Chapter 1: Introduction

1 Numerical Analysis

Definition

Numerical analysis is **a field of study** focused on developing and analyzing methods for solving mathematical problems that are difficult or impossible to solve exactly using traditional algebraic techniques. It involves studying the **accuracy, stability, and efficiency** of numerical methods.

2 Numerical Algorithm (Numerical Method)

Numerical Algorithm

A Numerical Algorithms are practical techniques used to perform **numerical computations**. providing an approximate solution and is used to solve problems involving large amounts of data. The algorithm is based on a well-defined iterative sequence, starting with an initial solution that progressively converges toward the desired result with each iteration.

$$\begin{cases} (U_n) & n \geq 0 \\ S_0 & \text{Initial Solution (Starting Point)} \end{cases}$$

3 Convergence Speed (Order of Convergence)

Convergence Speed

The number of iterations required to find the solution we are looking for:

- Linear Order: 1 (slow)
- Quadratic Order: 2 (faster)
- >> 2 (very fast)

4 Interpolation

Interpolation

Estimates the value between two known points of a function allowing for a smoother representation of the function's behavior.

5 Approximation

Approximation

Approximates the formula of a function from a set of values, with the objective of finding a simpler function that represents the general trend of the data, even if it doesn't pass through every point exactly.

6 Error

Error

An error represents the difference between the actual solution and the computed result. It indicates how far we are from the true solution. There are two cases:

- **Evaluation**: We know the exact solution, so we can directly calculate the error:

$$\boxed{E_r = |\overline{x} - x_{\text{app}}|}$$

- **Estimation**: We don't know the exact solution, so we only have an estimate of the error, based on the output of the algorithm:

$$\boxed{E_r = |\overline{x} - x_{\text{app}}| \leq \text{Algo}}$$

Where:

- E_r : The error value.
- $\overline{x}$: The exact solution.
- x_{app} : The approximate solution.
- Algo : The error value found by the algorithm.

7 Optimization

Optimization

Optimization in numerical algorithms refers to two things:

- **Error**: We aim to minimize the error in order to achieve the most accurate approximate solution.
- **Convergence Speed**: The higher the order of convergence, the less time the algorithm will take to converge to the solution we are looking for.

Chapter 2: Approached Resolution of Non-Linear Equation $f(x) = 0$

1 $f(x) = 0$ Equation

$f(x) = 0$

Let $\alpha \in \mathbb{R}$ such that $f(\alpha) = 0$.
Graphically, $f(x) = 0$ means that the function f intersects the x-axis.
In numerical methods, we approximate α using a sequence (x_i) such that:

$$\alpha \approx x_i, \quad \text{for some } i$$

Note

We use numerical methods only if the equation's degree is greater than 2 or if it is non-trivial, such as $3x - e^{-x} = 0$. For linear or quadratic equations, exact methods are sufficient, and numerical methods are not needed.

2 Sequence

Sequence

Each numerical algorithm (method) has their specific sequence that must convegre to the deseried solution

3 Initial Steps Of Any Numerical Algorithm

Initial Steps

Every numerical algorithm must first go through two crucial steps:

1. Determine the number of solutions.
2. Define the range of each solution within a closed, bounded, and continuous interval $[a, b]$.

Note

- **Terminal Phase:** In numerical analysis, we aim for real numerical values rather than exact expressions in fractional or functional form.
- **Differences Between Algorithms:** The main difference between numerical algorithms lies in how they approximate and compute solution values for each interval identified in the initial step.
- **Exercises That Do Not Mention the Initial Steps:** Even if an exercise does not explicitly state the need for the initial steps, we must always perform them, as they are crucial regardless of the algorithm used.

3.1 Finding the Number of Solutions

Number of Solutions

To determine the number of solutions, we first analyze the **monotonicity** of the function and construct its **variation table**. Then, we apply the **Intermediate Value Theorem (IVT)** corollary to determine the number of roots:

If $f : I \to \mathbb{R}$ is **continuous** on the **closed and bounded interval** $[a, b]$:

- $f(a) \cdot f(b) < 0$:
  - **No monotonicity** $\to$ At least one root exists.
  - **Strictly monotonic** $\to$ Exactly one root exists.
- $f(a) \cdot f(b) > 0$:
  - 0 or even number of solution

3.2 Range for Each Solution

Range

Sometimes, key points can be found by differentiating. If not, we can reduce the range by testing values, ideally until the interval length reaches 1 for better performance.

Example :

$$f(x) = e^{-x} - \ln(x) = 0$$

Finding Intervalle Of Definition

- e^{-x} is defined in $\mathbb{R}$
- $ln(x)$ is defined in $]0, +\infty[$

$$\boxed{D_f = \mathbb{R} \ \cap \ ]0, +\infty[\ = \ ]0, +\infty[}$$

Differentiate

Since f is the sum of two functions that are continuous and differentiable on $D_f \Rightarrow f$ is also continious and differentiable on D_f.

$$f'(x) = -e^{-x} - \frac{1}{x} = -(e^{-x} + \frac{1}{x})$$

Since $e^{-x} > 0$ and $\frac{1}{x} > 0 \quad \forall \quad x \in D_f =]0, +\infty[\quad \Rightarrow \quad e^{-x} + \frac{1}{x} > 0.$

Thus, $f'(x) < 0$ for all $x \in D_f$, meaning f' is strictly negative and f is strictly decreasing on D_f (Montonic).

