

Counterfactual inference in sequential experimental design

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Ongoing joint work with



Harvard John A. Paulson
School of Engineering
and Applied Sciences



Learning from interventions workshop, Simons Institute, Feb 14, 2022

Sequential decision making problems

- Online education: Enhance teaching strategies for better learning
- Online advertising: Update ads / placements to increase revenue
- **Mobile health:** Personalized app messages to promote healthy behavior

Physical activity



Wearable/trackers

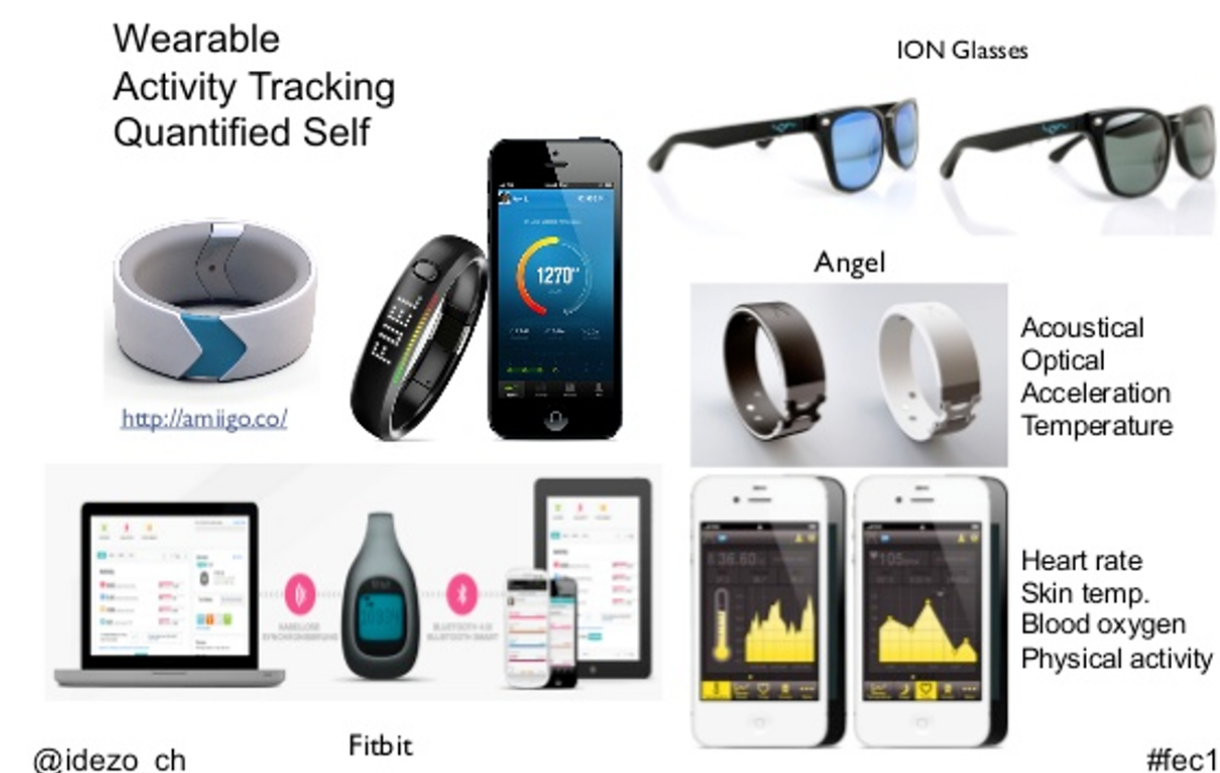


Image credits: Susan Murphy

Mobile health trial: A simplified but representative set-up

$Y_{i,t}^{(a)}$ = potential outcome of user i at time t under treatment a

Neyman-Rubin framework

Mobile health trial: A simplified but representative set-up

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For time $t = 1, 2, \dots, T$

For user $i = 1, \dots, N$


1. Randomly send a notification ($A_{i,t} = 1$) or not ($A_{i,t} = 0$) using a sampling policy $\pi_t \in [0, 1]^N$ with $\pi_t(i) = \mathbb{P}(A_{i,t} = 1 \mid \text{history}_{t-1})$

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2. Observe outcome $Z_{i,t} = Y_{i,t}^{(A_{i,t})} + \varepsilon_{i,t}$  noise

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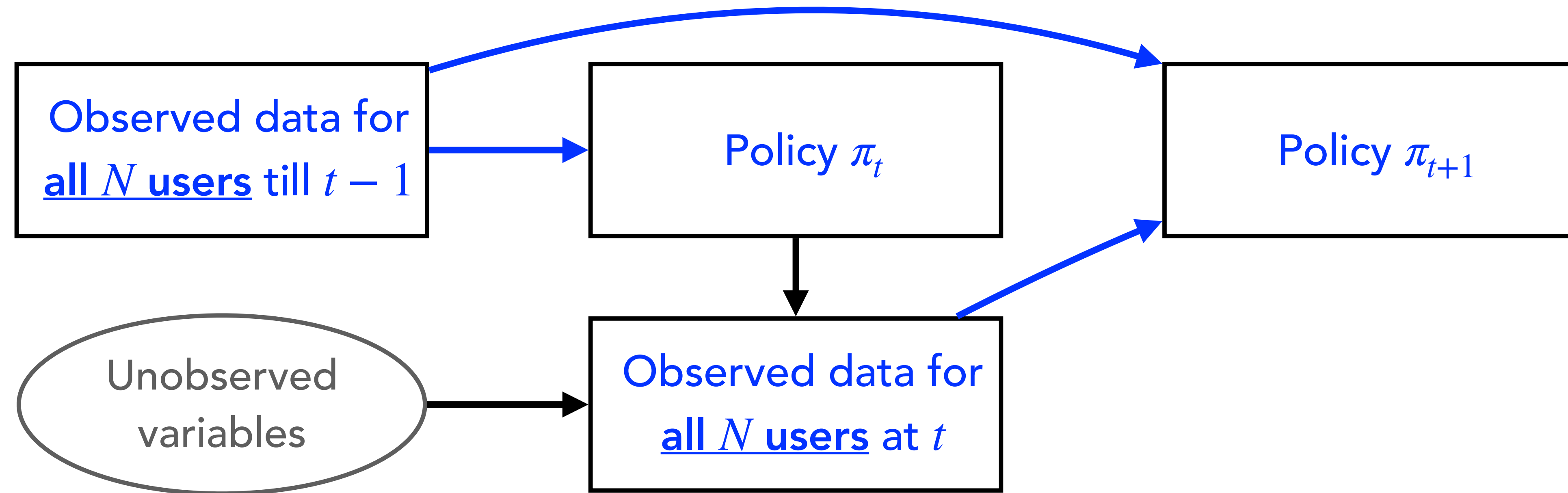
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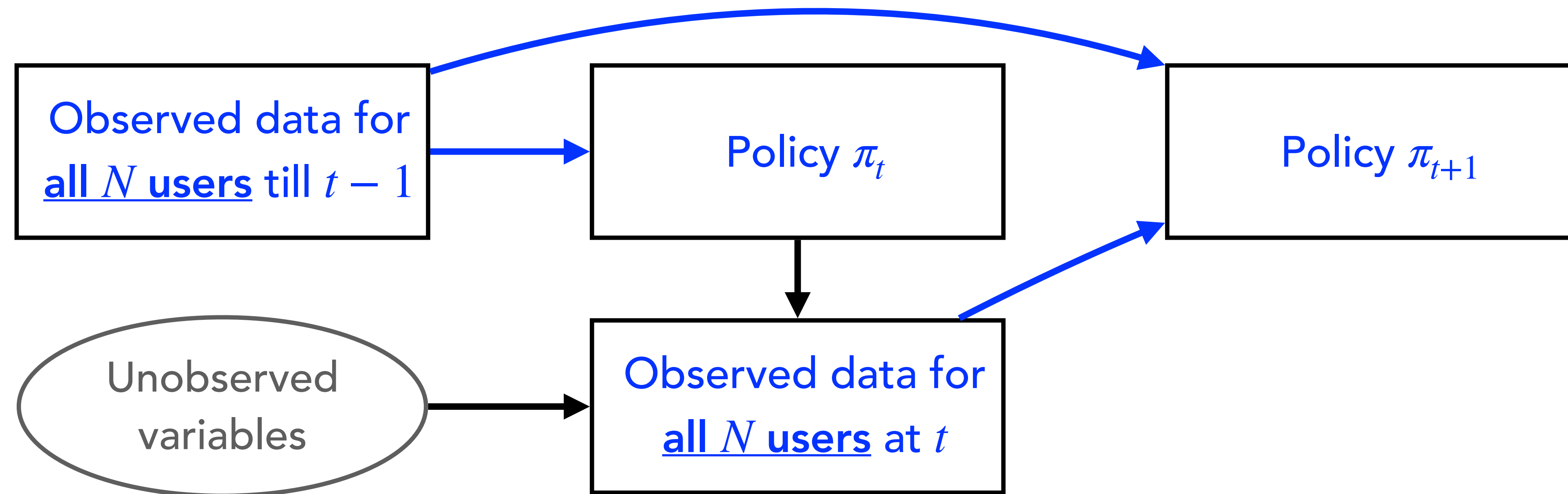
Update policy to π_{t+1} using data till time t

Sequential experimental design



- **Adaptive** policy to personalize to users
- Data **pooled during study** learn a good policy quickly

Sequential experimental design: After-study inference



- **Adaptive** policy to personalize to users
- Data **pooled during study** learn a good policy quickly
- After-study questions: **Was the treatment effective? On average? Heterogeneity across users? ...**

Our goal: Counterfactual inference, namely, estimation of all missing potential outcomes—a hard task due to **heterogeneity across users and time**

- can be used for generic after-study analyses, e.g., individual treatment effect $Y_{i,t}^{(1)} - Y_{i,t}^{(0)}$

Our goal: Counterfactual inference, namely, estimation of all missing potential outcomes—a hard task due to **heterogeneity across users and time**

Adaptivity and pooling of data across N users for policy π_t
makes after-study inference even more challenging

