MODULE - 6

PREDICATE LOGIC

Consider the following two statements

Every SCE student must study Physics
Jackson is a SCE student.
Therefore, Jackson must study Physics.

This cannot be expressed by propositional logic because non of the logical connectives are applicable here.
This kinds of problems are evaluated by predicate logic

A predicate is a statement that contains variables (predicate variables) that may be true or false depending on the values of the variable. We will denote the predicate P[variable]
For ex: i) John is a batchelor
Smith is a batchelor
Therefore John & smith are batchelor.

Here the predicate is "is a batchelor"
ii) P[x] = " x^2 is greater than x
iii) P(y) = y+2 is non negative

The domain of a predicate variable is the collection of all possible values that the variable may take to become a proposition.

20/10/19

The domain can be also called as the Universe or universe of discours.

The domain can be finite or infinite set.

For ex:-

→ Let $P(x,y) = x>y$ is a predicate in two variables. Let the universe of dicourse P be the set of integers. Then by applying the elements of the universe we can make this predicate a proposition.

ie, set of integers. $\{\ldots -3,-2,-1,0,1,2,3,\ldots\}$

$P(x,y) = x>y$

Take any two values from set,

let $x=1$, $y=2$.

$P(1,2) = 1>2$ which is false. Hence it is a proposition.

$x=-1$ $y=-5$

$P(-1,-5) = -1>-5$ whhs is true. Hence proposition

If there are more than one predicate variable in the given predicate. Then the universe of discouse may be the same or different for each variable.

→ $P(x,y,z)$ = x is born in city y in the year z.

In this example, the universe of discourse may not be given to you directly because from the arrangement of predicate, it is clear that x is taken from the universe of individuals, y is taken from universe of cities and z is taken from the universe of years.

But in another example, $P(x,y) = x+y=7$.

$P(x) = x>3$. In these predicates the universe of discourse must be clearly mention.

Now if the predicate is of one variable ie, $P(x)$ where x is the variable then we call $P(x)$ as <u>unary</u> predicate.

If the predicate is of two variables ie $P(x,y)$ then we call it as <u>binary</u> predicate.

And hence the predicate with 'n' variables ie $P(x_1, x_2, \ldots x_n)$ is called <u>n-array</u> predicate or <u>n-place</u> predicate.

<u>Note :-</u>

The predicate variables must be finite.

If $P(x_1, x_2, \ldots, x_n)$ is true for all values $c_1, c_2, c_3 \ldots c_n$ from the universe U then we say that $P(x_1, x_2, \ldots x_n)$ is <u>valid</u> in U.

If it is not all true, ie, for some values of $c_1, c_2, \dots c_n$ if the predicate $P(x_1, x_2, \dots x_n)$ is false then we say that P is <u>satisfiable</u> in U.

If for all values of $c_1, c_2 \dots c_n$ from the universe U, if the predicate $P(x_1, x_2, \dots x_n)$ is false then we call P as <u>unsatisfiable</u> in U

? Check whether the predicate is valid or not $P(x,y) = (x+y) > (x-y)$ where the universe is the set $\{1,2,3,4,5\}$ ~~6,7,8,9,10}~~.

A. In this we have to substitute the values of x & y from $U = \{1,2,3,4,5\}$

Let $x=1, y=2$

$P(1,2) = 3 > -2$; ~~P(1,2)~~ True

$P(1,3) = 4 > -2$; True

$P(1,4) = 5 > -3$; True

$P(1,5) = 6 > -4$; True

$P(2,3) = 5 > -1$; True

$P(2,4) = 6 > -2$; True

$P(2,5) = 7 > -3$; True

$P(3,4) = 7 > -1$; True

$P(3,5) = 8 > -2$; True

$P(4,5) = 9 > -1$; True.

$P(2,1) = 3 > 1$; True

$P(3,1) = 4 > 2$; True

$P(3,2) = 7 > 1$; True

$P(4,1) = 5 > 3$; True

$P(4,2) = 6 > 2$; True

$P(4,3) = 7 > 1$; True

$P(5,1) = 6 > 4$; True

$P(5,2) = 7 > 3$; True

$P(5,3) = 8 > 2$; True

$P(5,4) = 9 > 1$; True.

This predicate is a valid predicate.

Quantifiers

A quantifier is something that tells about the amount or quantity of the universe that satisfy the predicate.

There are two types of quantifications/quantifier.

i) Universal quantification / universal quantifier.

