Chapter 3

Transportation and Assignment Problem

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* Transportation Problem

→ It's a particular class of LPPs used to optimize (minimize) the transportation cost and allocate resources from 'm' source to 'n' destination.

* Mathematical Model of Traspotation problem

→ let us consider an example. A company has 3 production factories S_1, S_2 & S_3 with production capacity 7, 9 & 18 units per week of product, respectively. These units are to be shipped to 4 warehouses D_1, D_2, D_3 & D_4 with requirement of 5, 8, 7 and 14 units per week respectively. The transportation cost per unit betwn factories to warehouse are given below:

Factory \ ware house	D_1	D_2	D_3	D_4	Supply
S_1	19 x_{11}	30 x_{12}	50 x_{13}	10 x_{14}	7
S_2	70 x_{21}	30 x_{22}	40 x_{23}	60 x_{24}	9
S_3	40 x_{31}	8 x_{32}	70 x_{33}	20 x_{34}	18
Demand	5	8	7	14	34

Formulate this transportation problem as an LP model to minimize the total transportation cost.

Soln:

let x_{ij} = quanitity transported from factory i $(i = 1, 2, 3)$ (source) to a warehouse (Destination) j $(j = 1, 2, 3, 4)$

The transportation problem is stated as an LP model as follows:

- $\text{Min } Z = 19x_{11} + 30x_{12} + 50x_{13} + 10x_{14} + 70x_{21} + 30x_{22} + 40x_{23} + 60x_{24} + 40x_{31} + 8x_{32} + 70x_{33} + 20x_{34}$

- subject to the constraints

Supply:
$$x_{11} + x_{12} + x_{13} + x_{14} = 7$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 9$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 18$$

Demand:
$$x_{11} + x_{21} + x_{31} = 5$$
$$x_{12} + x_{22} + x_{32} = 8$$
$$x_{13} + x_{23} + x_{33} = 7$$
$$x_{14} + x_{24} + x_{34} = 14$$

and $x_{ij} \geq 0$ for $i = 1, 2, 3$ & $j = 1, 2, 3, 4$

- Decision variables = $m \times n$ (3×4) where, m = no. of rows (source) - 3
  n = no. of columns (destination) - 4
- Basic variables = $(m + n - 1)$
- Constraints = $m + n$.

- i : source → supply
- j : destination → demand

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→ Types of transportation problem

Transportation problem

- Unbalanced
  - Demand > supply
    $\Sigma D_j > \Sigma S_i$
    ↳ Add a dummy row with cost zero and supply = (demand - supply)
  - Demand < supply
    $\Sigma D_j < \Sigma S_i$
    ↳ Add a dummy column with cost zero and demand = (supply - demand)
- Balanced
  (Demand = supply)
  $\Sigma D_j = \Sigma S_i$

(110) (100)

Example 1 : Demand > supply

	D_1	D_2	D_3	D_4	supply
S_1	5	4	2	6	30
S_2	8	3	5	7	40
S_3	10	11	3	1	20
S_4	9	12	6	3	10
Demand	10	50	30	20	110 \ 100

→ Add dummy row & supply = 110 - 100 = 10

	D_1	D_2	D_3	D_4	supply
S_1	5	4	2	6	30
S_2	8	3	5	7	40
S_3	10	11	3	1	20
S_4	9	12	6	3	10
S_5	0	0	0	0	10
Demand	10	50	30	20	110

(100) < (110)

Example 2: Demand < supply

	D_1	D_2	D_3	D_4	supply
S_1	1	9	14	8	50
S_2	13	2	7	12	20
S_3	15	6	3	11	10
S_4	5	10	16	4	30
Demand	10	20	30	40	100 \ 110

→ Add dummy column, Demand = 110 - 100 = 10

	D_1	D_2	D_3	D_4	D_5	supply
S_1	1	9	14	8	0	50
S_2	13	2	7	12	0	20
S_3	15	6	3	11	0	10
S_4	5	10	16	4	0	30
Demand	10	20	30	40	10	110

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* Solution to Transportation problem

→ The solution of transportation problem can be obtained in two stages.

i, Initial solution

→ This can be obtained by using any one of three methods:

- North-West Corner Method (NWCM)
- Least cost Method (LCM)
- Vogel's Approximation Method (VAM)

ii, optimal solution

→ This can be obtained by using any one

- Modified Distribution (MODI) Method
- Stepping Stone Method

* North-West corner Method (NWCM)

→ Let us consider 1st example that we done before.

Q. Use NWCM to find optimal solution.

Soln: ① Start in cell (1,1) (north-west) & Allocate $x_{11} = \min(s_1, d_1)$ & subtract that from the corresponding demand & supply (whichever hits '0' 'exhaust' first)

	D_1	D_2	D_3	D_4	supply (S_i)
S_1	19 (5)	30	50	10	~~7~~ - ~~5~~ = 2
S_2	70	30	40	60	9
S_3	40	6	70	20	18
(D_j) Demand	0 ~~5~~ (5-5)	8	7	14	34

(ii) If S_1 becomes zero, strike that row & move down next cell (2,1)
If d_1 becomes zero, strike that column & move right to cell (1,2).

	D_2	D_3	D_4	Sup.
~~S_1~~	~~30~~ (2)	~~50~~	~~10~~	~~2~~ 0
S_2	30	40	60	9
S_3	6	70	20	18
Dmd.	~~8~~ 6	7	14	

(iii) Repeat until all D/S are allocated or becomes zero

	D_2	D_3	D_4	Sup.
S_2	30 (6)	40	60	~~9~~ 3
S_3	6	70	20	18
Dmd.	0 ~~6~~	7	14	

	D_3	D_4	sup
~~S_2~~	~~40~~ (3)	60	0 ~~3~~
S_3	70	20	18
DMd.	~~7~~ 4	14	

	D_3	D_4	Sup.
S_3	70 (4)	20 (14)	~~18~~ ~~14~~ 0
Dmd.	0 ~~4~~	~~14~~ 0	

- m = no. of rows
- n = no. of columns

Now, we have to ensure whether the initial solution is feasible or not.

No. of Allocations (occupied cell) = $m+n-1$

or, $6 = 3+4-1$

$\therefore 6 = 6$ (True)

Thus, the initial solution is feasible / non-degenerate.

$\therefore$ Total cost = Σ (unit cell cost $\times$ Allocated quantity)

$= 19\times 5 + 30\times 2 + 30\times 6 + 40\times 3 + 70\times 4 + 20\times 14$

$=$ Rs. 1015

This solution can be done directly as

	D_1	D_2	D_3	D_4	Supply
S_1	19 [5]	30 [2]	50	10	~~7~~ ~~2~~ 0
S_2	70	30 [6]	40 [3]	60	~~9~~ ~~3~~ 0
S_3	40	8	70 [4]	20 [14]	~~18~~ ~~14~~ 0
Demand	~~5~~ 0	~~8~~ ~~6~~ 0	~~7~~ ~~4~~ 0	~~14~~ 0	34

★ Example 2 (Unbalanced)

	D_1	D_2	D_3	Supply
S_1	4	8	8	76
S_2	16	24	16	82
S_3	8	16	24	77
Demand	72	102	41	215 / 235

Since, Demand < Supply, so, the problem is unbalanced.

Now, we have to add Dummy column, demand = supply − demand
$= 235 - 215$
$= 20$

	D_1	D_2	D_3	D_4	Supply
S_1	~~4~~ [72]	8 [4]	~~8~~	0	~~76~~ ~~4~~ 0
S_2	16	~~24~~ [82]	~~16~~	0	~~82~~ 0
S_3	8	16 [16]	24 [41]	0 [20]	~~77~~ ~~61~~ 20
Demand	~~72~~ 0	~~102~~ ~~98~~ ~~16~~ 0	~~41~~ 0	20	235

Here, No of allocation = $m+n-1$
$= 3+4-1 = 6$

$\therefore$ Total min. cost $= 4\times 72 + 4\times 8 + 24\times 82 + 16\times 16 + 24\times 41 + 0\times 20$

$=$ Rs. 3528

* Least cost Method (LCM)

(i) pick the cell having minimum unit cost c_{ij} & Allocate $x_{ij} = \min(s_i, d_j)$ & subtract that from corresponding supply s_i & demand d_j

	D_1	D_2	D_3	D_4	supply (S_i)
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8 (8)	70	20	~~18~~ 10
Demand (D_j)	5	~~8~~ 0	7	14	34

Note: If unit cost c_{ij} are same, choose any one of them as minimum.

