The Power of Two-Choices in reducing discrepancy

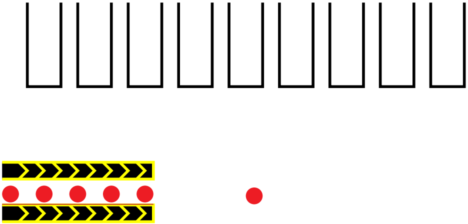
Ohad N. Feldheim

Based on joint works with:
Ori Gurel-Gurevich (HUJI)
Raaz Dwivdi & Aaditya Ramdas (Berkeley)

June 2017

Balls and bins model

Consider an online process in which N balls are randomly assigned, one by one to M bins.



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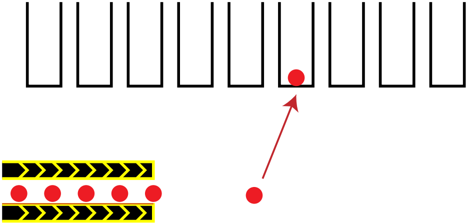

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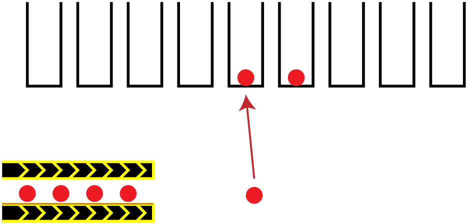

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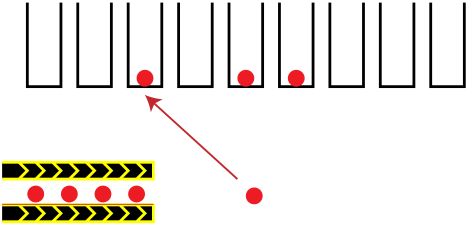

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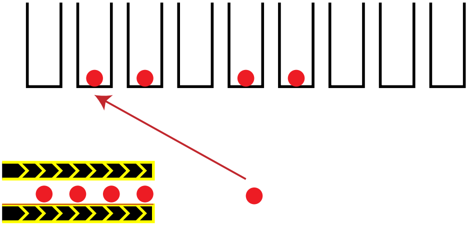

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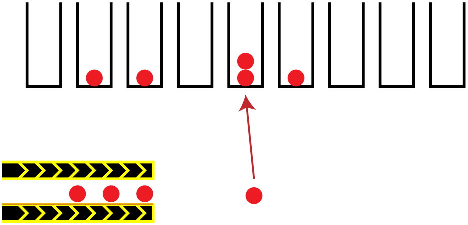

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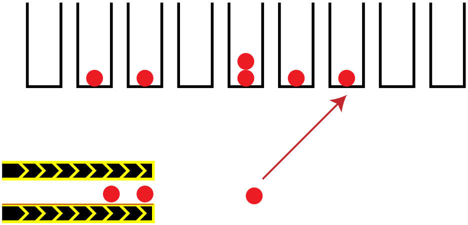

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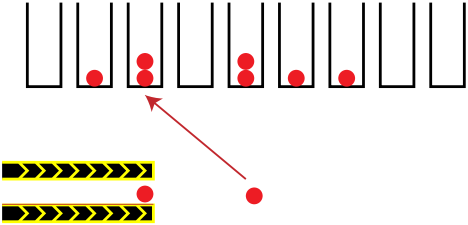

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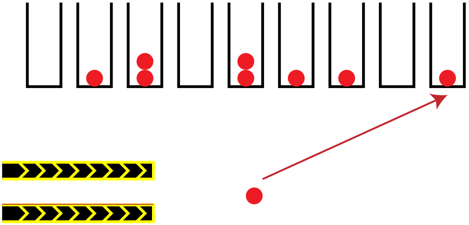

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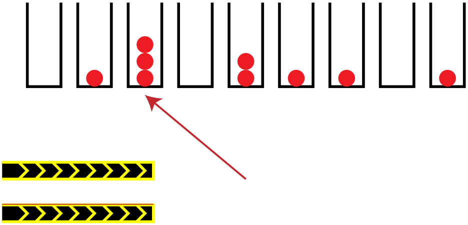

