Unit 3: Transportation and Assignment Problem (8 Hrs.)

Transportation Problems definition, linear form, Solution methods: North West comer method, least cost method, Vogel's approximation method. Degeneracy in transportation, Modified Distribution method, unbalanced problems and profit maximization problems. Transshipment Problems. Assignment Problem Structure and Solution: Short-Cut Method (Hungarian Method), Unbalanced Assignment Problem, Infeasible Assignment Problem, Maximization in an Assignment Problem, Crew Assignment Problem.

#Past Questions

1. Find the optimal solution for the following transportation problem using any methods. -2021

1	2	1	4	30
3	3	2	1	50
4	2	5	9	20
20	40	30	9	100

2. Write Hungarian algorithm to solve the assignment problem. -2021/2023
3. Given the transportation framework, find the optimal transportation cost. – 2023

From↓ | To→	A	B	C	Plant Capacity
W	40	80	80	55
X	160	-	160	25
Y	80	160	240	35
Requirement	35	45	35	115

4. A city corporation has decided to carry out Maintenance work in the city. There are 4 tasks to be completed using 5 contractors . Which contractors should be assigned those tasks? Solve the assignment problem to get best result. -2023 **long**

Contractor	Task 1	Task 2	Task 3	Task 4
A	19	24	29	25
B	17	27	30	29
C	19	28	31	28
D	20	12	28	29
E	20	25	31	26

5. Carew's machine shop has 4 machines on which 3 jobs have to be done. Each job can be assigned to one and only one machine. The cost(in Rs) of each job on each machine is given below:

	Machines			
Job	**P**	**Q**	**R**	**S**
A	45	60	70	80
B	20	32	42	47
C	25	37	47	55

Req: What are the job assignments which will minimize total cost?

6. Write short notes on (any two)
   a. Meaning of Degeneracy in Transportation Problem.

b. Assignment Problem Algorithm

7. 11. Jack Evan owns several trucks used to haul crushed stone to road project iun the country. The road contractor for whom jack hauls, N Teer, has given jack this schedule for next week:

Project	Requirement per week	Plant	Available per week
A	50	W	45
B	75	X	60
C	50	Y	60

Jack figures his cost from the crushing plant to each of the road projects to be these:

Cost information (in RS)

To /From	A	B	C
W	4	8	3
X	6	7	9
Y	8	2	5

Req: Compute Jack's optimal hauling schedule for next week and his transportation cost.

1. Transportation Problems

A **Transportation Problem** involves transporting goods from multiple **sources** (e.g., factories) to multiple **destinations** (e.g., warehouses) in such a way that:

- The **total transportation cost is minimized** (or profit is maximized),
- Supply and demand constraints are satisfied.

Mathematical (Linear) Formulation:

Let:

- m = number of sources
- n = number of destinations
- c_{ij} = cost of transporting one unit from source i to destination j
- x_{ij} = quantity transported from source i to destination j
- a_i = supply at source i
- b_j = demand at destination j

Objective function (minimize cost):

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to:

- $\sum_{j=1}^{n} x_{ij} = a_i$ (supply constraints)
- $\sum_{i=1}^{m} x_{ij} = b_j$ (demand constraints)
- $x_{ij} \geq 0$

The transportation problem can be stated in the following tabular form:

		Destinations					
		1	2	3	…	n	supply
Origins	1	(x_{11}) C_{11}	(x_{12}) C_{12}	(x_{13}) C_{13}	…	(x_{1n}) C_{1n}	a_1
	2	(x_{21}) C_{21}	(x_{22}) C_{22}	(x_{23}) C_{23}	…	(x_{2n}) C_{2n}	a_2
	⋮		⋮	⋮	⋮	⋮	⋮
	m	(x_{m1}) C_{m1}	(x_{m2}) C_{m2}	(x_{m3}) C_{m3}	…	(x_{mn}) C_{mn}	a_m
	demand	b_1	b_2	b_3	…	b_n	

Example 1: A company has three production factories $S1$, $S2$ and $S3$ with production capacity of 7, 9 and 18 units

(in 100s) per week of a product, respectively. These units are to be shipped to four warehouses $D1$, $D2$, $D3$ and $D4$ with requirement of 5, 8, 7 and 14 units (in 100s) per week, respectively. The transportation costs (in rupees) per unit between factories to warehouses are given in the table below:

	D1	D2	D3	D4	Supply (availability)
$S1$	19	30	50	10	7
$S2$	70	30	40	60	9
$S3$	40	8	70	20	18
Demand (requirement)	5	8	7	14	34

Formulate this transportation problem as an LP model to minimize the total transportation cost.

Solution: Let x_{ij} = number of units of the product to be transported from a production factory i (i = 1, 2, 3) to a warehouse j (j = 1, 2, 3, 4)

The transportation problem is stated as an LP model as follows:

Minimize (total transportation cost) $Z = 19\,x11 + 30\,x12 + 50\,x13 + 10\,x14 + 70\,x21 + 30\,x22 + 40\,x23 + 60\,x24 + 40\,x31 + 8\,x32 + 70\,x33 + 20\,x34$

subject to the constraints

$$\left.\begin{aligned} x11 + x12 + x13 + x14 &= 7 \\ x21 + x22 + x23 + x24 &= 9 \\ x31 + x32 + x33 + x34 &= 18 \end{aligned}\right\} \quad \text{(Supply)}$$

$$\left.\begin{aligned} x11 + x21 + x31 &= 5 \\ x12 + x22 + x32 &= 8 \\ x13 + x23 + x33 &= 7 \\ x14 + x24 + x34 &= 14 \end{aligned}\right\} \quad \text{(Demand)}$$

and $xij \geq 0$ for $i = 1, 2, 3$ and $j = 1, 2, 3$, and 4.

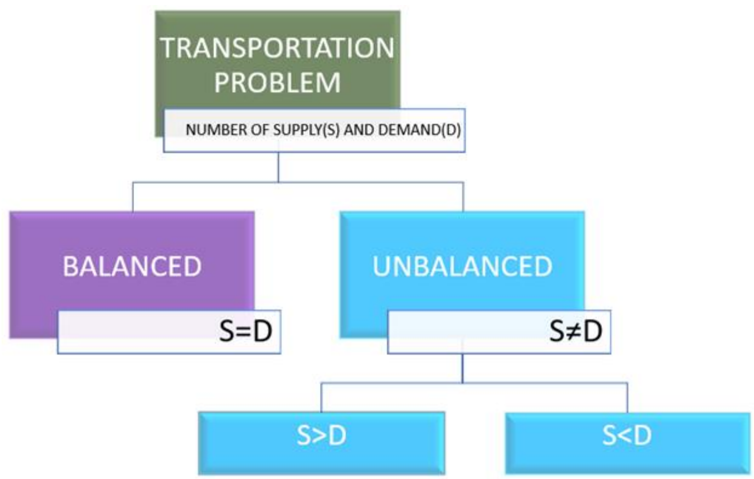
In the above LP model, there are $m \times n = 3 \times 4 = 12$ decision variables, xij and $m + n = 7$ constraints, where m are the number of rows and n are the number of columns in a general transportation table.

There are only ($m + n - 1$) basic variable out of $m \times n$

Therefore, BFS (Basic Feasible Solution) of Transportation Problem consist of at most ($m + n - 1$) positive variables other being zero.

Types of Transportation Problems:

- **Balanced:** Total supply equals total demand.
- **Unbalanced:** Total supply does not equal total demand. In this case, a dummy source or destination is added to balance the problem.



2. Solution Methods for Transportation Problem

A transportation problem can be solved by using two most familiar methods:

a. Simplex Method
b. Transportation Method

But our interest is in only transportation method i.e. we will discuss only transportation method.

