DISCRETE COMPUTATIONAL STRUCTURES

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3/8/2017
Thursday

MODULE - I

REVIEW OF ELEMENTARY SET THEORY

SET

A set is a well defined collection of objects. The objects are called elements or members of the set.

Note :-

- We use capital letters with/without subscripts to denote a set.
- Lowercase letters are used to denote the elements of the set.

There are two types of representation of set.

i) Roster method / Tabular form.

In this method we can list the elements in any order and enclosing them within curly braces

eg: $A = \{1, 3, 5\}$

ii) Set builder form

In this method we will be giving a description about ~~whether~~ the element ~~belongs to the set~~. So that we can identify the elements of the set.

eg: $A = \{x / x$ is an odd integer b/w 1 & 10.$\}$

Here in the above example "is an odd integer b/w 1 & 10" is the discription of the elements of the set. x is the representativ of the elements of set. The tabular form of above example is:

$$A = \{3, 5, 7, 9\}$$

Examples for a set

$N = \{1, 2, 3 \ldots\}$ set of natural numbers

$B = \{$ table, chair, pen, apple $\}$

$X = \{x : x$ is an vowel of English alphabet$\}$

Example for not a set

- Beautiful girls in the society
- Five eminet scientist in India.

> If an element p belongs to a set A then we write $p \in A$.
> The symbol $\in$ denotes "element of"

> If q is not an element of A then will denote as $q \notin A$

> We can write element of as included in or belongs to.

eg: $A = \{1, 2, 3, 4\}$

$1 \in A$

$2 \in A$

$6 \notin A$

Finite & Infinite Sets

The set which contains finite no: of elements is called as finite set. And a set with infinite no: of elements is called as infinite set.

eg: A = {1, 2, 3, 4, 5} finite

B = {1, 2, 3, ...} infinite

Cardinality

The no: of distinct elements of a set is called its cardinality. It is usually denoted by n, #, ||

eg: Let A = {1, 2, 3, 4}

n(A) = 4

B = {1, 2, 2, 3, 3, 4}

n(B) = 4

Since from the repeatation we will consider only 1

Equal Sets & Equivalent Sets.

Let A and B be two sets the sets A & B are called equal sets if set A and B have same elements.

The set A and B are said to be equivalent sets. If the cardinality of A and B are equal.

eg: A = {1, 2, 3} B = {1, 2, 3} ⟹ A = B

A = {1, 2, 3} B = {4, 5, 3} ⟹ A & B are equivalent sets since n(A) = n(B)

- Remark: Every equal sets are equivalent set.

Subset and Superset

Let A and B be two sets we call A is a subset of B or A is included in B if every element of A is an element of B & symbolically $A \subseteq B$. Then B is called the superset of A.

eg: $A = \{1, 2, 3, 4, 5\}$ $B = \{1, 2\}$

$B \subseteq A$

Remark: Every set is a subset of itself.

Empty Set or Null Set

A set which contains no elements is called an empty set or null set. It is usually denoted by ϕ or $\{\}$.

Remark:

The cardinality of the null set is zero.

Null set is the subset of every set.

Universal Set

A set is called a universal set if it includes every set under discussion. It is denoted by E or U.

eg: $A \{1, 2, 3\}$ and $B = \{a, b, c, d\}$

$U = \{1, 2, 3, a, b, c, d\}$

Power Set

For a set A the family of all subsets of A is called the powerset of A. It is denoted by $\wp(A)$ or P(A). The cardinality of powerset of A is $2^{n(A)}$

eg: Let $A = \{1,2,3,4\}$

$$P(A) = \{\phi, \{1,2,3,4\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$$

$$n(P(A)) = 2^4 = 16$$

Proper Subset or Proper inclusion

A set A is called a proper subset of B if A is subset of B. But A is not equal to B. It is denoted by $\subset$.

eg: Consider a set $A = \{1,2,3\}$ & $B = \{1,2,3,4,5\}$

Then A is a proper subset of B, symbolically, $A \subset B$

Operations on Set

1) Union

Let A & B to any two sets then the union of A & B denoted by '$A \cup B$' is defined as the set of all elements of A and the set of all elements of B and the common elements being taken once.

eg: Let $A = \{a, b, c, d\}$ $B = \{*, +, -, /\}$

Then $A \cup B = \{a, b, c, d, *, +, -, /\}$

2) Intersection

Let A & B be two sets then the intersection of A & B is denoted by '$A \cap B$' is a set of all elements common to both A & B.

eg: $A = \{1,2\}$ $B = \{2,4\}$.

$A \cap B = \{2\}$

3) Disjoint Sets

Let A & B be two sets which are said to be disjoint if $A \cap B = \phi$

4) Disjoint Collection

A collection of sets is called a disjoint collection if every pair of the set two at a time in the collection are disjoint.

The elements of a disjoint collection are said to be mutually disjoint sets.

eg: $A = \{1\}$ $B = \{a\}$ $C = \{*\}$

Then $A \cap B = \phi$, $B \cap C = \phi$ & $A \cap C = \phi$

then the family A, B, C is a disjoint collection and A, B, & C are called mutually disjoint sets.

5) Difference of Sets

Let A & B be two sets the difference of A with respect to B is the set of all elements that are in A but not in B. It is also called as relative compliment of B in A. It is denoted by A-B.

eg: Let A = {1,2,3}.
B = {4,5,3,6}
A-B = {1,2}   B-A = {4,5,3,6}

Note:

$A-B \neq B-A$

If A & B are equal sets then A-B and B-A are null se

6) Compliment set / Absolute Compliment

Let A be a set then compliment of A denoted by ~A or $\bar{A}$ is defined by U-A, where U is the universal set.

7) Symmetric difference / Boolean Sum

Let A & B be two sets then symmetric difference of A and B ~~be two sets then~~ denoted by either A+B, A⊕B, AΔB is defined as A+B = (A-B) ∪ (B-A)

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$A = \{1,2,3,4\}$

$B = \{1,2,a,b,c\}$

find the symmetric difference of A & B.

$A-B = \{3,4\}$

$B-A = \{a,b,c\}$

$(A-B) \cup (B-A) = \{3,4,a,b,c\}$

Venn Diagram

A venn diagram is the diagramatic representation of set operations.

Ordered Pairs

An ordered pair consists of two objects in a given fixed order. It is denoted by (,) or < , >

Note :-

The objects or elements to be ordered need not be distinct / different.

The equality of two ordered pairs. Let it be (x,y) & (u,v) is defined by $x=u$ & $y=v$

Cartesian Product

Let A and B be two sets, the cartesian product of A and B denoted by $A \times B$ is defined as $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$

ie, the set all ordered pairs in which the first element from A and second element from B

? Let $A = \{\alpha, \beta, \gamma\}$ $B = \{a, b, c, d, e\}$ find $A \times B$ & $B \times A$.