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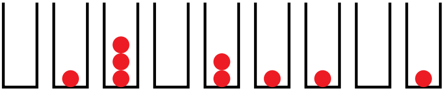

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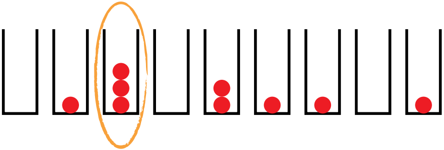

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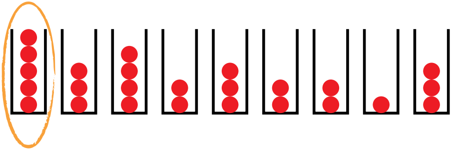

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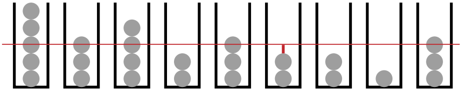

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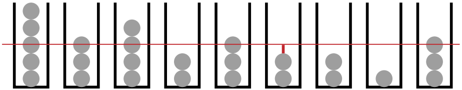

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- *Load balancing* is an effort to reduce these quantities. (possible with control over the distribution of the balls.)

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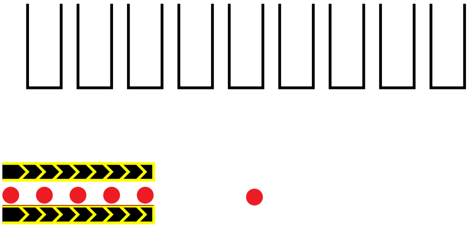

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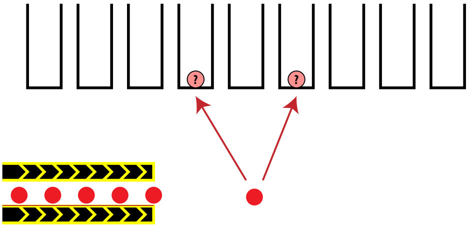

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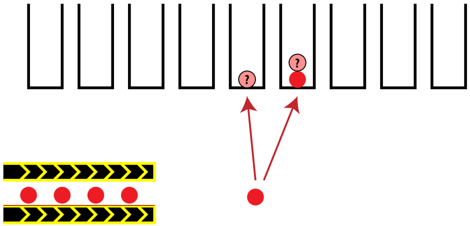

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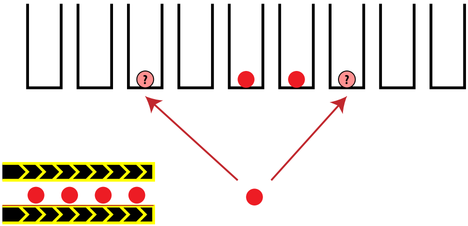

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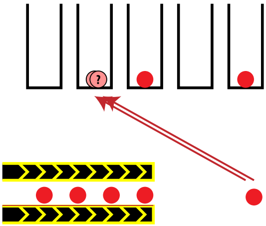

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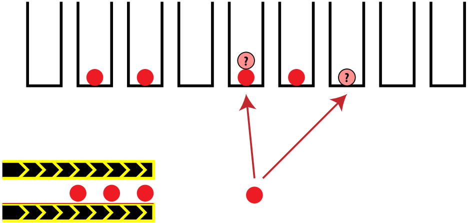

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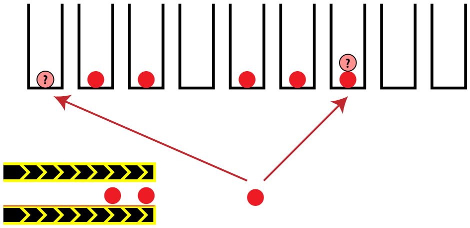

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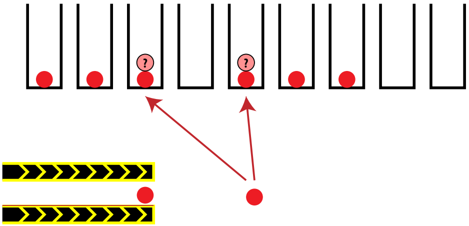