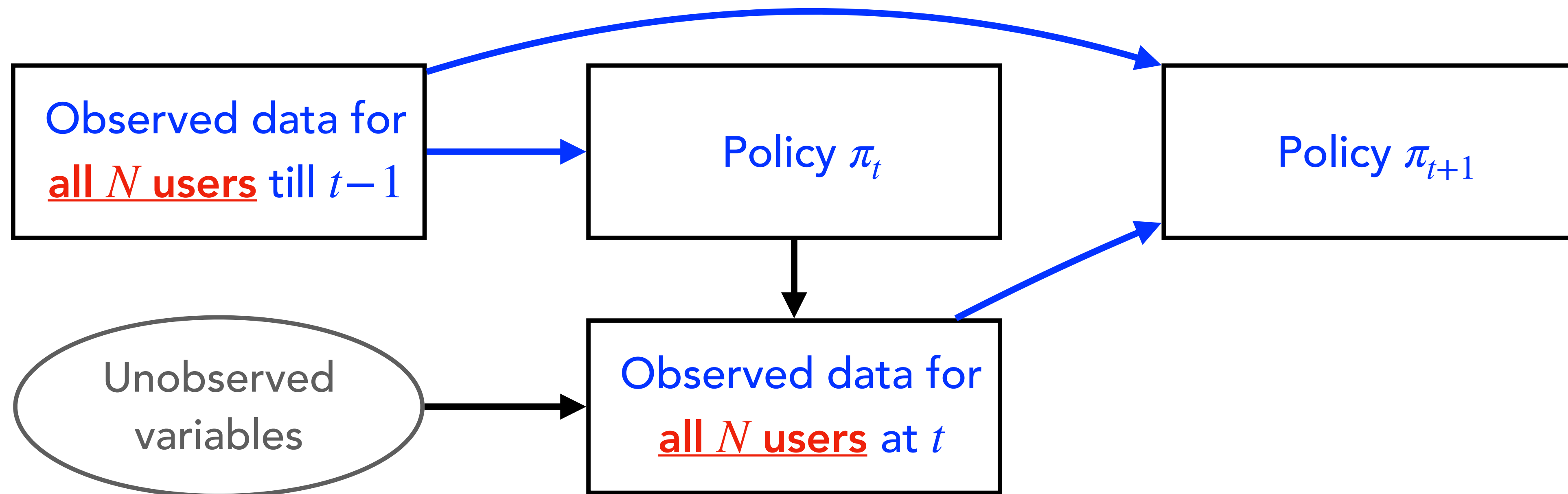
- can be used for generic after-study analyses, e.g., individual treatment effect $Y_{i,t}^{(1)} - Y_{i,t}^{(0)}$

Coming up next

- Settings considered in prior work
- Our contributions
 1. A non-parametric strategy for counterfactual estimation
 2. Distribution-free consistency under bilinear latent factor model
 3. Central limit theorem under a non-linear latent factor model

After-study inference:

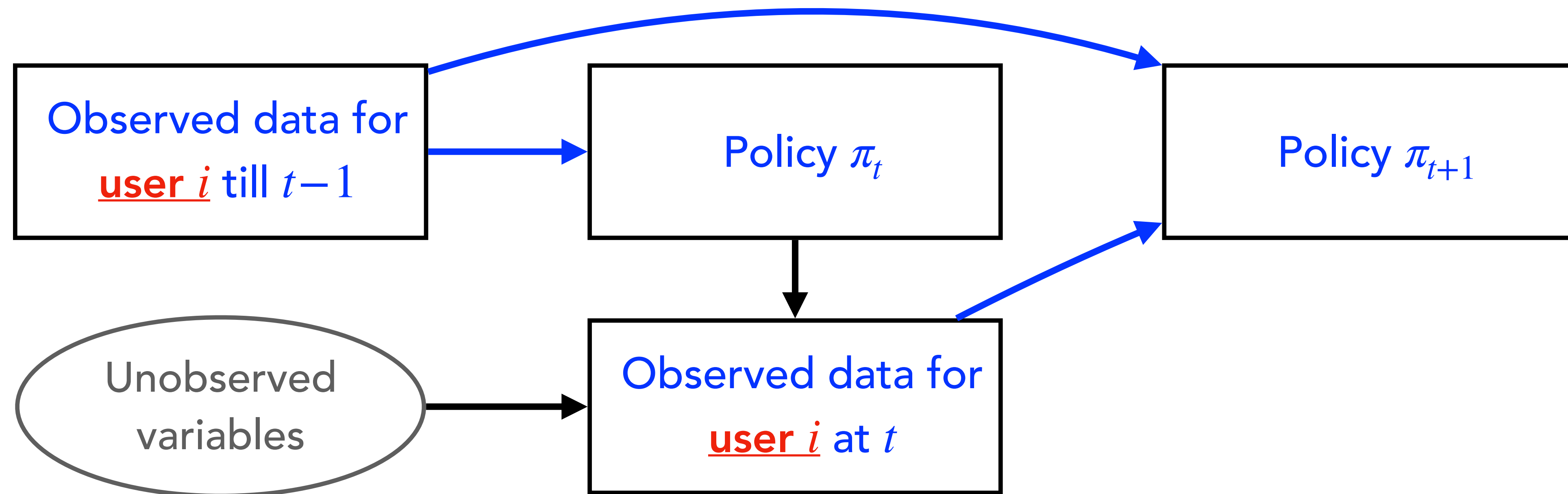
Our setting of sequential experiments



After-study inference:

Dynamic treatment regime in longitudinal studies

Policy adaptive for *a single fixed user over time* based on prior data, e.g., personalized clinical treatment [..., Robins 1986, 1987, ..., Murphy 2003, ...]



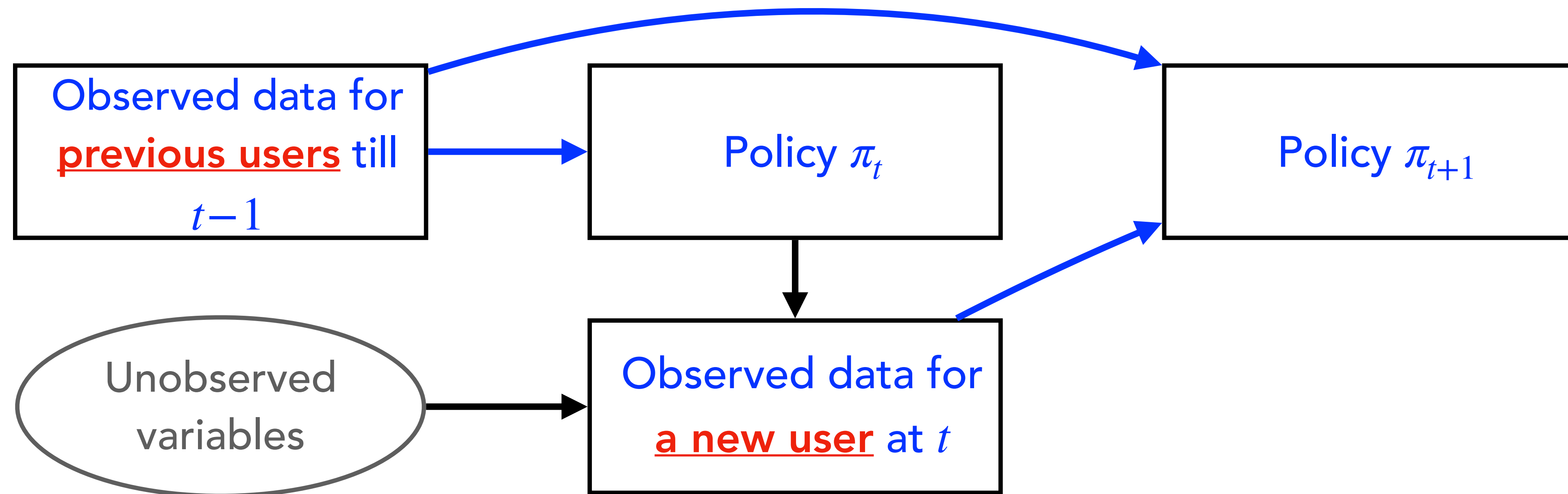
Experiments independent across users

After-study inference:

Policy evaluation for bandits

Adaptive policy with **i.i.d. users at each time**, e.g., online ads

[..., Zhang et al 2021, Hadad et al. 2021, Bibaut et al. 2021,...]

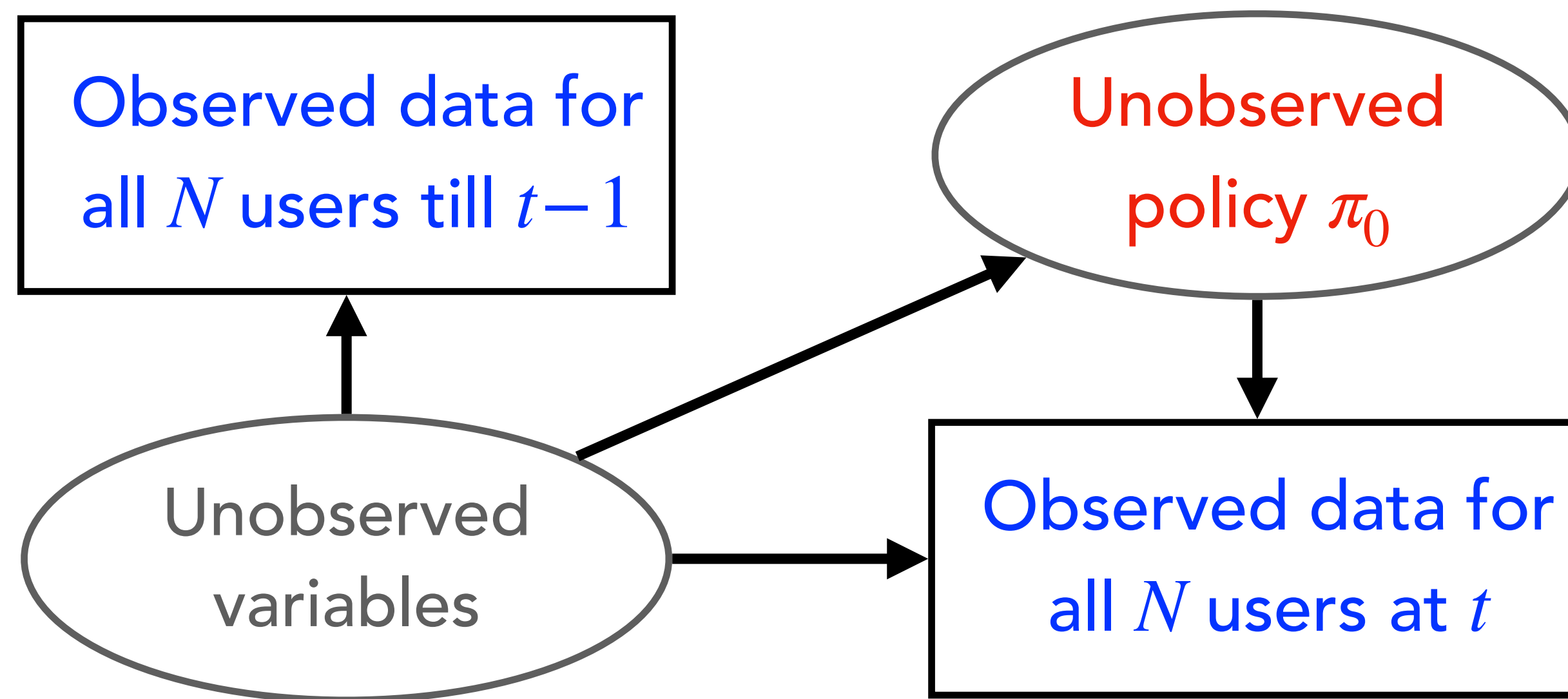


After-study inference:

Causal panel data

Observational settings with multiple users and multiple time, **treatment assigned only once per user (no real use of policy)**

[... Abadie et al. 2003, 2005, Chernozhukov et al. 2017, Athey et al. 2018, Agarwal et a. 2021 ...]