(ii) If s_i becomes 0, strike that row & if d_j becomes 0, strike that column & move to next remaining cell.

(iii) Repeat until all D/S becomes zero

	D_1	D_3	D_4	sup.
~~S_1~~	~~19~~	~~50~~	~~10~~ (7)	~~7~~ 0
S_2	70	40	60	9
S_3	40	70	20	10
Dmd	5	7	~~14~~ 7	

→

	D_1	D_3	D_4	sup.
S_2	70	40	60	9
S_3	40	70	20 (7)	~~10~~ 3
Dmd.	5	7	~~7~~ 0	

↓

	D_1	D_3	sup.
S_2	70	40 (7)	~~9~~ 2
S_3	40	70	3
	5	~~7~~ 0	

+ you can choose any value as min

←

	D_1	sup.
S_2	70 (2)	~~2~~ 0
S_3	40 (3)	~~3~~ 0
	~~5~~ ~~2~~ 0	

Here, the no. of allocated cells = 6 is equal to $m+n-1 = 3+4-1 = 6$

∴ The solution is feasible / non-degenerate.

∴ The total minimum cost $= 7\times10 + 2\times70 \times 7\times40 + 3\times40 + 8\times8 + 7\times20$ = Rs. 814 //

The solution can be done directly as:

	D_1	D_2	D_3	D_4	Supply
~~S_1~~	~~19~~	~~30~~	~~50~~	2 ~~10~~ [7]	~~7~~ 0
S_2	70 [2]	30	40 [7]	60	~~9~~ 2
~~S_3~~	~~40~~ [3]	1 ~~8~~ [8]	~~70~~	3 ~~20~~ [7]	18 ~~10~~ ~~3~~ 0
Demand	~~5~~ 2	~~8~~ 0	~~7~~ 0	~~14~~ ~~7~~ 0	

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* Vogel's Approximation Method (VAM)

→ This method gives better initial feasible solution, often closer to optimum.

(i) Compute penalty for each row & column (2nd smallest cost − smallest cost) & choose the row/column with highest penalty.

(ii) Find the cell that has least cost in selected row/column & Allocate $x_{ij} = \min(s_i, d_j)$ & subtract that from corresponding supply s_i & demand d_j

	D_1	D_2	D_3	D_4	Sup. (s_i)	penalty
S_1	19	30	50	10	7	9 (19−10)
S_2	70	30	40	60	9	10 (40−30)
S_3	40	8 (8)	70	20	~~18~~ 10	12 (20−8)
(d_j) Dmd	5	~~8~~ 0	7	14	34	
penalty	21	↑22	10	10		

highest penalty

(iii) If S_i becomes 0, strike that row, if d_j becomes 0, strike that column & move to next remaining cells

(iv) Repeat until all Demand & supply values are 0.

	D_1	D_3	D_4	sup.	penalty
S_1	19 (5)	50	10	~~7~~ 2	9
S_2	70	40	60	9	20
S_3	40	70	20	10	20
Dmd.	~~5~~ 0	7	14		
penalty	21	10	10		

	D_3	D_4	sup.	pen.
S_1	50	10	2	40
S_2	40	60	9	20
~~S_3~~	~~70~~	~~20~~ (10)	~~10~~ 0	~~50~~
Dmd.	7	~~14~~ 4		
pen.	10	10		

	D_3	D_4	sup.	per.
S_1	50	10 (2)	~~2~~ 0	40
S_2	40	60	9	20
Dmd.	7	~~4~~ 2		
Pen.	10	↑50		

	D_3	D_4	sup.	per.
S_2	40 (7)	60 (2)	~~9~~ ~~2~~ 0	
Dmd.	~~7~~ 0	~~2~~ 0		
pen.				

∴ Total cost = 8×8 + 19×5 + 10×2 + 20×10 + 40×7 + 60×2
= Rs. 779 ,,

This solution can be done directly as

	D_1	D_2	D_3	D_4	Sup.	Row penalties				
S_1	19 [5]	30	50	10 [2]	~~7~~ ~~2~~ 0	9	9	40	40	4
S_2	70	30	40 [7]	60 [2]	9	10	20	20	20	
S_3	40	8 [8]	70	20 [10]	~~18~~ ~~10~~ 0	12	20	50		3
Dmd.	~~5~~ 0	~~8~~ 0	7	~~14~~ ~~4~~ 2						
Column penalties	21	22↑	10	10						
	21↑		10	10						
			10	10						
			10	50↑						
	2	1								

* MODI (Modified Distribution) Method / (UV Method)

Algorithm

1) Find the initial basic feasible solution using any one of three methods.

2) For occupied cell, calculate u_i (for rows) and v_j (for column) using $C_{ij} = u_i + v_j$
   - Assign $u_i / v_j = 0$, having maximum no. of allocations.

3) For unoccupied cell, calculate $d_{ij} = C_{ij} - (u_i + v_j)$
   - If all $d_{ij} \geq 0$, current solution is optimal
   - If and $d_{ij} < 0$, choose the most negative and form a close loop (alternating '+' and '-' around the corner of occupied cell) to adjust flows.

4) Adjust allocation along that loop by the minimum allocation on the '-' position, reallocating flows to reduce total cost.

5) Repeat step 2 - 7 until optimal solution is obtained (i.e. $d_{ij} \geq 0$).

   - Identify the smallest allocation on a '-' corner, call it θ.
   - Add θ to all '+' corners, subtract θ from all '-' corners.
   - Recompute $u, v, \&\ d_{ij}$ until $d_{ij} \geq 0$

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Q. Find the optimal solution of the following problem using any methods.

i) Obtaining initial feasible solution from VAM

	D_1	D_2	D_3	D_4	Sup.	Row Penalties			
O_1	1 (20)	2	1 (10)	4	~~30~~ ~~10~~ 0	0	0	1	1 ←
O_2	3	3 (20)	2 (20)	1 (10)	~~50~~ 40	1	1	1	1
O_3	4	2 (20)	5	9	~~20~~	2	2	3 ←	
Dmd.	~~20~~ 0	~~40~~ 20	~~30~~ 20	~~10~~ 0	100				
Column penalties	2	0	1	3↑					
	2↑	0	1						
		0	1						
		1	1						

2   1

4   3

we choose this 1 because it has least value on Dmd/sup. among others: (10, 40, 20, 20)

Here: No. of occupied cell = 6 and

$m+n-1 = 3+4-1 = 6$ (both are equal).

so, the solution is non degenerate feasible solution.

∴ Total cost of initial solution is

$TC = 1\times20 + 3\times20 + 2\times20 + 2\times20 + 1\times10 + 1\times10$

$= 180$

ii) Testing of optimality of initial solution by MODI method

For Allocated (occupied) cell, calculate dual variables u_i & v_j using $c_{ij} = u_i + v_j$

$\Rightarrow \left\{ \begin{array}{l} u_i = c_{ij} - v_j \\ v_j = c_{ij} - u_i \end{array} \right.$

	D_1	D_2	D_3	D_4	
O_1	1 (20)		1 (10)		$u_1 = -1$
O_2		3 (20)	2 (20)	1 (10)	$u_2 = 0$ (since it has highest allocation i.e. 3)
O_3		2 (20)			$u_3 = -1$
	$v_1 = 2$	$v_2 = 3$	$v_3 = 2$	$v_4 = 1$	

Substituting $u_2 = 0$, we get

$c_{22} = u_2 + v_2 \Rightarrow 3 = 0 + v_2 \Rightarrow v_2 = 3$

$c_{32} = u_3 + v_2 \Rightarrow 2 = u_3 + 3 \Rightarrow u_3 = -1$

$c_{23} = u_2 + v_3 \Rightarrow 2 = 0 + v_3 \Rightarrow v_3 = 2$

$c_{24} = u_2 + v_4 \Rightarrow 1 = 0 + v_4 \Rightarrow v_4 = 1$

$c_{13} = u_1 + v_3 \Rightarrow 1 = u_1 + 2 \Rightarrow u_1 = -1$

$c_{11} = u_1 + v_1 \Rightarrow 1 = -1 + v_1 \Rightarrow v_1 = 2$

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(opportunity cost)

For Non Allocated (unoccupied) cell, calculate $d_{ij} = C_{ij} - (u_i + v_j)$

	D_1	D_2	D_3	D_4	
O_1	-	2 (2) = 0	-	4 (0) = 4	$u_1 = -1$
O_2	3 (2) = 1	-	-	-	$u_2 = 0$
O_3	4 (1) = 3	-	5 (1) = 4	9 (0) = 9	$u_3 = -1$
	$v_1 = 2$	$v_2 = 3$	$v_3 = 2$	$v_4 = 1$	

$d_{12} = c_{12} - (u_1 + v_2) = 2 - (-1 + 3) = 2 - 2 = 0$

$d_{21} = C_{21} - (u_2 + v_1) = 3 - (0 + 2) = 3 - 2 = 1$

$d_{31} = C_{31} - (u_3 + v_1) = 4 - 1 = 3$

⋮

Since, all the $d_{ij} \geq 0$, so, the transportation cost Rs. 180 is optimal.