A universal quantifier is a quantifier which have the meaning "for all", "for every", "for each", "for any", "for arbitrary". We use the symbol '$\forall$' to denote this ie, if are given the condition that the predicate $P(x)$ is true for every x in the universe U we can denote it by $\forall x \in U, P(x)$ is true.

Eg:-

The square of every real number is non negative can be represented by $\forall x \in R, x^2 \geq 0$ where R is the universe of discourse which is the set of real numbers and the predicate $P(x)$ is $x^2 \geq 0$. In other words we write this as U be the set of real numbers $x \in U$. $P(x)$ is $x^2 \geq 0$ $\therefore \forall x\ P(x)$

2) Existential quantifier / Existential Quantification

This is a quantifier which means "there exist", "there is atleast one", "for some". We use the symbol $\exists$ denote this.

ie, the predicate P(x) is true for atleast one value of x in the universe.

1) ? Write in the form of qualifier.

collection of 2 wheelers

U =

predicate = scooter

i) Every two wheeler is a scooter.

ii) There exist a lion who drinks coffee.

2) ? Check whether the predicate is satisfiable

$U = \{1, 2, 3, 5\}$ $P(x)$: x^2 is an even number.

1) i) The universe of dicourse is two wheelers.

~~Pa~~ 'x' is a scooter.

∴ The statement is represented by the universal quantifier. $\forall x \in U \; P(x)$

ii)

23/10/17

? Let the universe U = is set of integers. Consider the predicates $x < x+1$ $x = 1$ $x = x+1$ where $x \in U$.

Here by applying universal quantifier we have the following truth values

$\forall x\ [x < x+1]$ True

$\forall x\ [x = 3]$ False

$\forall x\ [x = x+1]$ False

By applying the existential quantifier we have

$\exists x\ [x < x+1]$; True.

$\exists x\ [x = 3]$; True.

$\exists x\ [x = x+1]$; False

In general there are two ways to make a predicate into proposition.

i) By assigning particular values to the predicate variables.

ii) By using quantifiers.

Free & Bound Variables.

A variable 'x' in each of the predicate is called a free variable. As 'x' varies over the universe the truth value of the statement may vary.

A variable in a quantified statement is said to be bound variable. ie, the variable will be connected by any of the quantifiers.

A quantified statement has a fixed truth value

Eg: $\forall x [P(x,y,z)]$ here x is the bound variable whereas y & z are free variables.

Let $y=2$ in the above predicate then the predicate becomes $\forall x [P(x,2,z)]$ here 'x' is the bound variable and 'z' is the free variable

Convertion of a Simple Quantified Statement into Compound Statement

$U = \{1,2,3\}$ then $\forall x [P(x)]$ means that $P(x)$ is true for every $x \in U$ and can be represented by $P(1) \wedge P(2) \wedge P(3)$

There exist $\exists x [P(x)]$ means $P(x)$ is true for some values of or. for atleast one value of $x \in U$ ie $P(1) \vee P(2) \vee P(3)$

Note:- We cannot interchange the universal quantifier & existential quantifier. But you can interchange the universal quantifier by itself and also existential quantifier by itself

Let the universe of discourse be the set of every married people.

$\forall x \exists y$ [x is married to y] means that for any 'x' there exist a person y to whom 'x' is married. and hence ~~there~~ this is true.

$\exists x \forall y$ [x is married to y] means that there exist a person y to whom every person 'x' is married and which is false. and hence interchange of order ~~ofor~~ different quantifiers are not allowed.

26/10/17.

Negation of a Quantified Statements.

$$\neg[\forall x\, P(x)] \equiv \exists x\, \neg P(x)$$

$$\neg[\exists x\, P(x)] = \forall x\, \neg P(x)$$

$$\neg[\exists x\, \neg P(x)] = \forall x\, \neg\neg P(x)$$

$$= \forall x\, P(x).$$

Note:-

The negation for quantified statements with n array predicates. we apply the rules of negations from left to right.

ie $\neg(\forall x \exists y\, P(x,y)] \equiv \exists x(\neg(\exists y\, P(x,y))$

$$\equiv \exists x \forall y \neg P(x,y)$$

Logical Equivalence of Predicates

$P(x)$ & $Q(x)$ be two predicates defined for a given universe..

$P(x)$ and $Q(x)$ are called logically equalent if by applying the values for the variable from the universe, we must get the same truth value for each predicate. ie

ie, $P(a) \leftrightarrow Q(a)$ is true for every value of 'a' in universe.