So what do we do?

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Reduce counterfactual inference to **sequential matrix completion**

- Fix treatment say 1, with $Y_{i,t}^\star \triangleq Y_{i,t}^{(1)}$
- $N \times T$ matrix of potential outcomes with missing at random entries

$$Z_{i,t} = \begin{cases} Y_{i,t}^\star + \varepsilon_{i,t} & \text{if } A_{i,t} = 1 \\ \text{unknown} & \text{if } A_{i,t} = 0 \end{cases} \quad \text{where } A_{i,t} = \text{Bernoulli}(\pi_t(i))$$

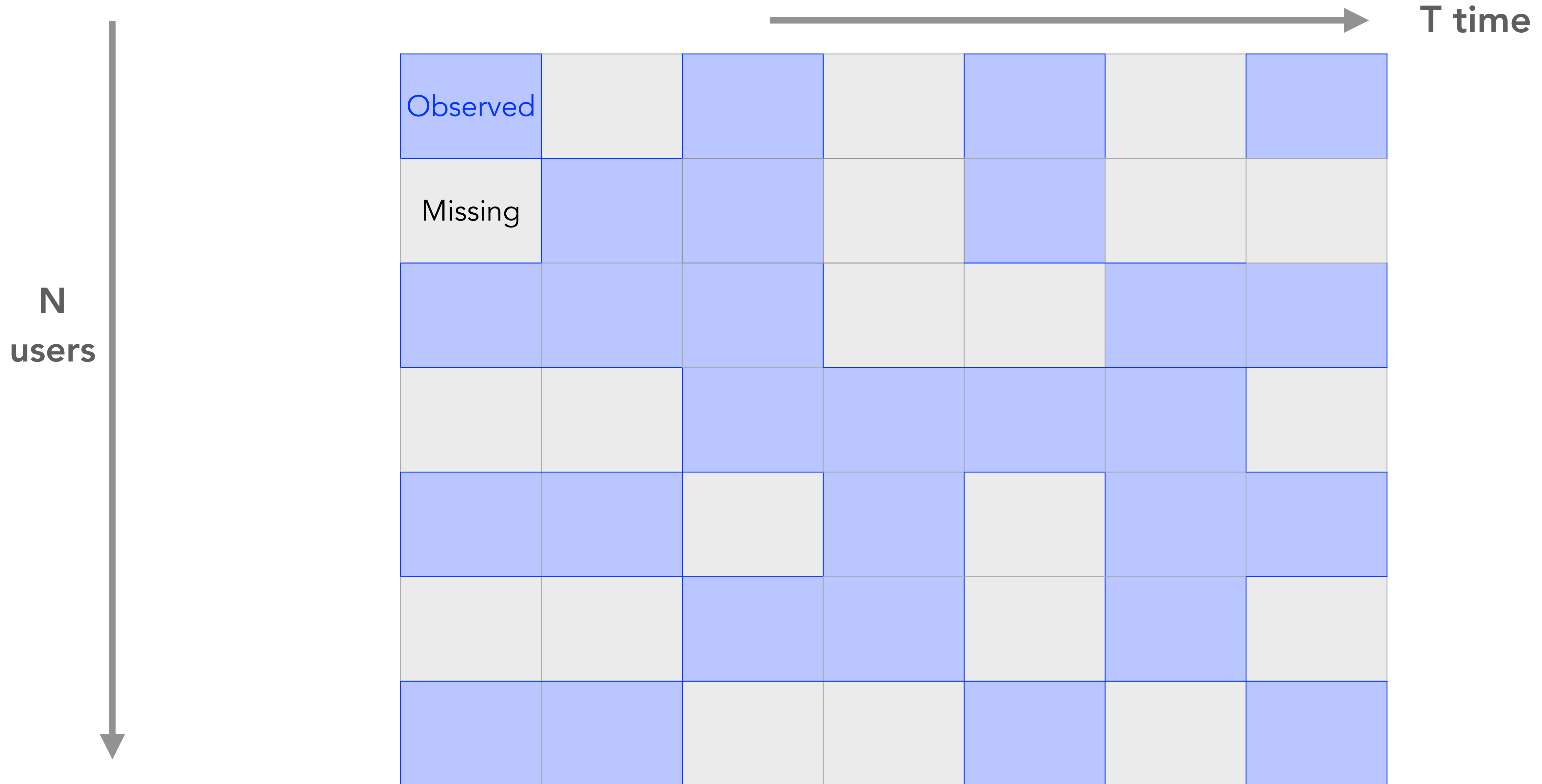
- **New goal:** Estimate missing entries $Y_{i,t}^\star$ (separately for each treatment)

Rest of the talk:

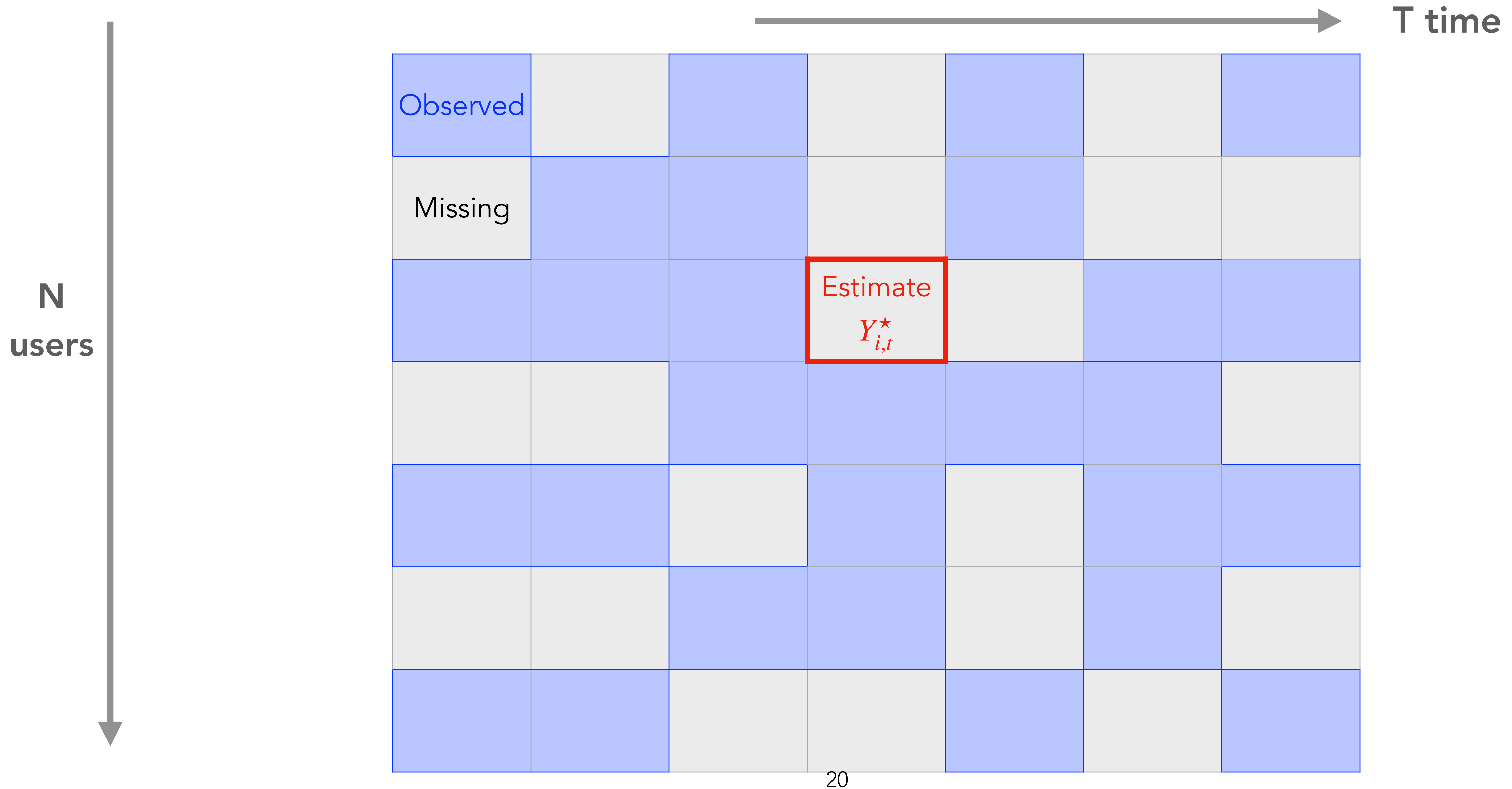
How to estimate $Y_{i,t}^\star$?

When will the estimate work?

