1 Opration Research

1.1 What's Operations Research?

Definition

Operations Research (**OR**) is an interdisciplinary field that uses mathematical models, statistical analysis, and optimization techniques to help solve complex decision-making problems and achieve the most efficient outcomes. it involves the use of several key techniques, including:

- **Optimization**: Finding the best solution based on given criteria.
- **Mathematical Modeling**: Representing real-world scenarios through mathematical equations and models.
- **Statistical Analysis**: Using data to analyze and predict outcomes.
- **Simulation**: Testing strategies in a controlled model environment.

The goal of Operations Research is to offer data-driven insights and methods that lead to more informed, optimized decisions.

1.2 Origin of Operations Research

Origin

Operations Research (**OR**) originated during World War II, when the British government assembled a team of analysts, scientists, engineers, and military officers to study complex operational problems such as : air defense, bombing strategies, convoy routing, and other crucial military operations . Using mathematical models and data analysis, the team simulated various scenarios to predict outcomes and recommend optimal decisions. The success of these methods in improving military strategy inspired other nations to adopt similar approaches.This eventually led to the formalization of OR as a scientific discipline after the war.

1.3 Types of Problems Treated by OR

Types of Problems

Operations Research (**OR**) focuses on solving real-world problems by finding the most optimal decisions.The types of problems OR addresses can generally be categorized as:

- **Maximization**: Achieving the highest possible value for an objective, such as maximizing profits, productivity, or efficiency.
  - Example: Maximizing a company's revenue by determining the most profitable product mix.
- **Minimization**: Reducing or minimizing undesirable factors, such as costs, time, or resource consumption.
  - Example: Minimizing the cost of materials in manufacturing while maintaining quality standards.
- **Optimization**: Finding the best possible solution from multiple alternatives, often involving both maximization and minimization aspects.
  - Example: Finding the shortest path in a transportation network or optimizing team roles in a project.

1.4 Algorithm Complexity

Algorithm Complexity

Algorithm complexity refers to the amount of computational resources an algorithm uses. These resources are typically categorized as:

- **Time Complexity**: The amount of time it takes for an algorithm to run, depending on the size of the input. It is commonly expressed using Big O notation (e.g., $O(n)$, $O(\log n)$), which describes the algorithm's growth rate as the input size increases.
- **Space Complexity**: The amount of memory or space an algorithm requires. This is influenced by the number and size of variables, data structures, and other memory-using elements.

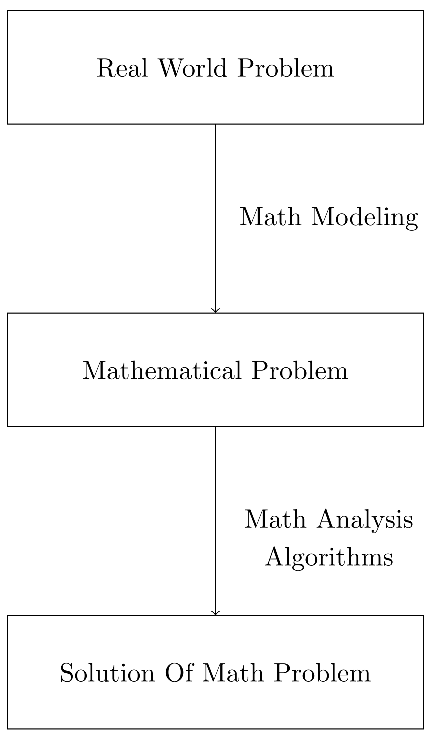
Minimizing both time and space complexity is crucial when developing efficient algorithms, as it leads to more optimal performance, especially for large-scale problems.

2 Linear Programming

2.1 What's Linear Programming ?

Definition

Linear programming is a sub-branch of optimization techniques. It involves modeling real-life problems as linear equations and inequalities and using specialized methods to find optimal solution(s), if they exist.



2.2 Models

2.2.1 Graph Model

This model is used when the objective function has two variables. It consists of converting all inequalities into equalities, drawing them as lines, and then identifying the feasible area where all conditions are met. We then sweep the objective function Z across the plot until we find the optimal solution(s).

Note

Solutions : When solving a linear program, the solution can be:

- **One or Multiple Optimal Solutions**: The feasible area is a polygon, and its vertex points are the possible optimal solutions.
- **Infinitely Many Solutions**: If the solution is unbounded, then the objective function will increase or decrease infinitely as we sweep the line.
- **No Solution**: If the feasible area is empty ($\{\emptyset\}$), it indicates that the system has contradictions.

