9) Solve the recurrence relation

$$2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n, \quad n \geq 0$$

$a_0 = 0$

$a_1 = 1$

$a_2 = 2$

$$2a_n = a_{n-1} + 2a_{n-2} - a_{n-3}, \quad n \geq 3.$$

$$2a_n - a_{n-1} - 2a_{n-2} + a_{n-3} = 0$$

$$2r^3 - r^2 - 2r + 1 = 0$$

$$r^2(2r-1) - (2r-1) = 0$$

$$(2r-1)(r^2-1) = 0$$

$2r - 1 = 0$ $\qquad (r+1)(r-1) = 0$

$\Rightarrow r = 1/2$ $\qquad r = -1, \ r = 1$

$$\Rightarrow a_n = c_1\left(\tfrac{1}{2}\right)^n + c_2(-1)^n + c_3(1)^n$$

$a_0 = 0.$

$0 = c_1 + c_2 + c_3.$

$a_1 = 1$

$1 = \frac{1}{2}c_1 - c_2 + c_3.$

$a_2 = 2$

$2 = \frac{1}{4}c_1 + c_2 + c_3.$

$a_0 = 0$

$\Rightarrow c_1 + c_2 + c_3 = 0$ — (1) $\qquad c_1 = -(c_2 + c_3)$

$a_1 = 1$

$\Rightarrow \frac{1}{2}c_1 - c_2 + c_3 = 1$ — (2)

$-\frac{8}{3} \quad \frac{1}{6} \quad \frac{5}{2}$

$a_2 = 2.$

$\frac{1}{4}c_1 + c_2 + c_3 = 2$ — (3)

$\Rightarrow c_1 = -\frac{8}{3} \qquad c_2 = \frac{1}{6}, \qquad c_3 = \frac{5}{2}$

$$\Rightarrow a_n = -\frac{8}{3}\left(\frac{1}{2}\right)^n + \frac{1}{6}(-1)^n + \frac{5}{2}(1)^n , \quad n \geq 3.$$

$a_0 = 0$
$a_1 = 1$
$a_2 = 2$

Q) $2a_n = 7a_{n-1} - 3a_{n-2}$, $a_0 = 2$. $n \geq 2$.
$a_1 = 5$.

Ans) $2a_n - 7a_{n-1} + 3a_{n-2} = 0$

$2r^2 - 7r + 3 = 0$

$2r^2 - 6r - r + 3 = 0$

$2r(r-3) - (r-3) = 0$

$(r-3)(2r-1) = 0$

$r = 3, \ \ r = 1/2$

$\Rightarrow a_n = c_1(3)^n + c_2\left(\frac{1}{2}\right)^n$

$a_0 = 2$

$\Rightarrow 2 = c_1 + c_2$ — (1)

$a_1 = 5$

$5 = 3c_1 + \frac{1}{2}c_2$ — (2)

$\Rightarrow c_1 = \frac{8}{5}, \ c_2 = \frac{2}{5}$

$$a_n = \frac{8}{5}(3)^n + \frac{2}{5}\left(\frac{1}{2}\right)^n$$

Q) $a_n = 2a_{n-1} + 2a_{n-2}$, $n \geq 2$, $a_0 = 1$, $a_1 = 3$.

Ans) $r^2 - 2r - 2 = 0$

$r = 1+\sqrt{3} \quad r = 1-\sqrt{3}$

$\Rightarrow a_n = c_1(1+\sqrt{3})^n + c_2(1-\sqrt{3})^n$

$a_0 = 1$

$\Rightarrow 1 = c_1 + c_2 \quad — (1)$ $\quad \Rightarrow (1+\sqrt{3}) = (1+\sqrt{3})c_1 + (1+\sqrt{3})c_2$

$a_1 = 3$ $\quad\quad 3 = (1+\sqrt{3})c_1 + (1-\sqrt{3})c_2$

$\Rightarrow 3 = (1+\sqrt{3})c_1 + (1-\sqrt{3})c_2 \quad — (2)$ $\quad 3 - 1 - \sqrt{3} = (\cancel{1} - \sqrt{3} - \cancel{1} - \sqrt{3})c_2$

$2 - \sqrt{3} = -2\sqrt{3}\,c_2$

$\Rightarrow c_2 = \dfrac{\sqrt{3}-2}{2\sqrt{3}}$

$\Rightarrow c_1 = 1 - \dfrac{(\sqrt{3}-2)}{2\sqrt{3}} = \dfrac{2\sqrt{3} - \sqrt{3} + 2}{2\sqrt{3}} = \dfrac{\sqrt{3}+2}{2\sqrt{3}}$

$\Rightarrow a_n = \dfrac{\sqrt{3}+2}{2\sqrt{3}}(1+\sqrt{3})^n + \dfrac{(\sqrt{3}-2)}{2\sqrt{3}}(1-\sqrt{3})^n$