Variation Table

x	0	1	2	$+\infty$
$f'(x)$		$-$		
$f(x)$	$+\infty$	$\searrow$		$-\infty$

$f(1) \approx 0.36 > 0$, $f(2) \approx -0.55 < 0$

IVT

f is continuous and strictly montonic in $I = [1, 2]$, and $f(1) \cdot f(2) < 0 \Rightarrow$ Exactly one root

4 Algorithm (Building The Sequence)

4.1 Dichotomy (Bisection)

Condition

This method is based on the **Intermediate Value Theorem (IVT)** and therefore requires the function f to be continuous. Additionally, we need a closed and bounded interval $[a, b]$ that contains a **single** root $\alpha \in [a, b]$.

4.1.1 Order Of Convergence

Order

The bisection method has a **linear order of convergence** : 1 , meaning the error decreases at a constant rate in each iteration.

4.1.2 Guaranteed Convergence

Guarantee

Even though the bisection method is slow, it always guarantees to converge to the desired solution.

4.1.3 Sequence

Sequence

For each iteration, we divide the interval $[a_n, b_n]$ into two equal sub-intervals, where $[a_0, b_0] = [a, b]$ and the midpoint is given by:

$$\boxed{x_n = \frac{a_n + b_n}{2} \quad , \quad \forall n \geq 0}$$

For each iteration, we determine the correct sub-interval for the next step:

- If $f(a_n) \cdot f(x_n) < 0$, then $\alpha \in [a_n, x_n]$, so we set:

$$[a_{n+1}, b_{n+1}] = [a_n, x_n]$$

- If $f(b_n) \cdot f(x_n) < 0$, then $\alpha \in [x_n, b_n]$, so we set:

$$[a_{n+1}, b_{n+1}] = [x_n, b_n]$$

4.1.4 Error Estimation

Error Estimation

$$\boxed{E_n = |\alpha - x_n| \leq \frac{b-a}{2^{n+1}} \ , \ \ \forall n \geq 0}$$

4.1.5 Tolerance

Tolerance

Tolerance is a fixed value set by the user to ensure that the error does not exceed a predefined bound, denoted by ϵ.

$$\boxed{E_n = |\alpha - x_n| \leq \frac{b-a}{2^{n+1}} \leq \epsilon \ , \ \ \forall n \geq 0}$$

4.1.6 Number Of Iterations

Number Of Iterations

$$\epsilon \geq \frac{b-a}{2^{n+1}}$$

$$\frac{2^{n+1}}{b-a} \geq \frac{1}{\epsilon}$$

$$2^{n+1} \geq \frac{b-a}{\epsilon}$$

$$\ln(2^{n+1}) \geq \ln\left(\frac{b-a}{\epsilon}\right)$$

$$(n+1)\ln(2) \geq \ln\left(\frac{b-a}{\epsilon}\right)$$

$$n \geq \frac{\ln\left(\frac{b-a}{\epsilon}\right)}{\ln(2)} - 1$$

$$\boxed{n = \left\lceil \frac{\ln\left(\frac{b-a}{\epsilon}\right)}{\ln(2)} - 1 \right\rceil}$$

4.1.7 Solution Intervalle

Solution Intervalle

$$|\alpha - x_n| \leq \epsilon$$

$$-\epsilon \leq \alpha - x_n \leq \epsilon$$

$$\boxed{x_n - \epsilon \leq \alpha \leq \epsilon + x_n}$$

Note

- The number of iterations is affected by the length of the interval $[a, b]$. By convention, it is better to choose an interval of length 1 for better performance.
- Length of an interval $[a, b]$: $b - a$.
- Midpoint (bisection) of an interval $[a, b]$: $\dfrac{a+b}{2}$.
- Some exercises may not provide the problem directly, requiring us to model it.

Example

approximate the value of $\sqrt[3]{80}$ with the tolerance $\epsilon = 10^{-1}$

Modelisation Of The Problem

$$x = \sqrt[3]{80} \Rightarrow x^3 = 80 \Rightarrow \boxed{f(x) = x^3 - 80}$$

Finding Intervalle Of Definition

$$\text{Since } f \text{ is a polynomial function} \Rightarrow \boxed{D_f = \mathbb{R}}$$

Differentiate

Since f is a polynomial function, it's continuous and differentiable on D_f.

$$f'(x) = 3x^2$$
$$f'(0) = 0$$

Since $f'(x) \geq 0$ on $D_f \Rightarrow f$ is increasing on D_f (monotonic)

Variation Table

x	$-\infty$		0		4	5		$+\infty$
$f'(x)$		$+$	0		$+$			
$f(x)$	$-\infty$			$\nearrow$				$+\infty$

$f(0) = -80$, which means that $\alpha \in [0, +\infty[$. We are going to shorten the interval by testing the points 4 and 5: $f(4) = -16$ and $f(5) = 45$.