Transportation Algorithm

The steps of the transportation algorithm are exact parallels of the simplex algorithm, they are:

Step 1: Determine a starting basic feasible solution, using any one of the following three methods

1. North West Corner Method
2. Least Cost Method
3. Vogel Approximation Method

Step 2: Determine the optimal solution using the following method

1. MODI (Modified Distribution Method) or UV Method.
2. Stepping Stone Method

Basic Feasible Solution of a Transportation Problem

a. North West Corner Method (NWCM):

<u>Algorithm:</u>

1. Start in cell $(1, 1)$ (north-west).
2. Allocate $x_{11} = \min(s_1, d_1)$.
3. Subtract that from the corresponding supply or demand (whichever hits zero "exhausts" first).
4. If s_1 becomes zero, move down to cell $(2, 1)$; if d_1 becomes zero, move right to $(1, 2)$.
5. Repeat until all supplies/demands are allocated.

Question: Use North-West Corner Method (NWCM) to find an initial basic feasible solution to the transportation problem using data of **Example 1.**

Solution: The cell (1,1) is the north-west corner cell in the given transportation table. The min values for row S1 and column D1 are compared i.e. min(s1,d1). The smaller of the two (5,7), i.e. 5, is assigned as the first allocation

	D1	D2	D3	D4	Supply	
S1	19 5	30	50	10	~~7~~	2
S2	70	30	40	60	9	
S3	40	8	70	20	18	
Demand	~~5~~	8	7	14	34	

	D2	D3	D4	Supply
~~S1~~	~~30~~ 2	~~50~~	~~10~~	~~2~~
S2	30	40	60	9
S3	8	70	20	18
Demand	~~8~~	7	14	
	6			

	D2	D3	D4	Supply	
S2	30 6	40	60	~~9~~	3
S3	8	70	20	18	
Demand	~~6~~	7	14		

	D3	D4	Supply
~~S2~~	~~40~~ 3	~~60~~	~~3~~
S3	70	20	18
Demand	~~7~~	14	
	4		

	D3	D4	Supply
S3	70 4	20 14	~~18~~ ~~14~~
Demand	~~4~~	~~14~~	

Once the procedure is over, count the number of positive allocations. These allocations (occupied cells) should be equal to $m + n - 1 = 3 + 4 - 1 = 6$. If yes, then solution is non-degenerate feasible solution, Otherwise degenerate solution.

The total transportation cost of the initial solution is obtained by multiplying the quantity *xij* in the occupied cells with the corresponding unit cost *cij* and adding all the values together. Thus, the total transportation cost of this solution is

Thus, Total cost = 5 × 19 + 2 × 30 + 6 × 30 + 3 × 40 + 4 × 70 + 14 × 20 = Rs 1,015

This problem can be done directly as:

Initial Solution by NWCM

Factory \ Warehouse	P	Q	R	S	Supply
A	19 $\boxed{5}$	30 $\boxed{2}$	50	10	S_1 = ~~7~~ 2
B	70	30 $\boxed{6}$	40 $\boxed{3}$	60	S_2 = ~~9~~ 3
C	40	8	70 $\boxed{4}$	20 $\boxed{14}$	S_3 = ~~18~~ 14
Demand	D_1 = ~~5~~	D_2 = ~~8~~ 6	D_3 = ~~7~~ 4	D_4 = ~~14~~	34

We have to be ensured whether the initial solution is feasible or not. Since the number of occupied cell i.e. allocated cell is equal to $m + n - 1 = 3 + 4 - 1 = 6$ where m is no. of row and no. of column in the transportation table, so the initial solution is feasible.

The total transportation cost of initial solution can be calculated as

$$\begin{aligned}\text{Total TC} &= \Sigma \text{ Unit Cell Cost} \times \text{Allocated Quantity.}\\ &= C_{11}X_{11} + C_{12}X_{12} + C_{22}X_{22} + C_{23}X_{23} + C_{33}X_{33} + C_{34}X_{34}\\ &= 19 \times 5 + 30 \times 2 + 30 \times 6 + 40 \times 3 + 70 \times 4 + 20 \times 14\\ &= \text{Rs. } 1015\end{aligned}$$

b. Least Cost Method (LCM): More efficient than NWCM.

Algorithm

Step-1:	Select the cell having minimum unit cost *cij* and Allocate xij= min(*si*,*dj*).
Step-2:	a. Subtract this min value from supply *si* and demand *dj*. b. If the supply *si* is 0, then cross (strike) that row and If the demand *dj* is 0 then cross (strike) that column. c. If min unit cost cell is same, then select any one of two.
Step-3:	Repeat these steps for all uncrossed rows and columns until all supply and demand values are 0

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8 8	70	20	~~18~~ 10
Demand	5	~~8~~	7	14	34

	D1	D3	D4	Supply
~~S1~~	~~19~~	~~50~~	~~10~~ 7	~~7~~
S2	70	40	60	9
S3	40	70	20	10
Demand	5	7	~~14~~ 7	

	D1	D3	D4	Supply
S2	70	40	60	9
S3	40	70	20 7	~~10~~ 3
Demand	5	7	~~7~~	

	D1	D3	Supply
S2	70	40 7	~~9~~ 2
S3	40	70	3
Demand	5	~~7~~	

	D1	Supply
S2	70 2	~~2~~
~~S3~~	~~40~~ 3	~~3~~
Demand	~~5~~ ~~2~~	

The total transportation cost of the initial solution by LCM is calculated as given below:

Total cost = 7 × 10 + 2 × 70 + 7 × 40 + 3 × 40 + 8 × 8 + 7 × 20 = Rs 814

#Note: The total transportation cost obtained by LCM is less than the cost obtained by NWCM.

This problem can be done directly as:

Initial Solution by LCM

Factory \ Warehouse	P	Q	R	S	Supply
A	19 ×	30 ×	50 ×	10 [7]	$S_1 = \cancel{7}$
B	70 [2]	30 ×	40 [7]	60 ×	$S_2 = \cancel{9}\ 2$
C	40 [3]	8 [8]	70 ×	20 [7]	$S_3 = \cancel{18}\ \cancel{10}\ \cancel{3}$
Demand	$D_1 = \cancel{5}\ \cancel{2}$	$D_2 = \cancel{8}$	$D_3 = \cancel{7}$	$D_4 = \cancel{14}\ \cancel{7}$	34

We have to be ensured whether the initial solution is feasible or not. Since the number of occupied cell i.e. allocated cell is equal to m + n –1 = 3 + 4 – 1 = 6 where m is no. of row and no. of column in the transportation table, so the initial solution is feasible.

The total transportation cost of initial solution can be calculated as

$$\text{Total TC} = \Sigma \text{ Unit Cell Cost} \times \text{Allocated Quantity.}$$
$$= C_{14}X_{14} + C_{21}X_{21} + C_{23}X_{23} + C_{31}X_{31} + C_{32}X_{32} + C_{34}X_{34}$$
$$= 10 \times 7 + 70 \times 2 + 40 \times 7 + 40 \times 3 + 8 \times 8 + 20 \times 7$$
$$= \text{Rs. } 814$$

Since, the total transportation cost of the initial solution obtained by LCM is less than that of NWCM, so LCM is more efficient than NWCM.

c. Vogel's Approximation Method (VAM): **Gives better initial feasible solution, often closer to optimum.**

This method is preferred over the NWCM and LCM, because the initial basic feasible solution obtained by this method is either optimal solution or very nearer to the optimal solution.

Algorithm:

1. For each row, compute row-penalty = (2nd smallest cost – smallest cost). Do likewise for each column.
2. Select the row or column with the highest penalty and find cell that has least cost in selected row or column.
   - If there is a tie in the values of penalties then select the cell where maximum allocation can be possible (or which has minimum demand or supply).
3. Allocate to the cell with **least cost** in that row/column (i.e. Allocate xij= min(*si*,*dj*).
4. Subtract this min value from supply *si* and demand *dj*.
   - If the supply *si* is 0, then cross (strike) that row .
   - If the demand *dj* is 0 then cross (strike) that column.
   - If supply and demand both are 0, then cross (strike) both row & column .
5. Update supplies/demands, recompute penalties on the reduced table.
6. Repeat these steps until all supply and demand values are 0.

