SIRS Epidemic Model

Parameter Estimation and Uncertainty Quantification

AIMS Rwanda — Mathematical Modelling III
Samuel, Aline, Honorine, Rabecca, Sylvester, Faida

Group 1 12th February, 2026

Introduction

Mathematical models help us understand and predict how infectious diseases spread. In this project, we study the SIRS model, where recovered individuals can lose immunity and return to the susceptible class. Since epidemiological data are often noisy, reliable parameter estimation requires more than simply fitting the model. We therefore, begin with least squares estimation to obtain point estimates, quantify uncertainty using a Jacobian-based approach, and apply Bayesian inference with MCMC. Finally, we carry on with parameter uncertainty to produce predictions with uncertainty bands and assess how reliable the model forecasts are.

The SIRS Model

People move through three stages:

Susceptible $\xrightarrow{\beta}$ **Infected** $\xrightarrow{\gamma}$ **Recovered** $\xrightarrow{\phi}$ **Susceptible**

Unlike the SIR model, immunity is temporary, so recovered individuals can become susceptible again.

Three unknown rates to estimate:

- β Transmission rate
- γ Recovery rate
- ϕ Waning-immunity rate

$N = 100$, $\kappa = 0$ (no births/deaths)
$S + I + R = N$ always

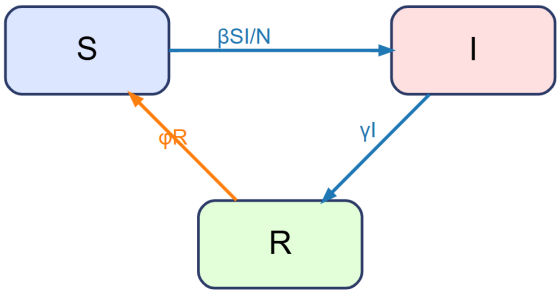

Figure 1: SIRS compartmental model flow diagram.

Data Generation

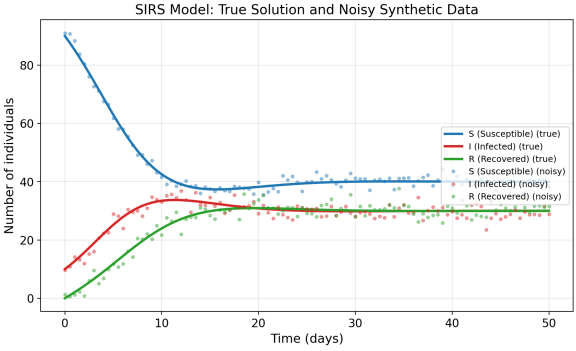

Figure 2: True SIRS trajectories and noisy observations

Table 1: True parameter values

Parameter	True value
β	0.500
γ	0.200
ϕ	0.200
S_0, I_0, R_0	90, 10, 0

The infection peaks around day 10 and then declines toward an endemic equilibrium.

Choosing κ

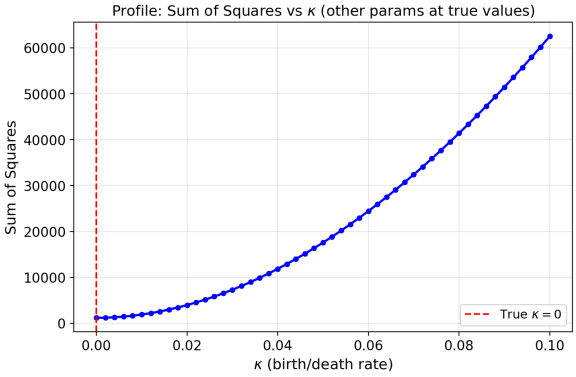

Figure 3: Sum of squares with κ

We tested whether including a birth/death rate κ improves the model fit.

The sum of squares is minimised near $\kappa = 0$ and increases as κ grows.

Sensitivity analysis on κ

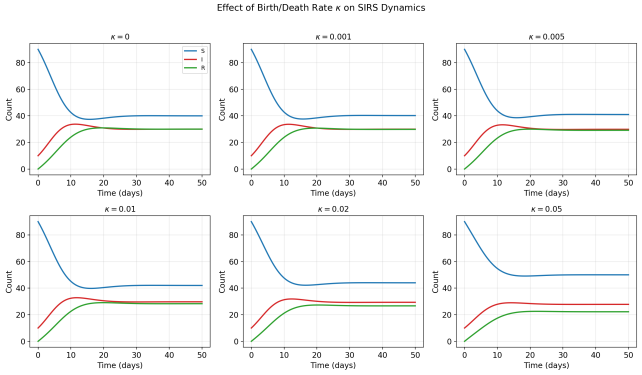

Figure 4: SIRS trajectories for six values of κ

Least Squares Fit

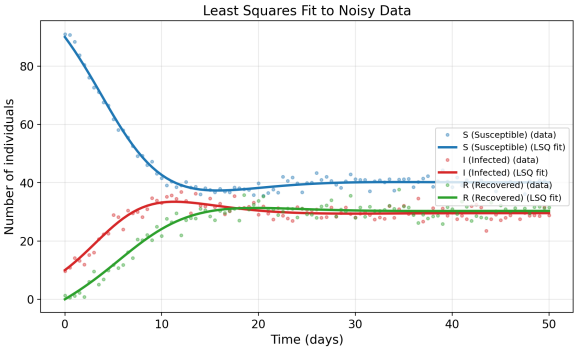

Figure 5: Best-fit model (lines) vs observed data (dots).

Table 2: Least-squares parameters vs true values.

	True	Estimated
β	0.500	**0.4965**
γ	0.200	**0.1992**
ϕ	0.200	**0.1940**

The estimates are close to the true values, suggesting a good fit.

Are the Residuals Well-Behaved?

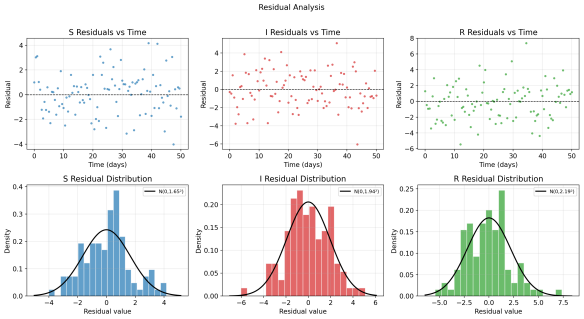

Figure 6: Top: residuals vs time. Bottom: histograms with Gaussian curve.

A good fit should have residuals that are:

- Random
- Centred around zero
- Similar spread over time
- Approximately normal

There is no clear pattern over time, and the histograms look close to a Gaussian curve. This is consistent with $\varepsilon \sim \mathcal{N}(0, \sigma^2)$.

Jacobian Analysis

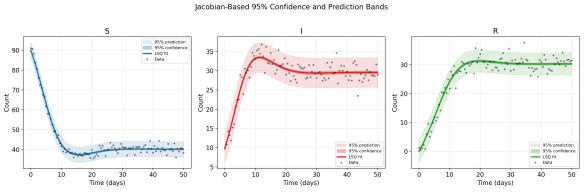

Figure 7: Dark: 95% confidence bands. Light: 95% prediction bands.

Table 3: Jacobian-based SE, t-values, and 95% CI.

	SE	t-value	95% CI
β	0.007	74.6	[0.483, 0.510]
γ	0.003	67.7	[0.193, 0.205]
ϕ	0.004	51.2	[0.187, 0.201]

$R^2 = 0.975 \quad \hat{\sigma}^2 = 3.80$

All t-values $\gg 2$, suggesting the parameters are well estimated.

MCMC Setup and Convergence

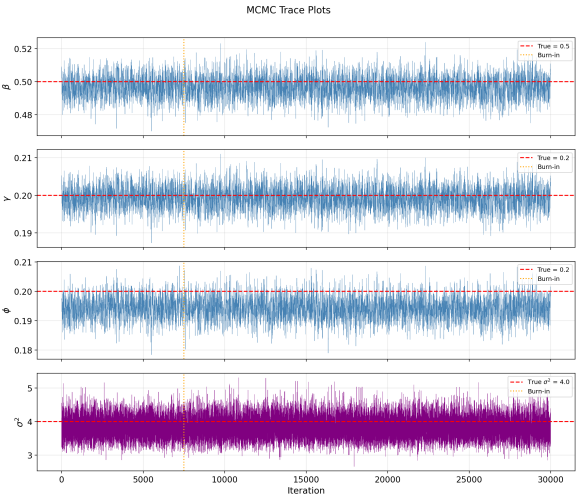

Figure 8: Trace plots

Trace plots look stable after burn-in and show good mixing.

Acceptance rate: **32%**

Autocorrelation Function

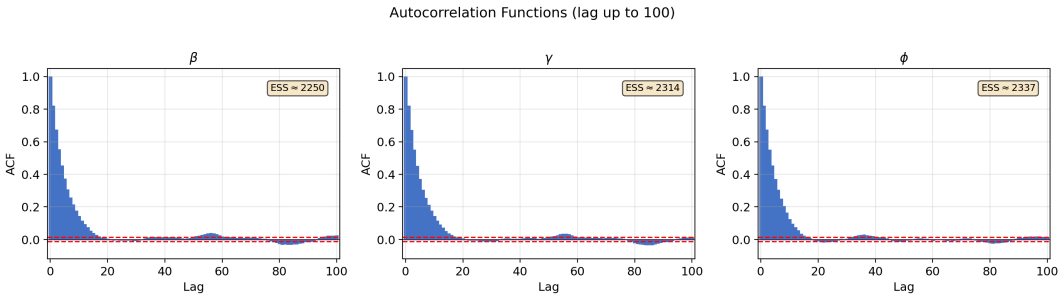

Figure 9: Autocorrelation functions

Posterior Distributions

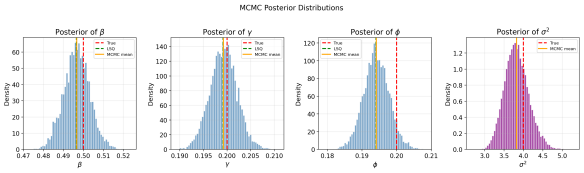

Figure 10: Posterior histograms

Table 4: Posterior summary

	True	Mean	2.5%	97.5%
β	0.50	0.497	0.484	0.510
γ	0.20	0.199	0.193	0.205
ϕ	0.20	0.194	0.187	0.202

Posterior distributions are approximately normal and close to the true values.

Pairwise Correlations in the Posterior

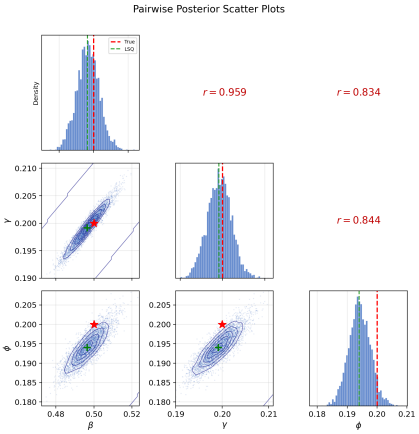

Figure 11: Pairwise posterior scatter plots.

This means β and γ move together,a higher transmission rate can be offset by a higher recovery rate, giving similar model curves.

Posterior Predictions and R_0

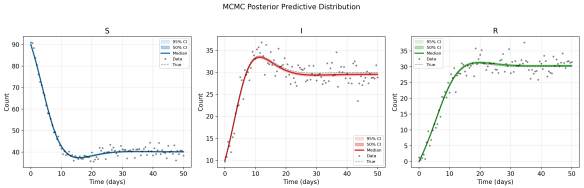

Figure 12: Posterior predictive trajectories

Basic reproduction number (with $\kappa = 0$):

$$R_0 = \frac{\beta}{\gamma}.$$

Table 5: Summary of R_0 from posterior samples.

Summary	Value
True R_0	2.50
Posterior mean	**2.49**
95% CrI	[2.47, 2.51]

Since $R_0 > 1$, sustained transmission is possible.

Additional Experiments using Adaptive Metropolis for MCMC

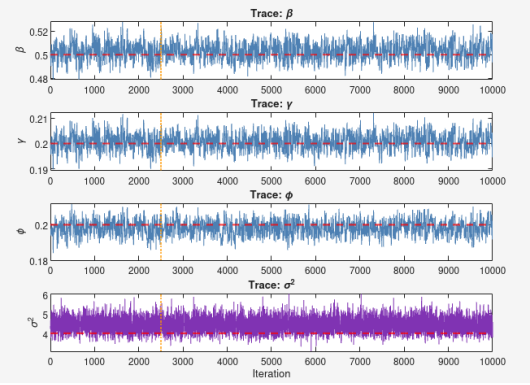

Figure 13: MCMC using Adaptive Metropolis

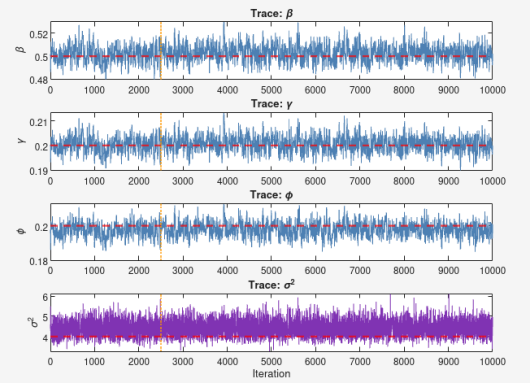

Figure 14: MCMC using Metropolis Hastings

Conclusions

Main result:

- The SIRS model fits well and recovers β, γ, ϕ close to the true values.
- Strong fit: $R^2 = 0.975$; residuals look random and roughly Gaussian.

Epidemic implication:

- $R_0 \approx 2.49$ with 95% CrI [2.47, 2.51].
- $R_0 > 1$ suggests sustained transmission is possible.

Why MCMC helps:

- Gives uncertainty ranges, not only one best estimate.
- Produces prediction bands and uncertainty in R_0.

This workflow can be used on real outbreak data to estimate parameters and make forecasts with uncertainty.

THANK YOU

References

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