2) $a_{n+2} = 4a_{n+1} - 4a_n, \quad n \geq 0, \quad a_0 = 1, \quad a_1 = 3$

$\Rightarrow r^2 - 4r + 4 = 0$

$\Rightarrow (r-2)^2 = 0$

$\Rightarrow r = 2, \; r = 2$

$\Rightarrow a_n = (c_1 + c_2 n)(2)^n.$

$a_0 = 1,$

$\Rightarrow 1 = (c_1)$

$a_1 = 3,$

$\Rightarrow 3 = (1 + c_2)2$

$= 2 + 2c_2$

$\Rightarrow c_2 = 1/2$

$\therefore a_n = (1 + \frac{1}{2}n)(2)^n = (2+n)2^{n-1}$

Q) $d_n = 2d_{n-1} - d_{n-2}$, $n \geq 2$.

$\cancel{d_0 =}$

$d_1 = 2$

$d_2 = 3$.

Ans) $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1, 1$

$\Rightarrow d_n = (C_1 + C_2 n)$

$d_1 = 2$

$\Rightarrow 2 = C_1 + 2C_2$

$d_2 = 3$

$3 = C_1 + 2C_2$

$2 = C_1 + C_2$

$1 = C_2$

$C_1 = 3 - 2C_2$

$= 3 - 2 = 1$

$\Rightarrow d_n = (1 + n)$

$\alpha + i\beta$

$\alpha - i\beta$.

$C_1 e^{\alpha + i\beta} + C_2 e^{\alpha - i\beta}$

$= e^{\alpha}(C_1 e^{i\beta} + C_2 e^{-i\beta})$

$= e^{\alpha}(C_1(\cos\beta$

Case-3

When roots are complex.

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)}$$

Let $a+ib$ and $a-ib$ be the roots.

$\Rightarrow a_n = c_1(a+ib)^n + c_2(a-ib)^n$

$a+ib = r(\cos\theta + i\sin\theta)$

$\pi + \frac{\pi}{4}$

$= \frac{5\pi}{4}$.

Q) $a_n + 2a_{n-1} + 2a_{n-2} = 0$, $n \geq 2$, $a_0 = 1$, $a_1 = 3$

$r^2 + 2r + 2 = 0$

$\Rightarrow r = -1 \pm i$

$r = -1+i$, $r = -1-i$.

$-1+i$

$0, 2\pi$.

$r = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{-1}\right)$

$= \tan^{-1}(-1)$

$= \pi - \tan^{-1}(1)$

$= \frac{-\pi}{4}$

$2\pi - \frac{\pi}{4}$

$\Rightarrow a_n = c_1(-1+i)^n + c_2(-1-i)^n$

$= c_1\sqrt{2}^n\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^n$

$+ c_2\sqrt{2}^n\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^n$.

$= c_1\sqrt{2}^n\left(\cos\frac{3n\pi}{4} + i\sin\frac{3n\pi}{4}\right) + c_2\sqrt{2}^n\left(\cos\frac{5n\pi}{4} - i\sin\frac{5n\pi}{4}\right)$

$= \sqrt{2}^n\left[c_1\left(\cos\frac{3n\pi}{4} + i\sin\frac{3n\pi}{4}\right) + c_2\left(\cos\frac{5n\pi}{4} - i\sin\frac{5n\pi}{4}\right)\right]$

$a_0 = 1$

$\Rightarrow 1 = c_1 + c_2$ — (1).

$a_1 = 3$

$3 = \sqrt{2}\left(c_1\frac{1}{\sqrt{2}} + c_2\frac{1}{\sqrt{2}} + c_1 i\frac{1}{\sqrt{2}} - c_2 i\frac{1}{\sqrt{2}}\right)$

$3 = (c_1 + c_2 + i(c_1 - c_2))$

$\Rightarrow 3 = -(c_1 - c_2) + i(c_1 - c_2)$

$3 = (c_1 - c_2)(-1 + i)$

$c_1 - c_2 = \frac{3}{-1+i} = \frac{-3-i}{2}$

$c_1 + c_2 = 1$

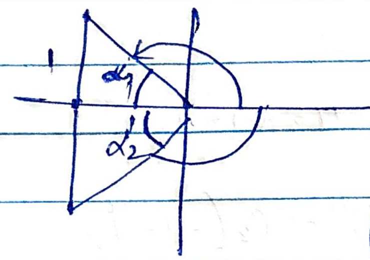
$2c_1 = \frac{3}{-1+i} + 1 = \frac{3-1+i}{-1+i}$