? Check the logical equivalence where

$U = \{1, 2, 3, 4\}$.

$P(x) : x^2 < 10$ $\quad$ $q(x) : 2x > x$

A. $P(1) = 1 < 10 - T$ $\quad$ $q(1) = 2 > 1 - T$

$P(2) = 4 < 10 - T$ $\quad$ $q(2) = 4 > 2 - T$

$P(3) = 9 < 10 - T$ $\quad$ $q(3) = 6 > 3 - T$

$P(4) = 16 < 10 - F$ $\quad$ $q(4) = 8 > 4 - T$

This is not logically equivalence

Logical Implication

Let $P(x)$ & $q(x)$ be two predicates defined on the universe if the implication $P(a) \rightarrow q(a)$ is true. for every 'a' in the universe we say that ~~$P(x) \rightarrow q(x)$~~.

$p(x)$ logically implies $q(x)$ denoted by $p(x) \Rightarrow q(x)$.

? $U = \{1, 2, 3, 4\}$ $p(x) = x^2 < 10$ $q(x) ; 2x > x$

check $p(x) \Rightarrow q(x)$

$p(1) = 1 < 10$ T $q(1) = 2 > 1$ T $p(1) \to q(1)$ is true
$p(2) = 4 < 10$ T $q(2) = 4 > 2$ T $p(2) \to q(2)$ "
$p(3) = 9 < 10$ T $q(3) = 6 > 3$ T $p(3) \to q(3)$ "
$p(4) = 16 < 10$ F $q(4) = 8 > 4$ T $p(4) \to q(4)$ true

$\therefore p(x) \Rightarrow q(x)$.

27/10/17

Theory of Inference - Validity of Arguments.

? For every integer n, n is even if it is divisible by 2.

A. The logical expression will be

Universe of discourse is the set of integers.

$x \in U$

$p(x)$: x is even

$q(x)$: x is divisible by 2.

$\therefore \forall x \, [q(x) \to p(x)]$

? All mathematics professors have studies calculus.

Universe of discourse is set collection of maths professors.

$x \in U$

$p(x)$: x have studies calculus.

∴ Logical expression is $\forall x \, P(x)$.

'or'

Let universe of discourse be collection of all people then the predicates $p(x)$: x is a mathematics professor, $q(x)$: x have studied calculus.

∴ The logical expression is

$$\forall x \, [p(x) \wedge q(x)]$$

? All mathematics professors have studied calculus Leena is a mathematics professor. Therefore Leena have studied calculus. here U cannot be maths pro becoz we cannot rep and state so U → people

A. Let the universe of discourse be collection of people. The predicates are.

$p(x)$: x is a mathematics professor.

$q(x)$: x have studies calculus.

l : Leena

The logical expression is

$$\forall x \, [p(x) \rightarrow q(x)]$$

$$\frac{p(l)}{\therefore \cancel{[p(l) \wedge q(x)]}}$$

$$\therefore q(l)$$

? All men are mortal.
Socrates is a men.
Therefore Socrates is mortal.

A. U is the collection of all people.
The predicates are.

$p(x)$: x is a men.
$q(x)$: x is ~~Socrates~~ mortal.
s : Socrates

The logical expression is.

$$\forall x\,[p(x) \rightarrow q(x)]$$
$$\frac{p(s)}{\therefore q(s)}$$

? One student in the class knowns how to make programs in JAVA and everyone who knows how to write programs in JAVA can get a high paying job. imply the conclusion someone in this class can get a high paying job.

$p(x)$: x is a student in the class
$q(x)$: x knows how to write progm in JAVA.
$r(x)$: x. ~~get~~ will get a high paying job.

$$\exists x \; [p(x) \to q(x)]$$
$$\forall x \; [q(x) \to r(x)]$$
$$\therefore \exists x \; [p(x) \to r(x)]$$

30/10/17

? Every computer science student needs a course in Maths. Salim is a CS student.
∴ Salim needs a course in Maths.

A. The logical expression are:-

$p(x)$: x is a cs student.
$q(x)$: x needs a course in Maths.
s : salim.

∴ The premise & the conclusion are.

$$\forall x \; [p(x) \to q(x)]$$
$$p(s)$$
$$\therefore q(s)$$

? A student in this class has not read the book. and everyone in this class passed the first examination imply the conclusion. someone who passed in the first examination has not read the book.

and means next premise

A. The predicates are.

$p(x)$: x is in this class.

$q(x)$: x has read the book.

