MA684 - Advanced Probability
The Fubini - Tonelli Theorems

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1 Preliminaries

Suppose we have two measure spaces $(\Omega_i, \mathcal{F}_i, \mu_i)$, $i = 1, 2$. We wish to construct a measure $\mu := \mu_1 \times \mu_2$ on a $\sigma-$algebra over the product space $\Omega_1 \times \Omega_2$ such that $\mu(A \times B) = \mu_1(A)\mu_2(B)$ for $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$.

We use the following definitions:

Definition 1.1.

1. $\Omega_1 \times \Omega_2 := \{(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$, the usual Cartesian product.
2. For $A \in \mathcal{F}_1, B \in \mathcal{F}_2$, we call $A \times B$ a measurable rectangle. We denote the collection of measurable rectangles as $\mathcal{C}$.
3. We denote the product $\sigma-$algebra corresponding to $\mathcal{F}_1$ and $\mathcal{F}_2$ over Ω_1 and Ω_2 as,
$$\mathcal{F}_1 \times \mathcal{F}_2 := \sigma{<}\{A \times B : A \in \mathcal{F}_1, B \in \mathcal{F}_2\}{>} = \sigma(\mathcal{C})$$
4. We call $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2)$ the product measurable space.

Lemma 1.1. $\mathcal{C}$ is a semi-algebra.

Proof.

(a) Clearly, $\phi \in \mathcal{C}$.

(b) Let $E, F \in \mathcal{C}$. So $E = A_1 \times B_1$ and $F = A_2 \times B_2$ for $A_1, A_2 \in \mathcal{F}_1$ and $B_1, B_2 \in \mathcal{F}_2$.
$$E \cap F = (A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2) \in \mathcal{C}$$
as $A_1 \cap A_2 \in \mathcal{F}_1$ and $B_1 \cap B_2 \in \mathcal{F}_2$. So, $\mathcal{C}$ is closed with respect to intersection.

(c) Also, $E^c = (A_1 \times B_1)^c = (A_1^c \times B_1) \cup (A_1 \times B_1^c) \cup (A_1^c \times B_1^c)$. Each set in the union is a member of $\mathcal{C}$ as both $\mathcal{F}_1$ and $\mathcal{F}_2$ are closed with respect to complementation. So, the complement of any set in $\mathcal{C}$ is the disjoint union of members of $\mathcal{C}$.

We have shown that all properties of a semi-algebra are satisfied. $\square$

Theorem 1.2. The set function $\mu : \mathcal{C} \to [0, \infty]$ defined by,

$$\mu(A \times B) = \mu_1(A)\mu_2(B)$$

is a measure on the semi-algebra $\mathcal{C}$.

Proof. (a) Clearly, $\mu(\phi) = 0$.

(b) Let $\{A_n \times B_n\}_{n \geq 1}$ be a countable collection of disjoint sets from $\mathcal{C}$ with union $\{A \times B\}$. Let $x \in \Omega_1, y \in \Omega_2$. So,

$$\mathbb{I}_A(x)\mathbb{I}_B(y) = \mathbb{I}_{A \times B}(x, y) = \sum_{n=1}^{\infty} \mathbb{I}_{A_n \times B_n}(x, y)$$

as $\bigcup_{n \geq 1}(A_n \times B_n) = A \times B$.

Consider,

$$\mu_1(A)\mathbb{I}_B(y) = \int_{\Omega_1} \mathbb{I}_A(x)\mathbb{I}_B(y) d\mu_1(x) = \int_{\Omega_1} \sum_{n=1}^{\infty} \mathbb{I}_{A_n \times B_n}(x, y) d\mu_1(x)$$

Using Monotone Convergence theorem,

$$\begin{aligned}\mu_1(A)\mathbb{I}_B(y) &= \sum_{n=1}^{\infty} \int_{\Omega_1} \mathbb{I}_{A_n \times B_n}(x, y) d\mu_1(x) = \sum_{n=1}^{\infty} \int_{\Omega_1} \mathbb{I}_{A_n}(x)\mathbb{I}_{B_n}(y) d\mu_1(x) \\ &= \sum_{n=1}^{\infty} \mu_1(A_n)\mathbb{I}_{B_n}(y)\end{aligned}$$

Now, this is an integrable function on the second measure space. Computing this integration using the Monotone Convergence theorem again,

$$\begin{aligned}\int_{\Omega_2} \mu_1(A)\mathbb{I}_B(y) d\mu_2(y) &= \sum_{n=1}^{\infty} \int_{\Omega_2} \mu_1(A_n)\mathbb{I}_{B_n}(y) d\mu_2(y) \\ &= \sum_{n=1}^{\infty} \mu_1(A_n)\mu_2(B_n) = \sum_{n=1}^{\infty} \mu(A_n \times B_n)\end{aligned}$$

by definition. This establishes countable additivity.