A. $A \times B = \{(\alpha,a), (\alpha,b), (\alpha,c), (\alpha,d), (\alpha,e), (\beta,a), (\beta,b), (\beta,c), (\beta,d), (\beta,e), (\gamma,a), (\gamma,b), (\gamma,c), (\gamma,d), (\gamma,e)\}$

$B \times A = \{(a,\alpha), (a,\beta), (a,\gamma), (b,\alpha), (b,\beta), (b,\gamma), (c,\alpha), (c,\beta), (c,\gamma), (d,\alpha), (d,\beta), (d,\gamma), (e,\alpha), (e,\beta), (e,\gamma)\}$

Note:-

→ $A \times B \neq B \times A$, but $n(A \times B) = n(B \times A)$

→ If A & B are two sets, then $n(A \times B) = n(A) \cdot n(B)$

RELATIONS

Relation can be refered to anything that connects to objects

Binary Relations

A binary relation denoted by R is the collection of all ordered pairs that satisfy some relational condition.

In other words, a binary operation R from A to B is the subset of cartesian product satisfying the relation.

We can represent relation in any of the three forms:

$x \in A$, $y \in B$, $R: A \to B$ such that

i) $(x,y) \in R$. where R is the relation

ii) xRy which is read as x is related to y.

iii) The relation set will be given in the tabular form.

? Let R is the relation from A to B where A is the set of natural numbers and the ~~reta~~ B is set of ~~w~~ even numbers and the relation is. $\leq$

A. $A = \{1, 2, 3, 4 \ldots\}$ $B = \{2, 4, 6, \ldots\}$

Here the relation is given to $\leq$

ie, $(x,y) \in R$ if $x \leq y$ where $x \in A$ & $y \in B$.

$R = \{(1,2), (1,4), (2,2) \ldots\ldots\}$

? Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{6, 7, 8, 9, 10\}$.

R is a relation from $B \to A$ defined by $>$

ie, $(x,y) \in R \Rightarrow x > y$ $x \in A$ & $y \in B$

A. $R = \{\phi\}$

? In above question $x \in B$ & $y \in A$.

$R = \{(6,1), (6,2), (6,3), (6,4), (6,5),$

(7,1), (7,2), (7,3), (7,4), (7,5),
(8,1), (8,2), (8,3), (8,4), (8,5),
(9,1), (9,2), (9,3), (9,4), (9,5),
(10,1), (10,2), (10,3), (10,4), (10,5) }

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Remark

For every $A \in U$, $A \times \phi = \phi$

Proof

Suppose that $A \times \phi \neq \phi$, that means there exist some ordered pair (atleast one) in $A \times \phi$. Let it be (a,b). By definition of cartesian product

if $(a,b) \in A \times \phi$

$$\Rightarrow a \in A \ \& \ b \in \phi$$

But $b \in \phi$ is impossible since ϕ is a null set. So assumption is wrong ie $A \times \phi \neq \phi$ is impossible

$\therefore$ if $A \in U$ then $A \times \phi = \phi$

Domain & Range of a Relation

Let R be a relation then the set D(R) called the domain the relation, is the collection of on x such that for some $y: (x,y) \in R$. Similary set range(R) is the set of all elements y such that for some x, $(x,y) \in R$ is called the range (R).

eg: In the above example ">" the domain(R) is $D(R) = \{6, 7, 8, 9, 10\}$

Range(R) $= \{1, 2, 3, 4, 5\}$

Inverse Relation

Let R be a relation from set A to B then the inverse of R is the relation from B to A and is given by $R^{-1} = \{(y,x) : (x,y) \in R\}$

? Let $A = \{1,2,3\}$ $B = \{6,7,8\}$. Let R be a relation from $A \to B$ defined by $x < y$ where $x \in A$ & $y \in B$

A. $R = \{(1,6),(1,7),(1,8),(2,6),(2,7),(2,8),(3,6),(3,7),(3,8)\}$

$R^{-1} = \{(6,1),(7,1),(8,1),(6,2),(7,2),(8,2),(6,3),(7,3),(8,3)\}$

$D(R) = \{1,2,3\}$ $\quad$ $Range(R) = \{6,7,8\}$

$D(R^{-1}) = \{6,7,8\}$ $\quad$ $Range(R^{-1}) = \{1,2,3\}$

Types of Relation & Their Properties

Void Relations (Empty Relation)

The relation R in a set A is called a void relation or empty relation if no element of set A is related to any element of set A.

ie $R = \phi$

eg: $A = \{1,2,5,8\}$

$R : A \to A$ defined by $x+y=1$

$R = \{\} \Rightarrow$ R is an empty relation.

Identity Relation | Universal Relation | Trivial Relation

For a given set A, $I = \{(a,a) : \forall a \in A\}$ is called the identity relation in A.

eg: $A = \{2,3,4\}$

$R : A \to A$ defined by $x = y$ $(x,y) \in A$

$R = \{(2,2), (3,3), (4,4)\}$ is an identity relation.

Symmetric Relation

A relation R is a set A is called symmetric relation if $\forall (x,y) \in A$ whenever xRy then yRx

eg: $A = \{1,2,3\}$

$R = \{(1,2), (1,3), (2,1), (3,1)\}$

This is a symmetric relation since the symmetric pairs (1,2) and (1,3) is present in R.

Reflexive Relation

The relation R in a set A is reflexive if for every element of $x \in A$ x related to x itself.

eg: $A = \{1,2,3,4\}$

Let $R = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,3), (4,4)\}$

This is reflexive relation but not a symmetric relation.

Transitive Relation

The relation R in a set A is transitive if $\forall (x,y,z) \in A$ whenever xRy & yRz then xRz

eg: $A = \{1,2\}$

$R = \{(1,1), (1,2), (2,1), (2,2)\}$

This relation is reflexive, symmetric & transitive

Antisymmetric Relation

A relation R on a set A is antisymmetric if $\forall (x,y) \in A$ whenever xRy and yRx then $x=y$

eg: $A = \{1,2,3\}$

$R = \{(1,1), (1,2), (1,3), (3,3)\}$

$A = \{1,2,3\}$

$R = \{(1,2), (2,1), (3,3)\}$

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Irreflexive Relation

A relation R on a set A is irreflexive if $\forall x \in A$ $(x,x) \notin R$

eg: $A = \{1,2,3\}$

$R = \{(1,2),(2,1),(3,2)\}$

It is irreflexive, not symmetric, not transitive, not antisymmetric

Equivalence Relation

A relation on a set A is called an equivalence relation if and only if it is reflexive, symetric, transitive.

? Let $A = \{1,2,3,4\}$ and R be a relation on A defined by $R = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(4,4)\}$. Check whether it is an equivalence relation. R is defined on A.