$$= \frac{2+i}{-1+i} = \frac{(2+i)(-1-i)}{2} = \frac{-2-2i-i+1}{2}$$

$$= \frac{-1-3i}{2}$$

$$c_1 = \frac{1-4i}{2} \qquad c_2 = \frac{1-(1-4i)}{2}$$

$$= \frac{1+4i}{2}.$$

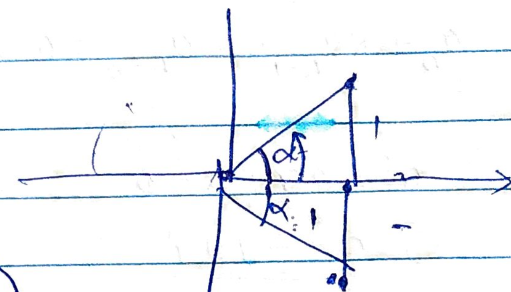


$-1+i$

$-1-i$

$\tan^{-1}\left|\frac{1}{1}\right|$

$= 45^\circ$



$(1+i)$

$(1-i)$

$\tan^{-1}\left|\frac{1}{1}\right|$

$\pi - 45^\circ$

$= \pi - \pi/4$

$3\pi/4$

$= 45^\circ$

-45°

$-(180 - \alpha_2)$

$-(180 - \pi/4)$

$-3\pi/4$

$(\pi/4) \quad (\pi/4)$

$(3\pi/4)$

$(5\pi/4)$

-8

$\textcircled{5}$

~~$a_n + 2a_{n-1} + 2a_{n-}$~~

$(-1+i) = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

$(-1-i) = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$

$a_n = c_1(-1+i)^n + c_2(-1-i)^n$

$= \sqrt{2}^n c_1\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)^n + \sqrt{2}^n c_2\left(\cos\left(\frac{3\pi}{4}\right) - i\sin\left(\frac{3\pi}{4}\right)\right)^n$

$= c_1\sqrt{2}^n\left(\cos\left(\frac{3n\pi}{4}\right) + i\sin\left(\frac{3n\pi}{4}\right)\right) + c_2\sqrt{2}^n\left(\cos\frac{3n\pi}{4} - i\sin\left(\frac{3n\pi}{4}\right)\right)$

$= \sqrt{2}^n\cos\left(\frac{3n\pi}{4}\right)(c_1+c_2) + i\sqrt{2}^n\sin\left(\frac{3\pi}{4}\right)(c_1 - c_2)$

$= \underline{\sqrt{2}^n\cos\left(\frac{3n\pi}{4}\right) + 4\sqrt{2}^n\cos\left(\frac{3n\pi}{4}\right)}$

$a_0 = 1$

$\Rightarrow 1 = c_1 + c_2$ — (1)

$c_1 + c_2 = \cancel{\frac{1-4i + 1+4i}{2}}$

$= 1$

$a_1 = 3.$

$3 = -1\,\cancel{(c_1+c_2)} + i\,(c_1 - c_2)$

$\therefore c_1 - c_2 = -4i$ — (2)

$c_1 + c_2 = 1$

$2c_1 = 1 - 4i$

$c_1 = \frac{1-4i}{2}$

$\Rightarrow 2c_2 = 1 + 4i$

$c_2 = \frac{1+4i}{2}$

Non Homoginious recurrence relation

$a_n + a_{n-1} + \dots = f(n)$

a^x

$\frac{1}{f(}$

$a_n = a_n^{(h)} + a_n^{(p)}$

homoginious (particular solution)

$a_n - 3a_{n-1} = 5(7^n)$ — ① $a_0 = 2$.

$a_n - 3a_{n-1} = 0$ — ②

$r - 3 = 0$

$r = 3$.

$a_n^h = c_1(3)^n$.

$a_n^{(p)} = A(7^n)$ (Assumption).

$a_{n-1} = A(7^{n-1})$

① ⇒ $A(7^n) - 3A(7^{n-1}) = 5(7^n)$

$7A - \cancel{21}\overset{3}{A} = 5 \times 7$.

$4A = 35$

$A = \frac{35}{4}$

$a_n = \frac{35}{4}(7)^n$

$a_n = c_1(3^n) + \frac{35}{4}(7^n)$

$2 = c_1 + \frac{35}{4}$

$c_1 = \frac{8 - 35}{4} = \frac{-27}{4}$

⇒ $a_n = \left(-\frac{27}{4}\right)(3^n) + \frac{35}{4}(7)^n = -\frac{1}{4}(3^{n+3}) + \frac{5}{4}(7)^{n+1}$

Q) $a_{n+1} - 2a_n = 1$, $a_0 = 0$, $n \geq 0$

$a_n - 2a_{n-1} = 1$, $a_0 = 0$, $n \geq 1$

$r - 2 = 1$

Pigeon hole Principle (Dirichlet's drawer principle)

If there are m pigeons with n pigeon holes, and $m > n$, there will be atleast one pigeon hole with two or more pigeons in it

Q) How many students must be there in a class to guarantee that two students got same marks in final marks, if exam is graded on a scale from 0 to 100 marks.

Ans) There are 101 possible marks. So if there are 102 students, we can garuntee atleast two of them gets the same score

Generalised PHP

If there are N objects, placed into k boxes, then, there is atleast one box containing, $\lceil N/k \rceil$ objects in it.

Q) What is the minimum no. of students required in the class to be sure that, atleast six will recieve the