Permutation groups, lattices and orthogonal block structures

Rosemary A. Bailey and Peter J. Cameron
University of St Andrews
with Marina Anagnostopoulou-Merkouri (Bristol)

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How it began

In the summer of 2022, Marina had a research internship in the department to work with Peter. Having finished before the money ran out, we looked at a new property of finite permutation groups which we called pre-primitivity. The idea was that pre-primitivity and quasiprimitivity were independent but together were equivalent to primitivity.

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In the next academic year, Marina had a STARIS internship, and we looked more generally at properties of transitive imprimitive permutation groups, which led to the work we describe here.

Combinatorial considerations about permutation groups

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Our aim was to do something similar for transitive but imprimitive groups.

What is a Latin square?

Definition

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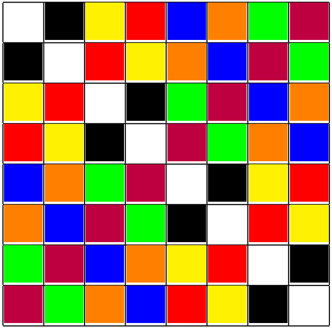
A Latin square of order 8



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Example

If Ω is the set of cells in a Latin square, then there are five natural uniform partitions of Ω:

- R each part is a row;
- C each part is a column;
- L each part consists of the those cells with a given letter;
- U the universal partition, with a single part;
- E the equality partition, whose parts are singletons.

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This can be read "Π_1 refines Π_2" or "Π_2 is coarser than Π_1".

Hasse diagrams

Given a collection $\mathcal{P}$ of partitions of a set Ω, we can show them on a Hasse diagram.

- Draw a dot for each partition in $\mathcal{P}$.
- If $\Pi_1 \prec \Pi_2$ then put Π_2 higher than Π_1 in the diagram.
- If $\Pi_1 \prec \Pi_2$ but there is no Π_3 in $\mathcal{P}$ with $\Pi_1 \prec \Pi_3 \prec \Pi_2$ then draw a line from Π_1 to Π_2.

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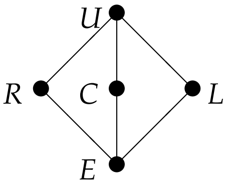
Here is the Hasse diagram for a Latin square.



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As noted, partitions can also be regarded as equivalence relations. The composition of two relations R_1 and R_2 is the relation $R_1 \circ R_2$ consisting of all pairs (α, β) for which there exists γ such that $(\alpha, \gamma) \in R_1$ and $(\gamma, \beta) \in R_2$. Two relations R_1 and R_2 commute if $R_1 \circ R_2 = R_2 \circ R_1$.

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Orthogonal block structures

Here is an alternative definition of Latin square.

Definition

A Latin square is a set $\{R, C, L\}$ of pairwise commuting uniform partitions of a set Ω which satisfy $R \wedge C = R \wedge L = C \wedge L = E$ and $R \vee C = R \vee L = C \vee L = U$.

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So Latin squares are OBS.

Properties

Proposition

- *Let* Π_1 *and* Π_2 *be equivalence relations. Then* $\Pi_1 \vee \Pi_2 = \Pi_1 \circ \Pi_2$ *if and only if* Π_1 *and* Π_2 *commute.*

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An OBS which is a distributive lattice is called a poset block structure or PBS. More on these later!

Statisticians at Rothamsted

Here are some of the statisticians who have worked at the agricultural research station at Rothamsted.

Ronald Fisher	1919–1933	then UCL, then Cambridge
Frank Yates	1931–1968	
Oscar Kempthorne	1941–1946	then Ames, Iowa
Desmond Patterson	1947–1967	then Edinburgh
John Nelder	1968–1984	previously National Vegetable Research Station
Rosemary Bailey	1981–1990	
Robin Thompson	1997–now	previously Edinburgh, now emeritus

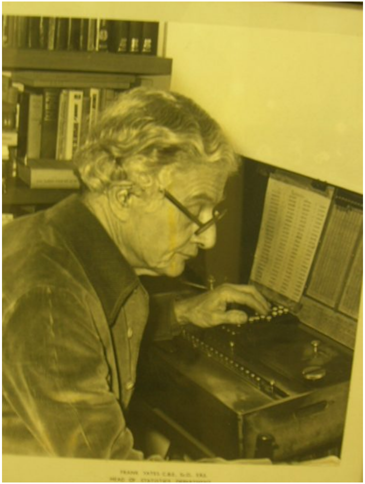
Ronald Fisher

Trivial OBS (only U and E).

Blocks containing plots.

A rectangle with one plot in each Row-Column intersection.

Frank Yates



Many more OBS, including

- blocks containing plots containing subplots
- several rectangles
- a rectangle with subplots
- several rectangles with subplots.