	D1	D2	D3	D4	supply	Row Penalty
S1	19	30	50	10	7	19-10=9
S2	70	30	40	60	9	40-30=10
S3	40	8 8	70	20	~~18~~ 10	20-8=12
Demand	5	~~8~~	7	14	34	
Column penalty	40-19=21	30-8=22	50-40=10	20-10=10		

	D1	D3	D4	supply	Row Penalty
S1	19 5	50	10	~~7~~ 2	19-10=9
S2	70	40	60	9	60-40=20
S3	40	70	20	10	40-20=20
Demand	~~5~~	7	14		
Column penalty	40-19=21	50-40=10	20-10=10		

	D3	D4	supply	Row Penalty
S1	50	10	2	50-10=40
S2	40	60	9	60-40=20
~~S3~~	~~70~~	~~20~~ 10	~~10~~	~~70-20=50~~
Demand	7	~~14~~ 4		
Column penalty	50-40=10	20-10=10		

	D3	D4	supply	Row Penalty
~~S1~~	~~50~~	~~10~~ 2	~~2~~	~~50-10=40~~
S2	40	60	9	60-40=20
Demand	7	~~4~~ 2		
Column penalty	50-40=10	60-10=50		

	D3	D4	supply	Row Penalty
S2	40	60 2	~~9~~ 7	60-40=20
Demand	7	~~2~~		
Column penalty	40	60		

	D3	supply	Row Penalty
S2	40 7	~~7~~	60-40=20
Demand	~~7~~		
Column penalty	40		

The total transportation cost associated with this method is:

Total cost = 5 × 19 + 2 × 10 + 7 × 40 + 2 × 60 + 8 × 8 + 10 × 20 = Rs 779

This problem can be done directly as:

Initial Solution by VAM

Factory \ Warehouse	P	Q	R	S	Supply	Cost difference			
A	[5] 19	× 30	× 50	[2] 10	$S_1 = \cancel{7}\ \cancel{2}$	9	9	40	40
B	× 70	× 30	[7] 40	[2] 60	$S_2 = \cancel{9}\ \cancel{2}$	10	20	20	20
C	× 40	[8] 8	× 70	[10] 20	$S_3 = \cancel{18}\ \cancel{10}$	12	20	← 50	–
Demand	$D_1 = \cancel{5}$	$D_2 = \cancel{8}$	$D_3 = \cancel{7}$	$D_4 = \cancel{14}\ 4\ \cancel{2}$	34				
Cost Difference	21	22 ↑	10	10					
	21↑	–	10	10					
	–	–	10	10					
	–	–	10	50↑					

3. Modified Distribution Method (MODI or u-v Method):

Algorithm

MODI Method Steps (Rule)	
Step-1:	Find an initial basic feasible solution using any one of the three methods NWCM, LCM or VAM.
Step-2:	Find *ui* and *vj* for rows and columns. To start a. assign 0 to *ui* or *vj* where maximum number of allocations in a row or column respectively. b. Calculate other *ui*'s and *vj*'s using *cij*=*ui*+*vj*, for all occupied cells.
Step-3:	For all unoccupied cells, calculate *dij*=*cij*-(*ui*+*vj*), .
Step-4:	Check the sign of *dij* a. If *dij*≥0, then current basic feasible solution is optimal and stop this procedure. b. If any *dij*<0, then the given solution is not an optimal solution and further improvement in the solution is possible.
Step-5:	Select the unoccupied cell with the largest negative value of *dij*, and included in the next solution.
Step-6:	Draw a closed path (or loop) from the unoccupied cell (selected in the previous step). The right angle turn in this path is allowed only at occupied cells and at the original unoccupied cell. Mark (+) and (-) sign alternatively at each corner, starting from the original unoccupied cell.

Step-7:	1. Select the minimum value from cells marked with (-) sign of the closed path. 2. Assign this value to selected unoccupied cell (So unoccupied cell becomes occupied cell). 3. Add this value to the other occupied cells marked with (+) sign. 4. Subtract this value to the other occupied cells marked with (-) sign.
Step-8:	Repeat Step-2 to step-7 until optimal solution is obtained. This procedure stops when all *dij*≥0 for unoccupied cells.

4. What If $\Delta < 0$?

If you had found some $\Delta_{ij} < 0$, you would:

1. **Choose** the most negative Δ cell to **enter** the basis.
2. **Trace** the unique closed loop (alternating horizontal/vertical moves) through occupied cells plus that entering cell.
3. **Label** the loop corners "+" at the entering cell, then alternate "–" on the next corner, "+" on the next, and so on.
4. **Identify** the smallest allocation on a "–" corner; call it θ.
5. **Add** θ to all "+" corners, **subtract** θ from all "–" corners.
6. **Recompute** u, v, and all Δ.

Repeat until no negative Δ remains.

Q. Find the optimal solution for the following transportation problem using any methods. -2021

1	2	1	4	30
3	3	2	1	50
4	2	5	9	20
20	40	30	10	100

Solution:

i) Obtaining initial feasible solution from VAM

	D_1	D_2	D_3	D_4	Sup.	Row Penalties				
O_1	1 (20)	2	1 (10)	4	~~30~~ ~~10~~ 0	1	1	1	1 ←	4
O_2	3	3 (20)	2 (20)	1 (10)	~~50~~ 40	1	1	1	1	
O_3	4	2 (20)	5	9	~~20~~	2	2	3 ←		3
Dmd.	~~20~~ 0	~~40~~ 20	~~30~~ 20	~~10~~ 0	100 / 100					
Column penalties	2	1	1	3 ↑						
	2 ↑	1	1							
		1	1							
		1	1							
	2			1						

→ we choose this 1 because it has value least on Dmd/sup. among others.

Here: No. of occupied cell = 6 and

$m+n-1 = 3+4-1 = 6$ (both are equal).

so, the solution is non degenerate feasible solution.

∴ Total cost of initial solution is

$TC = 1\times20 + 3\times20 + 2\times20 + 2\times20 + 1\times10 + 1\times10$

$= 180$

ii) Testing of optimality of initial solution by MODI method

For Allocated (occupied) cell, calculate dual variables u_i & v_j using $c_{ij} = u_i + v_j$

$\Rightarrow \begin{cases} u_i = c_{ij} - v_j \\ v_j = c_{ij} - u_i \end{cases}$

	D_1	D_2	D_3	D_4	
O_1	1 (20)		1 (10)		$u_1 = -1$
O_2		3 (20)	2 (20)	1 (10)	$u_2 = 0$ (since it has highest allocation i.e. 3)
O_3		2 (20)			$u_3 = -1$
	$v_1 = 2$	$v_2 = 3$	$v_3 = 2$	$v_4 = 1$	

Substituting $u_2 = 0$, we get

$c_{22} = u_2 + v_2 \Rightarrow 3 = 0 + v_2 \Rightarrow v_2 = 3$

$c_{32} = u_3 + v_2 \Rightarrow 2 = u_3 + 3 \Rightarrow u_3 = -1$

$c_{23} = u_2 + v_3 \Rightarrow 2 = 0 + v_3 \Rightarrow v_3 = 2$

$c_{24} = u_2 + v_4 \Rightarrow 1 = 0 + v_4 \Rightarrow v_4 = 1$

$c_{13} = u_1 + v_3 \Rightarrow 1 = u_1 + 2 \Rightarrow u_1 = -1$

$c_{11} = u_1 + v_1 \Rightarrow 1 = -1 + v_1 \Rightarrow v_1 = 2$

For Non-Allocated (unoccupied) cell, calculate $d_{ij} = c_{ij} - (u_i + v_j)$

	D_1	D_2	D_3	D_4	
O_1	-	2 (2) = 0	-	4 (0) = 4	$u_1 = -1$
O_2	3 (2) = 1	-	-	-	$u_2 = 0$
O_3	4 (1) = 3	-	5 (1) = 4	9 (0) = 9	$u_3 = -1$
	$v_1 = 2$	$v_2 = 3$	$v_3 = 2$	$v_4 = 1$	

$d_{12} = c_{12} - (u_1 + v_2) = 2 - (-1+2) = 2 - 2 = 0$

$d_{21} = c_{21} - (u_2 + v_1) = 3 - (0+2) = 3 - 2 = 1$

$d_{31} = c_{31} - (u_3 + v_1) = 4 - 1 = 3$

⋮

Since, all the $d_{ij} \geq 0$, so, the transportation cost Rs. 180 is optimal.