Q2 Find the optimal solution of the problem that is already solved in VAM method example.

Soln:

Since, initial feasible solution is already calculated in previous example using VAM method.

Now, Testing optimality of initial solution

For Allocated cell $(C_{ij} = u_i + v_j) \Rightarrow u_i / v_j = C_{ij} - v_j / u_i$

	D_1	D_2	D_3	D_4	
S_1	19 (5)	30	50	10 (2)	$u_1 = 10$
S_2	70	30	40 (7)	60 (2)	$u_2 = 60$
S_3	40	8 (8)	70	20 (10)	$u_3 = 20$
	$v_1 = 9$	$v_2 = -12$	$v_3 = -20$	$v_4 = 0$	

(19-10) (8-20) (40-60)

$\Rightarrow C_{11} - u_1$ $\Rightarrow C_{31} - u_3$ $\Rightarrow C_{23} - u_2$

For unallocated cell $(d_{ij} = C_{ij} - (u_i + v_j)) \geq 0$

	D_1	D_2	D_3	D_4	
S_1	-	30 (-2)	50 (10) → $u_1 + v_3$		$u_1 = 10$
S_2	70 (69)	30 (48)	-	-	$u_2 = 60$
S_3	40 (29)	-	70 (0)	-	$u_3 = 20$
	$v_1 = 9$	$v_2 = -12$	$v_3 = -20$	$v_4 = 0$	

At (2,2) cell $d_{22} = 30 - 48 = -12 < 0$

Now, Allocate θ at (2,2) cell.

Note: If there are others -ve values also, then choose the most -ve one cell to allocate value.

	D_1	D_2	D_3	D_4
S_1	19 (5)	–	–	10 (2) $2-2$
S_2	–	30 +θ (2)	40 (7)	60 (2) −θ
S_3	–	8 (8) −θ $8-2$	–	20 (10) +θ $10+2$

→ this cell becomes unallocated

now,

θ = minimum allocate value among all −ve position on closed path

i.e. $\theta = \min(8, 2) \Rightarrow 2$ (subtract 2 from all (−) & add to (+))

The modified solution is

• $d_{12} = 30-(-2) = 32$, $d_{13} = 50-18 = 32$ (do same for all d_{ij})

	D_1	D_2	D_3	D_4	
S_1	19 (5) u_i+v_j	30 (−2)	50 (8)	10 (2)	$u_1 = 0$
S_2	70 (51)	30 (2)	40 (7)	60 (42)	$u_2 = 32$
S_3	40 (29)	8 (6)	70 (18)	20 (12)	$u_3 = 10$
	$v_1 = 19$	$v_2 = -2$	$v_3 = 8$	$v_4 = 10$	

Since all $d_{ij} \geq 0$, so the final solution is arrived.

Also,
No. of allocation = 6 &
$m+n-1 = 3+4-1 = 6$
So, the soln is non-degenerate.

∴ Total cost $= 19\times5 + 30\times2 + 8\times6 + 40\times7 + 10\times2 + 20\times12$
$= 743$

* Degeneracy in Transportation

→ A basic feasible solution in a transportation problem must have exactly '$m+n-1$' allocations.

• If it has less than '$m+n-1$' allocation, it is degenerate.

Resolving Degeneracy

• Assign a very small quantity ε (epsilon) ≈ 0 to an unoccupied cell (at very low cost) that doesn't form a closed loop with current allocations until you have $(m+n-1)$ occupied cells.

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Example:

Q, Find the optimal solution of the following problem using any methods.

	D_1	D_2	D_3	Supply
S_1	8	5	6	120
S_2	15	10	12	80
S_3	3	9	10	80
Demand	150	80	50	

Soln

• Obtaining Initial feasible solution using VAM

	D_1	D_2	D_3	Supply	Row penalty		
S_1	~~8~~ 70	5	~~6~~ 50	~~120~~ ~~50~~ 0	1	1	1
S_2	15	10 80	12	80	2	2	2
S_3	3 80	9	10	~~80~~ 0	6 ←		
Demand	~~150~~ ~~70~~ 0	80	~~50~~ 0				
Column penalty	5	4	4				
	↑7	5	6				
		5	↑6				

3 1 2 3

Total cost = $8\times70 + 3\times80 + 10\times80 + 6\times50$ = Rs. 1900

• Testing optimality by MODI method

since, no. of occupied cell = 4, which is less than $m+n-1 = 3+3-1 = 5$

so, the solution is degenerate.

Therefore, we assign a very small quantity ϵ (epsilon) ≈ 0 to an unoccupied cell having least cell cost where loop cannot be formed i.e. cell (1,2)

→ This will update the no. of allocations back to $m+n-1 = 5$

	D_1	D_2	D_3	Supply	
S_1	8 70	5 ϵ	6 50	120	$u_1 = 0$
S_2	15	10 80	12	80	$u_2 = 5$
S_3	3 80	9	10	80	$u_3 = -5$
Dmd.	150	80	50		
	$v_1 = 8$	$v_2 = 5$	$v_3 = 6$		

No closed loop formed

For allocated cell find u_i & v_j using ($c_{ij} = u_i + v_j$)

For unallocated cell, find $d_{ij} = c_{ij} - (u_i + v_j)$

	D_1	D_2	D_3	
S_1	-	-	-	$u_1 = 0$
S_2	15^{13}	-	12^{11}	$u_2 = 5$
S_3	-	9^{0}	10^{1}	$u_3 = -5$
	$v_1 = 8$	$v_2 = 5$	$v_3 = 6$	

$d_{21} = 15 - 13 = 2$

$d_{23} = 12 - 11 = 1$

$d_{32} = 9 - 0 = 9$

$d_{33} = 10 - 1 = 9$

Since, all $d_{ij} \geq 0$. so final optimal solution is arrived.

$\therefore$ Total min. TC $= 8 \times 70 + 5 \times 0 + 6 \times 50 + 10 \times 80 + 3 \times 80$

$= 1900$ // $(\epsilon = 0)$

Q2, Solve the TP & obtain the optimal solution.

	D_1	D_2	D_3	Supply
S_1	2	2	3	10
S_2	4	1	2	15
S_3	1	3	1	40
Demand	20	15	30	65

Soln

- Obtaining initial feasible solution using VAM

	D_1	D_2	D_3	Supply	Row penalty	
S_1	2 (10)	2	3	10	0	1
S_2	4	1 (15)	2	~~15~~ 0	1	-
S_3	1 (10)	3	1 (30)	~~40~~ 10	0	0
Demand	20	~~15~~ 0	~~30~~ 0			
Column penalty	1	1 ↑	1			
	1		2 ↑			
		1	2			

$\therefore$ Total cost $= 10 \times 2 + 15 \times 1 + 10 \times 1 + 30 \times 1 = $ Rs. 75

- Testing optimality using MODI

Since, no. of occupied cell = 4, which is less than $m+n-1 = 3+3-1 = 5$

So, the solution is degenerate.

Therefore we add $\epsilon \approx 0$ to an unoccupied cell having least cost.

Here, cell (1,2) & cell (2,3) both has least cost (i.e. 2)

Choose any arbitrarily which do not form loop. Let S_1D_2.

The updated table is

	D_1	D_2	D_3	
S_1	2 (10)	2 (ε)	3 (2)	$u_1 = 1$
S_2	4 (1)	1 (15)	2 (1)	$u_2 = 0$
S_3	1 (10)	3 (1)	1 (30)	$u_3 = 0$
	$v_1 = 1$	$v_2 = 1$	$v_3 = 1$	

Since all $d_{ij} \geq 0$. so the solution is optimal

$\therefore$ Total min. $TC = 10\times2 + 2\times0 + 1\times15 + 1\times10 + 1\times30 = Rs.\ 75$

Q3. Find the optimal solution of the following problems.

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	10	2	3	15	9	35
S_2	5	10	15	2	4	40
S_3	15	5	14	7	15	20
S_4	20	15	13	25	8	30
Demand	20	20	40	10	35	125

Soln

obtaining initial feasible solution using VAM

	D_1	D_2	D_3	D_4	D_5	Supply	Row penalty				
1 S_1	10	2	3 (35)	15	9	~~35~~ 0	1	-	-	-	-
5 S_2	5 (20)	10	15	2 (10)	4 (10)	~~40~~ ~~20~~ ~~10~~ 0	2	2	2	6	11
4 S_3	15	5 (20)	14	7	15	~~20~~ 0	2	2	2	9	-
S_4	20	15	13 (5)	25	8 (25)	30	5	5	5	5	5
Demand	~~20~~ 0	~~20~~ 0	~~40~~ 5	~~10~~ 0	~~35~~ 25						
Column penalty	5	3	10	5	4						
	10	5	1	5	4						
	-	5	1	5	4						
	-	5	1	-	4						
	-	-	2	-	4						
	2	4		3							

$\therefore$ Total cost $= 5\times20 + 3\times35 + 5\times20 + 2\times10 + 4\times10 + 8\times25 + 13\times5$

$= Rs.\ 630$

Since, no. of occupied cell $= 7$ which is less than $m+n-1 = 8$

$\therefore$ we Add $\varepsilon \approx 0$ to (1,2) cell having least cost among unoccupied cell.

• Testing optimality using MODI method

The updated table is

	D_1	D_2	D_3	D_4	D_5	For Allocated cell find u_i & v_j ($c_{ij} = u_i + v_j$)
S_1	10	2 (ε)	3 (35)	15	9	$u_1 = -6$
S_2	5 (20)	10	15	2 (10)	4 (10)	$u_2 = 0$
S_3	15	5 (20)	14	7	15	$u_3 = -3$
S_4	20	15	13 (5)	25	8 (25)	$u_4 = 4$
	$v_1 = 5$	$v_2 = 8$	$v_3 = 9$	$v_4 = 2$	$v_5 = 4$	

For unallocated cell, find $d_{ij} = c_{ij} - (u_i + v_j)$

	D_1	D_2	D_3	D_4	D_5	
S_1	$10^{(-1)}$	-	-	15^{-4}	9^{-2}	$u_1 = -6$
S_2	-	10^{8}	15^{9}	-	-	$u_2 = 0$
S_3	15^{2}	-	14^{6}	7^{-1}	15^{1}	$u_3 = -3$
S_4	20^{9}	15^{12}	-	25^{6}	-	$u_4 = 4$
	$v_1 = 5$	$v_2 = 8$	$v_3 = 9$	$v_4 = 2$	$v_5 = 4$	

$d_{11} = 10 - (-1) = 11$

$d_{14} = 15 - (-4) = 19$

$d_{15} = 9 - (-2) = 11$

$d_{22} = 10 - 8 = 2$

$d_{23} = 15 - 9 = 6$

. . . .

Since, all $d_{ij} \geq 0$. so the solution is optimal . . . .

$\therefore$ Total min TC $= 5\times20 + 2\times0 + 3\times35 + 5\times20 + 2\times10 + 4\times10$
$+ 8\times25 + 13\times5$
$=$ Rs. 630 //

* Unbalanced Transportation problem

→ It occurs when Total supply ≠ total demand.

Make it balance by adding a dummy row or column

- Add dummy destination/column if (Supply > Demand)
- Add dummy source (row) if (demand > supply)
- Assign zero cost to dummy cells

Q. Determine the minimum transportation cost from the following matrix.

	D_1	D_2	D_3	D_4	Supply
S_1	45	60	45	30	70
S_2	35	15	35	35	60
S_3	30	25	45	55	90
Demand	60	40	60	20	180 / 220

Soln

Here, Total Demand $(\Sigma D) = 180$

Total supply $(\Sigma S) = 220$

Since, $\Sigma D \neq \Sigma S$, so, the given problem is unbalanced.

To make it balanced, we add dummy destination (D_5) with demand $= 220 - 180 = 40$ & 0 cost to dummy cells.

1) Obtaining initial solution using VAM

	D_1	D_2	D_3	D_4	D_5	Supply	Row Penalty			
S_1	45	60	45 (10)	30 (20)	0 (40)	~~70~~ ~~30~~ 10	30	15	15	0
S_2	35	15 (40)	35 (20)	35	0	~~60~~ 20	15	20	0	0
S_3	30 (60)	25	45 (30)	55	0	~~90~~ 30	25	5	15	15
Demand	~~60~~ 0	~~40~~ 0	60	~~20~~ 0	~~40~~ 0	220 / 220				
Column penalty	5	10	10	5	0					
	5	10	10	5						
	5		10	5						
	5		10							
	4	2		3	1					

∴ Total cost $= 45 \times 10 + 30 \times 20 + 0 \times 40 + 15 \times 40 + 35 \times 20 + 30 \times 60 + 45 \times 30$

$=$ Rs. 5500

Since, No. of allocation $= 7$ is equal to $m+n-1 = 3+5-1 = 7$

2) Testing optimality by MODI method

For allocated cell, calculate dual variables u_i & v_j using $c_{ij} = u_i + v_j$

	D_1	D_2	D_3	D_4	D_5	
S_1	-	-	45 (10)	30 (20)	0 (10)	$u_1 = 0$
S_2	-	15 (40)	35 (20)	-	-	$u_2 = -10$
S_3	30 (60)	-	45 (30)	-	-	$u_3 = 0$
	$v_1 = 30$	$v_2 = 25$	$v_3 = 45$	$v_4 = 30$	$v_5 = 0$	

For unallocated cell, calculate opportunity cost

$d_{ij} = c_{ij} - (u_i + v_j)$

	D_1	D_2	D_3	D_4	D_5	
S_1	45^{30}	60^{25}	-	-	-	$u_1 = 0$
S_2	35^{20}	-	-	35^{20}	0^{-10}	$u_2 = -10$
S_3	-	25^{25}	-	55^{30}	0^{0}	$u_3 = 0$
	$v_1 = 30$	$v_2 = 25$	$v_3 = 45$	$v_4 = 30$	$v_5 = 0$	

Since, all $d_{ij} \geq 0$, so the solution is optimal

$\therefore$ Total min. TC = Rs. 5500

* Maximization Transportation Problem

→ For this type of problem, first we have to convert the maximization problem into minimization by subtracting each entry from the highest cell value.

Q. Solve the given TP to maximize the profit

	D_1	D_2	D_3	D_4	Supply
S_1	15	51	42	33	23
S_2	80	42	26	81	44
S_3	90	40	66	60	33
Demand	23	31	16	30	

Soln

Since, the problem is maximization, so convert it into minimization by subtracting all the element from the highest element (i.e. 90)

So, the updated table is:

	D_1	D_2	D_3	D_4	Supply
S_1	75	39	48	57	23
S_2	10	48	64	9	44
S_3	0	50	24	30	33
Demand	23	31	16	30	100

1, Obtaining initial feasible solution using VAM

	D_1	D_2	D_3	D_4	Supply	Row penalty			
S_1	75	39 (23)	48	57	23	9	18	18	18
S_2	10 (6)	48 (8)	64	9 (30)	~~44~~ ~~38~~ 8	1	1	1	39
S_3	0 (17)	50	24 (16)	30	~~33~~ ~~17~~ 0	24	30		
Demand	~~23~~ ~~6~~ 0	31	~~16~~ 0	~~30~~ 0					
Column penalty	10	9	24 ↑	21					
	10	9		21					
	65 ↑	9		48					
		9		48 ↑					
	3		1	4			2		

Since, no. of allocation = 6 is equal to $m+n-1 = 3+4-1 = 6$

So, the solution is non-degenerate feasible solution.

2, Testing optimality by modi method

For allocated cell, calculate dual variables u_i & v_j using $C_{ij} = u_i + v_j$ & for unallocated cell, calculate $d_{ij} = C_{ij} - (u_i + v_j)$

	D_1	D_2	D_3	D_4	
S_1	75 [1]	39 (23)	48 [25]	57 [0]	$u_1 = -9$
S_2	10 (6)	48 (8)	64 [34]	9 (30)	$u_2 = 0$
S_3	0 (17)	50 [38]	24 (16)	30 [-1]	$u_3 = -10$
	$v_1 = 10$	$v_2 = 48$	$v_3 = 34$	$v_4 = 9$	

$d_{11} = 75 - 1 = 74$

$d_{13} = 48 - 25 = 23$

$d_{14} = 57 - 0 = 57$

$d_{23} = 64 - 34 = 30$

$d_{32} = 50 - 38 = 12$

$d_{34} = 30 - (-1) = 31$

Since, all $d_{ij} \geq 0$. so the solution is optimal.

Thus, the maximization table becomes

	D_1	D_2	D_3	D_4
S_1	15	51 (23)	42	33
S_2	80 (6)	42 (8)	26	81 (30)
S_3	90 (17)	40	66 (16)	60

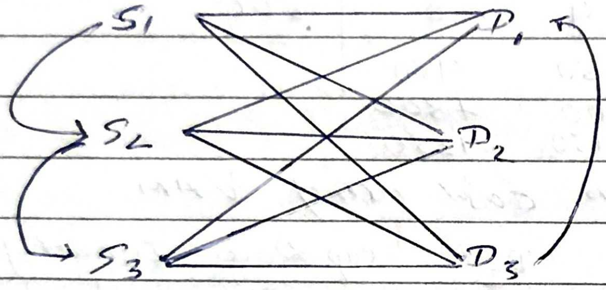
$\therefore$ The maximum profit $= 51 \times 23 + 80 \times 6 + 42 \times 8 + 90 \times 17 + 81 \times 30 + 66 \times 16$

$=$ Rs. 7005 ..

* Transhipment problems

→ In basic transportation problem, goods flow directly from source to a destination. But,
In a transhipment problem, goods may be routed through intermediate points (transhipment points), potentially involving multiple stages of shipping to reach their final destination.

Note: The intermediate points can be source & destination themselves.



Example:

Q1 The following is the transhipment problem with 4 source and 2 destinations. The supply values of the sources S_1, S_2, S_3 & S_4 are 100, 200, 150 & 350 units. The demand values of destination D_1 & D_2 are 350 & 450 units. Transportation cost per unit betwn different source and destinations are given in the following table. Solve the transhipment problem

Source \ Destination	S_1	S_2	S_3	S_4	D_1	D_2
S_1	0	4	20	5	25	12
S_2	10	0	6	10	5	20
S_3	15	20	0	8	45	7
S_4	20	25	10	0	30	6
D_1	20	18	60	15	0	10
D_2	10	25	30	23	4	0

Soln

① Check the problem is balanced or not

Total Supply = 100 + 200 + 150 + 350 = 800

Total Demand = 350 + 450 = 800

∴ The problem is balanced

② Add the values of total supply / demand to all the rows & column.

	S_1	S_2	S_3	S_4	D_1	D_2	Supply
S_1	0	4	20	5	25	12	100 + 800 = 900
S_2	10	0	6	10	5	20	200 + 800 = 1000
S_3	15	20	0	8	45	7	150 + 800 = 950
S_4	20	25	10	0	30	6	350 + 800 = 1150
D_1	20	18	60	15	0	10	800
D_2	10	25	30	23	4	0	800
Demand	800	800	800	800	350 +800 = 1150	450 +800 = 1250	

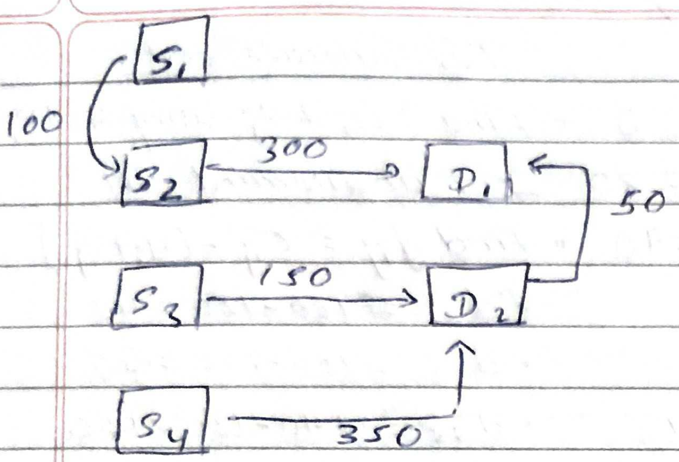
③ Find the transportation cost using VAM

		S_1	S_2	S_3	S_4	D_1	D_2	Sup.	Row penalty						
8	S_1	0 (800)	4 (100)	20	5	25	12	~~900~~ ~~100~~ 0	4	1	1	1	8	8	8
	S_2	10	0 (700)	6	10	5 (300)	20	~~1000~~ 300	5	5	5	5	5	5	5
6	S_3	15	20	0 (800)	8	45	7 (150)	~~950~~ ~~150~~ 0	7	7	← 7	1	13	← 13	
5	S_4	20	25	10	0 (800)	30	6 (350)	~~1150~~ ~~350~~ 0	6	6	6	← 6	← 19		
2	D_1	20	18	60	15	0 (800)	10	~~800~~ 0	10	← 10					
	D_2	10	25	30	23	4 (50)	0 (750)	~~800~~ 50	4	4	4	4	4	4	4
	Demand	~~800~~ 0	~~800~~ 700	~~800~~	~~800~~ 0	~~1150~~ 350	~~1250~~ ~~900~~ ~~750~~ 0								
		10 ↑	4	6	5	4	6								
	Column		4	6	5	4	6								
	penalty		4	6	5	1	6								
			4		5	1	6								
			4			1	6								
			4			1	7								
		1	4	3	4	1	12 ↑								
			9				7								

④ Draw the shipping pattern

- The allocations in the main diagonal cells are to be ignored

so, the diagrammatic representation of optimal shipping pattern is as:



* Past questions solution

2023 . Q. NO. 5 : Find the optimal transportation cost

To / From	A	B	C	capacity
W	40	80	80	55
X	160	-	160	25
Y	80	160	240	35
Req.	35	45	35	115

Soln

1) obtaining initial feasible solution from VAM

	A	B	C	Sup.	Row penalty	
W	40	80 (45)	80 (10)	55	40	0
~~X~~	160	-	160 (25)	~~25~~ 0	0	160
~~Y~~	80 (35)	160	240	~~35~~ 0	80	
Dmd.	~~35~~ 0	45	~~35~~ 10	115		
Column penalty	40	80	80			
		80	80			

The min. Total TC = 80×45 + 80×10 + 160×25 + 80×35 = 11200

Here, no. of occupied cell = 4 which is less than m+n-1 = 3+3-1 = 5

∴ The solution is degenerate.

To resolve it, we add ε≈0 to unoccupied cell having least cost where loop cannot be formed (i.e. cell (1,1))

2) Testing optimality using MODI method

	A	B	C	
W	40 (0)	80 (45)	80 (10)	$u_1 = 0$
X	160 [120]	-	160 (25)	$u_2 = 80$
Y	80 (35)	160 [120]	240 [120]	$u_3 = 40$
	$v_1 = 40$	$v_2 = 80$	$v_3 = 80$	

For Allocate cell
- Find u_i & v_j using $u_i + v_j = c_{ij}$

For unallocated cell
- Find $d_{ij} = c_{ij} - (u_i + v_j)$
- $d_{21} = 160 - 120 = 40$
- $d_{32} = 160 - 120 = 40$
- $d_{33} = 240 - 120 = 120$

Since, all $d_{ij} \geq 0$.
So, the solution is optimal.

$\therefore$ Total min. TC = $40 \times 0 + 80 \times 45 + 80 \times 10 + 80 \times 35 + 160 \times 25$
= Rs. 11200 //

2024. Q. No. 11. long

1) Obtaining initial feasible solution from VAM

	A	B	C	Supply	Row penalty			
W	4	8	3 (45)	~~450~~	1	1	1	3
X	6 (40)	7 (15)	9 (5)	60	1	1	1	
Y	8	2 (60)	5	~~600~~	3	3		2
Z	0 (10)	0	0	~~100~~	0			1
Demand	~~50~~ 40	~~75~~ 15	~~505~~	175				
	4	2	3					
Column	2	5	4					
penalty	2	1	5					

$\therefore$ Total cost = $3 \times 45 + 6 \times 40 + 7 \times 15 + 9 \times 15 + 2 \times 60 + 0 \times 10 = 642$.

Here, no. of allocation = $m + n - 1 = 6$

$\therefore$ The solution is non-degenerate.

2) Testing optimality using MODI method

	A	B	C	
W	4 [0]	8 [1]	3 (45)	$u_1 = -6$
X	+ 6 (40)	7 (15)	− 9 (5)	$u_2 = 0$
Y	8 [1]	2 (60)	5 [4]	$u_3 = -5$
Z	− 0 (10)	0 [1]	+ 0 [3]	$u_4 = -6$
	$v_1 = 6$	$v_2 = 7$	$v_3 = 9$	

	A	B	C
W			
X			
Y			
Z			

At (4,2) & (4,3) cell d_{ij} is −ve.

So, choose the most −ve one (i.e. (4,3) cell)
where, $d_{43} = -3$

New, Now allocate θ at (4,3) cell
$\therefore \theta = \min(5, 10) = 5$

- Resources - man, machine, worker, professor, vehicles, time slots, etc.
- tasks - jobs, classroom, people, counter, etc.

Date: ....

Page: ....

* Assignment Problem

→ It is a special type of LPP that deals with optimally assigning a set of resources to a set of tasks.

- Its objective is to minimize total cost or time or maximize total profit.
- It is a special case of transportation problem where
  - No. of source = No. of destinations
  - Each task is assigned to one agent and vice versa

* Types of Assignment Problem

1, Balanced

- No. of workers = No. of jobs

2, Unbalanced

- No. of workers ≠ No. of jobs

→ Add dummy row (resource) / column (job) to make it balance

* Hungarian Method

2021/2023 2024 # Algorithm:

Step 1: Check whether the problem is blanced or not. If not make it balance by adding dummy rows / columns

Step 2: Row reduction : subtract min. value from each row.

Step 3: Column reduction : Subtract min. value from each column.

Step 4: Make a assignment in opportunity cost table.

- Draw a minimum no. of lines that will cross all the 0's in the reduced cost matrix. In order to draw minimum no. of lines, check no. of '0' in 1st row, if there is only one '0' make a box over it & draw a line through colum, if there is more than one '0' skip that row & go to next row & continue the process until all 0's are covered.

Step 5: If no. of lines = no. of row / column, solution is optimal. If not:

Step 6: Adjust matrix : • subtract the smallest uncovered value from each uncovered elements & add it to the elements covered twice and repeat step 4.

Q.1, Solve the following Assignment problem

	J_1	J_2	J_3
P_1	10	20	30
P_2	20	10	40
P_3	50	30	20

soln

Here, No. of rows = No. of column, so the problem is balanced.

i) Row reduction: Subtract min value from each row

	J_1	J_2	J_3	
P_1	0	10	20	(-10)
P_2	10	0	30	(-10)
P_3	30	10	0	(-20)

ii) Column reduction: Subtract min value from each column.

	J_1	J_2	J_3
P_1	0	10	20
P_2	10	0	30
P_3	30	10	0
	(-0)	(-0)	(-0)

iii) Make a assignment (Each worker is assigned only one job)

	J_1	J_2	J_3
P_1	$\boxed{0}$	10	20
P_2	10	$\boxed{0}$	30
P_3	30	10	$\boxed{0}$

Since, No. of assignment = no. of rows / column = 3
So, the solution is optimal.

Hence, the total assigning cost is given by

	costs
P_1 - J_1	10
P_2 - J_2	10
P_3 - J_3	20
Total min. cost	

Q.2 Solve the following problem to find the optimal assignment schedule of salesman to counters that minimize total cost.

Salesman	Counters		
	C_1	C_2	C_3
S_1	9	7	10
S_2	18	11	19
S_3	13	14	12

Soln

Here, no. of rows = no. of columns = 3, so, the problem is balanced.

i) Row Reduction:

	C_1	C_2	C_3	
S_1	2	0	3	(−7)
S_2	7	0	8	(−11)
S_3	1	2	0	(−12)

ii) Column Reduction:

	C_1	C_2	C_3
S_1	1	0	3
S_2	6	0	8
S_3	0	2	0
	(−1)	(−0)	(−0)

iii) Make a assignment

	C_1	C_2	C_3
S_1	1	$\boxed{0}$	3
S_2	6	$\cancel{0}$	8
S_3	0	2	0

Here we are not able to assign counters for S_2, so, readjusting the matrix.

iv) Draw a set of horizontal & vertical lines to cover all the '0'
- subtract smallest uncovered value from uncovered elements.
- Add it to the elements covered twice
- Make a assignment again.

	C_1 (smallest)	C_2	C_3
S_1	1 −1	0	3 −1
S_2	6 −1	0	8 −1
S_3	0	2 +1	0

	C_1	C_2	C_3
S_1	$\boxed{0}$	$\cancel{0}$	2
S_2	5	$\boxed{0}$	7
S_3	$\cancel{0}$	3	$\boxed{0}$

Since, no. of assignment = no. of rows / column.

so, the solution is optimal

Salesman	Counter	Cost
S_1	C_1	9
S_2	C_2	11
S_3	C_3	12
Total min. cost		32

Q.3. Four professors are each capable of teaching any one of the 4 diff. subjects. But each professor should be assigned only one subject. Find the schedule so as to minimize the total subject preparation time for all subjects per professor.

Professors	S_1	S_2	S_3	S_4
P_1	2	10	9	7
P_2	15	4	14	8
P_3	13	14	16	11
P_4	3	15	13	8

i, Row Reduction

	S_1	S_2	S_3	S_4
P_1	0	8	7	5
P_2	11	0	10	4
P_3	2	3	5	0
P_4	0	12	10	5

ii, Column Reduction

	S_1	S_2	S_3	S_4	
P_1	$\boxed{0}$	8	2	5	
P_2	11	$\boxed{0}$	5	4	
P_3	2	3	$\boxed{0}$	0	3
P_4	~~0~~	12	5	5	
	1	2			

Here, no. of allocation = 3 ≠ no. of rows / column

so, the solution is not optimal.

Now, readjust the matrix subtracting smallest uncovered value from uncovered elements & adding it to elements covered twice.

	S_1	S_2	S_3	S_4	
P_1	0	8	$\boxed{0}$	3	4
P_2	11	$\boxed{0}$	3	2	
P_3	4	5	0	$\boxed{0}$	3
P_4	$\boxed{0}$	12	3	3	
	2	1			

Here, no. of allocation = no. of rows / column = 4

∴ The optimal solution is

professor	subject	time
P_1	S_3	9
P_2	S_2	4
P_3	S_4	11
P_4	S_1	3
Total min. Time		27 hrs

* Unbalanced Assignment Problem

Q.1, Solve the following Assignment problem

Worker \ Task	T_1	T_2	T_3
W_1	19	36	25
W_2	23	37	16
W_3	45	30	25
W_4	28	40	30

Soln

Here, no. of rows = 5, no. of columns = 3

$\therefore$ no. of rows $\neq$ no. of column. so, the given problem is unbalanced.

To make it balance, we add dummy column/Task (T_4)

i, Row Reduction

-) same table will formed as all row has '0' min. value.

	T_1	T_2	T_3	T_4
W_1	19	36	25	0
W_2	23	37	16	0
W_3	45	30	25	0
W_4	28	40	30	0

ii, Column reduction

	T_1	T_2	T_3	T_4
W_1	$\boxed{0}$	6	9	0
W_2	4	7	$\boxed{0}$	0
W_3	26	$\boxed{0}$	9	0
W_4	9	10	14	$\boxed{0}$

Since, no. of allocation = no. of rows/column.

So, the optimal solution is

Worker	Task	Time
W_1	T_1	19
W_2	T_3	16
W_3	T_2	30
W_4	T_4	0
Total min. time		65 hrs

Q, 2, Solve the following Assignment problem

Jobs \ Machines	M_1	M_2	M_3	M_4	M_5
J_1	10	11	4	2	8
J_2	7	11	10	14	12
J_3	5	6	9	12	14
J_4	13	15	11	10	7

Soln

Here, no. of rows ≠ no. of columns. so, the given problem is unbalanced. To make it balance, Add dummy row (J_5)

	M_1	M_2	M_3	M_4	M_5
J_1	10	11	4	2	8
J_2	7	11	10	14	12
J_3	5	6	9	12	14
J_4	13	15	11	10	7
J_5	0	0	0	0	0

i, Row Reduction

	M_1	M_2	M_3	M_4	M_5	
J_1	8	9	2	$\boxed{0}$	6	
J_2	$\boxed{0}$	4	3	7	5	
J_3	0	1	4	7	9	
J_4	6	8	4	3	$\boxed{0}$	
J_5	0	$\boxed{0}$	0	0	0	4
	2			1	3	

ii, Column Reduction

⇒ Same table.

Here, no. of allocations = 4 which is not equal to no. of rows so, the ~~given~~ solution is not optimal

iii, Readjust matrix (subtract smallest uncovered value)

	M_1	M_2	M_3	M_4	M_5
J_1	8	8	1	$\boxed{0}$	6
J_2	$\boxed{0}$	3	2	7	5
J_3	0	$\boxed{0}$	3	7	9
J_4	6	7	3	3	$\boxed{0}$
J_5	1	0	$\boxed{0}$	1	1
	2	3	5	1	4

⇒

Job	Machines	Time
J_1	M_4	2
J_2	M_1	7
J_3	M_2	6
J_4	M_5	7
J_5	M_3	0
Total min. Time		22 hrs

∴ No. of allocations = no. of rows = 5

so, the solution is optimal

→ convert into minimization by subtracting each entry from the highest one.

* Maximization Assignment Problem

Q.1 Find solution of Assignment problem using Hungarian method to maximize total profit.

	P_1	P_2	P_3	P_4	P_5
J_1	5	11	10	12	4
J_2	2	4	6	3	5
J_3	3	12	5	14	6
J_4	6	14	4	11	7
J_5	7	9	8	12	5

Soln:

Here, the problem is of maximization type. so we have to convert it into minimization by subtracting each element from the highest element (i.e. 14)

	P_1	P_2	P_3	P_4	P_5	
J_1	9	3	4	2	10	←
J_2	12	10	8	11	9	←
J_3	11	2	9	0	8	←
J_4	8	0	10	3	7	←
J_5	7	5	6	2	9	←

Row Reduction:

	P_1	P_2	P_3	P_4	P_5
J_1	7	1	2	0	8
J_2	4	2	0	3	1
J_3	11	2	9	0	8
J_4	8	0	10	3	7
J_5	5	3	4	0	7
	↑	↑	↑	↑	↑

⇒ Column Reduction:

		P_1	P_2	P_3	P_4	P_5
	J_1	3	1	2	$\boxed{0}$	7
3	J_2	$\boxed{0}$	2	0	3	0
	J_3	7	2	9	0	7
	J_4	4	$\boxed{0}$	10	3	6
	J_5	1	3	4	0	6
			2		1	

Here, no. of allocation = 3 & no. of rows = 5, so the solution is not optimal. Thus, subtracting uncovered elements from smallest uncovered element.

	P_1	P_2	P_3	P_4	P_5	
J_1	2	1	1	$\boxed{0}$	6	
J_2	0	3	$\boxed{0}$	4	0	4
J_3	6	2	8	0	6	
J_4	3	$\boxed{0}$	9	3	5	
J_5	$\boxed{0}$	2	3	0	5	
	3	2		1		

Here, no. of allocation = 4 & no. of rows = 5. so the solution is still not optimal

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Again, subtracting uncovered elements from smallest uncovered element and adding to intersection cell of two lines.

	P_1	P_2	P_3	P_4	P_5	
J_1	2	1	$\boxed{0}$	0	5	5
J_2	1	4	0	5	$\boxed{0}$	4
J_3	6	2	7	$\boxed{0}$	5	
J_4	3	$\boxed{0}$	8	3	4	
J_5	$\boxed{0}$	2	2	0	4	
	3	2		1		

Since, no. of allocations = no. of rows = 5
so, the optimal solution is

Job	person	profit
J_1	P_3	10
J_2	P_5	5
J_3	P_4	14
J_4	P_2	14
J_5	P_1	7
Total max. profit		50 units

Q.2. Find the assignment of operators to jobs that will result in a maximum profit, which job should be declined.

Job \ operator	A	B	C	D	E
1	6.20	7.80	5.00	10.10	8.20
2	7.10	8.40	6.10	7.30	5.90
3	8.70	9.20	11.10	7.10	8.10
4	4.80	6.40	8.70	7.70	8.00

→ Multiplying by 10 ⇒

	A	B	C	D	E
1	62	78	50	101	82
2	71	84	61	73	59
3	87	92	111	71	81
4	48	64	87	77	80

Soln

Since, no. of rows = 4 & no. of columns = 5 . so, the given probem is unbalanced. so, Add dummy row to make it balanced. Also, the problem is maximization type, so we have to convert it into minimization by subtracting each elements from highest element (i.e. 111)

	A	B	C	D	E
1	49	33	61	10	29
2	40	27	50	38	52
3	24	19	0	40	30
4	63	47	24	34	31
5	0	0	0	0	0

⇒ Row Reduction:

	A	B	C	D	E
1	39	23	51	$\boxed{0}$	19
2	13	$\boxed{0}$	23	11	25
3	24	19	$\boxed{0}$	40	30
4	39	23	0	10	7
5	$\boxed{0}$	0	0	0	0 — 4
		2	3	1	

Column Reduction:

→ since all column has smallest element '0', so same table is obtained.

	A	B	C	D	E
1	32	23	51	$\boxed{0}$	12
2	6	$\boxed{0}$	23	11	18
3	17	19	$\boxed{0}$	40	23
4	32	23	0	10	$\boxed{0}$
5	$\boxed{0}$	7	7	7	0
	5	2	3	1	4

Here, no. of allocation = no. of rows = 5.
So, the optimal solution is

Job	operator	profit
1	D	10.10
2	B	8.40
3	C	11.10
4	E	8.00
~~5~~	~~A~~	
Total max profit		37.60

* Infeasible (Restricted) constrained Assignment Problem

→ Sometimes it may happen that a particular resource (man, machine, worker) cannot be assigned to perform a particular activity (job, counter), then the problem is infeasible.

In such case, the cost of performing that particular activity by a particular resource is either assumed to very large (∞) or neglect that particular cell cost by leaving it vacant (-) throughout the solution, so as to prohibit the entry into the final solution.

Q. Find the optimal assignment of following problem.

	m_1	m_2	m_3	m_4
w_1	12	3	6	∞
w_2	4	10	∞	5
w_3	7	2	8	9
w_4	∞	7	8	6

→ The sign (-) indicate that particular worker machine assignment is not permitted.

Row reduction:

	m_1	m_2	m_3	m_4
w_1	9	0	3	∞
w_2	0	6	∞	1
w_3	5	0	6	7
w_4	∞	1	2	0

⇒ Column Reduction:

	m_1	m_2	m_3	m_4
w_1	9	$\boxed{0}$	1	∞
w_2	$\boxed{0}$	6	∞	1
w_3	5	0	4	7
w_4	∞	1	$\boxed{0}$	0

Here, no. of allocations = 3 which is not equal to no. of rows. So, the solution is not optimal.

	m_1	m_2	m_3	m_4	
w_1	9	0	$\boxed{0}$	∞	4
w_2	$\boxed{0}$	6	∞	0	2
w_3	5	$\boxed{0}$	3	6	
w_4	∞	2	0	$\boxed{0}$	3
		1			

⇒

work	machines	cost
w_1	m_3	6
w_2	m_1	4
w_3	m_2	2
w_4	m_4	6
Total min. cost		18

∴ The optimal solution is

- you can also by leaving (–) as it is.

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Q2. Three workers are available to work with the machines and the respective cost in Rs. associated with each worker-machine assignment is given below. The vacant places in table represent the prohibited case i.e. inappropriate place for assignment.

Workers \ Machines	A	B	C	D
P	12	3	6	10
Q	4	11	-	5
R	-	2	10	9

soln

Here, no. of rows = 3 & no. of columns = 4.
So, the given problem is unbalanced.
To make it balance, we create dummy row/workers (S).

	A	B	C	D
P	12	3	6	10
Q	4	11	-	5
R	-	2	10	9
S	0	0	0	0

$\Rightarrow$ Row Reduction:

	A	B	C	D
P	9	$\boxed{0}$	3	7
Q	$\boxed{0}$	7	-	1
R	-	0	8	7
S	0	0	$\boxed{0}$	0

Since, the min element of each column is zero, so the column reduction is not necessary.

Here, no. of allocation = 3, and no. of rows = 4
so, the solution is not optimal.