$r(x)$: x has passed in the first examination.

The premise & conclusion are.

$$\exists x \, [p(x) \rightarrow \neg q(x)]$$
$$\underline{\forall x. \, [p(x) \rightarrow r(x)]}$$
$$\therefore \exists x. \, [r(x) \rightarrow \neg q(x)]$$

Inference Theory for Predicate calculus.

1) Rule of universal Specifications.

If a predicate becomes true for all replacements by the member of the given universe then that predicate is true for each specific individual. member in that universe ie,

if $\forall x \, p(x)$ is true. then we can conclude that $p(c)$ is true, for c is an arbitrary member of the universe

2) Rule of Universal Generalisation

If a predicate $p(x)$ is proved to be true when x is replaced by any arbitrarly choosen element c from our universe then the universal quantifier ~~A~~ $\forall x \, p(x)$ is true.

ie, If c is the arbitrary member of the universe and if p(c) is true then we conclude that $\forall x\, p(x)$ is true.

3) Rule of Existential Specification

This rule allows us to conclude that if $\exists x\, p(x)$ is true then p(c) is true where c is not an arbitrary member of the universe, but one among them for which p(c) is true

4) Rule of Existential Generalisation.

This rule is used to conclude that for a particular element c in the universe, if p(c) is true then $\exists x\, p(x)$ is true

31/10/19

Note :-

→ Universal specification is used to eliminate the universal quantifier from the quantified statement. Universal generalisation is used to introduce the universal quantifier into the statement.

→ Existential specification is used to elimate existential quantifier. And Existential Generalisation is used to introduce existential quantifier.

? Check the validity of following arguments.

Consider the predicates

$m(x)$: x is a Physics professor.

$c(x)$: x have done Physics lab.

r : Rashmi

The logical expression is:

$$\begin{array}{l} \forall x\,[m(x) \rightarrow c(x)] \\ m(r) \\ \hline \therefore\ c(r) \end{array}$$

A.

Steps	Reasons
1) $\forall x\,[m(x) \rightarrow c(x)]$	premise
2) $m(r) \rightarrow c(r)$	U. Specifications.
3) $m(r)$	premise
4) $c(r)$	2,3 by modus ponens

? $$\begin{array}{l} \forall x\,[P(x) \rightarrow Q(x)] \\ \forall x\,[R(x) \rightarrow \neg Q(x)] \\ \hline \forall x\,[R(x) \rightarrow \neg P(x)] \end{array}$$

Steps	Reasons
1) $\forall x\,[R(x) \rightarrow \neg Q(x)]$	premis.
2) $R(a) \rightarrow \neg Q(a)$	U.S.
3) $\forall x\,[P(x) \rightarrow Q(x)]$	Premise.
4) $P(a) \rightarrow Q(a)$	U.S.

5) $\neg Q(a) \rightarrow \neg P(a)$ contrapositive

6) $R(a) \rightarrow \neg P(a)$. 2&5 Law of syllogism

7) $\forall x \, [R(x) \rightarrow \neg P(x)]$ U.G.

Proof Technique

Direct Proof

Here we begin with the premise (hypothesis), continuing with a sequance of deduction we end with a conclusion

Q. For eg: If m is an even integer then prove that $m+7$ is odd integer. by direct proof method

A. Here the given hypothesis is m is an even integer we have to prove that $m+7$ is an odd integer.

Since m is even $\rightarrow m = 2k$, k is any integer.

Substituting $m = 2k$ in $m+7$ we have

$$m+7 = 2k+7$$
$$= (2k+6)+1$$
$$= 2(k+3)+1$$
$$= 2t+1 \quad ; \; t = k+3$$

Here $2t+1$ is an odd number since $2t$ is an even number. $\therefore$ $m+7$ is an odd numb

2/11/17.

Note:-

In many cases direct proof may not reach at a conclusion. then we use another methods for proving the theorems of the form $p \to q$ These are called indirect proofs. They are:-

1) Proof By Contraposition | Contrapositive Proof | Indirect Proof
2) Proof By Contridiction.
3) Proof By Counter example.
4) By Mathematical Induction.

1) Proof By Contraposition.

We know that the contrapositive of $p \to q$ is $\neg q \to \neg p$. So proving in this method we apply direct proof method to the statements. $\neg q \to \neg p$.

? If m is an even integer then $m+7$ is an odd integer by using contrapositive method

A. The contrapositive argument is.

"If $m+7$ is not odd integer then m is not an even integer. $(\neg q \to \neg p)$.

To prove this let $m+7$ is not an odd integer implies $m+7$ is an even integer $\to$ $m+7 = 2s$

where s is an integer $\rightarrow m = 2s - 7$

$= 2s - 8 + 1$

$= 2(s - 4) + 1$

$\rightarrow$ m is a odd integer $\rightarrow$ m is not even integer. Hence proved.

9. Prove that a perfect number is not prime by indirect proof method.

A. This statement can be rewritten as
"If x is a perfect number then x is not a prime number."
The contrapositive statement is
"If x is a prime number then x is not a perfect number."

A perfect number is a number whose divisors except the given number when added gives the given number.
eg: 6 divisors: 1, 2, 3, 6. exclude 6 & $1+2+3 = \underline{6}$
$\therefore$ 6 is a perfect number
18 is not a perfect number 1, 2, 3, 6, 9, 18
$1+2+3+6+9 \neq 18$.

Let x is a prime number then the only divisors of x are 1 & x
Leaving out x we will have the only divisor as 1 and 1 cannot be equal to x

∴ By the definition of perfect number x is not a perfect number. Hence proved.

2) <u>Proof By Contradiction</u>

In this method to prove $p \to q$ we will be assuming $\neg q$ (ie, the conclusion is false). And by deducing we will reach at a condition where some of our predefined statement is false. This will be happening since we have assumed a wrong argument.

? Prove that $\sqrt{2}$ is not a rational number by contradiction method.

A. Suppose that $\sqrt{2}$ is a rational number. Then by definition, of rational number, we have $\sqrt{2} = \frac{p}{q}$, where p & q are integers and p & q are relatively prime ie, there is no common divisors for p & q.

$\sqrt{2} = p/q$

On squaring, we have $2 = \frac{p^2}{q^2}$.

$\Rightarrow 2q^2 = p^2$ ie $p^2 = 2q^2$.

$\Rightarrow p^2$ is an even number.

$\Rightarrow p$ is an even number.

Then p can be written in the form
$p = 2k$, k is an integer.

Substituting,
$p = 2k$ in $p^2 = 2q^2$, we have

$$(2k)^2 = 2q^2 \Rightarrow 4k^2 = 2q^2$$
$$\Rightarrow q^2 = 4k^2$$
$\Rightarrow q^2$ is even number.
$\Rightarrow q$ is even number

ie, q can be written in the form $q = 2s$, where s is an integer.

$\therefore$ we have $p = 2k$ and $q = 2s$, where $k, s \in I$

ie, p and q have a common factor 2.
ie, p and q are not relatively prime.

This is contradiction.

$\therefore$ our assumption is false.

$\therefore$ $\sqrt{2}$ is not a rational number.

3/11/17

3) Proof By Counterexample

Suppose we want to prove that the statement $\forall x$. is false then. by this method we want to find an element x such that $p(x)$ is false. The number x

is said to be the counter example.

? Check whether the given statement is true or false with counter example method.

All prime numbers are odd

A. Consider the number 2 which is a prime number but not an odd number. Hence our statement is false. 2 is the counter example

4) Proof by Mathematical Induction

In this method we show that the result is true for $n=1$

Assume that the result is true for $n=k$

Then we will show that the result will be true for $n=k+1$ if so we conclude that the result is true for all natural numbers 'n'

? Using mathematical induction prove that if 's' is a finite set with n elements then s has 2^n subsets.

A. By mathematical induction first we have to prove that the given statement is true for $n=1$

For every set with one element it will definitely have only two subsets ie null sell $\{\phi\}$ & the set itself. Therefore the condition is true for $n=1$ ie, if a set 's' have only one element then it have two subsets.

Assume that the result is true for $n=k$ ie, if a set ~~A~~ s has k elements then it have 2^k subsets.

Finally we will prove that the statement is true for $n=k+1$ ie, we have to prove that if S is a set with $k+1$ elements then it has 2^{k+1} subsets.

Let $S=\{a_1, a_2, a_3, \ldots a_k, a_{k+1}\}$ with cardinality $k+1$

Let $S_1=\{a_1, a_2, a_3, \ldots a_k\}$ then $S=S_1 \cup \{a_{k+1}\}$

By assumption S_1 has 2^k subsets and $\{a_{k+1}\}$ has two subsets. Therefore in total S has $2^k \cdot 2$ subsets ie, 2^{k+1} subsets.

$\therefore$ The statement is true for $n=k+1$.

Hence the proof.