Q 2. Find Solution using Voggel's Approximation method, also find optimal solution using modi method,

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	

Solution:

Initial feasible solution is

<table>
<tr><th></th><th>D1</th><th>D2</th><th>D3</th><th>D4</th><th>Supply</th><th>Row Penalty</th></tr>
<tr><td>S1</td><td>19(5)</td><td>30</td><td>50</td><td>10(2)</td><td>~~7~~ ~~2~~ 0</td><td>9 | 9 | 40 | 40</td></tr>
<tr><td>S2</td><td>70</td><td>30</td><td>40(7)</td><td>60(2)</td><td>9</td><td>10 | 20 | 20 | 20</td></tr>
<tr><td>S3</td><td>40</td><td>8(8)</td><td>70</td><td>20(10)</td><td>~~18~~ ~~10~~ 0</td><td>12 | 20 | 50 | --</td></tr>
<tr><td>Demand</td><td>~~5~~ 0</td><td>~~8~~ 0</td><td>7</td><td>~~14~~ ~~4~~ 2</td><td></td><td></td></tr>
<tr><td>Column Penalty</td><td>21
21
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--</td><td>22
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--
--</td><td>10
10
10
10</td><td>10
10
10
50</td><td></td><td></td></tr>
</table>

The minimum total transportation cost $=19\times5+10\times2+40\times7+60\times2+8\times8+20\times10=779$

Here, the number of allocated cells = 6 is equal to m + n - 1 = 3 + 4 - 1 = 6

∴ This solution is non-degenerate

Optimality test using modi method...

Find u_i and v_j for all occupied cells(i,j), where $c_{ij} = u_i + v_j$

Substituting, $v_4 = 0$, we get

$c_{14} = u_1 + v_4 \Rightarrow u_1 = c_{14} - v_4 \Rightarrow u_1 = 10 - 0 \Rightarrow u_1 = 10$

$c_{11} = u_1 + v_1 \Rightarrow v_1 = c_{11} - u_1 \Rightarrow v_1 = 19 - 10 \Rightarrow v_1 = 9$

$c_{24} = u_2 + v_4 \Rightarrow u_2 = c_{24} - v_4 \Rightarrow u_2 = 60 - 0 \Rightarrow u_2 = 60$

$c_{23} = u_2 + v_3 \Rightarrow v_3 = c_{23} - u_2 \Rightarrow v_3 = 40 - 60 \Rightarrow v_3 = -20$

$c_{34} = u_3 + v_4 \Rightarrow u_3 = c_{34} - v_4 \Rightarrow u_3 = 20 - 0 \Rightarrow u_3 = 20$

$c_{32} = u_3 + v_2 \Rightarrow v_2 = c_{32} - u_3 \Rightarrow v_2 = 8 - 20 \Rightarrow v_2 = -12$

	*D*1	*D*2	*D*3	*D*4	Supply	*ui*
*S*1	19 **(5)**	30	50	10 **(2)**	7	*u*1=10
*S*2	70	30	40 **(7)**	60 **(2)**	9	*u*2=60
*S*3	40	8 **(8)**	70	20 **(10)**	18	*u*3=20
Demand	5	8	7	14		
vj	*v*1=9	*v*2=-12	*v*3=-20	*v*4=0		

Find *dij* for all unoccupied cells(i,j), where *dij*=*cij*-(*ui*+*vj*)

$$d_{12} = c_{12} - \left(u_1 + v_2\right) = 30 - (10 - 12) = 32$$

$$d_{13} = c_{13} - \left(u_1 + v_3\right) = 50 - (10 - 20) = 60$$

$$d_{21} = c_{21} - \left(u_2 + v_1\right) = 70 - (60 + 9) = 1$$

$$d_{22} = c_{22} - \left(u_2 + v_2\right) = 30 - (60 - 12) = -18$$

$$d_{31} = c_{31} - \left(u_3 + v_1\right) = 40 - (20 + 9) = 11$$

$$d_{33} = c_{33} - \left(u_3 + v_3\right) = 70 - (20 - 20) = 70$$

*D*1	*D*2	*D*3	*D*4		Supply	*ui*
*S*1	19 **(5)**	30 [32]	50 [60]	10 **(2)**	7	*u*1=10
*S*2	70 [1]	30 [-18]	40 **(7)**	60 **(2)**	9	*u*2=60
*S*3	40 [11]	8 **(8)**	70 [70]	20 **(10)**	18	*u*3=20
Demand	5	8	7	14		
vj	*v*1=9	*v*2=-12	*v*3=-20	*v*4=0		

Now choose the minimum negative value from all *dij* (opportunity cost) = *d*22 = [-18]

and draw a closed path from *S*2*D*2 → unoccupied cell (selected in the previous step)..

Closed path is *S*2*D*2→*S*2*D*4→*S*3*D*4→*S*3*D*2

Closed path and plus/minus sign allocation...

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19 **(5)**	30 [32]	50 [60]	10 **(2)**	7	$u_1=10$
S_2	70 [1]	30 [-18] **(+θ)**	40 **(7)**	60 **(2)** **(-θ)**	9	$u_2=60$
S_3	40 [11]	8 **(8)** **(-θ))**	70 [70]	20 **(10)** **(+θ)**	18	$u_3=20$
Demand	5	8	7	14		
v_j	$v_1=9$	$v_2=-12$	$v_3=-20$	$v_4=0$		

Here, Minimum allocated value among all negative position **(-)** on closed path = 2
Subtract 2 from all **(-)** and Add it to all **(+)**

	D_1	D_2	D_3	D_4	Supply
S_1	19 **(5)**	30	50	10 **(2)**	7
S_2	70	30 **(2)**	40 **(7)**	60	9
S_3	40	8 **(6)**	70	20 **(12)**	18
Demand	5	8	7	14	

Recompute u_i , v_j and d_{ij} until dij ≤ 0
Find u_i and v_j for all occupied cells(i,j), where $c_{ij}=u_i+v_j$

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19 **(5)**	30	50	10 **(2)**	7	$u_1=0$
S_2	70	30 **(2)**	40 **(7)**	60	9	$u_2=32$
S_3	40	8 **(6)**	70	20 **(12)**	18	$u_3=10$
Demand	5	8	7	14		
v_j	$v_1=19$	$v_2=-2$	$v_3=8$	$v_4=10$		

Find d_{ij} for all unoccupied cells(i,j), where $d_{ij}=c_{ij}-(u_i+v_j)$

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19 **(5)**	30 [32]	50 [42]	10 **(2)**	7	$u_1=0$
S_2	70 [19]	30 **(2)**	40 **(7)**	60 [18]	9	$u_2=32$
S_3	40 [11]	8 **(6)**	70 [52]	20 **(12)**	18	$u_3=10$
Demand	5	8	7	14		
v_j	$v_1=19$	$v_2=-2$	$v_3=8$	$v_4=10$		

Since all $d_{ij} \geq 0$. So final optimal solution is arrived.

