Exercise 1: Primal To Dual

Find the dual for each primal

$$\max\ Z = 2x_1 + 4x_2 + 3x_3 \qquad \max\ Z = 10x_1 + 14x_2 \qquad \max\ Z = 3x_1 + x_2 - 2x_3$$

$$\begin{cases} 3x_1 + 4x_2 + 2x_3 \le 60 \\ 2x_1 + x_2 + 2x_3 \le 40 \\ x_1 + 3x_2 + 2x_3 \le 80 \\ x_1, x_2, x_3 \ge 0 \end{cases} \qquad \begin{cases} x_1 + x_2 \ge 12 \\ x_1 \ge 8 \\ x_2 \le 6 \\ x_1, x_2 \ge 0 \end{cases} \qquad \begin{cases} x_1 + 2x_2 \ge 10 \\ 3x_1 - x_2 + x_3 = 7 \\ x_1 + 3x_2 \le 8 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

Solution

Primal 1

$$\max\ Z = 2x_1 + 4x_2 + 3x_3$$

$$\begin{cases} 3x_1 + 4x_2 + 2x_3 \le 60 \\ 2x_1 + x_2 + 2x_3 \le 40 \\ x_1 + 3x_2 + 2x_3 \le 80 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 4 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 60 \\ 40 \\ 80 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}, \quad A^T = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

Dual 1

$$\min\ Z = 60y_1 + 40y_2 + 80y_3$$

$$\begin{cases} 3y_1 + 2y_2 + y_3 \geq 2 \\ 4y_1 + y_2 + 3y_3 \geq 4 \\ 2y_1 + 2y_2 + 2y_3 \geq 3 \\ y_1, y_2, y_3 \geq 0 \end{cases}$$

Primal 2

$$\max\ Z = 10x_1 + 14x_2$$

$$\begin{cases} x_1 + x_2 \geq 12 \\ x_1 \geq 8 \\ x_2 \leq 6 \\ x_1, x_2 \geq 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} -x_1 - x_2 \leq -12 \\ -x_1 \leq -8 \\ x_2 \leq 6 \\ x_1, x_2 \geq 0 \end{cases}$$

$$X = \begin{bmatrix} x_1 & x_2 \end{bmatrix}, \quad C = \begin{bmatrix} 10 & 14 \end{bmatrix}, \quad b = \begin{bmatrix} -12 \\ -8 \\ 6 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Dual 2

$$\min\ Z = -12y_1 - 8y_2 + 6y_3$$

$$\begin{cases} -y_1 - y_2 \geq 10 \\ -y_1 + y_3 \geq 14 \\ y_1, y_2 \geq 0 \end{cases}$$

Primal 3

$$\max\ Z = 3x_1 + x_2 - 2x_3$$

$$\begin{cases} x_1 + 2x_2 \geq 10 \\ 3x_1 - x_2 + x_3 = 7 \\ x_1 + 3x_2 \leq 8 \\ x_1, x_2, x_3 \geq 0 \end{cases} \implies \begin{cases} -x_1 - 2x_2 \leq -10 \\ 3x_1 - x_2 + x_3 \leq 7 \\ 3x_1 - x_2 + x_3 \geq 7 \\ x_1 + 3x_2 \leq 8 \\ x_1, x_2, x_3 \geq 0 \end{cases} \implies \begin{cases} -x_1 - 2x_2 \leq -10 \\ 3x_1 - x_2 + x_3 \leq 7 \\ -3x_1 + x_2 - x_3 \leq -7 \\ x_1 + 3x_2 \leq 8 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} -10 \\ 7 \\ -7 \\ 8 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -2 & 0 \\ 3 & -1 & 1 \\ -3 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} -1 & 3 & -3 & 1 \\ -2 & -1 & 1 & 3 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

Dual 3

$$\min\ Z = 10y_1 + 7y_2 + -7y_3 + 8y_4$$

$$\begin{cases} -y_1 + 3y_2 - 3y_3 + y_4 \geq 3 \\ -2y_1 - y_2 + y_3 + 3y_4 \geq 1 \\ y_2 - y_3 \geq -2 \\ y_1, y_2, y_3, y_4 \geq 0 \end{cases}$$

Exercise 2: Slackness Theorem

1. Find the dual of the below primal :

$$\min\ Z = 3x_1 + x_2 + x_3$$

$$\begin{cases} x_1 + 2x_2 \geq 8 \\ 3x_1 - x_2 + x_3 \geq 6 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

2. Solve the dual
3. Using the slackness theorem find solution of primal from solution of the dual

Solution

Primal

$$\min\ Z = 3x_1 + x_2 + x_3 \quad \Longrightarrow \quad \max\ -Z = -3x_1 - x_2 - x_3$$

$$\begin{cases} x_1 + 2x_2 \geq 8 \\ 3x_1 - 2x_2 + x_3 \geq 6 \\ x_1, x_2, x_3 \geq 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} -x_1 - 2x_2 \leq -8 \\ -3x_1 + 2x_2 - x_3 \leq -6 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