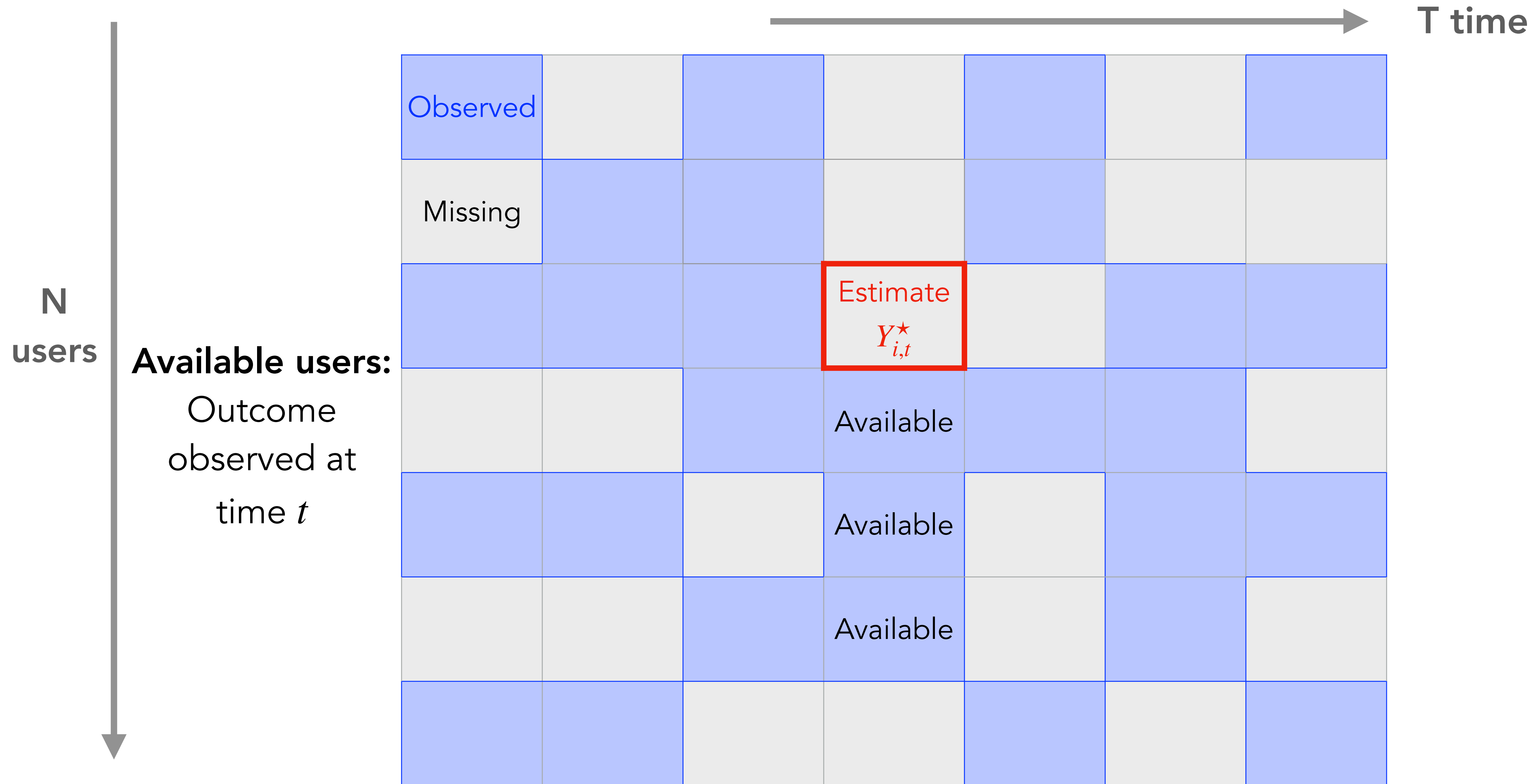
Estimating $Y_{i,t}^\star$ via a variant of nearest neighbors



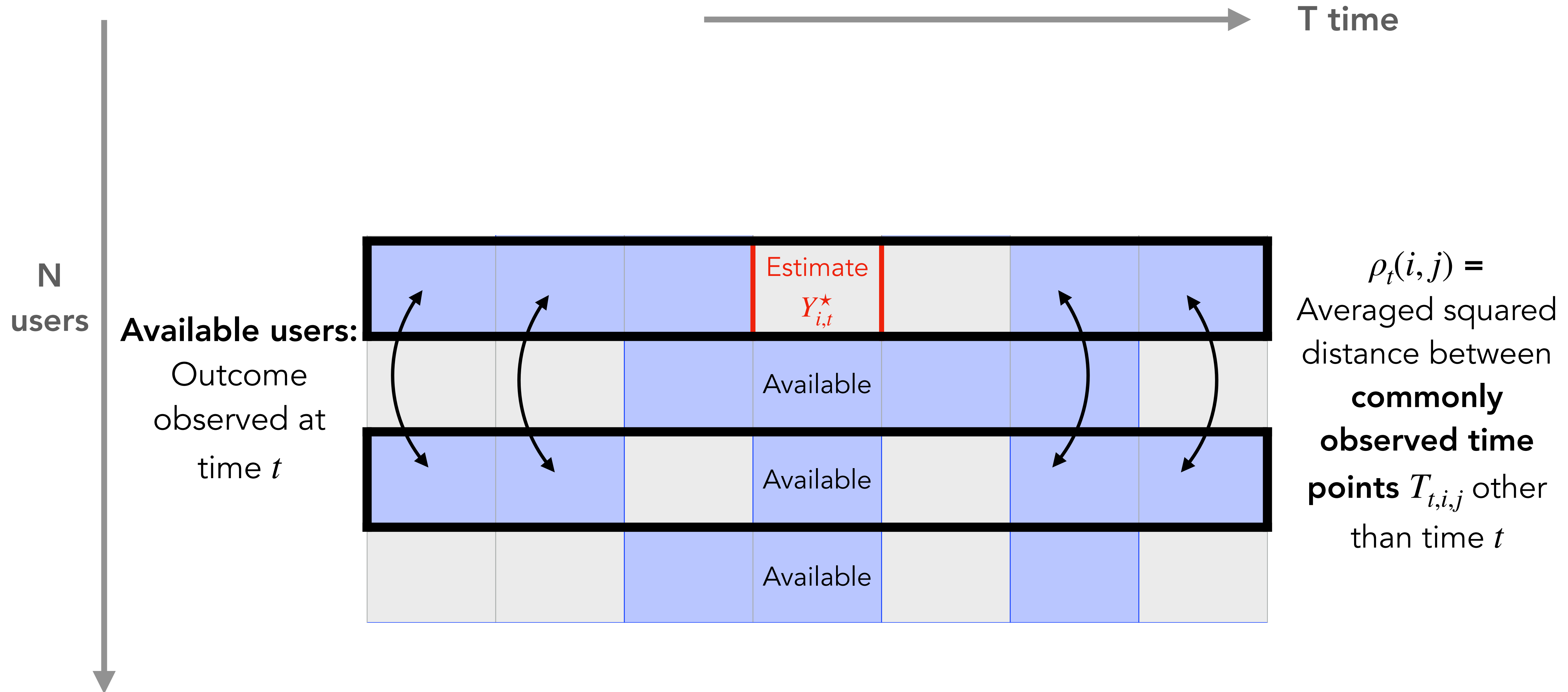
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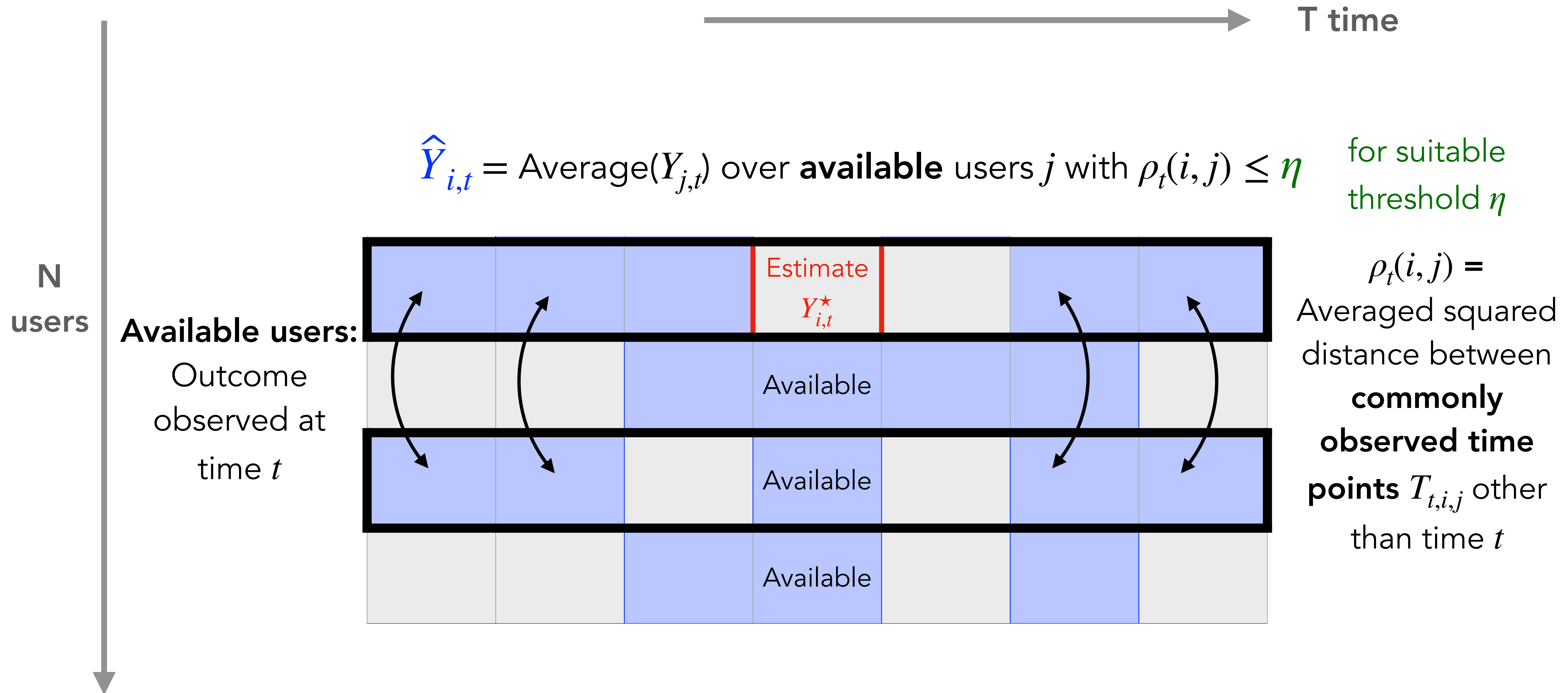
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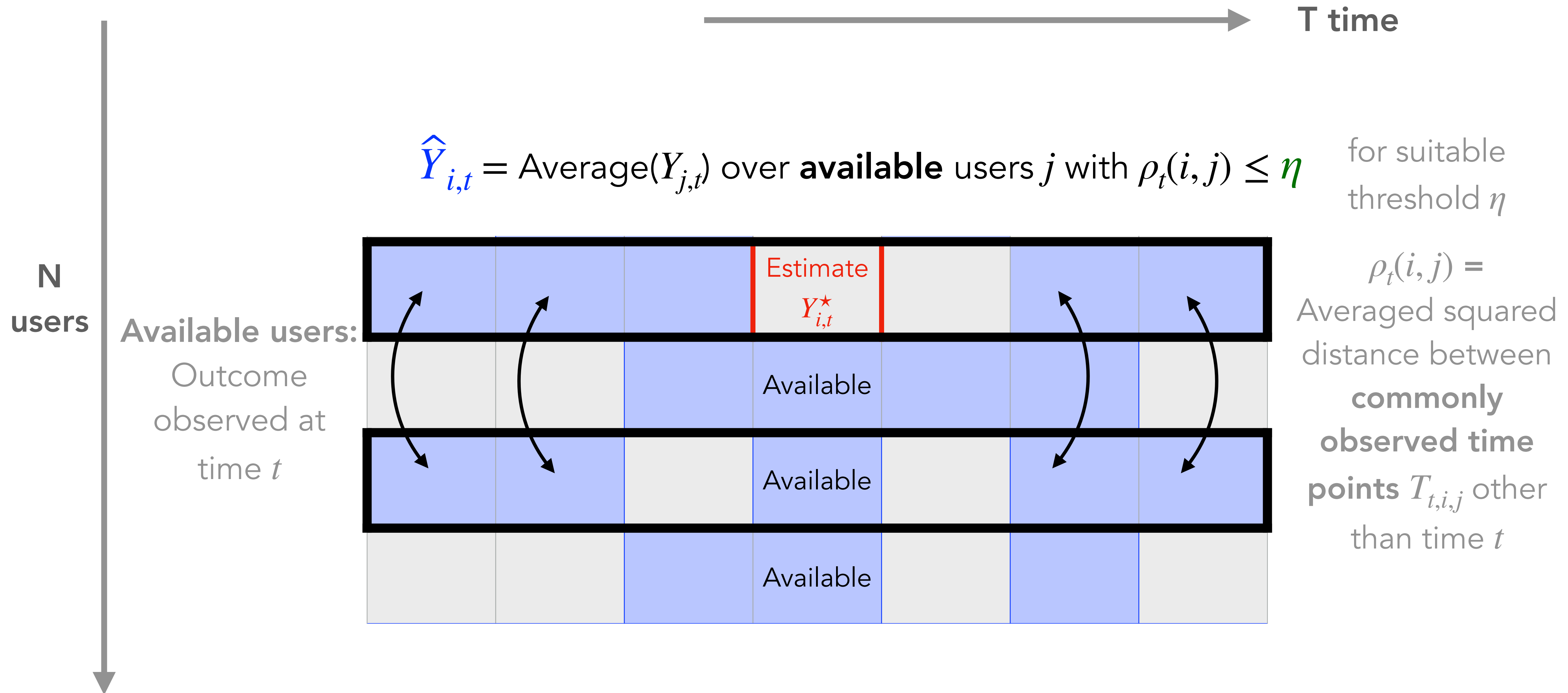
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Estimating $Y_{i,t}^\star$ via a variant of nearest neighbors



