Chapter 4

- Queue = Waiting line
- memoryless → future probability only depends on current state, not past history.

Queuing Theory

→ A queue is a line or list of customers who remain waiting for getting certain goods or services from a service center.

- A queue is formed if the arrival rate of customers is greater than the service rate.

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* Queuing Theory

→ Queuing Theory is a branch of mathematics that studies waiting lines (or queues), to analyze and optimize their performance.

- Its main objective is to minimize and make balance betn cost of waiting and service costs.
- Applications: Healthcare, Bank, traffic flow, supermarket, etc.

* Elements of queuing system / process.

→ Queuing system can be completely described by

1, The Input (Arrival pattern)
- Describes how customers arrive
- Often modeled using Poisson distribution (random arrivals)
- Inter-arrival time is exponentially distributed

2, The Service Mechanism (Service pattern)
- Describe how customers are served.
- Often modeled using exponential distribution (memoryless service time)

3, The queue discipline - rule according which customers selected.
- FIFO (eg. supermarket, ATM, call center
- LIFO (eg: warehouse, elevator (lifts), loading/unloading Trucks)
- SIRO - Service in Random order (eg: Lottery ticket, post-office)
- Priority (eg: Hospital emergency, Resturant, government services)

4, Customer's behavior
- patience : Stay in a line in discipline till get served
- Impatience
  - Balking : Refuse to join queue seeing its length.
  - Reneging : leave the queue, due to impatient.
  - Jockeying : move from one line to a shorter line.

- λ = mean arrival rate (no. of customers arriving per unit time)
- μ = mean service rate (no. of customers served per unit time)

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* Operating characteristics of queue system

1) System Length : The average no. of customers in queue + service. $\left(L_s = \dfrac{\lambda}{\mu - \lambda} \right)$

2) Queue length : Avg. no. of customers in queue.

- $L_q = \dfrac{\lambda^2}{\mu(\mu - \lambda)}$

3) Waiting time in system : Avg. time spent by customer in the system (waiting time + service time)

- $W_s = \dfrac{1}{\mu - \lambda}$

4) Waiting time in queue : Avg. time spent by customer in waiting line/queue.

- $W_q = \dfrac{\lambda}{\mu(\mu - \lambda)}$

5) Service utilization (Traffic Intensity) : proportion of time that a server actually spends with a customer.

- $\rho = \dfrac{\lambda}{\mu}$

- It is also known as probability that service centre is actually busy (i.e. % of time server is busy)

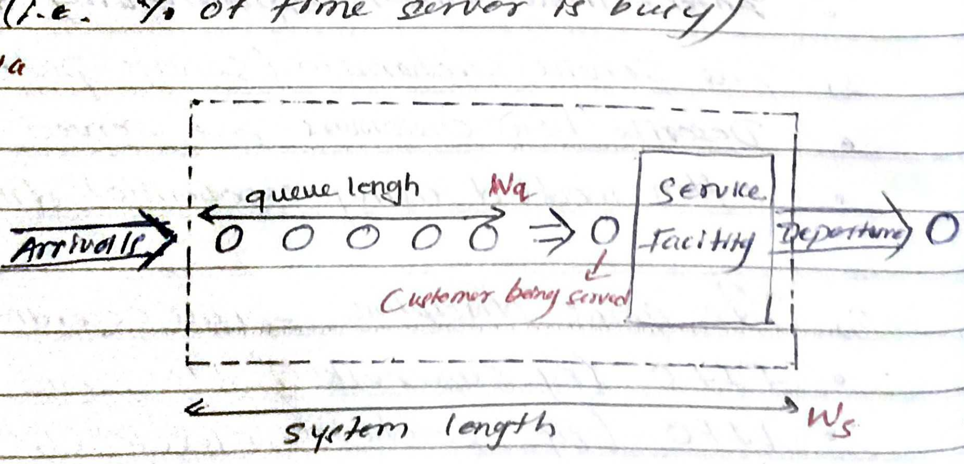
* Tricks for formula

- $W_q = \dfrac{\lambda}{\mu} . W_s$

- $L_s = \lambda . W_s$

- $L_q = \lambda . W_q$

  or, $\dfrac{\lambda}{\mu} . L_s$



- probability that service centre is idle / no customer in the system $(P_0) = 1 - \rho$