A. For equivalence relation we have to check reflexivity, symmetry, and transitive.

i) Reflexive

Here in this relation every element of A is related to itself and hence reflexivity is attained

ii) Symmetry

Here in this R for every pair the symmetric pairs are also present. So symmetry is attained.

iii) Transitive

Here $\forall$ pair if a & b are related and b & c are related $\Rightarrow$ a & c are related. Hence transitive

So in general it is an equivalence equation.

? Let $X = \{1,2,3,4,5,6,7\}$. R is a relation on x defined by $R = \{(x,y) / x-y$ is divisible by $3\}$

$$R = \{ (1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7)$$
$$(1,4), (4,1), (2,5), (5,2), (3,6), (6,3), (4,7), (7,4),$$
$$(1,7), (7,1) \}$$

For equivalence R reflexivity, symmetry & transitivity must be attained.

Reflexive

Here in this relation every element of A is related to itself and hence reflexivity is attained.

Symmetry

Here in this R for every pair symmetric pairs are also present. So symmetry is attained.

Reflexive

For any $a \in X$, $a-a=0$ which is divisible by 3. So by definition of relation every element of A is connected itself which means it is reflexive

Symmetry

For any $(a,b) \in X$ if $(a-b)$ divisible by 3 then clearly $(b-a)$ divisible by 3 (In this case number will be same only sign changes

Hence it is symmetric

Transitive

For any $(a,b,c) \in X$

For transitive we have to check if aRb & bRc then aRc

Here aRb ⇒ $a-b$ is divisible by 3

⇒ $a-b$ is a multiple of 3

bRc ⇒ $b-c$ is divisible by 3

⇒ $b-c$ is a multiple of 3

Our aim is to check $a-c$ is divisible by 3 or $a-c$ is multiple of 3

$$a-c = (a-b) + (b-c)$$

$$= a-b+b-c$$

⇒ $a-c$ is a multiple of 3 since $(a-b)+(b-c)$ is a multiple of 3.

Thus it is an equivalence relation.

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Relational Matrix & Graph of a Relation.

We can represent the relation from a set X to Y in three ways.

i) Matrix form

ii) Arrow diagram

iii) Graphical method

i) By Relational Matrix. / Matrix of Relation.

Step 1: Let $X = \{x_1, x_2, \cdots, x_m\}$

$Y = \{y_1, y_2, \cdots, y_n\}$ be any two sets

Consider a relation R from $X \rightarrow Y$

Step 2 : Construct a table from row entries at X and column entries at Y. If $x_i R y_i$ where $x_i \in X$ and $y_i \in Y$ then ~~the~~ enter 1 in the i^{th} row and j^{th} column. If $x_k \not R y_l$ then we enter zero to k^{th} row & l^{th} column

Step 3 : Form the matrix from the above table containing only 1 and 0.

? Let $X = \{1, 2, 3\}$ and $Y = \{x, y, z\}$ a relation is from $X \rightarrow Y$. defined by $R = \{(1,y), (1,z), (3,y)\}$. Represent this in matrix form.

A. $X = \{1, 2, 3\}$ $\quad\quad Y = \{x, y, z\}$

$R = \{(1,y), (1,z), (3,y)\}$

X \ Y	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0

The matrix form is.

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

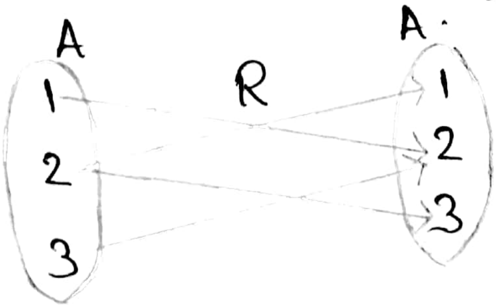
ii) By Arrow Diagram

In this we write down the element of set A and the elements of set B in two disjoint disk, and then draw an arrow from $a \in A$ ~~&~~ to $b \in B$ if and only if $(a,b) \in R$ and the relation is from $A \to B$.

? Consider $A = \{1,2,3\}$ R is a relation on A with relation, $R = \{(1,2),(2,1),(3,2),(2,3)\}$. Draw arrow diagram

A. $A = \{1,2,3\}$

$R = \{(1,2),(2,1)(3,2),(2,3)\}$



iii) By Graphical Method / Directed Graph method.

A relation can be represented pictorically or diagrammatically by a graph. In this method the relation is taken from a finite set to itself ~~infinite sets~~.

Let R be a relation in a set $X = \{x_1, x_2, \cdots x_m\}$. Then the elements of X are represented by points or ~~circles~~ circles called nodes/vertices. The vertices corresponding to x_i & x_j must be labelled according to the given relation.

ie, if $x_i R x_j$ we connect x_i & x_j by a directed arc or a directed line.

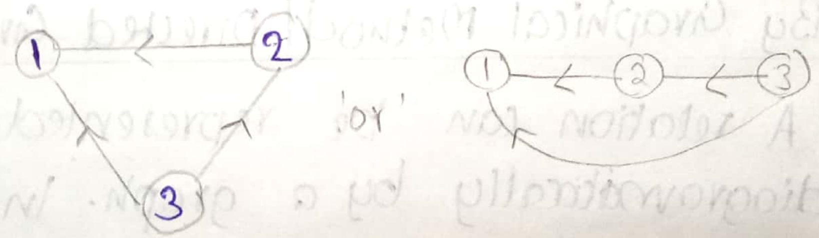
The relation is reflexive we get a loop.
ie, $x_i R x_i \Rightarrow$ there is an arc from x_i to x_i which is called loop.

If the relation is symmetric then we will be having two arcs in the opposite direction. For the sake of simplicity we can draw one line with 2 arrow signs in opposite direction.

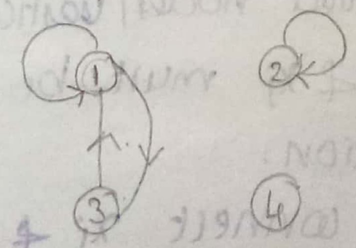
? $X = \{1,2,3\}$ R is a relation on X defined by $R = \{(x,y) \mid x > y\}$. Draw the directed graph of this relation.

A. $X = \{1,2,3\}$

$R = \{(2,1),(3,1),(3,2)\}$



? $X = \{1,2,3,4\}$ R is a relation on X defined by $R = \{(1,1),(2,2),(3,1),(1,3)\}$ ~~(4,4)~~. Draw this in directed graph.



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Partition.

Let S be a given set and $A = \{A_1, A_2, \ldots, A_m\}$ where each $A_1, A_2 \ldots A_m$ are sets. A is said to be a covering of S if it satisfies two conditions

i) Each A_i where $i = 1, 2, \ldots m$ is a non empty subset of S.

$A_i \subseteq S$ & $A_i \neq \phi$.

ii) $A_1 \cup A_2 \cup \ldots \cup A_m = S$

In this case each A_i $i = 1$ to n is the power. of S. Let S. be a given set then $A = \{A_1, A_2, A_3 \ldots A_m\}$ be a family of set.

then if it satisfies the following properties.