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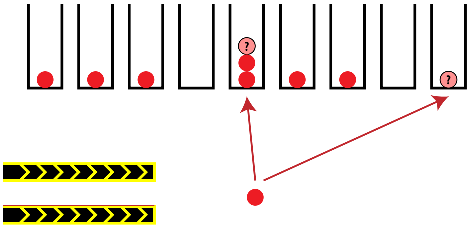

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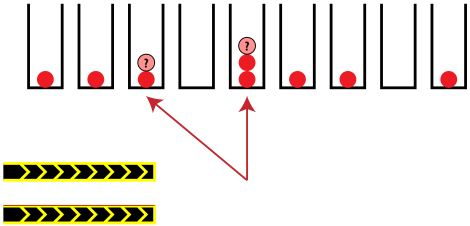

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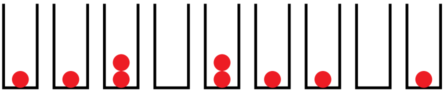

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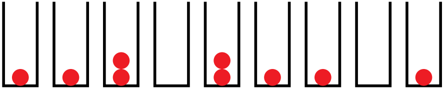

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Balls	no-choice max	2-choices max	no-choice typ. dev.	2-choices typ. dev.
M	$\frac{\log M}{\log\log M}$	$\frac{\log\log M}{2}$	O(1)	O(1)
$N >> M$	$\Theta\left(\sqrt{\frac{N \log M}{M}}\right)$	$\log M$	$\Theta\left(\sqrt{\frac{N}{M}}\right)$	O(1)

Power of two choices - remarks

This observation had many applications

- Server load-balancing
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- If balls keep appearing and dying at rate 1 the phenomenon persists (Luczak & McDiarmid '05)
- However if one can't keep track of the number of balls per bin (due to having $M^{1-\epsilon}$ bits of memory), then no asymptotic improvement over no-choice is possible (Alon, Gurel-Gurevich, Lubetzky '09)

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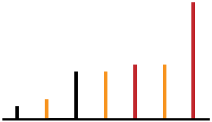

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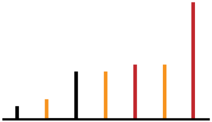

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However, even Kakutani offers no significant improvement in terms of interval variation.

Convergence of 2-Max interval partition process

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In a sense, corresponds to Alon, Gurel-Gurevich, Lubetzky.

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Power of two-choices in regulating interval discrepancy (F. & Gurel-Gurevich 2016+)

In a power of two-choices process on $\mathcal{U}[0,1]$, the chooser can obtain

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- Cf. lower bound of $\frac{C\log N}{N}$, no choice estimate $\sqrt{\frac{C\log N}{N}}$.

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Discrete counterpart

For N balls on $\mathcal{U}([M])$, a probabilistic retry strategy obtains

$$\Pr\left(\max_{a<b\in[M]} |\mu^n([a,b]) - \mu([a,b])| > \Delta \log^3 M\right) \leq Ce^{-c\Delta}.$$

Stochastic point of view on the power of two choices

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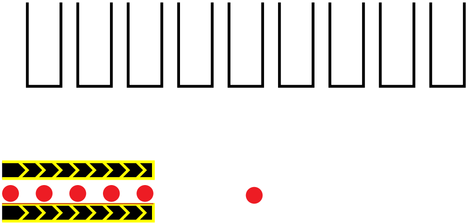

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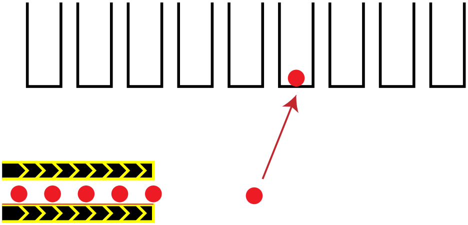

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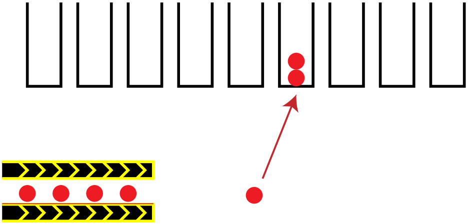

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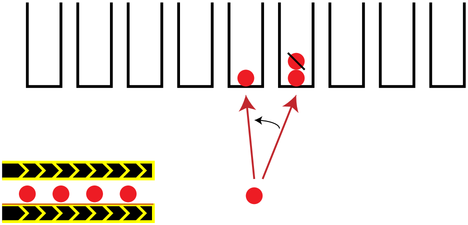