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* Kendall's Notation

→ Generally, any queuing model may be completely specified in the following symbolic form $(a/b/c):(d,e)$ where

- a → arrival distribution (M, D, G)
- b → service time / departures distribution (M, D, G)
- c → no. of service channels (servers) < 1 - ~~single server~~, s - multi server
- d → capacity of the system i.e max no. of customer allowed — finite(N), Infinite(∞)
- e → queue discipline (FIFO/LIFO/SIRO)

common symbols for a & b :

- M → Markovian (Arrival time follow poisson Distribution and service time follows an exponential distribution)
- D → Deterministic (constant)
- G → General (arbitary) departure distribution
- E_k → Earling distribution of order k
- GI → General independent arrival distribution

symbol used for c

- 1 - single server/channel
- s - multiple server
- ∞ - infinite server

symbol used for d

- ∞ → infinite no. of customers
- N → finite no. of customers (or. capacity of system is finite)

Examples:

i, (M/M/1) : (∞/FIFO)

ii, (M/M/1) : (N/FIFO)

iii, (M/M/s) : (∞/FIFO)

iv, (M/M/s) : (N/FIFO)

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* Classification of Queueing Model

The queuing models are classified as follows :

1, Single server Queuing Models

- Model-I : (M/M/1) : (∞/FIFO)

Here, 1st M → poisson arrivals (λ)
2nd M → Exponential service (μ)
1 → single server
∞ → Infinite capacity of system
FIFO → First in first out discipline

E.g. : A single ATM in a 24-hour bank lobby

- Arrivals : Random customers (poisson process)
- Service : Exponential transaction times
- Queue : No limit on waiting customers
- Discipline : customers served in arrival order.

- Model-II : (M/M/1) : (N/FIFO)

→ This denotes poisson arrivals, Exponential service times, single server, limited no. of customers (finite capacity of system), FCFS discipline

Eg: Doctor's appointment at Hospital, Hair Sallon,

2, Multi server Queuing models

- Model-III : (M/M/S) : (∞/FIFO)

→ This denotes poisson arrivals, Exponential service times, Multiple server, infinite queue & FIFO discipline

Eg: Airport check-In counters, call center, Bank with 4 Tellers

- Model-IV : (M/M/S) : (N/FIFO)

→ This denotes poisson arrivals, Exponential service times, Multiple server, finite queue & FIFO discipline.

Eg: University (multiple faculties with limited seat), Billing counter of Hospital for Health Insurance patient, vaccination Center (5 station, Max 30 people)

- Exponentially distributed means probability of an event occurring in future is independent of how long it has already been since the last event.

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Single Server/channel Queuing Model (M/M/1)

→ It is a queuing model where the arrivals follow a Poisson distribution, service times are exponentially distributed and there is only one server.

Assumptions:

- No. of customers arriving in a time interval 't' follows a poisson distribution with parameter λ.
- The interval betn any two successive arrival is exponentially distributed with parameter λ.
- The time take to complete a single service is exponentially distributed with parameter μ
- The no. of server is one.
- The population and queue size can be infinity
- The order of service is assumed to be FCFS.

Key Formulas:

- Utilization factor / Traffic intensity $(\rho) = \lambda/\mu$
- Average waiting Time in system/service $(W_s) = \frac{1}{\mu-\lambda}$
- Average waiting time in the queue $(W_q) = \rho . W_s = \frac{\lambda}{\mu(\mu-\lambda)}$
- Avg. no. of customer in the system/system length $(L_s) = W_s.\lambda = \frac{\lambda}{\mu-\lambda}$
- Avg. no. ~~of~~ of customer in the queue/queue length $(L_q) = W_q.\lambda = \frac{\lambda^2}{\mu(\mu-\lambda)}$
- Avg. no. of customer in non-empty queue/length of the non-empty queue $= \frac{\mu}{\mu-\lambda}$ or $\frac{1}{1-\rho}$
- Probability of no customer in the System/system is idle $(p_0) = 1-\rho$
- probability of 'n' customer in system $(P_n) = (1-\rho)\rho^n$
- Probability of no customer in queue and a customer is being served $(P_1) = (1-\rho)\rho^1$
- probability of having 'n' customers in the queue $= (1-\rho)\rho^{n+1}$
- probability of having at least 'r' customer in system $p(n \geq r) = \rho^r$
- probability of having more than 'r' customer on system $p(n > r) = \rho^{r+1}$

- poisson = arrival rate (λ)
- exponential = service rate (μ)

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Q.1, Customers arrive at a box office with one ticket window according to a Poission's input process with mean rate of 30 / hr. The time required to serve a customer has an exponential distribution with mean 90 seconds.