Intermediate Value Theorem (IVT)

Since f is continuous and strictly monotonic in $I = [4, 5]$, and $f(4) \cdot f(5) < 0$, $\Rightarrow$ There exists exactly one root in the interval $[4, 5]$.

Number Of Iteration

$$n = \left\lceil \frac{\ln\left(\dfrac{b-a}{\epsilon}\right)}{\ln(2)} - 1 \right\rceil = \left\lceil \frac{\ln\left(\dfrac{5-4}{10^{-1}}\right)}{\ln(2)} - 1 \right\rceil = \boxed{\lceil 2.3 \rceil = 3}$$

Bisection

Iterration 1

$$x_0 = \frac{a_0 + b_0}{2} = \frac{4+5}{2} = \frac{9}{2} = 4.5$$

$$f(x_0) = f(4.5) = 11.125$$

$$f(x_0).f(a_0) < 0 \Rightarrow [a_1, b_1] = [a_0, x_0] = [4, 4.5]$$

Iterration 2

$$x_1 = \frac{a_1 + b_1}{2} = \frac{4+4.5}{2} = \frac{17}{4} = 4.25$$

$$f(x_1) = f(4.25) \approx -3.2$$

$$f(x_1).f(b_1) < 0 \Rightarrow [a_2, b_2] = [x_1, b_1] = [4.25, 4.5]$$

Iterration 3

$$x_2 = \frac{a_2 + b_2}{2} = \frac{4.5+4.25}{2} = \frac{35}{8} = 4.375$$

$$f(x_2) = f(4.375) \approx 3.74$$

$$f(x_2).f(a_2) < 0 \Rightarrow [a_3, b_3] = [a_2, x_2] = [4.25, 4.375]$$

Iterration 4

$$x_3 = \frac{a_3 + b_3}{2} = \frac{4.25+4.375}{2} = \frac{69}{16} = 4.3125$$

Error Estimation

$$E_3 = \frac{b-a}{2^{3+1}} = \frac{5-1}{2^4} = \frac{1}{16} = \boxed{0.0625 < \epsilon = 10^{-1}}$$

Solution Interval

$$x_3 - \epsilon \leq \alpha \leq x_3 + \epsilon$$
$$\boxed{4.3125 - 10^{-1} \leq \alpha \leq 4.3125 + 10^{-1}}$$

Conclusion

$$\boxed{\alpha = \sqrt[3]{80} \approx x_3 = 4.3125}$$

4.2 Fixed Point φ

Fixed Point

Let f be a continuous function on $[a, b]$ and $\exists!\alpha \in [a, b]$ such that:

$$f(\alpha) = 0 \iff \varphi(\alpha) = \alpha.$$

This method consists of transforming the equation $f(x) = 0$ into:

$$\boxed{\varphi(x) = x}$$

Our goal is to find a suitable function φ related to f , we will learn about the conditions on φ in the next section. Graphically, this corresponds to the intersection of φ with the first bisector $y = x$.

Note

- A function f may have more than one possible choice for φ.
- Each root of f corresponds to a different function φ.

Example

$$f(x) = e^x - 2x - 1 = 0$$

$$\begin{gathered} e^x - 1 = 2x \\ e^x = 2x + 1 \\ \boxed{x = \ln(2x + 1) = \varphi_1} \end{gathered}$$

$$\begin{gathered} -2x - 1 = -e^x \\ -2x = -e^x + 1 \\ \boxed{x = \frac{e^x - 1}{2} = \varphi_2} \end{gathered}$$

4.3 Conditions for Choosing the Right φ

Conditions

As seen in the previous example, a function f can have more than one possible φ. The necessary conditions for choosing a suitable φ are:

- **Stability**
- **Contraction**

4.3.1 Stability

Stability

Let φ be defined on $[a, b]$, φ is stable if and only if :

$$\varphi([a, b]) \subseteq [a, b]$$

To check whether this is true, we need the table of variations of φ. By differentiating it (φ'), we can determine the critical points and analyze the behavior of φ. Then, we need to verify if the maximum and minimum values of φ on $[a, b]$ lie within $[a, b]$.

4.3.2 Contraction

Contraction

Let φ be defined on $[a, b]$, φ is contracted if $\exists k \in]0, 1[$ such that

$$|\varphi(x) - \varphi(y)| \leq k \cdot |x - y| \ \ \forall x, y \in [a, b]$$

To check constraction of φ we use the following result :
If φ is defined in $[a, b]$ and $\varphi \in C^1$:

$$\sup_{x \in [a,b]} |\varphi'(x)| < 1 \Longrightarrow \text{Contractive}$$

$$\sup_{x \in [a,b]} |\varphi'(x)| \geq 1 \Longrightarrow \text{Not Contractive}$$

We need to study variation of φ' to retrieve the min and max in $[a, b]$ to get the sup value.