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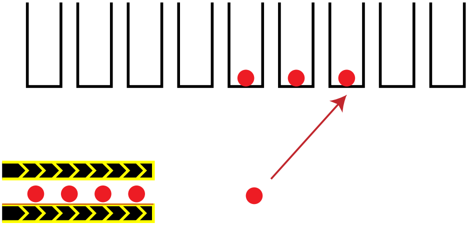

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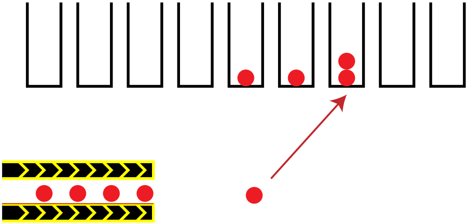

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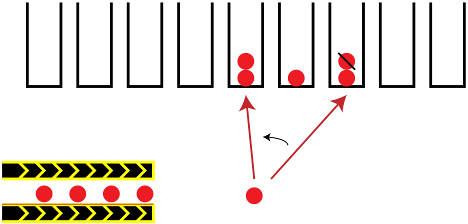

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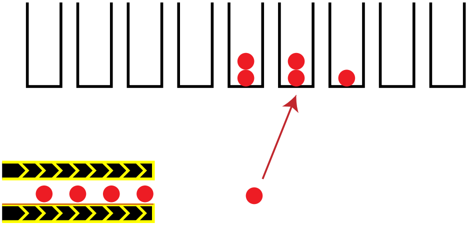

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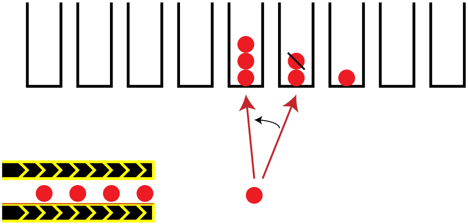

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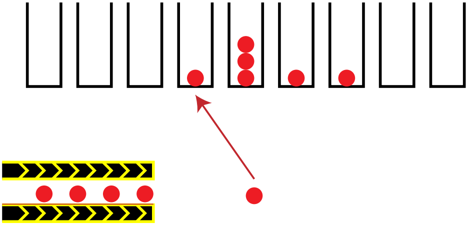

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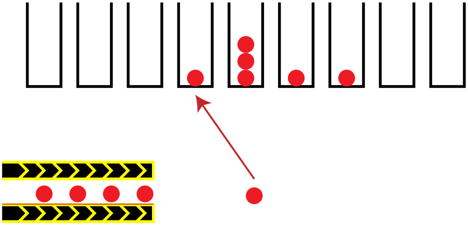

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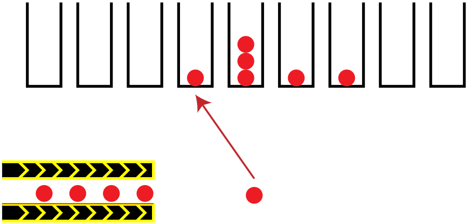

Equivalent to being oblivious to one of the two bins in the two choices setup. Asymptotic discrepancy like two-choices when $\#$balls $\gg$ $\#$ bins.

Stochastic Approach to Two-choices: Regulating Balls and Bins

Large family of one-retry distributions

What kind of distributions could be realized using one retry?

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The probability of a random bin to be re-rolled is $\sum_{i=1}^{M} \frac{f(x)}{M} = \frac{1}{2}$.
Hence the probability that x is chosen is now

$$\frac{1 - f(x)}{M} + \frac{1}{2M} = g(x)$$

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Self regulating poisson point process

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i.e. such a process has typical fluctuation $O(\theta)$.