Find the average

a, Length of service (L_s)

b, Queue Length (L_q)

c, waiting time in queue (W_q)

d, Time spent by customer in the system

soln

Given here,

Mean arrival rate (λ) = 30 customer / hr

Mean service rate (μ) = 1 customer / 90 sec

$= \frac{1}{90} \times 3600 = 40$ customer/hr

now,

a, $L_s = \frac{\lambda}{\mu - \lambda} = \frac{30}{40-30} = 3$

b, $L_q = \frac{\lambda}{\mu} \cdot L_s = \frac{30}{40} \times 3 = \frac{9}{4}$

c, $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{30}{40(40-30)} = \frac{3}{40}$ hr

d, $W_s = \frac{1}{\mu - \lambda} = \frac{1}{40-30} = \frac{1}{10}$ hr

Q2, The time spent by a repairman on his jobs has an exponential distribution with mean 30 mins. If he repairs sets in the order in which they come in, and if the arrival of sets is approximately poisson with an average rate of 10 per 8 hours day, what is the repairman's expected idle time each day? How many jobs are ahead of the average set just brought in?

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Soln:

Given here,

mean service rate (μ) = 1 service / 30 min

= 2 service / hr

= 16 services / 8 hrs

mean arrival rate (λ) = 10 sets / 8 hour

now,

a, Repairman's expected idle time (P_0) $= 1 - \rho$

$= 1 - \frac{\lambda}{\mu}$

$= 1 - \frac{\cancel{10}\,5}{\cancel{16}\,8}$

$\therefore$ The expected idle time/8hrs $= \frac{3}{8} \times 8$ hrs

$= 3$ hrs

b, Length of the system (L_s) $= \frac{\lambda}{\mu - \lambda}$

$= \frac{10}{16-10} = \frac{10}{6} = 1\,^2/_3$ sets.

Q.3. Arrivals at a telephone booth are considered to be poisson, with an average time of 10 minutes between one arrival and the next. The length of a phonecall assumed to be distributed exponentially, with mean 3 mins. Find the following:

a) What is the probability that person arriving at a booth will have to wait? / fraction of a day that the phone will be in use/busy?

b) What is the avg. length of queues that form from time to time?

c) The telephone department will install a second booth when convinced that an arrival would expect to have to wait atleast 3 mins for the phone. By how much time must the flow of arrivals be increased in order to justify a second booth?

d, Find the avg. no of units in the system

Soln

Given here,

mean arrival rate (λ) = 1 person / 10 mins

= 6 persons / hr. (×6)

mean service rate (μ) = 1 person / 3 mins

= 20 persons / hr (3×20 = 60)

now,

a, probability that a person will have to wait / Server utilization

$(\rho) = \frac{\lambda}{\mu} = \frac{6}{20} = 0.3$

b, Avg. length of non-empty queue

$(L_n) = \frac{\mu}{\mu - \lambda} = \frac{20}{20-6} = \frac{20}{14} = 1.43$ person

c, Here, for 2nd booth installation, arrival would have to wait at least 3 mins

i.e. W_q = 3 mins $= \frac{3}{60} = \frac{1}{20}$ hr

~~Now, for~~ additional arrival

$$W_q = \frac{\lambda'}{\mu(\mu - \lambda')}$$

or, $\frac{1}{\cancel{20}} = \frac{\lambda'}{\cancel{20}(20-\lambda')}$

or, $20 - \lambda' = \lambda'$

or, $20 = 2\lambda'$

$\therefore \lambda' = 10$

Thus, arrival must be increased by $= \lambda' - \lambda$

$= 10 - 6$

$= 4$ persons / hr

d, Avg. no of unit in the system $(L_s) = \frac{\lambda}{\mu - \lambda}$

$= \frac{6}{20-6} = \frac{6}{14}$

$= 0.43$ person

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Q.4. A radio machine on an average finds 5 customers coming to his shop every hour for repairing their radio sets. He disposes of each of them within 10 mins on an average. The arrival and servicing times follow poisson and exponential distribution respectively. In the light of the above facts determine:

a, proportion of time during which his shop remain empty.
b, The avg. no. of customers in his system and queue.
c, The avg. time spent by a customer in the queue and the service as well.
d, The probability of finding at least 5 customers in his shop.