Non-parametric estimate: Agnostic to policy and model!



**Will such a non-parametric estimate work for sequential
experimental design?**

Yes, under a latent factor model!
and suitable conditions

Latent factor model

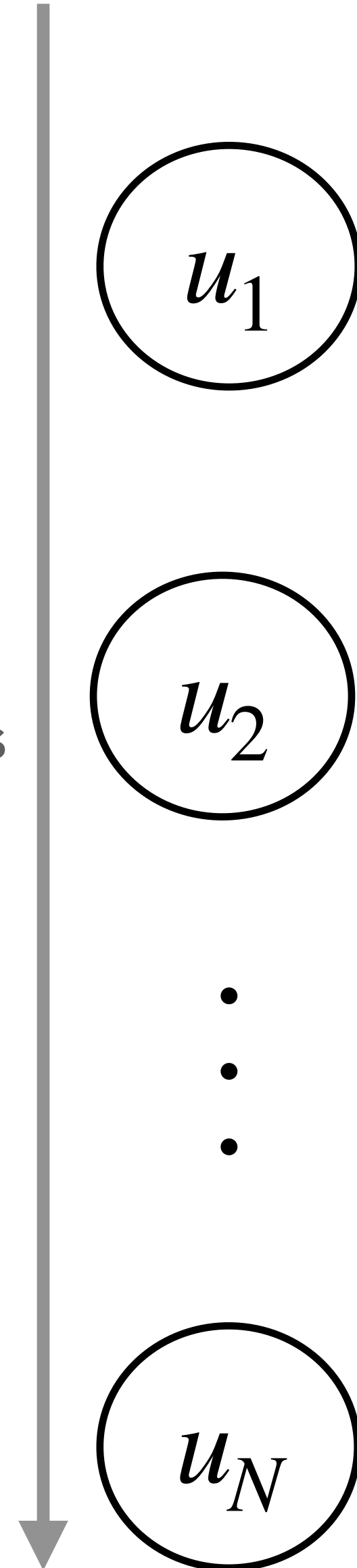
$$Y_{i,t}^\star \triangleq Y_{i,t}^{(1)} \triangleq f(u_i, v_t)$$