$\square$

Now that we have seen $\mathcal{C}$ is a semi-algebra and have a measure on it, we may extend the measure to its natural algebra $\mathcal{A}$ and further use the Caratheodory extension theorem to get a complete measure on a $\sigma-$algebra containing the product $\sigma-$algebra. However, computing $\mu(A)$ for arbitrary A in such a $\sigma-$algebra and computing integrals of arbitrary measurable functions is not straightforward in this case. So, we look at an alternate approach that gives a direct way of evaluating these quantities.

To this end, we first consider the following definitions,

Definition 1.2. (a) Let $A \in \mathcal{F}_1 \times \mathcal{F}_2$. Then, the ω_1-section of A for $\omega_1 \in \Omega_1$, is defined by,

$$A_{1\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in A\}$$

Similarly we define the ω_2-section as,

$$A_{2\omega_2} = \{\omega_1 \in \Omega_1 : (\omega_1, \omega_2) \in A\}$$

(b) Let $f : \Omega_1 \times \Omega_2 \to \Omega_3$ be a $\langle \mathcal{F}_1 \times \mathcal{F}_2, \mathcal{F}_3 \rangle$measurable function where $(\Omega_3, \mathcal{F}_3)$ is a measurable space. We define the ω_1 and ω_2 sections of f as the functions,

$$f_{1\omega_1}(\omega_2) = f(\omega_1, \omega_2), \quad \omega_2 \in \Omega_2$$

$$f_{2\omega_2}(\omega_1) = f(\omega_1, \omega_2), \quad \omega_1 \in \Omega_1$$

Clearly, $f_{1\omega_1} : \Omega_2 \to \Omega_3$ and $f_{2\omega_2} : \Omega_1 \to \Omega_3$.

Lemma 1.3. $f : \Omega_1 \times \Omega_2 \to \Omega_3$ be a $\langle \mathcal{F}_1 \times \mathcal{F}_2, \mathcal{F}_3 \rangle$measurable function where $(\Omega_3, \mathcal{F}_3)$ is a measurable space. We have the following properties,

1. For every $\omega_1 \in \Omega_1$, $A_{1\omega_1} \in \mathcal{F}_2$ and for every $\omega_2 \in \Omega_2$, $A_{2\omega_2} \in \mathcal{F}_1$.
2. For every $\omega_1 \in \Omega_1$, $f_{1\omega_1}$ is $\langle \mathcal{F}_2, \mathcal{F}_3 \rangle$measurable and for every $\omega_2 \in \Omega_2$, $f_{2\omega_2}$ is $\langle \mathcal{F}_1, \mathcal{F}_3 \rangle$measurable.

Proof. Let $\omega_1 \in \Omega_1$ be fixed. Consider $g : \Omega_2 \to \Omega_1 \times \Omega_2$ defined by,

$$g(\omega_2) = (\omega_1, \omega_2), \quad \omega_2 \in \Omega_2$$

Let $R \in \mathcal{C}$. So, $R = A \times B$ for $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Clearly,

$$g^{-1}(A \times B) = \begin{cases} B & \omega_1 \in A \\ \phi & \omega_1 \notin A \end{cases}$$

As both $B, \phi \in \mathcal{F}_2$, $g^{-1}(A \times B) \in \mathcal{F}_2$. That is, for all $R \in \mathcal{C}$, $g^{-1}(R) \in \mathcal{F}_2$. However, $\mathcal{F}_1 \times \mathcal{F}_2 = \sigma(\mathcal{C})$. Thus, we have shown that g is $\langle \mathcal{F}_2, \mathcal{F}_1 \times \mathcal{F}_2 \rangle$measurable.

So, for any $A \in \mathcal{F}_1 \times \mathcal{F}_2$, by definition, $A_{1\omega_1} = g^{-1}(A) \in \mathcal{F}_2$ because of g's measurability we just proved.

Similarly, $f_{1\omega_1} = f \circ g$. As f and g are measurable with respect to their spaces, we can say that their composition is measurable as well. That is, $f_{1\omega_1}$ is $\langle \mathcal{F}_2, \mathcal{F}_3 \rangle$measurable.

