2}R_2(\text{new})$$

Iteration-3		C_j	-2	-3	0	0	0
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2
x_1	-2	$\frac{5}{3}$	1	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$
x_3	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	1	$\frac{1}{3}$	$-\frac{2}{3}$
$Z = -\frac{10}{3}$		Z_j	-2	$\frac{4}{3}$	0	$\frac{2}{3}$	$\frac{2}{3}$
		$Z_j - C_j$	0	$\frac{13}{3}$	0	$\frac{2}{3}$	$\frac{2}{3}$
		Ratio	---	---	---	---	---

Since all $Z_j-C_j \geq 0$ and all $X_B \geq 0$ thus the current solution is the optimal solution.
Hence, optimal solution is arrived with value of variables as :
$x_1=5/3, x_2=0, x_3=1/3$
Max $Z=-10/3$
∴ Min $Z=10/3$

Example 2. Find solution using dual-simplex method

Max $Z = -2x_1 - x_2$

subject to

$-3x_1 - x_2 \leq -3$

$-4x_1 - 3x_2 \leq -6$

$-x_1 - 2x_2 \leq -3$

and $x_1, x_2 \geq 0$;

After introducing slack variables

Max $Z = -2x_1 - x_2 + 0S_1 + 0S_2 + 0S_3$

subject to

$-3x_1 - x_2 + S_1 = -3$

$-4x_1 - 3x_2 + S_2 = -6$

$-x_1 - 2x_2 + S_3 = -3$

and $x_1, x_2, S_1, S_2, S_3 \geq 0$

Iteration-1		C_j	-2	-1	0	0	0
B	**C_B**	**X_B**	**x_1**	**x_2**	**S_1**	**S_2**	**S_3**
S_1	0	-3	-3	-1	1	0	0
S_2	0	-6	-4	**(-3)**	0	1	0
S_3	0	-3	-1	-2	0	0	1
$Z=0$		**Z_j**	**0**	**0**	**0**	**0**	**0**
		Z_j-C_j	2	1	0	0	0
		Ratio=Z_j-C_j/S_2 and $S_2<0$	-0.5	-0.3333↑	---	---	---

Minimum negative X_B is -6 and its row index is 2. So, the leaving basis variable is S_2.
Maximum negative ratio is -0.3333 and its column index is 2. So, the entering variable is x_2.
∴ The pivot element is -3.
Entering $=x_2$, Departing $=S_2$, Key Element =-3

$$R_2(\text{new}) = R_2(\text{old}) \div (-3)$$

$$R_1(\text{new}) = R_1(\text{old}) + R_2(\text{new})$$

$$R_3(\text{new}) = R_3(\text{old}) + 2R_2(\text{new})$$

Iteration-2		C_j	-2	-1	0	0	0
B	**C_B**	**X_B**	**x_1**	**x_2**	**S_1**	**S_2**	**S_3**
S_1	0	-1	$\left(-\frac{5}{3}\right)$	0	1	$-\frac{1}{3}$	0
x_2	-1	2	$\frac{4}{3}$	1	0	$-\frac{1}{3}$	0
S_3	0	1	$\frac{5}{3}$	0	0	$-\frac{2}{3}$	1
$Z = -2$		**Z_j**	$-\frac{4}{3}$	-1	**0**	$\frac{1}{3}$	**0**
		$Z_j - C_j$	$\frac{2}{3}$	0	0	$\frac{1}{3}$	0
		Ratio $= \frac{Z_j - C_j}{S_{1,j}}$ and $S_{1,j} < 0$	-0.4 ↑	---	---	-1	---

Minimum negative X_B is -1 and its row index is 1. So, the leaving basis variable is S_1.

Maximum negative ratio is -0.4 and its column index is 1. So, the entering variable is x_1.

∴ The pivot element is -5/3.

Entering $=x_1$, Departing $=S_1$, Key Element $=-5/3$

$$R_1(\text{new}) = R_1(\text{old}) \times \left(-\frac{3}{5}\right)$$

$$R_2(\text{new}) = R_2(\text{old}) - \frac{4}{3} R_1(\text{new})$$

$$R_3(\text{new}) = R_3(\text{old}) - \frac{5}{3} R_1(\text{new})$$

Iteration-3		C_j	-2	-1	0	0	0
B	C_B	X_B	x_1	x_2	S_1	S_2	S_3
x_1	-2	$\frac{3}{5}$	1	0	$-\frac{3}{5}$	$\frac{1}{5}$	0
x_2	-1	$\frac{6}{5}$	0	1	$\frac{4}{5}$	$-\frac{3}{5}$	0
S_3	0	0	0	0	1	-1	1
$Z=-\frac{12}{5}$		Z_j	-2	-1	$\frac{2}{5}$	$\frac{1}{5}$	0
		$Z_j - C_j$	0	0	$\frac{2}{5}$	$\frac{1}{5}$	0
		Ratio	---	---	---	---	---

Since all $Z_j - C_j \geq 0$ and all $X_B \geq 0$ thus the current solution is the optimal solution.

Hence, optimal solution is arrived with value of variables as :
$x_1=35, x_2=6/5$

Max $Z=-12/5$

13. Economic Interpretation: Real-world meaning of optimal solution & resources

Linear Programming Problems (LPPs) are powerful mathematical tools used to optimize a desired outcome (like maximizing profit or minimizing cost) given a set of constraints (like limited resources, production capacities, or demand).

The **"Economic Interpretation"** of an LPP is all about understanding what the mathematical solution means in terms of real-world business decisions and resource allocation. The associated dual problem offers interesting economic interpretations of the limiting resource allocation model.

In economics, duality refers to the connection between a primal optimization problem (like maximizing profit) and its corresponding dual problem (like minimizing costs). The dual problem provides insights into the marginal value of resources used in the primal problem, often called "shadow prices".

Let's take the example of the furniture factory. Suppose you want to figure out how much profit you make by selling chairs and tables. This is called the "Primal" problem. Now, you also want to know what the value of the wood and the labor you used should be. This is called the "Dual" problem. Duality shows how these two problems are connected and how the solution of one provides information about the other.

Primal Problem:

- Typically involves maximizing a function (e.g., profit) subject to constraints (e.g., resource limitations).
- Represents a resource allocation problem where the goal is to find the optimal combination of activities (like production levels) to achieve the best outcome.

Dual Problem:

- Is constructed from the primal problem using specific rules.
- Often involves minimizing a function (e.g., cost of resources) subject to constraints.
- Provides an alternative perspective on the same economic situation, focusing on the value of the resources used.

Economic Interpretation:

- The dual problem's variables (often called dual variables or shadow prices) represent the marginal value of the corresponding resources in the primal problem.
- For example, if the primal problem is about maximizing profit by allocating raw materials and labor, the dual problem might be about minimizing the cost of those resources.
- The shadow price of a raw material in the dual problem would then represent the additional profit the company could make by having one more unit of that raw material available.
- The duality theorem essentially states that the maximum profit in the primal problem equals the minimum cost in the dual problem.

Example:

- A company produces two products using raw materials and labor.
- The primal problem is to maximize profit by determining the optimal production quantities.
- The dual problem could be to minimize the cost of raw materials and labor, given the production requirements.
- The shadow price of a particular raw material in the dual problem would indicate how much the company would be willing to pay for an extra unit of that material, given the existing production plan.

In essence, duality provides a powerful framework for understanding the relationships between resource allocation, costs, and the value of resources in economic decision-making.

Example Problem:

A company produces two products, A (x_1) and B (x_2), with the following constraints:
- **Machine Time**: Each unit of A requires 4 hours, and each unit of B requires 2 hours. Total available machine time is 80 hours.
- **Labor Hours**: Each unit of A requires 1 hour, and each unit of B requires 3 hours. Total available labor is 60 hours.
- **Demand Constraint**: At least 10 units of A must be produced.
- **Costs**: The cost of producing A is $20/unit, and B is $40/unit.

Objective: Minimize total production cost.

Mathematical Formulation (Primal LPP):

1. Primal Problem: Producer's Perspective (Cost Minimization)

Minimize $Z = 20x_1 + 40x_2$
Subject to:
$4x_1 + 2x_2 \geq 80$ (Machine Time)
$x_1 + 3x_2 \geq 60$ (Labor Hours)
$x_1 \geq 10$ (Demand Constraint)
$x_1, x_2 \geq 0$

Economic Interpretation of the Primal Problem:

1. Decision Variables (x_1, x_2): Quantities of products A and B to produce.
2. Objective Function: Minimizes total production cost.
3. Constraints:
   - Machine Time: Ensures at least 80 hours are used.
   - Labor Hours: Ensures at least 60 labor hours are used.
   - Demand Constraint: Ensures at least 10 units of A are produced.

2. Dual Problem: Resource Valuer's Perspective (Shadow Prices)

The dual problem assigns shadow prices (y_1, y_2, y_3) to each resource:
- y_1: Value of an additional hour of machine time.
- y_2: Value of an additional labor hour.
- y_3: Value of relaxing the demand constraint.

The corresponding dual LPP of primal LPP is

Maximize $W = 80y_1 + 60y_2 + 10y_3$
Subject to:
$4y_1 + y_2 + y_3 \leq 20$ (Constraint from x_1)
$2y_1 + 3y_2 \leq 40$ (Constraint from x_2)
$y_1, y_2, y_3 \geq 0$

Economic Interpretation of the Dual Problem:

1. Objective Function: Represents the total imputed value of resources.
2. Constraints:
   - $4y_1 + y_2 + y_3 \leq 20$: Resource cost for A must not exceed $20.
   - $2y_1 + 3y_2 \leq 40$: Resource cost for B must not exceed $40.
3. Shadow Prices:
   - y_1: Cost reduction from 1 more hour of machine time.

- y_2: Cost reduction from 1 more hour of labor.
- y_3: Cost reduction from reducing demand for A.

3. Optimal Solution and Economic Insights

Solving the Primal Problem:

Suppose the optimal solution is:
$x_1 = 15$, $x_2 = 10$, Min $Z = 20(15) + 40(10) = 700$

Constraint Usage:
- Machine Time: $4(15) + 2(10) = 80$ (fully used)
- Labor Hours: $15 + 3(10) = 45 < 60$ (slack = 15)
- Demand: $x_1 = 15 \geq 10$ (slack = 5)

Solving the Dual Problem:

Dual optimal solution (shadow prices):
$y_1 = 5$, $y_2 = 0$, $y_3 = 0$

Interpretation:
- $y_1 = 5$: 1 extra hour of machine time reduces cost by $5.
- $y_2 = 0$: Extra labor does not reduce cost.
- $y_3 = 0$: Reducing demand for A does not reduce cost.

4. Managerial Implications

1. Resource Allocation:
   - Invest in more machine time ($y_1 = 5$).
   - Do not hire more labor ($y_2 = 0$).
2. Cost Reduction:
   - Each additional machine hour saves $5.
3. Production Strategy:
   - Labor is underutilized; consider reallocating workforce.

Conclusion

- Primal Problem: Focuses on minimizing cost while meeting constraints.
- Dual Problem: Reveals marginal value (shadow prices) of resources.
- Economic Insight: Helps managers decide where to invest and cut costs.

This duality framework is widely used in supply chain optimization, production planning, and financial decision-making.

Term	Meaning
Optimal Solution	Best possible value of objective (profit/cost)
Slack Variable	Unused resource (e.g., time, material)
Surplus Variable	Amount exceeding minimum requirements
Shadow Price	Change in objective value for 1-unit resource change
Dual Variables	Implied worth/value of resources in the system