u_i : latent factor for user i

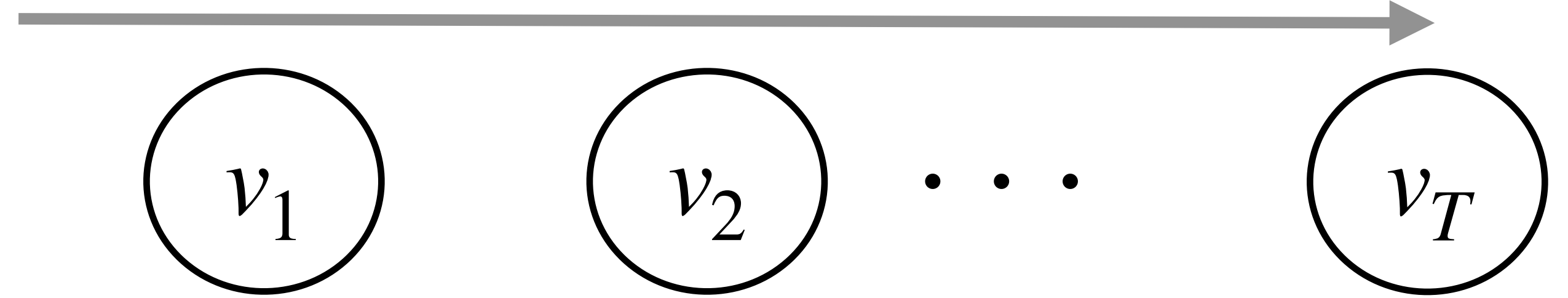
v_t : latent factor for time t

f : unknown (non) linear function

N
user
latent
factors



T time latent factors



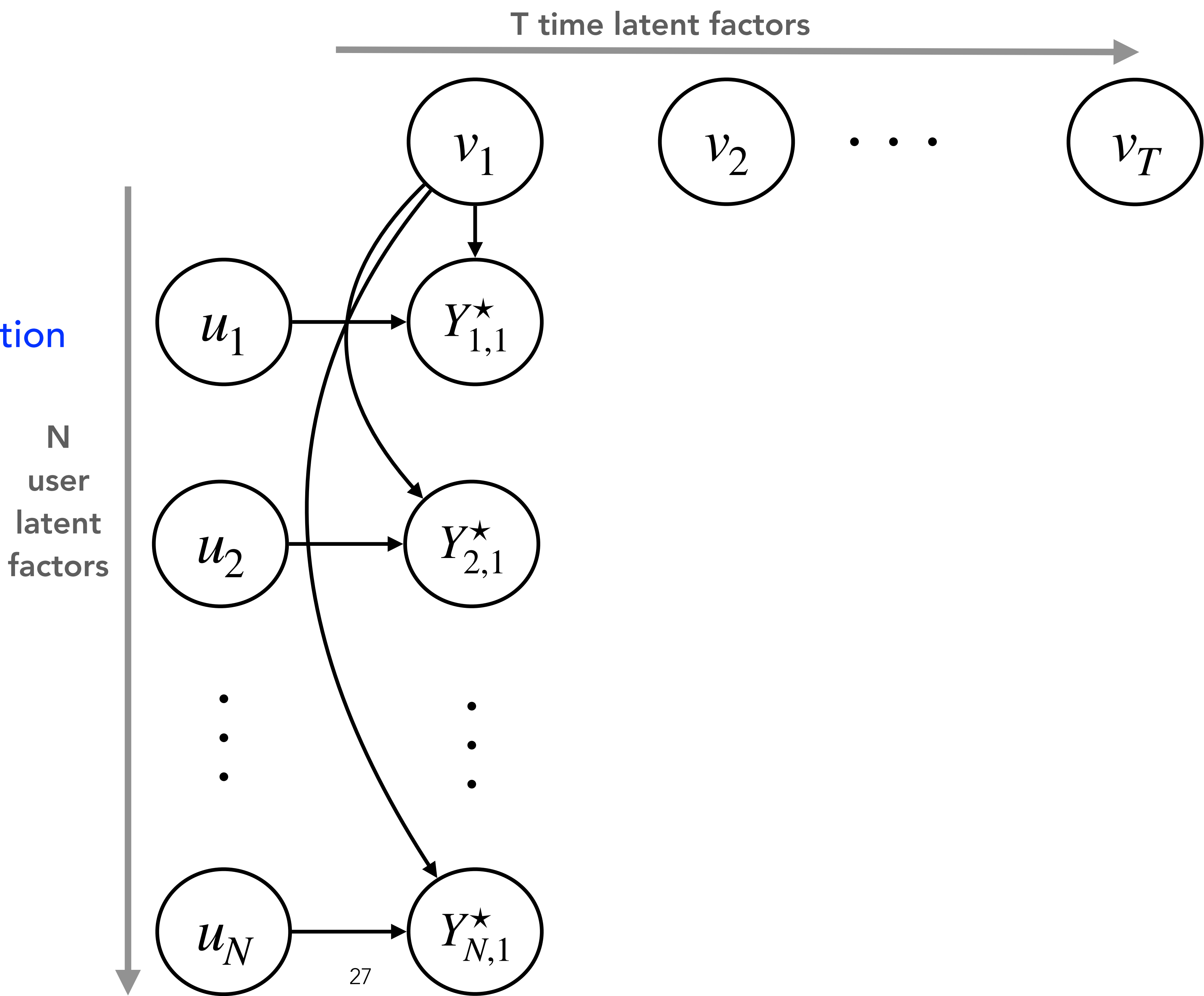
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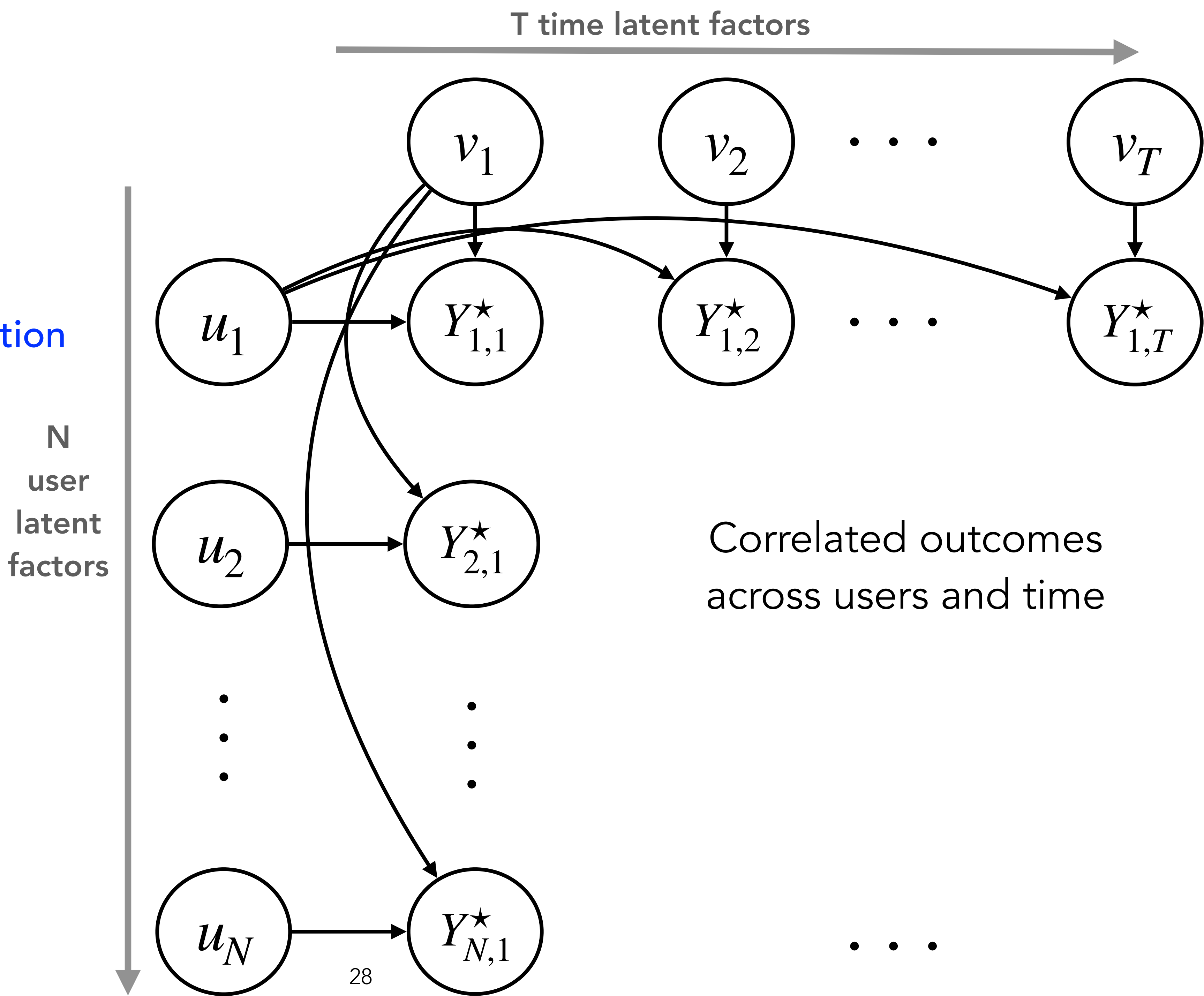
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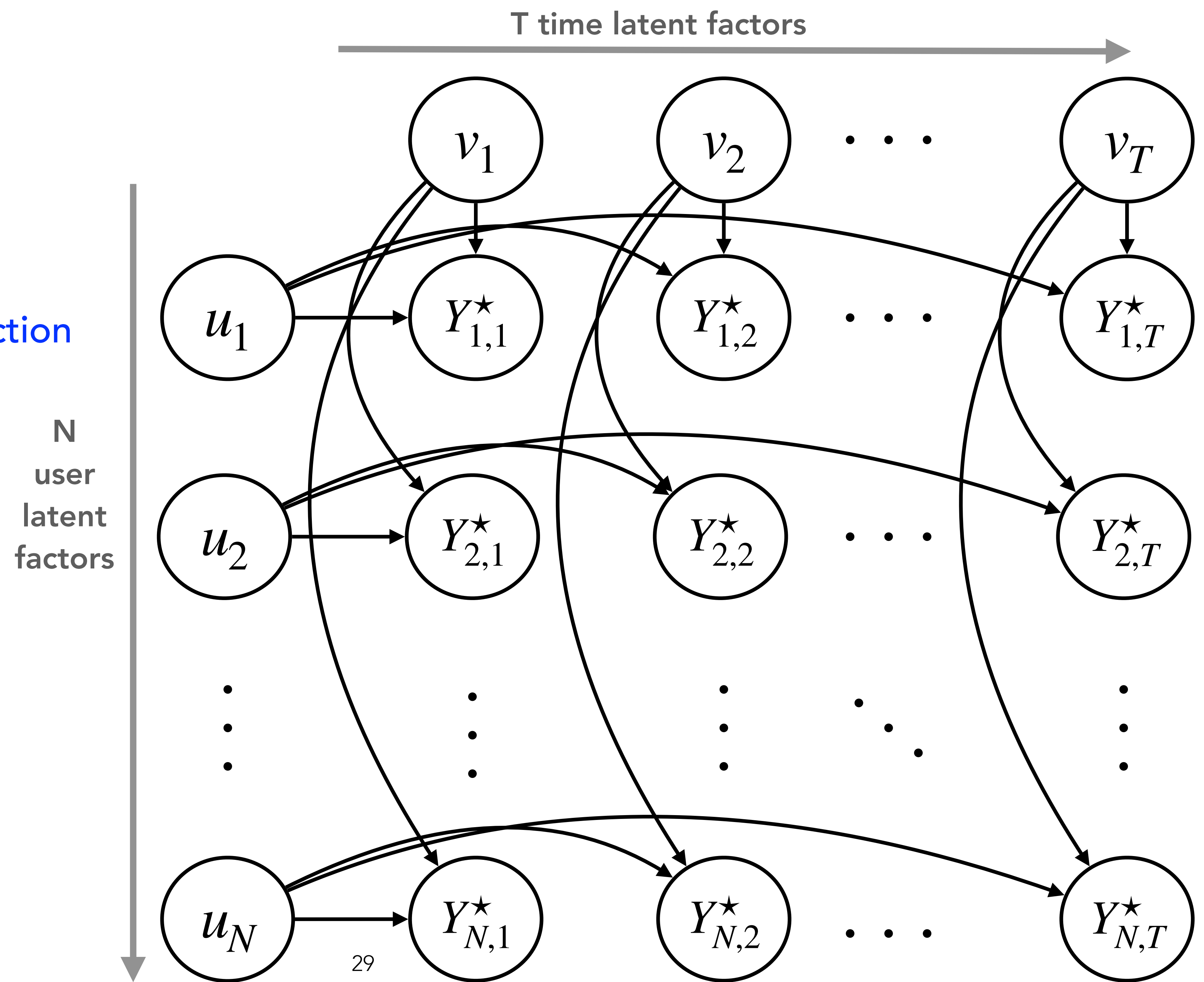
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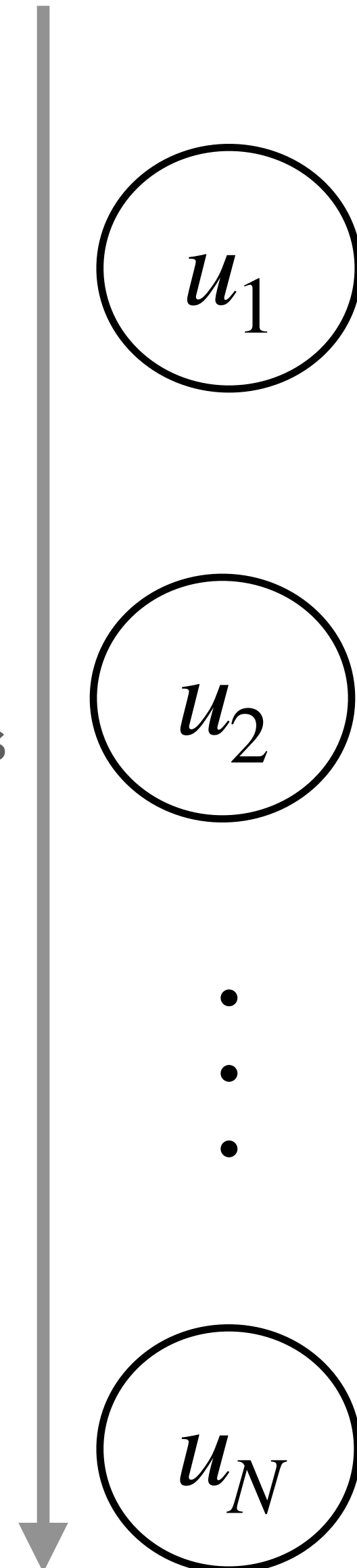
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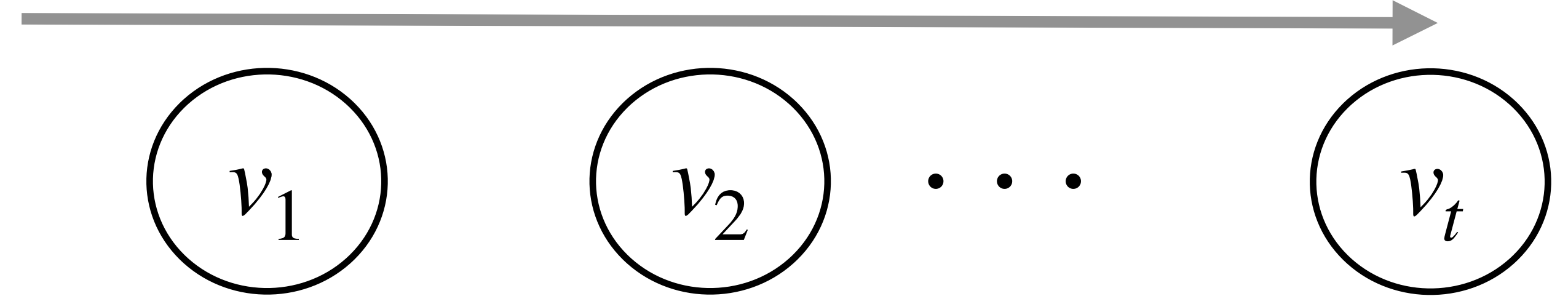
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N
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T time latent factors