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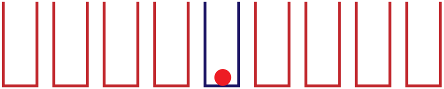

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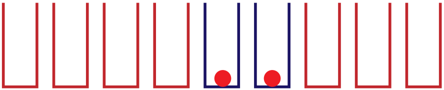

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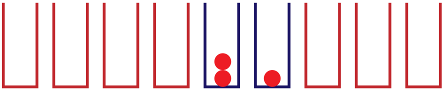

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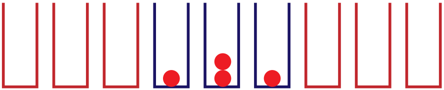

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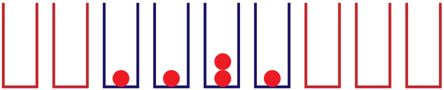

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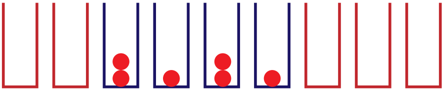

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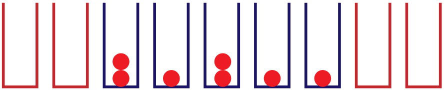

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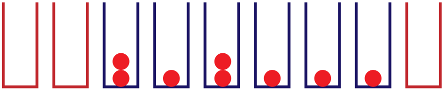

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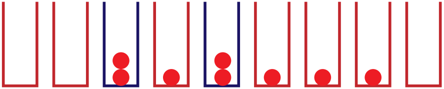

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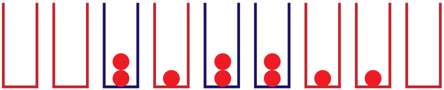

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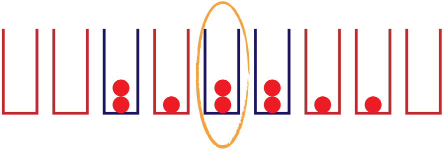

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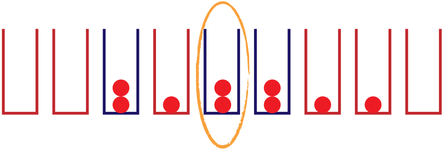

Among M such processes, the extremal fluctuation is $O(\theta \log M)$.

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All that is left is to show that the same holds after N balls were distributed. - i.e. show concentration of the stopping time.

Stochastic Approach to Two-choices:

Regulating Interval Partition

Balancing poisson point process

Consider two point processes X_t^0, X_t^1 with changing causal intensities $\lambda^0(t), \lambda^1(t)$ (which may depend on other variables), which satisfy

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Each $(a,b) \subset [0,1]$ has $(a,b) = \sum_{i=0}^{\lfloor \log_2 N \rfloor} I_i + R$ where I_i are diadic and $|R| < 1/N$.

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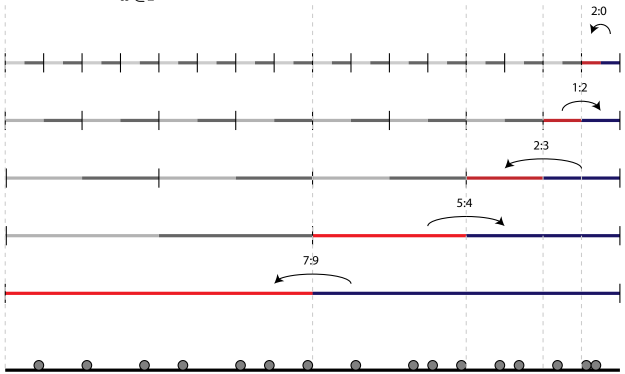

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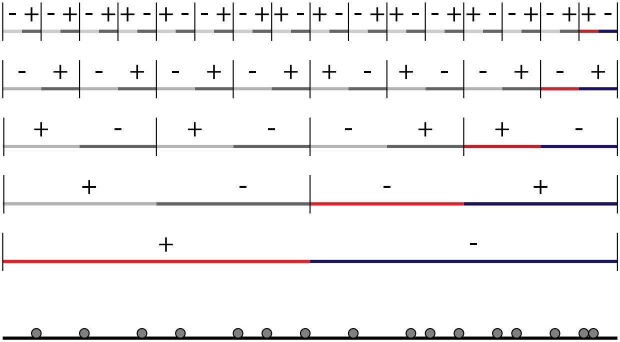

Future directions

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- Other spaces

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- Other spaces
- Other measures of discrepancy.

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- Reducing the power of the log.

THANK YOU.