4.4 Sequence

Sequence

Let φ be continuous, stable, and contractive on $[a, b]$, we have:

$\exists! \alpha \in [a, b]$ solution of $\varphi(x) = x$.

$\forall x_0 \in [a, b]$, the sequence x_n is defined $\forall n \geq 0$ by:

$$x_{n+1} = \varphi(x_n), \text{ which converges to } \alpha.$$

4.5 Error Estimation

Error Estimation

$$\boxed{|x_n - \alpha| \leq \frac{k^n}{1-k} \cdot |x_1 - x_0| \quad \forall n \geq 0}$$

4.6 Tolerance

Tolerance

$$\boxed{|x_n - \alpha| \leq \frac{k^n}{1-k} \cdot |x_1 - x_0| \leq \epsilon \quad \forall n \geq 0}$$

4.6.1 Number Of Iterations

Number Of Iterations

$$\epsilon \geq \frac{k^n}{1-k} \cdot |x_1 - x_0|$$

$$k^n \cdot |x_1 - x_0| \leq \epsilon \cdot (1-k)$$

$$k^n \leq \frac{\epsilon \cdot (1-k)}{|x_1 - x_0|}$$

$$\ln(k^n) \leq \ln\left(\frac{\epsilon \cdot (1-k)}{|x_1 - x_0|}\right)$$

$$n \cdot \ln(k) \leq \ln\left(\frac{\epsilon \cdot (1-k)}{|x_1 - x_0|}\right)$$

$$n \geq \frac{\ln\left(\frac{\epsilon \cdot (1-k)}{|x_1 - x_0|}\right)}{\ln(k)}$$

$$\boxed{n = \left\lceil \frac{\ln\left(\frac{\epsilon \cdot (1-k)}{|x_1 - x_0|}\right)}{\ln(k)} \right\rceil}$$

4.7 Proof

4.7.1 Existence & Uniqueness Of Solution α

Existence & Uniqueness

Existence:

Let the function h be continuous on $[a, b]$: $h(x) = \varphi(x) - x$. We have:

$$\begin{cases} h(a) = \varphi(a) - a \geq 0 \\ h(b) = \varphi(b) - b \leq 0 \end{cases}$$

From the Intermediate Value Theorem (IVT), h has at least one root $\alpha \in [a, b]$ such that $h(\alpha) = 0 \Longrightarrow \varphi(\alpha) = \alpha$.

Uniqueness:

Suppose there exists another root $\beta \in [a, b]$. Using the contraction property for α and β, we get:

$$|\varphi(\beta) - \varphi(\alpha)| \leq k \cdot |\beta - \alpha|$$
$$|\beta - \alpha| \leq k \cdot |\beta - \alpha|$$

Contradiction, since $k \in]0, 1[$, then:

$$|\beta - \alpha| > k \cdot |\beta - \alpha|$$

Therefore, the solution is unique.

4.7.2 Convergence to α

Convergence

Since φ is stable and continuous on $[a, b]$, we have:

$$\forall n \geq 0, \quad x_n \in [a, b]$$

Assume that x_n converges to some limit $l \in [a, b]$.

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \varphi(x_n)$$
$$l = \varphi(\lim_{n\to\infty} x_n)$$
$$\boxed{l = \varphi(l)}$$

Thus, $l = \alpha$.

Now, we will prove that x_n indeed converges to the fixed point α. Since φ is contractive, we obtain:

$$|x_n - \alpha| = |\varphi(x_{n-1}) - \varphi(\alpha)| \leq k \cdot |x_{n-1} - \alpha|$$
$$|x_{n-1} - \alpha| = |\varphi(x_{n-2}) - \varphi(\alpha)| \leq k \cdot |x_{n-2} - \alpha|$$
$$k \cdot |x_{n-1} - \alpha| \leq k \cdot k \cdot |x_{n-2} - \alpha|$$
$$k \cdot |x_{n-1} - \alpha| \leq k^2 \cdot |x_{n-2} - \alpha|$$
$$|x_n - \alpha| \leq k \cdot |x_{n-1} - \alpha| \leq k^2 \cdot |x_{n-2} - \alpha| \leq \cdots \leq k^n \cdot |x_0 - \alpha|$$
$$\boxed{0 \leq |x_n - \alpha| \leq k^n \cdot |x_0 - \alpha|}$$

Taking the limit as $n \to \infty$, we note that $k^n \to 0$ since $k < 1$, therefore:

$$\lim_{n\to\infty} |x_n - \alpha| = 0$$
$$\boxed{\lim_{n\to\infty} x_n = \alpha}$$

Thus, x_n converges to α.