- Examples include:

- $Y_{i,\cdot}^\star \sim$ Gaussian process with covariance kernel \mathbf{k}

$u_i =$ Gaussian vector

$v_t =$ Eigenfunctions of \mathbf{k}

$$f(u, v) = \langle u, v \rangle$$

- Also a sub-class of exchangeable data

Theoretical guarantees

$$f(u, v) = \langle u, v \rangle$$

- Consider any (i, t) with enough nearest neighbors j satisfying the conditions

- **“diverse latent-time factors”**: $\frac{1}{T_{t,i,j}} \sum_{t' \in T_{t,i,j}} v_{t'} v_{t'}^\top \succeq \lambda I_d$ for $\lambda > 0$

$T_{t,i,j}$ = commonly observed time points other than t ; used to compute distance $\rho_t(i, j)$

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- **“non-adversarial noise”** across $T_{t,i,j} \cup \{t\}$: behave roughly like iid/mixing process

$f(u, v) = \langle u, v \rangle$: A distribution-free consistency for any (i, t)

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- **“diverse latent-time factors”**: $\frac{1}{T_{t,i,j}} \sum_{t' \in T_{t,i,j}} v_{t'} v_{t'}^\top \geq \lambda I_d$ for $\lambda > 0$

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- **“non-adversarial noise”** across $T_{t,i,j} \cup \{t\}$: behave roughly like iid/mixing process
- For suitable scaling of threshold η & mild conditions on arbitrarily dependent policy, given user i with latent factor u_i , we have

$$\hat{Y}_{i,t} \rightarrow Y_{i,t}^\star \text{ for any } t \text{ as } N, T \rightarrow \infty$$

$f(u, v) = \langle u, v \rangle$: Advantage of bilinearity

Key intuition:

If latent-time factors $\{v_t\}$ “diverse enough”, and noise “non-adversarial”

$$\rho_t(i, j) \leq \eta \quad \implies \quad \|u_i - u_j\|_2 \lesssim \sqrt{\frac{\eta}{\lambda}} \quad \implies \quad |Y_{i,t}^\star - Y_{j,t}^\star| \lesssim \|v_t\|_2 \cdot \sqrt{\frac{\eta}{\lambda}}$$

Lipschitz f

Consider a **non-linear Lipschitz** f , and suppose

- $u_j \sim_{iid} \mathbb{P}_{user}, v_t \sim_{iid} \mathbb{P}_{time},$
- $\epsilon_{j,t} \sim_{iid} \mathbb{P}_{noise}, \mathbb{E}[\epsilon_{j,t}] = 0, \mathbb{E}[\epsilon_{j,t}^2] = \sigma^2$
- π_t depends on *all users'* history till $t - 1$; treatments $\{A_{j,t}\}$ assigned independently given the history

Lipschitz f : Central limit theorem for sequential estimation of $Y_{i,T}^\star$

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Under regularity conditions, given any user i at last time T with *number of neighbors $N_{i,T}$*

$$\sqrt{N_{i,T}}(\hat{Y}_{i,T} - Y_{i,T}^\star) \Rightarrow \mathcal{N}(0, \sigma^2) \quad \text{as } N, T \rightarrow \infty \text{ together at suitable rates}$$

Lipschitz f : Non-asymptotic expected squared error bound

$$\mathbb{E} \left[(\hat{Y}_{i,T} - Y_{i,T}^\star)^2 \mid u_i \right] \lesssim (\eta - 2\sigma^2) + \frac{D^2(1 + \gamma_{i,T})}{p_{\min,T}^2 \sqrt{T-1}} + \frac{\sigma^2}{p_{\min,T} \Phi_i N}$$

Bias due
to η

Concentration of
neighbor distance

Effective noise
variance

Lipschitz f : Non-asymptotic expected squared error bound

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$$p_{\min,T} = \min_{t,j} \pi_t(j)$$

min probability of sampling any entry

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$$p_{\min,T} = \min_{t,j} \pi_t(j) \quad \text{min probability of sampling any entry}$$

$$\Phi_i = \mathbb{P}_u \left(\mathbb{E}_v [f(u_i, v) - f(u, v)]^2 \leq \eta/2 - \sigma^2 \right) \quad \text{Probability of sampling a nearest neighbor}$$

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$$\gamma_{i,T} = \sup_{j \neq i, t < T} \left| \mathbb{E} \left[\sum_{t'=t+1}^{T-1} A_{i,t'} A_{j,t'} \mid \text{history}_t \right] - \mathbb{E} \left[\sum_{t'=t+1}^{T-1} A_{i,t'} A_{j,t'} \mid \text{history}_{t-1} \right] \right|$$

Cumulative future dependency of adaptive policies on one column

Lipschitz f : Non-asymptotic expected squared error bound

$$\mathbb{E} \left[(\hat{Y}_{i,T} - Y_{i,T}^\star)^2 \mid u_i \right] \lesssim (\eta - 2\sigma^2) + \frac{D^2(1 + \gamma_{i,T})}{p_{\min,T}^2 \sqrt{T-1}} + \frac{\sigma^2}{p_{\min,T} \Phi_i N}$$

Bias due
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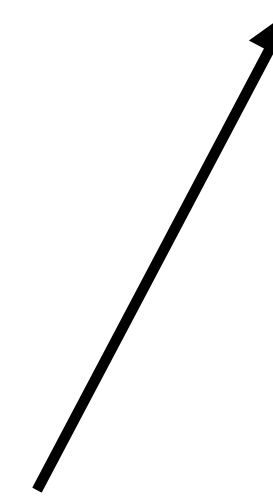
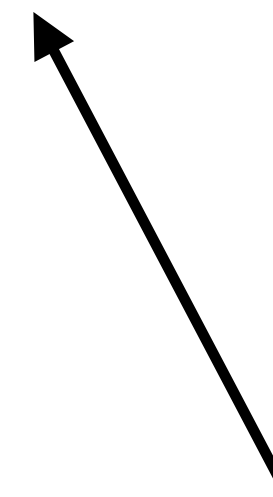
Concentration of
neighbor distance

Effective noise
variance

- Compared to Li et al 2017's guarantee for matrix completion with missing completely at random (MCAR) entries—an additional factor of $\gamma_{i,T}/p_{\min,T}$

Regularity conditions needed for the central limit theorem

$$\mathbb{E} \left[(\hat{Y}_{i,T} - Y_{i,T}^\star)^2 \mid u_i \right] \lesssim (\eta - 2\sigma^2) + \frac{D^2(1 + \gamma_{i,T})}{p_{\min,T}^2 \sqrt{T-1}} + \frac{\sigma^2}{p_{\min,T} \Phi_i N}$$



“Bias” terms go to zero after multiplying by number of neighbors $N_{i,t}$:

- $\eta - 2\sigma^2$ goes to zero fast enough
- $N_{i,t}$ can **not** grow faster than $p_{\min,T}^2 \sqrt{T}$ (cap number of nearest neighbors)

The denominator (min number of nearest neighbors) goes to ∞

Summary:

Counterfactual inference in sequential experimental design

For $t = 1, \dots, T$
For $i = 1, \dots, N$
 $A_{i,t} = \text{Bernoulli}(\pi_t(i))$
 $Z_{i,t} = \begin{cases} Y_{i,t}^\star + \varepsilon_{i,t} & \text{if } A_{i,t} = 1 \\ \text{unknown} & \text{if } A_{i,t} = 0 \end{cases}$
Update policy to π_{t+1} using all users' data

Policy & model
agnostic nearest
neighbor estimate
 $\hat{Y}_{i,t}$ for $Y_{i,t}^\star$

+

Latent factor model
 $Y_{i,t}^\star = f(u_i, v_t)$

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Bilinear f



Distribution-free **consistency**

$$\hat{Y}_{i,t} \rightarrow Y_{i,t}^\star$$

for any user i & any time t

Lipschitz non-
 linear f



Central limit theorem

$$\sqrt{N_{i,T}} (\hat{Y}_{i,T} - Y_{i,T}^\star) \Rightarrow \mathcal{N}(0, \sigma^2)$$

for any user i at last time T

Summary:

Counterfactual inference in sequential experimental design

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Central limit theorem

$$\sqrt{N_{i,T}} (\hat{Y}_{i,T} - Y_{i,T}^\star) \Rightarrow \mathcal{N}(0, \sigma^2)$$

for any user i at last time T

Just a starting point...Future work:

- Impact on downstream tasks
- Use of covariates/contexts
- Temporal structure
- Information across treatments

Preprint on arxiv coming today evening! (Title same as this slide's title)

Additional slides

Explicit non-asymptotic bound for the bilinear case

$f(u, v) = \langle u, v \rangle$: A deterministic error bound for any (i, t)

- $T_{t,i,j}$ = commonly observed time points between i and j other than t
(used to compute distance $\rho_t(i, j)$)
- $N_{i,t}$ = nearest neighbors for (i, t)
- Deterministic error bound:

$$(\hat{Y}_{i,t} - Y_{i,t}^\star)^2 \lesssim \frac{\|v_t\|_2^2}{\lambda} \left(\eta - c\sigma^2 + \frac{\langle \text{noise}, \{v_t\} \rangle}{\min_{j \in N_{i,t}} T_{t,i,j}} \right) + \left(\frac{\sum_{j \in N_{i,t}} \text{noise}_{j,t}}{|N_{i,t}|} \right)^2$$