	D_1	D_2	D_3	D_4	Supply
S_1	19 **(5)**	30	50	10 **(2)**	7
S_2	70	30 **(2)**	40 **(7)**	60	9
S_3	40	8 **(6)**	70	20 **(12)**	18
Demand	5	8	7	14	

∴ The minimum total transportation cost $= 19 \times 5 + 10 \times 2 + 30 \times 2 + 40 \times 7 + 8 \times 6 + 20 \times 12 = 743$

4. Degeneracy in Transportation:

- A **basic feasible solution** in a transportation problem must have exactly m+n−1 allocations.
- If it has **less than m+n−1** allocations, it is **degenerate**.
- To resolve, insert "zero-quantity" or ε allocations into some empty(unoccupied) cells (at very low cost) that **does not form a closed loop** with current allocations until you have m+n−1 occupied cells.

Example

Find Solution using Voggel's Approximation method, also find optimal solution using modi method,

	D1	D2	D3	Supply
S1	8	5	6	120
S2	15	10	12	80
S3	3	9	10	80
Demand	150	80	50	

Solution:

Initial feasible solution is

	D_1	D_2	D_3	Supply	Row Penalty
S_1	8**(70)**	5	6**(50)**	120	1 | 1 | 1
S_2	15	10**(80)**	12	80	2 | 2 | 2
S_3	3**(80)**	9	10	80	6 | -- | --
Demand	150	80	50		
Column Penalty	5 7 --	4 5 5	4 6 6		

The minimum total transportation cost $= 8 \times 70 + 6 \times 50 + 10 \times 80 + 3 \times 80 = 1900$

Here, the number of allocated cells = 4, which is one less than to m + n - 1 = 3 + 3 - 1 = 5

∴ This solution is degenerate

To resolve degeneracy, add a very small quantity ε (epsilon)~0 to an unoccupied cell, having least cell cost that **does not form a closed loop** with current allocations i.e. cell(1,2) = 5

	D_1	D_2	D_3	Supply
S_1	8 **(70)**	5 **(ε)**	6 **(50)**	120
S_2	15	10 **(80)**	12	80
S_3	3 **(80)**	9	10	80
Demand	150	80	50	

No closed loop formed

Optimality test using modi method...

Find u_i and v_j for all occupied cells(i,j), where $c_{ij}=u_i+v_j$

	D_1	D_2	D_3	Supply	u_i
S_1	8 **(70)**	5 **(ε)**	6 **(50)**	120	$u_1=0$
S_2	15	10 **(80)**	12	80	$u_2=5$
S_3	3 **(80)**	9	10	80	$u_3=-5$
Demand	150	80	50		
v_j	$v_1=8$	$v_2=5$	$v_3=6$		

Find d_{ij} for all unoccupied cells(i,j), where $d_{ij}=c_{ij}-(u_i+v_j)$

$$d_{21} = c_{21} - \left(u_2 + v_1\right) = 15 - (5 + 8) = 2$$

$$d_{23} = c_{23} - \left(u_2 + v_3\right) = 12 - (5 + 6) = 1$$

$$d_{32} = c_{32} - \left(u_3 + v_2\right) = 9 - (-5 + 5) = 9$$

$$d_{33} = c_{33} - \left(u_3 + v_3\right) = 10 - (-5 + 6) = 9$$

	D_1	D_2	D_3	Supply	u_i
S_1	8 **(70)**	5 **(ε)**	6 **(50)**	120	$u_1=0$
S_2	15 [2]	10 **(80)**	12 [1]	80	$u_2=5$
S_3	3 **(80)**	9 [9]	10 [9]	80	$u_3=-5$
Demand	150	80	50		
v_j	$v_1=8$	$v_2=5$	$v_3=6$		

Since all $d_{ij} \geq 0$.So final optimal solution is arrived.

∴ The minimum total transportation cost $=8\times70+5\times0+6\times50+10\times80+3\times80=1900$

5. Unbalanced Transportation Problem:

- When **total supply ≠ total demand**.
- Make it balanced by adding a **dummy row or column**:
  - Add dummy **destination(column)** if supply > demand.
  - Add dummy **source(row)** if demand > supply.
  - Assign **zero cost** to dummy cells.

Example
Find Solution using Voggel's Approximation method

	D1	D2	D3	Supply
S1	4	8	8	76
S2	16	24	16	82
S3	8	16	24	77
Demand	72	102	41	

Solution:

Here Total Demand = 215 is less than Total Supply = 235. So, we add a dummy demand constraint with 0-unit cost and with allocation 20.

Now, the modified table is

	D_1	D_2	D_3	D_{dummy}	Supply
S_1	4	8	8	0	76
S_2	16	24	16	0	82
S_3	8	16	24	0	77
Demand	72	102	41	20	

Initial feasible solution is

<table>
<tr><th></th><th>D_1</th><th>D_2</th><th>D_3</th><th>D_{dummy}</th><th>Supply</th><th>Row Penalty</th></tr>
<tr><td>S_1</td><td>4</td><td>8(76)</td><td>8</td><td>0</td><td>~~76~~ 0</td><td>4 | 4 | -- | --</td></tr>
<tr><td>S_2</td><td>16</td><td>24(21)</td><td>16(41)</td><td>0(20)</td><td>~~82~~ ~~62~~ 21</td><td>16 | 0 | 0 | 8</td></tr>
<tr><td>S_3</td><td>8(72)</td><td>16(5)</td><td>24</td><td>0</td><td>~~77~~ 5</td><td>8 | 8 | 8 | 8</td></tr>
<tr><td>Demand</td><td>~~72~~ 0</td><td>~~102~~ 26</td><td>~~41~~ 0</td><td>~~20~~ 0</td><td></td><td></td></tr>
<tr><td>Column Penalty</td><td>4
4
8 ↑
--</td><td>8
8 ↑
8
8</td><td>8
8
8
8 ↑</td><td>0
--
--
--</td><td></td><td></td></tr>
</table>

The minimum total transportation cost $=8\times76+24\times21+16\times41+0\times20+8\times72+16\times5=2424$

Here, the number of allocated cells = 6 is equal to m + n - 1 = 3 + 4 - 1 = 6

∴ This solution is non-degenerate

6. Profit Maximization Problems:

- If objective is to **maximize profit**, convert it into a **minimization problem**:
    - Subtract each profit value from the **maximum profit value**.
    - Solve using standard transportation method.

Find Solution using Voggel's Approximation method (Maximization)

	D1	D2	D3	Supply
S1	8	5	6	120
S2	15	10	12	80
S3	3	9	10	80
Demand	150	80	50	

Solution:

Problem is Maximization, so convert it to minimization by subtracting all the elements from max element (15)

	D_1	D_2	D_3	Supply
S_1	7	10	9	120
S_2	0	5	3	80
S_3	12	6	5	80
Demand	150	80	50	

Initial feasible solution is

	D_1	D_2	D_3	Supply	Row Penalty
S_1	7**(70)**	10**(50)**	9	~~120~~ ~~50~~	2 | 2 | 1 | 10 ←
S_2	0**(80)**	5	3	~~80~~ 0	3 | -- -- --
S_3	12	6**(30)**	5**(50)**	~~80~~ 30	1 | 1 | 1 | 6 |
Demand	~~150~~ ~~70~~ 0	~~80~~ 30	~~50~~		
Column Penalty	7↑ 5↑ -- --	1 4 4 4	2 4 4↑ --		

The maximum profit $=8\times70+5\times50+15\times80+9\times30+10\times50=2780$ (Σ Cell Profit(of Max) $\times$ Quantity allocated)

Here, the number of allocated cells = 5 is equal to $m + n - 1 = 3 + 3 - 1 = 5$

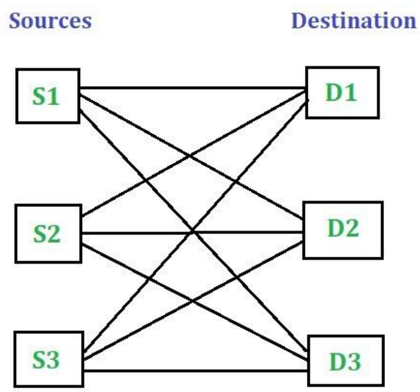
$\therefore$ This solution is non-degenerate