1) Each A_i, $i = 1$ to n is a non empty subset of S.

2) $A_1 \cup A_2 \cup \ldots \cup A_m = S$.

3) Each A_i is mutually disjoint ie for $i \neq j$ $A_i \cap A_j = \phi$

If this is the case then A is called the partition of S. Here each A_i, $i = 1$ to m are called the blocks of the partitions.

? Let $S = \{a, b, c\}$ Consider the following collection

$A = \{\{a,b\}, \{b,c\}\}$ $B = \{\{a\}, \{a,c\}\}$

$C = \{\{a\}, \{b,c\}\}$ $D = \{\{a,b,c\}\}$

$E = \{\{a\}, \{b\}, \{c\}\}$ $F = \{\{a\}, \{a,b\}, \{a,\cancel{b}\}\}$

Find out which of the following are covering and partion of S.

A a) Consider $S = \{a, b, c\}$

$A = \{\{a,b\}, \{b,c\}\}$.

Let $A_1 = \{a,b\}$ and $A_2 = \{b,c\}$

For covering, we have to check 2 conditions.

1) Given A_1 & $A_2 \subseteq S \neq \phi$

2) $A_1 \cup A_2 = \{a,b,c\} = S$.

$\therefore$ A is covering of S.

To check the partition,

$A_1 \cap A_2 = \{b\} \neq \phi$ violates the condition,

hence A is not a partition of S.

$\therefore$ Given family A is only a covering of S not a partition.

b) $B = \{\{a\}, \{a,c\}\}$

$B_1 = \{a\}$ $B_2 = \{a,c\}$

1) B_1 & $B_2 \subseteq S \neq \phi$.

2) $B_1 \cup B_2 = \{a,c\}$.

$\therefore \Rightarrow$ B not covering & not partions.

c) $C = \{\{a\}, \{b,c\}\}$

$C_1 = \{a\}$ $C_2 = \{b,c\}$

1) C_1 & $C_2 \subseteq S \neq \phi$

2) $C_1 \cup C_2 = \{a,b,c\} = S$.

3) $C_1 \cap C_2 = \phi$.

$\Rightarrow$ C is covering & partition.

d) $D = \{\{a,b,c\}\}$

$D_1 = \{a,b,c\}$ $D_2 =$

1) $D_1 \subseteq S \neq \phi$.

2) $D_1 = S$.

3) Intersection always $= \phi$

$\Rightarrow$ D is covering & partition.

e) $E = \{\{a\},\{b\},\{c\}\}$

$E_1 = \{a\}$ $E_2 = \{b\}$ $E_3 = \{c\}$

1) E_1 & E_2 & $E_3 \subseteq S \neq \phi$

2) $E_1 \cup E_2 \cup E_3 = \{a,b,c\} = S$.

3) $E_1 \cap E_2 \cap E_3 = \phi$

$\Rightarrow$ E is covering & partition

f) $F = \{\{a\},\{a,b\},\{a,c\}\}$

$F_1 = \{a\}$ $F_2 = \{a,b\}$ $F_3 = \{a,c\}$.

1) F_1 & F_2 & $F_3 \subseteq S \neq \phi$.

2) $F_1 \cup F_2 \cup F_3 = \{a,b,c\} = S$.

3) $F_1 \cap F_2 \cap F_3 = \{a\} \neq \phi$.

F is covering & not partition.

From the above example we can conclude that
i) For any finite set the smallest partition consists of singleton elements of the set.
ii) The largest partition consist of the block containing only one element ie the main set.
iii) Every partition is a covering but every covering is not a partition.

12/8/2017

Equivalence Class.

Suppose R is an equivalence relation on a set S for each 'a' ∈ S, let the equivalence class of 'a' denoted by $[a]_R$. It is the set of all elements of S to which 'a' is related under R ie,

$$[a]_R = \{y / (a,y) \in R\}$$

The members of the equivalence class are called the representative of the equivalence class.

The collection of all equivalence classes of elements of S under an equivalence relation R is called the quotient of S by R and is denoted by S/R.

? Let X = {a, b, c, d, e} and R = {(a,a), (b,b), (a,b), (b,a), (c,c), (d,d), (e,e), (d,e), (e,d)} is a relation on S. Find the equivalence classes and hence the quotient set if it exist.

A : Step 1 : Checking for equivalence relation.

It is reflexive, because $\forall$ element of X is related to itself.
It is symmetric, since the pairs (a,b) & (d,e) have their symmetric pairs (b,a) & (e,d) in R.
It is transitive, since if you take xRy & yRz then we can find xRz in R where $x, y, z \in X$

$\therefore$ Given R is an equivalence relation.

Step 2 : Finding equivalence class for every element of X

$[a]_R = \{a, b\}$

$[b]_R = \{b, a\}$

$[c]_R = \{c\}$

$[d]_R = \{d, e\}$

$[e]_R = \{e, d\}$

Step 3 : The quotient set.

$X|R = \{[a]_R, [b]_R, [c]_R, [d]_R, [e]_R\}$

$= \{a, b, c, d, e\}$

Remark :-

We can generate an equivalence relation from a partition. For that.

Step 1 : First we name with capital letters the blocks of the given partition. ie, if X is the given set and C is a partition then we will be considering the names of blocks of C as $C_1, C_2, C_3 \ldots C_n$

Step 2 : For any $a \in X$ we have to find a set or block, Let it be $C_1 \in C$ such that $a \in C_1$ but it doesnot belongs to any other blocks $C_2, C_3 \ldots C_n$.

Step 3 : Take the cartesian product of the corresponding block to itself $C_1 \times C_2$.

Step 4 : The equivalence relation R is the union of all cartesian products.

? Let $X = \{a, b, c, d, e\}$ and let $C = \{\{a,b\}, \{c\}, \{d,e\}\}$ be the partition. Find the equivalence relations to this partition.

A. Step 1 : $X = \{a, b, c, d, e\}$ and $C = \{\{a,b\}, \{c\}, \{d,e\}\}$

Let $C_1 = \{a, b\}$, $C_2 = \{c\}$, $C_3 = \{d, e\}$

Step 2 : Let $a \in X$, then $a \in C_1$ but $a \notin C_2$ & C_3

Hence $C_1 \times C_2 = \{(a,a), (b,b), (a,b), (b,a)\}$

$b \in X$ & $b \in C_1$ but $b \notin C_2$ & C_3

hence $C_1 \times C_2 = \{(a,a), (b,b), (a,b), (b,a)\}$

$c \in X$ & $c \in C_2$ but $c \notin C_1$ & C_3

hence $C_2 \times C_2 = \{c, c\}$

$d \in X$ & $d \in C_3$ but $d \notin C_1$ & C_2

hence $C_3 \times C_3 = \{(d,d), (e,e), (d,e), (e,d)\}$

$e \in X$ & $e \in C_3$ but $e \notin C_1, C_2$

hence $C_3 \times C_3 = \{(d,d), (e,e), (d,e), (e,d)\}$

Step 3 : Setting the relation S.

$S = (C_1 \times C_1) \cup (C_2 \times C_2) \cup (C_3 \times C_3)$

$= \{(a,a), (b,b), (a,b), (b,a), (c,c), (d,d), (e,e), (e,d), (d,e)\}$

17/8/17

FUNCTIONS

A function from $X \rightarrow Y$ is defined as a relation from $X \rightarrow Y$ such that every element of X is related to exactly one element in Y.

We usually denote the functions by lowercase letters.
eg: h, g, x etc.

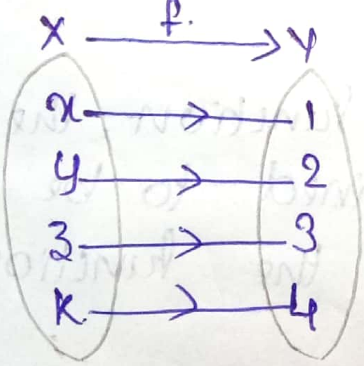
Let $f : X \rightarrow Y$ be a function, the domain of the function is defined to be the set X
The co-domain of the function is defined to be the set Y
Consider an element $a \in X$ & $b \in Y$. If the element a is related to element b, then we call 'b' as an image of 'a' under f
In that case 'a' is called the pre image of 'b' under 'f'
The range of a function 'f' is the collection of all images under 'f'

We normally represents the functions by arrow diagram where the given sets are represented by circles/disks & if the elements of the sets are related then we will be drawing an arrow b/w them.

? Let $X = \{x, y, 3, k\}$ and $Y = \{1, 2, 3, 4\}$. Let 'f': $X \to Y$ determine which of the following are functions. Justify your answer draw the arrow diagram. Find domain, range & co-domain of the function

i) $f = \{(x,1), (y,2), (3,3), (k,4)\}$

It is a function since every element of X is related to Y and it have only one image.



Domain $= \{x, y, 3, k\}$

Range $= \{1, 2, 3, 4\}$

Codomain $= \{1, 2, 3, 4\}$

ii) $g = \{(x,1),(y,1),(k,4)\}$.

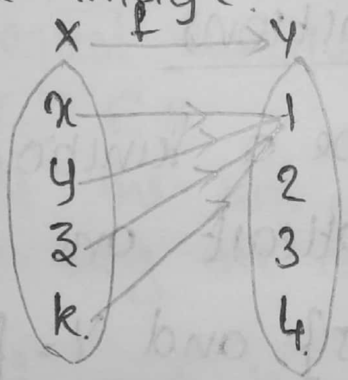
'g' is not a function, since every element of X is not related to Y.

iii) $h = \{(x,1),(x,2),(x,3),(x,4)\}$

'h' is not a function, since every element of X is not related to Y and the element 'x' has more than one image.

iv) $l = \{(x,1),(y,1),(k,1),(z,1)\}$

It is a function since every element of X has exactly one image.



Domain $= \{x,y,z,k\}$

Range $= \{1\}$

Co-domain $= \{1,2,3,4\}$

<u>Remark:</u>

Codomain of a function need not be equal to range.

TYPE OF FUNCTION

1. Injective (One-to-one)

A mapping $f: X \to Y$ is called injective if the distinct elements of X are mapped to distinct elements of Y ie every element in X must have unique image in Y

eg: Let $X = \{x, y, z, k\}$ $Y = \{1, 2, 3, 4\}$ $f: X \to Y$ is a function defined by $f = \{(x,1), (y,2), (z,3), (k,4)\}$

2. Surjective / Onto functions

Let $f: X \to Y$ be a function then if each $\in Y$ must have atleast one preimage in X.

eg: Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{a, b, c, d\}$ $f: X \to Y$ defined by

This is not one-one but onto.

Remark: If the function is onto the range of f is equal to co-domain.

If range(f) $\neq$ codomain we call that functions as into functions.

3. Bijective Function

Let $f : X \to Y$ be a function then if 'f' is one-one & onto.

eg: Let $X = \{x, y, z, k\}$ & $Y = \{1, 2, 3, 4\}$ $f : X \to Y$.

defined by $f = \{(x,1), (y,2), (z,3), (k,4)\}$

18/8/17 Equal functions

Consider two functions f & g from set $X \to Y$ is called equal functions if and only if $f(a) = g(a) \ \forall$ 'a' $\in X$

If this is not the case for atleast one element in X then they are called unequal functions.

? Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$. Let $f : X \to Y$ $g : X \to Y$ and $h : X \to Y$ be defined as

$f = \{(1,a), (2,a), (3,c)\}$ $g = \{(1,b), (2,a), (3,c)\}$

$h = \{(1,a), (2,a), (3,c)\}$

Which of the above are equal functions.

A. $f = \{(1,a), (2,a), (3,c)\} \Rightarrow f(1) = a \ ; \ f(2) = a \ ; \ f(3) = c.$

$g = \{(1,b), (2,a), (3,c)\} \Rightarrow g(1) = b \ ; \ g(2) = a \ ; \ g(3) = c$

$h = \{(1,a), (2,a), (3,c)\} \Rightarrow h(1) = a \ ; \ h(2) = a \ ; \ h(3) = c.$

Here $f \neq g$ since $f(1) = a \neq g(1) = b$.

$g \neq h$ since $g(1) = b \neq h(1) = a$.

$f = h$ since $f(1) = h(1)$; $f(2) = h(2)$; $f(3) = h(3)$

Identity Function

Consider any A. Let the function $f: A \to A$ is said to be identity function if each element of set A has image on itself.
ie, $f(a) = a \ \forall a \in A$

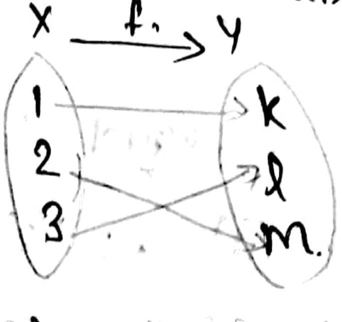
~~Note :-~~

Inverse of a Function / Invertible function.

A function $f: X \to Y$ is called invertible or it posses inverse if and only if 'f' is a bijective function.

? Let $X = \{1, 2, 3\}$ and $Y = \{k, l, m\}$ $f: X \to Y$ defined by $f = \{(1,k), (2,m), (3,l)\}$. Check whether f is invertible or not.

A.



Here f is one-one & onto and hence it is invertible.

$\therefore f^{-1} : Y \rightarrow X$ defined by

$f^{-1} = \{(k,1), (m,2), (l,3)\}$

Composition of function

Consider functions $f: A \rightarrow B$ & $g: B \rightarrow C$. The composition of 'f' with 'g' is the function from $A \rightarrow C$ defined by $g \circ f(x) = g(f(x)) \; \forall x \in A$

? $X = \{1,2,3\}$ $Y = \{a,b\}$ $Z = \{5,6,7\}$

$f: X \rightarrow Y$ $g: Y \rightarrow Z$ defined by

$f = \{(1,a), (2,a), (3,b)\}$ $g = \{(a,5), (b,7)\}$

Find the composition $g \circ f$.

A. $f(1) = a$; $f(2) = a$; $f(3) = b$

$g(a) = 5$; $g(b) = 7$,

$g \circ f : X \rightarrow Z$ defined by

$g \circ f(1) = g(f(1)) = g(a) = 5$

$g \circ f(2) = g(f(2)) = g(a) = 5$

$g \circ f(3) = g(f(3)) = g(b) = 7$.

? $X = \{1,2,3\}$ f, g, h, s be functions from $X \rightarrow X$ defined by $f = \{(1,2), (2,3), (3,1)\}$
$g = \{(1,2), (2,1), (3,3)\}$ $h = \{(1,1), (2,2), (3,1)\}$
$s = \{(1,1), (2,2), (3,3)\}$

Find :-

i) $f \circ g$
ii) $g \circ f$
iii) $s \circ g$
iv) $g \circ s$
v) $s \circ s$
vi) $f \circ s$

A.

$f(1) = 2$	$f(2) = 3$	$f(3) = 1$
$g(1) = 2$	$g(2) = 1$	$g(3) = 3$
$h(1) = 1$	$h(2) = 2$	$h(3) = 1$
$s(1) = 1$	$s(2) = 2$	$s(3) = 3$

i)
$$f \circ g(1) = f(g(1)) = f(2) = 3$$
$$f \circ g(2) = f(g(2)) = f(1) = 2$$
$$f \circ g(3) = f(g(3)) = f(3) = 1$$

ii)
$$g \circ f(1) = g(f(1)) = g(2) = 1$$
$$g \circ f(2) = g(f(2)) = g(3) = 3$$
$$g \circ f(3) = g(f(3)) = g(1) = 2$$

iii)
$$s \circ g(1) = s(g(1)) = s(2) = 2$$
$$s \circ g(2) = s(g(2)) = s(1) = 1$$
$$s \circ g(3) = s(g(3)) = s(3) = 3$$

iv)
$$g \circ s(1) = g(s(1)) = g(1) = 2$$
$$g \circ s(2) = g(s(2)) = g(2) = 1$$
$$g \circ s(3) = g(s(3)) = g(3) = 3$$

v) $s \circ s(1) = s(s(1)) = s(1) = 1$

$s \circ s(2) = s(s(2)) = s(2) = 2$

$s \circ s(3) = s(s(3)) = s(3) = 3$

we have to write... set

vi) $f \circ s(1) = f(s(1)) = f(1) = 2$

$f \circ s(2) = f(s(2)) = f(2) = 3$

$f \circ s(3) = f(s(3)) = f(3) = 1$

21/8/17

? fohog from $x \to x$ is defined by.

$f \circ h \circ g(x) = f \circ h(g(x))$

$f \circ h \circ g(1) = f \circ h(g(1)) = f \circ h(2) = f(h(2)) = f(2) = \underline{\underline{3}}$

$f \circ h \circ g(2) = f \circ h(g(2)) = f \circ h(1) = f(h(1)) = f(1) = \underline{\underline{2}}$

$f \circ h \circ g(3) = f \circ h(g(3)) = f \circ h(3) = f(h(3)) = f(1) = \underline{\underline{2}}$

Countable & Uncountable Sets.

Two sets A & B are said to equipotent or to have the same number of elements or same cardinality if there exist a one-to-one & an onto mapping on a function $f: A \to B$

Any set which is equipotent to the set of natural number is called denumerable.

ie, there exist a bijection from the set A to the set of natural numbers then A is called denumerable set.

A set called countable if it is finite/denumerable

Otherwise the set is uncountable.

eg: set of natural numbers.

eg: set of even numbers with $f(x) = 2x$; $x \in$ set of even numbers:

Set of real numbers is uncountable.

Partial Oder Relation

Consider a relation R on a set P satisfying the properties

i) R is reflexive

ii) R is antisymmetric

iii) R is transitive

Then R is called a partial order relation.

The set P together with a partial order relation is called a partial ordered set or poset.

The partial order relation is denoted by '$\leq$' and a poset is denoted by $(P, \leq)$

? Verify whether the set of natural numbers form a poset under the relation $\leq$.

A. For proving poset we have to check the conditions.

Reflexive : Since every element of N related to itself under the relation '$\leq$' (mainly =)

Antisymmetric: It is antisymmetric since we can only find either xRy or yRx.

ie, symmetric pair never occur

Transitive : The set N is transitive, since whenever xRy & yRz clearly xRz. where $x, y, z \in N$.

$\therefore$ N is a poset.

? Consider a set $A = \{4, 9, 16, 36\}$ is the relation 'divides' is a partial order.

A. $R = \{\cancel{(4,16)}, \cancel{(4,36)}, \cancel{(9,36)}, (16,4), (36,4), (36,9), (4,4), (9,9), (16,16), (36,36)\}$

Reflexive: since every element of A is related to itself, R is reflexive.

Antisymmetric : for every xRy & yRx ; $x \neq y$ where $x, y \in A$. Hence R is ~~not~~ antisymmetric for eg: $(4,16)$ & $(16,4) \in R$ but $16 \neq 4$.

Transitive : for every xRy & yRz , xRz where $x, y, z \in A$. Hence R is transitive.

Since R is not antisymmetric it is ~~not~~ a partial order. since it is reflexive, antisymmetric & transitive

? Consider $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$. Can you form the ordered pairs that satisfies the condition divisibility.

A. $R = \{(1,1), (2,1), (2,2), (3,1), (3,3), (5,1), (5,5), (6,1), (6,2), (6,3), (6,6), (30,30), (10,1), (10,2), (10,5), (10,10), (15,1), (15,3), (15,5), \cancel{(15,10)}, (15,15), (30,1), (30,5), (30,10), (30,15)\}$

Reflexive : Since every element is divisible by itself. $(x,x) \in R \; \forall x \in A$. So R is reflexive.

Transitive : When xRy & yRz ; xRz where $x,y,z \in A$. Hence R is transitive.

Antisymmetric: It is antisymmetric since we can only find either xRy or yRx where $x,y \in A$.

22/8/17

Comparable elements.

Consider an ordered set A, two elements a & b of set A are called comparable. if a & b are related. If they are not related they are called non-comparable elements.

? Consider $A = \{1,2,3,5,6,10,15,30\}$ is ordered by divisibility. Determine all the comparable & non-comparable pairs of elements of A.

A. The comparable pair of elements are.

$\{\{1,2\}, \{1,3\} \{1,5\} \{1,6\} \{1,10\} \{1,15\} \{1,30\}, \{2,6\}$
$\{2,10\} \{2,30\} \{3,6\} \{3,15\} \{3,30\} \{5,10\} \{5,15\} \{5,30\}$
$\{6,30\} \{10,30\} \{15,30\}\}$

Non comparable elements.

$\{\{2,3\}, \{2,5\}, \{2,15\}, \{3,5\}, \{3,10\}, \{5,6\}, \{6,10\}, \{6,15\}, \{10,15\}\}$

Total Order Relation | Linearly Ordered Relation

Consider an ordered set A, the set A is called totally ordered set if every pair of elements in A are comparable.

? Consider the set $I = \{1,2,3,\ldots\}$ is ordered by divisibility. Determine whether each of the following subsets of I are linearly ordered or not.

i) $\{2,4,8\}$

The possible pairs are $\{2,4\}, \{4,8\}$ & $\{2,8\}$. We know that all these pairs satisfy divisibility & hence it is a totally ordered set.

ii) $\{3,6,9,11\}$

The possible pairs are $\{3,6\}$ $\{3,9\}$ $\{3,11\}$ $\{6,9\}$ $\{6,11\}$ $\{9,11\}$

Here $\{3,11\}$ is not divisible ie, it is not comparable. hence it is not a totally ordered set.

iii) $\{1\}$

Here only one element so no need to check for divisibility, it is alway comparable. Hence it is totally ordered set.

iv) $\{2,4,6,8,10,\ldots\}$

The set is not totally ordered since every pair is not comparable.

Q. Show that the relation '<' defined on set of natural numbers is neither an equivalence relation nor an partial order relation but is a total order relation

A. The given set of natural numbers under the relation '<' is neither an equivalence relation nor a partial order relation since $(N, <)$ doesn't satisfy the reflexivity property.
This is a totally ordered set since every pair of this set are comparable under '<'

Hasse Diagram

In a partially ordered set $(A, \leq)$ under some relation. An element $y \in A$ is said to cover an element $x \in A$ if $x \leq y$ & if there doesn't exist any element $z \in A$ such that $x \leq z$ & $z \leq y$
The pictorial representation of partial order relation is called Hasse diagram.

Procedure for drawing Hasse Diagram

1) Each element is represented by small circle/dot.
2) The circle of $x \in A$ is drawn below the circle of $y \in A$ if y is a cover of x or y is directly related to x.
3) If there is any element in b/w x & y satisfying the relation, the line cannot be drawn b/w x & y.

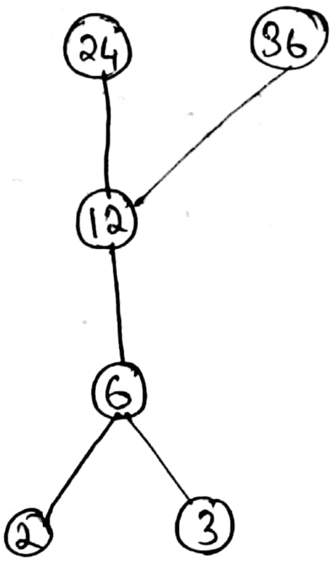
24/8/17

? Let $X = \{2, 3, 6, 12, 24, 36\}$ and the relation is divisibility. Draw the Hasse diagram with this relation.

A. $R = \{(2,6), (2,12), (2,24), (2,36), (3,6), (3,12), (3,24), (3,36), (6,12), (6,24), (6,36), (12,24), (12,36), (2,2), (3,3), (6,6), (12,12), (24,24), (36,36)\}$

This is a partial order relation.

Hasse Diagram.



? Consider the set $A = \{4, 5, 6, 7\}$ Let the relation be $\leq$. Draw the Hasse diagram.

A. $R = \{(4,5), (4,6), (4,7), (5,6), (4,4), (5,7), (5,5), (6,6), (6,7), (7,7)\}$

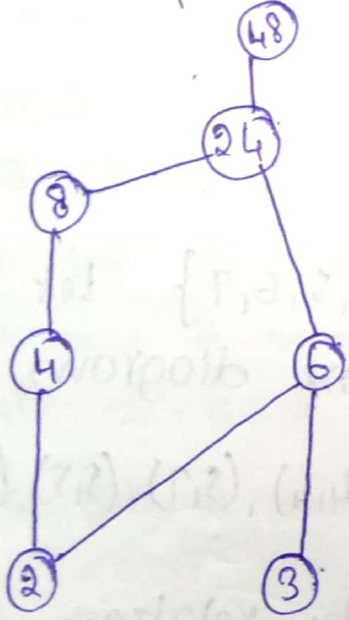
This is a partial order relation.

Hasse Diagram

? $A = \{2, 3, 4, 6, 8, 24, 48\}$ ordered by divisibility is a poset. Draw the Hasse diagram.

$R = \{(2,2), (3,3), (4,4), (6,6), (8,8), (24,24), (48,48), (2,4), (2,6), (2,8), (2,24), (2,48), (3,6), (3,24), (3,48), (4,8), (4,24), (4,48), (6,24), (8,24), (8,48), (24,48)\}$

This is a partial order relation.



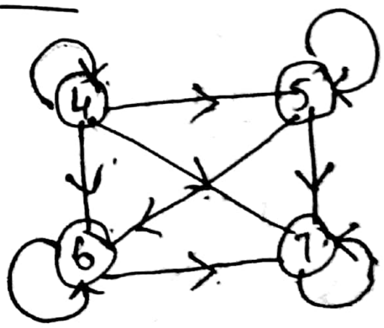
29/8/2017

Procedure To Convert a Directed Graph of a Relation to its equivalent Hasse Diagram.

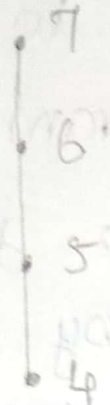
1) The vertices in the Hasse diagrams are denoted by points rather than by circles.
2) Since a partial order relation is reflexive, each vertex must be related to itself. But in Hasse diagram these edges must be deleted.
3) A partial order is transitive. So in Hasse diagram eliminate all the edges that are implied by the transitive property.
4) If a vertex 'a' is connected to vertex 'b' by an edge, then the vertex 'b' appears above the vertex 'a'. And therefore the arrows may be omitted in the Hasse diagram.

? Consider the set $A = \{4,5,6,7\}$ and let
$R = \{(4,5),(4,6),(4,7),(4,4),(5,5),(5,6),(5,7),(6,6),(6,7),(7,7)\}$
on A. Draw the directed graph & hence draw the Hasse diagram.

A. <u>Directed Graph</u>



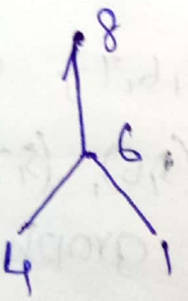
Hasse Diagram.



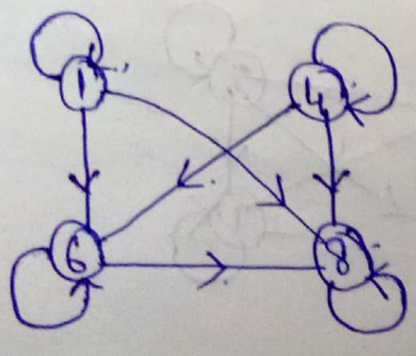
- Here we have deleted the loops in the directed graph, removed the arrows, the vertices are represented by dots and we have deleted the edges (4,7), (5,7) & (4,6) that resembles transitive property

? Draw the directed graph of the relation. Determine by the Hasse diagram defined on set A. $A = \{1,4,6,8\}$ as shown below. 1

A.



$R = \{(4,4), (1,1), (6,6), (8,8), (4,6), (1,6), (6,8), (4,8), (1,8)\}$



Elements of Posets

Maximal Element

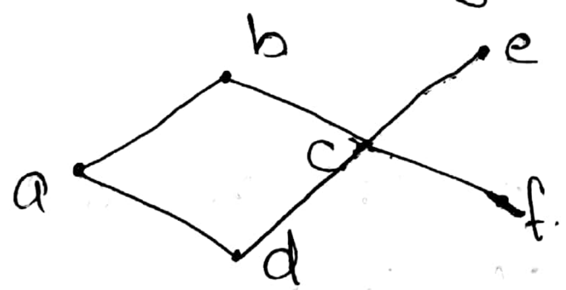
An element $x \in A$ is called a maximal element of A. If there is no element 'c' in A such that $x \leq c$ ($x \leq c \Rightarrow x$ is partially related to c)

Minimal Element

An element $y \in A$ is called a minimal element of A. If there is no element 'c' in A such that $c \leq y$

Remark:
There can be more that one maximal/minimal elements. It is not unique.

? Determine all the maximal & minimal elements of the poset shown by the Hasse diagram.



A. Here the maximal elements are b & e. and the minimal elements are d & f.

? Let $A = \{2, 3, 4, 6, 8, 24, 48\}$ with ordering divisibility. Determine all the maximal & minimal elements of A.

{2,2} {4,4}

{2,4} {2,6} {2,8} {2,24} {2,48} {3,6} {3,24} {3,48}

{4,8} {4,24} {4,48} {6,24} {6,48} {8,24} {8,48} {24,48}

Maximal. 48

Min : 2,3

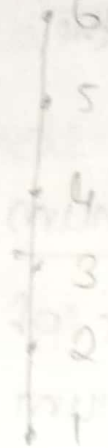
Greatest Element

An element $x \in A$ is called the greatest element of A if for all $y \in A$, $y \leq x$.

Least Element

An element $y \in A$ is called the least element of A if $\forall\ b \in A$, $y \leq b$.

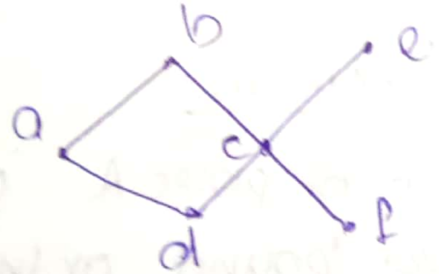
? Consider a poset $A = \{1,2,3,4,5,6\}$ be ordered as below. Find the greatest element & least element of A.



Greatest element = 6, least element = 1.

<u>Remark</u>:-

The greatest element & least element if they exist are unique.

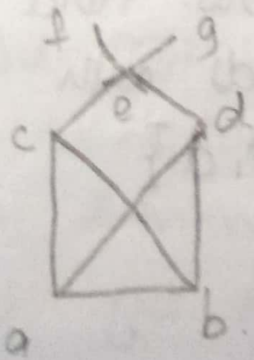


No greatest & least element. Since b & e are not related & d & f are not related.

<u>Upper Bound</u>

14/9/2021 Consider B be a subset of poset A. An element $x \in A$ is called an upper bound of B if $y \leq x, \forall y \in B$.

For ex:



Lower Bound

Consider B be a subset of the poset A an element $z \in A$ is called the lower bound of B, if $z \leq x \;\forall\; x \in B$

Least Upper Bound / Suprimum

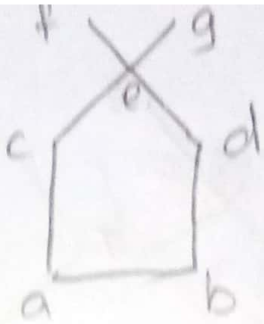
Consider B be a subset of poset A. An element $x \in A$ is called a suprimum of B denoted by LUB(B) or Sup(B), if x is the upper bound of B and x' is any other upper bound of B then $x \preceq x'$ [Supremum implies the least of all upper bounds]

Greatest Lower Bound / Infimum

Consider B be a subset of poset A. 'y' is said to be the greatest lower bound or infimum of B if y is the lower bound of B and if y' is any other lower bound of B then $y' \leq y$ [Infimum implies the greatest of all lower bounds]

Examples

Consider set $A = \{a, b, c, d, e, f, g\}$ The hasse diagram is given below. Find the upper bounds & lower bounds with their supremum & infimum for $B = \{c, e, d\}$



A. The upper bounds of B are e,f & g

∴ Sup(B) = e

Lower bounds of B = a, b

No infimum

? Determine the least upper bound, infimum for the set B = {a, b, c} whose Hasse diagram is given below.

A. Least Upper bound of B ~~& Infimum~~

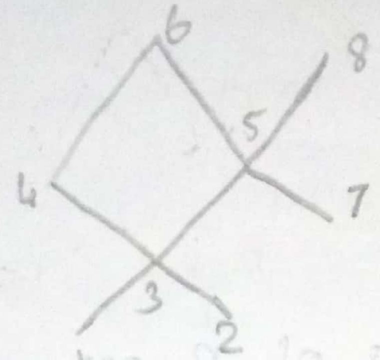
Upper bound of B = c, d, e

Sup(B) = C

Lower bound of B = k.

Infimum = k.

? Consider the poset A = {1,2,3,4,5,6,7,8} be ordered as below. B = {3,4,5}. Find the supremum & infimum



Upper bound is 5, 6.

Sup (B) = 5

Lower bound = 3

LATTICES

A lattice is a poset in which every pair has a supremum & infimum

Join

Consider a poset L under the order ≤ Let a, b ∈ L. Then supremum of a & b is called join of a & b and is denoted by a ⊕ b or a ∨ b

Meet

Consider the poset L under the order ≤. Let a, b ∈ L. The infimum of a, b is called the meet of a & b is denoted by a ∗ b or a ∧ b.

Remark :

The lattice is denoted by (L, ∗, ⊕)