4.7.3 Error

Error

Since φ is contractive, we obtain:

$$|x_{n+1} - x_n| = |\varphi(x_n) - \varphi(x_{n-1})| \leq k \cdot |x_n - x_{n-1}|$$

$$|x_{n+1} - x_n| \leq k \cdot |x_n - x_{n-1}| \leq k^2 \cdot |x_{n-1} - x_{n-2}| \leq \cdots \leq k^n \cdot |x_1 - x_0|$$

$$\boxed{|x_{n+1} - x_n| \leq k^n \cdot |x_1 - x_0|} \qquad (1)$$

Let $m \gg n$:

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - \cdots - x_{n+1} + x_{n+1} - x_n|$$

Using the triangle inequality $|a + b| \leq |a| + |b|$, we get:

$$\boxed{|x_m - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|} \qquad (2)$$

Replacing (1) in (2):

$$\begin{aligned} |x_m - x_n| &\leq k^{m-1} \cdot |x_1 - x_0| + k^{m-2} \cdot |x_1 - x_0| + \cdots + k^n \cdot |x_1 - x_0| \\ &\leq [k^{m-1} + k^{m-2} + \cdots + k^n] \cdot |x_1 - x_0| \\ &\leq k^n \cdot \left[\sum_{j=0}^{m-n-1} k^j\right] \cdot |x_1 - x_0| \end{aligned}$$

Using the geometric series formula for N terms with $r \neq 1$:

$$S = a + a \cdot r + a \cdot r^2 + \cdots + a \cdot r^{N-1}$$

$$S = a \cdot \frac{1 - r^N}{1 - r}$$

Setting $N = m - n$:

$$|x_m - x_n| \leq k^n \cdot \frac{1 - k^{m-n}}{1 - k} \cdot |x_1 - x_0|$$

As $m \to \infty$, x_m converges to α and k^{m-n} converges to 0:

$$\boxed{|x_n - \alpha| \leq \frac{k^n}{1 - k} \cdot |x_1 - x_0|}$$

Note

When a function φ does not satisfy the required conditions, there are two possible cases:

- **Incorrect interval** $[a, b]$**:** The function φ may have issues at one or both endpoints. In such cases, choosing a slightly smaller interval $[a', b']$ can resolve the problem.
- **Incorrect function** φ**:** The issue may lie not in the interval but in the function itself when $\inf_{x \in [a,b]} |\varphi'(x)| \geq 1$.

Sometimes, there may be multiple functions φ that satisfy all the conditions on $[a, b]$. In such cases, we choose the one with the smallest contraction factor, as it ensures faster convergence:

$$\min \left(\sup_{x \in [a,b]} |\varphi_n'(x)| \right)$$

Example:

$$f(x) = x + 1 - \frac{x \cdot e^x}{3} \quad , \quad D_f = \mathbb{R} \quad , \quad \epsilon = 10^{-4}$$

Differentiation

f is continuous and differentiable on D_f.

$$f'(x) = 1 - \frac{1}{3}(e^x + x \cdot e^x)$$

$$\boxed{f'(x) = 1 - \frac{e^x}{3}(1 + x)}$$

Since f' is ambiguous, we can't study its sign; therefore, we need to analyze f''. f' is continuous and differentiable on D_f.

$$f''(x) = -\frac{1}{3}(e^x + e^x + x \cdot e^x)$$

$$\boxed{f''(x) = \frac{-e^x}{3}(2 + x)}$$

$$\boxed{f''(-2) = 0}$$

Variation Table

x	$-\infty$		-2		β		$+\infty$
$f''(x)$		$+$	0		$-$		
$f'(x)$	1	$\nearrow$	1.045		$\searrow$		$-\infty$
$f'(x)$			$+$		0	$-$	
$f(x)$	$-\infty$		$\nearrow$		$f(\beta)$	$\searrow$	$-\infty$

We evaluate $f(x)$ at specific points starting from -2:

$$f(-2) = -0.90 < 0, \quad f(-1) = 0.1226 > 0, \quad f(1) = 1.09 > 0, \quad f(2) = -1.92 < 0$$

By the **Intermediate Value Theorem (IVT)**, since $f(x)$ is monotonic continious and changes sign in the intervals $[-2, -1]$ and $[1, 2]$, there exist two real roots:

$$\boxed{\alpha_1 \in [-2, -1], \quad \alpha_2 \in [1, 2]}$$

Finding φ Functions

$$x + 1 - \frac{x \cdot e^x}{3} = 0$$

$$\boxed{x = \frac{x \cdot e^x}{3} - 1 = \varphi_1}$$

$$x(1 - \frac{e^x}{3}) + 1 = 0$$

$$x(1 - \frac{e^x}{3}) = -1$$

$$\boxed{x = \frac{1}{\frac{e^x}{3} - 1} = \frac{1}{\frac{e^x - 3}{3}} = \frac{3}{e^x - 3} = \varphi_2}$$

