Chapter 2

Linear Programming Problem (LPP)

→ In any business, our main objective is profit maximization or cost minimization. So here inside the model of LPP 1st we have the objective function:

i.e. Max $z = x+2y$ or/ Min $z = 2x+3y$ → objective function

Also, in business we have limited resources like money, time, materials, machinery, etc and these resources which we connect with the business model is called constraint which is always written in the less than or greater than format

i.e. $2x+3y \geq 5$ → constraints
$x+y \leq 7$

And all these resources are always positive

i.e. $x, y \geq 0$ → non-negative variable

* LP (Linear programming)

→ LP is a mathematical technique used in OR to find the best possible outcome (such as maximum profit or min cost) in a given mathematical model, where the relationships are linear.

→ LP is a mathematical technique to find the best optimal solution for problems involving the allocation of limited resources to meet specific objectives.

* Terminology used in LPP / components of LPP

1. Decision variables: variables that decide the output (eg. $x_1, x_2, \ldots$) $x, y, \ldots$
2. Objective function: A mathematical expression that defines what you want to optimize (e.g. max profit, min cost)
3. Constraints: Limitations or restrictions on the resources available. It should be linear equations (=) or inequalities ($<, >, \leq$)
4. Non-negative restriction: Variable must be ≥ 0, can't be negative.

Eg: → Max $z = x+2y$

Subject to:

$2x+3y \geq 5$
$x+y \leq 7$
and $x, y \geq 0$

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* Mathematical formulation of LPP

→ It is the process of expressing a real-world optimization problem using mathematical equations and inequalities.

* Steps involved:

1, Identify decision variables (unknowns to be determined)
2, Write an objective function (maximize / minimize)
3, Formulate the constraints (inequalities / equations)
4, Include non-negativity restriction

• Example 1: A company produces tables and chairs. Profit per table = Rs. 50, profit per chair = Rs. 30. Total wood available = 400 kg, Each table requires 20 kg & chair 10 kg. Max labor hour = 80, table requires 4 hrs & chair 2 hrs. Formulate the problem as a LPP, to determine how many tables & chairs required to produce maximum profit.

→ Soln

	tables	chair	Total
wood	20	10	400
labor	4	2	80

Define decision variable → let, x = no. of tables
y = no. of chairs

Establish objective function → In order to maximize profit, we establish objective function as

profit $z = 50x + 30y$

Since, each table requires 20 kg and chair requires 10 kg of total wood available. so, 1st constraint can be expressed as:

Formulate constraints → $20x + 10y \leq 400$

Similarly, for labour hour

$4x + 2y \leq 80$

Non-negative restriction → Also, x & y must be ≥ 0, since there is no -ve production. So, we can write $x \geq 0$ and $y \geq 0$

Above problem can be summarized as:

Maximize profit $z = 50x + 30y$

subject to constraints: $20x + 10y \leq 400$

$4x + 2y \leq 80$

$x, y \geq 0$

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Example 2:

A firm manufactures 3 products A, B, C. Time to manufacture product A is twice that for B and trice that for C and to be produced in the ratio 3:4:5. The relevant data is give in the following table. If the whole raw material is engagged in manufacturing product A, 1600 units of this product can be produced. There is demand for atleast 300, 250, 200 units of products A, B and C and the profit earned per unit is Rs. (50, 40, 70) respectively. Formulate the problem as LPP.

Raw Material	Reqmt / unit of product (kg)			Total (kg) Availability
	A	B	C	
P	6	5	9	5000
Q	4	7	8	6000

→ Soln:

Let, x_1 = no. of product A produced (units)

x_2 = no. of " B "

x_3 = " C "

In order to maximize profit, we setup objective function as:

Maximize $Z = 50x_1 + 40x_2 + 70x_3$

According to Raw material requirement table

Since, per unit of product A require ~~6kg~~ of raw material 'P' and corresponding requirement of product B and C are 5 and 9 kg. So the 1st constraints can be expressed as

→ $6x_1 + 5x_2 + 9x_3 \le 5000$

||ly, to for raw material Q, constraint is

→ $4x_1 + 7x_2 + 8x_3 \le 6000$

Demand constraints

$x_1 \ge 300$

$x_2 \ge 250$

$x_3 \ge 200$

non-negativity constraints

production cannot be -ve, so

$x_1 \ge 0$

$x_2 \ge 0$

$x_3 \ge 0$

Thus, the final LPP model is

Maximize $Z = 50x_1 + 40x_2 + 70x_3$

subject to :

$6x_1 + 5x_2 + 9x_3 \le 5000$

$4x_1 + 6x_2 + 8x_3 \le 6000$

$x_1 \ge 300$

$x_2 \ge 250$

$x_3 \ge 200$

$x_1, x_2, x_3 \ge 0$

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Example 3 :

A chair manufacturing company produces two types of chairs, A and B, by using 3 machines, X, Y, Z. The time required for each chair on each machine and the maximum time available on are given below :

Machines	Time required for each type		Max. time available per week in hrs
	A	B	
X	6	8	120
Y	8	4	100
Z	12	4	144

The profit on the A and B are rupees 500 & 300 respectively. What combn of pains should be produced to obtain max profit.

Soln:

Let, x = no. of chairs of type A

y = no. of chairs of type B

Objective function :

Maximize $Z = 500x + 300y$

Subject to constraints :

$$\begin{aligned} 6x + 8y &\le 120 \\ 8x + 4y &\le 100 \\ 12x + 4y &\le 144 \\ x, y &\ge 0 \end{aligned} \Rightarrow \begin{aligned} 3x + 4y &\le 60 \\ 2x + y &\le 25 \\ 3x + y &\le 36 \\ x, y &\ge 0 \end{aligned}$$

* Methods for solving LPP

- Graphical Method (two variables only)
- Simplex method (more than two variables)

1) Graphical Method

→ It is a technique used to solve optimization problems, specially those with two decision variables, by visualizing the problems on a graph.

* Steps involves:

- plot constraints as lines on a graph.
- Identify the feasible region satisfying all constraints.
- Determine the corner points (vertices) of the feasible region
- Evaluate the objective function at each corner point.
- Find the optimal solution (corner points giving max/min value)

- Example 1:

