Mate1009 Algebra — Lecture 2

Mārtiņš Kālis
Notes partially adapted from
K. R. Mathews "Elementary linear algebra"

17 February 2021

1 Systems of linear equations

First, let us review what we've learned in the first lecture. Consider the following scenario:

Example 1.0.1. An investor has €10'000 to invest, and three investment options, each yielding an expected 1%, 10% and 15% annually. The higher the expected return, the higher the risk. To limit exposure, the investor wants to invest only €5'000 in the two riskier options. However she still wants the expected return to be €600 at the end of the year. How should she allocate her funds?

Solution. First, we can represent the problem as a system of linear equations (x_1, x_2, x_3 are the amount of euros invested in the three options):

$$\begin{cases} x_1 + x_2 + x_3 = 10000 \\ x_2 + x_3 = 5000 \\ 0.01x_1 + 0.1x_2 + 0.15x_3 = 600 \end{cases}$$

If we restrict ourselves to simple financial tools, we should also ensure that the investments are non-negative numbers (i.e., $x_i \geq 0$ for $i = 1, 2, 3$), but let's ignore that for now. We could also choose a better representation (e.g. we can notice that $x_2 + x_3 = 5000$ implies $x_1 = 5000$) and/or we could solve this system using tools learned in school. But for illustration purposes, we could try to use the tools from the first lecture, i.e. the augmented matrix and elementary row operations. The following is an augmented matrix for the system.

$$\begin{bmatrix} 1 & 1 & 1 & 10000 \\ 0 & 1 & 1 & 5000 \\ 0.01 & 0.1 & 0.15 & 600 \end{bmatrix}$$

We could apply the following elementary row operations:

1. $R_3 \to 100 \cdot R_3$
2. $R_1 \to R_1 - R_2$

3. $R_3 \to R_3 - R_1$
4. $R_3 \to R_3 - 10 \cdot R_2$
5. $R_3 \to R_3/5$
6. $R_2 \to R_2 - R_3$

This gets us to the reduced row-echelon form matrix for the system:

$$\begin{bmatrix} 1 & 0 & 0 & 5000 \\ 0 & 1 & 0 & 4000 \\ 0 & 0 & 1 & 1000 \end{bmatrix}$$

Keeping in mind the system this matrix represents, we can read the solutions right off the matrix. We can also write down the system explicitly to make it easier:

$$\begin{cases} x_1 = 5000 \\ x_2 = 4000 \\ x_3 = 1000 \end{cases}$$

The investor should invest €5'000 in the first option, €4'000 in the second and €1'000 in the third.

We took the following steps to solve the problem:

1. Translate the problem to a system of linear equations.
2. Translate that to the augmented matrix of the system.
3. Transform the matrix to the reduced row-echelon form using elementary row operations.
4. Read the solution off the reduced row-echelon matrix.

The second and fourth steps should be straight-forward. The first one involves some irreducible difficulty. What about the third step? We found a solution in this case. Is there an algorithm for the general case? Yes, there is, and we will study it in the next subsection.

Problem 1.0.1. An investor has €10'000 to invest, and three investment options, each yielding an expected 1%, 1%, 10% and 15% annually (i.e., an additional 1% option). The higher the expected return, the higher the risk. To limit exposure, the investor wants to invest only €5'000 in the two riskier options. However she still wants the expected return to be €600 at the end of the year. How should she allocate her funds?

1.3 Gauss-Jordan algorithm

Gauss-Jordan algorithm is a process that from matrix A produces a row equivalent matrix B that is in reduced row-echelon form. As we saw previously, if A is the augmented matrix of a system of linear equations, B gives an easy way to solve this given system.

Notation. If a matrix is in reduced row–echelon form, it is useful to denote the column numbers in which the leading entries 1 occur, by $c_1, c_2, \dots, c_r$, with the remaining column numbers being denoted by $c_{r+1}, \dots, c_n$, where r is the number of non–zero rows (or equivalently — the number of leading entries). For example, in the following 4×6 matrix, we have $r = 3$ columns with leading entries, their numbers are $c_1 = 2, c_2 = 4, c_3 = 5$ and $n - r = 3$ columns without leading entries with numbers $c_4 = 1, c_5 = 3, c_6 = 6$.

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

With this motivation and notation, we can move forward with the algorithm itself. This consists of the following broad steps:

1. Find the first non–zero column moving from left to right (column c_1) and select a non–zero entry from this column. By interchanging rows, if necessary, ensure that the first entry in this column is non–zero. Divide row 1 by a_{1c_1} ($R_1 \to R_1/a_{1c_1}$) thereby converting a_{1c_1} to 1. For each non–zero element a_{ic_1}, $i > 1$, (if any) in column c_1, add $-a_{ic_1}$ times row 1 to row i ($R_i \to R_i - a_{ic_1} R_1$), thereby ensuring that all elements in column c_1, apart from the first, are zero.

2. If the matrix obtained at step 1 has its $2^{\text{nd}}, \dots, m^{\text{th}}$ rows all zero, the matrix is in reduced row–echelon form. Otherwise suppose that the first column which has a non–zero element in the rows below the first is column c_2. Then $c_1 < c_2$. By interchanging rows below the first, if necessary, ensure that a_{2c_2} is non–zero. Then convert a_{2c_2} to 1 ($R_2 \to R_2/a_{2c_2}$) and by adding suitable multiples of row 2 to the remaining rows, where necessary, ensure that all remaining elements in column c_2 are zero.

3. Continue in a similar manner for rows $3, 4, \dots, r$ and columns $c_3, c_4, \dots, c_r$.

The process is repeated and will eventually stop after r steps, either because we run out of rows, or because we run out of non–zero columns. In general, the final matrix will be in reduced row–echelon form and will have r non–zero rows, with leading entries 1 in columns $c_1, \dots, c_r$, respectively.

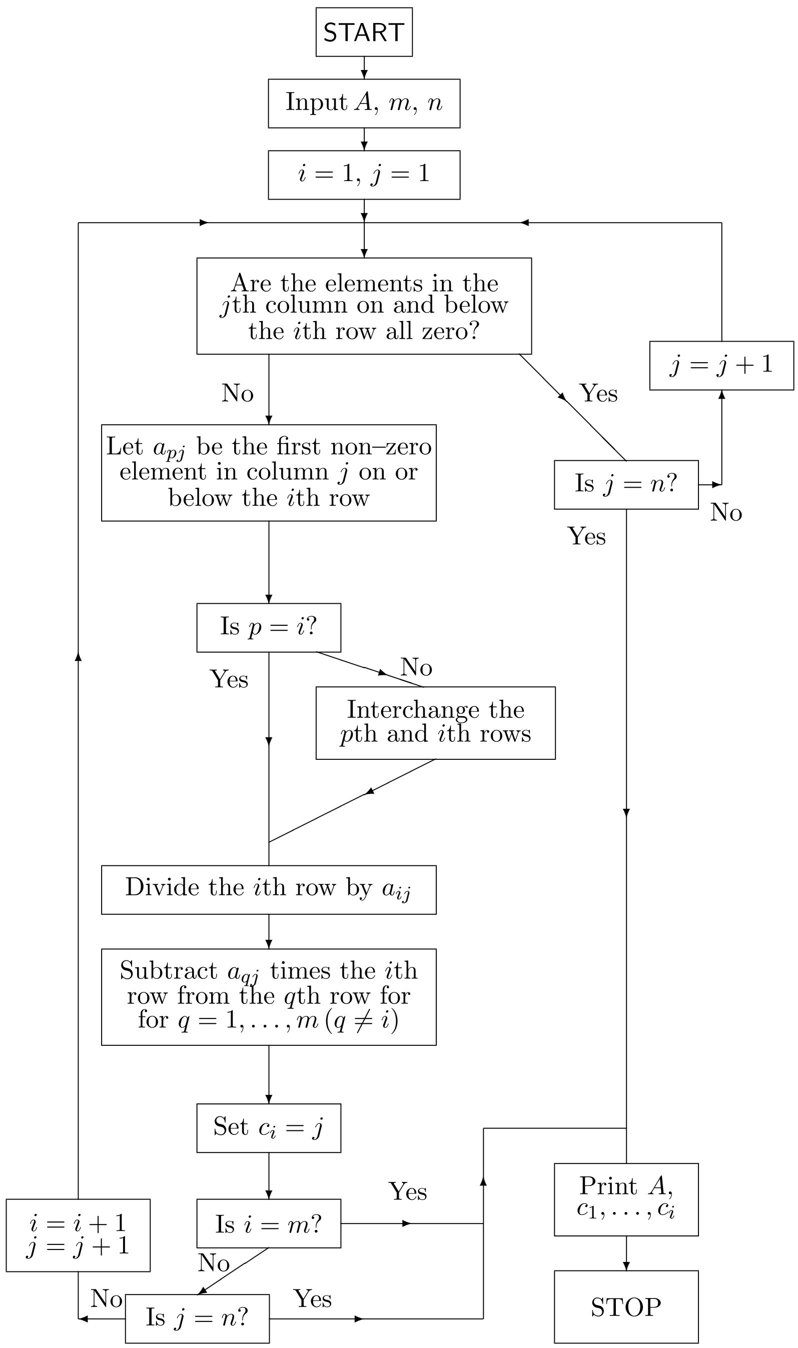

Figure 1: Figure form K. R. Mathews "Elementary linear algebra"

Example 1.3.1.

$$\begin{bmatrix} 0 & 0 & 4 & 0 \\ 2 & 2 & -2 & 5 \\ 5 & 5 & -1 & 5 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \begin{bmatrix} 2 & 2 & -2 & 5 \\ 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \end{bmatrix}$$

$$R_1 \to \frac{1}{2} R_1 \begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \end{bmatrix}$$

$$R_3 \to R_3 - 5R_1 \begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & -\frac{15}{2} \end{bmatrix}$$

$$R_2 \to \frac{1}{4} R_2 \begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -\frac{15}{2} \end{bmatrix}$$

$$\begin{cases} R_1 \to R_1 + R_2 \\ R_3 \to R_3 - 4R_2 \end{cases} \begin{bmatrix} 1 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{15}{2} \end{bmatrix}$$

$$R_3 \to -\frac{2}{15} R_3 \begin{bmatrix} 1 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \to R_1 - \frac{5}{2} R_3 \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Remark. It is possible to show that a given matrix is row-equivalent to *precisely one* matrix which is in reduced row-echelon form.

A flow-chart for the Gauss-Jordan algorithm from K. R. Mathews "Elementary linear algebra" is presented in figure 1. An implementation of the algorithm in Python is available on git.

Problem 1.3.1. Find reduced row-echelon forms which are row-equivalent to the following matrices and write down the elementary row operations used. You may use more efficient steps than the algorithm suggests, if you spot them.

(a) $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix}$

1.4 Systematic solution of linear systems

Suppose a system of m linear equations in n variables $x_1, \ldots, x_n$ has augmented matrix A and that A is row–equivalent to a matrix B which is in reduced row–echelon form, via the Gauss–Jordan algorithm. Then A and B are $m \times (n+1)$. Suppose that B has r non–zero rows and that the leading entry 1 in row i occurs in column number c_i, for $1 \leq i \leq r$. Then

$$1 \leq c_1 < c_2 < \cdots < c_r \leq n+1.$$

Also assume that the remaining column numbers are $c_{r+1}, \ldots, c_{n+1}$, where

$$1 \leq c_{r+1} < c_{r+2} < \cdots < c_n \leq n+1.$$

Case 1: $c_r = n+1$. The system is inconsistent. For the last row of B is $[0, 0, \ldots, 1]$, and the corresponding equation is

$$0x_1 + 0x_2 + \cdots + 0x_n = 1,$$

which has no solutions. Consequently the original system has no solutions.

Case 2: $c_r \leq n$. The system of equations corresponding to the non–zero rows of B is consistent. First notice that $r \leq n$ here.

If $r = n$, then $c_1 = 1, c_2 = 2, \ldots, c_n = n$ and

$$B = \begin{bmatrix} 1 & 0 & \ldots & 0 & d_1 \\ 0 & 1 & \ldots & 0 & d_2 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 1 & d_n \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

There is a unique solution $x_1 = d_1, x_2 = d_2, \ldots, x_n = d_n$.

If $r < n$, there will be more than one solution (infinitely many for real numbers). All solutions are obtained by taking the variables $x_{c_1}, \ldots, x_{c_r}$ as dependent variables and using the r equations corresponding to the non–zero rows of B to express these variables in terms of the remaining independent variables $x_{c_{r+1}}, \ldots, x_{c_n}$, which can take on arbitrary values:

$$\begin{aligned} x_{c_1} &= b_{1,n+1} - b_{1,c_{r+1}} x_{c_{r+1}} - \cdots - b_{1,c_n} x_{c_n} \\ &\vdots \\ x_{c_r} &= b_{r,n+1} - b_{r,c_{r+1}} x_{c_{r+1}} - \cdots - b_{r,c_n} x_{c_n} \end{aligned}$$

In particular, taking $x_{c_{r+1}} = 0, \ldots, x_{c_{n-1}} = 0$ and $x_{c_n} = 0, 1$ respectively, produce at least two solutions.

Example 1.4.1. Solve the following system:

$$\begin{aligned} x + y &= 0 \\ x - y &= 1 \\ 4x + 2y &= 1 \end{aligned}$$

Solution. The augmented matrix of the system is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

which is row-equivalent to

$$B = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

We read off the unique solution $x = \frac{1}{2}, y = -\frac{1}{2}$.
(Here $n = 2, r = 2, c_1 = 1, c_2 = 2$. Also $c_r = c_2 = 2 < 3 = n + 1$ and $r = n$.)

Example 1.4.2. Solve the system

$$\begin{aligned} 2x_1 + 2x_2 - 2x_3 &= 5 \\ 7x_1 + 7x_2 + x_3 &= 10 \\ 5x_1 + 5x_2 - x_3 &= 5 \end{aligned}$$

Solution. The augmented matrix of the system is

$$A = \begin{bmatrix} 2 & 2 & -2 & 5 \\ 7 & 7 & 1 & 10 \\ 5 & 5 & -1 & 5 \end{bmatrix}$$

which is row-equivalent to

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We read off the inconsistency for the original system.
(Here $n = 3, r = 3, c_1 = 1, c_2 = 3$. Also $c_r = c_3 = 4 = n + 1$.)

Example 1.4.3. Solve the system

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ x_1 + x_2 - x_3 &= 2 \end{aligned}$$

Solution. The augmented matrix of the system is

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

which is row-equivalent to

$$B = \begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & -1 & \frac{1}{2} \end{bmatrix}$$

The complete solution is $x_1 = \frac{3}{2}, x_2 = \frac{1}{2} + x_3$, with x_3 arbitrary.
(Here $n = 3, r = 2, c_1 = 1, c_2 = 2$. Also $c_r = c_2 = 2 < 4 = n + 1$ and $r < n$.)

Example 1.4.4. Solve the system

$$\begin{array}{lcl} 6x_3 + 2x_4 - 4x_5 - 8x^6 & = & 8 \\ 3x_3 + x_4 - 2x_5 - 4x^6 & = & 4 \\ 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 & = & 2 \\ 6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 & = & 1 \end{array}$$

Solution. The augmented matrix of the system is

$$A = \begin{bmatrix} 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 6 & -9 & 0 & 11 & -19 & 3 & 1 \end{bmatrix}$$

which is row-equivalent to

$$B = \begin{bmatrix} 1 & -\frac{3}{2} & 0 & \frac{11}{6} & -\frac{19}{6} & 0 & \frac{1}{24} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & 0 & \frac{5}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The complete solution is

$$\begin{aligned} x_1 &= \tfrac{1}{24} + \tfrac{3}{2}x_2 - \tfrac{11}{6}x_4 + \tfrac{19}{6}x_5 \\ x_3 &= \tfrac{5}{3} - \tfrac{1}{3}x_4 + \tfrac{2}{3}x_5 \\ x_6 &= \tfrac{1}{4} \end{aligned}$$

with x_2, x_4, x_5 arbitrary.
(Here $n = 6, r = 3, c_1 = 1, c_2 = 3, c_3 = 6$. Also $c_r = c_3 = 6 < 7 = n + 1$ and $r < n$.)

Example 1.4.5. Find the number t for which the following system is consistent and solve the system for this value of t.

$$\begin{array}{lcl} x + y & = & 2 \\ x - y & = & 0 \\ 3x - y & = & t \end{array}$$

Solution. The augmented matrix of the system is

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 3 & -1 & t \end{bmatrix}$$

which is row-equivalent to the simpler matrix

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & t-2 \end{bmatrix}$$

Hence if $t \neq 2$ the system is inconsistent. If $t = 2$ the system is consistent and we can read off the solution $x = 1, y = 1$ from B.

Example 1.4.6. For which values of a and b does the following system has (i) no solution, (ii) a unique solution, (iii) infinitely many solutions?

$$\begin{array}{rcl} x - 2y + 3z & = & 4 \\ 2x - 3y + az & = & 5 \\ 3x - 4y + 5z & = & b \end{array}$$

Solution. The augmented matrix of the system is

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 2 & -3 & a & 5 \\ 3 & -4 & 5 & b \end{bmatrix}$$

After elementary row operations $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_3 \to R_3 - 2R_2$ we obtain the following row equivalent matrix B:

$$B = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 0 & -2a+8 & b-6 \end{bmatrix}$$

Case 1. $a \neq 4$. Then $-2a + 8 \neq 0$ and we see that B can be reduced to a matrix of the form $\begin{bmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & \frac{b-6}{-2a+8} \end{bmatrix}$ and we have the unique solution $x = u, y = v, z = (b-6)/(-2a+8)$.

Case 2. $a = 4$. Then

$$B = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & b-6 \end{bmatrix}$$

If $b \neq 6$ we get no solution, whereas if $b = 6$ then B is row equivalent to the reduced row-echelon form

$$B = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We read off the complete solution $x = -2 + z, y = -3 + 2z$ with z arbitrary.

Problem 1.4.1. Find the value of t for which the following system is consistent and solve the system for this value of t.

$$\begin{array}{rcl} x + y & = & 1 \\ tx + y & = & t \\ (1+t)x + 2y & = & 3 \end{array}$$

Problem 1.4.2. For which values of a does the following system have (i) no solutions (ii) exactly one solution (iii) infinitely many solutions?

$$\begin{aligned}
x + 2y - 3z &= 4 \\
3x - y + 5z &= 2 \\
4x + y + (a^2 - 14)z &= a + 2
\end{aligned}$$