

A substitute for the non-existent affine plane of order 6

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Algebra and Combinatorics seminar,
University of St Andrews, 21 February 2018

Joint work with Peter Cameron (University of St Andrews)
and Tomas Nilson (Mid-Sweden University)

Abstract

A Latin square of order n can be used to make an incomplete-block design for n^2 treatments in $3n$ blocks of size n . The cells are the treatments, and each row, column and letter defines a block. Any pair of treatments concur in 0 or 1 blocks, and it is known that the block design is optimal for these parameters.

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I will describe how a series of mistakes and wrong turnings in a different research project led to an answer.

1. Square lattice designs.
2. Triple arrays and sesqui-arrays.
3. How the new designs were discovered, part I.
4. Resolvable designs for 36 treatments in blocks of size 6.
5. How the new designs were discovered, part II.

Square lattice designs.

Square lattice designs for 16 treatments in 2–4 replicates

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
<i>B</i>	<i>A</i>	<i>D</i>	<i>C</i>
<i>C</i>	<i>D</i>	<i>A</i>	<i>B</i>
<i>D</i>	<i>C</i>	<i>B</i>	<i>A</i>

α	β	γ	δ
γ	δ	α	β
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3. If $r = 2$ then STOP.
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Cheng and Bailey (1991) showed that these designs are **optimal** among block designs of this size, even over non-resolvable designs.

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Most of the literature on square lattice designs is by people who have never heard of nets, and vice versa.

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So $\mu_A \leq 1$, and a design maximizing μ_A , for given values of r and k and number of treatments, is **A-optimal**.

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Patterson and Williams (1976) used computer search to find a design for 36 treatments in 4 replicates of blocks of size 6. All pairwise treatment concurrences are in $\{0, 1, 2\}$.

The value of its A-criterion μ_A is 0.836, which compares well with the unachievable upper bound of 0.840.

Triple arrays and sesqui-arrays.

Triple arrays

Triple arrays were introduced independently by Preece (1966) and Agrawal (1966), and later named by McSorley, Phillips, Wallis and Yucas (2005).

They are row-column designs with r rows, c columns and v letters, satisfying the following conditions.

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- (A6) The number of letters common to any two columns is the non-zero constant $r(k - 1) / (c - 1)$.

A triple array with $r = 4$, $c = 9$, $v = 12$ and $k = 3$

- (A4) The number of letters common to any row and column is $k = 3$.
- (A5) The number of letters common to any two rows is the non-zero constant $c(k - 1)/(r - 1) = 6$.
- (A6) The number of letters common to any two columns is the non-zero constant $r(k - 1)/(c - 1) = 1$.

Sterling and Wormald (1976) gave this triple array.

<i>D</i>	<i>H</i>	<i>F</i>	<i>L</i>	<i>E</i>	<i>K</i>	<i>I</i>	<i>G</i>	<i>J</i>
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If letters are blocks, rows are levels of treatment factor $T1$, columns are levels of treatment factor $T2$, and there is no interaction between $T1$ and $T2$, then this is a good design.

Sesqui-arrays are a weakening of triple arrays

Cameron and Nilson introduced the weaker concept of sesqui-array by dropping the condition on pairs of columns. They are row-column designs with r rows, c columns and v letters, satisfying the following conditions.

- (A1) There is exactly one letter in each row-column intersection.
- (A2) No letter occurs more than once in any row or column.
- (A3) Each letter occurs k times, where $k > 1$ and $vk = rc$.
- (A4) The number of letters common to any row and column is k .
- (A5) The number of letters common to any two rows is the non-zero constant $c(k - 1) / (r - 1)$.

How the new designs were discovered, part I.

The story: Part I

Consider designs with $n + 1$ rows, n^2 columns and $n(n + 1)$ letters. Triple arrays have been constructed for $n \in \{3, 4, 5\}$ by Agrawal (1966) and Sterling and Wormald (1976); for $n \in \{7, 8, 11, 13\}$ by McSorley, Phillips, Wallis and Yucas (2005). There are values of n , such as $n = 6$, for which a BIBD for n^2 treatments in $n(n + 1)$ blocks of size n does not exist.

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TN found a general construction, using a pair of mutually orthogonal Latin squares of order n . So this works for all positive integers n except for $n \in \{1, 2, 6\}$.