$$\frac{x \cdot e^x}{3} = x + 1$$

$$\boxed{x = \frac{3(x+1)}{e^x} = \varphi_3}$$

$$e^x = \frac{3(x+1)}{x}$$

$$\boxed{x = \ln\left(\frac{3(x+1)}{x}\right) = \varphi_4}$$

Checking φ_1

D_{φ_1}

$$\boxed{D_{\varphi_1} = \mathbb{R}}$$

φ_1'

$$\varphi_1'(x) = \frac{1}{3}(x \cdot e^x + e^x)$$

$$\boxed{\varphi_1'(x) = \frac{e^x}{3}(x+1)}$$

$$\boxed{\varphi_1'(-1) = 0}$$

φ_1''

$$\varphi_1''(x) = \frac{1}{3}(e^x + e^x + x \cdot e^x)$$

$$\boxed{\varphi_1''(x) = \frac{e^x}{3}(x+2)}$$

$$\boxed{\varphi_1''(-2) = 0}$$

Variation Table

x	-2		-1		1		2
$\varphi_1''(x)$	0			$+$			
$\varphi_1'(x)$	-0.045	$\nearrow$	0	$\rightarrow$	1.81	$\rightarrow$	7.98
$\varphi_1'(x)$		$-$	0		$+$		
$\varphi_1(x)$	-1.09	$\searrow$	-1.12	$\nearrow$	-0.09	$\rightarrow$	5.9

$\inf\limits_{x\in[1,2]} |\varphi_1'(x)| = 1.81 > 1 \Longrightarrow \varphi_1$ is disqualified for the intervalle [1,2]

$\varphi_1([-2,-1]) \subseteq [-2,-1] \Longrightarrow \varphi_1$ is stable on $[-2,-1]$

$\varphi_1 \in C^1$ and $\sup\limits_{x\in[-2,-1]} |\varphi_1'(x)| = 0.045 < 1 \Longrightarrow \varphi_1$ is contractive on $[-2,-1]$

$$\boxed{\varphi_1 \text{ Corresponds to the root } \alpha_1 \in [-2,-1]}$$

Checking φ_2

D_{φ_2}

$$\boxed{D_{\varphi_2} = \mathbb{R} - \{\ln(3)\}}$$

φ_2'

$$\boxed{\varphi_2'(x) = \frac{-3e^x}{(e^x-3)^2}}$$

$$\boxed{\forall x \in D_{\varphi_2} \quad \varphi_2' < 0}$$

$\underline{\varphi_2''}$

$$\varphi_2''(x) = -3 \cdot \frac{e^x(e^x-3)^2 - e^x(2e^x(e^x-3)^{2-1})}{(e^x-3)^4}$$

$$\varphi_2''(x) = -3 \cdot \frac{e^x(e^x-3)((e^x-3)-2e^x)}{(e^x-3)^4}$$

$$\varphi_2''(x) = -3 \cdot \frac{e^x(-3-e^x)}{(e^x-3)^3}$$

$$\boxed{\varphi_2''(x) = 3 \cdot \frac{e^x(3+e^x)}{(e^x-3)^3}}$$

$$\boxed{\forall x > \ln(3) \quad \varphi_2'' > 0}$$

$$\boxed{\forall x < \ln(3) \quad \varphi_2'' < 0}$$

Variation Table

x	-2	-1	1	$\ln(3)$	2
$\varphi_2''(x)$		$-$		‖	$+$
$\varphi_2'(x)$	-0.04 ↘	-0.15 →	-102.75 →	$-\infty$ ‖ $-\infty$ ↗	-1.15
$\varphi_2'(x)$		$-$		‖	$-$
$\varphi_2(x)$	-1.04 ↘	-1.35 →	-10.6 →	$-\infty$ ‖ $+\infty$ ↘	0.68

$\inf_{x\in[1,\ln(3)[} |\varphi_2'(x)| = 102.75 > 1 \Longrightarrow \varphi_2$ is disqualified for the intervalle $[1, \ln(3)[$

$\inf_{x\in]\ln(3),2]} |\varphi_2'(x)| = 1.15 > 1 \Longrightarrow \varphi_2$ is disqualified for the intervalle $]\ln(3), 2]$

$\varphi_2([-2,-1]) \subseteq [-2,-1] \Longrightarrow \varphi_2$ is stable on $[-2,-1]$

$\varphi_2 \in C^1$ and $\sup_{x\in[-2,-1]} |\varphi_2'(x)| = 0.15 < 1 \Longrightarrow \varphi_2$ is contractive on $[-2,-1]$

$$\boxed{\varphi_2 \text{ Corresponds to the root } \alpha_1 \in [-2,-1]}$$

Both φ_1 and φ_2 satisfy the conidtion on [-2,-1] , we will choose the φ with the smallest contraction factor

$$\min\left(\sup_{x\in[-2,-1]}|\varphi_1'(x)|, \sup_{x\in[-2,-1]}|\varphi_2'(x)|\right) = \min(0.045, 0.15) = 0.045 = \sup_{x\in[-2,-1]}|\varphi_1'(x)|$$