We may similarly prove the analogous results for the ω_2-sections. $\square$

For $A \in \mathcal{F}_1 \times \mathcal{F}_2$, as we have just shown, $A_{1\omega_1} \in \mathcal{F}_2$ and $A_{2\omega_2} \in \mathcal{F}_1$. So, $\mu_2(A_{1\omega_1})$ and $\mu_1(A_{2\omega_2})$ are well-defined, which are functions of ω_1 and ω_2 respectively. If these functions are $\mathcal{F}_1$ and $\mathcal{F}_2$ measurable respectively, then we could define the measures over $\mathcal{F}_1 \times \mathcal{F}_2$,

$$\mu_{12}(A) = \int_{\Omega_1} \mu_2(A_{1\omega_1}) d\mu_1(\omega_1) \tag{1}$$

$$\mu_{21}(A) = \int_{\Omega_2} \mu_1(A_{2\omega_2}) d\mu_2(\omega_2) \tag{2}$$

(We have that for a space $(\Omega, \mathcal{F}, \mu)$ and any $\mathcal{F}$ measurable f, $\mu'(A) = \int_A f d\mu$ defines a measure over $(\Omega, \mathcal{F})$)

Note that if $A \in \mathcal{C}$, then $\mu_{12}(A) = \mu_{21}(A) = \mu_1(A_1)\mu_2(A_2)$ where $A = A_1 \times A_2$. So, if we can show that μ_{12} and μ_{21} are indeed measures and if we can show that there is a unique product measure, μ on $\mathcal{F}_1 \times \mathcal{F}_2$ such that $\mu(A \times B) = \mu_1(A)\mu_2(B), A \in \mathcal{F}_1, B \in \mathcal{F}_2$, then we can say that μ, μ_{12} and μ_{21} all coincide for sets in the product $\sigma-$algebra. This will allow us to find the measure of any set with the integrals (1) or (2). We state the following theorem which solidifies these ideas.

Theorem 1.4. Suppose the spaces $(\Omega_i, \mathcal{F}_i, \mu_i), i = 1, 2$ are $\sigma-$finite. Then,

1. For all $A \in \mathcal{F}_1 \times \mathcal{F}_2$, the functions $\mu_2(A_{1\omega_1})$ and $\mu_1(A_{2\omega_2})$ are $\mathcal{F}_1$ and $\mathcal{F}_2$ measurable functions respectively.

2. The measures μ_{12} and μ_{21} define measures over $\mathcal{F}_1 \times \mathcal{F}_2$ such that $\mu_{12}(A) = \mu_{21}(A)$ for all $A \in \mathcal{F}_1 \times \mathcal{F}_2$.

3. Also, μ_{12} and μ_{21} are the only measures μ such that,

   $$\mu(E \times F) = \mu_1(E)\mu_2(F)$$

   for all $E \times F \in \mathcal{C}$.

Remark 1.1. Note that the $\sigma-$finiteness assumption cannot be dropped. For example, let $\Omega_1 = \Omega_2 = [0,1]$ and $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{B}([0,1])$. Let μ_1 be the Lebesgue measure and μ_2 be the counting measure over $\mathcal{B}([0,1])$ and $A = \{(\omega_1, \omega_2) : \omega_1 = \omega_2\}$. Note that $(\Omega_2, \mathcal{F}_2, \mu_2)$ is not $\sigma-$finite as $[0,1]$ is uncountable. Clearly, $\mu_2(A_{1\omega_1}) = 1$ and $\mu_1(A_{2\omega_2}) = 0$ for any $\omega_1, \omega_2 \in [0,1]$.

So, $\mu_{12}(A) = \int_{\Omega_1} \mu_2(A_{1\omega_1}) d\mu_1(\omega_1) = 1$ and $\mu_{21}(A) = \int_{\Omega_2} \mu_1(A_{2\omega_2}) d\mu_2(\omega_2) = 0$.

This allows us now to define the product measure space as follows,

Definition 1.3. The unique measure in Theorem 1.4 is called the product measure, denoted by $\mu_1 \times \mu_2$. We call $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mu_1 \times \mu_2)$ the product measure space.

In the next section, we discuss the integration of measurable functions $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ with respect to $\mu_1 \times \mu_2$ and justify iterated integration with the Fubini-Tonelli theorems.

2 Fubini-Tonelli theorems

Let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a $\langle \mathcal{F}_1 \times \mathcal{F}_2, \mathcal{B}(\mathbb{R})\rangle$measurable function. In this section, we cover the conditions under which we may split the integral of f over $\Omega_1 \times \Omega_2$ into iterated integrals over its sections. That is,

(In the following discussion, we assume $(\Omega_i, \mathcal{F}_i, \mu_i)$ $i = 1,2$ are $\sigma-$finite measure spaces use the notation $\mu = \mu_1 \times \mu_2$)

$$\int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2) d\mu(\omega_1, \omega_2) = \int_{\Omega_1} \left[\int_{\Omega_2} f(\omega_1, \omega_2) d\mu_2(\omega_2) \right] d\mu_1(\omega_1)$$

or

$$\int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2) d\mu(\omega_1, \omega_2) = \int_{\Omega_2} \left[\int_{\Omega_1} f(\omega_1, \omega_2) d\mu_1(\omega_1) \right] d\mu_2(\omega_2)$$

The conditions under which the above equalities hold are given by the Fubini and Tonelli theorems which we state below.

Theorem 2.1. (Tonelli's Theorem) Let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}_+$ be a non-negative $\langle \mathcal{F}_1 \times \mathcal{F}_2, \mathcal{B}(\mathbb{R})\rangle$measurable function. Then,

1. The functions $g_1 : \Omega_1 \to \bar{\mathbb{R}}$ and $g_2 : \Omega_2 \to \bar{\mathbb{R}}$ defined below are $\langle \mathcal{F}_1, \mathcal{B}(\bar{\mathbb{R}})\rangle$and $\langle \mathcal{F}_2, \mathcal{B}(\bar{\mathbb{R}})\rangle$measurable respectively.

$$g_1(\omega_1) = \int_{\Omega_2} f(\omega_1, \omega_2) d\mu_2(\omega_2)$$
$$g_2(\omega_2) = \int_{\Omega_1} f(\omega_1, \omega_2) d\mu_1(\omega_1)$$

2. Iterated integrals are equal to the full integrals. That is,

$$\int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2) d\mu(\omega_1, \omega_2) = \int_{\Omega_1} g_1(\omega_1) d\mu_1(\omega_1) = \int_{\Omega_2} g_2(\omega_2) d\mu_2(\omega_2)$$

Proof. First, consider $f = \mathbb{I}_A$ for $A \in \mathcal{F}_1 \times \mathcal{F}_2$. Clearly,

$$g_1(\omega_1) = \int_{\Omega_2} \mathbb{I}_A d\mu_2(\omega_2) = \int_{A_{1\omega 1}} d\mu_2(\omega_2) = \mu_2(A_{1\omega_1})$$

which was shown to be $\langle \mathcal{F}_1, \mathcal{B}(\bar{\mathbb{R}}) \rangle$measurable in Theorem 1.4. We may use the same idea for g_2 as well.

Again, from Theorem 1.4, we see that,

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2) d\mu(\omega_1, \omega_2) &= \int_{\Omega_1} \mu_2(A_{1\omega_1}) d\mu_1(\omega_1) = \int_{\Omega_2} \mu_1(A_{2\omega_2}) d\mu_2(\omega_2) \\ &= \mu(A) \end{aligned}$$

By linearity of integrals, we can show that the relations hold for simple non-negative functions as well.

Now, for any general non-negative measurable function f, there exists a sequence of non-negative simple functions $\{f_n\}$ such that $f_n(\omega_1, \omega_2) \uparrow f(\omega_1, \omega_2) \quad \forall (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$.

Let $g_{1n}(\omega_1) = \int_{\Omega_2} f_n(\omega_1, \omega_2) d\mu_2(\omega_2)$. Clearly $\{g_{1n}\}$ is non-decreasing and $\langle \mathcal{F}_1, \mathcal{B}(\bar{\mathbb{R}}) \rangle$measurable as well. Using MCT, we get,

$$\begin{aligned} \lim_{n \to \infty} g_{1n} &= \lim_{n \to \infty} \int_{\Omega_2} f_n(\omega_1, \omega_2) d\mu_2(\omega_2) \\ &= \int_{\Omega_2} \lim_{n \to \infty} f_n(\omega_1, \omega_2) d\mu_2(\omega_2) \\ &= \int_{\Omega_2} f(\omega_1, \omega_2) d\mu_2(\omega_2) = g_1(\omega_1) \end{aligned}$$

So, g_1 is $\langle \mathcal{F}_1, \mathcal{B}(\bar{\mathbb{R}}) \rangle$measurable as it is the limit of $\langle \mathcal{F}_1, \mathcal{B}(\bar{\mathbb{R}}) \rangle$measurable functions. Following exactly similar arguments we can show that g_2 is $\langle \mathcal{F}_2, \mathcal{B}(\bar{\mathbb{R}}) \rangle$measurable as well.

Finally, using MCT,

$$\int_{\Omega_1\times\Omega_2} f(\omega_1,\omega_2)d\mu = \lim_{n\to\infty}\int_{\Omega_1\times\Omega_2} f_n(\omega_1,\omega_2)d\mu$$

However, f_n is simple, so the iterated integrals are equal to the full integral as we had shown previously. That is,

$$\begin{aligned}\int_{\Omega_1\times\Omega_2} f(\omega_1,\omega_2)d\mu &= \lim_{n\to\infty}\int_{\Omega_1\times\Omega_2} f_n(\omega_1,\omega_2)d\mu\\ &= \lim_{n\to\infty}\int_{\Omega_1} g_{1n}(\omega_1)d\mu_1(\omega_1) = \lim_{n\to\infty}\int_{\Omega_2} g_{2n}(\omega_2)d\mu_2(\omega_2)\end{aligned}$$

As the $\{g_{in}\}$ are non-decreasing and have limits g_i for $i = 1, 2$, we may use the MCT to finally obtain,

$$\begin{aligned}\int_{\Omega_1\times\Omega_2} f(\omega_1,\omega_2)d\mu &= \lim_{n\to\infty}\int_{\Omega_1} g_{1n}(\omega_1)d\mu_1(\omega_1) = \lim_{n\to\infty}\int_{\Omega_2} g_{2n}(\omega_2)d\mu_2(\omega_2)\\ &= \int_{\Omega_1}\lim_{n\to\infty} g_{1n}(\omega_1)d\mu_1(\omega_1) = \int_{\Omega_2}\lim_{n\to\infty} g_{2n}(\omega_2)d\mu_2(\omega_2)\\ &= \int_{\Omega_1} g_1(\omega_1)d\mu_1(\omega_1) = \int_{\Omega_2} g_2(\omega_2)d\mu_2(\omega_2)\end{aligned}$$

Which is the desired result. □

While Tonelli's theorem shows that iterated integrals are equal to the full integrals for non-negative measurable functions, Fubini's theorem states that it is in general true for functions in $L^1(\Omega_1\times\Omega_2, \mathcal{F}_1\times\mathcal{F}_2, \mu = \mu_1\times\mu_2)$, i.e, functions such that $\int_{\Omega_1\times\Omega_2}|f|d\mu < \infty$.

Theorem 2.2. (Fubini's Theorem) Let $f \in L^1(\Omega_1\times\Omega_2, \mathcal{F}_1\times\mathcal{F}_2, \mu)$. Then there exist sets $B_i \in \mathcal{F}_i$, $i = 1, 2$ such that,

1. $\mu_i(\Omega_i\backslash B_i) = 0$ for $i = 1, 2$
2. For any $\omega_1 \in B_1$, $f(\omega_1,\cdot) \in L^1(\Omega_2, \mathcal{F}_2, \mu_2)$ and for any $\omega_2 \in B_2$, $f(\cdot,\omega_2) \in L^1(\Omega_1, \mathcal{F}_1, \mu_1)$.
3. The function g_1 defined by,

$$g_1(\omega_1) = \begin{cases}\int_{\Omega_2} f(\omega_1,\omega_2)d\mu_2(\omega_2) & \omega_1 \in B_1\\ 0 & \omega_1 \notin B_1\end{cases}$$

is $\mathcal{F}_1$ measurable.

4. The function g_2 defined by,

$$g_2(\omega_2) = \begin{cases} \int_{\Omega_1} f(\omega_1, \omega_2) d\mu_1(\omega_1) & \omega_2 \in B_2 \\ 0 & \omega_2 \notin B_2 \end{cases}$$

is $\mathcal{F}_2$ measurable.

5. The functions g_1 and g_2 satisfy,

$$\int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2) d\mu(\omega_1, \omega_2) = \int_{\Omega_1} g_1(\omega_1) d\mu_1(\omega_1) = \int_{\Omega_2} g_2(\omega_2) d\mu_2(\omega_2)$$

That is, Fubini's theorem states that if f is integrable ($f \in L^1$), then the sectional functions of f are well-defined almost everywhere and the iterated integrals are equal to the full integral.

Proof. As $f \in L^1(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mu)$,

$$\int_{\Omega_1 \times \Omega_2} |f(\omega_1, \omega_2)| d\mu(\omega_1, \omega_2) < \infty$$

Using Tonelli's theorem (as $|f| > 0$),

$$\int_{\Omega_1 \times \Omega_2} |f(\omega_1, \omega_2)| d\mu(\omega_1, \omega_2) = \int_{\Omega_1} \left(\int_{\Omega_2} |f(\omega_1, \omega_2)| d\mu_2(\omega_2) \right) d\mu_1(\omega_1) < \infty$$

So, we know that $\mu_1(\Omega_1 \backslash B_1) = 0$ where $B_1 = \{\omega_1 : \int_{\Omega_2} |f(\omega_1, \cdot)| d\mu_2(\omega_2) < \infty\}$. That is, $f(\omega_1, \cdot) \in L^1(\Omega_2, \mathcal{F}_2, \mu_2)$ for $\omega_1 \in B_1$. So, we have shown points (1) and (2) to hold for one of the sectional functions. Exactly similar arguments for the other section will give us the result for it as well.

As both f^+ and f^- are non-negative, by Tonelli's theorem, the functions

$$g_{11}(\omega_1) = \int_{\Omega_2} f^+(\omega_1, \omega_2) d\mu_2(\omega_2)$$

and

$$g_{12}(\omega_1) = \int_{\Omega_2} f^-(\omega_1, \omega_2) d\mu_2(\omega_2)$$

are $\mathcal{F}_1$ measurable and,

$$\int_{\Omega_1} g_{11}(\omega_1) d\mu_1(\omega_1) = \int_{\Omega_1 \times \Omega_2} f^+(\omega_1, \omega_2) d\mu(\omega_1, \omega_2) \tag{3}$$

$$\int_{\Omega_1} g_{12}(\omega_1) d\mu_1(\omega_1) = \int_{\Omega_1 \times \Omega_2} f^-(\omega_1, \omega_2) d\mu(\omega_1, \omega_2) \tag{4}$$

Clearly, g_1 as defined in point 3 can be written as, $g_1 = (g_{11} - g_{12})\mathbb{I}_{B_1}$. The RHS is $\mathcal{F}_1$ measurable so g_1 is $\mathcal{F}_1$ measurable as well. Similar arguments hold for g_2. This proves points 3 and 4.

Further from (3) and (4),

$$\begin{aligned}
\int_{\Omega_1} |g_1(\omega_1)| d\mu_1(\omega_1) &\leq \int_{\Omega_1} g_{11} d\mu_1(\omega_1) + \int_{\Omega_1} g_{12} d\mu_1(\omega_1) \\
&= \int_{\Omega_1 \times \Omega_2} f^+(\omega_1, \omega_2) d\mu(\omega_1, \omega_2) + \int_{\Omega_1 \times \Omega_2} f^-(\omega_1, \omega_2) d\mu(\omega_1, \omega_2) \\
&= \int_{\Omega_1 \times \Omega_2} |f(\omega_1, \omega_2)| d\mu(\omega_1, \omega_2) < \infty
\end{aligned}$$

That is, g_1 is integrable (similarly, g_2 as well).

So, noting that $\mu_1(\Omega_1 \backslash B_1) = 0$ and using (3) and (4),

$$\begin{aligned}
\int_{\Omega_1} g_1(\omega_1) d\mu_1(\omega_1) &= \int_{\Omega_1} ((g_{11} - g_{12})\mathbb{I}_{B_1})(\omega_1) d\mu_1(\omega_1) \\
&= \int_{\Omega_1} g_{11}(\omega_1)\mathbb{I}_{B_1}(\omega_1) d\mu_1(\omega_1) - \int_{\Omega_1} g_{12}(\omega_1)\mathbb{I}_{B_1}(\omega_1) d\mu_1(\omega_1) \\
&= \int_{\Omega_1} g_{11}(\omega_1) d\mu_1(\omega_1) - \int_{\Omega_1} g_{12}(\omega_1) d\mu_1(\omega_1) \\
&= \int_{\Omega_1 \times \Omega_2} f^+(\omega_1, \omega_2) d\mu(\omega_1, \omega_2) - \int_{\Omega_1 \times \Omega_2} f^-(\omega_1, \omega_2) d\mu(\omega_1, \omega_2) \\
&= \int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2) d\mu(\omega_1, \omega_2)
\end{aligned}$$

which proves point 5 for $i = 1$. Exactly similar arguments hold to prove 5 for $i = 2$. $\square$