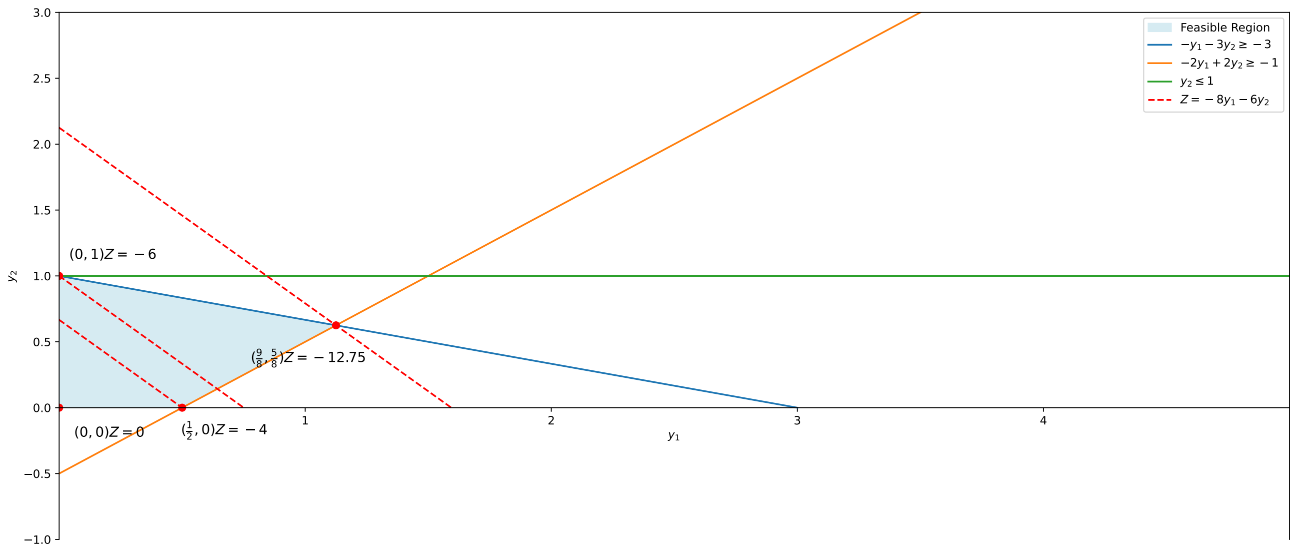
$$X = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}, \quad C = \begin{bmatrix} -3 & -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -8 \\ -6 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -2 & 0 \\ -3 & 2 & -1 \end{bmatrix}, \quad A^T = \begin{bmatrix} -1 & -3 \\ -2 & 2 \\ 0 & -1 \end{bmatrix}$$

Dual

$$\min Z = -8y_1 - 6y_2$$

$$\begin{cases} -y_1 - 3y_2 \geq -3 \\ -2y_1 + 2y_2 \geq -1 \\ -y_2 \geq -1 \\ y_1, y_2 \geq 0 \end{cases}$$



$$\text{solution is } \boxed{(x_1, x_2) = (\frac{9}{8}, \frac{5}{8})}$$

Primal

$X^* = \begin{bmatrix} x_1^* & x_2^* & x_3^* \end{bmatrix}$

$\max\ -Z = -3x_1 - x_2 - x_3$

$$\begin{cases} -x_1 - 2x_2 \leq -8 \\ -3x_1 + 2x_2 - x_3 \leq -6 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Dual

$Y^* = \begin{bmatrix} \frac{9}{8} & \frac{5}{8} \end{bmatrix}$

$\min\ Z = -8y_1 - 6y_2$

$$\begin{cases} -y_1 - 3y_2 \geq -3 \\ -2y_1 + 2y_2 \geq -1 \\ -y_2 \geq -1 \\ y_1, y_2 \geq 0 \end{cases}$$

$$\begin{cases} x_1^*(-(\frac{9}{8}) - 3(\frac{5}{8}) + 3) = 0 \\ x_2^*(-2(\frac{9}{8}) + 2(\frac{5}{8})) + 1 = 0 \\ x_3^*(-(\frac{5}{8}) + 1) = 0 \end{cases} \implies \begin{cases} x_1^*(0) = 0 \\ x_2^*(0) = 0 \\ x_3^*(\frac{3}{8}) = 0 \end{cases} \implies \begin{cases} x_1^* > 0 \\ x_2^* > 0 \\ x_3^* = 0 \end{cases}$$

$$\begin{cases} y_1^*(-8 - x_1^* - 2x_2^*) = 0 \\ y_2^*(-6 - 3x_1^* + 2x_2^* - x_3^*) = 0 \end{cases} \implies \begin{cases} \frac{9}{8}(-8 - x_1^* - 2x_2^*) = 0 \\ \frac{5}{8}(-6 - 3x_1^* + 2x_2^*) = 0 \end{cases} \implies \begin{cases} -8 - x_1^* - 2x_2^* = 0 \\ -6 - 3x_1^* + 2x_2^* = 0 \end{cases}$$

$$\begin{cases} x_1^* + 2x_2^* = 8 \\ 3x_1^* - 2x_2^* = 6 \end{cases} \implies \begin{cases} x_1^* = \frac{7}{2} \\ x_2^* = \frac{9}{4} \end{cases}$$

$$\boxed{X^* = \begin{bmatrix} \frac{7}{2} & \frac{9}{4} & 0 \end{bmatrix}}$$