Checking φ_3

D_{φ_3}

$$\boxed{D_{\varphi_3} = \mathbb{R}}$$

φ_3'

$$\varphi_3'(x) = 3 \cdot \frac{e^x - (x+1)\cdot e^x}{e^{2x}}$$

$$\varphi_3'(x) = 3 \cdot \frac{1 - (x+1)}{e^x}$$

$$\boxed{\varphi_3'(x) = \frac{-3x}{e^x}}$$

$$\boxed{\varphi_3'(0) = 0}$$

$$\boxed{\forall x > 0 \quad \varphi_3' < 0}$$

$$\boxed{\forall x < 0 \quad \varphi_3' > 0}$$

φ_3''

$$\varphi_3''(x) = -3 \cdot \frac{e^x - xe^x}{e^{2x}}$$

$$\varphi_3''(x) = -3 \cdot \frac{1-x}{e^x}$$

$$\boxed{\varphi_3''(x) = \frac{3(x-1)}{e^x}}$$

$$\boxed{\varphi_3''(1) = 0}$$

$$\boxed{\forall x > 1 \quad \varphi_3'' > 0}$$

$$\boxed{\forall x < 1 \quad \varphi_3'' < 0}$$

Variation Table

x	-2	-1	0	1	1.75	2
$\varphi_3''(x)$		$-$		0	$+$	
$\varphi_3'(x)$	44.33	8.15	0	-1.1	-0.91	-0.81
$\varphi_3'(x)$		$+$	0	$-$		
$\varphi_3(x)$	22.16	0	3	2.2	1.43	1.21

$\inf_{x\in[-2,-1]} |\varphi_3'(x)| = 8.15 > 1 \Longrightarrow \varphi_3$ is disqualified for the intervalle [-2,-1]

$\inf_{x\in[1,2]} |\varphi_3'(x)| = 0.81 < 1 \Longrightarrow \varphi_3$ is qualified for the intervalle [1,2]

$\varphi_3([1,2]) \nsubseteq [1,2] \Longrightarrow \varphi_3$ is not stable on $[1,2]$

$\varphi_3 \in C^1$ and $\sup_{x\in[1,2]} |\varphi_3'(x)| = 1.1 > 1 \Longrightarrow \varphi_3$ is not contractive on $[1,2]$

The issue is on the extremity 1 , we'll take a new intervalle [1.75,2] :

$\varphi_3([1.75,2]) \subseteq [1.75,2] \Longrightarrow \varphi_3$ is stable on $[1.75,2]$

$\varphi_3 \in C^1$ and $\sup_{x\in[1,2]} |\varphi_3'(x)| = 0.91 < 1 \Longrightarrow \varphi_3$ is contractive on $[1.75,2]$

$$\boxed{\varphi_3 \text{ Corresponds to the root } \alpha_2 \in [1.75,2]}$$

Checking φ_4

D_{φ_4}

$$\boxed{D_{\varphi_4} =]0,+\infty[}$$

$\underline{\varphi_4'}$

$$\varphi_4'(x) = \frac{3 \cdot \left(\frac{x+1}{x}\right)'}{3 \cdot \frac{x+1}{x}}$$

$$\varphi_4'(x) = \frac{\frac{x-(x+1)}{x^2}}{\frac{x+1}{x}}$$

$$\varphi_4'(x) = \frac{\frac{-1}{x}}{x+1}$$

$$\boxed{\varphi_4'(x) = \frac{-1}{x(x+1)}}$$

$$\boxed{\forall x \in D_{\varphi_4} \quad \varphi_4' < 0}$$

$\underline{\varphi_4''}$

$$\boxed{\varphi_4''(x) = \frac{2x+1}{x^2+x}}$$

$$\boxed{\forall x \in D_{\varphi_4} \quad \varphi_4'' > 0}$$

Variation Table

x	0		1.45		1.75		2
$\varphi_4''(x)$	$\Vert$		$+$				
$\varphi_4'(x)$	$\Vert$ $-\infty$	$\nearrow$	-0.28	$\longrightarrow$	-0.2	$\longrightarrow$	-0.16
$\varphi_4'(x)$	$\Vert$		$-$				
$\varphi_4(x)$	$\Vert$ $+\infty$	$\searrow$	1.62	$\longrightarrow$	1.55	$\longrightarrow$	1.5

$\inf\limits_{x\in[1.75,2]} |\varphi_4'(x)| = 0.16 < 1 \Longrightarrow \varphi_4$ is qualified for the intervalle [1.75,2]

$\varphi_4([1.75, 2]) \nsubseteq [1.75, 2] \Longrightarrow \varphi_4$ is not stable on $[1.75, 2]$
$\varphi_4 \in C^1$ and $\sup\limits_{x\in[1.75,2]} |\varphi_4'(x)| = 0.2 < 1 \Longrightarrow \varphi_4$ is contractive on $[1, 2]$