Max $z = 3x_1 + 4x_2$

$x_1 + 2x_2 \le 4$ →

x_1	0	4
x_2	2	0

$3x_1 + 2x_2 \le 6$ →

x_1	0	2
x_2	3	0

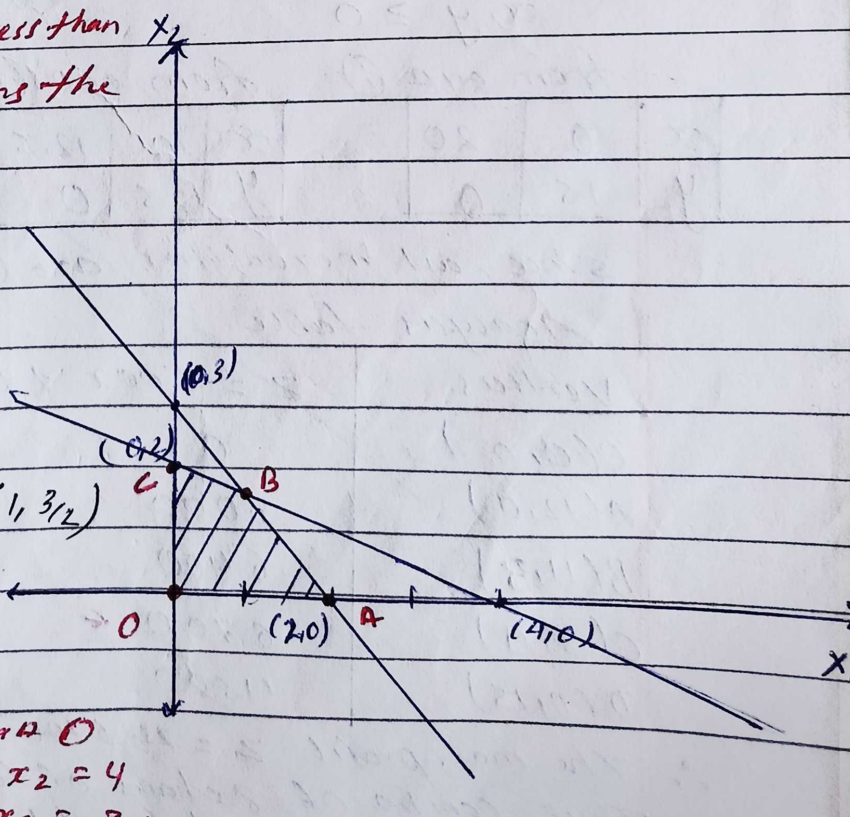
$x_1, x_2 \ge 0$

Note: When inequalities are less than $(<, \le)$, the graph contains the origin

Analysis table

vertices	$z = 3x_1 + 4x_2$
O(0,0)	0
A(2,0)	6
B(1,3/2)	9
C(0,2)	8

$\therefore$ Max $z = 9$ at $(1, 3/2)$



solving for vertex B

$x_1 + 2x_2 = 4$

$3x_1 + 2x_2 = 6$

$-2x_1 = -2$

$\therefore x_1 = 1$

& from eq ①

$1 + 2x_2 = 4$

$\therefore x_2 = 3/2$

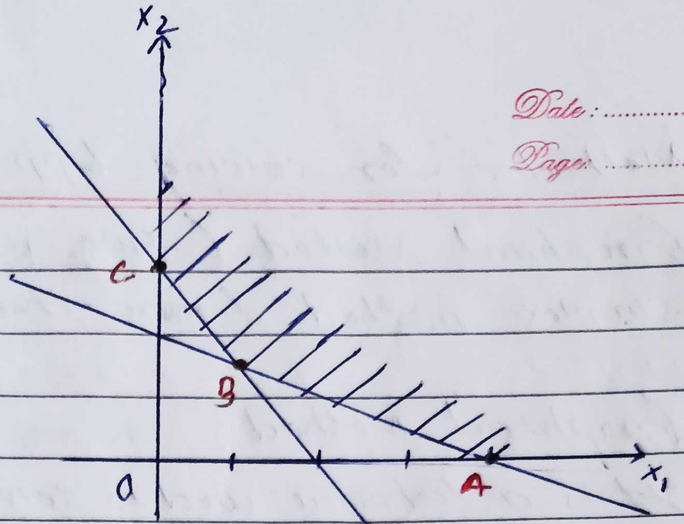
- Example 2 :

Min $Z = 3x_1 + 4x_2$

$x_1 + 2x_2 \geq 4$

$3x_1 + 2x_2 \geq 6$

$x_1, x_2 \geq 0$



Sol^n

Since, inequalities are greater than $(>, \geq)$, so the graph doesn't contain origin.

Analysis table

Vertices	$Z = 3x_1 + 4x_2$
A(4,0)	Z at A = 12
B(1, 3/2)	Z at B = 9
C(0,3)	Z at C = 12

$\therefore$ Z is minimum at $x_1 = 1$ and $x_2 = 3/2$

2023 ✓ Example 3 : continue of previous example 3

Max $Z = 500x + 300y$

$3x + 4y \leq 60$ —— (i)

$2x + y \leq 25$ —— (ii)

$3x + y \leq 36$ —— (iii)

$x, y \geq 0$

from eqn (i)

x	0	20
y	15	0

from eqn (ii)

x	0	12.5
y	25	0

from eqn (iii)

x	0	12
y	36	0

Since, all inequalities are $(\leq)$ so, graph contains origin.

Analysis table

Vertices	$Z = 500x + 300y$
O(0,0)	0
A(12,0)	6000
B(11,3)	6400
C(8,9)	6700 ←
D(0,15)	4500

$\therefore$ The max. profit Z = Rs. 6700 occur at x = 8 & y = 9

Thus, combo of 8 chairs of type A & 9 chairs of type B should be produced to obtain max profit.

solving for vertex B

As it is touched by eqn (ii) & (iii)

so, $2x + y = 25$

$\underline{-3x + y = -36}$

$+x = +11$

$\therefore x = 11$ & $y = 3$

solving for vertex C

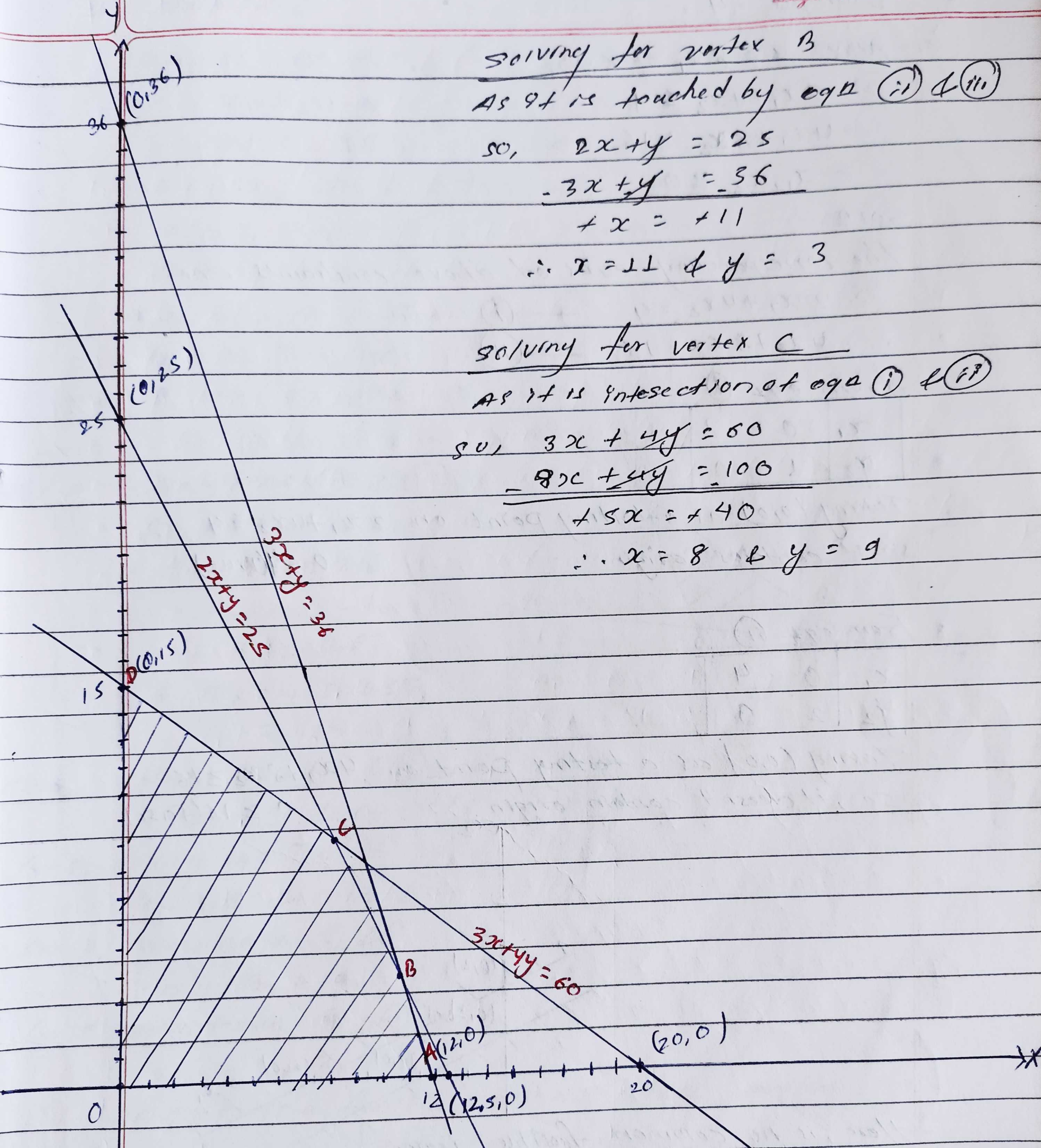
As it is intesection of eqn (i) & (ii)

so, $3x + 4y = 60$

$\underline{-8x + 4y = -100}$

$+5x = +40$

$\therefore x = 8$ & $y = 9$



Example - 4 :