John Nelder: Crossing and Nesting

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Nesting $\mathcal{P}_2$ within $\mathcal{P}_1$ gives the set $\mathcal{P}_1/\mathcal{P}_2$ of partitions $\{\Pi_1 \times U_2 : \Pi_1 \in \mathcal{B}_1\} \cup \{E_1 \times \Pi_2 : \Pi_2 \in \mathcal{B}_2\}$.

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Iterated crossing and nesting gives simple orthogonal block structures.

Desmond Patterson

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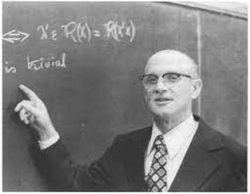
I did not believe him then, but, looking back, I can see that his approach did not incorporate Nelder's ideas until much later.

Oscar Kempthorne's papers



Then my colleague Robin Thompson gave me a 1961 technical report (long, but in typescript) by Oscar Kempthorne and his colleagues in Ames. This developed essentially the same ideas as Nelder's: lattices of partitions using some of the partitions in a Cartesian lattice (not necessarily with all coordinates having the same number of values, for example, the rows and columns of a rectangle).

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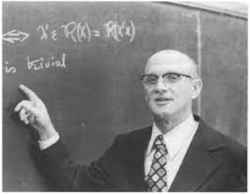

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Later I learnt that Kempthorne was furious that Nelder had "stolen" his ideas. I believe that they simply developed them independently, building on the work of Fisher and Yates. In those days, it took much longer for ideas to circulate widely.

Putting the bits together

One morning, I came into work after drinking too much in the pub the previous evening. I realised that my brain was not capable of serious work, so I gave it the apparently simple task of matching Nelder's block structures with those of Kempthorne. Slowly, I worked through dimensions 1, 2 and 3.

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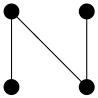
The next day, with a clear head, I realised that Kempthorne's approach always gives more possibilities than Nelder's in dimensions at least 4.

SOBS ⇒ poset, but not vice versa

crossing

nesting

not from SOBS



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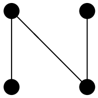

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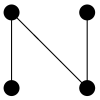

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Crossing and nesting give a similar formula in the statistical software R for use in analysis of variance.
"(Fields/Plots) $\times$ Year" becomes "(Fields/Plots) $*$ Year".

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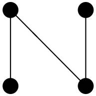

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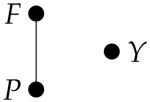
When Terry Speed and RAB combined the two approaches in 1982, we called the structures poset block structures.

How do we define PBS?

(Fields/Plots) $\times$ Year



Same Field, same Plot $\prec$ Same Field

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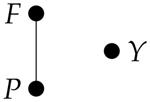

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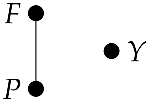

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Fortunately, my co-authors came up with a clever solution.

Not necessarily same Year $\prec$ Not necessarily same Plot or Year

$$\{Y\} \subset \{P, Y\}$$

Definition of Poset Block Structure

Let $(M, \sqsubseteq)$ be a partially ordered set.

Definition

A down-set in M is a subset D of M with the property that, if $m \in D$ and $m' \sqsubset m$, then $m' \in D$.

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Now we define a partition Π_D for each down-set D of M . This is done as follows.

Definition

Elements $(\alpha_1, \ldots, \alpha_N)$ and $(\beta_1, \ldots, \beta_N)$ are in the same part of Π_D if and only if $\alpha_i = \beta_i$ for all i with $m_i \notin D$.

Easy consequences

$$E = \Pi_{\emptyset} \text{ and } U = \Pi_M.$$

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If D_1 and D_2 are down-sets of M then

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In June 1988 I took time out from a 2-week conference in Minneapolis to visit Kempthorne. He was very friendly, and said that he much appreciated my work on PBS.

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$$(\omega f)_i = \omega_i(\omega \pi^i f_i) \quad \text{for } i = 1, \ldots, N.$$

Automorphism groups of block structures

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We saw earlier that Latin squares give rise to OBSs (consisting of the two partitions E and U and the row, column, and letter partitions). It is known that almost all Latin squares have trivial automorphism group.

Some comments on generalised wreath products

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3. If the poset $(M, \sqsubseteq)$ can be made by iterated crossing and nesting (as in a simple orthogonal block structure) then the GWP can be made by iterating the corresponding direct and wreath products.
4. If the poset $(M, \sqsubseteq)$ cannot be made in this way, then neither can the GWP.

Permutation groups

Proposition

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If the invariant partitions for G form a chain, then they commute pairwise. We do not know a weaker lattice property that forces the partitions to commute. Even requiring the lattice to be Boolean (isomorphic to the lattice of subsets of a set) does not suffice for this.

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We say that the transitive group G has the OB property if the invariant partitions commute (and so form an orthogonal block structure). It has the PB property if the lattice is distributive (and so is a poset block structure).