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Later, RAB found a simpler version of TN's construction, that needs a Latin square of order n but not orthogonal Latin squares. So $n = 6$ is covered. If this had been known earlier, PJC would not have found the nice design for $n = 6$.

Resolvable designs for 36 treatments in blocks of size 6.

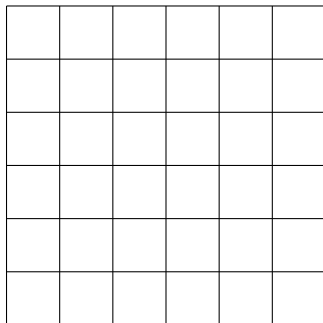
The Sylvester graph and its starfish

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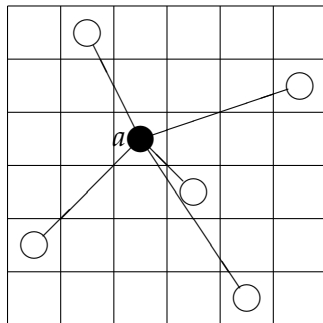
The vertices can be thought of as the cells of a 6×6 grid.



The Sylvester graph and its starfish

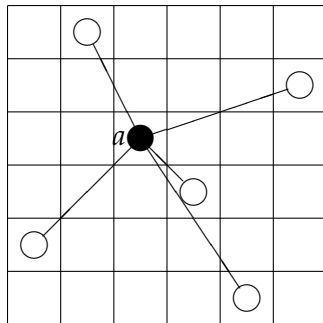
The Sylvester graph Σ is a graph on 36 vertices with valency 5. It has a transitive group of automorphisms, so it looks the same from each vertex.

The vertices can be thought of as the cells of a 6×6 grid.



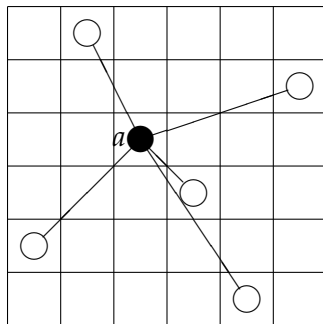
At each vertex a , the *starfish* $S(a)$ defined by the 5 edges at a has 6 vertices, one in each row and one in each column.

Pedantic naming



When I started to explain these ideas,
I called this set of six vertices the **spider** centred at a .

Pedantic naming



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I called this set of six vertices the **spider** centred at a .
PJC pointed out that spiders usually have more than five legs,
whereas some starfish have five.

A real starfish



Starfish whose centres are in the same column

		b			
					c
		a			

If there is an edge from a to c and an edge from b to c then the starfish $S(c)$ has two vertices in the third column.

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If there is an edge from a to c and an edge from b to c then the starfish $S(c)$ has two vertices in the third column. This cannot happen, so the starfish $S(a)$ and $S(b)$ have no vertices in common. So, for any one column, the 6 starfish centred on vertices in that column do not overlap, and so they give a single replicate of 6 blocks of size 6.

The galaxy of starfish centered on column 3

D	A	B^*	C	E	F
F	E	C^*	B	D	A
E	B	A^*	D	F	C
B	F	D^*	A	C	E
A	C	E^*	F	B	D
C	D	F^*	E	A	B

Constructing resolved designs with r replicates

For $r = 2$ or $r = 3$:

- Replicate 1 the blocks are the rows of the grid
- Replicate 2 the blocks are the columns of the grid
- Replicate 3 the blocks are the starfish of one particular column

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So if we use the galaxies of starfish of two or more columns then some treatment concurrences will be bigger than 1.

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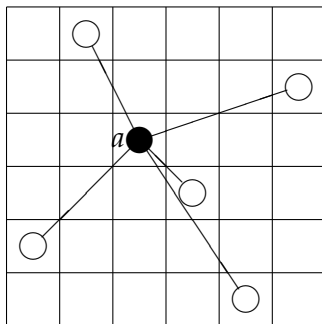
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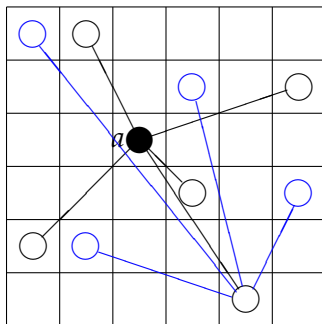
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The fine details of which designs we chose will be shown later.