Max $z = 6x_1 - 4x_2$

$2x_1 + 4x_2 \leq 4$

$4x_1 + 8x_2 \geq 16$

$x_1, x_2 \geq 0$

Soln

The corresponding eqns of above constraints are

$2x_1 + 4x_2 = 4$ —— (i)

$4x_1 + 8x_2 = 16$ —— (ii)

From eqn (i)

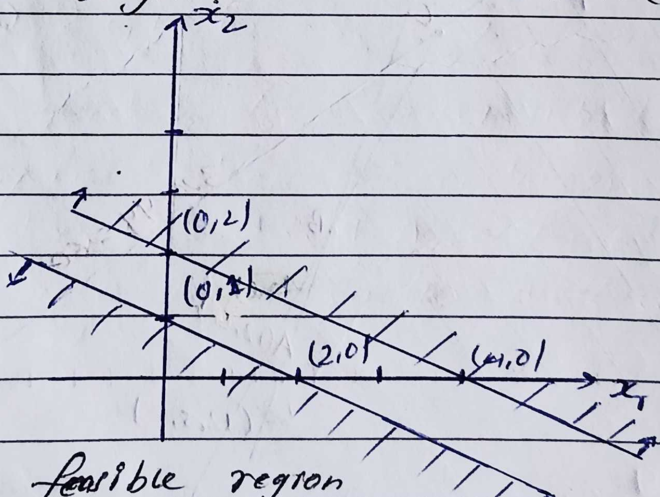
x_1	0	2
x_2	1	0

Taking (0,0) as a testing point on $2x_1 + 4x_2 \leq 4$,
so, it contains origin. $0 \leq 4$ (True)

From eqn (ii)

x_1	0	4
x_2	2	0

Taking (0,0) as a testing point on $4x_1 + 8x_2 \geq 16$
so, it doesn't contain origin. $0 \geq 16$ (False)



Here, is no common feasible region
so, these type of problem has no solution

Example 5 :

$$\text{Min } z = 2x_1 + 3x_2$$

$x_1 + x_2 \le 4$

$6x_1 + 2x_2 \ge 8.$

$x_1 + 5x_2 \ge 4$

$x_1 \le 3, \; x_2 \le 3$

$x_1 \ge 0, \; x_2 \ge 0$

$\Rightarrow$ The corresponding eqns are :

$x_1 + x_2 = 4$ —— (i)

$6x_1 + 2x_2 = 8$ —— (ii)

$x_1 + 5x_2 = 4$ —— (iii)

$x_1 = 3$ —— (iv)

$x_2 = 3$ —— (v)

From eqn (i)

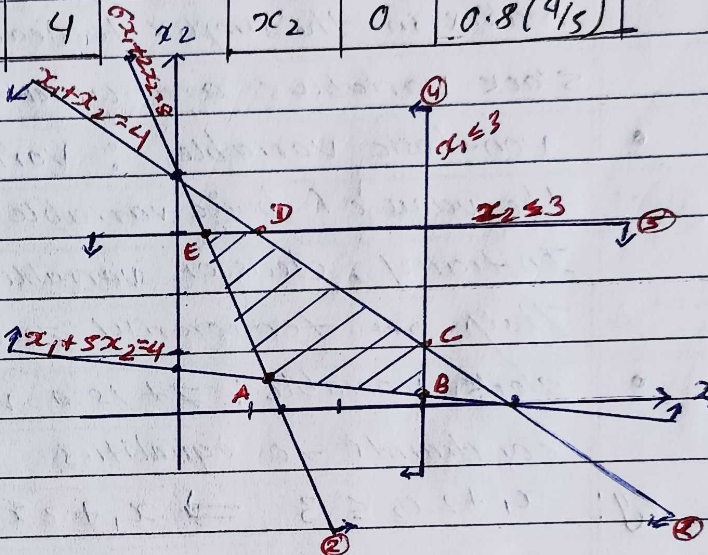
x_1	4	0
x_2	0	4

From eqn (ii)

x_1	4/3 (1.3)	0
x_2	0	4

from eqn (iii)

x_1	4	0
x_2	0	0.8(4/5)



Now,

for A, solving eqn (ii) & (iii)

$$6x_1 + 2x_2 = 8$$
$$-\; 6x_1 + 30x_2 = 24$$
$$-28x_2 = -16$$
$$\therefore x_2 = 0.57$$

& $x_1 = 4 - 5 \times 0.57$
$= 1.14$

For B, solving eqn (iii) & (iv)

$x_1 = 3$ & $x_2 = 1/5$

for C, solving eqn (i) & (iv)

$x_1 = 3$ & $x_2 = 1$

for D, solving eqn (i) & (v)

$x_2 = 3, \; x_1 = 1$

for E, solving eqn (ii) & (v)

$x_2 = 3, \; x_1 = 1/3$

$\therefore$ ABCDE is a feasible region whose vertices are

Vertices	$z = 2x_1 + 3x_2$
A (1.14, 0.57)	4 ←
B (3, 1/5)	66
C (3, 1)	9
D (1, 3)	11
E (1/3, 3)	9.66

$\therefore$ Min $z = 4$ at $x_1 = 1.14$ & $x_2 = 0.57$

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2) Simplex Method

→ It is a systematic algorithm used to solve linear programming problems, particularly those with more than two decision variables, by iteratively moving from one feasible corner point to another until the optimal solution is found.

* Terminologies used for solving LPP by simplex method

- Basic variables (BV) : Variables that form an identity matrix in the simplex tableau are basic variables. Initially, slack variables are usually basic variables.
- Non-basic variable : Variables set to zero to calculate the value of basic variable are non basic variable. Initially, decision variables (x_1, x_2) are non-basic. They are temporarily zero but may enter the basic later.
- Slack variable : It is a variable added to convert "$\leq$" constraints to equalities

Eg: $x_1 + 2x_2 \leq 3 \Rightarrow x_1 + 2x_2 + S_1 = 3$

- Surplus variable : It is a variable subtracted to convert "$\geq$" constraints to equalities

eg: $2x_1 + 3x_2 \geq 7 \Rightarrow 2x_1 + 3x_2 - S_2 = 7$

- Artificial variable : It is a temporarily introduced variable to maintain feasibility in '$\geq$' or '$=$' constraints.

Eg: $2x_1 + 3x_2 \geq 7 \Rightarrow 2x_1 + 3x_2 - S_2 + A = 7$

Used in Big M or Two-phase simplex Methods.

- C_j row : coeff. of variable in objective function (Z) ∴ $C_j = [5, 3, 0, 0]$
- Z_j row : For each column, $Z_j = \Sigma(C_B \times \text{column entries})$
- C_B : objective coeff. of current B.V. (initially $C_B = [0, 0]$
- b : constant terms on the RHS of constraints $b = [15, 10]$

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- Example 1 :

⇒ Max $Z = 5x_1 + 3x_2$

• Min Ratio : $\frac{\text{Const }(b)}{\text{column entries}}$

Subject to : $3x_1 + 5x_2 \leq 15$

$5x_1 + 2x_2 \leq 10$

$x_1, x_2 \geq 0$

Introducing slack variables in given LPP :

$3x_1 + 5x_2 + s_1 + 0 \cdot s_2 = 15$

$5x_1 + 2x_2 + 0 \cdot s_1 + s_2 = 10$

$Z = 5x_1 + 3x_2 + 0 \cdot s_1 + 0 \cdot s_2$

Developing initial simplex tableau

simplex table 1

		C_j	5	3	0	0	
C_B	B.V.	b	x_1	x_2	s_1	s_2	Min Ratio (b/x_1)
0	s_1	15	3	5	1	0	$15/3 = 5$
0	s_2	10	5	2	0	1	$10/5 = 2$ ← leaving variable
	Z_j		0	0	0	0	
	$Z_j - C_j$		-5	-3	0	0	
			↑ Entering variable				

Here, largest -ve $(Z_j - C_j)$ is -5 and its column index is 1 (x_1).
so, the entering variable is x_1.

Also, the min Ratio is 2 & its row index is 2 (i.e. s_2). so, the leaving variable is s_2. (Means s_2 is replaced by x_1 in B.V.)

∴ The key (pivot) element is 5

Now, we have to make key element 1 & corresponding element of key column as 0.

Key row → $R_2(\text{new}) = R_2(\text{old}) \div 5$

→ $R_1(\text{new}) = R_1(\text{old}) - 3 \times R_2(\text{new})$

simplex table 2

		C_j	5	3	0	0	
C_B	B.V.	b	x_1	x_2	s_1	s_2	Min Ratio
0	s_1 ←	9	0	19/5	1	-3/5	$9 \times \frac{5}{19} = 2.3$ ←
5	x_1	2	1	2/5	0	1/5	$2 \times \frac{5}{2} = 5$
	Z_j		5	2	0	1	
	$Z_j - C_j$		0	-1	0	1	
				↑			

-ve largest $Z_j - C_j$ is -1 and its column index is 2 (i.e. x_2)
so, the entering variable is x_2.
Min Ratio is 2.3 and its row index is 1 (i.e. S_1)
so, the leaving variable is S_1

Thus, entering $= x_2$, leaving $= S_1$ & key element $= \frac{19}{5}$

Applying, $R_1(new) = R_1(old) \times \frac{5}{19}$

$R_2(new) = R_2(old) - \frac{2}{5} \times R_1(new)$

simplex table 3

		C_j	5	3	0	0	
C_B	B.V.	b	x_1	x_2	S_1	S_2	Min Ratio
3	x_2	45/19	0	1	5/19	-3/19	
5	x_1	20/19	1	0	-2/19	5/19	
	Z_j		5	3	5/19	16/19	
	$Z_j - C_j$		0	0	5/19	16/19	

since, all $Z_j - C_j \geq 0$, optimal solution is arrived with value of variables as:

$$x_1 = \frac{20}{19}, \quad x_2 = \frac{45}{19}$$

$$\therefore \text{Max } Z = 5 \times \frac{20}{19} + 3 \times \frac{45}{19}$$

$$= \frac{235}{19} \;//$$

Note: In min Ratio -ve value is not considered

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- Example 2:

→ Max $Z = 3x_1 + 2x_2 + 5x_3$

Subject to:

$x_1 + x_2 + x_3 \le 9$

$2x_1 + 3x_2 + 5x_3 \le 30$

$2x_1 - x_2 - x_3 \le 8$

$x_1, x_2, x_3 \ge 0$

Soln

Introducing slack variables in given LPP:

$Z = 3x_1 + 2x_2 + 5x_1 + 0.s_1 + 0.s_2 + 0.s_3$

$x_1 + x_2 + x_3 + s_1 + 0.s_2 + 0.s_3 = 9$

$2x_1 + 3x_2 + 5x_3 + 0.s_1 + s_2 + 0.s_3 = 30$

$2x_1 - x_2 - x_3 + 0.s_1 + 0.s_2 + s_3 = 8$

Constructing initial simplex tableau

simplex table 1

	c_j		3	2	5	0	0	0	
C_B	B.V.	b	x_1	x_2	x_3	s_1	s_2	s_3	Min Ratio
0	s_1	9	1	1	1	1	0	0	9/1 = 9
0	s_2	30	2	3	5	0	1	0	30/5 = 6 ←
0	s_3	8	2	-1	-1	0	0	1	8/-1 = -8
	Z_j		0	0	0	0	0	0	
	$Z_j - C_j$		-3	-2	-5 ↑	0	0	0	

Here, -ve minimum $Z_j - C_j$ is -5 & its column index is 3 (i.e. x_3)

so, entering variable is x_3

+ve minimum ratio is 6 & its row index is 2 i.e. (S_2).

so, S_2 is leaving variable & key element is 5

(key row) $R_2(new) \to R_2(old) \div 5$, $R_1(new) \to R_1(old) - R_2(new)$

$R_3(new) \to R_3 + R_2(new)$

simplex table 2

	c_j		3	2	5	0	0	0	
C_B	B.V.	b	x_1	x_2	x_3	s_1	s_2	s_3	Min Ratio
0	s_1	3	3/5	2/5	0	1	-1/5	0	$3 \times 5/3 = 5$ ←
5	x_3	6	2/5	3/5	1	0	1/5	0	$6 \times 5/2 = 15$
0	s_3	14	12/5	-2/5	0	0	1/5	1	$14 \times 5/12 = 5.8$
	$Z_j - C_j$		-1 ↑	1	0	0	1	0	

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Negative minimum $Z_j - C_j$ is -1 & its column index is 1
so, entering variable is x_1
Minimum Ratio is 5 & its row index is 1.
so, leaving variable is S_1
$\therefore$ key element = 3/5
applying, $R_1(new) \rightarrow R_1(old) \times \frac{5}{3}$

$R_2(new) \longrightarrow R_2(old) - \frac{2}{5} R_1(new)$

$R_3(new) \longrightarrow R_3(old) - \frac{12}{5} R_1(new)$

simplex table 3

C_B	B.V.	C_j / b	3 / x_1	2 / x_2	5 / x_3	0 / S_1	0 / S_2	0 / S_3	Min Ratio
3	x_1	5	1	2/3	0	5/3	-4/3	0	
5	x_3	4	0	1/3	1	-2/3	1/3	0	
0	S_3	2	0	-2	0	-4	1	1	
		Z_j	3	11/3	5	5/3	2/3	0	
		$Z_j - C_j$	0	5/3	0	5/3	2/3	0	

Since, all $Z_j - C_j \geq 0$, so optimal solution is arrived with value of variable as:

$x_1 = 5$

$x_2 = 0$ (since there is no x_2 in table)

$x_3 = 4$

$\therefore$ Max $Z = 3 \times 5 + 2 \times 0 + 5 \times 4$

$= 35$

- M → very large +ve number
  - For Max → $-M$
  - For Min → M

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* Big M method

→ It is method to handle "≥" & "=" constraints in simplex method by introducing artificial variables that are heavily penalized in the objective.

- "≥" → Require surplus & artificial variables
- "=" → Require artificial variables

Note: Artificial variables are not real decision variable, they help to start the simplex process but should be removed from the final solution.

- Example 1

Max $Z = 2x_1 + 3x_2$

basic variable

Sub. to : $x_1 + x_2 \le 4$ (slack) → $x_1 + x_2 + S_1 = 4$

$2x_1 + x_2 \ge 6$ (surplus) → $2x_1 + x_2 - S_2 + A = 6$

Artificial variable

$x_1, x_2 \ge 0$

Introducing slack, surplus & artificial variables, we have

$$Z = 2x_1 + 3x_2 + 0.S_1 + 0.S_2 - MA$$

$$x_1 + x_2 + S_1 + 0.S_2 + 0.A = 4$$

$$2x_1 + x_2 + 0.S_1 - S_2 + A = 6$$

Constructing initial simplex tableau

Simplex table 1

you can remove this column in next table ✗

		c_j	2	3	0	0	$-M$	
C_B	B.V.	b	x_1	x_2	S_1	S_2	A	Min Ratio
0	S_1	4	1	1	1	0	0	4
$-M$	A	6	2	1	0	-1	1	3 ←
		Z_j	$-2M$	$-M$	0	M	$-M$	
		$Z_j - C_j$	$-2M-2$ ↑	$-M-3$	0	M	0	

- Since, Negative minimum $Z_j - C_j$ is $-2M-2$ & its column index is 1 (i.e. x_1). So, the entering variable is x_1.
  Min. Ratio is 3 & its row index is 2 (i.e. A). So, the leaving variable is A.
  ∴ The key element is 2

Note: After the Artificial variable is removed from B.V., you can also remove its corresponding column.