Direction of Increase/Decrease:

- **Both Positive** ($a > 0$, $b > 0$): Since both coefficients are positive, Z increases as x_1 and x_2 increase, and decreases as they decrease. The direction of increase is towards the right, and the direction of decrease is towards the left.
- **Both Negative** ($a < 0$, $b < 0$): The opposite of the positive case. Here, Z increases as x_1 and x_2 decrease, and decreases as they increase. The direction of increase is towards the left, while the direction of decrease is towards the right.
- **Different Signs** (a and b have opposite signs): The direction is determined by the coefficient with the larger absolute value, $\max(|a|, |b|)$. If this coefficient is positive, the direction of increase follows the same pattern as when both coefficients are positive. If this coefficient is negative, the direction of increase follows the pattern for both negative coefficients.

Example1 : Diet Problem

The Goal is to minimize food cost but to meet the minimum daily nutrition requirement

Food	Units	Protein	Vit c	Iron	Price
Apples	1 med	0.4	6	0.4	8
Banana	1 med	1.2	10	0.6	10

Variables Definition:

Let x_1 be the number of Daily Unit Appels.

Let x_2 be the number of Daily Unit Banana.

Constraint:

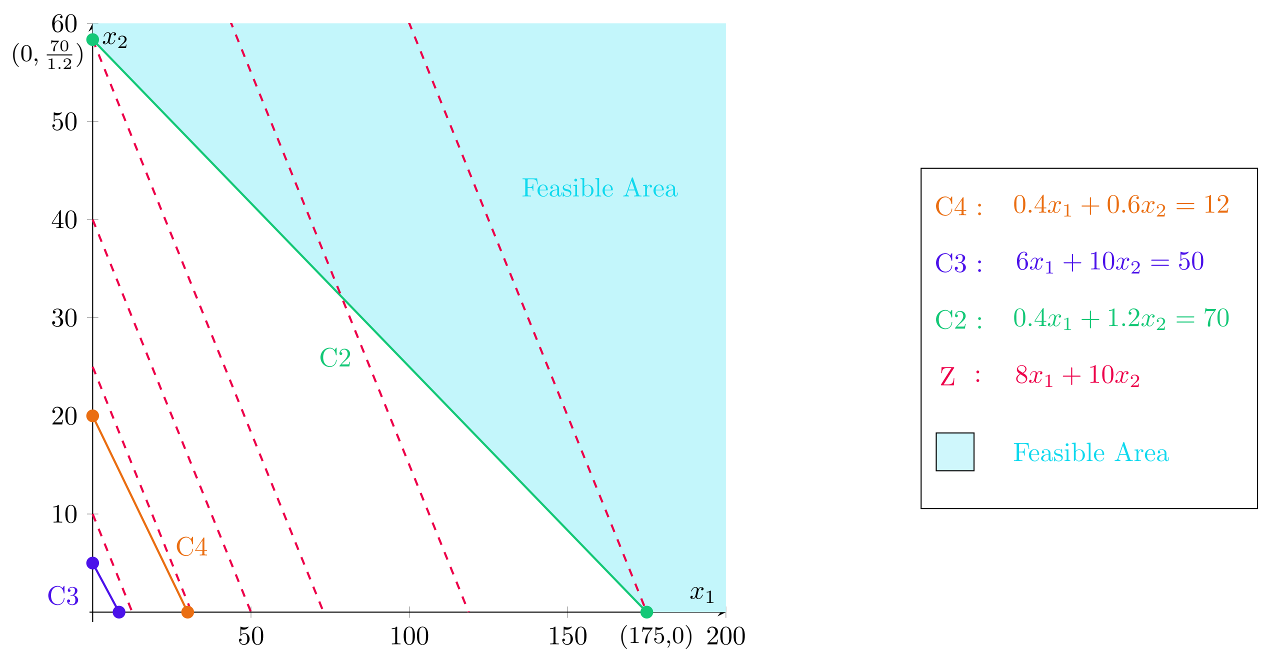
$$
\begin{cases}
\forall x_1, x_2 \geq 0 \quad \text{(Non-negative number of food item) ...C1} \\
0.4x_1 + 1.2x_2 \geq 70 \quad \text{(Minimum Protein Daily) ...C2} \\
6x_1 + 10x_2 \geq 50 \quad \text{(Minimum Vitamine c Daily) ...C3} \\
0.4x_1 + 0.6x_2 \geq 12 \quad \text{(Minimum Iron Daily) ...C4}
\end{cases}
$$

Objective Function

$$f(x_i) = Z = 8x_1 + 10x_2$$

The goal is to minimize food cost by minmizing $f(x_i)$, while meeting the minimum daily nutrition .