Gaussian process as a bilinear latent factor model

Latent factor model

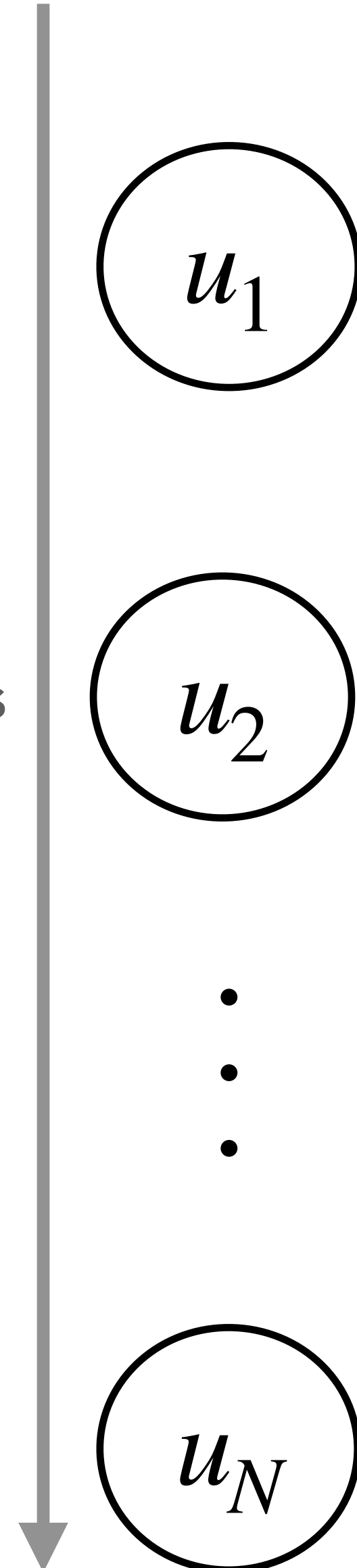
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u_i : latent factor for user i

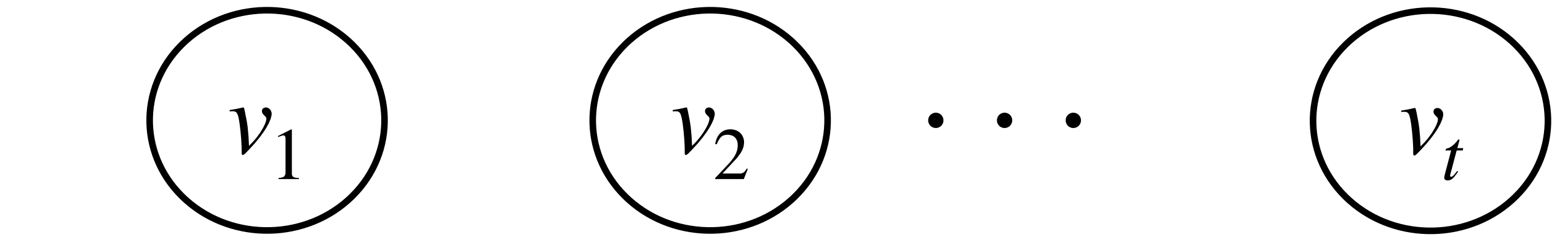
v_t : latent factor for time t

f : unknown (non) linear function

user
latent
factors



time latent factors



- $Y_{i,\cdot}^\star \sim$ Gaussian process with mean m in the reproducing kernel Hilbert space of the covariance kernel \mathbf{k} , which has eigenfunctions ϕ_j then

$$f(u, v) = \langle u, v \rangle,$$

$$u_i = (\langle m, \phi_j \rangle)_{j=1}^r + \mathcal{N}(0, I_r), \text{ and}$$

$$v_t = (\phi_j(x_t))_{j=1}^r.$$

Gaussian process as a latent factor model

- If each user's data is a sample from $\mathcal{GP}(0, \mathbf{k})$ where \mathbf{k} is Mercer's kernel such that

$$\mathbf{k}(t_1, t_2) = \sum_{\ell=1}^{\infty} \lambda_{\ell} \phi_{\ell}(t_1) \phi_{\ell}(t_2),$$

where $\lambda_{\ell}, \phi_{\ell}$ denote eigenvalue-eigenfunctions with $\{\phi_{\ell}\}$ orthonormal

- Then for $\xi_{i,\ell} \sim_{iid} \mathcal{N}(0,1)$, we have

$$Y_{i,t} = \sum_{\ell=1}^{\infty} \xi_{i,\ell} \sqrt{\lambda_{\ell}} \phi_{\ell}(t) \text{ almost surely } \implies Y_{i,t} = f(u_i, v_t) = \langle u_i, v_t \rangle$$

for $u_i = (a_1, a_2, \dots) + (\xi_{i,1}, \xi_{i,2}, \dots)$, and $v_t = (\sqrt{\lambda_1} \phi_1(t), \sqrt{\lambda_2} \phi_2(t), \dots)$

Example: Exchangeable data

- The latent factor model also holds if the matrix $\{Y_{i,t}^\star + \varepsilon_{i,t}, i = 1, \dots, N, t = 1, \dots, T\}$ for a sub-class of exchangeable data i.e., exchangeable under row and column permutations
- See Sec II.C Li et al. 2017

Mathematical description of nearest neighbors

Nearest neighbors estimate for $Y_{i,t}^\star$

- **Input:** Partially observed matrix; **Output:** Estimate of noiseless entry (i, t)
- **Algorithm:** Compute
 - Available neighbors at time $t = \{j : A_{j,t} = 1\}$
 - Good neighbors for user i at time $t = \{j : \rho_t(i, j) \leq \eta\}$ where

$$\rho_t(i, j) = \frac{1}{\sum_{t' \neq t} A_{i,t'} A_{j,t'}} \sum_{t' \neq t} (Z_{i,t'} - Z_{j,t'})^2 A_{i,t'} A_{j,t'}$$

- $\hat{Y}_{i,t}$ = Simple average of $Y_{j,t}$ over $\{j : A_{j,t} = 1 \text{ and } \rho_t(i, j) \leq \eta\}$.

Further details about prior work

Data collected with a fixed policy: Off policy evaluation

- Set-up considers either
 - i.i.d. users, e.g., multi-armed bandits
 - or, one user over time, e.g., Markov decision proces
 - but **not** multiple users over multiple time
- **Quantities of interest:** Average reward under alternative policy, estimated using IPW-based estimates, switch estimators etc.

[..., Li et al 2015, Wang et al. 2021, Ma et al. 2021,...]

Diverse mobile health applications

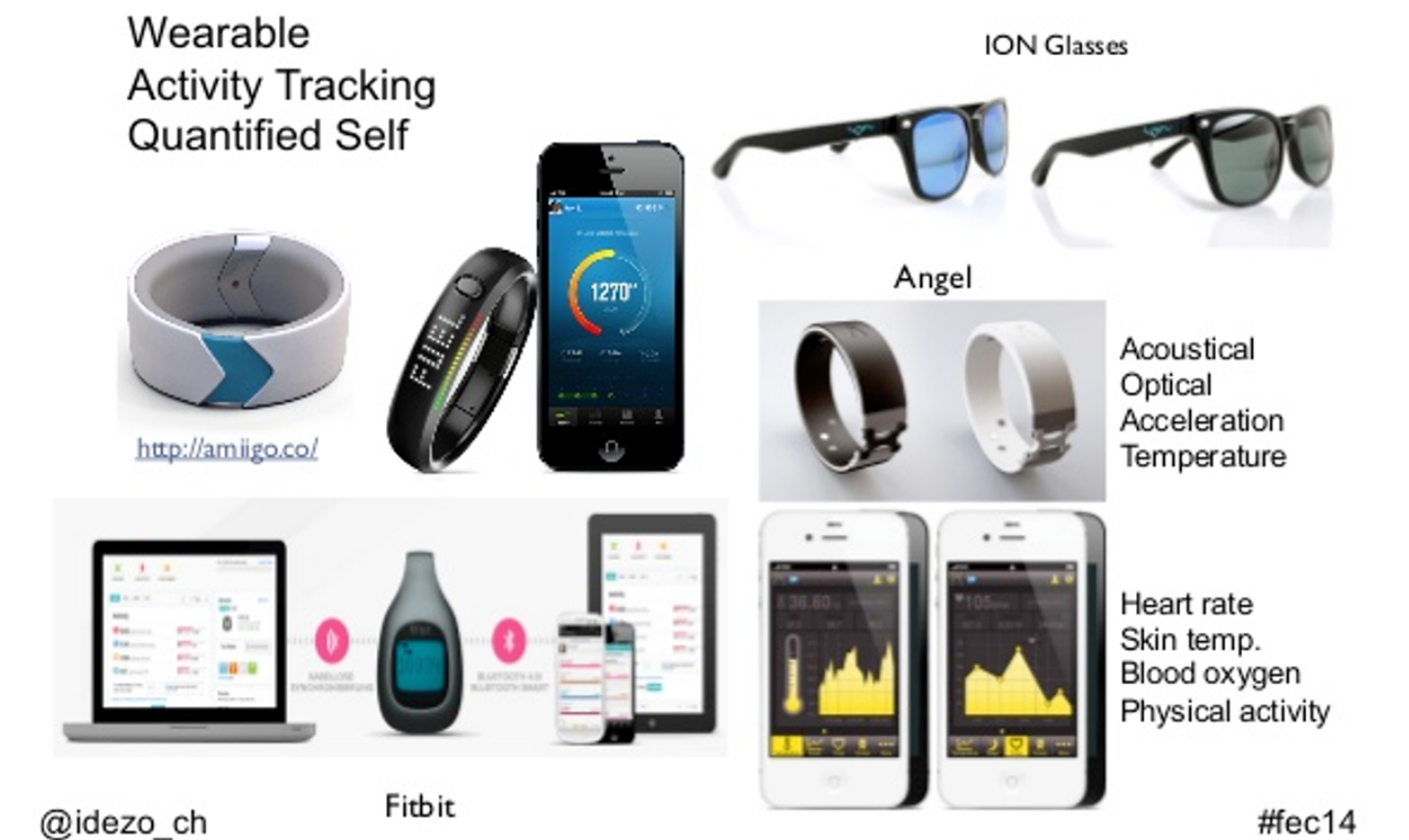
Smoking addiction



Well-being



Wearable/trackers



Binge drinking



Recovery support



Physical activity