- $R_{2(new)} \rightarrow \frac{R_{2(old)}}{2}$, • $R_{1(new)} \rightarrow R_{1(old)} - R_{2(new)}$

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simplex table 2

	C_j		2	3	0	0	
C_B	B.V.	b	x_1	x_2	s_1	s_2	Min Ratio
0	s_1	1	0	1/2	1	1/2	2 ← leaving
2	x_1	3	1	1/2	0	-1/2	6
		Z_j	2	1	0	-1	
		$Z_j - c_j$	0	-2	0	-1	

↑ Entering

- $R_{1(new)} \rightarrow R_{1(old)} \times 2$
- $R_{2(new)} \rightarrow R_{2(old)} - \frac{1}{2}R_{1(new)}$

simplex table 3

		c_j	2	3	0	0
C_B	B.V.	b	x_1	x_2	s_1	s_2
3	x_2	2	0	1	2	1
2	x_1	2	1	0	-1	-1
		Z_j	2	3	4	1
		$Z_j - c_j$	0	0	4	1

$\because$ All $(Z_j - c_j) \geq 0$, so optimal solution is obtained where, $x_1 = 2$, $x_2 = 2$

$\therefore$ Max $Z = 2 \times 2 + 3 \times 2$

$= 10$

Checking for Artificial variable

$A = 6 - 2x_1 - x_2 + S_2$

$= 6 - 4 - 2 + 0$

$= 0$ (feasible)

- Example 2: (Note: If you keep Min as it is, then do $+M \cdot A$)

Min $Z = x_1 + x_2$ $\Rightarrow$ Max $Z' = -x_1 - x_2$

Sub to : $2x_1 + x_2 \geq 4$ $\rightarrow 2x_1 + x_2 - S_1 + A_1 = 4$

$x_1 + 7x_2 \geq 7$ $\rightarrow x_1 + 7x_2 - S_2 + A_2 = 7$

$x_1, x_2 \geq 0$

Including surplus & artificial variables, we have

$Z' = -x_1 - x_2 + 0.S_1 + 0.S_2 - M.A_1 - M.A_2$

$2x_1 + x_2 - S_1 + 0.S_2 + A_1 + 0.A_2 = 4$

$x_1 + 7x_2 + 0.S_1 - S_2 + 0.A_1 + A_2 = 7$

Remove it

		c_j	-1	-1	0	0	-M	-M	
C_B	B.V.	b	x_1	x_2	S_1	S_2	A_1	A_2	Min Ratio
-M	A_1	4	2	1	-1	0	1	0	4/1=4
-M	A_2	7	1	7	0	-1	0	1	7/7=1 ←
		Z_j	-3M	-8M	M	M	-M	-M	
		$Z_j - c_j$	-3M-1	-8M-1	M	M	0	0	

↑

- $R_2(new) \rightarrow R_2(old) \div 7$
- $R_1(new) \rightarrow R_1(old) - R_2(new)$

C_B	B.V.	C_j / b	-1 / x_1	-1 / x_2	0 / s_1	0 / s_2	$\cancel{-M}$ / A_1	Min Ratio
-M	A_1	3	$13/7$	0	-1	$1/7$	1	$21/13 = 1.6 \leftarrow$
-1	x_2	1	$1/7$	1	0	$-1/7$	0	7
		Z_j	$\frac{-13M}{7} - \frac{1}{7}$	1	M	$\frac{-M}{7} + \frac{1}{7}$	-M	
		$Z_j - C_j$	$\frac{-13M+6}{7}$ ↑	0	M	$\frac{-M+1}{7}$	0	

Leaving = A_1, Entering = x_1, Key = $13/7$.

- $R_1(new) \rightarrow R_1(old) \times 7/13$
- $R_2(new) \rightarrow R_2(old) - 1/7\, R_1(new)$

C_B	B.V.	C_j / b	-1 / x_1	-1 / x_2	0 / s_1	0 / s_2
-1	x_1	$21/13$	1	0	$-7/13$	$1/13$
-1	x_2	$10/13$	0	1	$1/13$	$-2/13$
		Z_j	-1	-1	$6/13$	$1/13$
		$Z_j - C_j$	0	0	$6/13$	$1/13$

Since, All $Z_j - C_j \geq 0$

So, optimal solution is obtained.

Where $x_1 = \frac{21}{13}$, $x_2 = \frac{10}{13}$

$\therefore \min Z = x_1 + x_2$

$= \frac{21}{13} + \frac{10}{13} = \frac{31}{13}$ //

Checking for Artifical variable

$A_1 = 4 - 2x_1 - x_2 + s_1$

$= 4 - 2 \times \frac{21}{13} - \frac{10}{13}$

$= 0$

$||^{ly},\ A_2 = 0$ (feasible)

Note: We can do the same solution using Two phase method Unlike, Big M method, which penalizes artificial variables using a large constant M, Two-phase method splits the process into:

- Phase I: Minimize sum of artificial variables.
- Phase II: Use the feasible solution from phase I to solve original problem(objective).

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* Two phase Method

Phase-I

Introducing surplus & artificial variables, we have,

$\text{Max } z' = -A_1 - A_2$

$2x_1 + x_2 - s_1 + A_1 = 4$

$x_1 + 7x_2 - s_2 + A_2 = 7$

		c_j	0	0	0	0	-1	-1	
C_B	B.V.	b	x_1	x_2	s_1	s_2	A_1	A_2	Min Ratio
-1	A_1	4	2	1	-1	0	1	0	4/1=4
-1	A_2	7	1	7	0	+1	0	1	7/7=1
		Z_j	-3	-8	1	1	0	0	
		$Z_j - C_j$	-3	-8	1	1	0	0	

perform similar row operations as before

		c_j	0	0	0	0	-1	
C_B	B.V.	b	x_1	x_2	s_1	s_2	A_1	Min Ratio
-1	A_1	3	13/7	0	-1	1/7	1	1.62
0	x_2	1	1/7	1	0	-1/7	0	7
		$Z_j - C_j$	-13/7	0	1	-1/7	0	

		c_j				
C_B	B.V.	b	x_1	x_2	s_1	s_2
0	x_1	21/13	1	0	-7/13	1/13
0	x_2	10/13	0	1	1/13	-2/13
		$Z_j - C_j$	0	0	0	0

Phase-II

We eliminate the artificial variables & change objective function for the original. i.e. $Z' = -x_1 - x_2 + 0.s_1 + 0.s_2$

		c_j	-1	-1	0	0
C_B	B.V.	b	x_1	x_2	s_1	s_2
-1	x_1	21/13	1	0	-7/13	1/13
-1	x_2	10/13	0	1	1/13	-2/13
	$Z_j - C_j$		0	0	6/13	1/13

since, all $Z_j - C_j \geq 0$, optimal solution is obtained

where $x_1 = \frac{21}{13}$, $x_2 = \frac{10}{13}$ $\therefore$ Min $Z = \frac{31}{13}$ //

→ having more than set of basic solutions that can minimize or maximize the required objective function.

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* Multiple optimal solution / Alternate optimum

→ In the final simplex table when all $Z_j - C_j$ imply optimal solution (for Max all $Z_j - C_j \geq 0$ & for Min $Z_j - C_j \leq 0$)

but, if $Z_j - C_j = 0$ for some non-basic variable column, then this indicate there are more than 1 optimal solution. Thus by entering this variable into the basic, we may obtain another alternative optimal solution.

- Example :

Max $Z = 4x_1 + 3x_2$

sub to: $8x_1 + 6x_2 \leq 25$

$3x_1 + 4x_2 \leq 15$

$x_1, x_2 \geq 0$

Adding slack variables S_1 & S_2 in give LPP, we have

$Z = 4x_1 + 3x_2 + 0.S_1 + 0.S_2$

$8x_1 + 6x_2 + S_1 + 0.S_2 = 25$

$3x_1 + 4x_2 + 0.S_1 + S_2 = 15$

Simplex table 1

		C_j	4	3	0	0	
C_B	B.V.	b	x_1	x_2	S_1	S_2	Ratio
0	S_1	25	8	6	1	0	25/8 →
0	S_2	15	3	4	0	0	5
		Z_j	0	0	0	0	
		$Z_j - C_j$	-4 ↑	-3	0	0	

- $R_1(new) \rightarrow R_1(old) \div 8$
- $R_2(new) \rightarrow R_2(old) - 3 \times R_1(new)$

Simplex table 2

		C_j	4	3	0	0	
C_B	B.V.	b	x_1	x_2	S_1	S_2	Ratio
4	x_1	25/8	1	3/4	1/8	0	
0	S_2	45/8	0	7/4	-3/8	1	
		$Z_j - C_j$	0	0	1/2	0	

since, all $Z_j - C_j \geq 0$, so optimal solution is arrived.

where, $x_1 = \frac{25}{8}$, $x_2 = 0$ $\therefore$ Max $Z = \frac{25}{2}$ //

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Check for multiple optimal solution

In the final simplex tableau, we look at the non-basic variables (those not in the basic) and see if any of them have: $Z_j - C_j = 0$

In our case, the final row of $Z_j - C_j$ was

	x_1	x_2	s_1	s_2
$Z_j - C_j$	0 (basic)	0 (non-basic)	$1/2$	0

$\because$ x_2 is a non-basic variable (value = 0) & $Z_j - C_j = 0$, it means Multiple optimal solution exists.

Thus, by entering x_2 into the basis, we can obtain another alternative optimal solution.

Then, ratio column will take value $25/6$ & $45/14 \rightarrow$ min ratio

$\therefore$ $x_2 \rightarrow$ Enter, $S_2 \rightarrow$ leave & $7/4 \rightarrow$ key element

- $R_2(\text{new}) \rightarrow R_2(\text{old}) \times \frac{4}{7}$

- $R_1(\text{new}) \rightarrow R_1(\text{old}) - \frac{3}{4} \times R_2(\text{new})$

Simplex table 3

		C_j	4	3	0	0
CB	B.V.	b	x_1	x_2	s_1	s_2
4	x_1	$5/7$	1	0	$2/7$	$-3/7$
3	x_2	$45/14$	0	1	$-3/14$	$4/7$
	$Z_j - C_j$		0	0	$1/2$	0

Since, all $Z_j - C_j \geq 0$, so optimal solution is again obtained as:

$$x_1 = \frac{5}{7}, \quad x_2 = \frac{45}{14} \quad \& \quad \text{Max } Z = \frac{25}{2}$$

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* Unbounded solution

→ It occours when key (pivot) column is either negative or zero (≤ 0)

- Example:

$$\text{Max } Z = 4x_1 + 3x_2$$

Sub to: $x_1 - 6x_2 \leq 5$

$3x_1 \leq 11$

$x_1, x_2 \geq 0$

Soln

Introducing slack variable s_1 & S_2 in give LPP

$$Z = 4x_1 + 3x_2 + 0.s_1 + 0.S_2$$

$$x_1 - 6x_2 + S_1 + 0.S_2 = 5$$

$$3x_1 + 0.x_2 + 0.S_1 + S_2 = 11$$

Simplex table 1

		C_j	4	3	0	0	
C_B	B.V.	b	x_1	x_2	s_1	S_2	Ratio
0	S_1	5	1	-6	1	0	5
0	S_2	11	3	0	0	1	11/3 →
		$Z_j - C_j$	-4 ↑	-3	0	0	

- $R_2(new) \longrightarrow R_2(old) \div 3$
- $R_1(new) \longrightarrow R_1(old) - R_2(new)$

Simplex table 2

		C_j	4	3	0	0	
C_B	B.V.	b	x_1	x_2	S_1	S_2	Ratio
0	S_1	4/3	0	-6	1	-1/3	-2/9
4	x_2	11/3	1	0	0	1/3	∞
		$Z_j - C_j$	0	-3 ↑	0	4/3	

Here, -3 is the largest -ve ($Z_j - C_j$) value. so, x_2 should enter the basic but the values in key column (x_2) are -ve & zero.

so, this variable cannot be entered into the basic, because for minimum Ratio -ve value is not considered and zero value would resuet to ∞ (infinity)

Hence, the solution of given Lpp is unbounded.

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* Infeasible solution

→ If there is no any solution that satisfies all the constraints, then it is called Infeasible solution.

- In final simplex table when all $(z_j - c_j)$ imply optimal solution but at least one artificial variable present in the basis with positive value, then the solution becomes infeasible or no solution.

- Example:

Max $Z = 3x_1 + 2x_2$

sub to: $x_1 + x_2 \le 8 \rightarrow x_1 + x_2 + S_1 = 8$

$x_2 \ge 10 \rightarrow x_2 - S_2 + A = 10$

and $x_1, x_2 \ge 0$

Introducing slack, surplus & artificial variables

$Z = 3x_1 + 2x_2 + 0.S_1 + 0.S_2 - M.A$

$x_1 + x_2 + S_1 + 0.S_2 + 0.A = 8$

$0.x_1 + x_2 + 0.S_1 - S_2 + A = 10$

Simplex table 1

		c_j	3	2	0	0	$-M$	
C_B	B.V.	b	x_1	x_2	S_1	S_2	A	Ratio
0	S_1	8	1	1	1	0	0	8 →
$-M$	A	10	0	1	0	-1	1	10
		Z_j	0	$-M$	0	M	$-M$	
		$Z_j - c_j$	-3	$-M-2$ ↑	0	M	0	

- $R_2 \rightarrow R_2 - R_1$

Simplex table 2

		c_j	3	2	0	0	$-M$
C_B	B.V.	b	x_1	x_2	S_1	S_2	A
2	x_2	8	1	1	1	0	0
$-M$	A	2	-1	0	-1	-1	1
		Z_j	$2+M$	2	$2+M$	M	$-M$
		$Z_j - c_j$	$M-1$	0	$M+2$	M	0

Since, all $Z_j - c_j \ge 0$, so the solution is optimum
But, still artificial variable appears in B.V. with positive value 2.

• For every primal LPP, a corresponding dual exists.

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* Primal and Dual problems

Relationship

	Primal/Dual		Dual/primal
Objective:	Max	$\longleftrightarrow$	Min
constraints:	$\le$	$\longleftrightarrow$	$\ge$
	$=$	$\longleftrightarrow$	unrestricted
	Objective coefficients	$\longleftrightarrow$	RHS values
	No. of variables	$\longleftrightarrow$	No. of constraints
Variables:	$x_1, x_2, \ldots, x_n$	$\longrightarrow$	$y_1, y_2, \ldots, y_n$ or $w_1, w_2, \ldots, w_n$
	Row	$\longleftrightarrow$	column

Eg:

3 variables

$$\text{Max. } Z = \boxed{10x_1 + 7x_2 + 5x_3}$$

subject to:

2 constraints

$$4x_1 + 8x_2 + 9x_3 \le 15$$
$$3x_1 + 4x_2 - 6x_3 \le 18$$

and $x_1, x_2, x_3 \ge 0$

RHS constant

$\Leftrightarrow$

2 variables

$$\text{Min. } Z = \boxed{15w_1 + 18w_2}$$

subject to:

$$4w_1 + 3w_2 \ge 10$$
$$8w_1 + 4w_2 \ge 7$$
$$9w_1 - 6w_2 \ge 5$$

and $w_1, w_2 \ge 0$

3 constraints

* steps of formulating dual problems

step 1: Convert the problem in canonical form, making all
- Max = '$\le$' type
- Min = '$\ge$' type

Step 2: change the objective function, Max. in primal into Min. in dual and vice-versa

Step 3: The no. of variable in primal will be no. of constraints in dual and vice-versa

Step 4: cost coefficients in objective function of the primal will be RHS constant of the constraints in dual and vice-versa.

Step 5: For formulating constraints we consider the transpose of matrix (i.e. row is changed into column & vice-versa)

In above example

$$\begin{array}{ccc} & & \\ \end{array}
\begin{pmatrix} 4 & 8 & 9 \\ 3 & 4 & -6 \end{pmatrix}^T = \begin{pmatrix} 4 & 3 \\ 8 & 4 \\ 9 & -6 \end{pmatrix}$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \qquad\qquad \begin{array}{cc} w_1 & w_2 \end{array}$$

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* Example 1 (primal to dual conversion)

Max $Z = x_1 + 2x_2 + x_3$

Subject to:

$2x_1 + x_2 - x_3 \le 2$

$-2x_1 + x_2 - 5x_3 \ge -6$

$4x_1 + x_2 + x_3 \le 6$

$x_1, x_2, x_3 \ge 0$

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & -1 & 5 \\ 4 & 1 & 1 \end{pmatrix}^T = \begin{pmatrix} 2 & 2 & 4 \\ 1 & -1 & 1 \\ -1 & 5 & 1 \end{pmatrix}$$

→ **Canonical form:** (since objective fxn is Max. so, all constraints should be '$\le$' type)

Max $Z = x_1 + 2x_2 + x_3$

Subject to:

$$\left.\begin{array}{l} 2x_1 + x_2 - x_3 \le \\ 2x_1 - x_2 + 5x_3 \le \\ 4x_1 + x_2 + x_3 \le \end{array}\right. \boxed{\begin{array}{c} 2 \\ 6 \\ 6 \end{array}}$$

→ This is changed into objective coefficients in dual

→ RHS constant

$x_1, x_2, x_3 \ge 0$

Dual form: (since, there are 3 constraints in primal so, there should be 3 dual variables)

Let w_1, w_2 & w_2 be dual variable

Min $Z = 2w_1 + 6w_2 + 6w_2$

subject to:

Transpose of primal constraints ⇒

$$\left[\begin{array}{l} 2w_1 + 2w_2 + 4w_3 \ge \\ w_1 - w_2 + w_2 \ge \\ -w_1 + 5w_2 + w_3 \ge \end{array}\right] \left[\begin{array}{c} 1 \\ 2 \\ 1 \end{array}\right] \Rightarrow$$

coefficient of objective function in primal

* Example 2

Min $Z = 2x_2 + 5x_3$

Sub to: $x_1 + x_2 \ge 2$

$2x_1 + x_2 + 6x_3 \le 6$

$x_1 - x_2 + 3x_3 = 4$ → $x_1 - x_2 + 3x_3 \ge 4$; $x_1 - x_2 + 3x_3 \le 4$

$x_1, x_2, x_3 \ge 0$

→ **canonical form:**

Min $Z = 0x_1 + 2x_2 + 5x_3$ → (3 variables in primal) + 4 constraints

sub to:

$$\left.\begin{array}{l} x_1 + x_2 + 0.x_3 \ge 2 \\ -2x_1 - x_2 - 6x_3 \ge -6 \\ x_1 - x_2 + 3x_3 \ge 4 \\ -x_1 + x_2 - 3x_3 \ge -4 \end{array}\right]$$

$x_1, x_2, x_3 \ge 0$

Dual form:

let w_1, w_2, w_3 & w_4 be dual variable

$$Max\ Z = 2w_1 - 6w_2 + 4w_3 - 4w_4$$

sub to :

$$w_1 - 2w_2 + w_3 - w_4 \le 0$$
$$w_1 - w_2 - w_3 + w_4 \le 2$$
$$-6w_2 + 3w_3 - 3w_4 \le 5$$
$$w_1, w_2, w_3, w_4 \ge 0$$

~~This~~ can be re-arranged as, assuming $(w_3 - w_4 = w_3')$

Thus, $Max\ Z = 2w_1 - 6w_2 + 4w_3'$

sub to :

$$w_1 - 2w_2 + w_3' \le 0$$
$$w_1 - w_2 - w_3' \le 2$$
$$-6w_2 + 3w_3' \le 5$$
$$w_1, w_2 \ge 0 \ \& \ w_3' \text{ unrestricted}$$

* Example 3:

$Max\ Z = x + 2y$

sub to : $2x + 3y \ge 4$

$3x + 4y = 5 \rightarrow 3x + 4y \le 5, \quad 3x + 4y \ge 5$

$x \ge 0$, y unrestricted

→ Since, variable y is unrestricted, it can be expressed as

$y = y' - y''$, $y', y'' \ge 0$

canonical form:

$$Max\ Z = x + 2y' - 2y''$$

sub to :

$$-2x - 3y' + 3y'' \le -4$$
$$3x + 4y' + 4y'' \le 5$$
$$-3x - 4y' + 4y'' \le -5$$
$$x, y', y'' \ge 0$$

Dual form:

let w_1, w_2 & w_3 be dual variables

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$Min.\ Z = -4w_1 + 5w_2 - 5w_2$

Sub to: $-2w_1 + 3w_2 - 3w_3 \geq 1$

$-3w_1 + 4w_2 - 4w_3 \geq 2$

$3w_1 - 4w_2 + 4w_3 \geq -2$

Rearrange the dual form.

let $w_2 - w_3 = w_2'$

Thus, $Min.\ Z = -4w_1 + 5w_2'$

Sub to: $-2w_1 + 3w_2' \geq 1$

$-3w_1 + 4w_2' \geq 2$

$3w_1 - 4w_2' \geq -2$

$w_1 \geq 0$, w_2' unrestricted

This can be re-arranged as: $-3w_1 + 4w_2' = 2$

* Dual Simplex Method

Algorithm

Step 1: Formulate the problem

i, If the objective function is (Min.) type then change into (Max.) → No need to convert if already in Max.

ii, Change all "$\geq$" constraints to "$\leq$" multiplying by -1.

iii, Transform every '$\leq$' constraints into '$=$' constraints by adding a slack variable and assign a 0 cost coefficient in objective function.

Step 2: Find initial basic solution by setting zero value to the decision variables.

Step 3: Test for optimality

i, If all the values of $b \geq 0$ and $Z_j - C_j \geq 0$ then the solution is optimal, terminate the process

ii, If any $b < 0$, then select the most negative b and this row is called key row

iii, Find Max Ratio $= \dfrac{Z_j - C_j}{\text{key row}}$ and key row < 0, the corresponding column is key column.

iv, Find the new solution table.

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- Example:

Solve by dual simplex method

$\min z = 5x_1 + 6x_2$

sub to, $x_1 + x_2 \geq 2$

$4x_1 + x_2 \geq 4$

$x_1, x_2 \geq 0$

Soln

In order to apply the dual simplex method, convert Min z to Max z and all '$\geq$' constraint to '$\leq$' multiply by -1

Then the given LPP becomes:

$\text{Max } z = -5x_1 - 6x_2$

subject to, $-x_1 - x_2 \leq -2$

$-4x_1 - x_2 \leq -4$

Introducing slack variables s_1 & s_2 on given LPP, we get

$\text{Max } z = -5x_1 - 6x_2 + 0.s_1 + 0.s_2$

Sub to: $-x_1 - x_2 + s_1 + 0.s_2 = -2$

$-4x_1 - x_2 + 0.s_1 + s_2 = -4$

An initial basic feasible solution is given by.

$x_1 = x_2 = 0, \quad s_1 = -2, \quad s_2 = -4$

		c_j	-5	-6	0	0
C_B	B.V.	b	x_1	x_2	s_1	s_2
0	s_1	-2	-1	-1	1	0
0	s_2	-4 ←	-4	-1	0 ×	1 × → Not <0
	$Z_j - c_j$		5	6	0	0
$\left(\frac{Z_j - c_j}{s_2}\right)$	Max Ratio & $s_2 < 0$		$\frac{5}{-4}$ ↑	$\frac{6}{-1}$	...	...

Here, -4 is the most $-$ve b, so, s_2 is leaving variable

Max $-$ve Ratio is $-5/4$, so, x_1 is entering variable

& Key element is -4.

- $R_2(new) \rightarrow R_2(old) \div -4$
- $R_1(new) \rightarrow R_1(old) + R_2(new)$

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Simplex table 2

C_B	B.V.	C_j / b	-5 / x_1	-6 / x_2	0 / s_1	0 / s_2
0	s_1	-1	0	-3/4	1	-1/4 → key row
-5	x_1	1	1	1/4	0	-1/4
	$z_j - c_j$		0	19/4	0	5/4
$\left(\frac{z_j - c_j}{s_1}\right)$	Max Ratio		--	-19/3	--	-5 ↑ key column

- $R_1(new) \rightarrow R_1(old) \times -4$
- $R_2(new) \rightarrow R_2(old) + \frac{1}{4} R_1(new)$

Simplex Table 3

C_B	B.V.	C_j / b	-5 / x_1	-6 / x_2	0 / s_1	0 / s_2
0	s_2	4	0	3	-4	1
-5	x_1	2	1	1	-1	0
		$z_j - c_j$	0	1	5	0

Since, all $z_j - c_j \geq 0$ and also $b \geq 0$, the solution is optimum.

Where, $x_1 = 2$, $x_2 = 0$

$\therefore$ Max $Z = -5 \times 2 - 6 \times 0$

$= -10$

$\therefore$ Min $Z = -$ Max Z

$= 20$ //

Next Process

→ You can also solve this problem, first by converting it into dual form and using simplex method for solving dual problem.

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Dual form of given LPP:

Let w_1 & w_2 be dual variables, then given LPP becomes

Max $Z = 2w_1 + 4w_2$

sub to: $w_1 + 4w_2 \le 5$

$w_1 + w_2 \le 6$

$w_1, w_2 \ge 0$

Introducing slack variables s_1 & s_2 in given LPP, we get

$Z = 2w_1 + 4w_2 + 0.s_1 + 0.s_2$

$w_1 + 4w_2 + s_1 + 0.s_2 = 5$

$w_1 + w_2 + 0.s_1 + s_2 = 6$

Simplex Table 1

		C_j	2	4	0	0	
C_B	B.V.	b	w_1	w_2	s_1	s_2	Min Ratio
0	s_1	5	1	4	1	0	5/4
0	s_2	6	1	1	0	1	6/1
		$Z_j - C_j$	-2	-4	0	0	

- $R_1(new) \rightarrow R_1(old) \div 4$
- $R_2(new) \rightarrow R_2(old) - R_1(new)$

Simplex Table 2

		C_j	2	4	0	0	
C_B	B.V.	b	w_1	w_2	s_1	s_2	Min Ratio
4	w_2	5/4	1/4	1	1/4	0	5
0	s_2	19/4	3/4	0	-1/4	1	19/3
		$Z_j - C_j$	-1	0	1	0	

Simplex Table 3

		C_j	2	4	0	0
C_B	B.V.	b	w_1	w_2	s_1	s_2
2	w_1	5	1	4	1	4
0	s_2	1	0	-3	-1	-3
		$Z_j - C_j$	0	4	2	8

Since, all $Z_j - C_j \ge 0$, the solution is optimum.

where, $w_1 = 5$ & $w_2 = 0$

$\therefore$ Max $Z = 2 \times 5 + 4 \times 0 = 10$

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* Economic Interpretation

→ Linear programming problems (LPPs) are widely used in economics and business to optimize resource allocation, production planning, cost minimization and profit maximization. The 'primal' and 'dual' formulations provide complementary economic insights.

Example 1:

Let us take example of furniture factory. Suppose you want to figure out how much profit you make by selling chairs and tables. This is called "primal" problem. Now, you also want to know what the value of wood and the labor you used should be. This is called the "Dual" problem. Duality shows how these two problems are connected and how the solution of one provides information about the other.

Let's breakdown this example in more details:

* Resources (limited):
  - wood : you have 100 units of wood available per week.
  - Labor : " 80 hours of labor "

* production Requirements and profits
1) Chair : Requires 2 units of wood & 1 hour of labor & yields a profit of \$ 50 per chair.
2) Table : Requires 3 units of wood & 2 hours of labor & yields a profit of \$75 per table.

- your goal is to decide how many chairs and tables to produce each week to maximize your total profit.

Soln : (Mathematical Formulation)

1) Primal LPP (The production problem)

→ The primal problem is the original LPP that you set up to directly solve your business objective. Here, the primal problem is about maximizing profit by choosing production quantities.

	(x_1) chair	(x_2) table	Total availability
wood	2	3	100
labor	1	2	80

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* Decision variable:

- let x_1 = No. of chairs to produce
- let x_2 = No. of Tables to produce

* objective function : we want to maximize total profit (Z)

- $\text{Max } Z = 50x_1 + 75x_2$

* constraints (Resource Limitation)

- wood constraints : Total wood used for (chairs + table) cannot exceed the available 100 units of wood.

=) i.e. $2x_1 + 3x_2 \leq 10$

- Labor constraints : Total labor " " 80 hrs of labor

=) $x_1 + 2x_2 \leq 80$

- Non-negativity constraints : you can't produce a -ve no. of chairs or tables.

=) $x_1 \geq 0, \ x_2 \geq 0$

so, the primal LPP is :

$\text{Max } Z = 50x_1 + 75x_2$

subject to :

$2x_1 + 3x_2 \leq 100$

$x_1 + 2x_2 \leq 80$

$x_1, x_2 \geq 0$

Economic Interpretation of the primal problem

- objective function : Represents the total profit you want to maximize.
- Decision variables (x_1, x_2) : Represent the quantities of product (chairs, table) you need to produce.
- constraints : Represent the limited resources (wood, labor) that restrict your production.

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2, Dual LPP (The Resource Valuation problem)

→ The dual problem is a related problem derived from the original (primal) LPP. Every primal LPP has a unique dual LPP, and optimal solutions of both problems are interconnected.

- This is a different perspective on the same economic situation. Instead of focusing on producing goods, the dual problem focuses on valuing the scarce resources
- The dual problem assigns shadow prices (y_1, y_2) to each resource (wood, labor).

let y_1 = Imputed value / shadow price per unit of wood
y_2 = " " per hour of labor

∴ The corresponding Dual LPP is:

Min $W = 100y_1 + 80y_2$

subject to:

$2y_1 + y_2 \geq 50$ (chair value constraint)

$3y_1 + 2y_2 \geq 75$ (Table value constraint).

$y_1, y_2 \geq 0$

Economic Interpretation of the Dual problem:

- Objective function: Represent the total (marginal) imputed value of resource that the renter aims to minimize
- Dual variables / shadow prices (y_1, y_2): Represents the shadow prices or marginal value of each unit of the scarce resources (wood, labor). They tell you how much an extra unit of that resource would increase your maximum profit.
- Constraints: The value assigned to the resource for producing a product shouldn't exceed profit gained from that product. The RHS values 50 & 75 are per-unit profits from the primal problem.

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Example 2 :

· A company produces two products, A (x_1) and B (x_2), with the following constraints:

- Machine Time: Each unit of A requires 4 hours & each unit of B requires 2 hours. Total available machine time is 80 hours.
- Labor Hours: Each unit of A requires 1 hour, and each unit of B requires 3 hours. Total available labor is 60 hours.
- Demand constraint: At least 10 units of A must be produced.
- Costs: The cost of producing A is \$20/unit and B is \$40/unit.

objective: Minimize total production cost.

Soln:

1, Primal LPP: producer's perspective (cost Minimization)

→ $\text{Min } Z = 20x_1 + 40x_2$

Sub to:

$4x_1 + 2x_2 \geq 80$ (machine Time)

$x_1 + 3x_2 \geq 60$ (Labor Hours)

$x_1 \geq 10$ (Demand constraint)

$x_1, x_2 \geq 0$

Economic Interpretation of the primal problem

- Decision variable (x_1, x_2)

→ Represent quantities of product A and B to produce.

- Objective Function ($Z = 20x_1 + 40x_2$)

→ Minimize total production cost

- constraints

→ Ensure the company uses at least total availability

2, Dual LPP : Resource value's perspective (shadow prices)

→ The dual problem assigns shadow prices (y_1, y_2, y_3) to each resource (machine time, labor, demand).

let, y_1 = value of an additional hour of machine time
y_2 = value of an additional labor hour
y_3 = value of releaxing the demand constraint

∴ The corresponding dual LPP is

Max $W = 80y_1 + 60y_2 + 10y_3$

Subject to:

$4y_1 + y_2 + y_3 \le 20$
$2y_1 + 3y_2 \le 40$
$y_1, y_2, y_3 \ge 0$

Economic Interpretation of the Dual problem

- Dual variables / Shadow prices (y_1, y_2, y_3)
- y_1 : If machine time increases by 1 hour, the cost decreases by y_1.
- y_2 : If labor increased by 1 hour, the cost decreases by y_2.
- y_3 : If the minimum demand for A decreases by 1 unit, the cost decreases by y_3

- objective function

→ Represent the total imputed value of the resources. The company want to maximize the value derived from its resources.

- constraints

→ Ensure that the value of resource used to produce one unit of product (A & B) shouldn't exceed its production cost (\$20 & \$40)