Problem : Find the minimum of $f(x_i)$ subject to the contraints



Possible Solutions	$(0, \frac{70}{1.2})$	(175,0)
Objective Function	$\frac{700}{1.2} \approx 583.33$	1400

Solution

The blue area in the plot represents the feasible region, so the optimal solution(s) must be within this area. Since the objective function Z increases towards the right (due to both coefficients being positive) and we want to minimize Z, we need to sweep the objective function line towards the feasible area , and the first intresection between the objective function and one of the vertex points is the min the optimal solution . Therefore,the optimal solution is $(0, \frac{70}{1.2})$ with $Z = 8 \times 0 + 10 \times \frac{70}{1.2} \approx 583.33$.

Example2 : Blending Model

similar to the previous example , this time we have a farm that needs to feed their chicken , there are 2 feeds , we need to minimize cost of the feeds while meeting the minimum nutrient requirement

Feed	Nut_A	Nut_B	Nut_C	Cost
1	3	7	3	10
2	2	2	6	4
Min Require	60	84	72	

Variables Definition:

Let x_1 be the number of Feed 1.

Let x_2 be the number of Feed 2.

Constraint:

$$\begin{cases} \forall x_1, x_2 \geq 0 \quad \text{(Non-negative number of feeds) ...C1} \\ 3x_1 + 2x_2 \geq 60 \quad \text{(Minimum Nut}_A \text{ Daily) ...C2} \\ 7x_1 + 2x_2 \geq 84 \quad \text{(Minimum Nut}_B \text{ Daily) ...C3} \\ 3x_1 + 6x_2 \geq 72 \quad \text{(Minimum Nut}_C \text{ Daily) ...C4} \end{cases}$$

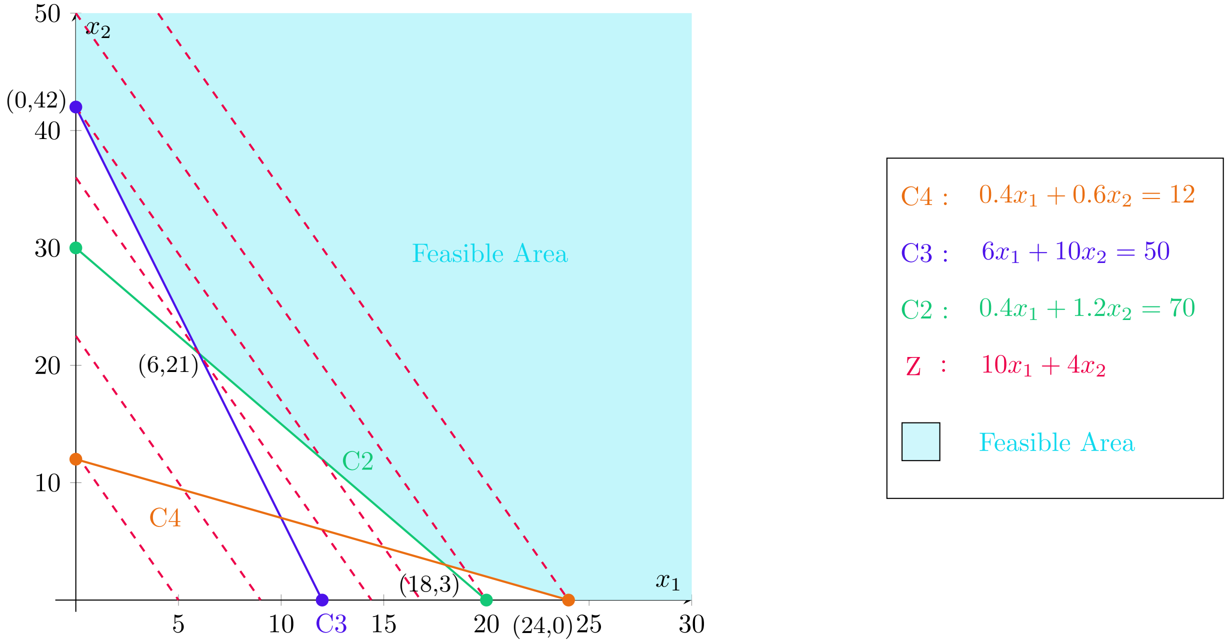
Objective Function

$$f(x_i) = Z = 10x_1 + 4x_2$$

The goal is to minimize feeds cost by minmizing $f(x_i)$, while meeting the minimum daily nutrition .

Problem : Find the minimum of $f(x_i)$ subject to the contraints

Possible Solutions	(0,42)	(6,21)	(18,3)	(24,0)
Objective Function	168	144	192	240



Solution

The blue area in the plot represents the feasible region, so the optimal solution(s) must be within this area. Since the objective function Z increases towards the right (due to both coefficients being positive) and we want to minimize Z, we need to sweep the objective function line towards the feasible area , and the first intresection between the objective function and one of the vertex points is the min the optimal solution . Therefore,the optimal solution is $(6, 21)$ with $Z = 10 \times 6 + 4 \times 21 = 144$.

If The Prices Changed

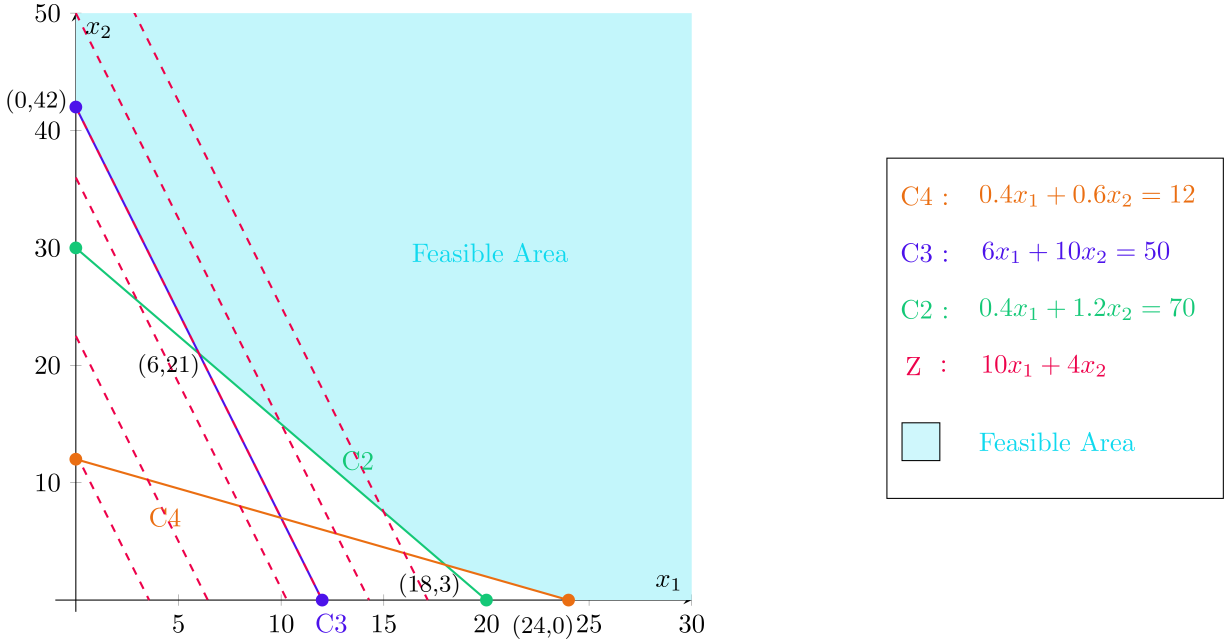
the price of feed1 is now 14 and feed2 remains the same 4 , the constraints don't change meaning the feasible region also stays the same and the possible solutions (coordinates of the vertex points of the polygon) only thing that changes is the objective function

New Objective Function

$$f(x_i) = Z = 14x_1 + 4x_2$$

The goal is to minimize feeds cost by minmizing $f(x_i)$, while meeting the minimum daily nutrition .

Possible Solutions	(0,42)	(6,21)	(18,3)	(24,0)
Objective Function	168	168	264	336



Observation

- The value of the minimum or maximum of the objective function is unique, but there can be multiple coordinate solutions.
- The line through (0, 42) and (6, 21) represents a contour line, indicating that the value of Z is constant along this line.
- If P_1 and P_2 are both optimal solutions, they must be adjacent boundary corners of the feasible region.
- When there are multiple solutions, they form a segment of coordinates on the boundary (edge) of the feasible region, where each coordinate in that segment represents an optimal solution. In linear programming, we typically focus on the vertex points.

Note

Difference Between Multiple & Infinite Solutions:

One might wonder about the difference between these two terms since even multiple solutions have an infinite number of solutions. The difference is that in infinite solutions, the solution is unbounded, unlike multiple solutions, which have their solutions on a finite segment. But as mentioned before, even though multiple solutions have infinite solutions, we focus only on the boundary corner points (vertex points).