The issue is on the extremity 1.75 , we'll take a new intervalle [1.45,2] :
$\varphi_4([1.45, 2]) \subseteq [1.45, 2] \Longrightarrow \varphi_4$ is stable on $[1.45, 2]$
$\varphi_4 \in C^1$ and $\sup\limits_{x\in[1.45,2]} |\varphi_4'(x)| = 0.28 < 1 \Longrightarrow \varphi_4$ is contractive on $[1.45, 2]$

$$\boxed{\varphi_4 \text{ Corresponds to the root } \alpha_2 \in [1.45, 2]}$$

We choose φ_4 over φ_3 because its contraction factor is smaller

$\underline{\alpha_1}$ **:**

We have $\varphi_1 = \dfrac{x \cdot e^x}{3} - 1$, $\alpha_1 \in [-2, -1]$ and $k = 0.0045$
We take $x_0 = 0$ therfore $x_1 = \varphi_1(x_0) = \varphi_1(0) = -1$

<u>Number Of Iteration :</u>

$$n = \left\lceil \frac{\ln\left(\dfrac{\epsilon \cdot (1-k)}{|x_1 - x_0|}\right)}{\ln(k)} \right\rceil = \left\lceil \frac{\ln\left(\dfrac{10^{-4} \cdot (1-0.045)}{|-1|}\right)}{\ln(0.045)} \right\rceil = \lceil 2.9 \rceil = 3$$

$$x_2 = \varphi_1(x_1) = \varphi_1(-1) = -1.122$$
$$x_3 = \varphi_1(x_2) = \varphi_1(-1.122) = -1.119$$

$\underline{\alpha_2 \text{ :}}$

We have $\varphi_4 = \dfrac{\ln(3(x+1)}{x}$, $\alpha_2 \in [1.45, 2]$ and $k = 0.28$
We take $x_0 = 1.5$ therfore $x_1 = \varphi_4(x_0) = \varphi_4(1.5) = 1.6$

Number Of Iteration :

$$n = \left\lceil \frac{\ln\left(\dfrac{\epsilon \cdot (1-k)}{|x_1 - x_0|}\right)}{\ln(k)} \right\rceil = \left\lceil \frac{\ln\left(\dfrac{10^{-4} \cdot (1-0.28)}{|0.1|}\right)}{\ln(0.28)} \right\rceil = \lceil 5.68 \rceil = 6$$

$$x_2 = \varphi_4(x_1) = \varphi_4(1.6) = 1.584$$
$$x_3 = \varphi_4(x_2) = \varphi_4(1.584) = 1.5879$$
$$x_4 = \varphi_4(x_3) = \varphi_4(1.5879) = 1.5870$$
$$x_5 = \varphi_4(x_4) = \varphi_4(1.587) = 1.58726$$
$$x_6 = \varphi_4(x_5) = \varphi_4(1.58726) = 1.5872$$

4.8 Newton's Method

Newton's Method

Let f be a continuous function on $[a, b]$ with a unique root $\alpha \in [a, b]$ satisfying $f(\alpha) = 0$ and $f'(\alpha) \neq 0$.

Then the Newton iteration is defined for all $n \geq 0$ by $\begin{cases} x_0 \in [a, b] \\ x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \end{cases}$

This method is based on the first-order Taylor expansion of f.

Graphically it corresponds to the intersection of the tangent line to f at $(x_n, f(x_n))$ with the line $y = x$.

4.9 Conditions

Conditions

let f be a function $\in C^2$ on $[a, b]$ that verifies all of the above conditions :

1. $f(a) \cdot f(b) < 0$
2. $f'(x) \neq 0 \ \forall x \in [a, b]$
3. f'' keeps a constant sign in $[a, b]$
4. $\dfrac{|f(c)|}{|f'(c)|} \leq b - a$ where $c = \begin{cases} a & \text{if } |f'(a)| < |f'(b)| \\ b & \text{if } |f'(b)| < |f'(a)| \end{cases}$

4.10 Error Estimation

Error Estimation

$$\boxed{|x_{n+1} - \alpha| \leq \frac{M}{2m} \cdot |x_{n+1} - x_n|^2 \quad \text{where } M = \sup_{x \in [a,b]} |f''(x)| \text{ and } m = \inf_{x \in [a,b]} |f'(x)| \ \ \forall n \geq 0}$$

4.11 Tolerance

Tolerance

$$\boxed{|x_{n+1} - \alpha| \leq \frac{M}{2m} \cdot |x_{n+1} - x_n|^2 \leq \epsilon \quad \text{where } M = \sup_{x \in [a,b]} |f''(x)| \text{ and } m = \inf_{x \in [a,b]} |f'(x)| \;\; \forall n \geq 0}$$

4.11.1 Number of Iterations

Number of Iterations

We can't directly calculate the number of iterations because the error-estimation formula does not explicitly involve n, so we have to compute the error at each iteration until it is greater than ϵ.

Example: