Mate1009 Algebra — Lecture 1

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Notes partially adapted from
K. R. Mathews "Elementary linear algebra"

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0 Overview

0.1 Course structure

The course will be remote on MS Teams. If, for some reason, you cannot get in touch with me on MS Teams, you can send me an email to martins.kalis.df@lu.lv. The preferred method is either a public question in the MS Teams course channel or private chat on MS Teams.

The lectures will take place on Wednesdays at 12:30–14:10 on MS Teams. I will try to record them, but I don't guarantee their availability. However, the lecture notes will definitely be available. I prefer that you join the lectures real-time, but that is not a requirement.

I have not set a time for office hours. Just let me know in a private chat if you need any help. We can then sort it out in the chat or arrange a time for a video call.

0.2 Grading

Your grade will consist of:

1. 25% (almost) weekly homework problems. The due date for each will usually be the next lecture. Grades for submissions after the due date will be reduced by half.

2. 25% mid-term exam (most likely during the 24 March lecture).

3. 50% final exam (June; date and time to be determined).

For both exams you will be expected to turn on your camera. You may use any non-electronic notes, books or print-outs, but no electronic resources. I might ask additional questions about your solutions after the exam.

To pass the course you need to score at least 40% on both exams, and at least 40% in total. To get the grade 10 for the course, you need to solve and submit the "honors option" problems. These will be published throughout the semester.

During the lecture we will also have unmarked quizzes — quick questions to check your understanding.

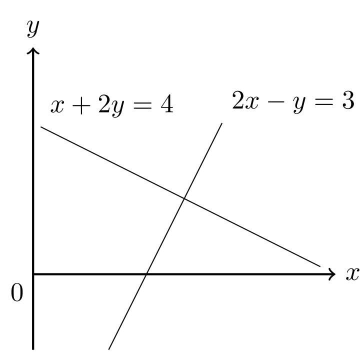
0.3 Contents of the course

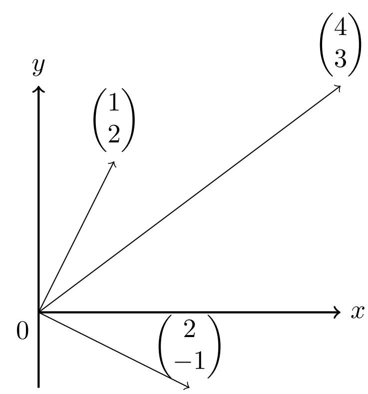
1. Systems of linear equations. Gauss-Jordan elimination.
2. Determinants. Cramer's rule. Determinant properties. Laplace expansion.
3. Complex numbers. Trigonometric form. De Moivre's formula. Roots.
4. Matrix algebra. Matrix inverse. Gauss-Jordan elimination revisited.
5. Algebra of polynomials. Euclidean algorithm. Greatest common divisor. Polynomial roots. Lagrange interpolation.
6. Fields, rings and groups.

1 Systems of linear equations

The simplest non-trivial case of a system of linear equations is two unknowns and two equations. This simple example shows two ways we can visualise the solution of this system of equations — either as an intersection of two lines or addition of vectors.

$$\begin{cases} x + 2y = 4 \\ 2x - y = 3 \end{cases} \qquad\qquad \begin{pmatrix} 1 \\ 2 \end{pmatrix} x + \begin{pmatrix} 2 \\ -1 \end{pmatrix} y = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$





There are several directions we might go to solve this system of linear equations:

- use an advanced calculator, e.g. wolframalpha.com [practical up to a point; does not give insight in the process],
- express one variable with the other, substitute, solve [gets unwieldy for more variables],
- add multiples of one equation to eliminate a variable from the other equation [great].

This simple example also allows to illustrate some categories of systems of linear equations according to their solutions:

- A system might have exactly one solution, as is the case with the given system. The solution is $x = 2, y = 1$. Notice that on the left graph the lines cross at point $(2, 1)$, and on the right graph $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ will lead to vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Such a system is called *consistent* and *independent*.
- A system might have infinitely many solutions — try removing any of the equations from the given system. Such a system is called *consistent* and *dependent*.
- A system might have no solutions (try adding any equation that does not have $(2, 1)$ as its solution to the given system, e.g. $x - y = 0$). Such a system is called *inconsistent*.

Example 1.0.1. *Solve the equation* $2x + 3y = 6$.

Solution. The equation $2x+3y = 6$ is equivalent to $2x = 6-3y$ or $x = 3-3/2y$, where y is arbitrary. So there are infinitely many solutions.

Example 1.0.2. *Solve the system*

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 0 \end{aligned}$$

Solution. We subtract the second equation from the first to get $2y = 1$ and $y = 1/2$. Then $x = 1/2 - z$, where z is arbitrary. Again there are infinitely many solutions.

Example 1.0.3. *Find a polynomial of the form* $y = a_0 + a_1x + a_2x^2$ *which passes through points* $(-1, 2), (1, 5), (2, 1)$.

Solution. When x has the values $-1, 1, 2$, then y takes values $2, 5, 1$ correspondingly, and we get three equations with variables a_0, a_1, a_2:

$$\begin{aligned} a_0 - a_1 + a_2 &= 2 \\ a_0 + a_1 + a_2 &= 5 \\ a_0 + 2a_1 + 4a_2 &= 1 \end{aligned}$$

with unique solutions $a_0 = 16/3, a_1 = 3/2, a_2 = -11/6$. So the required polynomial is $y = 16/3 + 3/2x - 11/6x^2$.

1.1 Representing linear equations

A *linear equation* with n variables is an equation in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n, b$ are given real numbers. For small values of n we can visualise the solution to such an equation (the following assumes that none of the $a_1, \ldots, a_n$ are equal to 0):

- for $n = 1$ such an equation describes a point on a line,
- for $n = 2$ such an equation describes a line in a plane,
- for $n = 3$ such an equation describes a plane in a space,
- for $n > 3$ visualisation gets tough.

A *system* of m linear equations with n variables is a set of linear equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

We wish to determine if such a system has a solution, that is to find out if there exist numbers $x_1, x_2, \ldots, x_n$ which satisfy each of the equations simultaneously. We say that the system is *consistent* if it has one or more solutions. Otherwise the system is called *inconsistent*. A consistent system is called *independent* if it has exactly one solution. Otherwise it is called *dependent*.

The following matrix is called the *coefficient matrix* of the system:

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22} & \ldots & a_{2n}\\ \vdots & & & \vdots\\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

The following matrix is called the *augmented matrix* of the system (the separating vertical line is optional):

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1\\ a_{21} & a_{22} & \ldots & a_{2n} & b_2\\ \vdots & & & \vdots & \vdots\\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right]$$

1.2 Solving linear equations — Gauss-Jordan algorithm

Definition 1.2.1 (Row-echelon form)**.** A matrix is in *row-echelon* form if

1. all zero rows (if any) are at the bottom of the matrix and
2. if two successive rows are non-zero, the second row starts with more zeros than the first (moving from left to right).

For example, the following matrix is in row-echelon form:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The following matrix is not in row-echelon form:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The *zero* matrix of any size is always in row-echelon form.

Definition 1.2.2 (Reduced row-echelon form)**.** A matrix is in *reduced row-echelon form* if:

1. it is in row-echelon form,
2. the leading (leftmost non-zero) entry in each non-zero row is 1,
3. all other elements of the column in which the leading entry 1 occurs are zeros.

For example the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

are in reduced row-echelon form, whereas the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are not in reduced row-echelon form, although they are in row-echelon form. The *zero* matrix of any size is always in reduced row-echelon form.

Definition 1.2.3 (Elementary row operations)**.** Three types of *elementary row operations* can be performed on matrices:

1. Interchanging two rows:
   $R_i \leftrightarrow R_j$ interchanges rows i and j.
2. Multiplying a row by a non-zero scalar:
   $R_i \to tR_i$ multiplies row i by the non-zero scalar t.
3. Adding a multiple of one row to another row:
   $R_j \to R_j + tR_i$ adds t times row i to row j.

Definition 1.2.4 (Row equivalence)**.** Matrix A is *row-equivalent* to matrix B is B is obtained from A by a sequence of elementary row operations.

Example 1.2.1.

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \\ R_2 \to R_2 + 2R_3 &\begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 5 \\ 1 & -1 & 2 \end{bmatrix} \\ R_2 \leftrightarrow R_3 &\begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 2 \\ 4 & -1 & 5 \end{bmatrix} \\ R_1 \to 2R_1 &\begin{bmatrix} 2 & 4 & 0 \\ 1 & -1 & 2 \\ 4 & -1 & 5 \end{bmatrix} = B \end{aligned}$$

Thus matrix A is row equivalent to matrix B. Clearly B is also row equivalent to A, by performing the inverse row-operations:

$$R_1 \to \frac{1}{2}R_1, R_2 \leftrightarrow R_3, R_2 \to R_2 - 2R_3$$

on B.

It is not difficult to prove that if A and B are row-equivalent augmented matrices of two systems of linear equations, then the two systems have the same solution sets — a solution of one of the systems is a solution of the other. For example, the systems whose augmented matrices are A and B in the example above are respectively

$$\begin{cases} x + 2y &= 0 \\ 2x + y &= 1 \\ x - y &= 2 \end{cases} \qquad \text{and} \qquad \begin{cases} 2x + 4y &= 0 \\ x - y &= 2 \\ 4x - y &= 5 \end{cases}$$

and these systems have precisely the same solutions.

Problem 1.2.1. Which of the following matrices are in reduced row-echelon form?

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$