	A	B	C	D
P	9	$\boxed{0}$	2	6
Q	$\boxed{0}$	7	-	0
R	-	0	7	6
S	1	1	$\boxed{0}$	0

$\Rightarrow$

	A	B	C	D
P	7	0	$\boxed{0}$	4
Q	$\boxed{0}$	8	-	0
R	-	$\boxed{0}$	5	4
S	1	2	0	$\boxed{0}$

→ not optimal.

∴ The optimal solution is

workers	machines	costs
P	C	6
Q	A	4
R	B	2
S	D	0
Total min. cost		12

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* Multiple optimal solutions in assignment problem

→ While making an assignment in reducted matrix, it is possible to have two or more way to assign 0's. In such case there may be an alternate optimal solution exists with same optimal value.

Example:

	P	Q
A	$\boxed{0}$	0
B	0	$\boxed{0}$

→ Assignment: A − P, B − Q

Alternative solution

	P	Q
A	0	$\boxed{0}$
B	$\boxed{0}$	0

→ Assignment: A − Q, B − P

The 1st solution is A−P & B−Q. The alternative solution can be obtained by interchanging the assignment of A & B amongthemselves i.e. the alternative solution is A−Q & B−P.

Q. Find the optimal assignment of following problem.

Jobs \ Machines	A	B	C	D
P	90	120	140	160
Q	40	65	85	95
R	50	75	95	110

⇒

	A	B	C	D
P	90	120	140	160
Q	40	65	85	95
R	50	75	95	110
S	0	0	0	0

Row Reduction:

	A	B	C	D
P	$\boxed{0}$	30	50	70
Q	0	25	45	55
R	0	25	45	60
S	0	$\boxed{0}$	0	0

→ Not optimal

⇒

	A	B	C	D
P	$\boxed{0}$	5	25	45
Q	0	$\boxed{0}$	20	30
R	0	0	20	35
S	25	0	$\boxed{0}$	0

→ Not optimal

	A	B	C	D
P	$\boxed{0}$	5	5	25
Q	0	$\boxed{0}$	0	10
R	0	0	$\boxed{0}$	15
S	45	20	0	$\boxed{0}$

∴ The optimal soln is

	Costs
P − A	90
Q − B	65
R − C	95
S − D	0
	230

Alternative soln

Q − C	85
R − B	75

Here, Alternative solution can be obtained by interchanging assignment of Q & R.

* Crew Assignment problem

→ It is a problem solving technique, used by various Airjet companies to assign the work to their crew members, to minimize time from one state/city to other.

Example: Minimum layover of hours betw flight : 8 hrs

Flight No.	Ktm - pokhara Depart	Arrival	Flight NO.	pokhara - ktm Depart	Arrival
1	7 AM	9 AM	101	9 AM	11 AM
2	9 AM	11 AM	102	10 AM	12 noon
3	1:30 PM	3:30 PM	103	3:30 pm	5:30 PM
4	7:30 PM	9:30 PM	104	8 PM	10 PM

Ktm Depart
7 AM
9 AM
1:30pm
7:30pm

soln

1, crew based at ktm (want go back ktm)

	101	102	103	104
1	24	25	6.5	11
2	22	23	28.5	9
3	17.5	18.5	24	28.5
4	11.5	12.5	18	22.5

(9-9) (9-10) (9-3.30) (9-8)
(11-9) (11-10) (11-3.30) (11-8)
" "
" "

2, crew based at pokhara (want to go back pokhara)

	101	102	103	104
1	20	19	13.5	9
2	22	21	15.5	11
3	26.5	25.5	20	15.5
4	8.5	7.5	26	21.5

(11-7) (11-9) (11-1:30) (11-7:30)
"
"
"

3, compare two table's corresponding element & select min. value.

	101	102	103	104
1	20^{P}	19^{P}	6.5^{K}	9^{P}
2	$22^{K/P}$	21^{P}	15.5^{P}	9^{K}
3	17.5^{K}	18.5^{K}	20^{P}	15.5^{P}
4	8.5^{P}	7.5^{P}	18^{K}	21.5^{P}

4, Row Reduction

	101	102	103	104
1	13.5	12.5	0	2.5
2	13	12	6.5	0
3	2	3	4.5	0
4	1	0	10.5	14

5, Column Reduction

	101	102	103	104
1	12.5	12.5	$\boxed{0}$	2.5
2	12	12	6.5	$\boxed{0}$
3	1	3	4.5	0
4	$\boxed{0}$	0	10.5	14

No. of allocation = 3 , no. of rows = 4
so, the solution is not optimal

6, Readjust matrix

	101	102	103	104
1	11.5	11.5	$\boxed{0}$	2.5
2	11	11	6.5	$\boxed{0}$
3	$\boxed{0}$	2	4.5	0
4	0	$\boxed{0}$	11.5	15

No. of allocations = no. of rows = 4
so, the optimal solution is

	hours
1 - 103	6.5 k
2 - 104	9 k
3 - 101	17.5 k
4 - 102	7.5 p
	40.5 hrs

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* Past Questions Solution

2023. Q. No. 10. Long

A city cooperation has decided to carry out maintenance work in the city. There are 4 tasks to be completed using 5 contractors. Which of these contractors should be assigned those tasks? Solve the assignment problem to get best result.

Contractor	Task 1	Task 2	Task 3	Task 4
A	19	24	29	25
B	17	27	30	29
C	19	28	31	28
D	20	12	28	29
E	20	25	31	26

Soln

Here, no. of rows = 5, no. of column = 4. so the problem is unbalanced. To make it balanced, we add dummy column/Task (Task 5)

Contractor \ Tasks	1	2	3	4	5
A	19	24	29	25	0
B	17	27	30	29	0
C	19	28	31	28	0
D	20	12	28	29	0
E	20	25	31	26	0

$\Rightarrow$ Column Reduction:

	1	2	3	4	5
A	2	12	1	$\boxed{0}$	0
B	$\boxed{0}$	15	2	4	0
C	2	16	3	3	$\boxed{0}$
D	3	$\boxed{0}$	0	4	0
E	3	13	3	1	0

Since, min. element of each row is '0'. so row reduction is not necessary.

Here, no. of allocation = 4, no. of rows = 5, so the solution is not optimal.

	1	2	3	4	5
A	2	12	1	$\boxed{0}$	1
B	$\boxed{0}$	15	2	4	1
C	1	15	2	2	$\boxed{0}$
D	3	$\boxed{0}$	0	4	1
E	2	12	2	0	0
	2			1	3

→ Not optimal

$\Rightarrow$

	1	2	3	4	5
A	2	11	$\boxed{0}$	0	1
B	$\boxed{0}$	14	1	4	1
C	1	14	1	2	$\boxed{0}$
D	4	$\boxed{0}$	0	5	2
E	2	11	1	$\boxed{0}$	0

→ optimal

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The optimal solution is

Contractor	Task	cost
A	3	29
B	1	17
C	5	0
D	2	12
E	4	26
Total min. cost		84

2024. Q. No. 6 short

Carew's machine shop has 4 machines on which 3 jobs have to be done. Each job can be assigned to one and only one machine. The cost (in Rs) of each job on each machine is given below.

Job \ Machines	P	Q	R	S
A	45	60	70	80
B	20	32	42	47
C	25	37	47	55

soln

Here, no. of rows = 3, no. of column = 4, so the given problem is unbalanced. To make it balance we add dummy row/Job (D)

	P	Q	R	S
A	45	60	70	80
B	20	32	42	47
C	25	37	47	55
D	0	0	0	0

⇒ Row Reduction:

	P	Q	R	S
A	$\boxed{0}$	15	25	35
B	0	12	22	27
C	0	12	22	30
D	~~0~~	$\boxed{0}$	0	0

→ not optimal

	P	Q	R	S
A	$\boxed{0}$	3	13	23
B	0	$\boxed{0}$	10	15
C	0	0	10	18
D	12	~~0~~	$\boxed{0}$	0

→ not optimal

⇒

	P	Q	R	S	
A	$\boxed{0}$	3	3	13	
B	0	$\boxed{0}$	0	5	
C	0	0	$\boxed{0}$	8	
D	22	10	0	$\boxed{0}$	2
	1	3	4		

Here, Alternate soln exists betwn B & C.

so, the optimal solution is

Job	Machines	cost
A	P	45
B	Q	32
C	R	47
x D	S	0
Total min. cost		124