More properties of the Sylvester graph



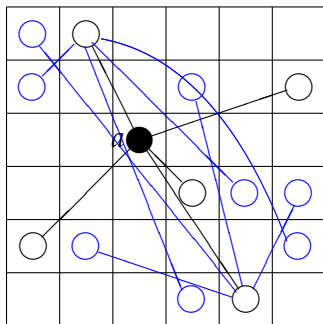
More properties of the Sylvester graph



Vertices at distance 2 from a are all in rows and columns different from a .

The Sylvester graph has no triangles

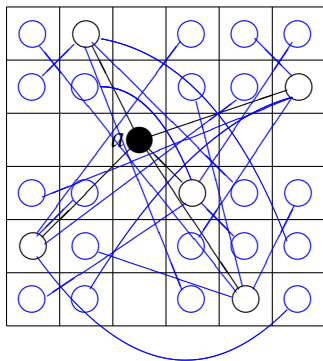
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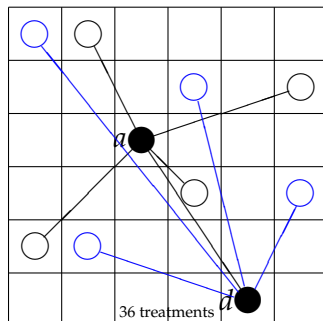
This implies that, if a is any vertex, the vertices at distance 2 from vertex a are precisely those vertices which are not in the starfish $S(a)$ or the row containing a or the column containing a .

Consequence I: concurrences

If a is any vertex, the vertices at distance 2 from vertex a are precisely those vertices which are not in the starfish $S(a)$ or the row containing a or the column containing a .

Consequence

If we make each starfish into a block, then the only way that distinct treatments a and d can occur together in more than one block is for vertices a and d to be joined by an edge so that they both occur in the starfish $S(a)$ and $S(d)$.



Consequence II: association scheme

If a is any vertex, the vertices at distance 2 from vertex a are precisely those vertices which are not in the starfish $S(a)$ or the row containing a or the column containing a .

Consequence

The four binary relations:

- ▶ different vertices in the same row;
- ▶ different vertices in the same column;
- ▶ vertices joined by an edge in the Sylvester graph Σ ;
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form an association scheme.

So, for any incomplete-block design which is partially balanced with respect to this association scheme, the information matrix has five eigenspaces, which we know (in fact, they have dimensions 1, 5, 5, 9 and 16), so it is straightforward to calculate the eigenvalues and hence the canonical efficiency factors.

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The unachievable upper bound given by the non-existent square lattice design is $\mu_A = 0.8537$.

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The unachievable upper bound given by the non-existent square lattice design is $A = 0.8571$.

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The non-existent design consisting of a balanced design in 7 replicates with one more replicate adjoined would have $A = 0.8547$.

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4	0.8380	0.8341	0.8258	0.836	0.8393	0.8400
5	0.8453	0.8422	0.8383		0.8464	0.8485
6	0.8498	0.8473	0.8442		0.8510	0.8537
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Highlighted entries correspond to partially balanced designs. Our designs have the small advantage that a late decision to add or drop a replicate leaves a design in the same series.

How the new designs were discovered, part II.

Back to the sesqui-arrays

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How do we take the one with 7 replicates and turn its dual into a 7×36 sesqui-array with 42 letters?

The story: Part II

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*	1	2	3	4	5	6	← sets of six letters
1	*	1	1	1	1	1	
2	2	*	2	2	2	2	
3	3	3	*	3	3	3	
4	4	4	4	*	4	4	
5	5	5	5	5	*	5	
6	6	6	6	6	6	*	

Forestry to the rescue

Later, PJC: The only hope of putting this right is to permute the letters in each column. I need 6 permutations. Each fixes the first row and one other. The rest of each permutation gives a circle on the other 5 rows, and I want these circles to have every row following each other row exactly once.

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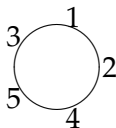
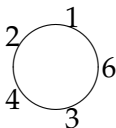
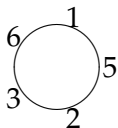
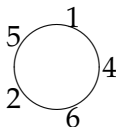
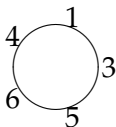
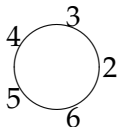
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We are also comparing our designs to Emlyn Williams's.

Having the same value of μ_A does not imply isomorphism of the block designs, even when the concurrence graphs are isomorphic.

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Triple arrays

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