Soln:

Given here,

mean arrival rate (λ) = 5 customer per hour

mean service rate (μ) = 1 customer per 10 mins

= 6 customer per hour

now,

a, $P_0 = 1 - \rho = 1 - \frac{\lambda}{\mu} = 1 - \frac{5}{6} = \frac{1}{6}$

b, Avg. no. of customer in system $(L_s) = \frac{\lambda}{\mu - \lambda} = \frac{5}{6-5} = 5$ customers

" " queue $(L_q) = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{5}{6} \times 5 = 4.17$ customers

c, Avg. time spent by customer in service $(W_s) = \frac{1}{\mu - \lambda} = \frac{1}{6-5} = 1$ hr

" " in queue $(W_q) = W_s . \frac{\lambda}{\mu} = 1 \times \frac{5}{6} = \frac{5}{6}$ hr

d, probability of finding atleast 5 customers in shop

$p(n \geq 5) = \rho^5 = \left(\frac{\lambda}{\mu}\right)^5 = \left(\frac{5}{6}\right)^5 = 0.402$ //

$\lambda = 20$ customer / hour
$\mu = 24$ customer / hour

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Q.5. A departmental store has a single cashier. During the rush hours, customers arrive at the rate of 20 customers per hour. The avg. no. of customers that can be processed by the cashier is 24 per hour. Assume that the conditions for the use of single channel queuing model apply. What is the

i, Utilization factor / Traffic intensity / probability that service is busy

$\rightarrow \rho = \frac{\lambda}{\mu} = \frac{20}{24} = 0.833$

ii, probability that cashier is idle / no customer in system

$\rightarrow P_0 = 1 - \rho = 1 - 0.833 = 0.167$

iii, Avg. no. of customer in queuing system / system length

$\rightarrow L_s = \frac{\lambda}{\mu - \lambda} = \frac{20}{24-20} = \frac{20}{4} = 5$ customers

iv, Avg. no of customer in the queue / Queue length

$\rightarrow L_q = \frac{\lambda}{\mu} \cdot L_s = 0.833 \times 5 = 4.16 \simeq 4$ customers

v, Avg. time spent by customer in the system / waiting time in system

$\rightarrow W_s = \frac{1}{\mu - \lambda} = \frac{1}{4} = 0.25$ hr $= 15$ mins

vi, Avg. time spent by customer in the waiting line / waiting time in the queue

$\rightarrow W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\lambda}{\mu} \times W_s = \frac{\overset{5}{\cancel{20}}}{24} \times \frac{1}{\cancel{4}} = 12.5$ mins

- $p(n \geq r) = \rho^r$
- $P(n=r) = (1-\rho)\rho^r$
- $p(n > r) = \rho^{r+1}$

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vii) What is the probability that there are at least two customers in the counter?

→ $p(n \geq 2) = \rho^2 = (0.833)^2$

viii) What is the probability that there is no customer waiting to be served?

→ $P_0 + P_1 = (1-\rho)\rho^0 + (1-\rho)\rho^1$
$= (1-\rho) + (1-\rho).\rho$
$= 0.167 + 0.167 \times 0.833$

ix) What is ~~the~~ the probability that a customer is being served and nobody is waiting?

→ $P_1 = (1-\rho)\rho^1$
$= 0.167 \times 0.833$

x) probability of having 4 customers in the system:

→ $P_4 = (1-\rho)\rho^4$

xi) probability of finding more than 3 customers in the system.

→ $p(n > 3) = \rho^{3+1}$

xii) The probability that there are at most 2 customers in the system.

→ $p(n \leq 2) = P_0 + P_1 + P_2$
$= (1-\rho) + (1-\rho)\rho + (1-\rho)\rho^2$

xiii) What is the avg. length of non-empty queue?

→ $L_n = \frac{\mu}{\mu - \lambda}$