7. Transshipment Problems:

Transshipment Model is a model which comes under the transportation problem. There are two types of transshipment problem:

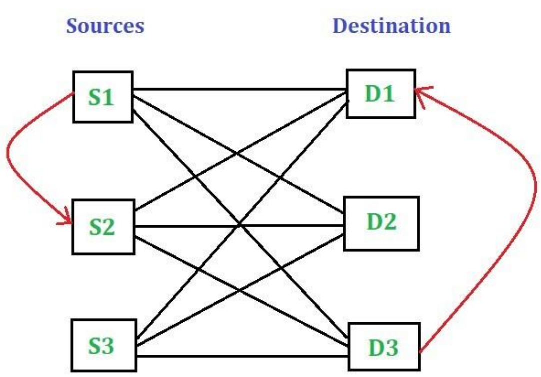
- With sources and destination acting as transient (i.e. intermediary) nodes
- With some transient nodes between sources and destination

Look at the given transportation problem diagram.



In a basic transportation problem, the shipment moves from one particular source to another particular destination, maybe from S1 to D1, S1 to D2, S1 to D3, S2 to D1, S2 to D2 and so on.

Transshipment problem with sources and destination acting as transient (i.e. intermediary)nodes



In this method, the shipment passes through one or more intermediary node before it reaches its desired destination. This method allows the shipment to pass from one source to another source and from one destination to another destination before it reaches the desired destination.

Note: The intermediary nodes can be sources and destinations themselves.

STEPS TO TRANS-SHIPMENT PROBLEM

- Check whether problem is Balance/Unbalanced
- Add the value 'N' (total supply/demand) to all the rows and columns
- Find out transportation cost by using VAM
- Draw the shipping pattern

Example:

Consider the following transshipment problem involving 4 sources and 2 destinations. The supply value of the sources S1, S2, S3 and S4 are 200 units, 250 units, 300 units and 450 units respectively. The demand value for destinations D1 and D2 are 600 units and 600 units respectively. The transportation cost per unit between different sources and destinations are summarized in the following table.

Solve the transshipment problem.

Sources \ Destinations	S_1	S_2	S_3	S_4	D_1	D_2
S_1	0	6	24	7	24	10
S_2	10	0	6	12	5	20
S_3	15	20	0	8	45	7
S_4	18	25	10	0	30	6
D_1	15	20	60	15	0	10
D_2	10	25	25	23	4	0

Solution:

1. **Check whether it is balanced or not.**

   Total supply = 200 + 250 + 300 + 450 = 1200

   Total demand = 600 + 600 = 1200

In this case the problem is balanced. In case the problem were not balanced we could add dummy row or column to make it balance.

Sources \ Destinations	S_1	S_2	S_3	S_4	D_1	D_2	Supply
S_1	0	6	24	7	24	10	200
S_2	10	0	6	12	5	20	250
S_3	15	20	0	8	45	7	300
S_4	18	25	10	0	30	6	450
D_1	15	20	60	15	0	10	
D_2	10	25	25	23	4	0	
Demand:					600	600	1200

2. **Add the value of total supply/demand to all rows and columns.**

Sources \ Destinations	S_1	S_2	S_3	S_4	D_1	D_2	Supply
S_1	0	6	24	7	24	10	200 + 1200 = 1400
S_2	10	0	6	12	5	20	250+ 1200 = 1450
S_3	15	20	0	8	45	7	300+ 1200 = 1500
S_4	18	25	10	0	30	6	450+ 1200 = 1650
D_1	15	20	60	15	0	10	1200
D_2	10	25	25	23	4	0	1200
Demand:	1200	1200	1200	1200	600 + 1200 = 1800	600 + 1200 = 1800	1200

3. **Find out the total transportation cost using VAM**

   After solving the transportation problem using VAM, we get the following solution,

Destinations

Sources	S1	S2	S3	S4	D1	D2	Supply
S1	[1200] 0	[200] 6	24	7	24	10	1400
S2	10	[1000] 0	6	12	[450] 5	20	1450
S3	15	20	[1200] 0	8	45	[300] 7	1500
S4	18	25	10	[1200] 0	30	[450] 6	1650
D1	15	20	60	15	[1200] 0	10	1200
D2	10	25	25	23	[150] 4	[1050] 0	1200
Demand:	1200	1200	1200	1200	1800	1800	

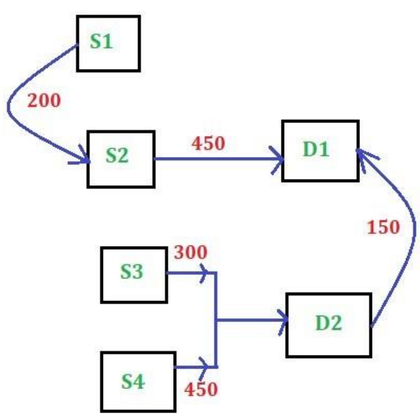
- Just ignore the zero cost cells and calculate the transportation cost.

  Total transportation cost is = (200 * 6) + (450 * 5) + (300 * 7) + (450 * 6) + (150 * 4) = 8850

4. **Draw the shipping pattern.**

   Note: The allocations in the main diagonal cells are to be ignored.

   To draw the shipping diagram first draw the four sources and two destinations as shown below:



8. Assignment Problems

It is a special type of LPP that deals with optimally assigning a set of resource to set of tasks.

- A **special case of transportation problem** where:
  - Number of sources = number of destinations.
  - Each task is assigned to one agent and vice versa.
- Objective: **Minimize total cost/time or maximize total profit**.

Assignment Problem Structure and Solution:

<u>Structure :</u>

	Job (activity)			
Worker (resource)	J_1	J_2	J_3	Supply
W_1	c_{11} (x_{11})	c_{12} (x_{12})	c_{13} (x_{13})	1
W_2	c_{21} (x_{21})	c_{22} (x_{22})	c_{23} (x_{23})	1
W_3	c_{31} (x_{31})	c_{32} (x_{32})	c_{33} (x_{33})	1
Demand	1	1	1	

Here, c_{ij} represents the cost of assignment of worker 'i' to job 'j' and
x_{ij} represents the assignment of worker 'i' to job 'j' such that

$$x_{ij} = \begin{cases} 1 & \text{if worker i is assigned to job j} \\ 0 & \text{otherwise} \end{cases}$$

General Mathematical Model

$$\text{minimize} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \cdot x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ (worker availability)}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ (job requirement)}$$

and $x_{ij} \geq 0$ for all I =1,2, …,n and j=1,2, …, n.

Solution :

- Simplex method
- Transportation problem method
- Hungarian method

The Hungarian method is preferred for assignment problems because it's more efficient and simpler than using the simplex method or solving it as a transportation problem.

Notes :

- Number of workers = Number of jobs.
- Each worker is assigned only one job.
- Each worker is independently capable for handling any job.
- Assigning criteria is either minimizing cost or maximizing profit.

Types of Assignment Problem

- **Balanced Assignment Problem:** The number of tasks equals the number of agents (e.g., assigning 5 workers to 5 jobs). In this case, a perfect matching is possible.

- **Unbalanced Assignment Problem:** The number of tasks and agents are not equal (e.g., assigning 7 workers to 5 jobs). A dummy task or agent is introduced to balance the problem, with zero or negligible cost/benefit associated with it.