Commuting subgroups

Let H and K be subgroups of a group G, with $H \leq K$. The corresponding interval in the subgroup lattice consists of all subgroups X for which $H \leq X \leq K$; it is a lattice, with $X \wedge Y = X \cap Y$ and $X \vee Y = \langle X, Y \rangle$.

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Let G be transitive on Ω, and G_α the stabiliser of $\alpha \in \Omega$. There is a natural isomorphism between the lattice of G-invariant partitions of Ω and the interval from G_α to G: the part of Π containing α is the the orbit containing α of the corresponding subgroup. Moreover, partitions corresponding to X and Y commute if and only if $XY = YX$: we say that X and Y commute (to avoid confusion with permutations). Thus:

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G has the OB property if and only if the subgroups containing G_α commute pairwise.

In particular, a regular permutation group has the OB property if and only if all its subgroups commute pairwise. These groups were determined by Iwasawa.

Pre-primitivity

My earlier paper with Marina studied the following property. A transitive permutation group G is pre-primitive if every G-invariant partition is the orbit partition of a subgroup of G. We can take this subgroup to be the full stabiliser of the partition, and so it is a normal subgroup of G. Since normal subgroups commute, we see:

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Proposition

A pre-primitive group has the OB property.

For the record: A permutation group is quasiprimitive if every non-trivial normal subgroup is transitive; thus, as mentioned earlier, a group is primitive if and only if it is pre-primitive and quasiprimitive.

Generalised wreath products

Our first main theorem is the following:

Theorem

A generalised wreath product of primitive permutation groups $(G_m : m \in M)$ is pre-primitive and has the OB property; it has the PB property if and only if there do not exist incomparable elements $m_1, m_2 \in M$ such that G_{m_1} and G_{m_2} are cyclic of the same prime order.

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The reason is that, if G and H are primitive, then the only non-trivial invariant partitions of $G \times H$ correspond to orbits of G and of H, unless $G = H = C_p$ for some prime p, in which case there are $p + 1$ invariant partitions. (Recall that $G \times H$ is the g.w.p. of G and H over the poset consisting of two incomparable elements.)

The Krasner–Kaloujnine theorem

This is the following well-known result:

Theorem

Let G be a transitive imprimitive permutation group, with non-trivial invariant partition Π. Then G is naturally embeddable in the wreath product $H \wr K$, where H is the permutation group induced on a part of Π by its setwise stabiliser, and K the permutation group induced on the set of parts of Π by G.

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Is there an extension to our situation? The answer is yes …

First attempt

Let G be a transitive group with the PB property, and let $\Lambda(G)$ be the lattice of G-invariant partitions. Then Λ is isomorphic to the lattice of down-sets in the poset M, whose elements can be recovered as the non-E join-indecomposable (JI) elements of Λ. If Π is the partition corresponding to $m \in M$, then there is a unique maximal partition Π^- below Π, and we could define G_m to be the stabiliser of a part of Π acting on the set of parts of Π^- below it.

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Unfortunately this does not work. The symmetric group S_6 has an outer automorphism, so acts in different ways on two sets of size 6. Let Ω be their Cartesian product. The only non-trivial partitions for G on Ω are given by the coordinate projections, and the stabiliser of a part acts on it as $\mathrm{PGL}(2,5)$. But S_6 is not a subgroup of $\mathrm{PGL}(2,5) \times \mathrm{PGL}(2,5)$.

Second attempt

There is a way round this. Show that there is a unique maximal G-invariant partition Ψ such that $\Psi \wedge \Pi = \Pi^-$, and that Ψ is also JI. Then let G_m^* be the group induced by the stabiliser of a part of Ψ on the parts of Ψ^- it contains. Now $G_m^* \geq G_m$, and we have:

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Theorem

Let G be a permutation group on Ω with the PB property, and let M be the corresponding poset and G_m^* the group defined above for $m \in M$. Then G is naturally embedded in the generalised wreath product of the groups G_m^* over the poset M.

Varying the poset

The direct product $G_1 \times G_2$ of transitive groups is naturally embedded in the wreath product in either order; indeed, it is their intersection. Can we generalise this?

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- *The intersection of the g.w.p.s of $\mathcal{F}$ over $\sqsubseteq_1$ and $\sqsubseteq_2$ is the g.w.p. over their intersection.*
- *If $\sqsubseteq_1$ is contained in $\sqsubseteq_2$, then the g.w.p. over $\sqsubseteq_1$ is embedded in the g.w.p. over $\sqsubseteq_2$.*

Linear extensions

Let M be a totally ordered set, say $\{1 < 2 < \cdots < r\}$, and let G_i be a transitive permutation group for each $i \in M$. The generalised wreath product over this poset is simply the iterated wreath product

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Theorem

The generalised wreath product of a family $(G_m : m \in M)$ of transitive permutation groups over a poset $(M, \sqsubseteq)$ is the intersection of the iterated wreath products over all linear extensions of $(M, \sqsubseteq)$.

This is immediate from the preceding theorem.