Hungarian Method (Short-Cut Method):

1. Check whether the problem is balanced or not. If not make it balanced by adding dummy rows/columns.
2. **Row reduction:** Subtract minimum value from each row.
3. **Column reduction:** Subtract minimum value from each column.
4. **Cover all zeros** with minimum number of horizontal/vertical lines.
   - Check no. of zero in 1st row , if there is only one zero make a box over it and draw a vertical line through the column and if there is more than one zero skip that row and go to next row.
   - Repeat this process to all the rows and column and draw lines accordingly(through column for row and through row for column) until all zeros are covered.

5. If number of lines(allocations) = number of rows/columns, **optimal assignment** found.
6. If not, adjust matrix:
   - Subtract smallest uncovered value from uncovered elements.
   - Add it to the elements covered twice.
   - Repeat step 4.

Example:

Find Solution of Assignment problem using Hungarian method (MIN case)

Work\Job	I	II	III
A	6	3	5
B	5	9	2
C	5	7	8

Solution:

The number of rows = 3 and columns = 3. So, the given problem is balanced.

Step-1: Find out each row minimum element and subtract it from that row

	I	*II*	*III*	
A	3	0	2	(-3)
B	3	7	0	(-2)
C	0	2	3	(-5)

Step-2: Find out each column minimum element and subtract it from that column.

	I	II	III
A	3	0	2
B	3	7	0
C	0	2	3
	(-0)	(-0)	(-0)

Step-3: Make assignment in the opporunity cost table
(1) Rowwise cell (A, II) is assigned

(2) Rowwise cell (B, III) is assigned

(3) Rowwise cell (C, I) is assigned

Rowwise & columnwise assignment shown in table

	I	II	III	
A	3	[0]	2	
B	3	7	[0]	
C	[0]	2	3	

Step-4: Number of assignments = 3, number of rows = 3
Which is equal, so solution is optimal
Thus ,Optimal solution is

Work	Job	Cost
A	II	3
B	III	2
C	I	5
	Total	10

1. Find Solution of Assignment problem using Hungarian method (MIN case)

Work\Job	I	II	III	IV	V
A	10	5	13	15	16
B	3	9	18	13	6
C	10	7	2	2	2
D	7	11	9	7	12
E	7	9	10	4	12

Solution:
The number of rows = 5 and columns = 5. So, the given problem is balanced.

Step-1: Find out each row minimum element and subtract it from that row

	I	II	III	IV	V	
A	5	0	8	10	11	(-5)
B	0	6	15	10	3	(-3)
C	8	5	0	0	0	(-2)
D	0	4	2	0	5	(-7)
E	3	5	6	0	8	(-4)

Step-2: Find out each column minimum element and subtract it from that column.

	I	II	III	IV	V
A	5	0	8	10	11
B	0	6	15	10	3
C	8	5	0	0	0
D	0	4	2	0	5
E	3	5	6	0	8
	(-0)	(-0)	(-0)	(-0)	(-0)

Step-2: Make an assignment

	I	II	III	IV	V
A	5	[0]	8	10	11
B	[0]	6	15	10	3
C	8	5	[0]	~~0~~	~~0~~
D	~~0~~	4	2	[0]	5
E	3	5	6	~~0~~	8

Here, Number of assignments = 4, number of rows = 5
Which is not equal, so solution is not optimal.

Step-4: Subtract smallest uncovered value(i.e. 2) from uncovered elements & Add it to the elements covered twice (i.e. 8, 5 & 0 of row C).

	I	*II*	*III*	*IV*	*V*
A	5	[0]	6	10	7
B	[0]	6	13	10	1
C	10	7	0	2	[0]
D	0	4	[0]	0	3
E	3	5	4	[0]	6

Step-4: Number of assignments = 5, number of rows = 5
Which is equal, so solution is optimal

Optimal solution is

Work	Job	Cost
A	*II*	5
B	*I*	3
C	*V*	2
D	*III*	9
E	*IV*	4
	Total	23

8. Unbalanced Assignment Problem:

The number of agents(rows) is not equal to the number of tasks(columns).

Remedy: Add dummy rows or columns (agents or tasks) with zero-cost assignments to make the cost matrix square.

Example
Find Solution of Assignment problem using Hungarian method (MIN case)

Work\Job	I	II	III	IV
A	9	14	19	15
B	7	17	20	19
C	9	18	21	18
D	10	12	18	19
E	10	15	21	16

Solution:
The number of rows = 5 and columns = 4
Here given problem is unbalanced. So, add 1 dummy column to convert it into a balance.

	I	*II*	*III*	*IV*	*J*5
A	9	14	19	15	0
B	7	17	20	19	0
C	9	18	21	18	0
D	10	12	18	19	0
E	10	15	21	16	0

Step-1: Find out each row minimum element and subtract it from that row

→ Same table will be formed as each row has 0 as a minimum element.

Step-2: Find out each column minimum element and subtract it from that column.

	I	*II*	*III*	*IV*	J_5
A	2	2	1	0	0
B	0	5	2	4	0
C	2	6	3	3	0
D	3	0	0	4	0
E	3	3	3	1	0
	(-7)	(-12)	(-18)	(-15)	(-0)

Step-3: Make assignment in the opportunity cost table

	I	*II*	*III*	*IV*	J_5
A	2	2	1	[0]	0
B	[0]	5	2	4	0
C	2	6	3	3	[0]
D	3	[0]	0	4	0
E	3	3	3	1	0

Here, number of assignments = 4, number of rows = 5

Which is not equal, so solution is not optimal.

Step-4: Subtract smallest uncovered value (i.e. 1) from uncovered elements & Add it to the elements covered twice (i.e. 0 in row A , B and D).

	I	*II*	*III*	*IV*	J_5
A	2	2	1	[0]	1
B	[0]	5	2	4	1
C	1	5	2	2	[0]
D	3	[0]	~~0~~	4	1
E	2	2	2	~~0~~	~~0~~

Here again, number of assignments = 4, number of rows = 5

Which is not equal, so solution is not optimal.

Step-5: Subtract smallest uncovered value (i.e. 1) from uncovered elements & Add it to the elements covered twice (i.e. 4's and 1's in row B and D).

	I	*II*	*III*	*IV*	J_5
A	1	1	[0]	~~0~~	1
B	[0]	5	2	5	2
C	~~0~~	4	1	2	[0]
D	3	[0]	~~0~~	5	2
E	1	1	1	[0]	~~0~~

Here, Number of assignments = 5, number of rows = 5

Which is equal, so solution is optimal

Work	Job	Cost
A	*III*	19
B	*I*	7
C	J_5	0
D	*II*	12
E	*IV*	16
	Total	54

9. Maximization in Assignment Problem:

Convert to **minimization** by subtracting each element from the **maximum value** in the matrix.

Example

Find Solution of Assignment problem using Hungarian method (MAX case)

Work\Job	I	II	III	IV
A	42	35	28	21
B	30	25	20	15
C	30	25	20	15
D	24	20	16	12

Solution:

The number of rows = 4 and columns = 4. So, the given problem is balanced.
Here, the problem is of Maximization type and convert it into minimization by subtracting it from maximum value 42

	I	II	III	IV
A	0	7	14	21
B	12	17	22	27
C	12	17	22	27
D	18	22	26	30

Step-1: Find out the each row minimum element and subtract it from that row

	I	II	III	IV	
A	0	7	14	21	(-0)
B	0	5	10	15	(-12)
C	0	5	10	15	(-12)
D	0	4	8	12	(-18)

Step-2: Find out the each column minimum element and subtract it from that column.

	I	II	III	IV	
A	0	3	6	9	
B	0	1	2	3	
C	0	1	2	3	
D	0	0	0	0	
	(-0)	(-4)	(-8)	(-12)	

Step-3: Make assignment in the opportunity cost table

	I	*II*	*III*	*IV*
A	[0]	3	6	9
B	~~0~~	1	2	3
C	~~0~~	1	2	3
D	~~0~~	[0]	~~0~~	~~0~~

Number of assignments = 2, number of rows = 4

Which is not equal, so solution is not optimal.

Step-4: Subtract smallest uncovered value (i.e. 1) from uncovered elements & Add it to the elements covered twice (i.e. 0 in row D).

	I	*II*	*III*	*IV*
A	[0]	2	5	8
B	~~0~~	[0]	1	2
C	~~0~~	~~0~~	1	2
D	1	~~0~~	[0]	~~0~~

Number of assignments = 3, number of rows = 4

Which is not equal, so solution is not optimal.

Step-5: Subtract smallest uncovered value (i.e. 1) from uncovered elements & Add it to the elements covered twice (i.e. 1 & 0 in row D).

	I	*II*	*III*	*IV*
A	[0]	2	4	7
B	~~0~~	[0]	~~0~~	1
C	~~0~~	~~0~~	[0]	1
D	2	1	~~0~~	[0]

Number of assignments = 4, number of rows = 4

Which is equal, so Optimal solution is

Work	Job	Cost
A	I	42
B	II	25
C	III	20
D	IV	12
	Total	99

10. Infeasible (Restricted) Assignments

An infeasible assignment problem occurs when the restrictions or constraints in an assignment problem make it impossible to find a solution that satisfies all conditions. This means there isn't a way to assign all tasks to resources (or vice versa) without violating some constraint.

In simpler terms: Imagine you have a set of tasks and a set of workers. Some workers might be unable to perform certain tasks, or some tasks might be incompatible with certain workers. If these limitations are so strict that you can't find a way to assign all workers to tasks (or vice versa) without conflicts, then the problem is infeasible.

Sometimes it may happen that a particular resource cannot be assigned to a particular activity.
In such case cost is considered to be very large (say M or ∞), so as to prohibit the entry into the final solution.

Example
Find Solution of Assignment problem using Hungarian method (MAX case)

Work\Job	A	B	C	D	E
A	8	2	-	5	4
B	10	9	2	8	4
C	5	4	9	6	-
D	3	6	2	8	7
E	5	6	10	4	3

Solution:

The number of rows = 5 and columns = 5. So, the problem is balanced

	A	B	C	D	E
A	8	2	M	5	4
B	10	9	2	8	4
C	5	4	9	6	M
D	3	6	2	8	7
E	5	6	10	4	3

Here the problem is of Maximization type so convert it into minimization by subtracting it from maximum value 10

	A	B	C	D	E
A	2	8	M	5	6
B	0	1	8	2	6
C	5	6	1	4	M
D	7	4	8	2	3
E	5	4	0	6	7

Step-1: Find out each row minimum element and subtract it from that row

	A	B	C	D	E	
A	0	6	M	3	4	(-2)
B	0	1	8	2	6	(-0)
C	4	5	0	3	M	(-1)
D	5	2	6	0	1	(-2)
E	5	4	0	6	7	(-0)

Step-2: Find out the each column minimum element and subtract it from that column.

	A	B	C	D	E	
A	0	5	M	3	3	
B	0	0	8	2	5	
C	4	4	0	3	M	
D	5	1	6	0	0	
E	5	3	0	6	6	
	(-0)	(-1)	(-0)	(-0)	(-1)	

Step-3: Make assignment in the opportunity cost table

	A	B	C	D	E
A	[0]	5	M	3	3
B	~~0~~	[0]	8	2	5
C	4	4	[0]	3	M
D	5	1	6	[0]	~~0~~
E	5	3	~~0~~	6	6

Number of assignments = 4, number of rows = 5

Which is not equal, so solution is not optimal.

Step-4: Develop the new revised table by selecting the smallest element, among the cells not covered by any line (2). Subtract 2 from every element in the cell not covered by a line.
Add 2 to every element in the intersection cell of two lines.

	A	B	C	D	E
A	[0]	5	M	1	1
B	0	0	8	0	3
C	4	4	[0]	1	M
D	7	3	8	0	0
E	5	3	0	4	4

Number of assignments = 4, number of rows = 5

Which is not equal, so solution is not optimal. Again, repeat step-4

	A	B	C	D	E
A	[0]	4	M	0	0
B	1	[0]	9	0	3
C	4	4	0	[0]	M
D	8	3	9	0	[0]
E	5	2	[0]	3	3

Number of assignments = 5, number of rows = 5

Which is equal, so optimal solution is

Work	Job	Cost
A	A	8
B	B	9
C	D	6
D	E	7
E	C	10
	Total	40

11. Crew Assignment Problem:

- Assign crew members (drivers, pilots, etc.) to tasks (flights, trips) based on cost/profit/time.
- Can be solved using the **Hungarian Method**.

Example: Crew assignment problem

Delhi - Mumbai			Mumbai - Delhi		
Flight No	Departure	Arrival	Flight No	Departure	Arrival
1	7.00	8.00	101	8.00	9.00
2	8.00	9.00	102	09.00	10.00
3	13.00	14.00	103	12.00	13.00
4	18.00	19.00	104	17.00	18.00

Minimum layover of hours between flights : 5

Solution:

To determine optimal assignments, first we calculate layover times from the above time table.

Calculating values for table 1 (layover time)

1*st* Row

Cell-1 Arrival time (Mumbai) = 08:00, Departure time (Mumbai) = 08:00, Waiting time (difference) = 24

Cell-2 Arrival time (Mumbai) = 08:00, Departure time (Mumbai) = 09:00, Waiting time (difference) = 25

…………….

4*th* Row

Cell-1 Arrival time (Mumbai) = 19:00, Departure time (Mumbai) = 08:00, Waiting time (difference) = 13

……………….

Table-1 : Crew based at Delhi

	101	102	103	104
1	24	25	28	9
2	23	24	27	8
3	18	19	22	27
4	13	14	17	22

Calculating values for table 2 (layover time)

1*st* Row

Cell-1 Arrival time (Delhi) = 09:00, Departure time (Delhi) = 07:00, Waiting time (difference) = 22

Cell-2 Arrival time (Delhi) = 10:00, Departure time (Delhi) = 07:00, Waiting time (difference) = 21

……..

Table-2 : Crew based at Mumbai

	101	102	103	104	
1	22	21	18	13	
2	23	22	19	14	
3	28	27	24	19	
4	9	8	29	24	

The composite layover time matrix (Table-3) is obtained by selecting the smaller element from the two corresponding elements of Table-1 and Table-2. The layover time marked with (*) represents that the crew is based at Mumbai, otherwise based at Delhi.

	101	102	103	104
1	22 *	21 *	18 *	9
2	23 *	22 *	19 *	8
3	18	19	22	19 *
4	9 *	8 *	17	22

Step-1: Find out the each row minimum element and subtract it from that row

	101	102	103	104	
1	13	12	9	0	(-9)
2	15	14	11	0	(-8)
3	0	1	4	1	(-18)
4	1	0	9	14	(-8)

Step-2: Find out the each column minimum element and subtract it from that column.

	101	102	103	104	
1	13	12	5	0	
2	15	14	7	0	
3	0	1	0	1	
4	1	0	5	14	
	(-0)	(-0)	(-4)	(-0)	

Step-3: Make assignment in the opportunity cost table

	101	102	103	104
1	13	12	5	**[0]**
2	15	14	7	~~0~~
3	**[0]**	1	~~0~~	1
4	1	**[0]**	5	14

Number of assignments = 3, number of rows = 4

Which is not equal, so solution is not optimal.

Step-4: Develop the new revised table by selecting the smallest element, among the cells not covered by any line (i.e. 5).Subtract 5 from every element in the cell not covered by a line. Add 5 to every element in the intersection cell of two lines.

	101	102	103	104
1	8	7	[0]	~~0~~
2	10	9	2	[0]
3	[0]	1	~~0~~	6
4	1	[0]	5	19

Number of assignments = 4, number of rows = 4

Which is equal, so optimal solution is

Work	Job	Cost
1	103	18
2	104	8
3	101	18
4	102	8
	Total	52